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Representation Theory of Rational Cherednik Algebras of Type $\mathbb{Z}/l\mathbb{Z}$ via Microlocal Analysis

by

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Abstract

Based on the methods developed in [KR], we consider microlocalization of rational Cherednik algebras of type $\mathbb{Z}/l\mathbb{Z}$. Our goal is to construct irreducible modules and standard modules of these rational Cherednik algebras by using microlocalization. As a consequence, we obtain sheaves corresponding to holonomic systems with regular singularities.

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§1. Introduction

A symplectic reflection algebra is a noncommutative deformation of the smash product $\mathbb{C}[V]\#\Gamma$, introduced by [EG], where V is a symplectic vector space and Γ is a finite group generated by symplectic reflections on V. Sometimes, we identify the symplectic reflection algebra with its spherical subalgebra, a noncommutative deformation of $\mathbb{C}[V]^{\Gamma}$, because these algebras are Morita equivalent except for a certain choice of their parameters.

When the group Γ coincides with a complex reflection group and V coincides with $\mathfrak{h}\oplus\mathfrak{h}^*$ where $\mathfrak{h}$ is the reflection representation of Γ, the symplectic reflection algebra is sometimes called a *rational Cherednik algebra*. An important property of rational Cherednik algebras is that they have a triangular decomposition similar to complex semisimple Lie algebras. Via the triangular decomposition, we can introduce a certain subcategory of the category of modules, called the *category* $\mathcal{O}$. The category $\mathcal{O}$ is a highest weight category in the sense of [CPS]. Its standard modules

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and costandard modules are studied in [GGOR]. For each irreducible $\mathbb{C}\Gamma$-module $E \in \operatorname{Irr} \mathbb{C}\Gamma$, we have a corresponding standard module $\Delta(E)$. The standard module has a unique irreducible quotient $L(E)$ and any irreducible module in the category $\mathcal{O}$ is isomorphic to $L(E)$ for a certain E. One of fundamental problems of the representation theory of rational Cherednik algebras is to determine multiplicities $[\Delta(E) : L(F)]$ in the Grothendieck group for $E, F \in \operatorname{Irr} \mathbb{C}\Gamma$.

When the group Γ is a wreath product $(\mathbb{Z}/l\mathbb{Z}) \wr \mathfrak{S}_n$ of a cyclic group $\mathbb{Z}/l\mathbb{Z}$ and a symmetric group $\mathfrak{S}_n$, there is a close connection between rational Cherednik algebras and quiver varieties which are symplectic varieties introduced in [Na]. After the leading work of [GS1] and [GS2], [KR] constructed a microlocalization of rational Cherednik algebras of type $\mathfrak{S}_n$. The microlocalization is a kind of Deformation-Quantization algebra, called a W-algebra, on a quiver variety. [KR] introduced the notion of F-action on W-algebras and established an equivalence of categories between the category of finitely generated modules over a rational Cherednik algebra and the category of F-equivariant, good modules over a W-algebra. This equivalence is an analogue of the Beilinson–Bernstein correspondence for complex semisimple Lie algebras.

In [BK] and [BLPW], microlocalization of rational Cherednik algebras of type $\mathbb{Z}/l\mathbb{Z}$ was studied independently. As an application of the microlocalization of rational Cherednik algebras, we study the construction of irreducible modules and standard modules of these rational Cherednik algebras via microlocalization.

Let us describe the structure of this article.

In Section 2, we review fundamental properties of minimal resolutions of Kleinian singularities of type A. We construct Kleinian singularities and their resolutions X as quiver varieties of cyclic quivers. Moreover, we see that the structure of X as a toric variety gives us an affine open covering $X = \bigcup_{i=1}^{l} X_i$ such that $X_i \simeq \mathbb{C}^2$.

In Section 3, we review the general setting of W-algebras and construct the microlocalization $\widetilde{\mathscr{A}_c}$ of rational Cherednik algebra A_c on X. By [BK, Theorem 6.3] and [BLPW, Theorem 6.1], we have an equivalence of categories

$$\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{A}_c}) \to A_c\text{-mod}, \qquad \mathscr{M} \mapsto \mathrm{Hom}_{\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}, \mathscr{M}),$$

under certain conditions on the parameter c.

At the end of Section 3.3, we describe the structure of $\widetilde{\mathscr{A}_c}|_{X_i}$ explicitly on the affine open subset X_i for $i = 1, \dots, l$.

In Section 4, we briefly review the representation theory of rational Cherednik algebras. Its spherical subalgebra is isomorphic to A_c. We introduce the category $\mathcal{O}(A_c)$, and review the definition of its standard modules $\Delta_c(i)$ and irreducible modules $L_c(i)$.

In Section 5.1, we construct $\widetilde{\mathscr{A}_c}$-modules $\mathcal{M}_c^{\Delta}(i)$ for $i = 1, \dots, l$. These are F-equivariant, holonomic $\widetilde{\mathscr{A}_c}$-modules supported on certain Lagrangian subvarieties. We show that the corresponding A_c-modules $\mathrm{Hom}_{\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}, \mathcal{M}_c^{\Delta}(i))$ are isomorphic to the standard modules $\Delta_c(i)$.

In Section 5.2, we construct $\widetilde{\mathscr{A}_c}$-modules $\mathcal{L}_c(i)$ for $i = 1, \dots, l$. These are also F-equivariant, holonomic $\widetilde{\mathscr{A}_c}$-modules supported on certain Lagrangian subvarieties. Moreover, we show that $\mathcal{L}_c(i)$ are irreducible $\widetilde{\mathscr{A}_c}$-modules. At the end of Section 5.2, we determine the multiplicity $[\Delta_c(i) : L_c(j)]$ in the Grothendieck group of $\mathcal{O}(A_c)$ as a corollary of the construction of the $\widetilde{\mathscr{A}_c}$-modules $\mathcal{M}_c^{\Delta}(i)$ and $\mathcal{L}_c(j)$.

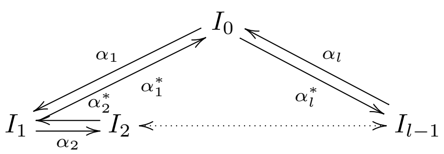
Finally, in the appendix, we explicitly construct global sections of the $\widetilde{\mathscr{A}_c}$-modules $\mathcal{M}_c^{\Delta}(i)$.

§2. Quiver varieties

In this section we review the definition and fundamental properties of quiver varieties without framing, which were introduced in [Kr].

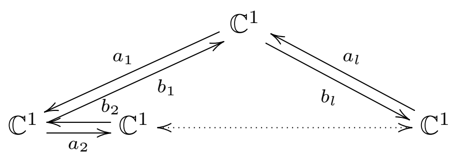
Let $Q = (I, E)$ be a cyclic quiver with vertices $I = \{I_i \mid i = 0, \dots, l-1\}$ and arrows $E = \{\alpha_i : I_{i-1} \to I_i \mid i = 1, \dots, l\}$. Let $\overline{Q} = (I, E \sqcup E^*)$ be a quiver with vertices I and arrows E and $E^* = \{\alpha_i^* : I_i \to I_{i-1}\}$. Throughout this paper, we regard indices of vertices and edges of Q and $\overline{Q}$ as integers modulo l, i.e. we regard $I_{l+i} = I_i$ and $\alpha_{l+i} = \alpha_i$.

$\overline{Q}$:



Set $\delta = (1, \dots, 1) \in (\mathbb{Z}_{\geq 0})^l$; we call δ a dimension vector. A *representation* of $\overline{Q}$ with dimension vector δ is a pair $(V, (a_i, b_i)_{i=1,\dots,l})$ of an I-graded vector space $V = \bigoplus_{i=0}^{l-1} V_i$ such that $\dim V_i = 1$ for all $i \in I$ and linear maps $a_i : V_{i-1} \to V_i$ and $b_i : V_i \to V_{i-1}$. Since $\dim V_i = 1$ for all i, we regard a_i and b_i as elements of $\mathbb{C}$. Let $GL(\delta) = \prod_{i=0}^{l-1} GL(V_i) \simeq (\mathbb{C}^*)^l$ be a reductive algebraic group acting on V. Let $G = PGL(\delta) = GL(\delta)/\mathbb{C}^*_{\mathrm{diag}} \simeq (\mathbb{C}^*)^{l-1}$ where $\mathbb{C}^*_{\mathrm{diag}}$ is the diagonal subgroup of $GL(\delta)$. Let $\mathfrak{g} = \mathrm{Lie}(G)$ be the Lie algebra of G. We have $\mathfrak{g} = (\bigoplus_{i=0}^{l-1} \mathbb{C})/\mathbb{C}_{\mathrm{diag}}$ where $\mathbb{C}_{\mathrm{diag}}$ is the diagonal Lie subalgebra of $\bigoplus_{i=0}^{l-1} \mathbb{C}$.

A representation of $\overline{Q}$:



Fix a parameter $\theta = (\theta_0, \dots, \theta_{l-1}) \in \mathbb{Z}^l$ such that $\theta_0 + \theta_1 + \cdots + \theta_{l-1} = 0$. Note that we regard indices of θ as integers modulo l, i.e. $\theta = (\theta_i)_{i \in \mathbb{Z}/l\mathbb{Z}}$, and $\theta_i + \theta_{i+1} + \cdots + \theta_{j-1}$ is well-defined for any $i, j \in \mathbb{Z}/l\mathbb{Z}$. We regard θ as a character of $\mathfrak{g}$.

A representation $(V, (a_i, b_i)_{i=1,\dots,l})$ is called θ-*semistable* if any I-graded subspace W of V which is stable under the action of $(a_i, b_i)_{i=1,\dots,l}$ satisfies the condition $\sum_{i=0}^{l-1} \theta_i \dim W_i \leq 0$.

Fix a parameter θ. Let $\widetilde{X}_\theta \subset \mathbb{C}^{2l}$ be the space of all θ-semistable representations,

$$\widetilde{X}_\theta = \{(a_i, b_i)_{i=1,\dots,l} \in \mathbb{C}^{2l} \mid (a_i, b_i)_{i=1,\dots,l} \text{ is } \theta\text{-semistable}\}.$$

The group $GL(\delta)$ acts on $\widetilde{X}_\theta$ by

$$GL(\delta) \times \widetilde{X}_\theta \to \widetilde{X}_\theta, \quad \big((g_i)_{i \in \mathbb{Z}/l\mathbb{Z}}, (a_j, b_j)_{j=1,\dots,l}\big) \mapsto (g_j g_{j-1}^{-1} a_j, g_{j-1} g_j^{-1} b_j)_{j=1,\dots,l}.$$

This action clearly factors through $G = PGL(\delta)$. Two points p, p' of $\widetilde{X}_\theta$ are called *S-equivalent* if the closures of their orbits intersect in $\widetilde{X}_\theta$.

Consider the following moment map with respect to the action of $GL(\delta)$ on $\widetilde{X}_\theta$:

$$\mu : \widetilde{X}_\theta \to \mathfrak{g}^* \subset \mathbb{C}^l, \quad (a_i, b_i)_{i=1,\dots,l} \mapsto (a_{i+1}b_{i+1} - a_i b_i)_{i=0,\dots,l-1}.$$

We consider the Hamiltonian reduction of $\widetilde{X}_\theta$ with respect to the moment map μ. The subset $\mu^{-1}(0) \subset \widetilde{X}_\theta$ is stable under the action of G.

Definition 2.1. The *quiver variety* of the quiver $\overline{Q}$ with dimension vector δ and stability parameter θ is a complex symplectic variety

$$X_\theta = \mu^{-1}(0)/\sim_{\mathrm{S}}$$

where $\sim_{\mathrm{S}}$ is S-equivalence.

We denote the S-equivalence class in X_θ containing $(a_i, b_i)_{i=1,\dots,l} \in \widetilde{X}_\theta$ by $[a_i, b_i]_{i=1,\dots,l}$.

