

# Ideals of Homogeneous Polynomials

by

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## Abstract

Given a surjective ideal of operators, we undertake a new general procedure to construct an ideal of polynomials. The relation with the ideal of polynomials obtained by the well-known method of composition is established.

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## §1. Introduction

Ever since Pietsch [16] initiated a research program on ideals of nonlinear operators, many authors have turned their attention to studying ideals of multilinear mappings. The transition from the multilinear setting to homogeneous polynomials occurred naturally.

Several strategies have been developed when trying to generalize ideals of operators to polynomials. A standard procedure is to mimic the property that characterizes the operators belonging to some operator ideal. For instance, (weakly) compact  $m$ -homogeneous polynomials are defined as those that map bounded sets to relatively (weakly) compact sets, copying the definition of a (weakly) compact operator. This technique, although useful in order to get the first basic examples, lacks the generality required in order to treat polynomial ideals in a systematic way. In this paper we undertake a study of a general method to generate an ideal of polynomials from an operator ideal  $\mathcal{I}$ . To accomplish this task, we introduce the concept of local  $\mathcal{I}$ -boundedness. We prove that the set of homogeneous polynomials that are locally  $\mathcal{I}$ -bounded form a Banach ideal of polynomials when endowed

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with a naturally defined norm. We consider only a particular case of the well-known method called *composition ideal* to generate polynomial ideals from a given operator ideal  $\mathcal{I}$ . This method consists in composing continuous  $m$ -homogeneous polynomials with operators belonging to  $\mathcal{I}$ . The composition ideal is formed by all polynomials  $P$  that factor as  $P = T \circ Q$ , where  $Q$  is a polynomial and  $T$  belongs to the ideal  $\mathcal{I}$  (see Section 2 for a formal description). From [19] it follows that compact and weakly compact polynomials form a composition ideal of polynomials. In [3] the present authors proved that  $p$ -compact homogeneous polynomials (mapping bounded sets to relatively  $p$ -compact sets) also form a composition ideal of polynomials.

By [19] any continuous  $m$ -homogeneous polynomial  $P$  is associated to a unique linear operator  $P_L$  called the *linearization* of  $P$ . This linearization yields another way to generate ideals of polynomials, by simply considering those continuous  $m$ -homogeneous polynomials whose linearization belongs to a given operator ideal  $\mathcal{I}$ . The polynomial ideal generated in this way turns out to be the same as the composition ideal of polynomials (see [6]).

After reviewing some preliminaries in Section 2, given a Banach operator ideal  $\mathcal{I}$ , in Section 3 we introduce the concept of local  $\mathcal{I}$ -boundedness and we construct the ideal  $\mathcal{P}_{\mathcal{I}}$  of  $\mathcal{I}$ -bounded polynomials. We prove that whenever  $\mathcal{I}$  is stable under the formation of symmetric tensor products then any continuous  $m$ -homogeneous polynomial maps  $\mathcal{I}$ -bounded sets to  $\mathcal{I}$ -bounded sets. As a consequence, we get a stronger ideal condition for  $\mathcal{P}_{\mathcal{I}}$ : the composition of an  $\mathcal{I}$ -bounded  $m$ -homogeneous polynomial with any continuous homogeneous polynomial is  $\mathcal{I}$ -bounded. In Section 4 we compare the ideal of  $\mathcal{I}$ -bounded polynomials with the composition ideal of polynomials. We show that the two ideals coincide whenever  $\mathcal{I}$  is surjective and satisfies Condition  $\Gamma$  that is introduced in [3].

## §2. Preliminaries and notation

In the following,  $E$ ,  $F$  and  $Z$  denote arbitrary (complex or real) Banach spaces. Let  $B_E$  denote the closed unit ball of  $E$ . If  $x \in E$  and  $\epsilon > 0$  then  $B_\epsilon(x)$  is the open ball of center  $x$  and radius  $\epsilon$ .

We denote by  $\mathcal{L}(^m E; F)$  the space of all continuous  $m$ -linear mappings from  $E \times \cdots \times E$  into  $F$ . Whenever  $m = 1$ ,  $\mathcal{L}(^1 E; F) = \mathcal{L}(E; F)$  coincides with the usual space of continuous linear operators, and for  $m = 0$  we agree that  $\mathcal{L}(^0 E; F)$  is the space of constant mappings from  $E$  to  $F$  which is identified with  $F$ . A mapping  $P : E \rightarrow F$  is a continuous  $m$ -homogeneous polynomial if there is  $A \in \mathcal{L}(^m E; F)$  such that  $P(x) = A(x, \dots, x)$  for all  $x \in E$ . Let  $\mathcal{P}(^m E; F)$  denote the space of

all continuous  $m$ -homogeneous polynomials from  $E$  to  $F$ , endowed with the usual sup norm. When  $F$  is the scalar field we simply write  $\mathcal{P}(^m E)$ .

By  $\widehat{\otimes}_\pi^{m,s} E$  and  $\widehat{\otimes}_{\pi_s}^{m,s} E$  we denote the  $m$ -fold completed symmetric tensor product of  $E$  endowed with the projective norm  $\pi$  and the projective  $s$ -tensor norm  $\pi_s$ , respectively. The projective norm  $\pi$  is well-known (see e.g. [20]) and the projective  $s$ -tensor norm  $\pi_s$  is defined by

$$\pi_s(z) = \inf \left\{ \sum_{j=1}^k |\lambda_j| \|x_j\|^m : k \in \mathbb{N}, z = \sum_{j=1}^k \lambda_j x_j \otimes \cdots \otimes x_j \right\}$$

for  $z \in \otimes^{m,s} E$  (see [12]). It is worth pointing out that the projective norm  $\pi$  induces a norm equivalent to the projective  $s$ -tensor norm  $\pi_s$  on the  $m$ -fold symmetric tensor product.

