Publ. RIMS Kyoto Univ. **49** (2013), 563–599
DOI 10.4171/PRIMS/113

Lévy Measure Density Corresponding to Inverse Local Time

by

Tomoko Takemura and Matsuyo Tomisaki

Abstract

We are concerned with the Lévy measure density corresponding to the inverse local time at the regular end point for a harmonic transform of a one-dimensional diffusion process. We show that the Lévy measure density is represented as the Laplace transform of the spectral measure corresponding to the original diffusion process, where the absorbing boundary condition is posed at the end point whenever it is regular.

2010 Mathematics Subject Classification: Primary 60G51; Secondary 60J55.
Keywords: inverse local time, Lévy measure, harmonic transformation.

§1. Introduction

Let s be a continuous increasing function on an open interval $I = (l_1, l_2)$, where $-\infty \le l_1 < l_2 \le \infty$, let m be a right continuous increasing function on I, and let k be a right continuous nondecreasing function on I. Let $\mathcal{G}_{s,m,k}$ be a one-dimensional diffusion operator on I with scale function s, speed measure m, and killing measure k. We denote by $\mathbb{D}_{s,m,k} = [X(t), P_x]$ the one-dimensional diffusion process on I with generator $\mathcal{G}_{s,m,k}$ and with end point l_i where the absorbing boundary condition is posed whenever l_i is (s,m,k)-regular $(i = 1, 2)$. For $\beta \ge 0$, let $\mathcal{H}_{s,m,k,\beta}$ be the set of all positive functions h satisfying $\mathcal{G}_{s,m,k}h = \beta h$. For $h \in \mathcal{H}_{s,m,k,\beta}$, we set

$$s_h(x) = \int_{(c_o,x]} h(y)^{-2}\, ds(y), \tag{1.1}$$

Communicated by T. Kumagai. Received July 21, 2012. Revised February 4, 2013.

T. Takemura: Department of Mathematics, Nara Women's University, Kita-Uoya Nishimachi, Nara, 630-8506 Japan;
e-mail: `sm18031@cc.nara-wu.ac.jp`
M. Tomisaki: Department of Mathematics, Nara Women's University, Kita-Uoya Nishimachi, Nara, 630-8506 Japan;
e-mail: `tomisaki@cc.nara-wu.ac.jp`

$$m_h(x) = \int_{(c_o,x]} h(y)^2\, dm(y), \tag{1.2}$$

where $c_o \in I$ is fixed arbitrarily. The diffusion operator $\mathcal{G}_{s_h,m_h,0}$ with scale function s_h, speed measure m_h, and the null killing measure is the harmonic transform of $\mathcal{G}_{s,m,k}$ based on $h \in \mathcal{H}_{s,m,k,\beta}$. Let $\mathbb{D}_{s_h,m_h,0}$ be the one-dimensional diffusion process on I with generator $\mathcal{G}_{s_h,m_h,0}$ and with the end point l_i being absorbing whenever it is $(s_h, m_h, 0)$-regular $(i = 1, 2)$.

When l_1 is $(s_h, m_h, 0)$-regular, it is possible to pose the reflecting boundary condition at l_1. We denote by $\mathbb{D}^*_{s_h,m_h,0} = [X(t), P_x^{(h*)}]$ such a diffusion process on I. Namely, the scale function and the speed measure are given by s_h and m_h, respectively, the killing measure is null, and l_1 is $(s_h, m_h, 0)$-regular and reflecting. We consider the local time $l^{(h*)}(t, \xi)$ for $\mathbb{D}^*_{s_h,m_h,0}$, that is,

$$\int_0^t f(X(u))\, du = \int_I l^{(h*)}(t, \xi)\, dm_h(\xi), \quad t > 0,$$

for bounded continuous functions f on I. Since $l^{(h*)}(t, \xi)$ is continuous and non-decreasing in t $P_x^{(h*)}$-a.s., the right continuous inverse function ${l^{(h*)}}^{-1}(t, \xi)$ exists. In particular, we denote by $\tau^{(h*)}(t)$ the inverse local time ${l^{(h*)}}^{-1}(t, l_1)$ at the end point l_1. Employing some results due to Itô and McKean (see [5, Section 6.2]), we find that, if $s_h(l_2) = \infty$, then $[\tau^{(h*)}(t), t \geq 0]$ is a Lévy process and there is a Lévy measure density $n^{(h*)}(\xi)$ such that

$$E_{l_1}^{(h*)}[e^{-\lambda \tau^{(h*)}(t)}] = \exp\left\{-t \int_0^\infty (1 - e^{-\lambda\xi}) n^{(h*)}(\xi)\, d\xi\right\},$$

where $E_{l_1}^{(h*)}$ stands for the expectation with respect to $P_{l_1}^{(h*)}$. The aim of this paper is to give a representation of $n^{(h*)}(\xi)$ in terms of data corresponding to the diffusion process $\mathbb{D}_{s,m,k}$. We state our results in Section 3 (see Theorem 3.2).

