

Periodicities of T-systems and Y-systems, Dilogarithm Identities, and Cluster Algebras II: Types C_r , F_4 , and G_2

by

Rei INOUE, Osamu IYAMA, Bernhard KELLER, Atsuo KUNIBA and
Tomoki NAKANISHI

Abstract

We prove the periodicities of the restricted T-systems and Y-systems associated with the quantum affine algebra of type C_r , F_4 , and G_2 at any level. We also prove the dilogarithm identities for these Y-systems at any level. Our proof is based on the tropical Y-systems and the categorification of the cluster algebra associated with any skew-symmetric matrix by Plamondon.

2010 Mathematics Subject Classification: Primary 13F60; Secondary 17B37.

Keywords: T-systems, Y-systems, cluster algebra, dilogarithm.

Contents

1	Introduction	44
2	Main results	44
3	Type C_r	50
4	Type F_4	66
5	Type G_2	74
6	Mutation equivalence of quivers	84
	References	84

Communicated by H. Nakajima. Received April 8, 2010.

R. Inoue: Faculty of Pharmaceutical Sciences, Suzuka University of Medical Science, Suzuka, 513-8670, Japan;

e-mail: reiiy@math.s.chiba-u.ac.jp

O. Iyama: Graduate School of Mathematics, Nagoya University, Nagoya, 464-8604, Japan;

e-mail: iyama@math.nagoya-u.ac.jp

B. Keller: Université Paris Diderot – Paris 7, UFR de Mathématiques, Institut de Mathématiques de Jussieu, UMR 7586 du CNRS, Case 7012, 2, place Jussieu, 75251 Paris Cedex 05, France;

e-mail: keller@math.jussieu.fr

A. Kuniba: Institute of Physics, University of Tokyo, Tokyo, 153-8902, Japan;

e-mail: atsuo@gokutan.c.u-tokyo.ac.jp

T. Nakanishi: Graduate School of Mathematics, Nagoya University, Nagoya, 464-8604, Japan;

e-mail: nakanisi@math.nagoya-u.ac.jp

§1. Introduction

This is a continuation of the paper [IIKKN]. In [IIKKN], we proved the periodicities of the restricted T-systems and Y-systems associated with the quantum affine algebra of type B_r at any level. We also proved the dilogarithm identities for these Y-systems at any level. Our proof was based on the tropical Y-systems and the categorification of the cluster algebra associated with any skew-symmetric matrix by Plamondon [P1, P2]. In this paper, using the same method, we prove the corresponding statements for types C_r , F_4 , and G_2 , thereby completing all the non-simply laced types.

The results are basically parallel to type B_r . Since the common method and the proofs of the statements for type B_r were described in [IIKKN] in detail, in this paper, we skip the proofs of most statements, and concentrate on presenting the results with emphasis on the special features of each case. Notably, the tropical Y-system at level 2, which is the core part in the entire method, is quite specific to each case.

While we try to make the paper as self-contained as possible, we also try to minimize duplication with [IIKKN]. Therefore, we have to ask the reader for tolerating numerous references to the companion paper [IIKKN] for the things which are omitted. In particular, basic definitions for cluster algebras are summarized in [IIKKN, Section 2.1].

The organization of the paper is as follows. In Section 2 we present the main results as well as the T-systems and Y-systems for each type. In Section 3 the results for type C_r are established. The key tropical Y-system at level 2 is described in detail in Section 3.6. In Section 4 the results for type F_4 are proved. In Section 5 we give the results for type G_2 . In Section 6 we list the known mutation equivalences of quivers corresponding to the T-systems and Y-systems.

§2. Main results

§2.1. Restricted T-systems and Y-systems of types C_r , F_4 , and G_2

Let X_r be the Dynkin diagram of type C_r , F_4 , or G_2 with rank r , and $I = \{1, \dots, r\}$ be the enumeration of the vertices of X_r :

$$\begin{array}{c}
 C_r \quad \circ \text{---} \circ \text{---} \dots \text{---} \circ \text{---} \circ \\
 \quad \quad 1 \quad 2 \quad \quad \quad r-1 \quad r \\
 F_4 \quad \circ \text{---} \circ \text{---} \circ \text{---} \circ \\
 \quad \quad 1 \quad 2 \quad 3 \quad 4 \\
 G_2 \quad \circ \text{---} \circ \\
 \quad \quad 1 \quad 2
 \end{array}$$

Let h and h^\vee be the Coxeter number and the dual Coxeter number of X_r , respectively. Then

$$(2.1) \quad \begin{array}{c|ccc} X_r & C_r & F_4 & G_2 \\ \hline h & 2r & 12 & 6 \\ \hline h^\vee & r+1 & 9 & 4 \end{array}$$

We define numbers t and t_a ($a \in I$) by

$$(2.2) \quad t = \begin{cases} 2, & X_r = C_r, F_4, \\ 3, & X_r = G_2, \end{cases} \quad t_a = \begin{cases} 1, & \alpha_a \text{ long root,} \\ t, & \alpha_a \text{ short root.} \end{cases}$$

For a given integer $\ell \geq 2$, we introduce a set of triplets

$$(2.3) \quad \mathcal{I}_\ell = \mathcal{I}_\ell(X_r) := \{(a, m, u) \mid a \in I; m = 1, \dots, t_a \ell - 1; u \in (1/t)\mathbb{Z}\}.$$

Definition 2.1 ([KNS]). Fix an integer $\ell \geq 2$. The *level ℓ restricted T-system* $\mathbb{T}_\ell(X_r)$ of type X_r (with the unit boundary condition) is the following system of relations for a family of variables $T_\ell = \{T_m^{(a)}(u) \mid (a, m, u) \in \mathcal{I}_\ell\}$, where $T_m^{(0)}(u) = T_0^{(a)}(u) = 1$, and furthermore, $T_{t_a \ell}^{(a)}(u) = 1$ (the *unit boundary condition*) if they occur in the right hand sides of the relations.

(Here and throughout the paper, $2m$ (resp. $2m+1$) on the left hand sides, for example, represents elements $2, 4, \dots$ (resp. $1, 3, \dots$.)

For $X_r = C_r$,

$$(2.4) \quad \begin{aligned} T_m^{(a)}(u - \frac{1}{2})T_m^{(a)}(u + \frac{1}{2}) &= T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u) + T_m^{(a-1)}(u)T_m^{(a+1)}(u) \\ &\quad (1 \leq a \leq r-2), \\ T_{2m}^{(r-1)}(u - \frac{1}{2})T_{2m}^{(r-1)}(u + \frac{1}{2}) &= T_{2m-1}^{(r-1)}(u)T_{2m+1}^{(r-1)}(u) \\ &\quad + T_{2m}^{(r-2)}(u)T_m^{(r)}(u - \frac{1}{2})T_m^{(r)}(u + \frac{1}{2}), \\ T_{2m+1}^{(r-1)}(u - \frac{1}{2})T_{2m+1}^{(r-1)}(u + \frac{1}{2}) &= T_{2m}^{(r-1)}(u)T_{2m+2}^{(r-1)}(u) \\ &\quad + T_{2m+1}^{(r-2)}(u)T_m^{(r)}(u)T_{m+1}^{(r)}(u), \\ T_m^{(r)}(u-1)T_m^{(r)}(u+1) &= T_{m-1}^{(r)}(u)T_{m+1}^{(r)}(u) + T_{2m}^{(r-1)}(u). \end{aligned}$$

For $X_r = F_4$,

$$(2.5) \quad \begin{aligned} T_m^{(1)}(u-1)T_m^{(1)}(u+1) &= T_{m-1}^{(1)}(u)T_{m+1}^{(1)}(u) + T_m^{(2)}(u), \\ T_m^{(2)}(u-1)T_m^{(2)}(u+1) &= T_{m-1}^{(2)}(u)T_{m+1}^{(2)}(u) + T_m^{(1)}(u)T_{2m}^{(3)}(u), \\ T_{2m}^{(3)}(u - \frac{1}{2})T_{2m}^{(3)}(u + \frac{1}{2}) &= T_{2m-1}^{(3)}(u)T_{2m+1}^{(3)}(u) \\ &\quad + T_m^{(2)}(u - \frac{1}{2})T_m^{(2)}(u + \frac{1}{2})T_{2m}^{(4)}(u), \\ T_{2m+1}^{(3)}(u - \frac{1}{2})T_{2m+1}^{(3)}(u + \frac{1}{2}) &= T_{2m}^{(3)}(u)T_{2m+2}^{(3)}(u) + T_m^{(2)}(u)T_{m+1}^{(2)}(u)T_{2m+1}^{(4)}(u), \\ T_m^{(4)}(u - \frac{1}{2})T_m^{(4)}(u + \frac{1}{2}) &= T_{m-1}^{(4)}(u)T_{m+1}^{(4)}(u) + T_m^{(3)}(u). \end{aligned}$$

For $X_r = G_2$,

$$\begin{aligned}
(2.6) \quad & T_m^{(1)}(u-1)T_m^{(1)}(u+1) = T_{m-1}^{(1)}(u)T_{m+1}^{(1)}(u) + T_{3m}^{(2)}(u), \\
& T_{3m}^{(2)}(u - \frac{1}{3})T_{3m}^{(2)}(u + \frac{1}{3}) = T_{3m-1}^{(2)}(u)T_{3m+1}^{(2)}(u) \\
& \quad + T_m^{(1)}(u - \frac{2}{3})T_m^{(1)}(u)T_m^{(1)}(u + \frac{2}{3}), \\
& T_{3m+1}^{(2)}(u - \frac{1}{3})T_{3m+1}^{(2)}(u + \frac{1}{3}) = T_{3m}^{(2)}(u)T_{3m+2}^{(2)}(u) \\
& \quad + T_m^{(1)}(u - \frac{1}{3})T_m^{(1)}(u + \frac{1}{3})T_{m+1}^{(1)}(u), \\
& T_{3m+2}^{(2)}(u - \frac{1}{3})T_{3m+2}^{(2)}(u + \frac{1}{3}) = T_{3m+1}^{(2)}(u)T_{3m+3}^{(2)}(u) \\
& \quad + T_m^{(1)}(u)T_{m+1}^{(1)}(u - \frac{1}{3})T_{m+1}^{(1)}(u + \frac{1}{3}).
\end{aligned}$$

Definition 2.2 ([KN]). Fix an integer $\ell \geq 2$. The level ℓ restricted Y -system $\mathbb{Y}_\ell(X_r)$ of type X_r is the following system of relations for a family of variables $Y_\ell = \{Y_m^{(a)}(u) \mid (a, m, u) \in \mathcal{I}_\ell\}$, where $Y_m^{(0)}(u) = Y_0^{(a)}(u)^{-1} = Y_{t_a \ell}^{(a)}(u)^{-1} = 0$ if they occur in the right hand sides in the relations:

For $X_r = C_r$,

$$\begin{aligned}
(2.7) \quad & Y_m^{(a)}(u - \frac{1}{2})Y_m^{(a)}(u + \frac{1}{2}) = \frac{(1 + Y_m^{(a-1)}(u))(1 + Y_m^{(a+1)}(u))}{(1 + Y_{m-1}^{(a)}(u)^{-1})(1 + Y_{m+1}^{(a)}(u)^{-1})} \\
& \quad (1 \leq a \leq r-2), \\
& Y_{2m}^{(r-1)}(u - \frac{1}{2})Y_{2m}^{(r-1)}(u + \frac{1}{2}) = \frac{(1 + Y_{2m}^{(r-2)}(u))(1 + Y_m^{(r)}(u))}{(1 + Y_{2m-1}^{(r-1)}(u)^{-1})(1 + Y_{2m+1}^{(r-1)}(u)^{-1})}, \\
& Y_{2m+1}^{(r-1)}(u - \frac{1}{2})Y_{2m+1}^{(r-1)}(u + \frac{1}{2}) = \frac{1 + Y_{2m+1}^{(r-2)}(u)}{(1 + Y_{2m}^{(r-1)}(u)^{-1})(1 + Y_{2m+2}^{(r-1)}(u)^{-1})}, \\
& Y_m^{(r)}(u-1)Y_m^{(r)}(u+1) = \frac{(1 + Y_{2m+1}^{(r-1)}(u))(1 + Y_{2m-1}^{(r-1)}(u))}{(1 + Y_{m-1}^{(r)}(u)^{-1})(1 + Y_{m+1}^{(r)}(u)^{-1})} \\
& \quad \times (1 + Y_{2m}^{(r-1)}(u - \frac{1}{2}))(1 + Y_{2m}^{(r-1)}(u + \frac{1}{2})).
\end{aligned}$$

For $X_r = F_4$,

$$\begin{aligned}
(2.8) \quad & Y_m^{(1)}(u-1)Y_m^{(1)}(u+1) = \frac{1 + Y_m^{(2)}(u)}{(1 + Y_{m-1}^{(1)}(u)^{-1})(1 + Y_{m+1}^{(1)}(u)^{-1})}, \\
& \quad (1 + Y_m^{(1)}(u))(1 + Y_{2m-1}^{(3)}(u))(1 + Y_{2m+1}^{(3)}(u)) \\
& Y_m^{(2)}(u-1)Y_m^{(2)}(u+1) = \frac{\times (1 + Y_{2m}^{(3)}(u - \frac{1}{2}))(1 + Y_{2m}^{(3)}(u + \frac{1}{2}))}{(1 + Y_{m-1}^{(2)}(u)^{-1})(1 + Y_{m+1}^{(2)}(u)^{-1})}, \\
& Y_{2m}^{(3)}(u - \frac{1}{2})Y_{2m}^{(3)}(u + \frac{1}{2}) = \frac{(1 + Y_m^{(2)}(u))(1 + Y_{2m}^{(4)}(u))}{(1 + Y_{2m-1}^{(3)}(u)^{-1})(1 + Y_{2m+1}^{(3)}(u)^{-1})},
\end{aligned}$$

$$(2.8) \quad \begin{aligned} Y_{2m+1}^{(3)}(u - \frac{1}{2})Y_{2m+1}^{(3)}(u + \frac{1}{2}) &= \frac{1 + Y_{2m+1}^{(4)}(u)}{(1 + Y_{2m}^{(3)}(u)^{-1})(1 + Y_{2m+2}^{(3)}(u)^{-1}), \\ \text{(cont.)} \quad Y_m^{(4)}(u - \frac{1}{2})Y_m^{(4)}(u + \frac{1}{2}) &= \frac{1 + Y_m^{(3)}(u)}{(1 + Y_{m-1}^{(4)}(u)^{-1})(1 + Y_{m+1}^{(4)}(u)^{-1}). \end{aligned}$$

For $X_r = G_2$,

$$(2.9) \quad \begin{aligned} Y_m^{(1)}(u-1)Y_m^{(1)}(u+1) &= \frac{(1 + Y_{3m-2}^{(2)}(u))(1 + Y_{3m+2}^{(2)}(u)) \\ &\quad \times (1 + Y_{3m-1}^{(2)}(u - \frac{1}{3}))(1 + Y_{3m-1}^{(2)}(u + \frac{1}{3})) \\ &\quad \times (1 + Y_{3m+1}^{(2)}(u - \frac{1}{3}))(1 + Y_{3m+1}^{(2)}(u + \frac{1}{3})) \\ &\quad \times (1 + Y_{3m}^{(2)}(u - \frac{2}{3}))(1 + Y_{3m}^{(2)}(u + \frac{2}{3})) \\ &\quad \times (1 + Y_{3m}^{(2)}(u))}{(1 + Y_{m-1}^{(1)}(u)^{-1})(1 + Y_{m+1}^{(1)}(u)^{-1}), \\ Y_{3m}^{(2)}(u - \frac{1}{3})Y_{3m}^{(2)}(u + \frac{1}{3}) &= \frac{1 + Y_m^{(1)}(u)}{(1 + Y_{3m-1}^{(2)}(u)^{-1})(1 + Y_{3m+1}^{(2)}(u)^{-1}), \\ Y_{3m+1}^{(2)}(u - \frac{1}{3})Y_{3m+1}^{(2)}(u + \frac{1}{3}) &= \frac{1}{(1 + Y_{3m}^{(2)}(u)^{-1})(1 + Y_{3m+2}^{(2)}(u)^{-1}), \\ Y_{3m+2}^{(2)}(u - \frac{1}{3})Y_{3m+2}^{(2)}(u + \frac{1}{3}) &= \frac{1}{(1 + Y_{3m+1}^{(2)}(u)^{-1})(1 + Y_{3m+3}^{(2)}(u)^{-1}). \end{aligned}$$

Let us write (2.4)–(2.6) in a unified manner

$$(2.10) \quad \begin{aligned} T_m^{(a)}(u - \frac{1}{t_a})T_m^{(a)}(u + \frac{1}{t_a}) \\ = T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u) + \prod_{(b,k,v) \in \mathcal{I}_\ell} T_k^{(b)}(v)^{G(b,k,v;a,m,u)}. \end{aligned}$$

Define the transposition ${}^tG(b, k, v; a, m, u) = G(a, m, u; b, k, v)$. Then (2.7)–(2.9) can be written as

$$(2.11) \quad Y_m^{(a)}(u - \frac{1}{t_a})Y_m^{(a)}(u + \frac{1}{t_a}) = \frac{\prod_{(b,k,v) \in \mathcal{I}_\ell} (1 + Y_k^{(b)}(v))^{G(b,k,v;a,m,u)}}{(1 + Y_{m-1}^{(a)}(u)^{-1})(1 + Y_{m+1}^{(a)}(u)^{-1})}.$$

§2.2. Periodicities

Definition 2.3. Let $\mathcal{T}_\ell(X_r)$ be the commutative ring over \mathbb{Z} with identity element, with generators $T_m^{(a)}(u)^{\pm 1}$ ($(a, m, u) \in \mathcal{I}_\ell$) and relations $\mathbb{T}_\ell(X_r)$ together with $T_m^{(a)}(u)T_m^{(a)}(u)^{-1} = 1$. Let $\mathcal{T}_\ell^\circ(X_r)$ be the subring of $\mathcal{T}_\ell(X_r)$ generated by $T_m^{(a)}(u)$ ($(a, m, u) \in \mathcal{I}_\ell$).

Definition 2.4. Let $\mathcal{Y}_\ell(X_r)$ be the semifield with generators $Y_m^{(a)}(u)$ ($(a, m, u) \in \mathcal{I}_\ell$) and relations $\mathbb{Y}_\ell(X_r)$. Let $\mathcal{Y}_\ell^\circ(X_r)$ be the multiplicative subgroup of $\mathcal{Y}_\ell(X_r)$.

generated by $Y_m^{(a)}(u)$, $1 + Y_m^{(a)}(u)$ ($(a, m, u) \in \mathcal{I}_\ell$). (Here we use the symbol $+$ instead of \oplus for simplicity.)

The first main result of the paper concern the periodicities of the T-systems and Y-systems.

Theorem 2.5 (Conjectured in [IIKNS]). *The following relations hold in $\mathcal{T}_\ell^\circ(X_r)$:*

- (i) *Half periodicity:* $T_m^{(a)}(u + h^\vee + \ell) = T_{t_a \ell - m}^{(a)}(u)$.
- (ii) *Full periodicity:* $T_m^{(a)}(u + 2(h^\vee + \ell)) = T_m^{(a)}(u)$.

Theorem 2.6 (Conjectured in [KNS]). *The following relations hold in $\mathcal{Y}_\ell^\circ(X_r)$:*

- (i) *Half periodicity:* $Y_m^{(a)}(u + h^\vee + \ell) = Y_{t_a \ell - m}^{(a)}(u)$.
- (ii) *Full periodicity:* $Y_m^{(a)}(u + 2(h^\vee + \ell)) = Y_m^{(a)}(u)$.