Given  $P \in \mathcal{P}(^m E; F)$ , we denote by  $\check{P}$  the unique symmetric continuous  $m$ -linear map  $\check{P} \in \mathcal{L}(^m E; F)$  such that  $\check{P}(x, \dots, x) = P(x)$  for all  $x \in E$ . The  $n$ -th *polarization constant* of the Banach space  $E$ , denoted by  $c(n, E)$ , is

$$c(n, E) = \inf \{ C > 0 : \|\check{P}\| \leq C \|P\| \text{ for all } P \in \mathcal{P}(^n E) \}.$$

It is well known that

$$1 \leq c(n, E) \leq \frac{m^m}{m!}$$

for every Banach space  $E$ .

Also,

$$\begin{aligned} P_L : \widehat{\otimes}_\pi^{n,s} E &\rightarrow F, & P_L(x \otimes \cdots \otimes x) &= P(x), \\ P_{L,s} : \widehat{\otimes}_{\pi_s}^{n,s} E &\rightarrow F, & P_{L,s}(x \otimes \cdots \otimes x) &= P(x) \end{aligned}$$

denote the linearization of  $P$ . If we consider the map  $\delta_m^E : E \rightarrow \otimes^{m,s} E$  given by  $\delta_m^E(x) = x \otimes \cdots \otimes x$  it is clear that  $P = P_L \circ \delta_m^E = P_{L,s} \circ \delta_m^E$ . The map  $\delta_m^E$  is continuous when  $\otimes^{m,s} E$  is endowed with either  $\pi$  or  $\pi_s$ . It is well-known that  $\|P_{L,s}\| = \|P\|$  and  $\|P_L\| = \|\check{P}\|$ . For the general theory of homogeneous polynomials and symmetric tensor products we refer to [11] and [12].

An *ideal of homogeneous polynomials* (or *polynomial ideal*)  $\mathcal{Q}$  is a subclass of the class of all continuous homogeneous polynomials between Banach spaces such that for all  $n \in \mathbb{N}$  and Banach spaces  $E$  and  $F$ , the components  $\mathcal{Q}(^n E, F) = \mathcal{P}(^n E, F) \cap \mathcal{Q}$  satisfy:

- (i)  $\mathcal{Q}(^n E, F)$  is a linear subspace of  $\mathcal{P}(^n E, F)$  which contains the  $n$ -homogeneous polynomials of finite type.
- (ii) The ideal property: if  $u \in \mathcal{L}(G, E)$ ,  $P \in \mathcal{Q}(^n E, F)$  and  $t \in \mathcal{L}(F, H)$ , then the composition  $t \circ P \circ u$  is in  $\mathcal{Q}(^n G, H)$ .

If  $\|\cdot\|_{\mathcal{Q}} : \mathcal{Q} \rightarrow \mathbb{R}^+$  satisfies

- (i')  $(\mathcal{Q}({}^n E; F), \|\cdot\|_{\mathcal{Q}})$  is a normed (Banach) space for all  $E, F$  and  $n$ ,
- (ii')  $\|P^n : \mathbb{K} \rightarrow \mathbb{K} : P^n(x) = x^n\|_{\mathcal{Q}} = 1$  for all  $n$ , and
- (iii') if  $u \in \mathcal{L}(G, E)$ ,  $P \in \mathcal{Q}({}^n E, F)$  and  $t \in \mathcal{L}(F, H)$ , then  $\|t \circ P \circ u\|_{\mathcal{Q}} \leq \|t\| \|P\|_{\mathcal{Q}} \|u\|^n$ ,

then  $[\mathcal{Q}, \|\cdot\|_{\mathcal{Q}}]$  is called a *normed (Banach) polynomial ideal*.

The case  $n = 1$  recovers the classical theory of (normed, Banach) operator ideals, for which the reader is referred to [7] and [9]. In particular we will use the notation  $[\mathcal{I}, \iota]$  for an arbitrary normed operator ideal, that is,  $\iota : \mathcal{I} \rightarrow \mathbb{R}^+$  satisfies (i'), (ii') and (iii') above.

The polynomial ideal  $\mathcal{Q}$  is said to be *closed* if each component  $\mathcal{Q}({}^n E; F)$  is a closed subspace of  $\mathcal{P}({}^n E; F)$ .

An operator ideal  $\mathcal{I}$  is *surjective* if  $T$  belongs to  $\mathcal{I}(E; F)$  whenever  $T \circ S \in \mathcal{I}(Z; F)$  for every surjection  $S \in \mathcal{L}(Z; E)$ . Here,  $E, F$  and  $Z$  are arbitrary Banach spaces. Equivalently,  $\mathcal{I}$  is surjective if  $T$  belongs to  $\mathcal{I}(E; F)$  whenever  $T(B_E) \subset U(B_Z)$  for some  $U \in \mathcal{I}(Z; F)$  (see [22]).