Let us consider the case of $\theta = 0 = (0, \dots, 0)$. For $(a_i, b_i)_{i=1,\dots,l} \in \mu^{-1}(0)$, we set $\bar{a} = \sqrt[l]{a_1 \cdots a_l}$, $\bar{b} = \sqrt[l]{b_1 \cdots b_l}$ such that $\bar{a}\bar{b} = a_1 b_1$. Then we have the following isomorphism of algebraic varieties:

$$X_0 \xrightarrow{\simeq} \mathbb{C}^2/(\mathbb{Z}/l\mathbb{Z}), \quad [a_i, b_i]_{i=1,\dots,l} \mapsto (\bar{a}, \bar{b}).$$

Note that the image of $(\bar{a}, \bar{b})$ in $\mathbb{C}^2/(\mathbb{Z}/l\mathbb{Z})$ is independent of the choice of root (cf. [Kr, Corollary 3.2]).

Proposition 2.2 ([Kr, Corollary 3.12]). *If a stability parameter* $\theta = (\theta_i)_{i=0,\dots,l-1}$ *satisfies* $\theta_i + \theta_{i+1} + \cdots + \theta_{j-1} \neq 0$ *for all* i, j $(i \neq j)$, *then* X_θ *is nonsingular and we have a minimal resolution of Kleinian singularities of type* A_{l-1}*:*

$$\pi_\theta : X_\theta \to X_0 \simeq \mathbb{C}^2/(\mathbb{Z}/l\mathbb{Z}).$$

In the rest of this paper, we fix a stability parameter θ satisfying the condition of Proposition 2.2. We denote X_θ by X, π_θ by π, etc. for simplicity.

Remark 2.3. Although all X_θ are isomorphic to one another as algebraic varieties, we use the explicit construction in Definition 2.1 to construct W-algebras in Section 3.3. Moreover, we give a condition for θ in order that W-affinity holds in Theorem 3.10.

One of the fundamental properties of X is that it is a toric variety with respect to the following action of the 2-dimensional torus $\mathbb{T}^2 = (\mathbb{C}^*)^2$:

$$(q_1, q_2)[a_i, b_i]_{i=1,\ldots,l} = [q_1 a_i, q_2 b_i]_{i=1,\ldots,l} \tag{1}$$

for $(q_1, q_2) \in \mathbb{T}^2$ and $[a_i, b_i]_{i=1,\ldots,l} \in X$. The following facts are easy to obtain from the general theory of toric varieties. We refer the reader to [Ku, Section 2] for proofs of these facts, or to [Fu] for the general theory of toric varieties.

The variety X has l $\mathbb{T}$-fixed points $p'_1, \ldots, p'_l$ where $p'_i = [a_j, b_j]_{j=1,\ldots,l}$ is given as follows:

$$\begin{aligned} &a_i = 0, && b_i = 0, && \\ &a_j = 0, && b_j \neq 0 && \text{if } \theta_i + \theta_{i+1} + \cdots + \theta_{j-1} < 0, \\ &a_j \neq 0, && b_j = 0 && \text{if } \theta_i + \theta_{i+1} + \cdots + \theta_{j-1} > 0. \end{aligned}$$

Note that our condition on the parameter θ ensures $\theta_i + \theta_{i+1} + \cdots + \theta_{j-1} \neq 0$ for all $i \neq j$.

Define an ordering $\rhd$ on the set of indices $\Lambda = \{1, \ldots, l\}$ by

$$i \rhd j \iff \theta_i + \cdots + \theta_{j-1} < 0.$$

By the condition on the stability parameter θ, the ordering $\rhd$ is a total ordering. Let $\eta_1, \ldots, \eta_l$ be the indices in Λ arranged so that

$$\eta_1 \rhd \cdots \rhd \eta_l. \tag{2}$$

Remark 2.4. Note that the order of $\eta_1, \ldots, \eta_i$ is reverse to the one of [Ku].

Set $p_i = p'_{\eta_i}$ for $i = 1, \ldots, l$. The explicit description of the point p_i is given as follows.

Lemma 2.5. *For $i = 1, \ldots, l$, the fixed point $p_i = p'_{\eta_i} = [a_j, b_j]_{j=1,\ldots,l}$ is given by*

$$\begin{aligned} &a_{\eta_i} = 0, && b_{\eta_i} = 0, && \\ &a_{\eta_j} = 0, && b_{\eta_j} \neq 0 && \textit{for } j > i, \\ &a_{\eta_j} \neq 0, && b_{\eta_j} = 0 && \textit{for } j < i. \end{aligned}$$

Let us consider the Lagrangian subvariety $\pi^{-1}(\{\bar{a}=0 \text{ or } \bar{b}=0\})$. It has $l+1$ irreducible components $D_0, D_1, \ldots, D_l$ such that $D_0, D_l \simeq \mathbb{C}^1$, $D_i \simeq \mathbb{P}^1$ for $1 \le i \le l-1$ and p_i is the unique intersection point of D_{i-1} and D_i. We can describe D_i explicitly as follows.

Lemma 2.6. *For $i = 1, \ldots, l$, the $\mathbb{T}$-divisor D_i is given by*

$$D_i = \overline{\left\{ [a_j, b_j]_{j=1,\ldots,l} \;\middle|\; \begin{array}{l} a_{\eta_j} = 0,\, b_{\eta_j} \neq 0 \text{ for } j > i \\ a_{\eta_j} \neq 0,\, b_{\eta_j} = 0 \text{ for } j \le i \end{array} \right\}}.$$

Similarly, D_0 is given by

$$D_0 = \overline{\{[a_j, b_j]_{j=1,\ldots,l} \mid a_j = 0,\, b_j \neq 0\}}.$$

The description as a toric variety gives us the following affine open covering of X:

$$X = \bigcup_{i=1}^{l} X_i, \qquad X_i = \left\{ [a_j, b_j]_{j=1,\ldots,l} \;\middle|\; \begin{array}{l} a_{\eta_j} \neq 0 \text{ for } j < i \\ b_{\eta_j} \neq 0 \text{ for } j > i \end{array} \right\}.$$

We define coordinate functions $\bar{x}_i$ (resp. $\bar{y}_i$) for $i = 1, \ldots, l$ on $\widetilde{X} \subset \mathbb{C}^{2l}$ by $\bar{x}_i((a_j, b_j)_{j=1,\ldots,l}) = a_i$ (resp. $\bar{y}_i((a_j, b_j)_{j=1,\ldots,l}) = b_i$). For $i = 1, \ldots, l$, let R_i be the following subring of $\mathbb{C}(\bar{x}_1, \ldots, \bar{x}_l, \bar{y}_1, \ldots, \bar{y}_l)$ which is isomorphic to a polynomial ring in two variables:

$$R_i = \mathbb{C}[\bar{f}_i, \bar{g}_i]$$

where

$$\bar{f}_i = \frac{\bar{x}_{\eta_1}\bar{x}_{\eta_2}\ldots\bar{x}_{\eta_i}}{\bar{y}_{\eta_{i+1}}\bar{y}_{\eta_{i+2}}\ldots\bar{y}_{\eta_l}},\; \bar{g}_i = \frac{\bar{y}_{\eta_i}\bar{y}_{\eta_{i+1}}\ldots\bar{y}_{\eta_l}}{\bar{x}_{\eta_1}\bar{x}_{\eta_2}\ldots\bar{x}_{\eta_{i-1}}} \in \mathbb{C}(\bar{x}_1, \ldots, \bar{x}_l, \bar{y}_1, \ldots, \bar{y}_l)$$

Then we have

$$X_i = \operatorname{Spec} R_i \simeq \mathbb{C}^2 = T^*\mathbb{C}^1.$$

Consider the symplectic form $\omega_{\widetilde{X}} = \sum_{i=1}^{l} d\bar{x}_i \wedge d\bar{y}_i$ on $\widetilde{X}$. It induces a symplectic form ω_X on X, and we have $\omega_X|_{X_i} = d\bar{f}_i \wedge d\bar{g}_i$.

For $i = 1, \ldots, l$, the fixed point p_i belongs to X_i. For $i = 1, \ldots, l-1$, we have $D_i \simeq \mathbb{P}^1 \subset X_i \cup X_{i+1}$ and there is an isomorphism $X_i \cup X_{i+1} \simeq T^*\mathbb{P}^1$. Note that $\bar{f}_i\bar{g}_{i+1} = 1$ on $X_i \cap X_{i+1}$.

§3. W-algebras

In this section, we recall the definition of W-algebras ($\hbar$-localized DQ-algebras), and construct a W-algebra on X by quantum Hamiltonian reduction. We introduce quantized symplectic coordinates of the W-algebra on X. In the rest of the paper, we consider complex manifolds equipped with the analytic topology. For a manifold M, we denote by $\mathcal{O}_M$ the sheaf of holomorphic functions on M.

§3.1. Definition of W-algebras

Let $\hbar$ be an indeterminate. Given $m \in \mathbb{Z}$, let $\mathscr{W}_{T^*\mathbb{C}^n}(m)$ be the sheaf of formal series $\sum_{k \geq -m} \hbar^k a_k$ ($a_k \in \mathcal{O}_{T^*\mathbb{C}^n}$) on the cotangent bundle $T^*\mathbb{C}^n$ of $\mathbb{C}^n$. We set $\mathscr{W}_{T^*\mathbb{C}^n} = \bigcup_m \mathscr{W}_{T^*\mathbb{C}^n}(m)$. We define a noncommutative $\mathbb{C}((\hbar))$-algebra structure on $\mathscr{W}_{T^*\mathbb{C}^n}$ by

$$f \circ g = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} \hbar^{|\alpha|} \frac{1}{\alpha!} \partial^\alpha_\xi f \cdot \partial^\alpha_x g$$

where, for a multi-power $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, we set $\alpha! = \alpha_1! \cdots \alpha_n!$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Note that $\mathscr{W}_{T^*\mathbb{C}^n}(0)$ is a $\mathbb{C}[[\hbar]]$-subalgebra of $\mathscr{W}_{T^*\mathbb{C}^n}$.

Let X be a complex symplectic manifold with symplectic form ω. A *W-algebra* on X is a sheaf $\mathscr{W}$ of $\mathbb{C}((\hbar))$-algebras such that for any point $x \in X$, there is an open neighbourhood U of x, a symplectic map $\varphi : U \to T^*\mathbb{C}^n$, and a $\mathbb{C}((\hbar))$-algebra isomorphism $\psi : \mathscr{W}|_U \xrightarrow{\sim} \varphi^{-1}\mathscr{W}_{T^*\mathbb{C}^n}$.

The following fundamental properties of any W-algebra $\mathscr{W}$ are listed in [KR].