Applying our results, we find some interesting facts. Let us consider the following diffusion generators on $(0, \infty)$:

$$\mathcal{L}_1 = \frac{1}{2}\frac{d^2}{dx^2} + \left\{\frac{1}{2x} + \sqrt{2\beta}\,\frac{K_\nu'(\sqrt{2\beta}\,x)}{K_\nu(\sqrt{2\beta}\,x)}\right\}\frac{d}{dx}, \tag{1.3}$$

$$\mathcal{L}_2 = \frac{1}{2}\frac{d^2}{dx^2} + \left\{-\frac{1}{2x} + 2\kappa x \frac{W'_{-\frac{\beta}{2\kappa}+\frac{\nu+1}{2},\frac{|\nu|}{2}}(\kappa x^2)}{W_{-\frac{\beta}{2\kappa}+\frac{\nu+1}{2},\frac{|\nu|}{2}}(\kappa x^2)}\right\}\frac{d}{dx}, \tag{1.4}$$

$$\mathcal{L}_3 = \frac{1}{2}\frac{d^2}{dx^2} + \left\{-\frac{1}{2x} + 2\kappa x \frac{W'_{-\frac{\beta}{2\kappa}-\frac{\nu+1}{2},\frac{|\nu|}{2}}(\kappa x^2)}{W_{-\frac{\beta}{2\kappa}-\frac{\nu+1}{2},\frac{|\nu|}{2}}(\kappa x^2)}\right\}\frac{d}{dx}, \tag{1.5}$$

where $-1 < \nu < 1$, $\kappa > 0$ and $\beta > 0$. $K_l(x)$ and $W_{k,l}(x)$ are the modified Bessel function and the Whittaker function, respectively (see Section 4 for their definitions). Let $\mathbb{D}_i^*$ be the diffusion process on $(0,\infty)$ with generator $\mathcal{L}_i$ $(i = 1,2,3)$. The end point 0 is regular for all $\mathbb{D}_i^*$. Therefore the inverse local time $\tau^*(t)$ at the end point 0 with reflecting boundary condition exists. Noting that $\mathcal{L}_1$ is a harmonic transform of a Bessel operator, and $\mathcal{L}_2$ and $\mathcal{L}_3$ are harmonic transforms of radial Ornstein–Uhlenbeck operators, by means of Theorem 3.2 below, we find that the Lévy measure densities $n_i^*(\xi)$ of $\tau^*(t)$ corresponding to $\mathbb{D}_i^*$ are

$$n_1^*(\xi) = C\xi^{-(|\nu|+1)}e^{-\beta\xi}, \tag{1.6}$$

$$n_2^*(\xi) = C\left(\frac{\kappa}{\sinh(\kappa\xi)}\right)^{|\nu|+1} e^{\{\kappa(\nu+1)-\beta\}\xi}, \tag{1.7}$$

$$n_3^*(\xi) = C\left(\frac{\kappa}{\sinh(\kappa\xi)}\right)^{|\nu|+1} e^{\{-\kappa(\nu+1)-\beta\}\xi}, \tag{1.8}$$

where $C = 2^{-(|\nu|+1)}\Gamma(|\nu|+1)$. In Section 2, we show a simple convergence theorem for a sequence of Lévy measure densities under the assumption that the scale functions and speed measures are convergent (see Theorem 2.4). Since the scale functions and speed measures of $\mathcal{L}_2$ and $\mathcal{L}_3$ converge to the scale function and speed measure of $\mathcal{L}_1$ as $\kappa \to 0$, respectively, Theorem 2.4 leads to

$$n_1^*(\xi) = \lim_{\kappa\to 0} n_2^*(\xi) = \lim_{\kappa\to 0} n_3^*(\xi).$$

This also directly follows from (1.6)–(1.8).