Recall that, given a normed operator ideal  $[\mathcal{I}, \iota]$ , there is a smallest surjective operator ideal  $\mathcal{I}^{\text{sur}}$  which contains  $\mathcal{I}$ . Moreover, a continuous linear operator  $T \in \mathcal{L}(E; F)$  belongs to  $\mathcal{I}^{\text{sur}}(E; F)$  if and only if  $T \circ Q_E \in \mathcal{I}$ , where  $Q_E : \ell_1(B_E) \rightarrow E$  is the canonical surjection. If we define  $\iota^{\text{sur}}(T) := \iota(T \circ Q_E)$  then  $[\mathcal{I}^{\text{sur}}, \iota^{\text{sur}}]$  is a normed operator ideal, which is Banach whenever  $[\mathcal{I}, \iota]$  is (cf. [7, Section 9.8]). The ideal  $[\mathcal{I}^{\text{sur}}, \iota^{\text{sur}}]$  is called the *surjective hull* of  $[\mathcal{I}, \iota]$ .

Given an operator ideal  $[\mathcal{I}, \iota]$ , the *composition ideal* of polynomials,  $\mathcal{I} \circ \mathcal{P}$ , consists of all homogeneous polynomials  $P$  between Banach spaces that can be factored as  $P = T \circ Q$  where  $Q$  is a homogeneous polynomial and  $T$  is a linear operator belonging to  $\mathcal{I}$ . For  $m \in \mathbb{N}$  and Banach spaces  $E$  and  $F$ , the usual composition norm  $\|\cdot\|_{\mathcal{I} \circ \mathcal{P}}$  of an  $m$ -homogeneous polynomial  $P \in \mathcal{I} \circ \mathcal{P}({}^m E; F)$  is given by

$$(2.1) \quad \|P\|_{\mathcal{I} \circ \mathcal{P}} := \inf\{\iota(T)\|Q\| : P = T \circ Q, Q \in \mathcal{P}({}^m E; G), T \in \mathcal{I}(G; F)\},$$

With this norm,  $\mathcal{I} \circ \mathcal{P}$  becomes a Banach polynomial ideal whenever  $[\mathcal{I}, \iota]$  is a Banach operator ideal (see [6]).

Let  $E$  be a Banach space, and  $1 \leq p, p' \leq \infty$  be conjugate exponents. We recall that a subset  $K$  of  $E$  is said to be *relatively  $p$ -compact* if for some sequence  $(x_n) \in \ell_p(E)$ ,  $K \subset \{\sum_n a_n x_n \mid (a_n) \in B_{\ell'_p}\}$ , where  $\ell_p(E)$  denotes the space of  $p$ -summable sequences in  $E$  endowed with its natural norm and  $B_{\ell'_p}$  denotes the closed unit ball of  $\ell'_p$ . An operator  $T \in \mathcal{L}(E; F)$  is  *$p$ -compact* if  $T(B_E)$  is relatively

$p$ -compact. In this case the norm  $k_p$  is defined as

$$k_p(T) = \inf\{\|(x_n)_n\|_p\},$$

where the infimum is taken over all sequences  $(x_n)_n \in \ell_p(F)$  fulfilling the above condition. The norm  $k_p$  was introduced by Sinha and Karn [21] and characterized by Delgado, Piñeiro and Serrano [8, Proposition 3.15]. We shall use  $[\mathcal{K}_p, k_p]$  to denote the Banach ideal formed by all  $p$ -compact operators. Pietsch [17] has recently revisited the theory of  $p$ -compact operators by using the general theory of operator ideals as developed in his monograph [15].

### §3. The $\mathcal{I}$ -ideal of homogeneous polynomials

Let  $[\mathcal{I}, \iota]$  be a normed operator ideal, and let  $F$  be a Banach space. Let  $C_{\mathcal{I}}(F)$  stand for the collection of subsets  $A \subset F$  such that  $A \subset T(B_Z)$  for some Banach space  $Z$  and some  $T \in \mathcal{I}(Z; F)$ . We will call any  $A \in C_{\mathcal{I}}(F)$  an  $\mathcal{I}$ -bounded set.

Notice that  $\mathcal{I}$ -bounded sets are bounded. The family  $C_{\mathcal{I}}(F)$  was first considered in [22] (see also [13]). The central role that it plays in our work justifies our coining the term ‘ $\mathcal{I}$ -bounded set’ to name its elements.

**Example 3.1.** Let  $E$  be a Banach space and  $A$  be a subset of  $E$ . Then  $A \in C_{\mathcal{K}_p}(E)$  if and only if  $A$  is relatively  $p$ -compact.

*Proof.* The necessity follows from the definition of  $C_{\mathcal{K}_p}(E)$ .

Conversely, if  $A$  is relatively  $p$ -compact then it is bounded. Hence, the operator  $T : \ell_1(A) \rightarrow E$  given by  $T((t_x)_{x \in A}) := \sum_{x \in A} t_x x$  is well-defined. As the closed absolutely convex hull of a relatively  $p$ -compact set is relatively  $p$ -compact,  $\bar{\Gamma}(A)$  is relatively  $p$ -compact. Since  $\bar{\Gamma}(A) = T(B_{\ell_1(A)})$ ,  $T$  belongs to  $\mathcal{K}_p(\ell_1(A); E)$ . From the inclusion

$$A \subset \bar{\Gamma}(A) = T(B_{\ell_1(A)})$$

we conclude that  $A \in C_{\mathcal{K}_p}(E)$ . □

The following well-known property of  $C_{\mathcal{I}}$  will allow us to define the  $\mathcal{I}$ -ideal of homogeneous polynomials: If  $A_1, \dots, A_n \in C_{\mathcal{I}}(F)$  and  $t_1, \dots, t_n$  are scalars, then  $\bigcup_{i=1}^n A_i$  and  $\sum_{i=1}^n t_i A_i$  belong to  $C_{\mathcal{I}}(F)$  (see [13, Proposition 3]).