1. The algebra $\mathscr{W}$ is a coherent and noetherian algebra.
2. $\mathscr{W}$ contains a canonical $\mathbb{C}[[\hbar]]$-subalgebra $\mathscr{W}(0)$ which is locally isomorphic to $\mathscr{W}_{T^*\mathbb{C}^n}(0)$ (via the maps ψ). We set $\mathscr{W}(m) = \hbar^{-m}\mathscr{W}(0)$.
3. We have a canonical $\mathbb{C}$-algebra isomorphism $\mathscr{W}(0)/\mathscr{W}(-1) \xrightarrow{\sim} \mathcal{O}_X$ (coming from the canonical isomorphism via the maps ψ). The corresponding morphism $\sigma_m : \mathscr{W}(m) \to \hbar^{-m}\mathcal{O}_X$ is called the *symbol map*.
4. We have
$$\sigma_0(\hbar^{-1}[f, g]) = \{\sigma_0(f), \sigma_0(g)\}$$
for any $f, g \in \mathscr{W}(0)$. Here $\{\cdot, \cdot\}$ is the Poisson bracket of X induced from the symplectic structure of X.
5. The canonical map $\mathscr{W}(0) \to \varprojlim_{m\to\infty} \mathscr{W}(0)/\mathscr{W}(-m)$ is an isomorphism.
6. A section a of $\mathscr{W}(0)$ is invertible in $\mathscr{W}(0)$ if and only if $\sigma_0(a)$ is invertible in $\mathcal{O}_X$.
7. Given a $\mathbb{C}((\hbar))$-algebra automorphism ϕ of $\mathscr{W}$, we can find locally an invertible section a of $\mathscr{W}(0)$ such that $\phi = \mathrm{Ad}(a)$. Moreover a is unique up to a scalar multiple. In other words, we have canonical isomorphisms
$$\begin{array}{ccc} \mathscr{W}(0)^\times/\mathbb{C}[[\hbar]]^\times & \xrightarrow[\mathrm{Ad}]{\sim} & \mathrm{Aut}(\mathscr{W}(0)) \\ \downarrow\wr & & \downarrow\wr \\ \mathscr{W}^\times/\mathbb{C}((\hbar))^\times & \xrightarrow[\mathrm{Ad}]{\sim} & \mathrm{Aut}(\mathscr{W}) \end{array}$$
8. Let v be a $\mathbb{C}((\hbar))$-linear filtration-preserving derivation of $\mathscr{W}$. Then there exists locally a section a of $\mathscr{W}(1)$ such that $v = \mathrm{ad}(a)$. Moreover a is unique up to a

scalar. In other words, we have an isomorphism

$$\mathscr{W}(1)/\hbar^{-1}\mathbb{C}[[\hbar]] \xrightarrow[\mathrm{ad}]{\sim} \mathrm{Der}_{\mathrm{filtered}}(\mathscr{W}).$$

9. If $\mathscr{W}$ is a W-algebra, then its opposite ring $\mathscr{W}^{\mathrm{opp}}$ is a W-algebra on X^{opp} where X^{opp} is the symplectic manifold with symplectic form $-\omega$.

A tuple $(f_1, \dots, f_n; g_1, \dots, g_n)$ of elements $f_i, g_i \in \mathscr{W}(0)$ is called *quantized symplectic coordinates* of $\mathscr{W}$ if $[f_i, f_j] = [g_i, g_j] = 0$ and $[g_i, f_j] = \hbar\delta_{ij}$.

For a $\mathscr{W}$-module $\mathscr{M}$, a $\mathscr{W}(0)$-*lattice of* $\mathscr{M}$ is a coherent $\mathscr{W}(0)$-submodule $\mathscr{M}(0)$ such that the canonical homomorphism $\mathscr{W} \otimes_{\mathscr{W}(0)} \mathscr{M}(0) \to \mathscr{M}$ is an isomorphism. We say that a $\mathscr{W}$-module $\mathscr{M}$ is *good* if, for any relatively compact open subset U of X, there is a coherent $\mathscr{W}(0)|_U$-lattice of $\mathscr{M}|_U$. We denote the category of $\mathscr{W}$-modules by $\mathrm{Mod}(\mathscr{W})$ and the full subcategory of good $\mathscr{W}$-modules by $\mathrm{Mod}^{\mathrm{good}}(\mathscr{W})$. Then $\mathrm{Mod}^{\mathrm{good}}(\mathscr{W})$ is an abelian subcategory of $\mathrm{Mod}(\mathscr{W})$.

Remark 3.1. For a $\mathscr{W}$-module $\mathscr{M}$ with a $\mathscr{W}(0)$-lattice $\mathscr{M}(0)$, the above property 6 implies that the support of $\mathscr{M}$ coincides with the support of the associated $\mathscr{O}_X$-module $\mathscr{M}(0)/\mathscr{M}(-1)$. We denote it by $\mathrm{Supp}\,\mathscr{M}$.

Next, we review the notion of F-actions.

Let X be a symplectic manifold with an action of $\mathbb{G}_{\mathrm{m}}$: $\mathbb{C}^* \ni t \mapsto T_t \in \mathrm{Aut}(X)$. We assume there exists a positive integer $m \in \mathbb{Z}_{>0}$ such that $T_t^*\omega = t^m\omega$ for all $t \in \mathbb{C}^*$.

An *F-action* with exponent m on $\mathscr{W}$ is an action of $\mathbb{G}_{\mathrm{m}}$ on the $\mathbb{C}$-algebra $\mathscr{W}$, $\mathscr{F}_t : T_t^{-1}\mathscr{W} \xrightarrow{\sim} \mathscr{W}$ for $t \in \mathbb{C}^*$, such that $\mathscr{F}_t(\hbar) = t^m\hbar$ and $\mathscr{F}_t(f)$ depends holomorphically on t for any $f \in \mathscr{W}$.

An F-action with exponent m on $\mathscr{W}$ extends to an F-action with exponent 1 on $\mathscr{W}[\hbar^{1/m}] = \mathbb{C}((\hbar^{1/m})) \otimes_{\mathbb{C}((\hbar))} \mathscr{W}$ given by $\mathscr{F}_t(\hbar^{1/m}) = t^1\hbar^{1/m}$.

Definition 3.2. A $\mathscr{W}[\hbar^{1/m}]$-*module with F-action* is a $\mathbb{G}_{\mathrm{m}}$-equivariant $\mathscr{W}[\hbar^{1/m}]$-module, i.e. there exist isomorphisms $\mathscr{F}_t : T_t^{-1}\mathscr{M} \xrightarrow{\sim} \mathscr{M}$ for $t \in \mathbb{C}^*$, and we assume that

1. $\mathscr{F}_t(u)$ depends holomorphically on t for any $u \in \mathscr{M}$;
2. $\mathscr{F}_t(fu) = \mathscr{F}_t(f)\mathscr{F}_t(u)$ for $f \in \mathscr{W}[\hbar^{1/m}]$ and $u \in \mathscr{M}$; and
3. $\mathscr{F}_t \circ \mathscr{F}_{t'} = \mathscr{F}_{tt'}$ for $t, t' \in \mathbb{C}^*$.

We denote by $\mathrm{Mod}_F(\mathscr{W}[\hbar^{1/m}])$ the category of $\mathscr{W}[\hbar^{1/m}]$-modules with F-action, and by $\mathrm{Mod}_F^{\mathrm{good}}(\mathscr{W}[\hbar^{1/m}])$ its full subcategory of good $\mathscr{W}[\hbar^{1/m}]$-modules with F-action. These are $\mathbb{C}$-linear abelian categories.

§3.2. Holonomic $\mathscr{W}$-modules

In this section, we review the notion of holonomic $\mathscr{W}$-modules introduced in [KS2]. The following proposition is due to [KS1].

Proposition 3.3 ([KS1, Prop. 2.3.17]). *For a good $\mathscr{W}$-module $\mathscr{M}$, $\operatorname{Supp}\mathscr{M}$ is involutive with respect to the Poisson bracket of X. In particular, $\dim\operatorname{Supp}\mathscr{M} \geq \dim X/2$.*

Definition 3.4 ([KS2]). We call a good $\mathscr{W}$-module $\mathscr{M}$ *holonomic* if $\operatorname{Supp}\mathscr{M}$ is a Lagrangian subvariety of X, i.e., $\dim\operatorname{Supp}\mathscr{M} = \dim X/2$.

Proposition 3.5. *The category $\mathrm{Mod}^{\mathrm{hol}}(\mathscr{W})$ of all holonomic $\mathscr{W}$-modules is an abelian subcategory of $\mathrm{Mod}^{\mathrm{good}}(\mathscr{W})$.*

The following lemma is obvious.

Lemma 3.6. *Let $\mathscr{M}$ be a good $\mathscr{W}$-module such that $\operatorname{Supp}\mathscr{M}$ is the disjoint union of subsets Z_1 and Z_2. Then there exist submodules $\mathscr{N}_1$, $\mathscr{N}_2$ of $\mathscr{M}$ with support Z_1, Z_2, respectively, and such that $\mathscr{M} = \mathscr{N}_1 \oplus \mathscr{N}_2$.*

Proof. Define

$$\mathscr{N}_i = \{m \in \mathscr{M} \mid \operatorname{Supp} m \subset Z_i\}$$

for $i = 1, 2$. Then the claim of the lemma follows immediately. □

In the present paper, we consider the case $\dim X = 2$.

Let $\bar{x}$, $\bar{\xi} \in \mathscr{O}_{T^*\mathbb{C}^1}$ be coordinate functions on $T^*\mathbb{C}^1$ defined by $\bar{x}((a,b)) = a$, $\bar{\xi}((a,b)) = b$ for $(a,b) \in T^*\mathbb{C}^1$. Let x, $\xi \in \mathscr{W}_{T^*\mathbb{C}^1}(0)$ be the standard quantized symplectic coordinates, that is, $[\xi, x] = \hbar$ and $\sigma_0(x) = \bar{x}$, $\sigma_0(\xi) = \bar{\xi}$.

For $\lambda \in \mathbb{C}$, let $\mathscr{M}_\lambda$ be the $\mathscr{W}_{T^*\mathbb{C}^1}$-module defined by

$$\mathscr{M}_\lambda = \mathscr{W}_{T^*\mathbb{C}^1}/\mathscr{W}_{T^*\mathbb{C}^1}(x\xi - \hbar\lambda).$$

Then $\mathscr{M}_\lambda$ is a holonomic $\mathscr{W}_{T^*\mathbb{C}^1}$-module supported on $\{\bar{x}\bar{\xi} = 0\} \subset T^*\mathbb{C}^1$. Let v_λ be the image of the constant section $1 \in \mathscr{W}_{T^*\mathbb{C}^1}$ in $\mathscr{M}_\lambda$.

Lemma 3.7. *For $m \in \mathbb{Z}$, we have the following isomorphism of $\mathscr{W}_{T^*\mathbb{C}^1}|_{\{\bar{x}\neq 0\}}$-modules:*

$$\mathscr{M}_\lambda|_{\{\bar{x}\neq 0\}} \to \mathscr{M}_{\lambda+m}|_{\{\bar{x}\neq 0\}}, \qquad v_\lambda \mapsto x^{-m} v_{\lambda+m}.$$

Obviously, the inverse homomorphism is given by $v_{\lambda+m} \mapsto x^m v_\lambda$.

A similar proposition holds globally on $T^*\mathbb{C}^1$. It is an analogue of a well-known fact about regular holonomic $\mathcal{D}_{\mathbb{C}^1}$-modules.

Proposition 3.8. *For any $\lambda \neq -1$, we have an isomorphism of $\mathscr{W}_{T^*\mathbb{C}^1}$-modules $\mathscr{M}_\lambda \simeq \mathscr{M}_{\lambda+1}$.*

Proof. Define homomorphisms of $\mathscr{W}_{T^*\mathbb{C}^1}$-modules

$$\phi : \mathscr{M}_\lambda \to \mathscr{M}_{\lambda+1}, \qquad v_\lambda \mapsto \hbar^{-1}\xi v_{\lambda+1},$$
$$\psi : \mathscr{M}_{\lambda+1} \to \mathscr{M}_\lambda, \qquad v_{\lambda+1} \mapsto \frac{1}{\lambda+1} x v_\lambda.$$

These homomorphisms are mutually inverse:

$$\phi \circ \psi(v_{\lambda+1}) = \phi\left(\frac{1}{\lambda+1} x v_\lambda\right) = \frac{\hbar^{-1}}{\lambda+1}(x \circ \xi) v_{\lambda+1} = v_{\lambda+1},$$
$$\psi \circ \phi(v_\lambda) = \psi(\hbar^{-1}\xi v_\lambda) = \frac{\hbar^{-1}}{\lambda+1}(\xi \circ x) v_\lambda = v_\lambda.$$

Therefore $\mathscr{M}_\lambda$ and $\mathscr{M}_{\lambda+1}$ are isomorphic. □

The following proposition is essential for the microlocal analysis of holonomic $\mathscr{W}_{T^*\mathbb{C}^1}$-modules. This is an analogue of a consequence of the classification theorem of simple holonomic systems (cf. [Ka, Proposition 8.36]).