§2.3. Dilogarithm identities

Let $L(x)$ be the *Rogers dilogarithm function*

$$(2.12) \quad L(x) = -\frac{1}{2} \int_0^x \left\{ \frac{\log(1-y)}{y} + \frac{\log y}{1-y} \right\} dy \quad (0 \leq x \leq 1).$$

We introduce the *constant version* of the Y-system.

Definition 2.7. Fix an integer $\ell \geq 2$. The *level ℓ restricted constant Y-system* $\mathbb{Y}_\ell^c(X_r)$ of type X_r is the following system of relations for a family of variables $Y_\ell^c = \{Y_m^{(a)} \mid a \in I; m = 1, \dots, t_a \ell - 1\}$, where $Y_m^{(0)} = Y_0^{(a)-1} = Y_{t_a \ell}^{(a)-1} = 0$ if they occur on the right hand sides of the relations.

For $X_r = C_r$,

$$(2.13) \quad \begin{aligned} (Y_m^{(a)})^2 &= \frac{(1 + Y_m^{(a-1)})(1 + Y_m^{(a+1)})}{(1 + Y_{m-1}^{(a-1)})(1 + Y_{m+1}^{(a-1)})} \quad (1 \leq a \leq r-2), \\ (Y_{2m}^{(r-1)})^2 &= \frac{(1 + Y_{2m}^{(r-2)})(1 + Y_m^{(r)})}{(1 + Y_{2m-1}^{(r-1)-1})(1 + Y_{2m+1}^{(r-1)-1})}, \\ (Y_{2m+1}^{(r-1)})^2 &= \frac{1 + Y_{2m+1}^{(r-2)}}{(1 + Y_{2m}^{(r-1)-1})(1 + Y_{2m+2}^{(r-1)-1})}, \\ (Y_m^{(r)})^2 &= \frac{(1 + Y_{2m-1}^{(r-1)})(1 + Y_{2m}^{(r-1)})^2(1 + Y_{2m+1}^{(r-1)})}{(1 + Y_{m-1}^{(r)-1})(1 + Y_{m+1}^{(r)-1})}. \end{aligned}$$

For $X_r = F_4$,

$$\begin{aligned}
(Y_m^{(1)})^2 &= \frac{1 + Y_m^{(2)}}{(1 + Y_{m-1}^{(1)-1})(1 + Y_{m+1}^{(1)-1})}, \\
(Y_m^{(2)})^2 &= \frac{(1 + Y_m^{(1)})(1 + Y_{2m-1}^{(3)})(1 + Y_{2m}^{(3)})^2(1 + Y_{2m+1}^{(3)})}{(1 + Y_{m-1}^{(2)-1})(1 + Y_{m+1}^{(2)-1})}, \\
(2.14) \quad (Y_{2m}^{(3)})^2 &= \frac{(1 + Y_m^{(2)})(1 + Y_{2m}^{(4)})}{(1 + Y_{2m-1}^{(3)-1})(1 + Y_{2m+1}^{(3)-1})}, \\
(Y_{2m+1}^{(3)})^2 &= \frac{1 + Y_{2m+1}^{(4)}}{(1 + Y_{2m}^{(3)-1})(1 + Y_{2m+2}^{(3)-1})}, \\
(Y_m^{(4)})^2 &= \frac{1 + Y_m^{(3)}}{(1 + Y_{m-1}^{(4)-1})(1 + Y_{m+1}^{(4)-1})}.
\end{aligned}$$

For $X_r = G_2$,

$$\begin{aligned}
(Y_m^{(1)})^2 &= \frac{(1 + Y_{3m-2}^{(2)})(1 + Y_{3m-1}^{(2)})^2(1 + Y_{3m}^{(2)})^3(1 + Y_{3m+1}^{(2)})^2(1 + Y_{3m+2}^{(2)})}{(1 + Y_{m-1}^{(1)-1})(1 + Y_{m+1}^{(1)-1})}, \\
(2.15) \quad (Y_{3m}^{(2)})^2 &= \frac{1 + Y_m^{(1)}}{(1 + Y_{3m-1}^{(2)-1})(1 + Y_{3m+1}^{(2)-1})}, \\
(Y_{3m+1}^{(2)})^2 &= \frac{1}{(1 + Y_{3m}^{(2)-1})(1 + Y_{3m+2}^{(2)-1})}, \\
(Y_{3m+2}^{(2)})^2 &= \frac{1}{(1 + Y_{3m+1}^{(2)-1})(1 + Y_{3m+3}^{(2)-1})}.
\end{aligned}$$

Proposition 2.8. *There exists a unique positive real solution of $\mathbb{Y}_\ell^c(X_r)$.*

Proof. The proof of [HIKKN, Proposition 1.8] is applicable. \square

The second main result of the paper is the dilogarithm identities conjectured by Kirillov [Ki, Eq. (7)], and properly corrected by Kuniba [Ku, Eqs. (A.1a), (A.1c)].

Theorem 2.9 (Dilogarithm identities). *Suppose that a family of positive real numbers $\{Y_m^{(a)} \mid a \in I; m = 1, \dots, t_a \ell - 1\}$ satisfies $\mathbb{Y}_\ell^c(X_r)$. Then*

$$(2.16) \quad \frac{6}{\pi^2} \sum_{a \in I} \sum_{m=1}^{t_a \ell - 1} L\left(\frac{Y_m^{(a)}}{1 + Y_m^{(a)}}\right) = \frac{\ell \dim \mathfrak{g}}{h^\vee + \ell} - r,$$

where \mathfrak{g} is the simple Lie algebra of type X_r .

The right hand side of (2.16) is equal to

$$(2.17) \quad \frac{r(\ell h - h^\vee)}{h^\vee + \ell}.$$

In fact, we prove a functional generalization of Theorem 2.9.

Theorem 2.10 (Functional dilogarithm identities). *Suppose that a family of positive real numbers $\{Y_m^{(a)}(u) \mid (a, m, u) \in \mathcal{I}_\ell\}$ satisfies $\mathbb{Y}_\ell(X_r)$. Then*

$$(2.18) \quad \frac{6}{\pi^2} \sum_{\substack{(a,m,u) \in \mathcal{I}_\ell \\ 0 \leq u < 2(h^\vee + \ell)}} L\left(\frac{Y_m^{(a)}(u)}{1 + Y_m^{(a)}(u)}\right) = 2tr(\ell h - h^\vee) = \begin{cases} 4r(2r\ell - r - 1), & C_r, \\ 48(4\ell - 3), & F_4, \\ 24(3\ell - 2), & G_2, \end{cases}$$

$$(2.19) \quad \frac{6}{\pi^2} \sum_{\substack{(a,m,u) \in \mathcal{I}_\ell \\ 0 \leq u < 2(h^\vee + \ell)}} L\left(\frac{1}{1 + Y_m^{(a)}(u)}\right) = \begin{cases} 4\ell(2r\ell - \ell - 1), & C_r, \\ 8\ell(3\ell + 1), & F_4, \\ 12\ell(2\ell + 1), & G_2. \end{cases}$$

The two identities (2.18) and (2.19) are equivalent to each other, since the sum of the right hand sides is equal to $2t(h^\vee + \ell)((\sum_{a \in I} t_a)\ell - r)$, which is the total number of $(a, m, u) \in \mathcal{I}_\ell$ in the region $0 \leq u < 2(h^\vee + \ell)$.

It is clear that Theorem 2.9 follows from Theorem 2.10.

§3. Type C_r

The C_r case is quite parallel to the B_r case. For the reader's convenience, we repeat most of the basic definitions and results from [IIKKN]. Most propositions are proved in a parallel manner to the B_r case, so that proofs are omitted. The properties of the tropical Y-system at level 2 (Proposition 3.10) are crucial and specific to C_r . Since its derivation is a little more complicated than in the B_r case, an outline of the proof is provided.

§3.1. Parity decompositions of T-systems and Y-systems

For a triplet $(a, m, u) \in \mathcal{I}_\ell$, we define the ‘parity conditions’ \mathbf{P}_+ and \mathbf{P}_- by

$$(3.1) \quad \mathbf{P}_+ : r + a + m + 2u \text{ is odd if } a \neq r; 2u \text{ is even if } a = r,$$

$$(3.2) \quad \mathbf{P}_- : r + a + m + 2u \text{ is even if } a \neq r; 2u \text{ is odd if } a = r.$$

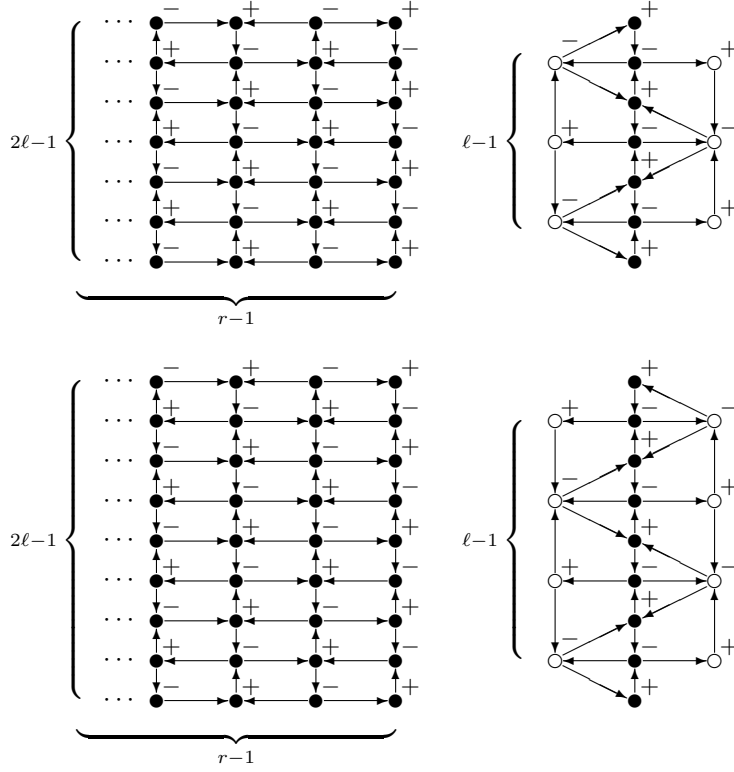


Figure 1. The quiver $Q_\ell(C_r)$ for ℓ even (top) and for ℓ odd (bottom), where we identify the rightmost column in the left quiver with the middle column in the right quiver.

We write, for example, $(a, m, u) : \mathbf{P}_+$ if (a, m, u) satisfies \mathbf{P}_+ . We have $\mathcal{I}_\ell = \mathcal{I}_{\ell+} \sqcup \mathcal{I}_{\ell-}$, where $\mathcal{I}_{\ell\varepsilon}$ is the set of all $(a, m, u) : \mathbf{P}_\varepsilon$.

Define $\mathcal{T}_\ell^\circ(C_r)_\varepsilon$ ($\varepsilon = \pm$) to be the subring of $\mathcal{T}_\ell^\circ(C_r)$ generated by $T_m^{(a)}(u)$ ($(a, m, u) \in \mathcal{I}_{\ell\varepsilon}$). Then we have $\mathcal{T}_\ell^\circ(C_r)_+ \simeq \mathcal{T}_\ell^\circ(C_r)_-$ via $T_m^{(a)}(u) \mapsto T_m^{(a)}(u + \frac{1}{2})$ and

$$(3.3) \quad \mathcal{T}_\ell^\circ(C_r) \simeq \mathcal{T}_\ell^\circ(C_r)_+ \otimes_{\mathbb{Z}} \mathcal{T}_\ell^\circ(C_r)_-.$$

For a triplet $(a, m, u) \in \mathcal{I}_\ell$, we set other ‘parity conditions’ \mathbf{P}'_+ and \mathbf{P}'_- by

$$(3.4) \quad \mathbf{P}'_+ : r + a + m + 2u \text{ is even if } a \neq r; 2u \text{ is even if } a = r,$$

$$(3.5) \quad \mathbf{P}'_- : r + a + m + 2u \text{ is odd if } a \neq r; 2u \text{ is odd if } a = r.$$

We have $\mathcal{I}_\ell = \mathcal{I}'_{\ell+} \sqcup \mathcal{I}'_{\ell-}$, where $\mathcal{I}'_{\ell\varepsilon}$ is the set of all $(a, m, u) : \mathbf{P}'_\varepsilon$. We also have

$$(3.6) \quad (a, m, u) : \mathbf{P}'_+ \Leftrightarrow (a, m, u \pm \frac{1}{t_a}) : \mathbf{P}_+.$$

Define $\mathcal{Y}_\ell^\circ(C_r)_\varepsilon$ ($\varepsilon = \pm$) to be the subgroup of $\mathcal{Y}_\ell^\circ(C_r)$ generated by $Y_m^{(a)}(u)$, $1 + Y_m^{(a)}(u)$ ($(a, m, u) \in \mathcal{I}'_{\ell\varepsilon}$). Then $\mathcal{Y}_\ell^\circ(C_r)_+ \simeq \mathcal{Y}_\ell^\circ(C_r)_-$ via $Y_m^{(a)}(u) \mapsto Y_m^{(a)}(u + \frac{1}{2})$, $1 + Y_m^{(a)}(u) \mapsto 1 + Y_m^{(a)}(u + \frac{1}{2})$, and

$$(3.7) \quad \mathcal{Y}_\ell^\circ(C_r) \simeq \mathcal{Y}_\ell^\circ(C_r)_+ \times \mathcal{Y}_\ell^\circ(C_r)_-.$$

§3.2. Quiver $Q_\ell(C_r)$

With type C_r and $\ell \geq 2$ we associate the quiver $Q_\ell(C_r)$ by Figure 1, where the rightmost column in the left quiver and the middle column in the right quiver are identified. Also, we assign an empty or filled circle \circ/\bullet and a sign $+/-$ to each vertex.

Let us choose the index set \mathbf{I} of the vertices of $Q_\ell(C_r)$ so that $\mathbf{i} = (i, i') \in \mathbf{I}$ represents the vertex in the i' th row (from the bottom) and the i th column (from the left) of the left quiver for $i = 1, \dots, r-1$, in the right column of the right quiver for $i = r$, and in the left column of the right quiver for $i = r+1$. Thus, $i = 1, \dots, r+1$, and $i' = 1, \dots, \ell-1$ if $i \neq r, r+1$, while $i' = 1, \dots, 2\ell-1$ if $i = r, r+1$. We use the natural notation \mathbf{I}° (resp. \mathbf{I}_+°) for the set of vertices \mathbf{i} with property \circ (resp. \circ and $+$), and so on. We have $\mathbf{I} = \mathbf{I}^\circ \sqcup \mathbf{I}^\bullet = \mathbf{I}_+^\circ \sqcup \mathbf{I}_-^\circ \sqcup \mathbf{I}_+^\bullet \sqcup \mathbf{I}_-^\bullet$.

We define composite mutations,

$$(3.8) \quad \mu_+^\circ = \prod_{\mathbf{i} \in \mathbf{I}_+^\circ} \mu_{\mathbf{i}}, \quad \mu_-^\circ = \prod_{\mathbf{i} \in \mathbf{I}_-^\circ} \mu_{\mathbf{i}}, \quad \mu_+^\bullet = \prod_{\mathbf{i} \in \mathbf{I}_+^\bullet} \mu_{\mathbf{i}}, \quad \mu_-^\bullet = \prod_{\mathbf{i} \in \mathbf{I}_-^\bullet} \mu_{\mathbf{i}}.$$

Note that they do not depend on the order of the product.

Let \mathbf{r} be the involution acting on \mathbf{I} by left-right reflection of the right quiver. Let ω be the involution acting on \mathbf{I} defined by, for even r , up-down reflection of the left quiver and 180° rotation of the right quiver; and for odd r , up-down reflection of the left and right quivers. Let $\mathbf{r}(Q_\ell(C_r))$ and $\omega(Q_\ell(C_r))$ denote the quivers induced from $Q_\ell(C_r)$ by \mathbf{r} and ω , respectively. For example, if there is an arrow $\mathbf{i} \rightarrow \mathbf{j}$ in $Q_\ell(C_r)$, then there is an arrow $\mathbf{r}(\mathbf{i}) \rightarrow \mathbf{r}(\mathbf{j})$ in $\mathbf{r}(Q_\ell(C_r))$. For a quiver Q , Q^{op} denotes the opposite quiver.

Lemma 3.1. *Let $Q = Q_\ell(C_r)$.*

(i) *We have a periodic sequence of mutations of quivers*

$$(3.9) \quad Q \xleftarrow{\mu_+^\bullet \mu_+^\circ} Q^{\text{op}} \xleftarrow{\mu_-^\bullet} \mathbf{r}(Q) \xleftarrow{\mu_+^\bullet \mu_-^\circ} \mathbf{r}(Q)^{\text{op}} \xleftarrow{\mu_-^\bullet} Q.$$

(ii) $\omega(Q) = Q$ if $h^\vee + \ell$ is even, and $\omega(Q) = \mathbf{r}(Q)$ if $h^\vee + \ell$ is odd.

§3.3. Cluster algebra and alternative labels

It is standard to identify a quiver Q with no loop and no 2-cycle with a skew-symmetric matrix B . We use the following convention for the direction of arrows:

$$(3.10) \quad i \rightarrow j \Leftrightarrow B_{ij} = 1.$$

(In this paper we only encounter the situation where $B_{ij} = -1, 0, 1$.) Let $B_\ell(C_r)$ be the skew-symmetric matrix corresponding to the quiver $Q_\ell(C_r)$. In the rest of the section, we set $B = (B_{ij})_{i,j \in \mathbf{I}} = B_\ell(C_r)$ unless otherwise mentioned.

Let $\mathcal{A}(B, x, y)$ be the *cluster algebra with coefficients in the universal semifield* $\mathbb{Q}_{\text{sf}}(y)$, where (B, x, y) is the initial seed [FZ2]. See also [IIKKN, Section 2.1] for the conventions and notations on cluster algebras we employ. (Here we use the symbol $+$ instead of \oplus in $\mathbb{Q}_{\text{sf}}(y)$, since it is the ordinary addition of subtraction-free expressions of rational functions of y .)

Definition 3.2. The *coefficient group* $\mathcal{G}(B, y)$ associated with $\mathcal{A}(B, x, y)$ is the multiplicative subgroup of the semifield $\mathbb{Q}_{\text{sf}}(y)$ generated by all the coefficients y'_i of $\mathcal{A}(B, x, y)$ together with $1 + y'_i$.

In view of Lemma 3.1 we set $x(0) = x$, $y(0) = y$ and define clusters $x(u) = (x_i(u))_{i \in \mathbf{I}}$ ($u \in \frac{1}{2}\mathbb{Z}$) and coefficient tuples $y(u) = (y_i(u))_{i \in \mathbf{I}}$ ($u \in \frac{1}{2}\mathbb{Z}$) by the sequence of mutations

$$(3.11) \quad \cdots \xleftarrow{\mu_-^\bullet} (B, x(0), y(0)) \xleftarrow{\mu_+^\bullet \mu_+^\circ} (-B, x(\frac{1}{2}), y(\frac{1}{2})) \\ \xleftarrow{\mu_-^\bullet} (\mathbf{r}(B), x(1), y(1)) \xleftarrow{\mu_+^\bullet \mu_-^\circ} (-\mathbf{r}(B), x(\frac{3}{2}), y(\frac{3}{2})) \xleftarrow{\mu_-^\bullet} \cdots,$$

where $\mathbf{r}(B) = B'$ is defined by $B'_{ij} = B_{\mathbf{r}(i)\mathbf{r}(j)}$.

