

Mori Contractions of Maximal Length

by

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Abstract

We prove a relative version of the theorem of Cho, Miyaoka and Shepherd-Barron: a Mori fibre space of maximal length is birational to a projective bundle.

2010 Mathematics Subject Classification: Primary 14E30; Secondary 14D06, 14J40, 14J45.
Keywords: Mori contraction, length of extremal rays, rational curves, degenerations of projective spaces.

§1. Introduction

§1.A. Motivation

Let X be a Fano manifold of dimension d with Picard number one, and denote by H the ample generator of the Picard group. Let $i(X) \in \mathbb{N}$ be such that $-K_X \equiv i(X)H$; by a classical theorem of Kobayashi and Ochiai [KO73] one has $i(X) \leq d+1$ and equality holds if and only if X is isomorphic to the projective space \mathbb{P}^d . If one tries to understand Fano manifolds of higher Picard number, the index $i(X)$ is less useful, since it can be equal to one even for very simple manifolds like $\mathbb{P}^d \times \mathbb{P}^{d+1}$. For these manifolds the pseudoindex

$$\min\{-K_X \cdot C \mid C \subset X \text{ a rational curve}\}$$

yields much more precise results, the most well-known being the theorem of Cho, Miyaoka and Shepherd-Barron:

Communicated by S. Mukai. Received January 31, 2012. Revised April 13, 2012, July 6, 2012, and July 27, 2012.

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Theorem 1.1 ([CMSB02, Cor. 0.3] [Keb02, Thm. 1.1]). *Let X be a projective manifold of dimension d such that for every curve $C \subset X$, we have $-K_X \cdot C \geq d + 1$. Then X is isomorphic to the projective space \mathbb{P}^d .*

Fano manifolds (or more generally Fano varieties with certain singularities) are important objects since they appear naturally in the minimal model program as the general fibres of Mori fibre spaces. However if one wants to get a more complete picture of Mori fibre spaces, one should also try to obtain some information on the special fibres. In the polarised setting we have a relative version of the Kobayashi–Ochiai theorem:

Theorem 1.2 ([Fuj87, Lemma 2.12], [Ion86]). *Let X be a manifold and $\varphi: X \rightarrow Y$ a projective, equidimensional morphism of relative dimension d onto a normal variety. Suppose that the general fibre F is isomorphic to \mathbb{P}^d and that there exists a φ -ample Cartier divisor A on X such that the restriction to F is isomorphic to the hyperplane divisor H . Then φ is the projectivisation of a vector bundle and A is a global hyperplane divisor.*

The existence of the φ -ample Cartier divisor A being a rather restrictive condition, the object of this paper is to replace it by a more flexible “numerical” hypothesis, i.e. we prove a relative version of Cho–Miyaoaka–Shepherd-Barron theorem, even for fibrations that are not equidimensional.

§1.B. Main results

Let X be a quasi-projective manifold and let $\varphi: X \rightarrow Y$ be an *elementary contraction*, i.e. a Mori contraction associated with an extremal ray R of X . The *length* of the extremal ray R (or of the elementary contraction) is defined as

$$l(R) := \min\{-K_X \cdot C \mid C \subset X \text{ a rational curve such that } [C] \in R\}.$$

Denote by $E \subset X$ an irreducible component of the φ -exceptional locus ($E = X$ for a contraction of fibre type), and let F be an irreducible component of any φ -fibre contained in E . Then by the Ionescu–Wiśniewski inequality [Ion86, Thm. 0.4], [Wiś91a, Thm. 1.1] one has

$$(1.1) \quad \dim E + \dim F \geq \dim X + l(R) - 1.$$

In this paper we investigate the case when (1.1) is an equality: examples of such a situation are given by projective bundles, i.e. smooth fibrations $\varphi: X \rightarrow Y$ that are locally trivial in the analytic topology with fibre $F \simeq \mathbb{P}^d$. If the contraction φ maps X onto a point, Theorem 1.1 says that $X = E = F$ is a projective

space; more generally Andreatta and Wiśniewski conjectured that F is a projective space regardless of the dimension of the target [AW97, Conj. 2.6]. We prove a strong version of this conjecture for Mori fibre spaces:

Theorem 1.3. *Let X be a quasi-projective manifold that admits an elementary contraction $\varphi: X \rightarrow Y$ of fibre type onto a normal variety Y such that the general fibre has dimension d . Suppose that the contraction has length $l(R) = d + 1$. If φ is equidimensional, it is a projective bundle. If φ is not equidimensional, there exists a commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{\mu'} & X \\ \varphi' \downarrow & & \downarrow \varphi \\ Y' & \xrightarrow{\mu} & Y \end{array}$$

such that μ and μ' are birational, X' and Y' are smooth, and $\varphi': X' \rightarrow Y'$ is a projective bundle.