Proposition 3.9. *Set $Z_1 = \{x = 0\}$ and $Z_2 = \{\xi = 0\}$. Note that the module $\mathscr{M}_\lambda$ is supported on the Lagrangian subvariety $Z_1 \cup Z_2$. Then:*

1. *For $\lambda \notin \mathbb{Z}$, $\mathscr{M}_\lambda$ is an irreducible $\mathscr{W}_{T^*\mathbb{C}^1}$-module.*
2. *For $\lambda \in \mathbb{Z}_{\geq 0}$, there exists a $\mathscr{W}_{T^*\mathbb{C}^1}$-submodule $\mathscr{N}$ of $\mathscr{M}_\lambda$ supported on Z_1 and such that* $\operatorname{Supp} \mathscr{M}_\lambda/\mathscr{N} = Z_2$ *on a neighbourhood of $\{x = \xi = 0\}$.*
3. *For $\lambda \in \mathbb{Z}_{<0}$, there exists a $\mathscr{W}_{T^*\mathbb{C}^1}$-submodule $\mathscr{N}$ of $\mathscr{M}_\lambda$ supported on Z_2 and such that* $\operatorname{Supp} \mathscr{M}_\lambda/\mathscr{N} = Z_1$ *on a neighbourhood of $\{x = \xi = 0\}$.*

Proof. The proof is similar to that of [Ka, Proposition 8.36]. □

§3.3. W-algebras $\widetilde{\mathscr{A}_c}$ on the quiver variety X

In this section, we define W-algebras on the quiver variety $X = X_\theta$ depending on a parameter $c = (c_0, \ldots, c_{l-1}) \in \mathbb{C}^l$ such that $c_0 + \cdots + c_{l-1} = 0$. To ensure the smoothness of X_θ, we assume that the stability parameter $\theta = (\theta_0, \ldots, \theta_{l-1})$ satisfies the condition of Proposition 2.2.

We denote the restriction of the canonical W-algebra $\mathscr{W}_{T^*\mathbb{C}^l}$ to $\widetilde{X} \subset T^*\mathbb{C}^l$ by $\mathscr{W}_{\widetilde{X}}$. Let $(x_1, \ldots, x_l; y_1, \ldots, y_l)$ $(x_i, y_i \in \mathscr{W}_{T^*\mathbb{C}^l})$ be the standard quantized symplectic coordinates: $[x_i, x_j] = [y_i, y_j] = 0$ and $[y_i, x_j] = \delta_{ij}\hbar$ for all i, j. The action of the reductive group G on $\widetilde{X}$ induces an action on $\mathscr{W}_{\widetilde{X}}$. We define the following homomorphism $\mu_{\widetilde{X}}$ of Lie algebras:

$$\mu_{\widetilde{X}} : \mathfrak{g} \to \mathscr{W}_{\widetilde{X}}(1), \qquad A_i \mapsto \hbar^{-1}(x_{i+1}y_{i+1} - x_i y_i).$$

We call $\mu_{\widetilde{X}}$ the *quantum moment map* with respect to the action of G. Fix a parameter $c = (c_0, \ldots, c_{l-1}) \in \mathbb{C}^l$ such that $c_0 + \cdots + c_{l-1} = 0$. We define a $\mathscr{W}_{\widetilde{X}}$-module $\mathscr{L}_c$ by

$$\mathscr{L}_c = \mathscr{W}_{\widetilde{X}} / \sum_{i=0}^{l-1} \mathscr{W}_{\widetilde{X}}(\mu_{\widetilde{X}}(A_i) + c_i) = \mathscr{W}_{\widetilde{X}} / \sum_{i=0}^{l-1} \mathscr{W}_{\widetilde{X}}(x_{i+1}y_{i+1} - x_i y_i + \hbar c_i).$$

The $\mathscr{W}_{\widetilde{X}}$-module $\mathscr{L}_c$ is a good $\mathscr{W}_{\widetilde{X}}$-module with a $\mathscr{W}_{\widetilde{X}}(0)$-lattice

$$\mathscr{L}_c(0) := \mathscr{W}_{\widetilde{X}}(0) / \sum_{i=0}^{l-1} \mathscr{W}_{\widetilde{X}}(0)(x_{i+1}y_{i+1} - x_i y_i + \hbar c_i).$$

Define a sheaf of algebras on X,

$$\mathscr{A}_c = (p_* \mathscr{E}nd_{\mathscr{W}_{\widetilde{X}}}(\mathscr{L}_c)^G)^{\mathrm{opp}}$$

where $p : \mu^{-1}(0) \to X$ is the projection. By [KR], $\mathscr{A}_c$ is a W-algebra on X. Set

$$\mathscr{A}_c(0) = \left(p_* \mathscr{E}nd_{\mathscr{W}_{\widetilde{X}}(0)}(\mathscr{L}_c(0))^G\right)^{\mathrm{opp}}.$$

Then $\mathscr{A}_c(0)$ is a canonical $\mathbb{C}[[\hbar]]$-subalgebra of $\mathscr{A}_c$.

Define an F-action on $\mathscr{W}_{\widetilde{X}}$ by $\mathscr{F}_t(x_i) = tx_i$, $\mathscr{F}_t(y_i) = ty_i$, and $\mathscr{F}_t(\hbar) = t^2\hbar$ for $t \in \mathbb{C}^*$. The corresponding $\mathbb{G}_\mathrm{m}$-action on $\widetilde{X}$ is given by $\mathbb{G}_\mathrm{m} \ni t \mapsto T_t \in \mathrm{Aut}(\widetilde{X})$, $T_t((a_i, b_i)_{i=1,\ldots,l}) = (ta_i, tb_i)_{i=1,\ldots,l}$. This action induces a $\mathbb{G}_\mathrm{m}$-action on the quiver variety X. Under the embedding $\mathbb{G}_\mathrm{m} \subset \mathbb{T}^2$, $t \mapsto (t, t)$, this action coincides with the action given by (1).

The F-action on $\mathscr{W}_{\widetilde{X}}$ induces an F-action with exponent 2 on $\mathscr{A}_c$. Then we set $\widetilde{\mathscr{A}_c} = \mathscr{A}_c[\hbar^{1/2}]$ and $\widetilde{\mathscr{A}_c}(0) = \mathscr{A}_c(0)[\hbar^{1/2}]$.

Set $A_c = (\mathrm{End}_{\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}))^{\mathrm{opp}}$. From [BK] or [BLPW], we have the following W-affinity of the algebra $\widetilde{\mathscr{A}_c}$:

Theorem 3.10 ([BK, Theorem 6.3], [BLPW, Theorem 6.1]). *Assume that we have $c_i + c_{i+1} + \cdots + c_{j-1} \neq 0$ for all $0 < i < j \leq l$, and that*

$$c_i + c_{i+1} + \cdots + c_{j-1} \in \mathbb{Z}_{\geq 0} \quad \textit{implies} \quad \theta_i + \theta_{i+1} + \cdots + \theta_{j-1} < 0,$$

i.e. $i \rhd j$. Then we have the following equivalence of categories:

$$\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{A}_c}) \simeq A_c\text{-mod}, \qquad \mathscr{M} \mapsto \mathrm{Hom}_{\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}, \mathscr{M}).$$

Its quasi-inverse functor is given by $M \mapsto \widetilde{\mathscr{A}_c} \otimes_{A_c} M$.

Remark 3.11. As we will see in Section 4, the algebra A_c is isomorphic to the spherical subalgebra of the rational Cherednik algebra of type $\mathbb{Z}/l\mathbb{Z}$.

In the rest of this paper, we assume the assumptions of Theorem 3.10.

Let $\mathbf{1}_c$ be the image of the constant section $1 \in \mathscr{W}_{\widetilde{X}}$ in $\mathscr{L}_c$. For a G-invariant section $f \in \mathscr{W}_{\widetilde{X}}$, a G-invariant endomorphism of $\mathscr{L}_c$ is uniquely defined by the right multiplication $g\mathbf{1}_c \mapsto gf\mathbf{1}_c \in \mathscr{E}nd_{\mathscr{W}_{\widetilde{X}}}(\mathscr{L}_c)$ for $g \in \mathscr{W}_{\widetilde{X}}$. By abuse of notation, we denote the image of the above G-invariant endomorphism in $\widetilde{\mathscr{A}_c}$ by the same symbol f.

Consider the global sections $x_1 \cdots x_l$, $y_1 \cdots y_l$, $x_i y_i$ of $\widetilde{\mathscr{A}_c}$ for $i = 1, \ldots, l$. Although they are not F-invariant, the global sections $\hbar^{-l/2} x_1 \cdots x_l$, $\hbar^{-l/2} y_1 \cdots y_l$, $\hbar^{-1} x_i y_i$ are elements of $A_c = (\mathrm{End}_{\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}))^{\mathrm{opp}}$. In $\widetilde{\mathscr{A}_c}$, we have the relations

$$x_{i+1} y_{i+1} - x_i y_i + \hbar c_i = 0 \quad \text{for } i = 1, \ldots, l.$$

Next, we consider the local structure of the W-algebra $\widetilde{\mathscr{A}_c}$ on the affine open subset X_i for $i = 1, \ldots, l$. Set $\widetilde{\mathscr{A}_{c,i}} = \widetilde{\mathscr{A}_c}|_{X_i}$. Recall the arrangement of the indices $\eta_1, \ldots, \eta_l$ in (2). We define local sections of $\widetilde{\mathscr{A}_c}$ on X_i by

$$f_i = (x_{\eta_1} \cdots x_{\eta_i}) \circ (y_{\eta_{i+1}} \cdots y_{\eta_l})^{-1}, \qquad g_i = (y_{\eta_i} \cdots y_{\eta_l}) \circ (x_{\eta_1} \cdots x_{\eta_{i-1}})^{-1}.$$

Note that x_{η_j} $(1 \le j \le i-1)$ and y_{η_k} $(i+1 \le k \le l)$ are invertible on $p^{-1}(X_i)$, and f_i and g_i are well-defined (see property 6 in Section 3.1). We have $f_i \circ g_i = x_{\eta_i} y_{\eta_i}$ and $g_i \circ f_i = x_{\eta_i} y_{\eta_i} + \hbar$. That is, $(f_i; g_i)$ are quantized symplectic coordinates of $\widetilde{\mathscr{A}_{c,i}}$ for $i = 1, \ldots, l$. Thus, as explained in [KR, 2.2.3], $\widetilde{\mathscr{A}_{c,i}}$ is isomorphic to $\mathscr{W}_{T^*\mathbb{C}^1}$ via $x \mapsto f_i$, $\xi \mapsto g_i$.

We have

$$y_1 \cdots y_l = g_i \circ (x_{\eta_1} y_{\eta_1}) \circ \cdots \circ (x_{\eta_{i-1}} y_{\eta_{i-1}}) \quad \text{on } X_i. \tag{3}$$

Moreover $g_{i+1} \circ f_i = f_i \circ g_{i+1} = 1$ in $\widetilde{\mathscr{A}_c}|_{X_i \cap X_{i+1}}$. Sometimes, we denote the section $g_{i+1}|_{X_i \cap X_{i+1}}$ by f_i^{-1}.

For $i = 1, \ldots, l-1$, we set

$$\tilde{c}_i = c_{\eta_i} + c_{\eta_i + 1} + \cdots + c_{\eta_{i+1} - 1}. \tag{4}$$

Under the assumption of Theorem 3.10, we have $\tilde{c}_i + \tilde{c}_{i+1} + \cdots + \tilde{c}_{j-1} \notin \mathbb{Z}_{\le 0}$ for $1 \le i < j \le l$. Then

$$x_{\eta_{i+1}} y_{\eta_{i+1}} - x_{\eta_i} y_{\eta_i} + \hbar \tilde{c}_i = 0. \tag{5}$$

§4. Rational Cherednik algebras and categories $\mathcal{O}$

In this section, we review the definition of, and fundamental facts about, rational Cherednik algebras of type $\mathbb{Z}/l\mathbb{Z}$ and their categories $\mathcal{O}$.