If  $T \in \mathcal{L}(E; F)$  is such that  $T(B_E)$  belongs to  $C_{\mathcal{I}}(F)$  then it cannot be concluded that  $T \in \mathcal{I}(E; F)$  in general. This condition actually characterizes surjective operator ideals. We omit the easy proof.

**Proposition 3.2.** *Let  $\mathcal{I}$  be an operator ideal. The following assertions are equivalent:*

- (i)  $\mathcal{I}$  is surjective.
- (ii) For any Banach spaces  $E$  and  $F$  and any  $T \in \mathcal{L}(E; F)$ ,  $T \in \mathcal{I}(E; F)$  if and only if  $T(B_E) \in C_{\mathcal{I}}(F)$ .

We shall say that an  $m$ -homogeneous polynomial  $P \in \mathcal{P}(^m E; F)$  is *locally  $\mathcal{I}$ -bounded* (or  $\mathcal{I}$ -bounded for short) if for every  $x \in E$  there exists a neighborhood  $V_x$  of  $x$  such that  $P(V_x) \in C_{\mathcal{I}}(F)$ . The set of all locally  $\mathcal{I}$ -bounded  $m$ -homogeneous polynomials from  $E$  to  $F$  is denoted by  $\mathcal{P}_{\mathcal{I}}(^m E; F)$ . When  $m = 1$  we write  $\mathcal{L}_{\mathcal{I}}(E; F) = \mathcal{P}_{\mathcal{I}}(^1 E; F)$ . It is easy to prove that  $P \in \mathcal{P}(^m E; F)$  is  $\mathcal{I}$ -bounded if and only if  $P(B_E) \in C_{\mathcal{I}}(F)$ .

It is clear that  $\mathcal{P}_{\mathcal{I}}(^m E; F)$  is a vector space. Let us endow it with a norm. Given  $P \in \mathcal{P}_{\mathcal{I}}(^m E; F)$ , we have  $P(B_E) \subset T(B_Z)$  for some Banach space  $Z$  and some  $T \in \mathcal{I}(Z; F)$ . Define

$$\|P\|_{\mathcal{I}} := \inf \iota(T)$$

where  $T$  varies among those operators in  $\mathcal{I}$  fulfilling the above condition.

Notice that

$$\mathcal{I}(E; F) \subset \mathcal{L}_{\mathcal{I}}(E; F) \subset \mathcal{L}(E; F)$$

and

$$\|T\| \leq \|T\|_{\mathcal{I}} \leq \iota(T)$$

for all  $T \in \mathcal{I}(E; F)$ .

By Proposition 3.2, it follows that  $\mathcal{I}(E; F) = \mathcal{L}_{\mathcal{I}}(E; F)$  if and only if  $\mathcal{I}$  is surjective.

**Proposition 3.3.**  $[\mathcal{P}_{\mathcal{I}}, \|\cdot\|_{\mathcal{I}}]$  is a normed ideal of polynomials.

*Proof.* The only condition that needs some attention is the ideal property. Let  $P \in \mathcal{P}_{\mathcal{I}}(^m E; F)$ ,  $u \in \mathcal{L}(G; E)$  and  $v \in \mathcal{L}(F; Z)$ . Clearly  $v \circ P \in \mathcal{P}_{\mathcal{I}}(^m E; Z)$  and  $\|v \circ P\|_{\mathcal{I}} \leq \|v\| \|P\|_{\mathcal{I}}$ .

On the other hand,

$$P(u(B_G)) \subset P(\|u\|B_E) = \|u\|^m P(B_E) \subset \|u\|^m T(B_H)$$

for some Banach space  $H$  and some  $T \in \mathcal{I}(H; F)$ . Then

$$\|P \circ u\|_{\mathcal{I}} \leq \iota(\|u\|^m T) = \|u\|^m \iota(T).$$

Taking the infimum we get

$$\|P \circ u\|_{\mathcal{I}} \leq \|u\|^m \|P\|_{\mathcal{I}}. \quad \square$$

**Theorem 3.4.**  $[\mathcal{P}_{\mathcal{I}}, \|\cdot\|_{\mathcal{I}}]$  is a Banach ideal of polynomials whenever  $[\mathcal{I}, \iota]$  is a Banach ideal of operators.

*Proof.* It suffices to prove that any absolutely convergent series in  $\mathcal{P}_{\mathcal{I}}({}^m E; F)$  is convergent. Let  $(P_n)_n$  be a sequence of polynomials in  $\mathcal{P}_{\mathcal{I}}({}^m E; F)$  such that  $M := \sum_{n=0}^{\infty} \|P_n\|_{\mathcal{I}} < \infty$ . As the induced usual norm  $\|\cdot\|$  in  $\mathcal{P}_{\mathcal{I}}({}^m E; F)$  is weaker than  $\|\cdot\|_{\mathcal{I}}$  and  $(\mathcal{P}({}^m E; F), \|\cdot\|)$  is complete it follows that  $\sum_{n=0}^{\infty} P_n$  converges to some  $P \in \mathcal{P}({}^m E; F)$ . Let us prove that  $P \in \mathcal{P}_{\mathcal{I}}({}^m E; F)$ .