Since X is smooth, Theorem 1.1 immediately implies that a general φ -fibre is isomorphic to \mathbb{P}^d . Our contribution is to prove that the condition on the length severely limits the possible degenerations of these projective spaces. The smoothness of X is not essential for these degeneration results (cf. Section 3), so one can easily derive analogues of Theorem 1.3 making some assumption on the singularities of the general fibre (e.g. isolated LCIQ singularities [CT07]).

If the contraction φ is birational, the situation is more complicated:

Theorem 1.4. *Let X be a quasi-projective manifold that admits a birational elementary contraction $\varphi: X \rightarrow Y$. Let $E \subset X$ be an irreducible component of the exceptional locus such that for a general fibre of $E \rightarrow \varphi(E) =: Z$ the inequality (1.1) is an equality. Let $\tilde{E} \rightarrow E$ be the normalisation of E and $\tilde{\varphi}: \tilde{E} \rightarrow \tilde{Z}$ be the fibration obtained by the Stein factorisation of $\tilde{E} \rightarrow E \rightarrow Z$. Then $\tilde{\varphi}$ is a projective bundle in codimension one. Moreover, there exists a commutative diagram*

$$\begin{array}{ccc} E' & \xrightarrow{\mu'} & \tilde{E} \\ \varphi' \downarrow & & \downarrow \tilde{\varphi} \\ Z' & \xrightarrow{\mu} & \tilde{Z} \end{array}$$

such that μ and μ' are birational, E' and Z' are smooth, and $\varphi': E' \rightarrow Z'$ is a projective bundle.

As in the case of a Mori fibre space, it is not hard to see that the general fibre of $\tilde{\varphi}: \tilde{E} \rightarrow \tilde{Z}$ is a projective space ([CMSB02, Rem. 12], cf. Lemma 2.2),

so the key point is to study its degenerations. While this result gives a rather precise description of the normalisation \tilde{E} and the fibration $\tilde{\varphi}$, it does not prove that an irreducible component of a general fibre of $\varphi|_E : E \rightarrow Z$ is a projective space. However, if the contraction φ is divisorial (so the inequality (1.1) simplifies to $\dim F \geq l(R)$), Andreatta and Occhetta [AO02, Thm. 5.1] proved that all the nontrivial fibres of φ have dimension $l(R)$ if and only if X is the blow-up of a manifold along a submanifold of codimension $l(R) + 1$. If one studies the proof of Kawamata's classification of smooth fourfold flips [Kaw89, Thm. 1.1] (which corresponds to the case $\dim X = 4$, $\dim E = \dim F = 2$, $l(R) = 1$) one sees that for flipping contractions the proof of the normality requires completely different techniques. We leave this interesting problem for future research.

§1.C. Further developments

Theorem 1.3 completely determines the equidimensional fibre type contractions of maximal length: they are projective bundles. If the fibration is not equidimensional it still gives a precise description of the Chow family defined by the fibration. It seems reasonable to expect that this description will allow one to deduce some information about higher-dimensional fibres. For example if X of dimension n maps onto a threefold Y , it is not hard to see that any fibre component of dimension $n - 2$ is normalised by \mathbb{P}^{n-2} . More generally we expect that the theory of varieties covered by high-dimensional linear spaces [Ein85], [Wiś91b], [BSW92], [ABW92], [Sat97], [NO11] can be applied in this context.

Another interesting line of investigation would be to prove that under additional assumptions the fibration φ is always equidimensional. We recall the following conjecture by Beltrametti and Sommese:

Conjecture 1.5 ([BS93], [BS95, Conj. 14.1.10]). *Let (X, L) be a polarised projective manifold of dimension n that is an adjunction theoretic scroll $\varphi: X \rightarrow Y$ over a normal variety Y of dimension m . If $n \geq 2m - 1$, then φ is equidimensional.*

Wiśniewski [Wiś91b, Thm. 2.6] proved this conjecture if L is very ample and $n \geq 2m$, but apart from partial results for low-dimensional Y , [BSW92], [Som86], [Tir10], this conjecture is very much open. We expect that Conjecture 1.5 even holds in the more general setting of Theorem 1.3.