Let $\mathbb{Z}/l\mathbb{Z} = \langle \gamma \rangle$ be a cyclic group with an action on $\mathbb{C}$ given by $\gamma \mapsto \zeta = \exp(2\pi\sqrt{-1}/l)$. Let $D(\mathbb{C}^*)$ be the algebra of algebraic differential operators on $\mathbb{C}^*$. Let z be the standard coordinate function on $\mathbb{C}$. Then we have $\mathbb{C} = \mathrm{Spec}[z]$ and $\mathbb{C}^* = \mathrm{Spec}\,\mathbb{C}[z, z^{-1}]$. The algebra $D(\mathbb{C}^*)$ is generated by $z^{\pm 1}$ and d/dz.

The action of $\mathbb{Z}/l\mathbb{Z}$ on $\mathbb{C}$ induces an action of $D(\mathbb{C}^*)$ given by $\gamma(z) = \zeta^{-1}z$, $\gamma(d/dz) = \zeta d/dz$. We denote by $D(\mathbb{C}^*) \# \mathbb{Z}/l\mathbb{Z}$ the smash product of $D(\mathbb{C}^*)$ and $\mathbb{Z}/l\mathbb{Z}$.

For a parameter $\kappa = (\kappa_1, \dots, \kappa_{l-1}) \in \mathbb{C}^{l-1}$, we define the *Dunkl operator* ∂_κ by

$$\partial_\kappa = \frac{d}{dz} + \frac{l}{z}\sum_{i=0}^{l-1} \kappa_i \mathbf{e}_i$$

where we regard $\kappa_0 = 0$ and let $\mathbf{e}_i = (1/l)\sum_{j=0}^{l-1} \zeta^{ij}\gamma^j$ be an idempotent of $\mathbb{C}(\mathbb{Z}/l\mathbb{Z})$ for $i = 0, 1, \dots, l-1$.

Definition 4.1 ([EG]). 1. The *rational Cherednik algebra* $H_\kappa = H_\kappa(\mathbb{Z}/l\mathbb{Z})$ is the subalgebra of $D(\mathbb{C}^*) \# \mathbb{Z}/l\mathbb{Z}$ generated by z, ∂_κ and γ.
2. The *spherical subalgebra* of H_κ is the algebra $\mathbf{e}_0 H_\kappa \mathbf{e}_0$.

The following proposition is an analogue of the triangular decomposition of semisimple Lie algebras.

Proposition 4.2 ([EG]). *We have the following isomorphisms of* $\mathbb{C}$*-linear spaces:*

$$H_\kappa \simeq \mathbb{C}[z] \otimes_{\mathbb{C}} \mathbb{C}(\mathbb{Z}/l\mathbb{Z}) \otimes_{\mathbb{C}} \mathbb{C}[\partial_\kappa] \quad \textit{and} \quad \mathbf{e}_0 H_\kappa \mathbf{e}_0 \simeq \mathbb{C}[z, \partial_\kappa]^{\mathbb{Z}/l\mathbb{Z}} = \mathbb{C}[z^l, z\partial_\kappa, \partial_\kappa^l].$$

The following isomorphism is essentially established by Holland in [Ho, Corollary 4.7]:

$$\begin{aligned} \mathbf{e}_0 H_\kappa \mathbf{e}_0 \to A_c, \quad \mathbf{e}_0 z^l \mathbf{e}_0 &\mapsto \hbar^{-l/2} x_1 \cdots x_l, \\ \mathbf{e}_0 \partial_\kappa^l \mathbf{e}_0 &\mapsto \hbar^{-l/2} y_1 \cdots y_l, \\ \mathbf{e}_0 z\partial_\kappa \mathbf{e}_0 &\mapsto \hbar^{-1} x_1 y_1, \end{aligned} \tag{6}$$

where

$$c = c(\kappa) = (c_i)_{i=0,1,\dots,l-1}, \qquad c_i = \kappa_i - \kappa_{i+1} - 1/l + \delta_{i,0}. \tag{7}$$

Remark 4.3. We consider the algebra A_c defined in Section 3.3 instead of the algebra $\mathfrak{A}^\lambda$ studied in [Ho]. As shown in [BK, Proposition 3.5], these two algebras are isomorphic.

Remark 4.4. In [Ho] and [Ku], the parameters $\lambda_i = -c_i$ are used for the quantum Hamiltonian reduction A_c.

Lemma 4.5 (cf. [Ku, Proposition 4.4]). *Assume* $c_i + c_{i+1} + \cdots + c_{j-1} \neq 0$ *for* $0 < i < j \leq l$. *Then the rational Cherednik algebra* H_κ *is Morita equivalent to its spherical subalgebra* $\mathbf{e}_0 H_\kappa \mathbf{e}_0 \simeq A_c$*, i.e. we have an equivalence of categories*

$$H_\kappa\text{-mod} \to (\mathbf{e}_0 H_\kappa \mathbf{e}_0)\text{-mod}, \qquad M \mapsto \mathbf{e}_0 M.$$

In the present paper, we assume the assumption of Lemma 4.5 holds.

The category $\mathcal{O}(H_\kappa)$ is the subcategory of H_κ-mod such that the Dunkl operator ∂_κ acts locally nilpotently on each module $M \in \mathcal{O}(H_\kappa)$.

Consider an irreducible $\mathbb{C}(\mathbb{Z}/l\mathbb{Z})$-module $\mathbb{C}\mathbf{e}_i$ for $i = 0, \dots, l$. We regard $\mathbb{C}\mathbf{e}_i$ as a $\mathbb{C}[\partial_\kappa] \# \mathbb{Z}/l\mathbb{Z}$-module by $\partial_\kappa \mathbf{e}_i = 0$. We define an H_κ-module by

$$\mathbf{\Delta}_\kappa(i) = H_\kappa \otimes_{\mathbb{C}[\partial_\kappa]\#\mathbb{Z}/l\mathbb{Z}} \mathbb{C}\mathbf{e}_i$$

called a *standard module*. By Proposition 4.2, we have

$$\mathbf{\Delta}_\kappa(i) = \mathbb{C}[z]\mathbf{e}_i \tag{8}$$

as a $\mathbb{C}$-linear space.

By the equivalence of Lemma 4.5, we have a subcategory $\mathcal{O}(A_c)$ of A_c-mod which is equivalent to the category $\mathcal{O}(H_\kappa)$. We call $\mathcal{O}(A_c)$ the *category $\mathcal{O}$ of A_c*. For $i = 1, \dots, l$, we define an A_c-module by

$$\Delta_c(i) = \mathbf{e}_0 \mathbf{\Delta}_\kappa(i)$$

where c is given by (7) and we regard $\mathbf{e}_l = \mathbf{e}_0$. The module $\Delta_c(i)$ is the *standard module* for $\mathcal{O}(A_c)$.

The following proposition is a list of fundamental and well-known facts about the category $\mathcal{O}(A_c)$ and the modules $\Delta_c(i)$ $(i = 1, \dots, l)$.

Proposition 4.6 ([GGOR]). *We have the following fundamental facts about the standard modules $\Delta_c(i)$:*

1. *For $i = 1, \dots, l$, the standard module $\Delta_c(i)$ has a unique irreducible quotient $L_c(i)$.*
2. *The irreducible modules $L_c(i)$ $(i = 1, \dots, l)$ are mutually nonisomorphic.*
3. *Any simple object in the category $\mathcal{O}(A_c)$ is isomorphic to $L_c(i)$ for some $i = 1, \dots, l$.*

Remark 4.7. Originally [GGOR] considered the category $\mathcal{O}(H_\kappa)$ of the rational Cherednik algebra H_κ, and not of its spherical subalgebra $\mathbf{e}_0 H_\kappa \mathbf{e}_0 \simeq A_c$.

By (8), we have

$$\Delta_c(i) = \mathbf{e}_0 \mathbf{\Delta}_\kappa(i) = \mathbf{e}_0 \mathbb{C}[z^l] z^{l-i} \mathbf{e}_i = \mathbb{C}[\hbar^{-l/2} x_1 \cdots x_l] e_i \tag{9}$$

as a $\mathbb{C}$-linear space where we denote $\mathbf{e}_0 z^{l-i} \mathbf{e}_i$ by e_i.

In A_c, we have

$$[\hbar^{-1} x_{\eta_1} y_{\eta_1}, \hbar^{-l/2} x_1 \cdots x_l] = \hbar^{-l/2} x_1 \cdots x_l,$$
$$[\hbar^{-1} x_{\eta_1} y_{\eta_1}, \hbar^{-l/2} y_1 \cdots y_l] = -\hbar^{-l/2} y_1 \cdots y_l.$$

For $i = 1, \dots, l$, the operator $\hbar^{-1} x_{\eta_1} y_{\eta_1}$ acts semisimply on the standard module $\Delta_c(\eta_i)$, i.e. $\Delta_c(\eta_i)$ is a direct sum of eigenspaces with respect to the action of

$\hbar^{-1}x_{\eta_1}y_{\eta_1}$. In fact, by direct calculation we have

$$\Delta_c(\eta_i) = \bigoplus_{m\in\mathbb{Z}_{\geq 0}} (\hbar^{-l/2}x_1\cdots x_l)^m e_{\eta_i},$$

and

$$(\hbar^{-1}x_{\eta_1}y_{\eta_1})\circ(\hbar^{-l/2}x_1\cdots x_l)^m e_{\eta_i} = (m+\tilde{c}_{\eta_1}+\tilde{c}_{\eta_2}+\cdots+\tilde{c}_{\eta_{i-1}})(\hbar^{-l/2}x_1\cdots x_l)^m e_{\eta_i}, \tag{10}$$

where $\tilde{c}_{\eta_j}$ is the parameter defined at (4).

Lemma 4.8. *We have*

$$\Delta_c(\eta_i) = A_c/(A_c(\hbar^{-1}x_{\eta_i}y_{\eta_i}) + A_c(\hbar^{-l/2}y_1\cdots y_l))$$

for $i=1,\dots,l$.

Proof. The standard module $\Delta_c(\eta_i)$ is cyclic with cyclic vector e_{η_i}. By (5) and (10), we have $\hbar^{-1}x_{\eta_i}y_{\eta_i}e_{\eta_i}=0$. Thus, we have the surjective homomorphism of A_c-modules

$$A_c/(A_c(\hbar^{-1}x_{\eta_i}y_{\eta_i}) + A_c(\hbar^{-l/2}y_1\cdots y_l)) \twoheadrightarrow \Delta_c(\eta_i), \qquad f\mapsto fe_{\eta_i}.$$

By Proposition 4.2 and (9), this homomorphism is an isomorphism. □

§5. Microlocal construction of modules

§5.1. Construction of the standard modules

In this section, we introduce $\widetilde{\mathscr{A}_c}$-modules $\mathcal{M}_c^{\Delta}(\eta_i)$ supported on Lagrangian subvarieties $D_i\cup D_{i+1}\cup\cdots\cup D_l$ of X. Moreover, we show that $\mathcal{M}_c^{\Delta}(\eta_i)$ is a counterpart of the standard module $\Delta_c(\eta_i)$ of A_c through the equivalence of Theorem 3.10.

Definition 5.1. For $1\leq i<i'\leq l$, a parameter $\lambda=(\lambda_j)_{j=i+1,\dots,i'}\in\mathbb{C}^{i'-i}$ is called *admissible* when $\lambda_j-\lambda_{j+1}-\tilde{c}_j\in\mathbb{Z}$ for $j=i,\dots,i'-1$, where we regard $\lambda_i=0$.