We next consider the following diffusion operator on $(0,a)$:

$$\mathcal{L}_4 = \frac{1}{2}\frac{d^2}{dx^2} - \sqrt{2\beta}\,\frac{e^{2\sqrt{2\beta}\,a} + e^{2\sqrt{2\beta}\,x}}{e^{2\sqrt{2\beta}\,a} - e^{2\sqrt{2\beta}\,x}}\frac{d}{dx}, \tag{1.9}$$

where $a > 0$ and $\beta > 0$. Let $\mathbb{D}_4^*$ be the diffusion process on $(0,a)$ with generator $\mathcal{L}_4$. The end point 0 is regular and the inverse local time $\tau^*(t)$ at the end point 0 with reflecting boundary condition exists. Since $\mathcal{L}_4$ is a harmonic transform of Brownian motion on $(0,a)$, by using Theorem 3.2 we find that the Lévy measure density of $\tau^*(t)$ is

$$n_4^*(\xi) = a^{-3}e^{-\beta\xi}\sum_{n=1}^{\infty}(n\pi)^2 e^{-(n^2\pi^2/2a^2)\xi}. \tag{1.10}$$

It is easy to see that $\mathcal{L}_4$ converges to $\tilde{\mathcal{L}}_4 = \frac{1}{2}\frac{d^2}{dx^2} - \sqrt{2\beta}\frac{d}{dx}$ as $a \to \infty$, in the sense that the scale function and the speed measure of $\mathcal{L}_4$ converge to those of $\tilde{\mathcal{L}}_4$. Therefore by Theorem 2.4, we find that

$$\lim_{a\to\infty} n_4^*(\xi) = \tilde{n}_4^*(\xi) = \sqrt{\pi/2}\,e^{-\beta\xi}\xi^{-3/2}, \tag{1.11}$$

where $\tilde{n}_4^*$ is the Lévy measure density of the inverse local time corresponding to $\tilde{\mathcal{L}}_4$. (1.11) also follows from (1.10).

In [2] and [3], C. Donati-Martin and M. Yor considered the diffusion processes $\mathbb{D}_i^*$ $(i = 1, 2)$, and showed (1.6), and (1.7) with $-1 < \nu < 0$. Their method is based on various transforms of Bessel processes. In [12] the second author gave an analytical proof for the representation (1.7) by using a harmonic transform. In this paper we generalize the analytical method used in [12] to obtain Theorem 3.2.

In Section 2 we summarize some definitions and facts needed below. Then we prove a convergence theorem for a sequence of Lévy measure densities. In Section 3 we state our main results and prove them. In Section 4 we present some interesting diffusion operators including the $\mathcal{L}_i$'s given by (1.3)–(1.5) and (1.9).

§2. Preliminaries

Let s, m, and k be the functions given at the beginning of the previous section. We sometimes use the same symbols s, m and k for the respective induced measures $ds(x)$, $dm(x)$ and $dk(x)$. For a function u on I, we set $u(l_i) = \lim_{x\to l_i,\, x\in I} u(x)$ if the limit exists, for $i = 1, 2$. We denote by $D_s u(x)$ the right derivative with respect to $s(x)$, that is, $D_s u(x) = \lim_{\varepsilon\downarrow 0}\{u(x+\varepsilon) - u(x)\}/\{s(x+\varepsilon) - s(x)\}$, provided it exists. Let us fix a point $c_o \in I$ arbitrarily and set

$$J_{\mu,\nu}(x) = \int_{(c_o,x]} d\mu(y) \int_{(c_o,y]} d\nu(z) \qquad \text{for } x \in I,$$

where μ and ν are Borel measures on I, and the integral $\int_{(a,b]}$ is read as $-\int_{(b,a]}$ if $a > b$. Following [4], we call the boundary point l_i

- (s, m, k)-*regular* if $J_{s,m+k}(l_i) < \infty$ and $J_{m+k,s}(l_i) < \infty$,
- an (s, m, k)-*exit point* if $J_{s,m+k}(l_i) < \infty$ and $J_{m+k,s}(l_i) = \infty$,
- an (s, m, k)-*entrance point* if $J_{s,m+k}(l_i) = \infty$ and $J_{m+k,s}(l_i) < \infty$,
- (s, m, k)-*natural* if $J_{s,m+k}(l_i) = \infty$ and $J_{m+k,s}(l_i) = \infty$.

§2.1. One-dimensional diffusion process $\mathbb{D}_{s,m,k}$

Let $D(\mathcal{G}_{s,m,k})$ be the space of all functions $u \in L^2(I, m)$ which have a continuous version u (we use the same symbol) satisfying the following conditions:

($\mathcal{G}$-i) There exist constants A, B and a function $f_u \in L^2(I, m)$ such that

$$u(x) = A + Bs(x) + \int_{(c_o,x]} \{s(x) - s(y)\} f_u(y)\, dm(y) + \int_{(c_o,x]} \{s(x) - s(y)\} u(y)\, dk(y), \qquad x \in I. \tag{2.1}$$

($\mathcal{G}$-ii) If l_i is (s,m,k)-regular, then $u(l_i)=0$ for each $i=1,2$.