Conjecture 1.6. *Let X be a projective manifold of dimension n that admits an elementary contraction $\varphi: X \rightarrow Y$ of fibre type onto a normal variety Y of dimension m . Suppose that the contraction has length $l(R) = n - m + 1$. If $n \geq 2m - 1$, then φ is equidimensional.*

§2. Notation and basic results

We work over the complex field \mathbb{C} . A *fibration* is a projective surjective morphism $\varphi: X \rightarrow Y$ with connected fibres between normal varieties such that $\dim X > \dim Y$. The φ -*equidimensional* (resp. φ -*smooth*) *locus* is the largest Zariski open subset $Y^* \subset Y$ such that for every $y \in Y^*$, the fibre $\varphi^{-1}(y)$ has dimension $\dim X - \dim Y$ (resp. has dimension $\dim X - \dim Y$ and is smooth).

An *elementary Mori contraction* of a quasi-projective manifold X is a morphism $\varphi: X \rightarrow Y$ with connected fibres onto a normal variety Y such that the anticanonical divisor $-K_X$ is φ -ample and the numerical classes of curves contracted by φ lie on an extremal ray $R \subset N_1(X)$.

We will use the notation of [Kol96, Ch.II]: if $U \rightarrow V$ is a variety U that is projective over some base V , we denote by $\text{RatCurves}^n(U/V)$ the space parameterising rational curves on U . If moreover $s: V \rightarrow U$ is a section, we denote by $\text{RatCurves}^n(s, U/V)$ the space parameterising rational curves passing through $s(V)$. In the case where V is a point and $x = s(V)$, we simply write $\text{RatCurves}^n(U)$ and $\text{RatCurves}^n(x, U)$.

Let F be a normal, projective variety of dimension d . If $\mathcal{H} \subset \text{RatCurves}^n(F)$ is an irreducible component parameterising a family of rational curves that dominates F , then for a general point $x \in F$ one has

$$\dim \mathcal{H}_x = \dim \mathcal{H} + 1 - d,$$

where $\mathcal{H}_x \subset \text{RatCurves}^n(x, F)$ parameterises the members of \mathcal{H} passing through x . If \mathcal{H}_x is proper, it follows from bend-and-break [Mor79] that $\dim \mathcal{H}_x \leq d - 1$. Thus in this case we have

$$(2.1) \quad \dim \mathcal{H} \leq 2d - 2.$$

Fix now an ample \mathbb{Q} -Cartier divisor A on F . Let $C \subset F$ be a rational curve such that

$$A \cdot C = \min\{A \cdot C' \mid C' \subset F \text{ is a rational curve}\};$$

then any irreducible component of $\mathcal{H} \subset \text{RatCurves}^n(F)$ containing C is proper [Kol96, Ch. II, Prop. 2.14], so it parameterises a family \mathcal{H} of rational curves that is unsplit in the sense of [CMSB02, Def. 0.2]. If moreover the irreducible component \mathcal{H} has the maximal dimension $2d - 2$, the family is *doubly dominant*, i.e. for every $x, y \in F$ there exists a member of the family \mathcal{H} that joins x and y . In this setting one can prove a version of Theorem 1.1 that includes normal varieties:

Theorem 2.1 ([CMSB02, Main Thm. 0.1]). *Let F be a normal, projective variety of dimension d and A an ample \mathbb{Q} -Cartier divisor on F . Suppose that there exists*

an irreducible component $\mathcal{H} \subset \text{RatCurves}^n(F)$ of dimension at least $2d - 2$ such that for a curve $[C] \in \mathcal{H}$ we have

$$A \cdot C = \min\{A \cdot C' \mid C' \subset F \text{ is a rational curve}\}.$$

Then F is isomorphic to a projective space \mathbb{P}^d and \mathcal{H} parameterises the family of lines on \mathbb{P}^d .

Remark. A proof of this statement following the strategy in [Keb02] can be found in [CT07, §4].

If F is not smooth, it is in general quite hard to verify the condition $\dim \mathcal{H} \geq 2d - 2$ (cf. [CT07]). Since we work with an ambient space that is smooth, things are much simpler:

Lemma 2.2. *Let X be a quasi-projective manifold that admits a proper birational morphism $\varphi: X \rightarrow Y$ such that $-K_X$ is φ -ample. Set*

$$l(\varphi) := \min\{-K_X \cdot C \mid C \subset X \text{ a rational curve such that } \varphi(C) = \text{pt}\}.$$

Let $E \subset X$ be an irreducible component of the exceptional locus such that

$$(2.2) \quad 2 \dim E - \dim \varphi(E) = \dim X + l(\varphi) - 1.$$

Let F be an irreducible component of a general fibre of $E \rightarrow \varphi(E)$. Then the normalisation of F is a projective space.