Definition 5.2. For $i=1,\dots,l$, fix an admissible parameter $\lambda=(\lambda_j)_{j=i+1,\dots,l}$. We define an $\widetilde{\mathscr{A}_c}$-module $\mathcal{M}_{c,\lambda}(\eta_i)$ by gluing local sheaves as follows:

$$\begin{aligned}
\mathcal{M}_{c,\lambda}(\eta_i)|_{X_i} &= \widetilde{\mathscr{A}_{c,i}}/\widetilde{\mathscr{A}_{c,i}}g_i,\\
\mathcal{M}_{c,\lambda}(\eta_i)|_{X_j} &= \widetilde{\mathscr{A}_{c,j}}/\widetilde{\mathscr{A}_{c,j}}(f_j\circ g_j-\hbar\lambda_j)\\
&= \widetilde{\mathscr{A}_{c,j}}/\widetilde{\mathscr{A}_{c,j}}(x_{\eta_j}y_{\eta_j}-\hbar\lambda_j) \quad (\text{for } j=i+1,\dots,l),\\
\mathcal{M}_{c,\lambda}(\eta_i)|_{X_j} &= 0 \quad (\text{for } j=1,\dots,i-1).
\end{aligned}$$

The gluing is given by

$$u_j = f_j^{\lambda_j - \lambda_{j+1} - \tilde{c}_j} u_{j+1} \quad \text{on } X_j \cap X_{j+1} \tag{11}$$

where u_j is the image of the constant section $1 \in \mathscr{A}_{c,j}$ in $\mathcal{M}_{c,\lambda}(\eta_i)|_{X_j}$ for $j = i, \dots, l$.

Note that we have

$$\mathcal{M}_{c,\lambda}(\eta_i)|_{X_j} \simeq \mathscr{M}_{\lambda_j}, \qquad u_j \mapsto v_{\lambda_j}, \tag{12}$$

under the isomorphism $\widetilde{\mathscr{A}_{c,j}} \simeq \mathscr{W}_{T^*\mathbb{C}^1}$.

Lemma 5.3. *The module $\mathcal{M}_{c,\lambda}(\eta_i)$ is a well-defined good $\widetilde{\mathscr{A}_c}$-module supported on the Lagrangian subvariety $D_i \cup D_{i+1} \cup \dots \cup D_l$.*

Proof. For $j = i+1, \dots, l-1$, set $\mathscr{N}_1 = \mathcal{M}_{c,\lambda}(\eta_i)|_{X_j}$ and $\mathscr{N}_2 = \mathcal{M}_{c,\lambda}(\eta_i)|_{X_{j+1}}$. By (5), we have

$$f_{j+1} \circ g_{j+1} - f_j \circ g_j + \hbar \tilde{c}_j = 0$$

on $X_j \cap X_{j+1}$. Thus,

$$\begin{aligned} \mathscr{N}_2|_{X_j \cap X_{j+1}} &= \widetilde{\mathscr{A}_c}|_{X_j \cap X_{j+1}} / \widetilde{\mathscr{A}_c}|_{X_j \cap X_{j+1}} (f_{j+1} \circ g_{j+1} - \hbar\lambda_{j+1}), \\ &= \widetilde{\mathscr{A}_c}|_{X_j \cap X_{j+1}} / \widetilde{\mathscr{A}_c}|_{X_j \cap X_{j+1}} (f_j \circ g_j - \hbar(\lambda_{j+1} + \tilde{c}_j)). \end{aligned}$$

Since λ is admissible, by Lemma 3.7, $\mathscr{N}_1|_{X_j \cap X_{j+1}}$ is isomorphic to $\mathscr{N}_2|_{X_j \cap X_{j+1}}$ via the map $u_j \mapsto f_j^{\lambda_j - \lambda_{j+1} - \tilde{c}_j} u_{j+1}$. For $j = i$, we set $\mathscr{N}_1 = \widetilde{\mathscr{A}_{c,i}} / \widetilde{\mathscr{A}_{c,i}}(f_i \circ g_i)$, and $\mathscr{N}_2 = \mathcal{M}_{c,\lambda}(\eta_i)|_{X_{i+1}}$. By the same argument for the case $j = i+1, \dots, l-1$, $\mathscr{N}_1|_{X_i \cap X_{i+1}}$ is isomorphic to $\mathscr{N}_2|_{X_i \cap X_{i+1}}$. By Proposition 3.9, this isomorphism induces an isomorphism between $\widetilde{\mathscr{A}_{c,i}} / \widetilde{\mathscr{A}_{c,i}} g_i|_{X_i \cap X_{i+1}}$ and $\mathscr{N}_2|_{X_i \cap X_{i+1}}$. As a consequence, $\mathcal{M}_{c,\lambda}(\eta_i)$ is well-defined.

Set $\mathscr{N}_1(0) = \widetilde{\mathscr{A}_c}(0)|_{X_j} u_j$ and $\mathscr{N}_2(0) = \widetilde{\mathscr{A}_c}(0)|_{X_{j+1}} u_{j+1}$. Clearly $\mathscr{N}_\alpha(0)$ is an $\widetilde{\mathscr{A}_c}(0)|_{X_j}$-lattice of $\mathscr{N}_\alpha$ for $\alpha = 1, 2$. Moreover, the above isomorphism between $\mathscr{N}_1|_{X_j \cap X_{j+1}}$ and $\mathscr{N}_2|_{X_j \cap X_{j+1}}$ induces an isomorphism of $\widetilde{\mathscr{A}_c}(0)|_{X_j \cap X_{j+1}}$-modules between $\mathscr{N}_1(0)|_{X_j \cap X_{j+1}}$ and $\mathscr{N}_2(0)|_{X_j \cap X_{j+1}}$. Thus we have an $\widetilde{\mathscr{A}_c}(0)$-lattice $\mathcal{M}_{c,\lambda}(0)$ of $\mathcal{M}_{c,\lambda}$, which is defined by $\mathcal{M}_{c,\lambda}(0)|_{X_j} = \widetilde{\mathscr{A}_c}(0)|_{X_j} u_j$. Therefore $\mathcal{M}_{c,\lambda}(\eta_i)$ is a good $\widetilde{\mathscr{A}_c}$-module. □

Lemma 5.4. *Fix $i \in \{1, \dots, l\}$. Take an arbitrary admissible parameter $\lambda = (\lambda_j)_{j=i+1,\dots,l} \in \mathbb{C}^{l-i}$ such that $\lambda \in (\mathbb{C} \setminus \mathbb{Z}_{<0})^{l-i}$ (resp. $\lambda \in (\mathbb{C} \setminus \mathbb{Z}_{\geq 0})^{l-i}$) and $\lambda' \in \lambda + (\mathbb{Z}_{\geq 0})^{l-i}$ (resp. $\lambda' \in \lambda + (\mathbb{Z}_{\leq 0})^{l-i}$). Then we have an isomorphism of $\widetilde{\mathscr{A}_c}$-modules*

$$\mathcal{M}_{c,\lambda}(\eta_i) \simeq \mathcal{M}_{c,\lambda'}(\eta_i).$$

Proof. We will prove the case where $\lambda \in (\mathbb{C} \setminus \mathbb{Z}_{<0})^{l-i}$ and $\lambda' \in \lambda + (\mathbb{Z}_{\geq 0})^{l-i}$. It is enough to show that the claim of the lemma holds when there exists an index $j \in \{i+1, \ldots, l\}$ such that $\lambda'_j + 1 = \lambda_j$ and $\lambda'_k = \lambda_k$ for $k \neq j$.

By (12), we have

$$\mathcal{M}_{c,\lambda}(\eta_i)|_{X_j} \simeq \mathscr{M}_{\lambda_j}, \qquad \mathcal{M}_{c,\lambda'}(\eta_i)|_{X_j} \simeq \mathscr{M}_{\lambda_j+1}.$$

Thus, there exists an isomorphism of $\widetilde{\mathscr{A}_{c,j}}$-modules $\mathcal{M}_{c,\lambda}(\eta_i)|_{X_j} \simeq \mathcal{M}_{c,\lambda'}(\eta_i)|_{X_j}$ by Proposition 3.8. For $k \neq j$, we have a trivial isomorphism of $\widetilde{\mathscr{A}_{c,k}}$-modules $\mathcal{M}_{c,\lambda}(\eta_i)|_{X_k} \simeq \mathcal{M}_{c,\lambda'}(\eta_i)|_{X_k}$. These isomorphisms induce an isomorphism of $\widetilde{\mathscr{A}_c}$-modules $\mathcal{M}_{c,\lambda}(\eta_i) \simeq \mathcal{M}_{c,\lambda'}(\eta_i)$.

The case where $\lambda \in (\mathbb{C} \setminus \mathbb{Z}_{\geq 0})^{l-i}$ and $\lambda' \in \lambda + (\mathbb{Z}_{\leq 0})^{l-i}$ is proved similarly. □

Next, we define $\widetilde{\mathscr{A}_c}$-modules $\mathcal{M}_c^{\Delta}(\eta_i)$, $\mathcal{M}_c^{\nabla}(\eta_i)$.

Definition 5.5. For an admissible parameter $\lambda \in (\mathbb{C} \setminus \mathbb{Z}_{\geq 0})^{l-i}$, we denote

$$\mathcal{M}_c^{\Delta}(\eta_i) = \mathcal{M}_{c,\lambda}(\eta_i).$$

Remark 5.6. For an admissible parameter $\lambda \in (\mathbb{C} \setminus \mathbb{Z}_{<0})^{l-i}$, we denote

$$\mathcal{M}_c^{\nabla}(\eta_i) = \mathcal{M}_{c,\lambda}(\eta_i).$$

The module $\mathcal{M}_c^{\nabla}(\eta_i)$ is an $\widetilde{\mathscr{A}_c}$-module (conjecturally) corresponding to a costandard module of A_c.

In the rest of this section, we show that the $\widetilde{\mathscr{A}_c}$-module $\mathcal{M}_c^{\Delta}(\eta_i)$ corresponds to the standard module $\Delta_c(\eta_i)$ via the equivalence of categories in Theorem 3.10, i.e. we have $\mathrm{Hom}_{\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}, \mathcal{M}_c^{\Delta}(\eta_i)) \simeq \Delta_c(\eta_i)$.

Theorem 5.7. *We have an isomorphism of $\widetilde{\mathscr{A}_c}$-modules $\mathcal{M}_c^{\Delta}(\eta_i) \simeq \widetilde{\mathscr{A}_c} \otimes_{A_c} \Delta_c(\eta_i)$. In other words, we have an isomorphism of A_c-modules*

$$\mathrm{Hom}_{\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}, \mathcal{M}_c^{\Delta}(\eta_i)) \simeq \Delta_c(\eta_i).$$

Proof. By Lemma 4.8, we have

$$\widetilde{\mathscr{A}_c} \otimes_{A_c} \Delta_c(\eta_i) \simeq \widetilde{\mathscr{A}_c}/(\widetilde{\mathscr{A}_c} x_{\eta_i} y_{\eta_i} + \widetilde{\mathscr{A}_c} y_1 \cdots y_l). \tag{13}$$

For $j = 1, \ldots, i$, by (5), on X_j we have

$$y_1 \cdots y_l = g_j \circ x_{\eta_1} y_{\eta_1} \circ \cdots \circ x_{\eta_{j-1}} y_{\eta_{j-1}} = g_j \circ \prod_{k=1}^{j-1} (f_j \circ g_j + \hbar(\tilde{c}_k + \cdots + \tilde{c}_{j-1})).$$

Since, $\hbar(\tilde{c}_k + \cdots + \tilde{c}_{i-1}) \in \mathbb{C}((\hbar))$ is a nonzero scalar for $k = 1, \ldots, i-1$, we have

$$\begin{aligned}
&\widetilde{\mathscr{A}_{c,j}}(f_j \circ g_j - \hbar(\tilde{c}_j + \cdots + \tilde{c}_{i-1})) \\
&\qquad + \widetilde{\mathscr{A}_{c,j}}\; g_j \circ \prod_{k=1}^{j-1} \{(f_j \circ g_j - \hbar(\tilde{c}_j + \cdots + \tilde{c}_{i-1})) - \hbar(\tilde{c}_k + \cdots + \tilde{c}_{j-1})\} \\
&= \widetilde{\mathscr{A}_{c,j}}(f_j \circ g_j - \hbar(\tilde{c}_j + \cdots + \tilde{c}_{i-1})) + \widetilde{\mathscr{A}_{c,j}} g_j \circ \prod_{k=1}^{j-1} \hbar(\tilde{c}_k + \cdots + \tilde{c}_{j-1}) \\
&= \widetilde{\mathscr{A}_{c,j}}(f_j \circ g_j - \hbar(\tilde{c}_j + \cdots + \tilde{c}_{i-1})) + \widetilde{\mathscr{A}_{c,j}} g_j.
\end{aligned}$$