For a pair $(\mathbf{i}, u) \in \mathbf{I} \times \frac{1}{2}\mathbb{Z}$, we define the parity conditions \mathbf{p}_+ and \mathbf{p}_- by

$$(3.12) \quad \mathbf{p}_+ : \begin{cases} \mathbf{i} \in \mathbf{I}_+^\circ \sqcup \mathbf{I}_+^\bullet, & u \equiv 0, \\ \mathbf{i} \in \mathbf{I}_-^\bullet, & u \equiv \frac{1}{2}, \frac{3}{2}, \\ \mathbf{i} \in \mathbf{I}_-^\circ \sqcup \mathbf{I}_+^\bullet, & u \equiv 1, \end{cases} \quad \mathbf{p}_- : \begin{cases} \mathbf{i} \in \mathbf{I}_+^\circ \sqcup \mathbf{I}_+^\bullet, & u \equiv \frac{1}{2}, \\ \mathbf{i} \in \mathbf{I}_-^\bullet, & u \equiv 0, 1, \\ \mathbf{i} \in \mathbf{I}_-^\circ \sqcup \mathbf{I}_+^\bullet, & u \equiv \frac{3}{2}, \end{cases}$$

where \equiv is equivalence modulo $2\mathbb{Z}$. We have

$$(3.13) \quad (\mathbf{i}, u) : \mathbf{p}_+ \Leftrightarrow (\mathbf{i}, u + \frac{1}{2}) : \mathbf{p}_-.$$

Each $(\mathbf{i}, u) : \mathbf{p}_+$ is a mutation point of (3.11) in the forward direction of u , and each $(\mathbf{i}, u) : \mathbf{p}_-$ is one in the backward direction of u . Notice that there are also

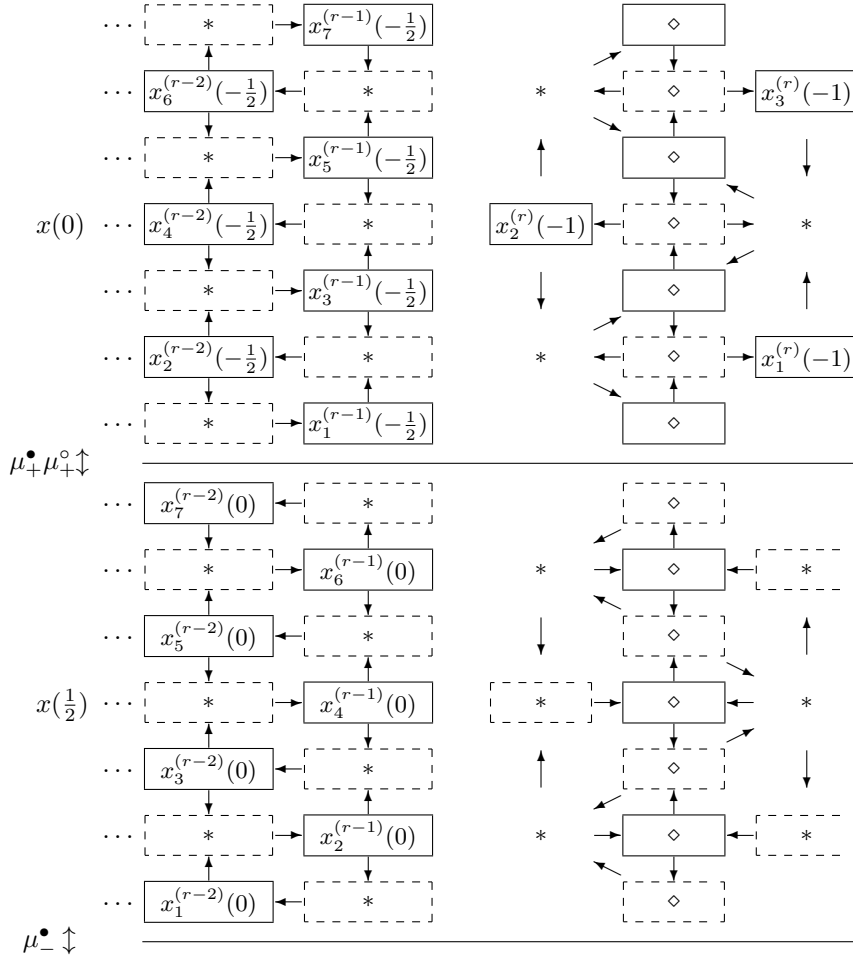


Figure 2. (Continued in Figure 3.) Labeling of cluster variables $x_i(u)$ by $\mathcal{I}_{\ell+}$ for C_r , $\ell = 4$. The variables framed by solid/dashed lines satisfy the condition $\mathbf{p}_+/\mathbf{p}_-$, respectively. The middle column in the right quiver (marked by \diamond) is identified with the rightmost column in the left quiver.

some (\mathbf{i}, u) which satisfy neither \mathbf{p}_+ nor \mathbf{p}_- , and are not mutation points of (3.11); explicitly, they are (\mathbf{i}, u) with $\mathbf{i} \in \mathbf{I}_+^\circ$, $u \equiv 1, \frac{3}{2} \pmod{2\mathbb{Z}}$, or with $\mathbf{i} \in \mathbf{I}_-^\circ$, $u \equiv 0, \frac{1}{2} \pmod{2\mathbb{Z}}$.

There is a correspondence between the parity condition \mathbf{p}_\pm here and $\mathbf{P}_\pm, \mathbf{P}'_\pm$ in (3.1) and (3.4).

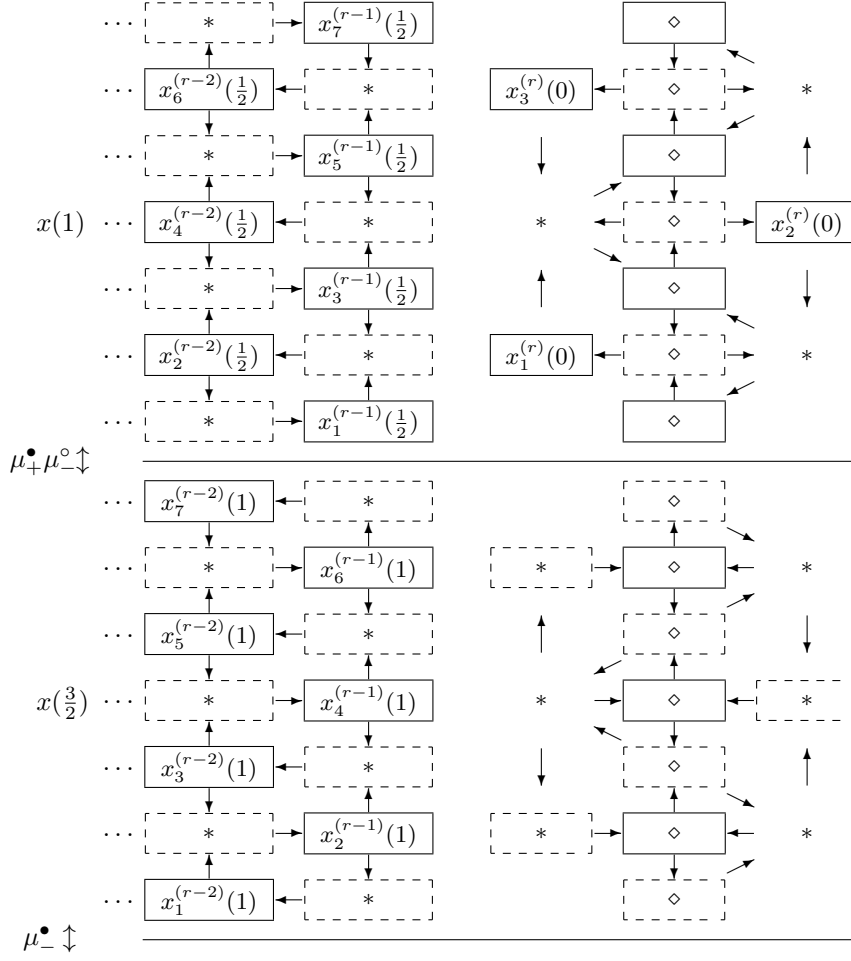


Figure 3. Continuation of Figure 2.

Lemma 3.3. *Below \equiv means equivalence modulo $2\mathbb{Z}$.*

(i) *The map $g : \mathcal{I}_{\ell_+} \rightarrow \{(\mathbf{i}, u) : \mathbf{p}_+\}$ given by*

$$(3.14) \quad (a, m, u - \frac{1}{t_a}) \mapsto \begin{cases} ((a, m), u), & a \neq r, \\ ((r+1, m), u), & a = r; m + u \equiv 0, \\ ((r, m), u), & a = r; m + u \equiv 1, \end{cases}$$

is a bijection.

(ii) The map $g' : \mathcal{I}'_{\ell_+} \rightarrow \{(\mathbf{i}, u) : \mathbf{p}_+ \text{ or } \mathbf{p}_-\}$ given by

$$(3.15) \quad (a, m, u) \mapsto \begin{cases} ((a, m), u), & a \neq r, \\ ((r+1, m), u), & a = r; m+u \equiv 0, \\ ((r, m), u), & a = r; m+u \equiv 1, \end{cases}$$

is a bijection.

We introduce alternative labels $x_{\mathbf{i}}(u) = x_m^{(a)}(u-1/t_a)$ ($(a, m, u-1/t_a) \in \mathcal{I}_{\ell_+}$) for $(\mathbf{i}, u) = g((a, m, u-1/t_a))$ and $y_{\mathbf{i}}(u) = y_m^{(a)}(u)$ ($(a, m, u) \in \mathcal{I}'_{\ell_+}$) for $(\mathbf{i}, u) = g'((a, m, u))$, respectively. See Figures 2–3.

§3.4. T-system and cluster algebra

The result in this subsection is completely parallel to the B_r case [IIKKN].

Let $\mathcal{A}(B, x)$ be the cluster algebra with trivial coefficients, where (B, x) is the initial seed [FZ2]. Let $\mathbf{1} = \{1\}$ be the *trivial semifield* and $\pi_{\mathbf{1}} : \mathbb{Q}_{\text{sf}}(y) \rightarrow \mathbf{1}$, $y_{\mathbf{i}} \mapsto 1$, be the projection. Let $[x_{\mathbf{i}}(u)]_{\mathbf{1}}$ denote the image of $x_{\mathbf{i}}(u)$ under the algebra homomorphism $\mathcal{A}(B, x, y) \rightarrow \mathcal{A}(B, x)$ induced by $\pi_{\mathbf{1}}$. It is called the *trivial evaluation*.

Recall that $G(b, k, v; a, m, u)$ is defined in (2.10).

Lemma 3.4. *The family $\{x_m^{(a)}(u) \mid (a, m, u) \in \mathcal{I}_{\ell_+}\}$ satisfies a system of relations*

$$(3.16) \quad x_m^{(a)}\left(u - \frac{1}{t_a}\right)x_m^{(a)}\left(u + \frac{1}{t_a}\right) = \frac{y_m^{(a)}(u)}{1 + y_m^{(a)}(u)} \prod_{(b, k, v) \in \mathcal{I}_{\ell_+}} x_k^{(b)}(v)^{G(b, k, v; a, m, u)} \\ + \frac{1}{1 + y_m^{(a)}(u)} x_{m-1}^{(a)}(u)x_{m+1}^{(a)}(u),$$

where $(a, m, u) \in \mathcal{I}'_{\ell_+}$. In particular, the family $\{[x_m^{(a)}(u)]_{\mathbf{1}} \mid (a, m, u) \in \mathcal{I}_{\ell_+}\}$ satisfies the T -system $\mathbb{T}_{\ell}(C_r)$ in $\mathcal{A}(B, x)$ after replacing $T_m^{(a)}(u)$ with $[x_m^{(a)}(u)]_{\mathbf{1}}$.

Definition 3.5. The T -subalgebra $\mathcal{A}_T(B, x)$ of $\mathcal{A}(B, x, y)$ associated with the sequence (3.11) is the subalgebra of $\mathcal{A}(B, x)$ generated by $[x_{\mathbf{i}}(u)]_{\mathbf{1}}$ ($(\mathbf{i}, u) \in \mathbf{I} \times \frac{1}{2}\mathbb{Z}$).

Theorem 3.6. *The ring $\mathcal{T}_{\ell}^{\circ}(C_r)_+$ is isomorphic to $\mathcal{A}_T(B, x)$ via the correspondence $T_m^{(a)}(u) \mapsto [x_m^{(a)}(u)]_{\mathbf{1}}$.*

§3.5. Y-system and cluster algebra

The result in this subsection is completely parallel to the B_r case [IIKKN].

Lemma 3.7. *The family $\{y_m^{(a)}(u) \mid (a, m, u) \in \mathcal{I}'_{\ell+}\}$ satisfies the Y -system $\mathbb{Y}_\ell(C_r)$ after replacing $Y_m^{(a)}(u)$ with $y_m^{(a)}(u)$.*

Definition 3.8. The Y -subgroup $\mathcal{G}_Y(B, y)$ of $\mathcal{G}(B, y)$ associated with the sequence (3.11) is the subgroup of $\mathcal{G}(B, y)$ generated by $y_{\mathbf{i}}(u)$ ($(\mathbf{i}, u) \in \mathbf{I} \times \frac{1}{2}\mathbb{Z}$) and $1 + y_{\mathbf{i}}(u)$ ($(\mathbf{i}, u) : \mathbf{p}_+$ or \mathbf{p}_-).

Theorem 3.9. *The group $\mathcal{Y}_\ell^{\circ}(C_r)_+$ is isomorphic to $\mathcal{G}_Y(B, y)$ via the correspondence $Y_m^{(a)}(u) \mapsto y_m^{(a)}(u)$ and $1 + Y_m^{(a)}(u) \mapsto 1 + y_m^{(a)}(u)$.*

§3.6. Tropical Y-system at level 2

The tropical semifield $\text{Trop}(y)$ is an abelian multiplicative group freely generated by the elements $y_{\mathbf{i}}$ ($\mathbf{i} \in \mathbf{I}$) with addition

$$(3.17) \quad \prod_{\mathbf{i} \in \mathbf{I}} y_{\mathbf{i}}^{a_{\mathbf{i}}} \oplus \prod_{\mathbf{i} \in \mathbf{I}} y_{\mathbf{i}}^{b_{\mathbf{i}}} = \prod_{\mathbf{i} \in \mathbf{I}} y_{\mathbf{i}}^{\min(a_{\mathbf{i}}, b_{\mathbf{i}})}.$$

Let $\pi_{\mathbf{T}} : \mathbb{Q}_{\text{sf}}(y) \rightarrow \text{Trop}(y)$, $y_{\mathbf{i}} \mapsto y_{\mathbf{i}}$, be the projection. Let $[y_{\mathbf{i}}(u)]_{\mathbf{T}}$ and $[\mathcal{G}_Y(B, y)]_{\mathbf{T}}$ denote the images of $y_{\mathbf{i}}(u)$ and $\mathcal{G}_Y(B, y)$ under the multiplicative group homomorphism induced by $\pi_{\mathbf{T}}$, respectively. They are called the *tropical evaluations*, and the resulting relations in the group $[\mathcal{G}_Y(B, y)]_{\mathbf{T}}$ form the *tropical Y-system*.

We say a (Laurent) monomial $m = \prod_{\mathbf{i} \in \mathbf{I}} y_{\mathbf{i}}^{k_{\mathbf{i}}}$ is *positive* (resp. *negative*) if $m \neq 1$ and $k_{\mathbf{i}} \geq 0$ (resp. $k_{\mathbf{i}} \leq 0$) for any \mathbf{i} .

The following properties of the tropical Y-system at level 2 will be the key in the entire method.

Proposition 3.10. *For $[\mathcal{G}_Y(B, y)]_{\mathbf{T}}$ with $B = B_2(C_r)$, the following facts hold:*

- (i) *Let $0 \leq u < 2$. For any $(\mathbf{i}, u) : \mathbf{p}_+$, the monomial $[y_{\mathbf{i}}(u)]_{\mathbf{T}}$ is positive.*
- (ii) *Let $-h^{\vee} \leq u < 0$.*
 - (a) *Let $\mathbf{i} = (i, 2)$ ($i \leq r - 1$), $(r, 1)$, or $(r + 1, 1)$. For any $(\mathbf{i}, u) : \mathbf{p}_+$, the monomial $[y_{\mathbf{i}}(u)]_{\mathbf{T}}$ is negative.*
 - (b) *Let $\mathbf{i} = (i, 1), (i, 3)$ ($i \leq r - 1$). For any $(\mathbf{i}, u) : \mathbf{p}_+$, the monomial $[y_{\mathbf{i}}(u)]_{\mathbf{T}}$ is positive for $u = -\frac{1}{2}h^{\vee}, -\frac{1}{2}h^{\vee} - \frac{1}{2}$ and negative otherwise.*
- (iii) *$y_{ii'}(2)$ equals $y_{i, 4-i'}^{-1}$ if $i \leq r - 1$, and $y_{ii'}^{-1}$ if $i = r, r + 1$.*
- (iv) *For even r , $y_{ii'}(-h^{\vee})$ equals $y_{ii'}^{-1}$ if $i \leq r - 1$, and $y_{2r+1-i, i'}^{-1}$ if $i = r, r + 1$. For odd r , $y_{ii'}(-h^{\vee}) = y_{ii'}^{-1}$.*

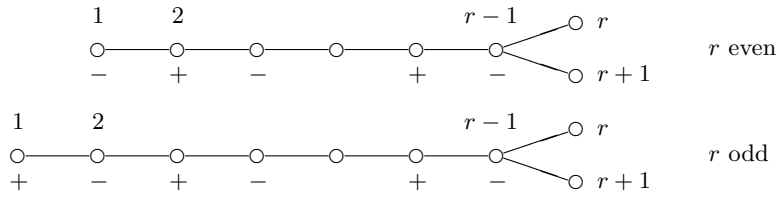
One can directly verify (i) and (iii) in the same way as in the B_r case [IIKKN, Proposition 3.2]. In the rest of this subsection we give the outline of the proof

of (ii) and (iv). Note that (ii) and (iv) can be proved independently for each variable y_i . (To be precise, we also need to ensure that no monomial is 1. However, this can be easily guaranteed, so that we do not describe the details here.) Below we separate the variables into two parts. Here is a brief summary of the results.

(1) *The D part.* The powers of $[y_i(u)]_{\mathbf{T}}$ in the variables $y_{i,2}$ ($i \leq r-1$) and $y_{r,1}, y_{r+1,1}$ are described by the root system of type D_{r+1} with a Coxeter-like transformation. It turns out that they are further described by (a subset of) the root system of type A_{2r+1} with the Coxeter transformation.

(2) *The A part.* The powers of $[y_i(u)]_{\mathbf{T}}$ in the variables $y_{i,1}$ and $y_{i,3}$ ($i \leq r-1$) are mainly described by the root system of type A_{r-1} with the Coxeter transformation.

3.6.1. D part. Let us consider the D part first. Let D_{r+1} be the Dynkin diagram of type D with index set $J = \{1, \dots, r+1\}$. We assign a sign $+/-$ to vertices of D_{r+1} (no sign for r and $r+1$) as inherited from $Q_2(C_r)$.