Proof. If $\varphi(E)$ has positive dimension, let $H \subset Y$ be a general hyperplane section and $X_H := \varphi^{-1}(H)$ its preimage. By adjunction the intersection numbers of $-K_X$ and $-K_{X_H}$ with the φ -exceptional curves agree, so we have $l(\varphi|_{X_H}) \geq l(\varphi)$. Let E_H be an irreducible component of $E \cap X_H$ that surjects onto an irreducible component of $\varphi(E) \cap H$. Applying the inequality (1.1) to the birational morphism¹ $\varphi|_{X_H}: X_H \rightarrow H$ and E_H we get

$$2 \dim E_H - \dim \varphi(E_H) \geq \dim X_H + l(\varphi|_{X_H}) - 1.$$

Since $\dim E_H = \dim E - 1$ and $\dim \varphi(E_H) = \dim \varphi(E) - 1$ the equality (2.2) implies that this inequality is an equality and $l(\varphi|_{X_H}) = l(\varphi)$.

Thus we can take general hyperplane sections on Y until F is an irreducible component of E , and the equality (2.2) simplifies to

$$2 \dim F = \dim X + l(\varphi) - 1.$$

¹The proof of [Wis91a, Thm. 1.1] does not use that the contraction is elementary.

Let $C \subset F \subset X$ be a rational curve passing through a general point $x \in F$ that has minimal degree with respect to $-K_X$. Then C belongs to an irreducible family $\mathcal{H} \subset \text{RatCurves}^n(F)$ of rational curves that dominates F and moreover \mathcal{H}_x is proper. Note now that any deformation of C in X is contracted by φ , hence contained in F . Thus we can estimate $\dim \mathcal{H}$ by applying the standard Riemann–Roch estimate [CMSB02, p. 14] for the dimension of deformation spaces on the manifold X :

$$\dim \mathcal{H} \geq -K_X \cdot C - 3 + \dim X \geq l(\varphi) - 3 + \dim X = 2 \dim F - 2.$$

By (2.1) we see that these inequalities are in fact equalities, in particular one has $-K_X \cdot C = l(\varphi)$. Thus if $\nu: \tilde{F} \rightarrow F$ is the normalisation and $\tilde{\mathcal{H}} \subset \text{RatCurves}^n(\tilde{F})$ the family obtained by lifting the members of \mathcal{H} , it satisfies the conditions of Theorem 2.1 with respect to the polarisation $A := -\nu^*K_X$. Consequently, we have $\tilde{F} \simeq \mathbb{P}^{\dim F}$ and $-\nu^*K_X \equiv l(R)H$ with H the hyperplane divisor.² \square

§3. Degenerations of \mathbb{P}^d

In this section we prove our main results on degenerations of \mathbb{P}^d satisfying a length condition.

Proposition 3.1. *Let X be a normal, quasi-projective variety and let $\varphi: X \rightarrow Y$ be an equidimensional fibration of relative dimension d onto a normal variety such that the general fibre F is isomorphic to \mathbb{P}^d . Let A be a φ -ample \mathbb{Q} -Cartier divisor, and let $e \in \mathbb{N}$ be such that $A|_F \equiv eH$ with H the hyperplane divisor. Suppose that the following length condition holds:*

$$(3.1) \quad A \cdot C \geq e \quad \forall C \subset X \text{ rational curve such that } \varphi(C) = \text{pt.}$$

Then all the fibres are irreducible and generically reduced. Moreover the normalisation of any fibre is a projective space.

Remark 3.2. In general the degeneration behaviour of projective spaces can be quite complicated, for example it depends in a subtle manner on the geometry of the total space. Consider $\varphi: X \rightarrow C$ a fibration from a normal variety X onto a smooth curve C such that all the fibres are integral and the general fibre is isomorphic to \mathbb{P}^d . If X is smooth (or at least factorial), then Tsen’s theorem implies that $X \rightarrow C$ is a \mathbb{P}^d -bundle ([NO07, Lemma 2.17], cf. also the proof of [DP10, Lemma 4]). This assumption cannot be weakened:

²Cf. [Zha91] for an argument assuming smoothness of the general fibre.

Example 3.3 ([AR12]). Let $W_4 \subset \mathbb{P}^6$ be the cone over the Veronese surface $R_4 \subset \mathbb{P}^5$ (that is the 2-uple embedding of \mathbb{P}^2 in \mathbb{P}^5). The blow-up of W_4 in the vertex P is isomorphic to the projectivised bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)) \rightarrow \mathbb{P}^2$; this desingularisation contracts a \mathbb{P}^2 with normal bundle $\mathcal{O}_{\mathbb{P}^2}(-2)$ onto the vertex P . In particular W_4 is a terminal \mathbb{Q} -factorial threefold and the canonical divisor K_{W_4} is not Gorenstein, but 2-Gorenstein.