By (2.1), f_u is uniquely determined as a function in $L^2(I,m)$ if it exists. The operator $\mathcal{G}_{s,m,k}$ from $D(\mathcal{G}_{s,m,k})$ into $L^2(I,m)$ is defined by $\mathcal{G}_{s,m,k}u=f_u$, and it is called the one-dimensional generalized diffusion operator with scale function s, speed measure m, and killing measure k. The condition ($\mathcal{G}$-ii) implies that the absorbing boundary condition is posed at the regular boundary.

We denote by $\mathbb{D}_{s,m,k}=[X(t),P_x]$ the one-dimensional diffusion process on I whose generator is $\mathcal{G}_{s,m,k}$ with domain $D(\mathcal{G}_{s,m,k})$, that is, the end point l_i is absorbing whenever it is (s,m,k)-regular $(i=1,2)$. Further we denote by $p(t,x,y)$ the transition probability density with respect to dm for $\mathbb{D}_{s,m,k}$, that is,

$$P_x(X(t)\in E)=\int_E p(t,x,y)\,dm(y),\qquad t>0,\ x\in I,\ E\in\mathcal{B}(I),$$

where $\mathcal{B}(I)$ stands for the set of all Borel subsets of I. If l_1 is (s,m,k)-regular, $p(t,x,y)$ is represented as

$$p(t,x,y)=\int_{[0,\infty)} e^{-\lambda t}\psi_o(x,\lambda)\psi_o(y,\lambda)\,d\sigma(\lambda),\qquad t>0,\ x,y\in I, \tag{2.2}$$

where $d\sigma(\lambda)$ is a Borel measure on $[0,\infty)$ satisfying

$$\int_{[0,\infty)} e^{-\lambda t}\,d\sigma(\lambda)<\infty,\qquad t>0, \tag{2.3}$$

and $\psi_o(x,\lambda)$, $x\in I$, $\lambda\ge 0$, is the solution of the integral equation

$$\psi_o(x,\lambda)=s(x)-s(l_1)+\int_{(l_1,x]}\{s(x)-s(y)\}\psi_o(y,\lambda)\{-\lambda\,dm(y)+dk(y)\}. \tag{2.4}$$

It is well known that (2.4) has a unique solution. If l_1 is not (s,m,k)-regular, $p(t,x,y)$ is not always representable as in (2.2) with $d\sigma(\lambda)$ satisfying (2.3) and $\psi_o(x,\lambda)$ the solution of an integral equation. If l_1 is an (s,m,k)-entrance point, we give a sufficient condition for $p(t,x,y)$ to have a representation (2.2). Let $\psi(x,\lambda)$, $x\in I$, $\lambda\ge 0$, be the solution of the integral equation

$$\psi(x,\lambda)=1+\int_{(l_1,x]}\{s(x)-s(y)\}\psi(y,\lambda)\{-\lambda\,dm(y)+dk(y)\}. \tag{2.5}$$

Proposition 2.1. *Assume that l_1 is an (s,m,k)-entrance point and*

$$\int_{(l_1,c_o]}\{s(c_o)-s(x)\}^2\,dm(x)<\infty. \tag{2.6}$$

Then $p(t,x,y)$ can be represented as in (2.2) with $d\sigma(\lambda)$ satisfying (2.3) and $\psi_o(x,\lambda)$ replaced by $\psi(x,\lambda)$ which is the solution of the integral equation (2.5).

We show Proposition 2.1 in Section 2.4.

We next record some estimates for the solution of (2.4) or (2.5). If l_1 is (s,m,k)-regular or an (s,m,k)-exit point, and $\psi_o(x,\lambda)$ is the solution of (2.4), then

$$|\psi_o(x,\lambda)| \le \{s(x)-s(l_1)\}\exp\biggl\{\int_{(l_1,x]} ds(y)\int_{(y,x]}(\lambda dm(z)+dk(z))\biggr\}, \tag{2.7}$$

$$|D_s\psi_o(x,\lambda)| \le \exp\biggl\{\int_{(l_1,x]} ds(y)\int_{(y,x]}(\lambda dm(z)+dk(z))\biggr\}. \tag{2.8}$$

If l_1 is an (s,m,k)-entrance point, and $\psi(x,\lambda)$ is the solution of (2.5), then

$$|\psi(x,\lambda)| \le \exp\biggl\{\int_{(l_1,x]}(\lambda dm(y)+dk(y))\int_{(y,x]} ds(z)\biggr\}, \tag{2.9}$$

$$|D_s\psi(x,\lambda)| \le \int_{(l_1,x]}(\lambda dm(y)+dk(y))\exp\biggl\{\int_{(l_1,x]}(\lambda dm(y)+dk(y))\int_{(y,x]} ds(z)\biggr\}. \tag{2.10}$$

It is easy to get (2.7) with (2.10), so we omit the proof.