Let $\Pi = \{\alpha_1, \dots, \alpha_{r+1}\}$, $-\Pi$, Φ_+ be the sets of simple roots, of negative simple roots, and of positive roots, respectively, of type D_{r+1} . Following [FZ1], we introduce the *piecewise-linear analogue* σ_i of the simple reflection s_i , acting on the set $\Phi_{\geq -1} = \Phi_+ \sqcup (-\Pi)$ of *almost positive roots* by

$$(3.18) \quad \begin{aligned} \sigma_i(\alpha) &= s_i(\alpha), \quad \alpha \in \Phi_+, \\ \sigma_i(-\alpha_j) &= \begin{cases} \alpha_j, & j = i, \\ -\alpha_j, & \text{otherwise.} \end{cases} \end{aligned}$$

Let

$$(3.19) \quad \sigma_+ = \prod_{i \in J_+} \sigma_i, \quad \sigma_- = \prod_{i \in J_-} \sigma_i,$$

where J_{\pm} is the set of vertices of D_{r+1} with the respective sign. We define

$$(3.20) \quad \sigma = \sigma_- \sigma_+ \sigma_{r+1} \sigma_- \sigma_+ \sigma_r.$$

Lemma 3.11. *The following facts hold:*

(I) *Let r be even.*

- (i) *For $i \leq r - 1$, $\sigma^k(-\alpha_i) \in \Phi_+$ ($1 \leq k \leq r/2$), $\sigma^{r/2+1}(-\alpha_i) = -\alpha_i$.*
- (ii) *For $i \leq r - 1$, $\sigma^k(\alpha_i) \in \Phi_+$ ($0 \leq k \leq r/2$), $\sigma^{r/2+1}(\alpha_i) = \alpha_i$.*
- (iii) *$\sigma^k(-\alpha_r) \in \Phi_+$ ($1 \leq k \leq r/2$), $\sigma^{r/2+1}(-\alpha_r) = -\alpha_{r+1}$.*
- (iv) *$\sigma^k(-\alpha_{r+1}) \in \Phi_+$ ($1 \leq k \leq r/2 + 1$), $\sigma^{r/2+2}(-\alpha_{r+1}) = -\alpha_r$.*
- (v) *The elements of Φ_+ appearing in (i)–(iv) exhaust the set Φ_+ , thereby providing the orbit decomposition of Φ_+ under σ .*

(II) *Let r be odd.*

- (i) *For $i \in J_+$, $\sigma^k(-\alpha_i) \in \Phi_+$ ($1 \leq k \leq r + 1$), $\sigma^{r+2}(-\alpha_i) = -\alpha_i$, $\sigma^{(r+1)/2}(-\alpha_i) = \alpha_i$.*
- (ii) *For $i \in J_-$, $\sigma^k(-\alpha_i) \in \Phi_+$ ($1 \leq k \leq r + 1$), $\sigma^{r+2}(-\alpha_i) = -\alpha_i$, $\sigma^{(r+3)/2}(-\alpha_i) = \alpha_i$.*
- (iii) *$\sigma^k(-\alpha_r) \in \Phi_+$ ($1 \leq k \leq (r + 1)/2$), $\sigma^{(r+3)/2}(-\alpha_r) = -\alpha_r$.*
- (iv) *$\sigma^k(-\alpha_{r+1}) \in \Phi_+$ ($1 \leq k \leq (r + 1)/2$), $\sigma^{(r+3)/2}(-\alpha_{r+1}) = -\alpha_{r+1}$.*
- (v) *The elements of Φ_+ appearing in (i)–(iv) exhaust the set Φ_+ , thereby providing the orbit decomposition of Φ_+ under σ .*

Proof. The statements are verified by explicitly calculating $\sigma^k(-\alpha_i)$ and $\sigma^k(\alpha_i)$. The examples for $r = 10$ (for even r) and 9 (for odd r) are given in Tables 1 and 2, respectively, where we use the notations

(3.21)

$$\begin{aligned} [i, j] &= \alpha_i + \cdots + \alpha_j \quad (1 \leq i < j \leq r), \quad [i] = \alpha_i \quad (1 \leq i \leq r), \\ \{i, j\} &= (\alpha_i + \cdots + \alpha_{r-1}) + (\alpha_j + \cdots + \alpha_{r+1}) \quad (1 \leq i < j \leq r + 1, i \leq r - 1), \end{aligned}$$

and $\{r + 1\} = \alpha_{r+1}$. In fact, it is not difficult to read off the general rule from these examples. \square

The orbits $\sigma(-\alpha_i)$ and $\sigma(\alpha_i)$ are further described by (a subset of) the *root system of type A_{2r+1}* . Let $\Pi' = \{\alpha'_1, \dots, \alpha'_{2r+1}\}$ and Φ'_+ be the sets of simple roots and of positive roots of type A_{2r+1} , respectively, with standard index set $J' = \{1, \dots, 2r + 1\}$. Define $J'_+ = \{i \in J' \mid i - r \text{ is even}\}$ and $J'_- = \{i \in J' \mid i - r \text{ is odd}\}$. We introduce the notations $[i, j]' = \alpha'_i + \cdots + \alpha'_j$ ($1 \leq i < j \leq 2r + 1$) and $[i]' = \alpha'_i$, parallel to (3.21). Let $O'_i = \{(\sigma')^k(-\alpha'_i) \mid 1 \leq k \leq r + 1\}$ be the orbit of $-\alpha'_i$ in Φ'_+ under $\sigma' = \sigma'_- \sigma'_+$, $\sigma'_\pm = \prod_{i \in J'_\pm} \sigma'_i$, where σ'_i is the piecewise-linear analogue of the simple reflection s'_i acting as in (3.18).

Table 1. The orbits $\sigma^k(-\alpha_i)$ and $\sigma^k(\alpha_i)$ in Φ_+ under σ of (3.20) for $r = 10$. The orbits of $-\alpha_i$ and α_i ($i \leq 8$), for example, $-\alpha_1 \rightarrow [2, 3] \rightarrow [6, 7] \rightarrow \dots \rightarrow -\alpha_1$ and $\alpha_1 \rightarrow [4, 5] \rightarrow [8, 9] \rightarrow \dots \rightarrow \alpha_1$, are aligned alternately. The orbits of $-\alpha_{10}$ and $-\alpha_{11}$, namely, $-\alpha_{10} \rightarrow \{7, 9\} \rightarrow \{3, 5\} \rightarrow \dots \rightarrow -\alpha_{11}$, and $-\alpha_{11} \rightarrow \{9, 11\} \rightarrow \{5, 7\} \rightarrow \dots \rightarrow -\alpha_{10}$, are aligned alternately. The numbers $-1, -2, \dots$ in the head line will be identified with the parameter u in (3.24).

		-1	-2	-3	-4	-5	-6	-7	-8	-9	-10	-11	
1 -	$-\alpha_1$	[1]	[2, 3]	[4, 5]	[6, 7]	[8, 9]	{11}	[9, 10]	[7, 8]	[5, 6]	[3, 4]	[1, 2]	$-\alpha_1$
2 +	α_2	[1, 3]	[2, 5]	[4, 7]	[6, 9]	{8, 11}	{9, 10}	[7, 10]	[5, 8]	[3, 6]	[1, 4]	[2]	$-\alpha_2$
3 -	$-\alpha_3$	[3]	[1, 5]	[2, 7]	[4, 9]	{6, 11}	{8, 9}	{7, 10}	[5, 10]	[3, 8]	[1, 6]	[2, 4]	α_3
4 +	α_4	[3, 5]	[1, 7]	[2, 9]	{4, 11}	{6, 9}	{7, 8}	{5, 10}	[3, 10]	[1, 8]	[2, 6]	[4]	$-\alpha_4$
5 -	$-\alpha_5$	[5]	[3, 7]	[1, 9]	{2, 11}	{4, 9}	{6, 7}	{5, 8}	{3, 10}	[1, 10]	[2, 8]	[4, 6]	α_5
6 +	α_6	[5, 7]	[3, 9]	{1, 11}	{2, 9}	{4, 7}	{5, 6}	{3, 8}	{1, 10}	[2, 10]	[4, 8]	[6]	$-\alpha_6$
7 -	$-\alpha_7$	[7]	[5, 9]	{3, 11}	{1, 9}	{2, 7}	{4, 5}	{3, 6}	{1, 8}	{2, 10}	[4, 10]	[6, 8]	α_7
8 +	α_8	[7, 9]	{5, 11}	{3, 9}	{1, 7}	{2, 5}	{3, 4}	{1, 6}	{2, 8}	{4, 10}	[6, 10]	[8]	$-\alpha_8$
9 -	$-\alpha_9$	[9]	{7, 11}	{5, 9}	{3, 7}	{1, 5}	{2, 3}	{1, 4}	{2, 6}	{4, 8}	{6, 10}	[8, 10]	α_9
	$-\alpha_{11}$	{9, 11}	{7, 9}	{5, 7}	{3, 5}	{1, 3}	{1, 2}	{2, 4}	{4, 6}	{6, 8}	{8, 10}	[10]	$-\alpha_{11}$
	$-\alpha_{10}$												$-\alpha_{10}$

Table 2. The orbit of $\sigma^k(-\alpha_i)$ in Φ_+ under σ of (3.20) for $r = 9$. The orbit of $-\alpha_i$ ($i \leq 8$), for example, $-\alpha_1 \rightarrow [3, 4] \rightarrow [7, 8] \rightarrow \dots \rightarrow \alpha_1 \rightarrow [1, 2] \rightarrow [5, 6] \rightarrow \dots \rightarrow -\alpha_1$, is aligned in a cyclic and alternating way. The orbits of $-\alpha_9$ and $-\alpha_{10}$, namely, $-\alpha_9 \rightarrow \{6, 8\} \rightarrow \{2, 4\} \rightarrow \dots \rightarrow -\alpha_9$, and $-\alpha_{10} \rightarrow \{8, 10\} \rightarrow \{4, 6\} \rightarrow \dots \rightarrow -\alpha_{10}$, are aligned alternatingly. The numbers $-1, -2, \dots$ in the head line will be identified with the parameter u in (3.24).

		-1	-2	-3	-4	-5	-6	-7	-8	-9	-10	
1 +	α_1	$-\alpha_1$	[1, 2]	[3, 4]	[5, 6]	[7, 8]	[8, 9]	[6, 7]	[4, 5]	[2, 3]	[1]	$-\alpha_1$
2 -		$-\alpha_2$	[2]	[1, 4]	[3, 6]	[5, 8]	{7, 10}	{8, 9}	[4, 7]	[2, 5]	[1, 3]	$-\alpha_2$
3 +	α_3	$-\alpha_3$	[2, 4]	[1, 6]	[3, 8]	{5, 10}	{7, 8}	{6, 9}	[4, 9]	[2, 7]	[1, 5]	$-\alpha_3$
4 -		$-\alpha_4$	[4]	[2, 6]	[1, 8]	{3, 10}	{5, 8}	{6, 7}	{4, 9}	[2, 9]	[1, 7]	$-\alpha_4$
5 +	α_5	$-\alpha_5$	[4, 6]	[2, 8]	{1, 10}	{3, 8}	{5, 6}	{4, 7}	{2, 9}	[1, 9]	[3, 7]	$-\alpha_5$
6 -		$-\alpha_6$	[6]	[4, 8]	{2, 10}	{1, 8}	{3, 6}	{4, 5}	{2, 7}	{1, 9}	[3, 9]	$-\alpha_6$
7 +	α_7	$-\alpha_7$	[6, 8]	{4, 10}	{2, 8}	{1, 6}	{3, 4}	{2, 5}	{1, 7}	{3, 9}	[5, 9]	$-\alpha_7$
8 -		$-\alpha_8$	[8]	{6, 10}	{4, 8}	{2, 6}	{1, 4}	{2, 3}	{1, 5}	{3, 7}	{5, 9}	$-\alpha_8$
	$-\alpha_{10}$	$-\alpha_9$	{8, 10}	{6, 8}	{4, 6}	{2, 4}	{1, 2}	{1, 3}	{3, 5}	{5, 7}	{7, 9}	$-\alpha_{10}$

Lemma 3.12. *Let*

$$(3.22) \quad \rho : \Phi_+ \rightarrow \bigsqcup_{i=1}^r O'_i$$

be the map defined by

$$(3.23) \quad \begin{aligned} [i, j] &\mapsto \begin{cases} [i, j]', & j - r \text{ odd}, \\ [2r + 2 - j, 2r + 2 - i]', & j - r \text{ even}, \end{cases} \\ \{i, j\} &\mapsto \begin{cases} [i, 2r + 2 - j]', & j - r \text{ odd}, \\ [j, 2r + 2 - i]', & j - r \text{ even}, \end{cases} \\ \{r + 1\} &\mapsto [r, r + 1]', \end{aligned}$$

where $[i] = [i, i]$. Then ρ is a bijection. Furthermore, under the bijection ρ , the action of σ is translated into the one of the square of the Coxeter element $s' = s'_- s'_+$ of type A_{2r+1} acting on Φ'_+ , where $s'_\pm = \prod_{i \in J'_\pm} s'_i$.

For $-h^\vee \leq u < 0$, define

$$(3.24) \quad \alpha_i(u) = \begin{cases} \sigma^{-u/2}(-\alpha_i), & i \in J_+, u \equiv 0, \\ \sigma^{-(u-1)/2}(\alpha_i), & i \in J_+, u \equiv -1, \\ \sigma^{-(2u-1)/4}(-\alpha_i), & i \in J_-, u \equiv -\frac{3}{2}, \\ \sigma^{-(2u+1)/4}(\alpha_i), & i \in J_-, u \equiv -\frac{1}{2}, \\ \sigma^{-u/2}(-\alpha_r), & i = r, u \equiv 0, \\ \sigma^{-(u-1)/2}(-\alpha_{r+1}), & i = r + 1, u \equiv -1, \end{cases}$$

where \equiv is equivalence modulo $2\mathbb{Z}$. Note that they correspond to the positive roots in Tables 1 and 2 with u being the parameter in the head lines. By Lemma 3.11 they are all the positive roots of D_{r+1} .

Lemma 3.13. *The family in (3.24) satisfies the recurrence relations*

$$(3.25) \quad \begin{aligned} \alpha_i(u - \tfrac{1}{2}) + \alpha_i(u + \tfrac{1}{2}) &= \alpha_{i-1}(u) + \alpha_{i+1}(u) \quad (1 \leq i \leq r - 2), \\ \alpha_{r-1}(u - \tfrac{1}{2}) + \alpha_{r-1}(u + \tfrac{1}{2}) &= \begin{cases} \alpha_{r-2}(u) + \alpha_r(u) & (u \text{ even}), \\ \alpha_{r-2}(u) + \alpha_{r+1}(u) & (u \text{ odd}), \end{cases} \\ \alpha_r(u - 1) + \alpha_r(u + 1) &= \alpha_{r-1}(u - \tfrac{1}{2}) + \alpha_{r-1}(u + \tfrac{1}{2}) \quad (u \text{ odd}), \\ \alpha_{r+1}(u - 1) + \alpha_{r+1}(u + 1) &= \alpha_{r-1}(u - \tfrac{1}{2}) + \alpha_{r-1}(u + \tfrac{1}{2}) \quad (u \text{ even}), \end{aligned}$$

where $\alpha_0(u) = 0$.

Proof. These relations are easily verified by using the explicit expressions of $\alpha_i(u)$. See Tables 1 and 2. The first two relations can also be obtained from Lemma 3.12 and [FZ2, Eq. (10.9)]. \square

Let us return to the proof of (ii) of Proposition 3.10 for the D part. For a monomial m in $y = (y_i)_{i \in \mathbf{I}}$, let $\pi_D(m)$ denote the specialization with $y_{i,1} = y_{i,3} = 1$ ($i \leq r-1$). For simplicity, we set $y_{i2} = y_i$ ($i \leq r-1$), $y_{r1} = y_r$, $y_{r+1,1} = y_{r+1}$, and also $y_{i2}(u) = y_i(u)$ ($i \leq r-1$), $y_{r1}(u) = y_r(u)$, $y_{r+1,1}(u) = y_r(u)$. We define the vectors $\mathbf{t}_i(u) = (t_i(u)_k)_{k=1}^{r+1}$ by

$$(3.26) \quad \pi_D([y_i(u)]_{\mathbf{T}}) = \prod_{k=1}^{r+1} y_k^{t_i(u)_k}.$$

We also identify each vector $\mathbf{t}_i(u)$ with $\alpha = \sum_{k=1}^{r+1} t_i(u)_k \alpha_k \in \mathbb{Z}\Pi$.

Proposition 3.14. *Let $-h^\vee \leq u < 0$. Then*

$$(3.27) \quad \mathbf{t}_i(u) = -\alpha_i(u)$$

for (i, u) as in (3.24), and

$$(3.28) \quad \pi_D([y_{i1}(u)]_{\mathbf{T}}) = \pi_D([y_{i3}(u)]_{\mathbf{T}}) = 1 \quad (i \leq r-1, r+i+2u \text{ even}).$$

Note that these formulas determine $\pi_D([y_{\mathbf{i}}(u)]_{\mathbf{T}})$ for any $(\mathbf{i}, u) : \mathbf{p}_+$.

Proof. We can verify the claim for $-2 \leq u \leq -\frac{1}{2}$ by direct computation. Then, by backward induction on u , one can establish the claim, together with the recurrence relations among $\mathbf{t}_i(u)$'s with (i, u) in (3.24),

$$(3.29) \quad \begin{aligned} \mathbf{t}_i(u - \tfrac{1}{2}) + \mathbf{t}_i(u + \tfrac{1}{2}) &= \mathbf{t}_{i-1}(u) + \mathbf{t}_{i+1}(u) \quad (1 \leq i \leq r-2), \\ \mathbf{t}_{r-1}(u - \tfrac{1}{2}) + \mathbf{t}_{r-1}(u + \tfrac{1}{2}) &= \begin{cases} \mathbf{t}_{r-2}(u) + \mathbf{t}_r(u) & (u \text{ even}), \\ \mathbf{t}_{r-2}(u) + \mathbf{t}_{r+1}(u) & (u \text{ odd}), \end{cases} \\ \mathbf{t}_r(u-1) + \mathbf{t}_r(u+1) &= \mathbf{t}_{r-1}(u - \tfrac{1}{2}) + \mathbf{t}_{r-1}(u + \tfrac{1}{2}) \quad (u \text{ odd}), \\ \mathbf{t}_{r+1}(u-1) + \mathbf{t}_{r+1}(u+1) &= \mathbf{t}_{r-1}(u - \tfrac{1}{2}) + \mathbf{t}_{r-1}(u + \tfrac{1}{2}) \quad (u \text{ even}). \end{aligned}$$

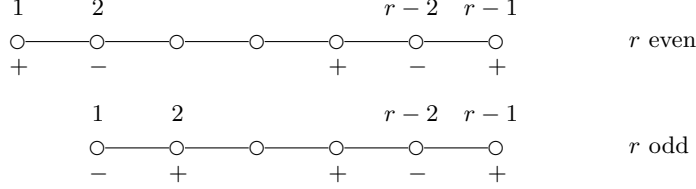
Note that (3.29) coincides with (3.25) under (3.27). To derive (3.29), one uses the mutations as in [IIKKN, Figure 6] (or the tropical version of the Y-system $\mathbb{Y}_2(C_r)$ directly) and the positivity/negativity of $\pi_D([y_{\mathbf{i}}(u)]_{\mathbf{T}})$ resulting from (3.27) and (3.28) by induction hypothesis. \square

Now (ii) and (iv) of Proposition 3.10 for the D part follow from Proposition 3.14.