The base locus of a general pencil of hyperplane sections of $W_4 \subset \mathbb{P}^6$ identifies to a smooth quartic curve $C \subset R_4$. If we denote by $\mu: X \rightarrow W_4$ the blow-up in C , it is a terminal, \mathbb{Q} -factorial, 2-Gorenstein threefold admitting a fibration $\varphi: X \rightarrow \mathbb{P}^1$ whose fibres are isomorphic to the members of the general pencil, in particular they are integral. Thus the general fibre F is isomorphic to $R_4 \simeq \mathbb{P}^2$, but the fibre F_0 corresponding to the hyperplane section through the vertex P is a cone over the quartic curve C . In particular it is not normalised by \mathbb{P}^2 .

Let us note that $A := K_X \otimes \mu^* \mathcal{O}_{W_4}(2)$ is a φ -ample \mathbb{Q} -Cartier divisor such that the restriction to a general fibre is numerically equivalent to the hyperplane divisor $H \subset \mathbb{P}^2$. However this divisor is not Cartier, so Theorem 1.2 does not apply. Let us also note that under the 2-uple embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$, a line is mapped onto a conic. If we degenerate the general fibre F to F_0 , these conics degenerate to a union of two lines $l_1 \cup l_2$ passing through the vertex of the cone F_0 . Since we have $A \cdot l_i = \frac{1}{2}$, the length condition (3.1) is not satisfied.

Proof of Proposition 3.1. By assumption the general fibre is a projective space. Let $\mathcal{H} \subset \text{RatCurves}^n(X/Y)$ be the unique irreducible component such that a general point corresponds to a line l contained in the general fibre F . We have $A \cdot l = e$ and by (3.1) one has $A \cdot C \geq e$ for every rational curve C contained in a fibre, so the variety \mathcal{H} is proper over the base Y [Kol96, Ch. II, Prop. 2.14]. Since the general fibre of $\mathcal{H} \rightarrow Y$ corresponds to the $2d - 2$ -dimensional family of lines in the projective space $\mathbb{P}^d \simeq F$, it follows by upper semicontinuity that for every $0 \in Y$, all the irreducible components of the fibre \mathcal{H}_0 have dimension at least $2d - 2$. Let $\mathcal{H}_{0,i}$ be the normalisation of such an irreducible component and $\mathcal{U}_{0,i} \rightarrow \mathcal{H}_{0,i}$ be the universal family over it. The image of the evaluation morphism $p: \mathcal{U}_{0,i} \rightarrow X$ is an irreducible component D_i of the set-theoretical fibre $(\varphi^{-1}(0))_{\text{red}}$, so it has dimension d . Let $\nu: \tilde{D}_i \rightarrow D_i$ be the normalisation; then the family of rational curves $\mathcal{H}_{0,i}$ lifts to \tilde{D}_i , so $\text{RatCurves}^n(\tilde{D}_i)$ has an irreducible component of dimension at least $2d - 2$. Since for any rational curve $C \subset D_i \subset X$ we have $A \cdot C \geq e$ by (3.1), the pull-back $\nu^* A$ also satisfies this inequality. Moreover $\nu^* A$ has degree exactly e on the rational curves parameterised by $\mathcal{H}_{0,i}$. Thus we conclude with Theorem 2.1 that \tilde{D}_i is isomorphic to a projective space.

We now suppose for contradiction that there exists a $0 \in Y$ such that the fibre $F_0 := \varphi^{-1}(0)$ is reducible or not generically reduced. Then we can decompose the cycle $[F_0]$ as

$$[F_0] = \sum_i a_i [D_i]$$

with $a_i \in \mathbb{N}$ and D_i the irreducible components of the set-theoretical fibre $(\varphi^{-1}(0))_{\text{red}}$. Since the degree is constant in a well-defined family of proper algebraic cycles [Kol96, Prop. 3.12], we have

$$e^d = [F] \cdot A^d = [F_0] \cdot A^d = \sum_i a_i ([D_i] \cdot A^d).$$

By assumption the sum on the right hand side is not trivial, so $[D_1] \cdot A^d < e^d$. Denote by $\nu: \mathbb{P}^d \rightarrow D_1$ the normalisation; then

$$(\nu^* A)^d = [D_1] \cdot A^d < e^d.$$

Thus we see that $\nu^* A \equiv bH$ with $0 < b < e$. Yet this is impossible, since it implies that for a general line $l \subset \mathbb{P}^d$, we have

$$A \cdot \nu(l) = \nu^* A \cdot l = b < e,$$

contradicting (3.1). □

Lemma 3.4. *Let $\varphi: X \rightarrow Y$ be an equidimensional fibration from a normal variety X onto a manifold Y . Suppose that φ is generically smooth and the general fibre F is a Fano manifold with Picard number one. If φ has irreducible and generically reduced fibres, then $-K_X$ is \mathbb{Q} -Cartier.*