For $\alpha \ge 0$ and $i=1,2$, let $g_i(\cdot,\alpha)$ be a function on I with the following properties:

$$g_i(x,\alpha) \text{ is positive and continuous in } x, \tag{2.11}$$

$$g_1(x,\alpha) \text{ is nondecreasing in } x, \tag{2.12}$$

$$g_2(x,\alpha) \text{ is nonincreasing in } x, \tag{2.13}$$

$$g_i(l_i,\alpha)=0 \text{ if } |s(l_i)|<\infty, \tag{2.14}$$

$$\begin{aligned} g_i(x,\alpha) &= g_i(c_o,\alpha)+D_s g_i(c_o,\alpha)\{s(x)-s(c_o)\} \\ &\quad + \int_{(c_o,x]}\{s(x)-s(y)\}g_i(y,\alpha)\{\alpha dm(y)+dk(y)\}, \qquad x\in I. \end{aligned} \tag{2.15}$$

Here $D_s g_i(x,\alpha)=\lim_{\varepsilon\downarrow 0}\{g_i(x+\varepsilon,\alpha)-g_i(x,\alpha)\}/\{s(x+\varepsilon)-s(x)\},\ i=1,2$. It is known that there exist functions with the above properties (see [5, Section 4.6]).

Combining some properties of $g_i(x,\alpha)$ given in [5, Section 4.6] with Lemma 3.2 of [11], we obtain the following.

Proposition 2.2. *Assume that k is not a null measure or $\alpha>0$. Then*

$$\biggl|\int_{(l_i,c_o]} g_i(x,\alpha)^{-2}\,ds(x)\biggr| = \infty, \tag{2.16}$$

$$\biggl|\int_{(l_i,c_o]} g_j(x,\alpha)^{-2}\,ds(x)\biggr| < \infty, \tag{2.17}$$

where $i,j=1,2$ and $i\ne j$.

Now we set $W(\alpha) = D_s g_1(x,\alpha) g_2(x,\alpha) - g_1(x,\alpha) D_s g_2(x,\alpha)$. Note that $W(\alpha)$ is a positive number independent of $x \in I$. We put

$$G(\alpha,x,y) = G(\alpha,y,x) = W(\alpha)^{-1} g_1(x,\alpha) g_2(y,\alpha)$$

for $\alpha > 0$, $x, y \in I$, $x \le y$. Then $G(\alpha,x,y)$ is the α-Green function corresponding to $\mathbb{D}_{s,m,k}$ and

$$G(\alpha,x,y) = \int_0^\infty e^{-\alpha t} p(t,x,y)\, dt, \qquad \alpha > 0,\ x, y \in I.$$

It is easy to see that, if $k \neq 0$, then $G(0,x,y)$ exists and is given by

$$G(0,x,y) = G(0,y,x) = W^{-1} g_1(x) g_2(y), \qquad x, y \in I,\ x \le y,$$

where $g_i(x) = g_i(x,0)$, $i = 1,2$, and $W = D_s g_1(x) g_2(x) - g_1(x) D_s g_2(x)$, which is a positive constant independent of $x \in I$. In the case $k = 0$, $G(0,x,y) \in (0,\infty)$ exists if and only if $|s(l_i)| < \infty$ for $i = 1$ or 2.

§2.2. Inverse local time

We next consider the case that the end point l_i is reflecting whenever it is (s,m,k)-regular $(i = 1,2)$. More precisely, let $D(\mathcal{G}^*_{s,m,k})$ be the space of all functions $u \in L^2(I,m)$ which have a continuous version u (we use the same symbol) satisfying ($\mathcal{G}$-i) and

($\mathcal{G}$-iii) If l_i is (s,m,k)-regular, then $D_s u(l_i) = 0$ for each $i = 1,2$.

(2.1) implies that f_u is uniquely determined as a function in $L^2(I,m)$ if it exists, in this case, too. The operator $\mathcal{G}^*_{s,m,k}$ from $D(\mathcal{G}^*_{s,m,k})$ into $L^2(I,m)$ is defined by $\mathcal{G}^*_{s,m,k} u = f_u$. The condition ($\mathcal{G}$-iii) implies that the reflecting boundary condition is posed at the regular boundary.