Let  $\epsilon > 0$ . By the definition of  $\|\cdot\|_{\mathcal{I}}$ , for each  $n$  there is a Banach space  $Z_{\epsilon}^n$  and an operator  $T_{\epsilon}^n \in \mathcal{I}(Z_{\epsilon}^n; F)$  such that

$$(3.1) \quad \|P_n\|_{\mathcal{I}} + \epsilon/2^n > \iota(T_{\epsilon}^n)$$

and

$$(3.2) \quad P_n(B_E) \subset T_{\epsilon}^n(B_{\epsilon}^n)$$

where  $B_{\epsilon}^n$  denotes the closed unit ball of  $Z_{\epsilon}^n$ . Let

$$\Lambda_1((Z_{\epsilon}^n)_n) := \left\{ (z_n)_n \in \prod_n Z_{\epsilon}^n : p_r((z_n)_n) = \sum_{n=0}^{\infty} r^n \|z_n\| < \infty \text{ for all } 0 < r < 1 \right\}$$

endowed with the topology generated by the family of seminorms  $\{p_r : 0 < r < 1\}$ .

Let

$$D := \{(z_n)_n \in \Lambda_1((Z_{\epsilon}^n)_n) : z_n \in B_{\epsilon}^n\}.$$

Denote by  $Z$  the linear span of  $D$  endowed with its Minkowski functional. Then  $Z$  is a Banach space and  $Z = \bigcup_{\lambda > 0} \lambda D$ . Define the map  $T : Z \rightarrow F$  by  $T(\lambda(z_n)_n) := \lambda \sum_{n=0}^{\infty} T_{\epsilon}^n(z_n)$  for any  $\lambda > 0$  and any  $(z_n)_n \in D$ . From the linearity of each  $T_{\epsilon}^n$  it follows that  $T(\lambda(z_n)_n) = T(\lambda'(z'_n)_n)$  whenever  $\lambda(z_n)_n = \lambda'(z'_n)_n$ . By (3.1),

$$\sum_{n=0}^{\infty} \|T_{\epsilon}^n(z_n)\| \leq \sum_{n=0}^{\infty} \|T_{\epsilon}^n\| \leq \sum_{n=0}^{\infty} \iota(T_{\epsilon}^n) \leq \sum_{n=0}^{\infty} (\|P_n\|_{\mathcal{I}} + \epsilon) = M + \epsilon < \infty.$$

Thus  $T$  is well-defined, and clearly it is linear and continuous.

We now prove that  $T \in \mathcal{I}(Z; F)$ . For each  $n$  define  $S_n : Z \rightarrow F$  by  $S_n(\lambda(z_j)_j) = \lambda T_{\epsilon}^n(z_n)$  for  $\lambda > 0$  and  $(z_n)_n \in D$ . If  $\pi_n : Z \rightarrow Z_n$  denotes the projection on the  $n$ th coordinate,  $S_n = T_{\epsilon}^n \circ \pi_n$ . Then  $S_n \in \mathcal{I}(Z; F)$  by the ideal property. Since

$$T((z_j)_j) = \sum_{n=0}^{\infty} T_{\epsilon}^n(z_n) = \sum_{n=0}^{\infty} S_n((z_j)_j)$$

and

$$\sum_{n=0}^{\infty} \iota(S_n) \leq \sum_{n=0}^{\infty} \iota(T_{\epsilon}^n) < \sum_{n=0}^{\infty} (\|P_n\|_{\mathcal{I}} + \epsilon/2^n) = M + \epsilon < \infty,$$

it follows from the hypothesis of  $[\mathcal{I}, \iota]$  being Banach that  $T = \sum_{n=0}^{\infty} S_n$  belongs to  $\mathcal{I}(Z; F)$ .

Let us see that  $P(B_E) \subset T(D)$ . Given  $x \in B_E$ , by (3.2) for each  $n$  there is  $z_n \in B_\epsilon^n$  such that  $P_n(x) = T_\epsilon^n(z_n)$ . Then

$$P(x) = \sum_{n=0}^\infty P_n(x) = \sum_{n=0}^\infty T_\epsilon^n(z_n) = T((z_n)_n).$$

As  $(z_n)_n \in D$ , it follows that  $P(B_E) \subset T(D)$ .

We have proved that  $P = \sum_{n=0}^\infty P_n$  belongs to  $\mathcal{P}_{\mathcal{I}}({}^m E; F)$ . □

Given  $T \in \mathcal{L}(E; F)$ , let  $\otimes_m T : \hat{\otimes}_{\pi_s}^{m,s} E \rightarrow \hat{\otimes}_\pi^{m,s} F$  be defined by

$$\otimes_m T \left( \sum_{i=1}^n \alpha_i x_i \otimes \cdots \otimes x_i \right) = \sum_{i=1}^n \alpha_i T(x_i) \otimes \cdots \otimes T(x_i)$$

and extended by continuity to the completions. An operator ideal  $[\mathcal{I}, \iota]$  is *stable* under the formation of symmetric tensor products if  $\otimes_m T : \hat{\otimes}_{\pi_s}^{m,s} E \rightarrow \hat{\otimes}_\pi^{m,s} F$  belongs to  $\mathcal{I}(\hat{\otimes}_{\pi_s}^{m,s} E; \hat{\otimes}_\pi^{m,s} F)$  whenever  $T \in \mathcal{I}(E; F)$  and, in this case,  $\iota(\otimes_m T) \leq C\iota(T)^m$  for some positive constant  $C$ .