Proof. Denote by H the ample generator of $\text{Pic}(F)$ and let A be a φ -ample Cartier divisor. Set $e \in \mathbb{N}$ such that $A|_F \equiv eH$, and $d \in \mathbb{N}$ such that $-K_F \equiv dH$. The reflexive sheaf $\mathcal{O}_X(eK_X + dA)$ is locally free in a neighbourhood of a general fibre F and the restriction $\mathcal{O}_X(eK_X + dA) \otimes \mathcal{O}_F$ is isomorphic to the structure sheaf \mathcal{O}_F . Up to replacing A by $A \otimes \varphi^* H$ with H a sufficiently ample Cartier divisor on Y we can suppose without loss of generality that

$$H^0(X, \mathcal{O}_X(eK_X + dA)) \simeq H^0(Y, \varphi_* \mathcal{O}_X(eK_X + dA)) \neq 0.$$

Thus we obtain a nonzero morphism

$$\mathcal{O}_X(-eK_X) \rightarrow \mathcal{O}_X(dA)$$

which is an isomorphism on the general fibre F . The vanishing locus is thus a Weil divisor $\sum k_i D_i$ such that for all i we have $\varphi(D_i) \subsetneq Y$. Since φ is equidimensional

and all the fibres are irreducible and generically reduced we see that $D_i = \varphi^*E_i$ with E_i a prime divisor on Y . Since Y is smooth, the divisor E_i is Cartier, hence $\sum k_i D_i = \sum k_i \varphi^*E_i$ is Cartier. Thus we have an isomorphism

$$\mathcal{O}_X(-eK_X) \simeq \mathcal{O}_X(dA - \sum k_i \varphi^*E_i).$$

The right hand side is Cartier, so K_X is \mathbb{Q} -Cartier. □

Proposition 3.5. *In the situation of Proposition 3.1, suppose that Y is smooth. Then the fibration φ is a projective bundle.*

Proof. We will proceed by induction on the relative dimension d , the case $d = 1$ is [Kol96, Ch. II, Thm. 2.8]. Taking general hyperplane sections on Y and arguing by induction on the dimension we can suppose without loss of generality that φ has at most finitely many singular fibres. The problem being local, we can suppose that $Y \subset \mathbb{C}^{\dim Y}$ is a polydisc around 0 and φ is smooth over $Y \setminus 0$. Since all the fibres are generically reduced, by Proposition 3.1 we can choose a section $s: Y \rightarrow X$ such that $s(0) \subset F_{0, \text{non-s}}$ where $F_0 := \varphi^{-1}(0)$.

As in the proof of Proposition 3.1, we denote by $\mathcal{H} \subset \text{RatCurves}^n(X/Y)$ the unique irreducible component such that a general point corresponds to a line l contained in the general fibre $F \simeq \mathbb{P}^d$. There exists a unique irreducible component $\mathcal{H}_s \subset \text{RatCurves}^n(s, X/Y)$ such that for general $y \in Y$ the lines in $\varphi^{-1}(y) \simeq \mathbb{P}^d$ passing through $s(y)$ are parameterised by \mathcal{H}_s . If we denote by $\psi: \mathcal{H}_s \rightarrow Y$ the natural fibration, its general fibre is isomorphic to a projective space \mathbb{P}^{d-1} . Let $q: \mathcal{U}_s \rightarrow \mathcal{H}_s$ be the universal family; then by [Kol96, Ch. II, Cor. 2.12] the fibration q is a \mathbb{P}^1 -bundle. Let $p: \mathcal{U}_s \rightarrow X$ be the evaluation morphism; then p is birational since this holds for the restriction to a general φ -fibre. The variety X being normal, we know by Zariski’s main theorem that p has connected fibres. The family of rational curves being unsplit, it follows from bend-and-break that p has finite fibres over $X \setminus s(Y)$. Thus we have an isomorphism

$$(3.2) \quad X \setminus s(Y) \simeq \mathcal{U}_s \setminus E,$$

where $E := (p^{-1}(s(Y)))_{\text{red}}$. Again by bend-and-break and connectedness of the fibres the algebraic set E is irreducible, so it is a prime divisor with a finite, birational morphism $q|_E: E \rightarrow \mathcal{H}_s$. Since \mathcal{H}_s is normal, we see by Zariski’s main theorem that $q|_E$ is an isomorphism and E is a q -section.