3.6.2. A part. The A part can be studied in a similar way to the D part. Therefore, we only present the result.

First we note that the quiver $Q_2(C_r)$ is symmetric under the exchange $y_{i,1} \leftrightarrow y_{i,3}$ ($i \leq r-1$). Thus, one can concentrate on the powers of $[y_{\mathbf{i}}(u)]_{\mathbf{T}}$ in the variables $y_{i,1}$ ($i \leq r-1$).

Let A_{r-1} be the Dynkin diagram of type A with index set $J = \{1, \dots, r-1\}$. We assign a sign $+/-$ to vertices (except for r) of A_{r-1} as inherited from $Q_2(C_r)$:



Let $\Pi = \{\alpha_1, \dots, \alpha_{r-1}\}$, $-\Pi$, Φ_+ be the sets of simple roots, of negative simple roots, and of positive roots, respectively, of type A_{r-1} . Again, we introduce the piecewise-linear analogue σ_i of the simple reflection s_i , acting on $\Phi_{\geq -1} = \Phi_+ \sqcup (-\Pi)$ as in (3.18). Let

$$(3.30) \quad \sigma_+ = \prod_{i \in J_+} \sigma_i, \quad \sigma_- = \prod_{i \in J_-} \sigma_i,$$

where J_{\pm} is the set of vertices of A_{r-1} with the respective sign. We define

$$(3.31) \quad \sigma = \sigma_- \sigma_+.$$

For a monomial m in $y = (y_{\mathbf{i}})_{\mathbf{i} \in \mathbf{I}}$, let $\pi_A(m)$ denote the specialization with $y_{i,2} = y_{i,3} = 1$ ($i \leq r-1$) and $y_{r,1} = y_{r+1,1} = 1$. We set $y_{i,1} = y_i$ ($i \leq r-1$).

We define the vectors $\mathbf{t}_{\mathbf{i}}(u) = (t_{\mathbf{i}}(u)_k)_{k=1}^{r-1}$ by

$$(3.32) \quad \pi_A([y_{\mathbf{i}}(u)]_{\mathbf{T}}) = \prod_{k=1}^{r-1} y_k^{t_{\mathbf{i}}(u)_k}.$$

We also identify each vector $\mathbf{t}_{\mathbf{i}}(u)$ with $\alpha = \sum_{k=1}^{r-1} t_{\mathbf{i}}(u)_k \alpha_k \in \mathbb{Z}\Pi$.

With these notations, the result for the A part is summarized as follows.

Proposition 3.15. *Let $-h^\vee \leq u < 0$. For $(\mathbf{i}, u) : \mathbf{p}_+$, $\mathbf{t}_{\mathbf{i}}(u)$ is given by, for $i \leq r-1$,*

$$\begin{aligned}
 \mathbf{t}_{i,1}(u) &= \begin{cases} -\sigma^{-u}(-\alpha_i), & i \in J_+, \\ -\sigma^{-(2u-1)/2}(-\alpha_i), & i \in J_-, \end{cases} \\
 \mathbf{t}_{i,2}(u) &= \begin{cases} -[2r+2-i+2u, r-1], & -\frac{1}{2}h^\vee \leq u < 0, \\ -[-1-i-2u, r-1], & -h^\vee \leq u < -\frac{1}{2}h^\vee, \end{cases} \\
 \mathbf{t}_{i,3}(u) &= 0,
 \end{aligned}
 \tag{3.33}$$

and

$$(3.34) \quad \begin{aligned} \mathbf{t}_{r1}(u) &= \begin{cases} -[r+2+2u, r-1], & -\frac{1}{2}h^\vee \leq u < 0, \\ -[-1-r-2u, r-1], & -h^\vee \leq u < -\frac{1}{2}h^\vee, \end{cases} \\ \mathbf{t}_{r+1,1}(u) &= \begin{cases} -[r+2+2u, r-1], & -\frac{1}{2}h^\vee \leq u < 0, \\ -[-1-r-2u, r-1], & -h^\vee \leq u < -\frac{1}{2}h^\vee, \end{cases} \end{aligned}$$

where $[i, j]$ equals $\alpha_i + \dots + \alpha_j$ if $i \leq j$ and 0 if $i > j$.

Note that $\mathbf{t}_{i1}(-h^\vee/2) = \alpha_{r-i}$ ($i \in J_-$ for r even and $i \in J_+$ for r odd) and $\mathbf{t}_{i1}(-h^\vee/2 - 1/2) = \alpha_{r-i}$ ($i \in J_+$ for r even and $i \in J_-$ for r odd), and that they are the only positive monomials in (3.33) and (3.34). Now (ii) and (iv) of Proposition 3.10 for the A part follow from Proposition 3.15.

This completes the proof of Proposition 3.10.

§3.7. Tropical Y-systems at higher levels

By the same method as for the B_r case [IIKKN, Proposition 4.1], one can establish the ‘factorization property’ of the tropical Y-system at higher levels. As a result, we obtain a generalization of Proposition 3.10.

Proposition 3.16. *For $[\mathcal{G}_Y(B, y)]_{\mathbf{T}}$ with $B = B_\ell(C_r)$, the following facts hold:*

- (i) *Let $0 \leq u < \ell$. For any $(\mathbf{i}, u) : \mathbf{p}_+$, the monomial $[y_{\mathbf{i}}(u)]_{\mathbf{T}}$ is positive.*
- (ii) *Let $-h^\vee \leq u < 0$.*
 - (a) *Let $\mathbf{i} \in \mathbf{I}^\circ$ or $\mathbf{i} = (i, i')$ ($i \leq r-1$, $i' \in 2\mathbb{N}$). For any $(\mathbf{i}, u) : \mathbf{p}_+$, the monomial $[y_{\mathbf{i}}(u)]_{\mathbf{T}}$ is negative.*
 - (b) *Let $\mathbf{i} = (i, i')$ ($i \leq r-1$, $i' \notin 2\mathbb{N}$). For any $(\mathbf{i}, u) : \mathbf{p}_+$, the monomial $[y_{\mathbf{i}}(u)]_{\mathbf{T}}$ is positive for $u = -\frac{1}{2}h^\vee, -\frac{1}{2}h^\vee - \frac{1}{2}$ and negative otherwise.*
- (iii) *$y_{ii'}(\ell)$ equals $y_{i, 2\ell-i'}^{-1}$ if $i \leq r-1$, and $y_{i, \ell-i'}^{-1}$ if $i = r, r+1$.*
- (iv) *For even r , $y_{ii'}(-h^\vee)$ equals $y_{ii'}^{-1}$ if $i \leq r-1$, and $y_{2r+1-i, i'}^{-1}$ if $i = r, r+1$. For odd r , $y_{ii'}(-h^\vee) = y_{ii'}^{-1}$.*

We obtain two important corollaries of Propositions 3.10 and 3.16.

Theorem 3.17. *For $[\mathcal{G}_Y(B, y)]_{\mathbf{T}}$ the following relations hold:*

- (i) *Half periodicity: $[y_{\mathbf{i}}(u + h^\vee + \ell)]_{\mathbf{T}} = [y_{\omega(\mathbf{i})}(u)]_{\mathbf{T}}$.*
- (ii) *Full periodicity: $[y_{\mathbf{i}}(u + 2(h^\vee + \ell))]_{\mathbf{T}} = [y_{\mathbf{i}}(u)]_{\mathbf{T}}$.*

Theorem 3.18. For $[\mathcal{G}_Y(B, y)]_{\mathbf{T}}$, let N_+ and N_- denote the total numbers of the positive and negative monomials, respectively, among $[y_{\mathbf{i}}(u)]_{\mathbf{T}}$ for $(\mathbf{i}, u) : \mathbf{p}_+$ in the region $0 \leq u < 2(h^\vee + \ell)$. Then

$$(3.35) \quad N_+ = 2\ell(2r\ell - \ell - 1), \quad N_- = 2r(2\ell r - r - 1).$$

We observe the symmetry (the *level-rank duality*) for the numbers N_+ and N_- under the exchange of r and ℓ .

§3.8. Periodicities and dilogarithm identities

Applying [IIKKN, Theorem 5.1] to Theorem 3.17, we obtain the periodicities:

Theorem 3.19. For $\mathcal{A}(B, x, y)$, the following relations hold:

- (i) *Half periodicity:* $x_{\mathbf{i}}(u + h^\vee + \ell) = x_{\omega(\mathbf{i})}(u)$.
- (ii) *Full periodicity:* $x_{\mathbf{i}}(u + 2(h^\vee + \ell)) = x_{\mathbf{i}}(u)$.

Theorem 3.20. For $\mathcal{G}(B, y)$, the following relations hold:

- (i) *Half periodicity:* $y_{\mathbf{i}}(u + h^\vee + \ell) = y_{\omega(\mathbf{i})}(u)$.
- (ii) *Full periodicity:* $y_{\mathbf{i}}(u + 2(h^\vee + \ell)) = y_{\mathbf{i}}(u)$.

Theorems 2.5 and 2.6 for C_r follow from Theorems 3.6, 3.9, 3.19, and 3.20. Furthermore, Theorem 2.10 for C_r is obtained from the above periodicities and Theorem 3.18 as in the B_r case [IIKKN, Section 6].

§4. Type F_4

The F_4 case is quite parallel to the B_r and C_r cases. We do not repeat the same definitions unless otherwise mentioned. Again, the properties of the tropical Y-system at level 2 (Proposition 4.7) are crucial and specific to F_4 .

§4.1. Parity decompositions of T-systems and Y-systems

For a triplet $(a, m, u) \in \mathcal{I}_\ell$, we reset the ‘parity conditions’ \mathbf{P}_+ and \mathbf{P}_- to be

$$(4.1) \quad \mathbf{P}_+ : 2u \text{ is even if } a = 1, 2; a + m + 2u \text{ is odd if } a = 3, 4,$$

$$(4.2) \quad \mathbf{P}_- : 2u \text{ is odd if } a = 1, 2; a + m + 2u \text{ is even if } a = 3, 4.$$

Then we have $\mathcal{T}_\ell^\circ(F_4)_+ \simeq \mathcal{T}_\ell^\circ(F_4)_-$ via $T_m^{(a)}(u) \mapsto T_m^{(a)}(u + \frac{1}{2})$ and

$$(4.3) \quad \mathcal{T}_\ell^\circ(F_4) \simeq \mathcal{T}_\ell^\circ(F_4)_+ \otimes_{\mathbb{Z}} \mathcal{T}_\ell^\circ(F_4)_-.$$

For a triplet $(a, m, u) \in \mathcal{I}_\ell$, we reset the parity conditions \mathbf{P}'_+ and \mathbf{P}'_- to be

$$(4.4) \quad \mathbf{P}'_+ : 2u \text{ is even if } a = 1, 2; a + m + 2u \text{ is even if } a = 3, 4,$$

$$(4.5) \quad \mathbf{P}'_- : 2u \text{ is odd if } a = 1, 2; a + m + 2u \text{ is odd if } a = 3, 4.$$

We have

$$(4.6) \quad (a, m, u) : \mathbf{P}'_+ \Leftrightarrow (a, m, u \pm \frac{1}{t_a}) : \mathbf{P}_+.$$

Also, we have $\mathcal{Y}_\ell^\circ(F_4)_+ \simeq \mathcal{Y}_\ell^\circ(F_4)_-$ via $Y_m^{(a)}(u) \mapsto Y_m^{(a)}(u + \frac{1}{2})$, $1 + Y_m^{(a)}(u) \mapsto 1 + Y_m^{(a)}(u + \frac{1}{2})$, and

$$(4.7) \quad \mathcal{Y}_\ell^\circ(F_4) \simeq \mathcal{Y}_\ell^\circ(F_4)_+ \times \mathcal{Y}_\ell^\circ(F_4)_-.$$

§4.2. Quiver $Q_\ell(F_4)$

With type F_4 and $\ell \geq 2$ we associate the quiver $Q_\ell(F_4)$ by Figure 4, where the right column in the left quiver and the middle column in the right quiver are

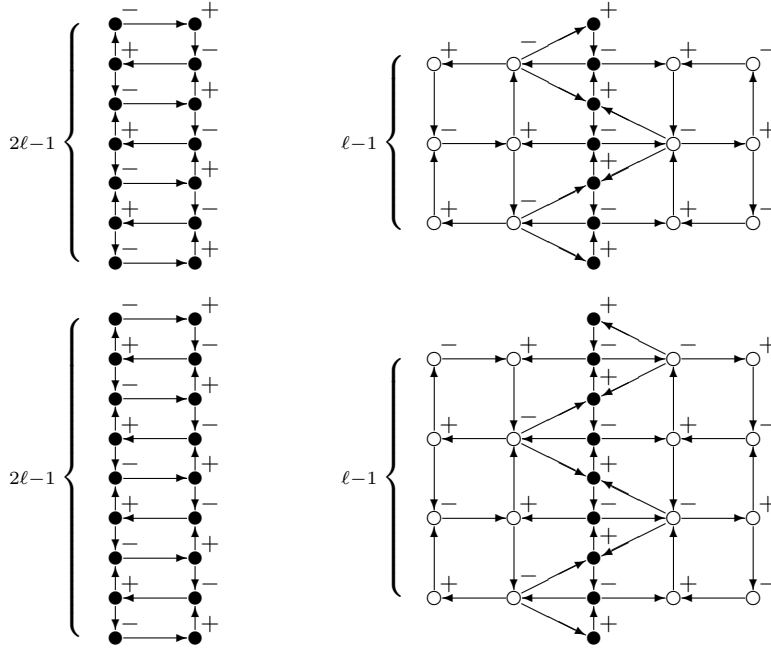


Figure 4. The quiver $Q_\ell(F_4)$ for ℓ even (top) and for ℓ odd (bottom), where we identify the right column in the left quiver with the middle column in the right quiver.

identified. Also, we assign an empty or filled circle \circ/\bullet and a sign $+/-$ to each vertex.

Let us choose the index set \mathbf{I} of the vertices of $Q_\ell(F_4)$ so that $\mathbf{i} = (i, i') \in \mathbf{I}$ represents the vertex in the i' th row (from the bottom) and the i th column (from the left) of the right quiver for $i = 1, 2, 3$, in the $(i-1)$ th column of the right quiver for $i = 5, 6$, and in the left column of the left quiver for $i = 4$. Thus, $i = 1, \dots, 6$, and $i' = 1, \dots, \ell - 1$ if $i = 1, 2, 5, 6$, while $i' = 1, \dots, 2\ell - 1$ if $i = 3, 4$.

Let \mathbf{r} be the involution acting on \mathbf{I} by left-right reflection of the right quiver. Let $\boldsymbol{\omega}$ be the involution acting on \mathbf{I} by up-down reflection of the left quiver and 180° rotation of the right quiver.

Lemma 4.1. *Let $Q = Q_\ell(F_4)$.*

- (i) *We have the same periodic sequence of mutations of quivers as in (3.9).*
- (ii) *$\boldsymbol{\omega}(Q) = Q$ if $h^\vee + \ell$ is even, and $\boldsymbol{\omega}(Q) = \mathbf{r}(Q)$ if $h^\vee + \ell$ is odd.*

§4.3. Cluster algebra and alternative labels

Let $B_\ell(F_4)$ be the skew-symmetric matrix corresponding to the quiver $Q_\ell(F_4)$. In the rest of this section, we set $B = (B_{\mathbf{ij}})_{\mathbf{i}, \mathbf{j} \in \mathbf{I}} = B_\ell(F_4)$ unless otherwise mentioned.

Let $\mathcal{A}(B, x, y)$ be the cluster algebra with coefficients in the universal semifield $\mathbb{Q}_{\text{sf}}(y)$, and $\mathcal{G}(B, y)$ be the coefficient group associated with $\mathcal{A}(B, x, y)$.

In view of Lemma 4.1 we set $x(0) = x$, $y(0) = y$ and define clusters $x(u) = (x_{\mathbf{i}}(u))_{\mathbf{i} \in \mathbf{I}}$ ($u \in \frac{1}{2}\mathbb{Z}$) and coefficient tuples $y(u) = (y_{\mathbf{i}}(u))_{\mathbf{i} \in \mathbf{I}}$ ($u \in \frac{1}{2}\mathbb{Z}$) by the sequence of mutations (3.11).

For a pair $(\mathbf{i}, u) \in \mathbf{I} \times \frac{1}{2}\mathbb{Z}$, we set the same parity condition \mathbf{p}_+ and \mathbf{p}_- as in (3.12). We have (3.13), and each $(\mathbf{i}, u) : \mathbf{p}_+$ is a mutation point of (3.11) in the forward direction of u , while each $(\mathbf{i}, u) : \mathbf{p}_-$ is one in the backward direction of u as before.

Lemma 4.2. *Below \equiv means equivalence modulo $2\mathbb{Z}$.*

- (i) *The map $g : \mathcal{I}_{\ell+} \rightarrow \{(\mathbf{i}, u) : \mathbf{p}_+\}$ defined by*

$$(4.8) \quad (a, m, u - \frac{1}{t_a}) \mapsto \begin{cases} ((a, m), u), & a = 1, 2; a + m + u \equiv 0, \\ ((7 - a, m), u), & a = 1, 2; a + m + u \equiv 1, \\ ((a, m), u), & a = 3, 4, \end{cases}$$

is a bijection.

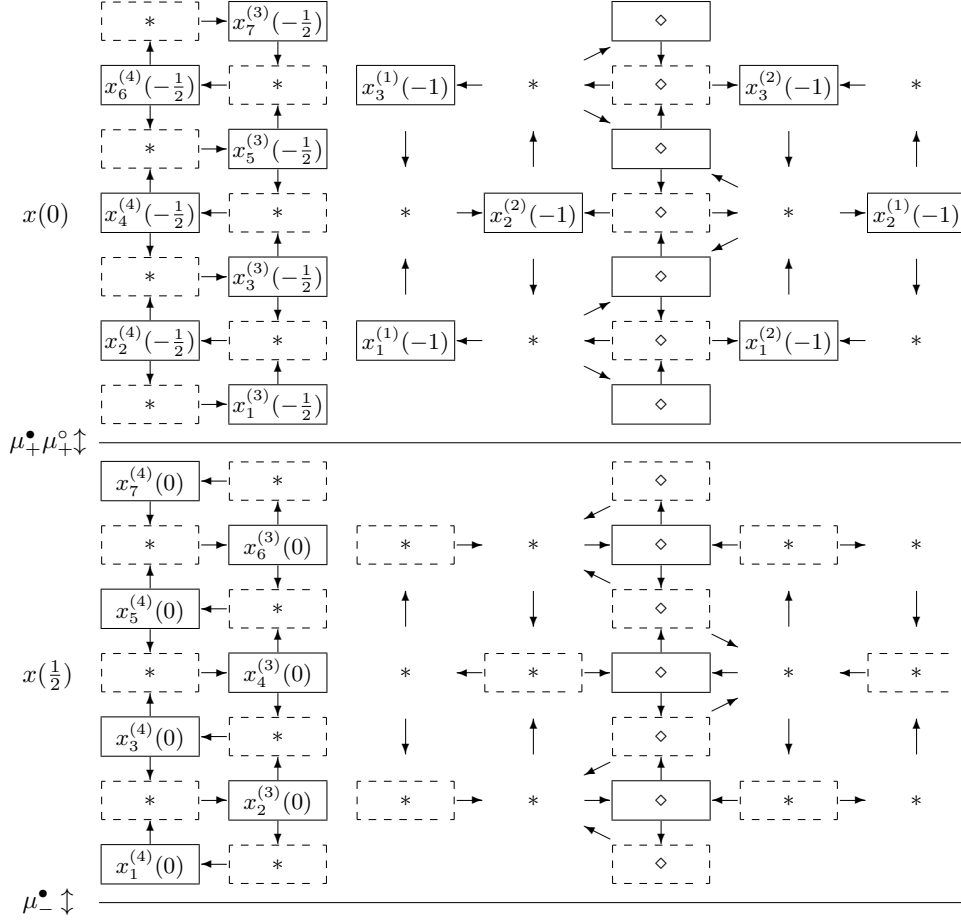


Figure 5. (Continued in Figure 6.) Labeling of cluster variables $x_i(u)$ by $\mathcal{I}_{\ell+}$ for F_4 , $\ell = 4$. The middle column in the right quiver (marked by \diamond) is identified with the right column in the left quiver.