In [3, Theorem 3.4] the authors prove that the ideal of  $p$ -compact operators is stable under the formation of symmetric tensor products.

**Theorem 3.5.** *If an operator ideal  $[\mathcal{I}, \iota]$  is stable under the formation of symmetric tensor products then any continuous  $m$ -homogeneous polynomial maps  $\mathcal{I}$ -bounded sets to  $\mathcal{I}$ -bounded sets.*

*Proof.* Let  $P \in \mathcal{P}({}^m E; F)$  and  $A \in C_{\mathcal{I}}(E)$ . Then  $A \subset T(B_Z)$  for some Banach space  $Z$  and some  $T \in \mathcal{I}(Z; E)$ . Using the commutativity of the diagram

$$\begin{array}{ccccc} Z & \xrightarrow{T} & E & \xrightarrow{P} & F \\ \delta_m^Z \downarrow & & \delta_m^E \downarrow & \nearrow P_L & \\ \hat{\otimes}_{\pi_s}^{m,s} Z & \xrightarrow{\otimes_m T} & \hat{\otimes}_\pi^{m,s} E & & \end{array}$$

we get

$$\begin{aligned} P(A) &\subset P \circ T(B_Z) = P_L \circ \delta_m^E \circ T(B_Z) = P_L \circ (\otimes_m T) \circ \delta_m^Z(B_Z) \\ &\subset P_L \circ (\otimes_m T)(B_{\hat{\otimes}_{\pi_s}^{m,s} Z}). \end{aligned}$$

By the stability assumption and the ideal property,  $P_L \circ (\otimes_m T) \in \mathcal{I}(\hat{\otimes}_{\pi_s}^{m,s} Z; F)$ . Hence  $P(A) \in C_{\mathcal{I}}(F)$ . □

As a consequence we get a stronger ideal condition for  $\mathcal{P}_{\mathcal{I}}$ .



**Theorem 3.6.** *Let  $\mathcal{I}$  be an operator ideal and let  $E, F, G$  and  $Z$  be Banach spaces.*

(i) *If  $P \in \mathcal{P}_{\mathcal{I}}({}^m E; F)$  and  $Q \in \mathcal{P}({}^l Z; E)$  then  $P \circ Q \in \mathcal{P}_{\mathcal{I}}({}^{m+l} Z; F)$ . In this case,*

$$\|P \circ Q\|_{\mathcal{I}} \leq \|P\|_{\mathcal{I}} \|Q\|_{\mathcal{I}}^m.$$

(ii) *If  $\mathcal{I}$  is stable under the formation of symmetric tensor products then  $R \circ P \in \mathcal{P}_{\mathcal{I}}({}^{mk} E; G)$  for any  $P \in \mathcal{P}_{\mathcal{I}}({}^m E; F)$  and any  $R \in \mathcal{P}({}^k F; G)$ . In this case, there exists  $K > 0$  such that*

$$\|R \circ P\|_{\mathcal{I}} \leq Kc(k, F) \|R\| \|P\|_{\mathcal{I}}^m,$$

where  $c(k, F)$  is the  $k$ -th polarization constant of  $F$ .

*Proof.* (i) Let  $T \in \mathcal{I}(H; F)$  be so that  $P(B_E) \subset T(B_H)$ . Since  $Q(B_Z) \subset \|Q\|_{B_E}$ ,

$$P \circ Q(B_Z) \subset P(\|Q\|_{B_E}) = \|Q\|_{B_E}^m P(B_E) \subset \|Q\|_{B_E}^m T(B_H).$$

Then  $P \circ Q \in \mathcal{P}_{\mathcal{I}}({}^{m+l} Z; F)$  and  $\|P \circ Q\|_{\mathcal{I}} \leq \|Q\|_{B_E}^m \iota(T)$ . Taking the infimum over  $T$  we conclude that  $\|P \circ Q\|_{\mathcal{I}} \leq \|P\|_{\mathcal{I}} \|Q\|_{\mathcal{I}}^m$ .

(ii) As  $P \in \mathcal{P}_{\mathcal{I}}({}^m E; F)$ , we have  $P(B_E) \in C_{\mathcal{I}}(F)$ . By Theorem 3.5,  $R(P(B_E)) \in C_{\mathcal{I}}(G)$ . Then  $R \circ P \in \mathcal{P}_{\mathcal{I}}({}^{mk} E; G)$ . However, in order to get the inequality for the norms, the above argument needs to be more precise: Let  $T \in \mathcal{I}(H; F)$  be so that  $P(B_E) \subset T(B_H)$ . Taking  $A = P(B_E)$  in the proof of Theorem 3.5, we get

$$R \circ P(B_E) \subset R_L \circ (\otimes_m T)(B_{\hat{\otimes}_{\pi_s}^{m,s} H}).$$

Since  $\otimes_m T \in \mathcal{I}(\hat{\otimes}_{\pi_s}^{m,s} H; \hat{\otimes}_{\pi}^{m,s} F)$  then

$$\|R \circ P\|_{\mathcal{I}} \leq \iota(R_L \circ (\otimes_m T)) \leq \|R_L\| \iota(\otimes_m T) \leq c(k, F) \|R\| K \iota(T)^m,$$

for a suitable  $K > 0$ . As  $T$  is arbitrary,

$$\|R \circ P\|_{\mathcal{I}} \leq Kc(k, F) \|R\| \|P\|_{\mathcal{I}}^m. \quad \square$$

#### §4. Comparing with the composition ideal of polynomials

An operator ideal  $\mathcal{I}$  is said to satisfy *Condition  $\Gamma$*  if the closed absolutely convex hull of any  $\mathcal{I}$ -bounded set is  $\mathcal{I}$ -bounded. Given a subset  $A$  of a Banach space  $E$ , the closed absolutely convex hull of  $A$  is denoted by  $\overline{\Gamma}(A)$ .