By Lemma 3.4 we know that $-K_X$ is \mathbb{Q} -Cartier, so $-K_{(\mathcal{U}_s \setminus E)}$ is \mathbb{Q} -Cartier. Since $\mathcal{U}_s \setminus E \rightarrow \mathcal{H}_s$ is locally trivial (it is a \mathbb{C} -bundle), it follows that $-K_{\mathcal{H}_s}$ is \mathbb{Q} -Cartier. Since $\mathcal{U}_s \rightarrow \mathcal{H}_s$ is a \mathbb{P}^1 -bundle, it now follows that $-K_{\mathcal{U}_s}$ is \mathbb{Q} -Cartier. The \mathbb{P}^1 -bundle $\mathcal{U}_s \rightarrow \mathcal{H}_s$ has a section E , so it is isomorphic to the projectivised

bundle $\mathbb{P}(V) \rightarrow \mathcal{H}_s$ with $V := \varphi_* \mathcal{O}_X(E)$. By the canonical bundle formula we have

$$(3.3) \quad K_{\mathcal{U}_s} \equiv q^*(K_{\mathcal{H}_s} + \det V) - 2E.$$

We claim that the following length condition holds:

$$(3.4) \quad -K_{\mathcal{H}_s} \cdot C \geq d \quad \forall C \subset \mathcal{H}_s \text{ rational curve such that } \psi(C) = \text{pt}.$$

Assuming this for the time being, let us see how to conclude: applying the induction hypothesis to ψ , we see that $\mathcal{H}_s \rightarrow Y$ is a \mathbb{P}^{d-1} -bundle. Since q is a \mathbb{P}^1 -bundle, we see that the central fibre of $\mathcal{U}_s \xrightarrow{\psi \circ q} Y$ is a \mathbb{P}^1 -bundle over \mathbb{P}^{d-1} that contracts a section onto a smooth point of F_0 . A classical argument [Mor79] shows that $F_0 \simeq \mathbb{P}^d$.

Proof of the claim. The claim is obvious for curves in the general fibres which are isomorphic to \mathbb{P}^{d-1} , so we can concentrate on curves in the central fibre $\mathcal{H}_0 := (\psi^{-1}(0))_{\text{red}}$. We set $\mathcal{U}_0 := q^{-1}(\mathcal{H}_0)$ and denote by $\nu: \tilde{\mathcal{H}}_0 \rightarrow \mathcal{H}_0$ the normalisation. Then $\mathcal{U}_0 \times_{\mathcal{H}_0} \tilde{\mathcal{H}}_0$ is normal and $\tilde{q}_0: \mathcal{U}_0 \times_{\mathcal{H}_0} \tilde{\mathcal{H}}_0 \rightarrow \tilde{\mathcal{H}}_0$ is a \mathbb{P}^1 -bundle. By Proposition 3.1 we already know that $F_{0,\text{red}}$ is normalised by \mathbb{P}^d , so we obtain a commutative diagram

$$\begin{array}{ccccc}
 & & & & \mathbb{P}^d \\
 & & & \nearrow \mu & \downarrow \\
 \mathcal{U}_0 \times_{\mathcal{H}_0} \tilde{\mathcal{H}}_0 & \xrightarrow{\tilde{\nu}} & \mathcal{U}_0 & \xrightarrow{p_0} & F_{0,\text{red}} \\
 \downarrow \tilde{q}_0 & & \downarrow q_0 & & \\
 \tilde{\mathcal{H}}_0 & \xrightarrow{\nu} & \mathcal{H}_0 & &
 \end{array}$$

Thus the curves parameterised by $\tilde{\mathcal{H}}_0$ are lines in \mathbb{P}^d , in particular $\tilde{\mathcal{H}}_0 \simeq \mathbb{P}^{d-1}$ and

$$\mathcal{U}_0 \times_{\mathcal{H}_0} \tilde{\mathcal{H}}_0 \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^{d-1}} \oplus \mathcal{O}_{\mathbb{P}^{d-1}}(-1))$$

with $\tilde{\nu}^{-1}(E \cap \mathcal{U}_0)$ corresponding to the exceptional section.