(ii) The map $g' : \mathcal{I}'_{\ell+} \rightarrow \{(\mathbf{i}, u) : \mathbf{p}_+\}$ defined by

$$(4.9) \quad (a, m, u) \mapsto \begin{cases} ((a, m), u), & a = 1, 2; a + m + u \equiv 0, \\ ((7 - a, m), u), & a = 1, 2; a + m + u \equiv 1, \\ ((a, m), u), & a = 3, 4, \end{cases}$$

is a bijection.

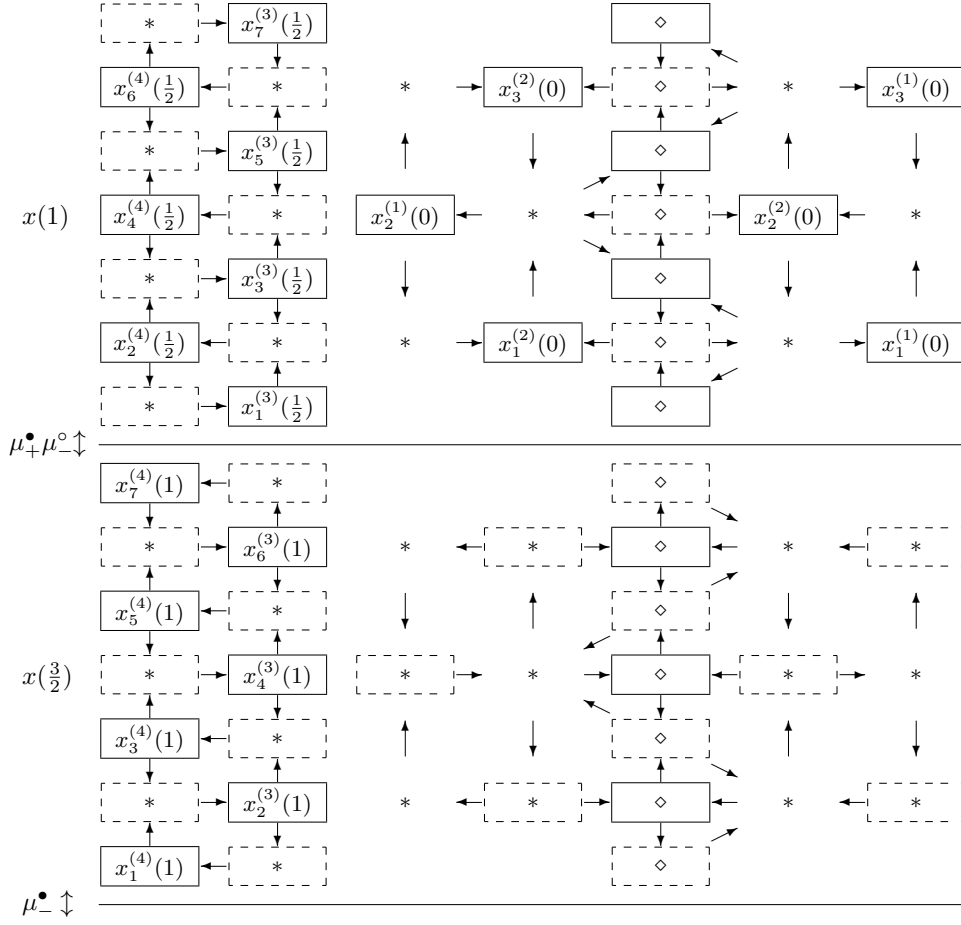


Figure 6. Continuation of Figure 5.

We introduce alternative labels $x_{\mathbf{i}}(u) = x_m^{(a)}(u - 1/t_a)$ ($(a, m, u - 1/t_a) \in \mathcal{I}_{\ell_+}$) for $(\mathbf{i}, u) = g((a, m, u - 1/t_a))$ and $y_{\mathbf{i}}(u) = y_m^{(a)}(u)$ ($(a, m, u) \in \mathcal{I}'_{\ell_+}$) for $(\mathbf{i}, u) = g'((a, m, u))$, respectively. See Figures 5–6.

§4.4. T-system and cluster algebra

The result in this subsection is completely parallel to the B_r and C_r cases.

Lemma 4.3. *The family $\{x_m^{(a)}(u) \mid (a, m, u) \in \mathcal{I}_{\ell_+}\}$ satisfies the system of relations (3.16) with $G(b, k, v; a, m, u)$ for $\mathbb{T}_{\ell}(F_4)$. In particular, the family $\{[x_m^{(a)}(u)]_{\mathbf{1}} \mid (a, m, u) \in \mathcal{I}_{\ell_+}\}$ satisfies the T-system $\mathbb{T}_{\ell}(F_4)$ in $\mathcal{A}(B, x)$ after replacing $T_m^{(a)}(u)$ with $[x_m^{(a)}(u)]_{\mathbf{1}}$.*

The T-subalgebra $\mathcal{A}_T(B, x)$ is defined as in Definition 3.5.

Theorem 4.4. *The ring $\mathcal{T}_\ell^\circ(F_4)_+$ is isomorphic to $\mathcal{A}_T(B, x)$ via the correspondence $Y_m^{(a)}(u) \mapsto [x_m^{(a)}(u)]_1$.*

§4.5. Y-system and cluster algebra

The result in this subsection is completely parallel to the B_r and C_r cases.

Lemma 4.5. *The family $\{y_m^{(a)}(u) \mid (a, m, u) \in \mathcal{I}'_{\ell+}\}$ satisfies the Y-system $\mathbb{Y}_\ell(F_4)$ after replacing $Y_m^{(a)}(u)$ with $y_m^{(a)}(u)$.*

The Y-subgroup $\mathcal{G}_Y(B, y)$ is defined as in Definition 3.8.

Theorem 4.6. *The group $\mathcal{Y}_\ell^\circ(F_4)_+$ is isomorphic to $\mathcal{G}_Y(B, y)$ via the correspondence $Y_m^{(a)}(u) \mapsto y_m^{(a)}(u)$ and $1 + Y_m^{(a)}(u) \mapsto 1 + y_m^{(a)}(u)$.*

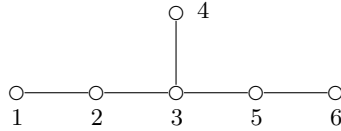
§4.6. Tropical Y-system at level 2

By direct computations, the following properties are verified.

Proposition 4.7. *For $[\mathcal{G}_Y(B, y)]_{\mathbf{T}}$ with $B = B_2(F_4)$, the following facts hold:*

- (i) *Let $0 \leq u < 2$. For any $(\mathbf{i}, u) : \mathbf{p}_+$, the monomial $[y_{\mathbf{i}}(u)]_{\mathbf{T}}$ is positive.*
- (ii) *Let $-h^\vee \leq u < 0$.*
 - (a) *Let $\mathbf{i} = (1, 1), (2, 1), (5, 1), (6, 1), (3, 2)$, or $(4, 2)$. For any $(\mathbf{i}, u) : \mathbf{p}_+$, the monomial $[y_{\mathbf{i}}(u)]_{\mathbf{T}}$ is negative.*
 - (b) *Let $\mathbf{i} = (3, 1), (3, 3), (4, 1)$, or $(4, 3)$. For any $(\mathbf{i}, u) : \mathbf{p}_+$, the monomial $[y_{\mathbf{i}}(u)]_{\mathbf{T}}$ is negative for $u = -\frac{1}{2}, -1, -\frac{3}{2}, -3, -\frac{7}{2}, -4, -\frac{11}{2}, -6, -\frac{13}{2}, -8, -\frac{17}{2}, -9$ and positive for $u = -2, -\frac{5}{2}, -\frac{9}{2}, -5, -7, -\frac{15}{2}$.*
- (iii) *$y_{i,i'}(2)$ equals $y_{i,i'}^{-1}$ if $i = 1, 2, 5, 6$, and $y_{i,4-i'}^{-1}$ if $i = 3, 4$.*
- (iv) *$y_{i,i'}(-h^\vee)$ equals $y_{7-i,i'}^{-1}$ if $i = 1, 2, 5, 6$, and $y_{i,i'}^{-1}$ if $i = 3, 4$.*

Also we have a description of the ‘core part’ of $[y_{\mathbf{i}}(u)]_{\mathbf{T}}$ for $-h^\vee \leq u < 0$, corresponding to the D part for C_r , in terms of the root system of E_6 . We use the following indexing of the Dynkin diagram E_6 :



Let $\Pi = \{\alpha_1, \dots, \alpha_6\}$, $-\Pi$, Φ_+ be the sets of simple roots, of negative simple roots, and of positive roots, respectively, of type E_6 . Let σ_i be the piecewise-linear analogue of the simple reflection s_i , acting on the set $\Phi_{\geq -1} = \Phi_+ \sqcup (-\Pi)$ of almost positive roots. We write $\sum_i m_i \alpha_i \in \Phi_+$ as $[1^{m_1}, 2^{m_2}, \dots, 6^{m_6}]$; furthermore, $[1^0, 2^1, 3^1, 4^1, 5^0, 6^0]$, for example, is abbreviated as $[2, 3, 4]$.

We define

$$(4.10) \quad \sigma = \sigma_3(\sigma_4\sigma_2\sigma_6)\sigma_3(\sigma_4\sigma_1\sigma_5).$$

Lemma 4.8. *The orbits under σ are:*

$$(4.11) \quad \begin{aligned} -\alpha_1 &\rightarrow [1, 2, 3] \rightarrow [2, 3, 4, 5, 6] \rightarrow [1, 2, 3^2, 4, 5] \rightarrow [5, 6] \rightarrow -\alpha_6, \\ -\alpha_2 &\rightarrow [2, 3] \rightarrow [1, 2^2, 3^2, 4, 5, 6] \rightarrow [1, 2^2, 3^3, 4^2, 5^2, 6] \\ &\quad \rightarrow [1, 2, 3^2, 4, 5^2, 6] \rightarrow [5] \rightarrow -\alpha_5, \\ -\alpha_3 &\rightarrow [2, 3, 4] \rightarrow [1, 2, 3^2, 4, 5, 6] \rightarrow [2, 3^2, 4, 5^2, 6] \rightarrow [1, 2, 3, 5] \rightarrow -\alpha_3, \\ \alpha_3 &\rightarrow [2, 3, 5, 6] \rightarrow [1, 2^2, 3^2, 4, 5] \rightarrow [1, 2, 3, 4, 5, 6] \rightarrow [3, 4, 5] \rightarrow \alpha_3, \\ -\alpha_4 &\rightarrow [2] \rightarrow [1, 2, 3, 4] \rightarrow [3, 4, 5, 6] \rightarrow [3, 5] \rightarrow -\alpha_4, \\ \alpha_4 &\rightarrow [3, 4] \rightarrow [3, 5, 6] \rightarrow [2, 3, 5] \rightarrow [1, 2] \rightarrow \alpha_4, \\ -\alpha_5 &\rightarrow [2, 3^2, 4, 5, 6] \rightarrow [1, 2^2, 3^3, 4, 5^2, 6] \rightarrow [1, 2^2, 3^2, 4, 5^2, 6] \\ &\quad \rightarrow [1, 2, 3, 4, 5] \rightarrow -\alpha_2, \\ -\alpha_6 &\rightarrow [6] \rightarrow [2, 3^2, 4, 5] \rightarrow [1, 2, 3, 5, 6] \rightarrow [2, 3, 4, 5] \rightarrow [1] \rightarrow -\alpha_1. \end{aligned}$$

In particular, these elements of Φ_+ exhaust the set Φ_+ , thereby providing the orbit decomposition of Φ_+ under σ .

For $-h^\vee \leq u < 0$, define

$$(4.12) \quad \alpha_i(u) = \begin{cases} \sigma^{-u/2}(-\alpha_i), & i = 1, 4, 5; u \equiv 0, \\ \sigma^{-(u-1)/2}(-\alpha_i), & i = 2, 6; u \equiv -1, \\ \sigma^{-(u-1)/2}(\alpha_4), & i = 4; u \equiv -1, \\ \sigma^{-(2u-1)/4}(-\alpha_3), & i = 3; u \equiv -\frac{3}{2}, \\ \sigma^{-(2u+1)/4}(\alpha_3), & i = 3; u \equiv -\frac{1}{2}, \end{cases}$$

where \equiv is equivalence mod $2\mathbb{Z}$. By Lemma 4.8 they are (all the) positive roots of E_6 .

For a monomial m in $y = (y_i)_{i \in \mathbf{I}}$, let $\pi_A(m)$ denote the specialization with $y_{31} = y_{33} = y_{41} = y_{43} = 1$. For simplicity, we set $y_{i1} = y_i$ ($i = 1, 2, 5, 6$), $y_{i2} = y_i$ ($i = 3, 4$), and also $y_{i1}(u) = y_i(u)$ ($i = 1, 2, 5, 6$), $y_{i2}(u) = y_i(u)$ ($i = 3, 4$). We

define the vectors $\mathbf{t}_i(u) = (t_i(u)_k)_{k=1}^6$ by

$$(4.13) \quad \pi_A([y_i(u)]_{\mathbf{T}}) = \prod_{k=1}^6 y_k^{t_i(u)_k}.$$

We also identify each vector $\mathbf{t}_i(u)$ with $\alpha = \sum_{k=1}^6 t_i(u)_k \alpha_k \in \mathbb{Z}\Pi$.

Proposition 4.9. *Let $-h^\vee \leq u < 0$. Then*

$$(4.14) \quad \mathbf{t}_i(u) = -\alpha_i(u)$$

for (i, u) as in (4.12).

§4.7. Tropical Y-systems at higher levels

Due to the factorization property, we obtain the following.

Proposition 4.10. *Let $\ell > 2$ be an integer. For $[\mathcal{G}_Y(B, y)]_{\mathbf{T}}$ with $B = B_\ell(F_4)$, the following facts hold.*

- (i) *Let $0 \leq u < \ell$. For any $(\mathbf{i}, u) : \mathbf{p}_+$, the monomial $[y_{\mathbf{i}}(u)]_{\mathbf{T}}$ is positive.*
- (ii) *Let $-h^\vee \leq u < 0$.*
 - (a) *Let $\mathbf{i} \in \mathbf{I}^\circ$ or $\mathbf{i} = (3, i'), (4, i')$ ($i' \in 2\mathbb{N}$). For any $(\mathbf{i}, u) : \mathbf{p}_+$, the monomial $[y_{\mathbf{i}}(u)]_{\mathbf{T}}$ is negative.*
 - (b) *Let $\mathbf{i} = (3, i'), (4, i')$ ($i' \notin 2\mathbb{N}$). For any $(\mathbf{i}, u) : \mathbf{p}_+$, the monomial $[y_{\mathbf{i}}(u)]_{\mathbf{T}}$ is negative for $u = -\frac{1}{2}, -1, -\frac{3}{2}, -3, -\frac{7}{2}, -4, -\frac{11}{2}, -6, -\frac{13}{2}, -8, -\frac{17}{2}, -9$ and positive for $u = -2, -\frac{5}{2}, -\frac{9}{2}, -5, -7, -\frac{15}{2}$.*
- (iii) *$y_{ii'}(\ell)$ equals $y_{i, \ell-i'}^{-1}$ if $i = 1, 2, 5, 6$, and $y_{i, 2\ell-i'}^{-1}$ if $i = 3, 4$.*
- (iv) *$y_{ii'}(-h^\vee)$ equals $y_{7-i, i'}^{-1}$ if $i = 1, 2, 5, 6$, and $y_{ii'}^{-1}$ if $i = 3, 4$.*

We obtain corollaries of Propositions 4.7 and 4.10.

Theorem 4.11. *For $[\mathcal{G}_Y(B, y)]_{\mathbf{T}}$ the following relations hold:*

- (i) *Half periodicity: $[y_{\mathbf{i}}(u + h^\vee + \ell)]_{\mathbf{T}} = [y_{\omega(\mathbf{i})}(u)]_{\mathbf{T}}$.*
- (ii) *Full periodicity: $[y_{\mathbf{i}}(u + 2(h^\vee + \ell))]_{\mathbf{T}} = [y_{\mathbf{i}}(u)]_{\mathbf{T}}$.*

Theorem 4.12. *For $[\mathcal{G}_Y(B, y)]_{\mathbf{T}}$, let N_+ and N_- denote the total numbers of the positive and negative monomials, respectively, among $[y_{\mathbf{i}}(u)]_{\mathbf{T}}$ for $(\mathbf{i}, u) : \mathbf{p}_+$ in the region $0 \leq u < 2(h^\vee + \ell)$. Then*

$$(4.15) \quad N_+ = 4\ell(3\ell + 1), \quad N_- = 24(4\ell - 3).$$

§4.8. Periodicities and dilogarithm identities

Applying [IIKKN, Theorem 5.1] to Theorem 4.11, we obtain the periodicities:

Theorem 4.13. *For $\mathcal{A}(B, x, y)$, the following relations hold:*

- (i) *Half periodicity: $x_{\mathbf{i}}(u + h^\vee + \ell) = x_{\omega(\mathbf{i})}(u)$.*
- (ii) *Full periodicity: $x_{\mathbf{i}}(u + 2(h^\vee + \ell)) = x_{\mathbf{i}}(u)$.*

Theorem 4.14. *For $\mathcal{G}(B, y)$, the following relations hold:*

- (i) *Half periodicity: $y_{\mathbf{i}}(u + h^\vee + \ell) = y_{\omega(\mathbf{i})}(u)$.*
- (ii) *Full periodicity: $y_{\mathbf{i}}(u + 2(h^\vee + \ell)) = y_{\mathbf{i}}(u)$.*

Theorems 2.5 and 2.6 for F_4 follow from Theorems 4.4, 4.6, 4.13, and 4.14. Furthermore, Theorem 2.10 for F_4 is obtained from the above periodicities and Theorem 4.12 as in the B_r case [IIKKN, Section 6].

§5. Type G_2

The G_2 case is mostly parallel to the former cases, but slightly different because the number t in (2.2) is three. Again, the properties of the tropical Y-system at level 2 (Proposition 5.9) are crucial and specific to G_2 .