**Example 4.1.** By [13, Proposition 3] any operator ideal which is surjective and closed satisfies Condition  $\Gamma$ . The most usual examples are the ideals of compact operators and weakly compact operators.

**Example 4.2.** For  $1 < p < \infty$ , the ideal  $[K_p, k_p]$  of all  $p$ -compact operators is surjective and satisfies Condition  $\Gamma$ , although  $K_p$  is not closed (see [2, 3]).

Let us consider the surjective hull  $\mathcal{N}^{\text{sur}}$  of the ideal  $\mathcal{N}$  of nuclear operators. By [7, Ex. 9.14],  $T \in \mathcal{L}(E; F)$  belongs to  $\mathcal{N}^{\text{sur}}(E; F)$  if and only if there is a sequence  $(y_n)_n \in \ell_1(F)$  such that

$$T(B_E) \subset \left\{ \sum_{n=1}^{\infty} \lambda_n y_n : |\lambda_n| \leq 1 \right\}.$$

This description shows that  $\mathcal{N}^{\text{sur}}$  coincides with  $K_1$ , the ideal of 1-compact linear operators.

**Example 4.3.** The ideal  $\mathcal{N}^{\text{sur}}$  satisfies Condition  $\Gamma$ .

*Proof.* Let  $A \in C_{\mathcal{N}^{\text{sur}}}(F)$ . Then there is a Banach space  $E$  and an operator  $T \in \mathcal{N}^{\text{sur}}(E; F)$  such that  $A \subset T(B_E)$ . Choose a sequence  $(y_n)_n \in \ell_1(F)$  such that

$$T(B_E) \subset \left\{ \sum_{n=1}^{\infty} \lambda_n y_n : |\lambda_n| \leq 1 \right\}.$$

Consider the nuclear map  $\phi_{(y_n)_n} : \ell^\infty \rightarrow F$  defined by  $\phi_{(y_n)_n}((\lambda_n)_n) = \sum_{n=1}^{\infty} \lambda_n y_n$ . Notice that

$$\phi_{(y_n)_n}(B_{\ell^\infty}) = \left\{ \sum_{n=1}^{\infty} \lambda_n y_n : |\lambda_n| \leq 1 \right\}.$$

As  $\phi_{(y_n)_n}$  is linear,  $\phi_{(y_n)_n}(B_{\ell^\infty})$  is absolutely convex. Let us see that  $\phi_{(y_n)_n}$  is weak\*-norm continuous. Let  $\epsilon > 0$ . The set

$$V := \left\{ (\lambda_n)_n \in \ell^\infty : \sum_{n=1}^{\infty} |\lambda_n| \|y_n\| < \epsilon \right\}$$

is a weak\* neighbourhood of 0. Since  $\phi_{(y_n)_n}(V) \subset \epsilon B_F$ ,  $\phi_{(y_n)_n}$  is weak\*-norm continuous.

Since  $B_{\ell^\infty}$  is weak\* compact, we see that  $\phi_{(y_n)_n}(B_{\ell^\infty})$  is norm compact in  $F$ . Hence,

$$\overline{\Gamma}(A) \subset \overline{\Gamma}(\phi_{(y_n)_n}(B_{\ell^\infty})) = \phi_{(y_n)_n}(B_{\ell^\infty}).$$

Since  $\phi_{(y_n)_n}$  trivially belongs to  $\mathcal{N}^{\text{sur}}(\ell^\infty; F)$ , we conclude that  $\overline{\Gamma}(A) \in C_{\mathcal{N}^{\text{sur}}}(F)$ . □

Other examples of surjective (not necessarily closed) operator ideals fulfilling Condition  $\Gamma$  can be easily constructed by considering the surjective hull of an operator ideal that enjoys Condition  $\Gamma$ , as is shown in the following result.

**Proposition 4.4.** *If an operator ideal  $\mathcal{I}$  satisfies Condition  $\Gamma$  then  $\mathcal{I}^{\text{sur}}$  also satisfies Condition  $\Gamma$ .*

*Proof.* Let  $\mathcal{I}$  be an operator ideal satisfying Condition  $\Gamma$ , and let  $F$  be a Banach space. Consider a set  $A \subset F$  such that  $A \in C_{\mathcal{I}^{\text{sur}}}(F)$ . We have to prove that  $\overline{\Gamma}(A) \in C_{\mathcal{I}^{\text{sur}}}(F)$ . Choose a Banach space  $Z$  and  $T \in \mathcal{I}^{\text{sur}}(Z; F)$  such that  $A \subset T(B_Z)$ . By the characterization of the surjective hull,  $T \circ Q_Z \in \mathcal{I}(\ell_1(B_Z); F)$ . Since  $B_Z \subset Q_Z(B_{\ell_1(B_Z)})$ , it follows that  $A \subset T(Q_Z(B_{\ell_1(B_Z)}))$ . Thus we see that  $A$  belongs to  $C_{\mathcal{I}}(F)$ . Since  $\mathcal{I}$  satisfies Condition  $\Gamma$ ,  $\overline{\Gamma}(A) \in C_{\mathcal{I}}(F)$ . The inclusion  $C_{\mathcal{I}}(F) \subset C_{\mathcal{I}^{\text{sur}}}(F)$  yields the result.  $\square$

Condition  $\Gamma$  has been used strongly in [4], where it is essential when working with non-closed ideals of operators. Let us see that  $\mathcal{P}_{\mathcal{I}}$  turns out to be the composition ideal of polynomials  $\mathcal{I} \circ \mathcal{P}$  whenever  $\mathcal{I}$  is a surjective ideal fulfilling Condition  $\Gamma$ .