Let $\mathbb{D}^*_{s,m,k} = [X(t), P^*_x]$ be the one-dimensional diffusion process on I whose generator is $\mathcal{G}^*_{s,m,k}$ with domain $D(\mathcal{G}^*_{s,m,k})$, that is, the end point l_i is reflecting whenever it is (s,m,k)-regular $(i = 1,2)$.

Now we assume that the killing measure is null, and l_1 is $(s,m,0)$-regular. We consider the local time $l(t,\xi)$, that is,

$$\int_0^t f(X(u))\, du = \int_I l(t,\xi)\, dm(\xi), \qquad t > 0,$$

for bounded continuous functions f on I. Since $l(t,\xi)$ is continuous and non-decreasing in t P^*_x-a.s., the right continuous inverse function $l^{-1}(t,\xi)$ exists. In particular, we denote by $\tau^*(t)$ the inverse local time $l^{-1}(t,l_1)$ at the end point l_1. The following result was obtained by Itô and McKean (see [5, Section 6.2]).

Proposition 2.3 ([5]). *Assume the following conditions hold:*

(2.18) *The killing measure is null.*

(2.19) *l_1 is $(s, m, 0)$-regular and reflecting.*

(2.20) $s(l_2) = \infty$, *or* l_2 *is $(s, m, 0)$-regular and reflecting.*

Then $[\tau^*(t),\, t \geq 0]$ is a Lévy process and there is a Lévy measure density $n^*(\xi)$ such that

$$E^*_{l_1}[e^{-\lambda\tau^*(t)}] = \exp\left\{-t\int_0^\infty (1 - e^{-\lambda\xi})n^*(\xi)\, d\xi\right\}, \tag{2.21}$$

$$n^*(\xi) = \lim_{x\to l_1} q^*(\xi, x)/\{s(x) - s(l_1)\}, \tag{2.22}$$

where $E^*_{l_1}$ stands for the expectation with respect to $P^*_{l_1}$,

$$\int_0^t q^*(\xi, x)\, d\xi = P^*_x(\sigma_{l_1} < t), \qquad x \in I,\, t > 0, \tag{2.23}$$

and σ_{l_1} is the first hitting time for l_1. In particular, if $s(l_2) = \infty$, then

$$n^*(\xi) = \lim_{x,y\to l_1} D_{s(x)}D_{s(y)}p(\xi, x, y) = \int_{[0,\infty)} e^{-\lambda\xi}\, d\sigma(\lambda), \tag{2.24}$$

where $p(t, x, y)$ is the transition probability density with respect to dm for $\mathbb{D}_{s,m,k}$, and $d\sigma(\lambda)$ is the Borel measure appearing in the representation (2.2) satisfying (2.3).

Proof. In Section 6.2 of [5], (2.21) with (2.22) is obtained under the assumption

$$P^*_x(\sigma_{l_1} < \infty) = 1, \qquad x \in I. \tag{2.25}$$

In view of Problem 4.6.6 of [5], (2.25) holds true if and only if (2.18) and (2.20) are satisfied. Thus we obtain (2.21) and (2.22).

Assume $s(l_2) = \infty$. Then

$$q^*(t, x) = \lim_{z\to l_1} p(t, z, x)/\{s(z) - s(l_1)\}, \qquad t > 0,\, x \in I. \tag{2.26}$$

Since $p(t, x, y)$ has the representation (2.2) and $\psi_o(x, \lambda)$ is the solution of (2.4), we have

$$\lim_{x\to l_1} D_s\psi_o(x, \lambda) = 1,$$
$$|D_s\psi_o(x, \lambda)| \leq \exp\{\lambda(s(x) - s(l_1))(m(x) - m(l_1))\},$$

by means of (2.8). Therefore we obtain (2.24) by virtue of (2.22), (2.26), and the dominated convergence theorem. □

We next give a simple convergence theorem for a sequence of Lévy measure densities. For $j = 0, 1, 2, \dots$, let $\mathbb{D}^{*j}$ be the diffusion process on $I^j = (l_1^j,\ l_2^j)$ with scale function s^j, speed measure m^j and no killing.