In order to check the length condition (3.4) it is sufficient to do this for a curve $C_0 \subset \mathcal{H}_0$ such that $C_0 = \nu(l_0)$ with l_0 a general line in $\tilde{\mathcal{H}}_0 \simeq \mathbb{P}^{d-1}$. We can lift l_0 to a subsection $l' \subset \mathbb{P}(\mathcal{O}_{\mathbb{P}^{d-1}} \oplus \mathcal{O}_{\mathbb{P}^{d-1}}(-1))$ that is disjoint from the exceptional section, so we obtain a curve $C' := \tilde{\nu}(l') \subset \mathcal{U}_0 \setminus E$ such that $C' \rightarrow C_0$ is birational. In particular we have

$$-K_{\mathcal{H}_s} \cdot C_0 = -q^* K_{\mathcal{H}_s} \cdot C',$$

so it is sufficient to show that the right hand side is equal to d . The rational curve C' does not meet E and $\mu(l')$ is a line, so C' corresponds to a point in \mathcal{H} that is not

in \mathcal{H}_s . Using the isomorphism (3.2) we can deform C' to a curve $C'' \subset \mathcal{U}_s$ that is a line in a general fibre $\varphi^{-1}(y) \simeq \mathbb{P}^d$ and which does not meet the point $s(y)$, so C'' is disjoint from E . Yet for such a curve the formula (3.3) restricted to a general fibre immediately shows that $-q^*K_{\mathcal{H}_s} \cdot C'' = d$. \square

Remark 3.6. Proposition 3.5 should be true without the assumption that Y is smooth. In fact this assumption is only needed to ensure via Lemma 3.4 that $-K_X$ (and by consequence $-K_{\mathcal{H}_s}$ and $-K_{\mathcal{U}_s}$) is \mathbb{Q} -Cartier. However our computations only use that these varieties are “relatively \mathbb{Q} -Cartier”, i.e. some multiple of the canonical divisor is linearly equivalent to a Cartier divisor plus some Weil divisors that are pull-backs from the base Y . We leave the technical details to the interested reader. (Note added in proof: this statement has now been proved by Araujo and Druel [AD12, Prop. 4.10].)

§4. Proofs of main results

Proof of Theorem 1.3. By Theorem 1.1 the general φ -fibre is a projective space.

If φ is equidimensional, then $A := -K_X$ satisfies the length condition (3.1) in Proposition 3.1 with $e = l(R)$. Thus we know that every fibre is generically reduced. In particular for every $y \in Y$ there exists a point $x \in \varphi^{-1}(y)$ such that the map between the Zariski tangent spaces

$$T_\varphi : T_{X,x} \rightarrow T_{Y,y}$$

has rank equal to $\dim Y$. Since X is smooth, this implies that Y is smooth at y . Conclude with Proposition 3.5.

Suppose now that φ is not equidimensional. Let now \bar{Y} be the closure of the φ -equidimensional locus in the relative Chow variety $\text{Chow}(X/Y)$ and let $\bar{X} \rightarrow \bar{Y}$ be the universal family. Let $Y' \rightarrow \bar{Y}$ be a desingularisation, and write X' for the normalisation of $\bar{X} \times_{\bar{Y}} Y'$. We denote by $\varphi' : X' \rightarrow Y'$ the natural fibre space structure and by $\mu' : X' \rightarrow X$ the birational morphism induced by the map $\bar{X} \rightarrow X$. By the rigidity lemma there exists a birational morphism $\mu : Y' \rightarrow Y$ such that $\mu \circ \varphi' = \varphi \circ \mu'$. Note also that the restriction of μ' to any φ' -fibre is finite, so the pull-back $A' := -(\mu')^*K_X$ is a φ' -ample Cartier divisor. The divisor A' on X' satisfies the length condition (3.1) in Proposition 3.1 with $e = l(R)$. Since Y' is smooth, we can conclude with Proposition 3.5. \square

Proof of Theorem 1.4. By Lemma 2.2 the general $\tilde{\varphi}$ -fibre is a projective space. Moreover if $\nu : \tilde{E} \rightarrow E \subset X$ denotes the normalisation, then $A := -\nu^*K_X$ (restricted to the $\tilde{\varphi}$ -equidimensional locus) satisfies the length condition (3.1) in Proposition 3.1 with $e = l(R)$. If $C \subset \tilde{Z}$ is a curve cut out by general hyperplane

sections, then C is smooth and the fibration $\tilde{\varphi}^{-1}(C) \rightarrow C$ is equidimensional, so it is a projective bundle by Proposition 3.5. Thus $\tilde{\varphi}$ is a projective bundle in codimension one.

The proof of the second statement is analogous to the proof of the second statement in Theorem 1.3. \square

Acknowledgements

We thank C. Araujo for sending us her inspiring preprint [AR12]. We thank M. Andreatta, S. Druel, H. Hamm, F. Han, P. Popescu-Pampu, M. Wolff and J. A. Wiśniewski for helpful communications.

A. Höring was partially supported by the A.N.R. project “CLASS”.

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