**Theorem 4.5.** *Let  $P \in \mathcal{P}({}^m E; F)$ .*

- (i) *If  $P_{L,s} \in \mathcal{L}_{\mathcal{I}}(\hat{\otimes}_{\pi_s}^{m,s} E; F)$  then  $P \in \mathcal{P}_{\mathcal{I}}({}^m E; F)$ .*
- (ii) *If  $\mathcal{I}$  satisfies Condition  $\Gamma$  then the converse to (i) holds: if  $P \in \mathcal{P}_{\mathcal{I}}({}^m E; F)$  then  $P_{L,s} \in \mathcal{L}_{\mathcal{I}}(\hat{\otimes}_{\pi_s}^{m,s} E; F)$ .*

*Proof.* (i) If  $P_{L,s} \in \mathcal{L}_{\mathcal{I}}(\hat{\otimes}_{\pi_s}^{m,s} E; F)$  then  $P_{L,s}(B_{\hat{\otimes}_{\pi_s}^{m,s} E}) \in C_{\mathcal{I}}(F)$ . As  $P(B_E) \subset P_{L,s}(B_{\hat{\otimes}_{\pi_s}^{m,s} E})$  it follows that  $P(B_E)$  is  $\mathcal{I}$ -bounded.

(ii) Since  $P(B_E)$  is  $\mathcal{I}$ -bounded, Condition  $\Gamma$  ensures that  $\overline{\Gamma}(P(B_E))$  also is. The chain

$$P_{L,s}(B_{\hat{\otimes}_{\pi_s}^{m,s} E}) = P_{L,s}(\overline{\Gamma}(\delta^m(B_E))) \subset \overline{\Gamma}(P_{L,s}(\delta^m(B_E))) = \overline{\Gamma}(P(B_E))$$

proves now that  $P_{L,s}(B_{\hat{\otimes}_{\pi_s}^{m,s} E})$  is  $\mathcal{I}$ -bounded too.  $\square$

The next result partially recovers [3, Theorem 3.1] in the case of  $p$ -compact homogeneous polynomials.

**Corollary 4.6.** *If  $\mathcal{I}$  is surjective and satisfies Condition  $\Gamma$  then  $\mathcal{P}_{\mathcal{I}}({}^m E; F) = \mathcal{I} \circ \mathcal{P}({}^m E; F)$  for all Banach spaces  $E$  and  $F$ , and  $\|\cdot\|_{\mathcal{I}} \leq \|\cdot\|_{\mathcal{I} \circ \mathcal{P}}$ .*

*Proof.* By Theorem 4.5,  $P \in \mathcal{P}_{\mathcal{I}}({}^m E; F)$  if and only if  $P_{L,s} \in \mathcal{L}_{\mathcal{I}}(\hat{\otimes}_{\pi_s}^{m,s} E; F)$ . By Proposition 3.2,  $\mathcal{L}_{\mathcal{I}}(\hat{\otimes}_{\pi_s}^{m,s} E; F) = \mathcal{I}(\hat{\otimes}_{\pi_s}^{m,s} E; F)$ . The equality  $\mathcal{P}_{\mathcal{I}}({}^m E; F) = \mathcal{I} \circ \mathcal{P}({}^m E; F)$  follows now from [6, Proposition 3.2]. Let  $P \in \mathcal{P}_{\mathcal{I}}({}^m E; F)$ . Then, by [6, Proposition 3.7],

$$\|P\|_{\mathcal{I}} \leq \|P_{L,s}\|_{\mathcal{I}} \leq \iota(P_{L,s}) = \|P\|_{\mathcal{I} \circ \mathcal{P}}. \quad \square$$

**Corollary 4.7.** *If  $\mathcal{I}$  is closed and surjective then  $\|P\|_{\mathcal{I}} = \|P\|$  for all  $P \in \mathcal{P}_{\mathcal{I}}({}^m E; F)$ .*

*Proof.* The inequality  $\|P\| \leq \|P\|_{\mathcal{I}}$  always holds. If  $\mathcal{I}$  is closed and surjective then by [13, Proposition 3],  $\mathcal{I}$  satisfies Condition  $\Gamma$ . If  $P \in \mathcal{P}_{\mathcal{I}}({}^m E; F)$ , then by Theorem 4.5 and Proposition 3.2,  $P_{L,s} \in \mathcal{L}_{\mathcal{I}}(\hat{\otimes}_{\pi_s}^{m,s} E; F) = \mathcal{I}(\hat{\otimes}_{\pi_s}^{m,s} E; F)$ . Consequently,

$$\|P\|_{\mathcal{I}} \leq \iota(P_{L,s}) = \|P_{L,s}\| = \|P\|$$

for all  $P \in \mathcal{P}_{\mathcal{I}}({}^m E; F)$ . □

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