Theorem 2.4. *Assume the following conditions hold:*

$$\begin{aligned} &\lim_{j\to\infty} l_i^j = l_i^0, \qquad i = 1, 2, \\ &\lim_{j\to\infty} s^j(x) = s^0(x), \qquad x \in I^0, \\ &\lim_{j\to\infty} m^j(x) = m^0(x), \qquad x \in \mathcal{C}(m^0), \end{aligned}$$

where $\mathcal{C}(m^0)$ stands for the set of continuity points of m^0. Further for $j=0,1,2,\dots$, assume that

$$l_1^j \textit{ is } (s^j, m^j, 0)\textit{-regular and reflecting}, \qquad s^j(l_2^j) = \infty.$$

Let n^{*j} be the Lévy measure density corresponding to the inverse local time at l_1^j for $\mathbb{D}^{*j}$. Then

$$\lim_{j\to\infty} n^{*j}(\xi) = n^{*0}(\xi), \qquad \xi > 0. \tag{2.27}$$

Proof. For $j = 0, 1, 2, \dots$, let $\mathbb{D}^j$ be the diffusion process on I^j with scale function s^j, speed measure m^j and no killing, and with l_1^j being absorbing. The transition probability density $p^j(t, x, y)$ with respect to dm^j is represented as

$$p^j(t, x, y) = \int_{[0,\infty)} e^{-\lambda t} \psi_o^j(x, \lambda) \psi_o^j(y, \lambda)\, d\sigma^j(\lambda), \qquad t > 0,\ x, y \in I^j,$$

where $d\sigma^j(\lambda)$ satisfies (2.3) with σ replaced by σ^j, and ψ_o^j is the solution of (2.4) with $s = s^j$, $m = m^j$, and $k = 0$. By Proposition 2.3, n^{*j} is represented as (2.24), that is,

$$n^{*j}(\xi) = \int_{[0,\infty)} e^{-\lambda\xi}\, d\sigma^j(\lambda).$$

In the same way as in the proof of Lemma 5.3 of [8], we obtain

$$\lim_{j\to\infty} \int_{[0,\infty)} e^{-\lambda\xi}\, d\sigma^j(\lambda) = \int_{[0,\infty)} e^{-\lambda\xi}\, d\sigma^0(\lambda),$$

which implies (2.27). □

§2.3. Harmonic transform of $\mathcal{G}_{s,m,k}$

For $\beta \geq 0$, let $h_\beta(\,\cdot\,)$ be a positive continuous function on I satisfying

$$h_\beta(x) = h_\beta(c_o) + D_s h_\beta(c_o)\{s(x) - s(c_o)\} + \int_{(c_o,x]} \{s(x) - s(y)\} h_\beta(y)\{\beta dm(y) + dk(y)\}, \qquad x \in I. \tag{2.28}$$

Such a function exists. Indeed, it can be represented as a linear combination of $g_i(\,\cdot\,,\beta)$, $i = 1, 2$.

Let $\mathcal{H}_{s,m,k,\beta}$ be the set of all positive functions h_β satisfying (2.28). For $h \in \mathcal{H}_{s,m,k,\beta}$, we consider s_h and m_h defined by (1.1) and (1.2), respectively. Let $\mathcal{G}_{s_h,m_h,0}$ be the diffusion operator with scale function s_h, speed measure m_h, and null killing measure, which is the harmonic transform of $\mathcal{G}_{s,m,k}$ based on $h \in \mathcal{H}_{s,m,k,\beta}$. Let $\mathbb{D}_{s_h,m_h,0}$ be the one-dimensional diffusion process on I with generator $\mathcal{G}_{s_h,m_h,0}$ and with the end point l_i being absorbing whenever it is $(s_h, m_h, 0)$-regular ($i = 1, 2$). Further we denote by $p^{(h)}(t, x, y)$ the transition probability density with respect to dm_h for $\mathbb{D}_{s_h,m_h,0}$. By Proposition 2.2 of [11],

$$p^{(h)}(t, x, y) = e^{-\beta t} p(t, x, y)/h(x)h(y), \qquad t > 0,\ x, y \in I. \tag{2.29}$$

If l_1 is $(s_h, m_h, 0)$-regular, $p^{(h)}(t, x, y)$ is represented as

$$p^{(h)}(t, x, y) = \int_{[0,\infty)} e^{-\lambda t} \psi_o^{(h)}(x, \lambda) \psi_o^{(h)}(y, \lambda)\, d\sigma^{(h)}(\lambda), \qquad t > 0,\ x, y \in I, \tag{2.30}$$

where $d\sigma^{(h)}(\lambda)$ is a Borel measure on $[0,\infty)$ satisfying

$$\int_{[0,\infty)} e^{-\lambda t}\, d\sigma^{(h)}(\lambda) < \infty, \qquad t > 0, \tag{2.31}$$

and $\psi_o^{(h)}(x, \lambda)$, $x \in I$, $\lambda \geq \beta$, is the solution of the integral equation (2.4) with s, m, k replaced by $s_h, m_h, 0$, respectively.

Remark 2.5. We can show $d\sigma^{(h)}(\lambda) = 0$ on $[0, \beta)$. The proof is the same as the proof of Proposition 2.1 below, so we omit it.