§5.1. Parity decompositions of T-systems and Y-systems

For a triplet $(a, m, u) \in \mathcal{I}_\ell$, we reset the parity conditions \mathbf{P}_+ and \mathbf{P}_- to be

$$(5.1) \quad \mathbf{P}_+ : a + m + 3u \text{ is even,}$$

$$(5.2) \quad \mathbf{P}_- : a + m + 3u \text{ is odd.}$$

Then we have $\mathcal{T}_\ell^\circ(G_2)_+ \simeq \mathcal{T}_\ell^\circ(G_2)_-$ via $T_m^{(a)}(u) \mapsto T_m^{(a)}(u + \frac{1}{3})$ and

$$(5.3) \quad \mathcal{T}_\ell^\circ(G_2) \simeq \mathcal{T}_\ell^\circ(G_2)_+ \otimes_{\mathbb{Z}} \mathcal{T}_\ell^\circ(G_2)_-.$$

For $(a, m, u) \in \mathcal{I}_\ell$, we reset the parity conditions \mathbf{P}'_+ and \mathbf{P}'_- to be

$$(5.4) \quad \mathbf{P}'_+ : a + m + 3u \text{ is odd,}$$

$$(5.5) \quad \mathbf{P}'_- : a + m + 3u \text{ is even.}$$

We have

$$(5.6) \quad (a, m, u) : \mathbf{P}'_+ \Leftrightarrow (a, m, u \pm \frac{1}{t_a}) : \mathbf{P}_+.$$

Also, we have $\mathfrak{Y}_\ell^\circ(G_2)_+ \simeq \mathfrak{Y}_\ell^\circ(G_2)_-$ via $Y_m^{(a)}(u) \mapsto Y_m^{(a)}(u + \frac{1}{3})$, $1 + Y_m^{(a)}(u) \mapsto 1 + Y_m^{(a)}(u + \frac{1}{3})$, and

$$(5.7) \quad \mathfrak{Y}_\ell^\circ(G_2) \simeq \mathfrak{Y}_\ell^\circ(G_2)_+ \times \mathfrak{Y}_\ell^\circ(G_2)_-.$$

§5.2. Quiver $Q_\ell(G_2)$

With type G_2 and $\ell \geq 2$ we associate the quiver $Q_\ell(G_2)$ by Figure 7, where the right columns in the three quivers are identified. Also we assign an empty or filled

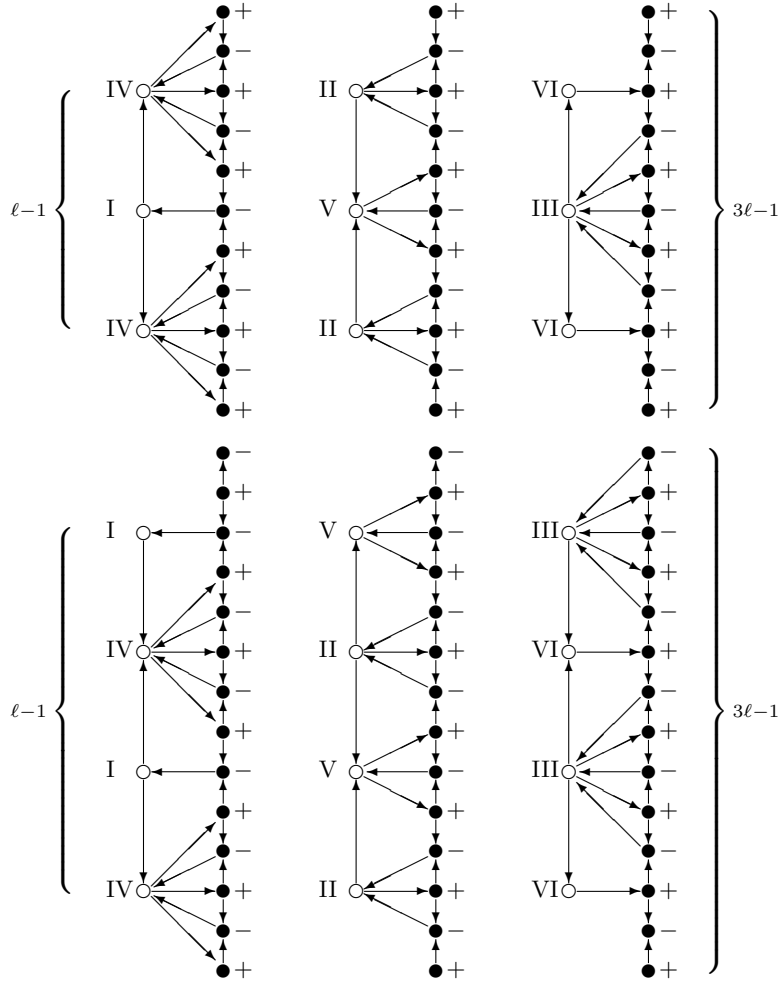


Figure 7. The quiver $Q_\ell(G_2)$ for ℓ even (top) and for ℓ odd (bottom), where we identify the right columns in the three quivers.

circle \circ/\bullet to each vertex; furthermore, we assign a sign $+/-$ to each vertex with \bullet , and one of the numbers $\text{I}, \dots, \text{VI}$ to each vertex with \circ .

Let us choose the index set \mathbf{I} of the vertices of $Q_\ell(G_2)$ so that $\mathbf{i} = (i, i') \in \mathbf{I}$ represents the vertex in the i' th row (from the bottom) and in the left column in the i th quiver (from the left) for $i = 1, 2, 3$, and in the right column in any quiver for $i = 4$. Thus, $i = 1, \dots, 4$, and $i' = 1, \dots, \ell - 1$ if $i \neq 4$, while $i' = 1, \dots, 3\ell - 1$ if $i = 4$.

For a permutation s of $\{1, 2, 3\}$, let ν_s be the permutation of \mathbf{I} such that $\nu_s(i, i')$ equals $(s(i), i')$ for $i \neq 4$, and $(4, i')$ for $i = 4$. Let ω be the involution acting on \mathbf{I} by up-down reflection. Let $\nu_s(Q_\ell(G_2))$ and $\omega(Q_\ell(G_2))$ denote the quivers induced from $Q_\ell(G_2)$ by ν_s and ω , respectively.

Lemma 5.1. *Let $Q = Q_\ell(G_2)$.*

(i) *We have a periodic sequence of mutations of quivers*

$$(5.8) \quad Q \xleftarrow{\mu_+^\bullet \mu_{\text{I}}^\circ} \nu_{(23)}(Q)^{\text{op}} \xleftarrow{\mu_-^\bullet \mu_{\text{II}}^\circ} \nu_{(312)}(Q) \xleftarrow{\mu_+^\bullet \mu_{\text{III}}^\circ} \nu_{(13)}(Q)^{\text{op}} \\ \xleftarrow{\mu_-^\bullet \mu_{\text{IV}}^\circ} \nu_{(231)}(Q) \xleftarrow{\mu_+^\bullet \mu_{\text{V}}^\circ} \nu_{(12)}(Q)^{\text{op}} \xleftarrow{\mu_-^\bullet \mu_{\text{VI}}^\circ} Q.$$

(ii) $\omega(Q) = Q$ if $h^\vee + \ell$ is even, and $\omega(Q) = \nu_{(13)}(Q)^{\text{op}}$ if $h^\vee + \ell$ is odd.

See Figures 8–10 for an example.

§5.3. Cluster algebra and alternative labels

Let $B_\ell(G_2)$ be the skew-symmetric matrix corresponding to the quiver $Q_\ell(G_2)$. In the rest of the section, we set $B = (B_{\mathbf{ij}})_{\mathbf{i}, \mathbf{j} \in \mathbf{I}} = B_\ell(G_2)$ unless otherwise mentioned.

Let $\mathcal{A}(B, x, y)$ be the cluster algebra with coefficients in the universal semifield $\mathbb{Q}_{\text{sf}}(y)$, and $\mathcal{G}(B, y)$ be the coefficient group associated with $\mathcal{A}(B, x, y)$.

In view of Lemma 5.1 we set $x(0) = x$, $y(0) = y$ and define clusters $x(u) = (x_{\mathbf{i}}(u))_{\mathbf{i} \in \mathbf{I}}$ ($u \in \frac{1}{3}\mathbb{Z}$) and coefficient tuples $y(u) = (y_{\mathbf{i}}(u))_{\mathbf{i} \in \mathbf{I}}$ ($u \in \frac{1}{3}\mathbb{Z}$) by the sequence of mutations

$$(5.9) \quad \cdots \xleftarrow{\mu_-^\bullet \mu_{\text{VI}}^\circ} (B, x(0), y(0)) \xleftarrow{\mu_+^\bullet \mu_{\text{I}}^\circ} (-\nu_{(23)}(B), x(\frac{1}{3}), y(\frac{1}{3})) \\ \xleftarrow{\mu_-^\bullet \mu_{\text{II}}^\circ} (\nu_{(312)}(B), x(\frac{2}{3}), y(\frac{2}{3})) \xleftarrow{\mu_+^\bullet \mu_{\text{III}}^\circ} (-\nu_{(13)}(B), x(1), y(1)) \\ \xleftarrow{\mu_-^\bullet \mu_{\text{IV}}^\circ} (\nu_{(231)}(B), x(\frac{4}{3}), y(\frac{4}{3})) \xleftarrow{\mu_+^\bullet \mu_{\text{V}}^\circ} (-\nu_{(12)}(B), x(\frac{5}{3}), y(\frac{5}{3})) \xleftarrow{\mu_-^\bullet \mu_{\text{VI}}^\circ} \cdots,$$

where $\nu_s(B) = B'$ is defined by $B'_{\nu_s(\mathbf{i})\nu_s(\mathbf{j})} = B_{\mathbf{ij}}$.

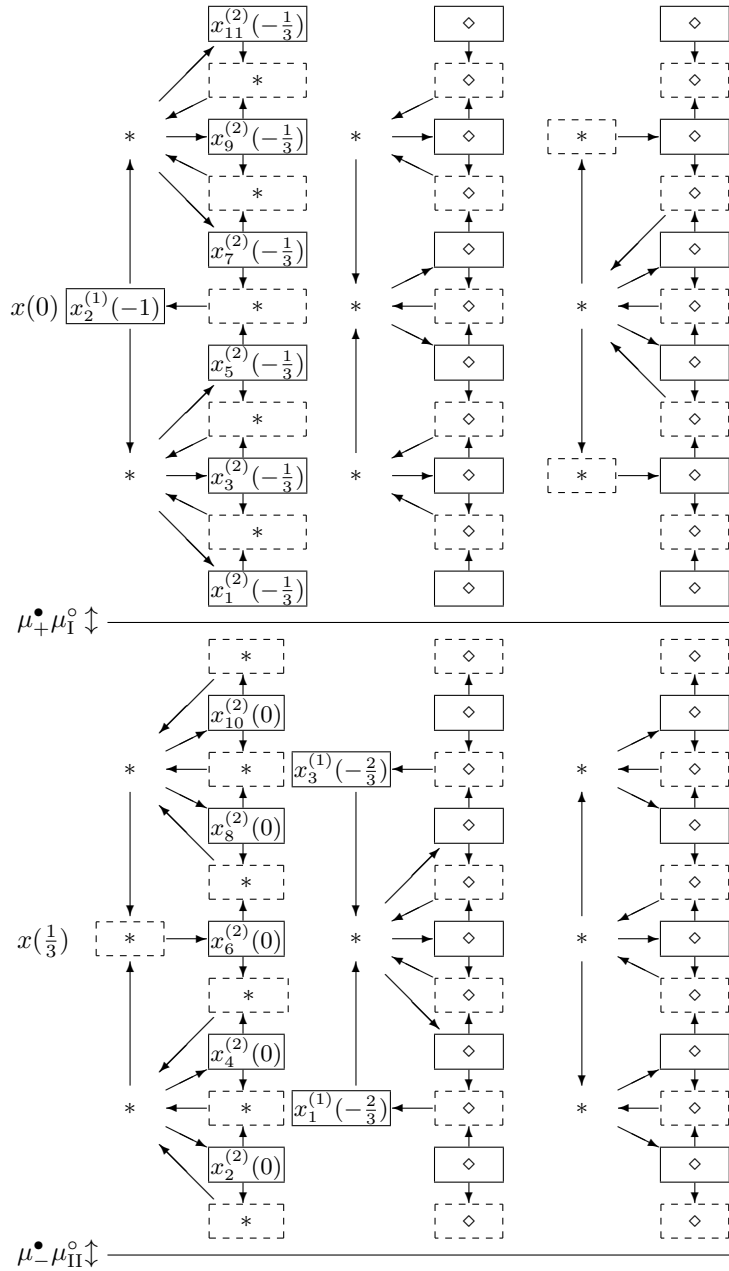


Figure 8. (Continued in Figures 9 and 10.) Labeling of cluster variables $x_i(u)$ by $\mathcal{I}_{\ell+}$ for G_2 , $\ell = 4$. The right columns in the middle and right quivers (marked by \diamond) are identified with the right column in the left quiver.

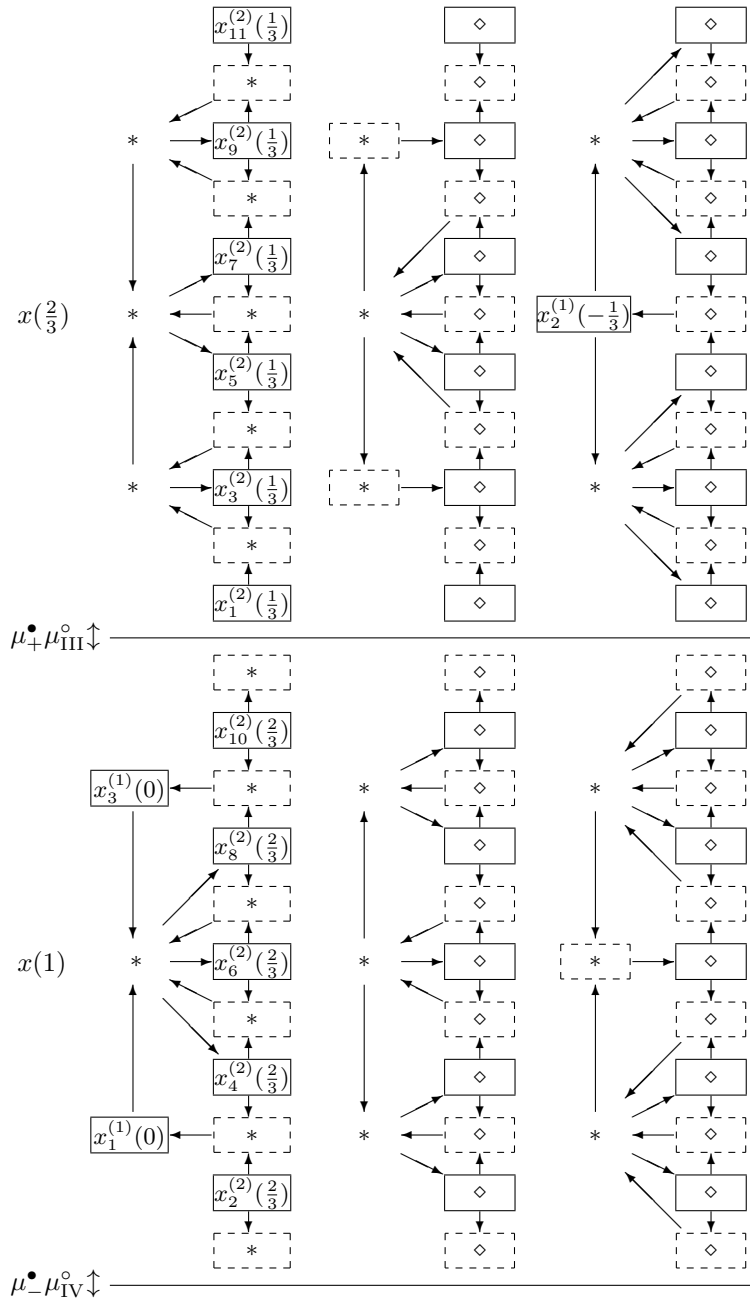


Figure 9. Continuation of Figure 8.

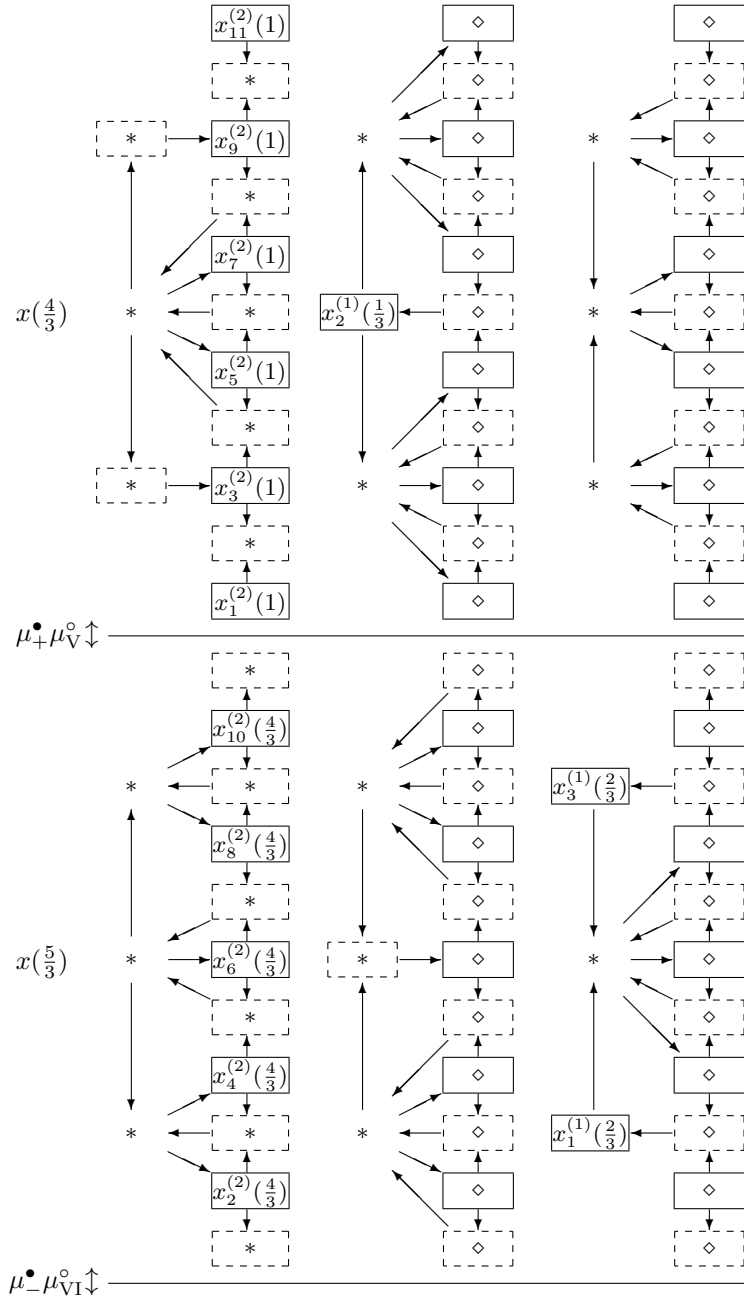


Figure 10. Continuation of Figure 9.