Therefore, on X_j, the isomorphism (13) reduces to

$$\begin{aligned}
\widetilde{\mathscr{A}_{c,j}} \otimes_{A_c} \Delta_c(\eta_i) &\simeq \widetilde{\mathscr{A}_{c,j}} \Big/ \Big\{ \widetilde{\mathscr{A}_{c,j}}(f_j \circ g_j - \hbar(\tilde{c}_j + \cdots + \tilde{c}_{i-1})) \\
&\qquad + \widetilde{\mathscr{A}_{c,j}} g_j \circ \prod_{k=1}^{j-1} (f_j \circ g_j - \hbar(\tilde{c}_k + \cdots + \tilde{c}_{j-1})) \Big\} \\
&= \widetilde{\mathscr{A}_{c,j}} / \{ \widetilde{\mathscr{A}_{c,j}}(f_j \circ g_j - \hbar(\tilde{c}_j + \cdots + \tilde{c}_{i-1})) + \widetilde{\mathscr{A}_{c,j}} g_j \} \\
&= \begin{cases} \widetilde{\mathscr{A}_{c,i}} / \widetilde{\mathscr{A}_{c,i}}\; g_i & \text{for } j = i, \\ 0 & \text{for } j = 1, \ldots, i-1. \end{cases}
\end{aligned}$$

For $j = i+1, \ldots, l$, on X_j, we have

$$y_1 \cdots y_l = g_j \circ (x_{\eta_1} y_{\eta_1}) \circ \cdots \circ (x_{\eta_i} y_{\eta_i}) \circ \cdots \circ (x_{\eta_{j-1}} y_{\eta_{j-1}})$$

by (3). Since $x_{\eta_k} y_{\eta_k}$ and $x_{\eta_i} y_{\eta_i}$ commute with each other, on X_j we have

$$\begin{aligned}
\widetilde{\mathscr{A}_{c,j}} \otimes_{A_c} \Delta_c(\eta_i) &\simeq \widetilde{\mathscr{A}_{c,j}} \Big/ \Big(\widetilde{\mathscr{A}_{c,j}} x_{\eta_i} y_{\eta_i} + \widetilde{\mathscr{A}_{c,j}} g_j \circ \Big(\prod_{k \neq i} x_{\eta_k} y_{\eta_k} \Big) \circ x_{\eta_i} y_{\eta_i} \Big) \\
&= \widetilde{\mathscr{A}_{c,j}} / \widetilde{\mathscr{A}_{c,j}} (x_{\eta_j} y_{\eta_j} + \hbar(\tilde{c}_i + \cdots + \tilde{c}_{j-1})) \quad \text{for } j = i+1, \ldots, l.
\end{aligned}$$

Note that $\lambda = (\lambda_j)_{j=i+1,\ldots,l} \in \mathbb{C}^{l-i}$ where $\lambda_j = -(\tilde{c}_i + \cdots + \tilde{c}_{j-1})$ is admissible and $\lambda \in (\mathbb{C} \setminus \mathbb{Z}_{\geq 0})^{l-i}$. Thus, the $\widetilde{\mathscr{A}_c}$-module $\widetilde{\mathscr{A}_c} \otimes_{A_c} \Delta_c(\eta_i)$ is clearly isomorphic to $\mathcal{M}_c^{\Delta}(\eta_i)$. □

§5.2. Construction of irreducible modules of $\widetilde{\mathscr{A}_c}$

In this subsection, we construct modules $\mathcal{L}_c(i)$ over the W-algebra $\widetilde{\mathscr{A}_c}$ for $i = 1, \ldots, l$, and show that they are irreducible. Under the equivalence of Theorem 3.10, $\mathrm{Hom}_{\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}, \mathcal{L}_c(i))$ is isomorphic to the irreducible module $L_c(i)$ over A_c defined in Section 4.

Fix $i \in \{1, \dots, l\}$. We denote by $\epsilon(i)$ the unique integer in $\{i+1, \dots, l+1\}$ such that $\tilde{c}_i + \tilde{c}_{i+1} + \dots + \tilde{c}_{\epsilon(i)-1} \in \mathbb{Z}$ and $\tilde{c}_i + \tilde{c}_{i+1} + \dots + \tilde{c}_{j-1} \notin \mathbb{Z}$ for any $i < j < \epsilon(i)$.

Definition 5.8. Fix an admissible parameter $\lambda = (\lambda_{i+1}, \dots, \lambda_{\epsilon(i)-1}) \in \mathbb{C}^{\epsilon(i)-i-1}$ where we regard $\lambda_{\epsilon(i)} = -1$. We define an $\widetilde{\mathscr{A}_c}$-module $\mathcal{L}_{c,\lambda}(\eta_i)$ by gluing local sheaves as follows:

$$\begin{aligned}
\mathcal{L}_{c,\lambda}(\eta_i)|_{X_i} &= \widetilde{\mathscr{A}_{c,i}}/\widetilde{\mathscr{A}_{c,i}}g_i, \\
\mathcal{L}_{c,\lambda}(\eta_i)|_{X_j} &= \widetilde{\mathscr{A}_{c,j}}/\widetilde{\mathscr{A}_{c,j}}(f_j \circ g_j - \hbar\lambda_j) \\
&= \widetilde{\mathscr{A}_{c,j}}/\widetilde{\mathscr{A}_{c,j}}(x_{\eta_j} y_{\eta_j} - \hbar\lambda_j) \quad (\text{for } j = i+1, \dots, \epsilon(i)-1), \\
\mathcal{L}_{c,\lambda}(\eta_i)|_{X_{\epsilon(i)}} &= \widetilde{\mathscr{A}_{c,\epsilon(i)}}/\widetilde{\mathscr{A}_{c,\epsilon(i)}} f_{\epsilon(i)}, \\
\mathcal{L}_{c,\lambda}(\eta_i)|_{X_j} &= 0 \quad (\text{for } j = 1, \dots, i-1, \epsilon(i)+1, \dots, l).
\end{aligned}$$

The gluing is given by

$$u_j = f_j^{\lambda_j - \lambda_{j+1} - \tilde{c}_j} u_{j+1} \quad \text{on } X_j \cap X_{j+1} \tag{14}$$

where u_j is the image of the constant function $1 \in \widetilde{\mathscr{A}_{c,j}}$ in $\mathcal{L}_{c,\lambda}(\eta_i)|_{X_j}$ for $j = i, \dots, \epsilon(i)$.

Remark 5.9. If $\tilde{c}_i + \tilde{c}_{i+1} + \dots + \tilde{c}_{j-1} \notin \mathbb{Z}$ for any $j = i+1, \dots, l$, we regard $\epsilon(i) = l+1$ and the definition of $\mathcal{L}_{c,\lambda}(\eta_i)$ is given by

$$\begin{aligned}
\mathcal{L}_{c,\lambda}(\eta_i)|_{X_i} &= \widetilde{\mathscr{A}_{c,i}}/\widetilde{\mathscr{A}_{c,i}}g_i, \\
\mathcal{L}_{c,\lambda}(\eta_i)|_{X_j} &= \widetilde{\mathscr{A}_{c,j}}/\widetilde{\mathscr{A}_{c,j}}(f_j \circ g_j - \hbar\lambda_j) \quad (\text{for } j = i+1, \dots, l), \\
\mathcal{L}_{c,\lambda}(\eta_i)|_{X_j} &= 0 \quad (\text{for } j = 1, \dots, i-1).
\end{aligned}$$

Clearly $\mathcal{L}_{c,\lambda}(\eta_i) \simeq \mathcal{M}_{c,\lambda}(\eta_i)$ in this case.

Note that we have an isomorphism of $\widetilde{\mathscr{A}_{c,j}}$-modules

$$\mathcal{L}_{c,\lambda}(\eta_i)|_{X_j} \simeq \mathscr{M}_{\lambda_j}, \quad u_j \mapsto v_{\lambda_j}, \tag{15}$$

for $j = i+1, \dots, \epsilon(i)-1$, under the isomorphism $\widetilde{\mathscr{A}_{c,j}} \simeq \mathscr{W}_{T^*\mathbb{C}^1}$.

The following lemmas are proved similarly to Lemmas 5.3 and 5.4 by using (15) instead of (12).

Lemma 5.10. *The module $\mathcal{L}_{c,\lambda}(\eta_i)$ is a well-defined good $\widetilde{\mathscr{A}_c}$-module supported on the Lagrangian subvariety $D_i \cup D_{i+1} \cup \dots \cup D_{\epsilon(i)}$.*

Proof. The well-definedness is proved similarly to Lemma 5.3.

Similarly to the proof of Lemma 5.3, there exists an $\widetilde{\mathscr{A}_c}(0)$-lattice of $\mathcal{L}_{c,\lambda}(\eta_i)$ given by $\mathcal{L}_{c,\lambda}(\eta_i)(0)|_{X_j} = \widetilde{\mathscr{A}_c}(0)|_{X_j} u_j$ for $j = i, \dots, \epsilon(i)$. Thus, $\mathcal{L}_{c,\lambda}(\eta_i)$ is a good $\widetilde{\mathscr{A}_c}$-module. □

Lemma 5.11. *For any admissible parameters $\lambda, \lambda' \in \mathbb{C}^{\epsilon(i)-i-1}$, we have an isomorphism of $\widetilde{\mathscr{A}_c}$-modules $\mathcal{L}_{c,\lambda}(\eta_i) \simeq \mathcal{L}_{c,\lambda'}(\eta_i)$.*

Proof. Note that $\lambda_j \notin \mathbb{Z}$ because λ is admissible and satisfies $\tilde{c}_i + \tilde{c}_{i+1} + \cdots + \tilde{c}_{j-1} \notin \mathbb{Z}$ for any $i < j < \epsilon(i)$. Thus this lemma is proved similarly to Lemma 5.4. □

By the above lemma, the $\widetilde{\mathscr{A}_c}$-module $\mathcal{L}_{c,\lambda}(\eta_i)$ is not, up to isomorphism, dependent on $\lambda \in \mathbb{C}^{\epsilon(i)-i-1}$.

Definition 5.12. We denote the $\widetilde{\mathscr{A}_c}$-module $\mathcal{L}_{c,\lambda}(\eta_i)$ by $\mathcal{L}_c(\eta_i)$.

In the rest of this subsection, we show that the $\widetilde{\mathscr{A}_c}$-module $\mathcal{L}_c(\eta_i)$ is irreducible.

For $i = 1, \dots, l$, the good $\widetilde{\mathscr{A}_c}$-module $\mathcal{L}_c(\eta_i)$ is supported on the Lagrangian subvariety $D_i \cup D_{i+1} \cup \cdots \cup D_{\epsilon(i)}$. Thus, $\mathcal{L}_c(\eta_i)$ is a holonomic module. The irreducibility of $\mathcal{L}_c(\eta_i)$ now follows immediately from Propositions 3.5 and 3.9.

Proposition 5.13. *The module $\mathcal{L}_c(\eta_i)$ is an irreducible $\widetilde{\mathscr{A}_c}$-module.*

Proof. Assume there exists a nonzero submodule $\mathcal{N}$ of $\mathcal{L}_c(\eta_i)$. By Proposition 3.5 and Lemma 3.6, $\operatorname{Supp}\mathcal{N} = D_j \cup D_{j+1} \cup \cdots \cup D_k$ for some $i \leq j \leq k \leq \epsilon(i)$. Assume $j \neq i$; then $\mathcal{L}_c(\eta_i)|_{X_j}$ is an $\widetilde{\mathscr{A}_{c,j}} \simeq \mathscr{W}_{T^*\mathbb{C}^1}$-module and it has a nontrivial $\mathscr{W}_{T^*\mathbb{C}^1}$-submodule $\mathcal{N}|_{X_j}$ supported on $\{x = 0\}$. On the other hand, by the definition of $\mathcal{L}_c(\eta_i)$, we have $\mathcal{L}_c(\eta_i)|_{X_j} \simeq \mathscr{M}_{\lambda_j}$ and $\lambda_j \notin \mathbb{Z}$. By Proposition 3.9, $\mathcal{L}_c(\eta_i)|_{X_j}$ is an irreducible $\mathscr{W}_{T^*\mathbb{C}^1}$-module, which contradicts the assumption. Thus we have $j = i$. Similarly, $k = \epsilon(i)$. Therefore $\mathcal{N} = \mathcal{L}_c(\eta_i)$, and thus $\mathcal{L}_c(\eta_i)$ is an irreducible $\widetilde{\mathscr{A}_c}$-module. □

Theorem 5.14. *For $i = 1, \dots, l$, we have*

$$\operatorname{Hom}_{\operatorname{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}, \mathcal{L}_c(\eta_i)) = L_c(\eta_i).$$

Proof. By Proposition 3.9 together with the definitions of $\mathcal{M}_c^{\Delta}(\eta_i)$ and $\mathcal{L}_c(\eta_i)$ (Definitions 5.2 and 5.8), $\mathcal{L}_c(\eta_i)$ is a quotient of $\mathcal{M}_c^{\Delta}(\eta_i)$. Applying the equivalence of Theorem 3.10, the A_c-module $\operatorname{Hom}_{\operatorname{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}, \mathcal{L}_c(\eta_i))$ is a quotient of $\operatorname{Hom}_{\operatorname{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}, \mathcal{M}_c^{\Delta}(\eta_i)) \simeq \Delta_c(\eta_i)$. Since $\mathcal{L}_c(\eta_i)$ is an irreducible $\widetilde{\mathscr{A}_c}$-module, the A_c-module $\operatorname{Hom}_{\operatorname{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}, \mathcal{L}_c(\eta_i))$ is an irreducible quotient of $\Delta_c(\eta_i)$. Therefore, it is isomorphic to $L_c(\eta_i)$. □

In [KS2, Sections 2 and 3], Kashiwara and Schapira introduced the notion of regular holonomic $\widetilde{\mathscr{A}_c}$-modules. By the definition, the full subcategory of regular holonomic $\widetilde{\mathscr{A}_c}$-modules is closed under extensions. Thus we have the following corollary.

Corollary 5.15. *For any A_c-module M in $\mathcal{O}(A_c)$, the corresponding $\widetilde{\mathscr{A}_c}$-module $\widetilde{\mathscr{A}_c} \otimes_{A_c} M$ is regular holonomic.*

Next, we discuss the decomposition of the standard modules of $\mathcal{O}(A_c)$ in the Grothendieck group of $\mathcal{O}(A_c)$.

Corollary 5.16. *In the Grothendieck group of $\mathcal{O}(A_c)$, we have*

$$[\Delta_c(\eta_i)] = \sum_{j:\, \tilde{c}_i + \cdots + \tilde{c}_{j-1} \in \mathbb{Z}} [L_c(\eta_j)].$$

Proof. By Proposition 4.6(3), we have

$$[\Delta_c(\eta_i)] = \sum_{j=1}^{l} n_j [L_c(\eta_j)]$$

for some $n_j \in \mathbb{Z}_{\geq 0}$. If $\operatorname{Supp} \mathcal{L}_c(\eta_j) \not\subset \operatorname{Supp} \mathcal{M}_c^{\Delta}(\eta_i) = D_i \cup \cdots \cup D_l$, we have $n_j = 0$. The modules $\mathcal{M}_c^{\Delta}(\eta_i)$ and $\mathcal{L}_c(\eta_j)$ are (at most) multiplicity-one on D_k, i.e. $\mathcal{M}_c^{\Delta}(\eta_i)(0)/\mathcal{M}_c^{\Delta}(\eta_i)(-1)$ and $\mathcal{L}_c(\eta_j)(0)/\mathcal{L}_c(\eta_i)(-1)$ are invertible $\mathcal{O}_{D_k}$-modules on $D_k \setminus \{p_k, p_{k+1}\}$. Thus we have

$$\sum_{j:\, \operatorname{Supp} \mathcal{L}_c(\eta_j) \cap D_k \neq \emptyset} n_j = 1 \quad \text{for } k = i, \ldots, l.$$

That is, $[\Delta_c(\eta_i)]$ is multiplicity-free in the Grothendieck group. Since $\mathcal{L}_c(\eta_j)$ is a unique irreducible module whose support is of the form $D_j \cup D_{j+1} \cup \cdots$, we have $n_j = 1$ for $j = i, \ldots, l$ such that $\tilde{c}_i + \cdots + \tilde{c}_{j-1} \in \mathbb{Z}$ by comparing the supports of $\mathcal{M}_c^{\Delta}(\eta_i)$ and $\mathcal{L}_c(\eta_i)$. □

Remark 5.17. We can also determine the multiplicity $[\Delta_c(\eta_i) : L_c(\eta_j)]$ in the Grothendieck group of $\mathcal{O}(A_c)$ algebraically in this case. The same result of Corollary 5.16 follows immediately from [Ku, Lemma 4.3].

Finally, we discuss the subcategory of $\operatorname{Mod}_F^{\text{good}}(\widetilde{\mathscr{A}_c})$ corresponding to the category $\mathcal{O}(A_c)$. Since a section f of the W-algebra $\widetilde{\mathscr{A}_c}$ is invertible if and only if its symbol $\sigma_0(f)$ is invertible in $\mathcal{O}_X$, $\hbar^{-1} y_1 \cdots y_l$ acts locally nilpotently on an A_c-module M if and only if $\operatorname{Supp} \widetilde{\mathscr{A}_c} \otimes_{A_c} M \subset \bigcup_{i=1}^{l} D_i$. Thus, as mentioned in [Mc, Remark 8.8.2], we have an equivalence of these subcategories:

$$\mathcal{O}(A_c) \simeq \operatorname{Mod}_{F, \bigcup_{i=1}^{l} D_i}^{\text{good}}(\widetilde{\mathscr{A}_c}),$$

where $\mathrm{Mod}^{\mathrm{good}}_{F,\bigcup_{i=1}^{l} D_i}(\widetilde{\mathscr{A}_c})$ is the full subcategory of $\mathrm{Mod}^{\mathrm{good}}_{F}(\widetilde{\mathscr{A}_c})$ whose modules are supported on $\bigcup_{i=1}^{l} D_i$. As a consequence of Corollary 5.15, good $\widetilde{\mathscr{A}_c}$-modules with F-action supported on $\bigcup_{i=1}^{l} D_i$ are automatically regular holonomic.

Appendix. Global sections of the standard modules

We can explicitly calculate global sections of $\mathcal{M}^{\Delta}(\eta_i)$. Fix an admissible parameter $\lambda = (\lambda_j)_{j=i+1,\dots,l} \in \mathbb{C}^{l-i}$. First, the restriction homomorphisms are given explicitly as follows:

$$\begin{aligned}
\mathrm{Res}_1 : \Gamma(X_j, \mathcal{M}_{c,\lambda}(\eta_i)) &\to \Gamma(X_j \cap X_{j+1}, \mathcal{M}_{c,\lambda}(\eta_i)),\\
f_j^m u_j &\mapsto f_j^m u_j \quad (m \in \mathbb{Z}_{\geq 0}),\\
g_j^m u_j &\mapsto C'_{-m,j} f_j^{-m} u_j \quad (m \in \mathbb{Z}_{>0}),\\
\mathrm{Res}_2 : \Gamma(X_{j+1}, \mathcal{M}_{c,\lambda}(\eta_i)) &\to \Gamma(X_j \cap X_{j+1}, \mathcal{M}_{c,\lambda}(\eta_i)),\\
g_{j+1}^m u_{j+1} &\mapsto f_j^{-m+\lambda_{j+1}+\tilde{c}_j-\lambda_j} u_j \quad (m \in \mathbb{Z}_{\geq 0}),\\
f_{j+1}^m u_{j+1} &\mapsto C_{m,j+1} f_j^{m+\lambda_{j+1}+\tilde{c}_j-\lambda_j} u_j \quad (m \in \mathbb{Z}_{\geq 0}),
\end{aligned}$$

where

$$\begin{aligned}
C_{m,j} &= \hbar^m (m+\lambda_j)(m+\lambda_j-1)\cdots(\lambda_j+1) \quad (m \in \mathbb{Z}_{\geq 0}),\\
C'_{m,j} &= \hbar^{-m} (m+\lambda_j+1)(m+\lambda_j+2)\cdots\lambda_j \quad (m \in \mathbb{Z}_{<0})
\end{aligned}$$

are scalar constants. For an index $j = i, \dots, l$ such that $\tilde{c}_i + \cdots + \tilde{c}_{j-1} \notin \mathbb{Z}$, we have $C_{m,j}, C'_{m,j} \neq 0$ for all m.

Assume $\lambda = (\lambda_j)_{j=i+1,\dots,l} \in (\mathbb{C} \setminus \mathbb{Z}_{\geq 0})^{l-i}$. For an index $j = i, \dots, l$ such that $\tilde{c}_i + \cdots + \tilde{c}_{j-1} \in \mathbb{Z}$, we have

$$\begin{aligned}
&C_{m,j} \neq 0 \quad (\text{for } m < -\lambda_j, \text{and } j = i, \dots, l-1),\\
&C'_{m,j} \neq 0 \quad (\text{for any } m, \text{and } j = i, \dots, l-1).
\end{aligned} \tag{16}$$

Now, we construct global sections of $\mathcal{M}_c^{\Delta}(\eta_i)$ explicitly. Fix $i = 1, \dots, l$ and $\lambda_{i+1}, \dots, \lambda_l$ such that $\lambda_j < -\tilde{c}_i - \tilde{c}_{i+1} - \cdots - \tilde{c}_{j-1}$ for all $j = i+1, \dots, l$.

For $j = i, \dots, l$ and $k = j, \dots, l$, set

$$m_{j,k} = -\lambda_k - \tilde{c}_{k-1} - \tilde{c}_{k-2} - \cdots - \tilde{c}_j.$$

Note that we have $m_{j,k} + \lambda_k + \tilde{c}_{k-1} - \lambda_{k-1} = m_{j,k-1}$. For $j = i, \dots, l$ such that $\tilde{c}_i + \cdots + \tilde{c}_{j-1} \in \mathbb{Z}$, take $m \in \mathbb{Z}$ such that $0 \leq m < \tilde{c}_j + \cdots + \tilde{c}_{\epsilon(j)-1}$ (we regard

$\tilde{c}_l = \infty$). Then we define a section

$$v_{j,m} = \begin{cases} (\hbar^{-l/2} f_l)^{m_{j,l}+m} u_l & \text{on } X_l, \\ \Big(\prod_{k=j'+1}^{l} C_{m_{j,k}+m,k} \Big) (\hbar^{-j'+l/2} f_{j'})^{m_{j,j'}+m} u_{j'} & \text{on } X_{j'} \ (j \le j' \le l), \\ 0 & \text{on } X_{j'} \ (j' \le j-1). \end{cases}$$

Note that $C_{m_{j,k}+m,k} \neq 0$ by (16), and $v_{j,m}$ is a well-defined global section. Moreover, because $v_{j,m}$ is an F-equivariant section, we can identify it with an F-equivariant homomorphism

$$\widetilde{\mathscr{A}_c} \ni 1 \mapsto v_{j,m} \in \mathcal{M}_c^{\Delta}(\eta_i)$$

in $\mathrm{Hom}_{\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}, \mathcal{M}_c^{\Delta}(\eta_i))$. It is clear that the vectors $\{v_{j,m}\}_{j,m}$ are linearly independent. By Theorem 5.7 and (9), $\{v_{j,m}\}_{j,m}$ is a basis of the $\mathbb{C}$-vector space $\mathrm{Hom}_{\mathrm{Mod}_F^{\mathrm{good}}(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}, \mathcal{M}_c^{\Delta}(\eta_i))$.

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