For a pair $(\mathbf{i}, u) \in \mathbf{I} \times \frac{1}{3}\mathbb{Z}$, we set the parity conditions \mathbf{p}_+ and \mathbf{p}_- to be

$$(5.10) \quad \mathbf{p}_+ : \begin{cases} \mathbf{i} \in \mathbf{I}_I^\circ \sqcup \mathbf{I}_+^\bullet, & u \equiv 0, \\ \mathbf{i} \in \mathbf{I}_{II}^\circ \sqcup \mathbf{I}_+^\bullet, & u \equiv \frac{1}{3}, \\ \mathbf{i} \in \mathbf{I}_{III}^\circ \sqcup \mathbf{I}_+^\bullet, & u \equiv \frac{2}{3}, \\ \mathbf{i} \in \mathbf{I}_{IV}^\circ \sqcup \mathbf{I}_+^\bullet, & u \equiv 1, \\ \mathbf{i} \in \mathbf{I}_V^\circ \sqcup \mathbf{I}_+^\bullet, & u \equiv \frac{4}{3}, \\ \mathbf{i} \in \mathbf{I}_{VI}^\circ \sqcup \mathbf{I}_+^\bullet, & u \equiv \frac{5}{3}, \end{cases} \quad \mathbf{p}_- : \begin{cases} \mathbf{i} \in \mathbf{I}_{VI}^\circ \sqcup \mathbf{I}_-^\bullet, & u \equiv 0, \\ \mathbf{i} \in \mathbf{I}_I^\circ \sqcup \mathbf{I}_+^\bullet, & u \equiv \frac{1}{3}, \\ \mathbf{i} \in \mathbf{I}_{II}^\circ \sqcup \mathbf{I}_-^\bullet, & u \equiv \frac{2}{3}, \\ \mathbf{i} \in \mathbf{I}_{III}^\circ \sqcup \mathbf{I}_+^\bullet, & u \equiv 1, \\ \mathbf{i} \in \mathbf{I}_{IV}^\circ \sqcup \mathbf{I}_-^\bullet, & u \equiv \frac{4}{3}, \\ \mathbf{i} \in \mathbf{I}_V^\circ \sqcup \mathbf{I}_+^\bullet, & u \equiv \frac{5}{3}, \end{cases}$$

where \equiv is equivalence modulo $2\mathbb{Z}$. We have

$$(5.11) \quad (\mathbf{i}, u) : \mathbf{p}_+ \Leftrightarrow (\mathbf{i}, u + \frac{1}{3}) : \mathbf{p}_-.$$

Each $(\mathbf{i}, u) : \mathbf{p}_+$ is a mutation point of (5.9) in the forward direction of u , and each $(\mathbf{i}, u) : \mathbf{p}_-$ is one in the backward direction of u .

Lemma 5.2. *Below \equiv means equivalence modulo $2\mathbb{Z}$.*

(i) *The map $g : \mathcal{I}_{\ell_+} \rightarrow \{(\mathbf{i}, u) : \mathbf{p}_+\}$ defined by*

$$(5.12) \quad (a, m, u - \frac{1}{t_a}) \mapsto \begin{cases} ((1, m), u), & a = 1; m + u \equiv 0, \\ ((2, m), u), & a = 1; m + u \equiv \frac{4}{3}, \\ ((3, m), u), & a = 1; m + u \equiv \frac{2}{3}, \\ ((4, m), u), & a = 2, \end{cases}$$

is a bijection.

(ii) *The map $g' : \mathcal{I}'_{\ell_+} \rightarrow \{(\mathbf{i}, u) : \mathbf{p}_+\}$ defined by*

$$(5.13) \quad (a, m, u) \mapsto \begin{cases} ((1, m), u), & a = 1; m + u \equiv 0, \\ ((2, m), u), & a = 1; m + u \equiv \frac{4}{3}, \\ ((3, m), u), & a = 1; m + u \equiv \frac{2}{3}, \\ ((4, m), u), & a = 2, \end{cases}$$

is a bijection.

We introduce alternative labels $x_{\mathbf{i}}(u) = x_m^{(a)}(u - 1/t_a)$ ($(a, m, u - 1/t_a) \in \mathcal{I}_{\ell_+}$) for $(\mathbf{i}, u) = g((a, m, u - 1/t_a))$ and $y_{\mathbf{i}}(u) = y_m^{(a)}(u)$ ($(a, m, u) \in \mathcal{I}'_{\ell_+}$) for $(\mathbf{i}, u) = g'((a, m, u))$, respectively. See Figures 8–10 for an example.

§5.4. T-system and cluster algebra

Lemma 5.3. *The family $\{x_m^{(a)}(u) \mid (a, m, u) \in \mathcal{I}_{\ell_+}\}$ satisfies the system of relations (3.16) with $G(b, k, v; a, m, u)$ for $\mathbb{T}_{\ell}(G_2)$. In particular, the family*

$\{[x_m^{(a)}(u)]_{\mathbf{1}} \mid (a, m, u) \in \mathcal{I}_{\ell+}\}$ satisfies the T -system $\mathbb{T}_{\ell}(G_2)$ in $\mathcal{A}(B, x)$ after replacing $T_m^{(a)}(u)$ with $[x_m^{(a)}(u)]_{\mathbf{1}}$.

Definition 5.4. The T -subalgebra $\mathcal{A}_T(B, x)$ of $\mathcal{A}(B, x)$ associated with the sequence (5.9) is the subring of $\mathcal{A}(B, x)$ generated by $[x_{\mathbf{i}}(u)]_{\mathbf{1}}$ ($(\mathbf{i}, u) \in \mathbf{I} \times \frac{1}{3}\mathbb{Z}$).

Theorem 5.5. The ring $\mathcal{T}_{\ell}^{\circ}(G_2)_+$ is isomorphic to $\mathcal{A}_T(B, x)$ via the correspondence $T_m^{(a)}(u) \mapsto [x_m^{(a)}(u)]_{\mathbf{1}}$.

§5.5. Y-system and cluster algebra

Lemma 5.6. The family $\{y_m^{(a)}(u) \mid (a, m, u) \in \mathcal{I}'_{\ell+}\}$ satisfies the Y -system $\mathbb{Y}_{\ell}(G_2)$ after replacing $Y_m^{(a)}(u)$ with $y_m^{(a)}(u)$.

Definition 5.7. The Y -subgroup $\mathcal{G}_Y(B, y)$ of $\mathcal{G}(B, y)$ associated with the sequence (5.9) is the subgroup of $\mathcal{G}(B, y)$ generated by $y_{\mathbf{i}}(u)$ ($(\mathbf{i}, u) \in \mathbf{I} \times \frac{1}{3}\mathbb{Z}$) and $1 + y_{\mathbf{i}}(u)$ ($(\mathbf{i}, u) : \mathbf{p}_+$ or \mathbf{p}_-).

Theorem 5.8. The group $\mathcal{Y}_{\ell}^{\circ}(G_2)_+$ is isomorphic to $\mathcal{G}_Y(B, y)$ via the correspondence $Y_m^{(a)}(u) \mapsto y_m^{(a)}(u)$ and $1 + Y_m^{(a)}(u) \mapsto 1 + y_m^{(a)}(u)$.

§5.6. Tropical Y-system at level 2

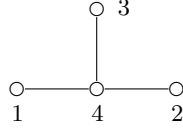
By direct computations, the following properties are verified.

Proposition 5.9. For $[\mathcal{G}_Y(B, y)]_{\mathbf{T}}$ with $B = B_2(G_2)$, the following facts hold:

- (i) Let $0 \leq u < 2$. For any $(\mathbf{i}, u) : \mathbf{p}_+$, the monomial $[y_{\mathbf{i}}(u)]_{\mathbf{T}}$ is positive.
- (ii) Let $-h^{\vee} \leq u < 0$.
 - (a) Let $\mathbf{i} = (1, 1), (2, 1), (3, 1),$ or $(4, 3)$. For any $(\mathbf{i}, u) : \mathbf{p}_+$, the monomial $[y_{\mathbf{i}}(u)]_{\mathbf{T}}$ is negative.
 - (b) Let $\mathbf{i} = (4, 1), (4, 2), (4, 4),$ or $(4, 5)$. For any $(\mathbf{i}, u) : \mathbf{p}_+$, the monomial $[y_{\mathbf{i}}(u)]_{\mathbf{T}}$ is negative for $u = -\frac{1}{3}, -\frac{2}{3}, -2, -\frac{7}{3}, -\frac{11}{3}, -4$, and positive for $u = -1, -\frac{4}{3}, -\frac{5}{3}, -\frac{8}{3}, -3, -\frac{10}{3}$.
- (iii) $y_{ii'}(2)$ equals $y_{ii'}^{-1}$ if $i \neq 4$, and $y_{4,6-i'}^{-1}$ if $i = 4$.
- (iv) $y_{ii'}(-h^{\vee}) = y_{ii'}^{-1}$.

Also we have a description of the core part of $[y_{\mathbf{i}}(u)]_{\mathbf{T}}$ in the region $-h^{\vee} \leq u < 0$ in terms of the root system of D_4 . We use the following indexing of the

Dynkin diagram D_4 :



Let $\Pi = \{\alpha_1, \dots, \alpha_4\}$, $-\Pi$, Φ_+ be the sets of simple roots, of negative simple roots, and of positive roots, respectively, of type D_4 . Let σ_i be the piecewise-linear analogue of the simple reflection s_i , acting on the set $\Phi_{\geq -1} = \Phi_+ \sqcup (-\Pi)$ of almost positive roots. We define

$$(5.14) \quad \sigma = \sigma_3 \sigma_4 \sigma_1 \sigma_4 \sigma_2 \sigma_4.$$

Lemma 5.10. *The orbits under σ are:*

$$(5.15) \quad \begin{aligned} -\alpha_1 &\rightarrow \alpha_1 + \alpha_3 + \alpha_4 \rightarrow \alpha_1 + \alpha_2 + \alpha_4 \rightarrow -\alpha_1, \\ -\alpha_2 &\rightarrow \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \rightarrow \alpha_2 + \alpha_4 \rightarrow -\alpha_2, \\ -\alpha_3 &\rightarrow \alpha_3 \rightarrow \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 \rightarrow -\alpha_3, \\ -\alpha_4 &\rightarrow \alpha_2 + \alpha_3 + \alpha_4 \rightarrow \alpha_4 \rightarrow \alpha_1 \\ &\rightarrow \alpha_2 \rightarrow \alpha_3 + \alpha_4 \rightarrow \alpha_1 + \alpha_4 \rightarrow -\alpha_4. \end{aligned}$$

In particular, these elements of Φ_+ exhaust the set Φ_+ , thereby providing the orbit decomposition of Φ_+ under σ .

For $-h^\vee \leq u < 0$, define

$$(5.16) \quad \alpha_i(u) = \begin{cases} \sigma^{-(u-1)/2}(-\alpha_1), & i = 1; u = -1, -3, \\ \sigma^{-(3u-1)/6}(-\alpha_2), & i = 2; u = -\frac{5}{3}, -\frac{11}{3}, \\ \sigma^{-(3u-5)/6}(-\alpha_3), & i = 3; u = -\frac{1}{3}, -\frac{7}{3}, \\ \sigma^{-(3u+2)/6}(\alpha_3 + \alpha_4), & i = 4; u = -\frac{2}{3}, -\frac{8}{3}, \\ \sigma^{-(3u+4)/6}(\alpha_1), & i = 4; u = -\frac{4}{3}, -\frac{10}{3}, \\ \sigma^{-u/2}(-\alpha_4), & i = 4; u = -2, -4. \end{cases}$$

By Lemma 5.10 these are (all the) positive roots of D_4 .

For a monomial m in $y = (y_i)_{i \in \mathbf{I}}$, let $\pi_D(m)$ denote the specialization with $y_{41} = y_{42} = y_{44} = y_{45} = 1$. For simplicity, we set $y_{i1} = y_i$ ($i \neq 4$), $y_{43} = y_4$, and also $y_{i1}(u) = y_i(u)$ ($i \neq 4$), $y_{43}(u) = y_4(u)$. We define the vectors $\mathbf{t}_i(u) = (t_i(u)_k)_{k=1}^4$ by

$$(5.17) \quad \pi_D([y_i(u)]_{\mathbf{T}}) = \prod_{k=1}^4 y_k^{t_i(u)_k}.$$

We also identify each vector $\mathbf{t}_i(u)$ with $\alpha = \sum_{k=1}^4 t_i(u)_k \alpha_k \in \mathbb{Z}\Pi$.

Proposition 5.11. *Let $-h^\vee \leq u < 0$. Then*

$$(5.18) \quad \mathbf{t}_i(u) = -\alpha_i(u)$$

for (i, u) as in (5.16).

§5.7. Tropical Y-systems of higher levels

Proposition 5.12. *Let $\ell > 2$ be an integer. For $[\mathcal{G}_Y(B, y)]_{\mathbf{T}}$ with $B = B_\ell(G_2)$, the following facts hold:*

- (i) *Let $0 \leq u < \ell$. For any $(\mathbf{i}, u) : \mathbf{p}_+$, the monomial $[y_{\mathbf{i}}(u)]_{\mathbf{T}}$ is positive.*
- (ii) *Let $-h^\vee \leq u < 0$.*
 - (a) *Let $\mathbf{i} \in \mathbf{I}^\circ$ or $\mathbf{i} = (4, i')$ ($i' \in 3\mathbb{N}$). For any $(\mathbf{i}, u) : \mathbf{p}_+$, the monomial $[y_{\mathbf{i}}(u)]_{\mathbf{T}}$ is negative.*
 - (b) *Let $\mathbf{i} = (4, i')$ ($i' \notin 3\mathbb{N}$). For any $(\mathbf{i}, u) : \mathbf{p}_+$, the monomial $[y_{\mathbf{i}}(u)]_{\mathbf{T}}$ is negative for $u = -\frac{1}{3}, -\frac{2}{3}, -2, -\frac{7}{3}, -\frac{11}{3}, -4$ and positive for $u = -1, -\frac{4}{3}, -\frac{5}{3}, -\frac{8}{3}, -3, -\frac{10}{3}$.*
- (iii) *$y_{ii'}(\ell)$ equals $y_{i, \ell-i'}^{-1}$ if $i \neq 4$, and $y_{4, 3\ell-i'}^{-1}$ if $i = 4$.*
- (iv) *$y_{ii'}(-h^\vee) = y_{ii'}^{-1}$.*

The following are corollaries of Propositions 5.9 and 5.12.

Theorem 5.13. *For $[\mathcal{G}_Y(B, y)]_{\mathbf{T}}$, the following relations hold:*

- (i) *Half periodicity: $[y_{\mathbf{i}}(u + h^\vee + \ell)]_{\mathbf{T}} = [y_{\omega(\mathbf{i})}(u)]_{\mathbf{T}}$.*
- (ii) *Full periodicity: $[y_{\mathbf{i}}(u + 2(h^\vee + \ell))]_{\mathbf{T}} = [y_{\mathbf{i}}(u)]_{\mathbf{T}}$.*

Theorem 5.14. *For $[\mathcal{G}_Y(B, y)]_{\mathbf{T}}$, let N_+ and N_- denote the total numbers of the positive and negative monomials, respectively, among $[y_{\mathbf{i}}(u)]_{\mathbf{T}}$ for $(\mathbf{i}, u) : \mathbf{p}_+$ in the region $0 \leq u < 2(h^\vee + \ell)$. Then*

$$(5.19) \quad N_+ = 6\ell(2\ell + 1), \quad N_- = 12(3\ell - 2).$$

§5.8. Periodicities and dilogarithm identities

Applying [IKKN, Theorem 5.1] to Theorem 5.13, we obtain the periodicities:

Theorem 5.15. *For $\mathcal{A}(B, x, y)$, the following relations hold:*

- (i) *Half periodicity: $x_{\mathbf{i}}(u + h^\vee + \ell) = x_{\omega(\mathbf{i})}(u)$.*
- (ii) *Full periodicity: $x_{\mathbf{i}}(u + 2(h^\vee + \ell)) = x_{\mathbf{i}}(u)$.*

Theorem 5.16. For $\mathcal{G}(B, y)$, the following relations hold:

- (i) *Half periodicity:* $y_{\mathbf{i}}(u + h^\vee + \ell) = y_{\omega(\mathbf{i})}(u)$.
- (ii) *Full periodicity:* $y_{\mathbf{i}}(u + 2(h^\vee + \ell)) = y_{\mathbf{i}}(u)$.

Theorems 2.5 and 2.6 for G_2 follow from Theorems 5.5, 5.8, 5.15, and 5.16. Furthermore, Theorem 2.10 for G_2 is obtained from the above periodicities and Theorem 5.14 as in the B_r case [IIKKN, Section 6].

§6. Mutation equivalence of quivers

Recall that two quivers Q and Q' are said to be *mutation equivalent*, denoted $Q \sim Q'$ here, if there is a quiver isomorphism from Q to some quiver obtained from Q' by successive mutations.

Below we present several mutation equivalent pairs of quivers $Q_\ell(X_r)$, though the list is not complete at all. For simply laced X_r , $Q_\ell(X_r)$ is the quiver defined as the square product $\vec{X}_r \square \vec{A}_{\ell-1}$ in [Ke, Section 8].

Proposition 6.1. We have the following mutation equivalences of quivers:

$$\begin{aligned}
 (6.1) \quad & Q_2(B_r) \sim Q_2(D_{2r+1}), \\
 & Q_2(C_3) \sim Q_3(D_4), \\
 & Q_2(F_4) \sim Q_3(D_5), \\
 & Q_3(C_2) \sim Q_4(A_3), \\
 & Q_\ell(G_2) \sim Q_\ell(C_3).
 \end{aligned}$$

References

- [FZ1] S. Fomin and A. Zelevinsky, Y-systems and generalized associahedra, *Ann. of Math.* **158** (2003), 977–1018. [Zbl 1057.52003](#) [MR 2031858](#)
- [FZ2] ———, Cluster algebras IV. Coefficients, *Compos. Math.* **143** (2007), 112–164. [Zbl 1127.16023](#) [MR 2295199](#)
- [IIKKN] R. Inoue, O. Iyama, B. Keller, A. Kuniba and T. Nakanishi, Periodicities of T-systems and Y-systems, dilogarithm identities, and cluster algebras I: Type B_r , *Publ. RIMS Kyoto Univ.* **49** (2013), 1–42.
- [IIKNS] R. Inoue, O. Iyama, A. Kuniba, T. Nakanishi and J. Suzuki, Periodicities of T-systems and Y-systems, *Nagoya Math. J.* **197** (2010), 59–174. [Zbl pre05704436](#) [MR 2649278](#)
- [Ke] B. Keller, Cluster algebras, quiver representations and triangulated categories, in *Triangulated categories*, T. Holm et al. (eds.), London Math. Soc. Lecture Note Ser. 375, Cambridge Univ. Press, 2010, 76–160. [Zbl 1215.16012](#) [MR 2681708](#)
- [Ki] A. N. Kirillov, Identities for the Rogers dilogarithm function connected with simple Lie algebras, *J. Soviet Math.* **47** (1989), 2450–2458. [Zbl 0685.33014](#) [MR 0947332](#)
- [Ku] A. Kuniba, Thermodynamics of the $U_q(X_r^{(1)})$ Bethe ansatz system with q a root of unity, *Nucl. Phys. B* **389** (1993), 209–244. [MR 1202213](#)

- [KN] A. Kuniba and T. Nakanishi, Spectra in conformal field theories from the Rogers dilogarithm, *Mod. Phys. Lett. A* **7** (1992), 3487–3494. [Zbl 1021.81842](#) [MR 1192727](#)
- [KNS] A. Kuniba, T. Nakanishi and J. Suzuki, Functional relations in solvable lattice models: I. Functional relations and representation theory, *Int. J. Modern Phys. A* **9** (1994), 5215–5266. [Zbl 0985.82501](#) [MR 1304818](#)
- [P1] P.-G. Plamondon, Cluster characters for cluster categories with infinite-dimensional morphism spaces, *Adv. Math.* **227** (2011), 1–39. [Zbl pre05880817](#) [MR 2782186](#)
- [P2] ———, Cluster algebras via cluster categories with infinite-dimensional morphism spaces, *Compos. Math.* **147** (2011), 1921–1954. [Zbl 1244.13017](#) [MR 2862067](#)