§2.4. Proof of Proposition 2.1

First we note the following.

Lemma 2.6. *Assume that l_1 is an (s, m, k)-entrance point. Then the following conditions are equivalent:*

(i) (2.6) *holds true.*

(ii) *There exists a* $\beta > 0$ *such that*

$$\int_{(l_1,c_o]} g_2(x,\beta)^2\,dm(x) < \infty.$$

(iii) *For any* $\beta > 0$,

$$\int_{(l_1,c_o]} g_2(x,\beta)^2\,dm(x) < \infty.$$

(iv) *There exist* $\beta > 0$ *and* $h \in \mathcal{H}_{s,m,k,\beta}$ *such that* $h(l_1) = \infty$ *and* $|m_h(l_1)| < \infty$.

(v) $|m_h(l_1)| < \infty$ *for any* $\beta > 0$ *and* $h \in \mathcal{H}_{s,m,k,\beta}$.

Proof. Since l_1 is an (s,m,k)-entrance point, $|m_h(l_1)| < \infty$ and $|D_s g_2(l_1,\beta)| \in (0,\infty)$ for $\beta > 0$ (see [5, Section 4.6]). Therefore (i)⇔(ii)⇔(iii) follows.

Let $h \in \mathcal{H}_{s,m,k,\beta}$. Since h can be represented as a linear combination of $g_i(\,\cdot\,,\beta)$, $i = 1,2$, and $g_2(l_1,\beta) = \infty$,

$$h(l_1) = \infty \;\Leftrightarrow\; h(x) = C_1 g_1(x,\beta) + C_2 g_2(x,\beta) \text{ with } C_1 \geq 0 \text{ and } C_2 > 0.$$

Therefore (ii)⇔(iv)⇔(v) follows. □

Remark 2.7. If $k \neq 0$ or $\beta > 0$, then

$$\int_{(l_1,c_o]} g_1(x,\beta)^2\,dm(x) < \infty, \qquad g_2(l_1,\beta) = \infty, \qquad |D_s g_2(l_1,\beta)| \in (0,\infty).$$

If $k = 0$ and $\beta = 0$, then

$$g_1(x,0) = C_1, \qquad g_2(x,0) = \begin{cases} C_2 & \text{if } s(l_2) = \infty, \\ C_2\{s(l_2) - s(x)\} & \text{if } s(l_2) < \infty, \end{cases}$$

where C_1 and C_2 are positive constants (see [5, Section 4.6]; see also [10, Proposition 2.1]). Therefore the lemma holds true with $\beta \geq 0$ in place of $\beta > 0$ in (iii)–(v). In particular, if $k \neq 0$ or $s(l_2) < \infty$, the lemma holds true with $\beta \geq 0$ in place of $\beta > 0$ in (ii)–(v).

Proof of Proposition 2.1. Assume that l_1 is an (s,m,k)-entrance point, and (2.6) holds. By Lemma 2.6, there exist $\beta > 0$ and $h \in \mathcal{H}_{s,m,k,\beta}$ such that $h(l_1) = \infty$ and $|m_h(l_1)| < \infty$. Then l_1 is $(s_h, m_h, 0)$-regular by Theorem 1.1 of [9]. Therefore $p^{(h)}(t,x,y)$ is represented as (2.30) with $d\sigma^{(h)}(\lambda)$ satisfying (2.31) and $\psi_o^{(h)}(x,\lambda)$, $x \in I$, $\lambda \geq 0$, the solution of the integral equation (2.4) with s, m, k replaced by $s_h, m_h, 0$, respectively. First we note that

$$\int_{[0,\beta)} d\sigma^{(h)}(\lambda) = 0. \tag{2.32}$$

Indeed, if $\int_{[0,\beta)} d\sigma^{(h)}(\lambda) > 0$, then by (2.29),

$$\begin{aligned} p(t,x,x) &= e^{\beta t} h(x)^2 p^{(h)}(t,x,x) \\ &\geq h(x)^2 \int_{[0,\beta)} e^{(\beta-\lambda)t} \psi_o^{(h)}(x,\lambda)^2 \, d\sigma^{(h)}(\lambda) \to \infty \qquad \text{as } t \to \infty. \end{aligned}$$

This contradicts the existence of $\lim_{t\to\infty} p(t,x,x) \in [0,\infty)$.

By (2.30) and (2.32), we have

$$p(t,x,y) = \int_{[0,\infty)} e^{-\lambda t} \widetilde{\psi}(x,\lambda) \widetilde{\psi}(y,\lambda) \, d\widetilde{\sigma}(\lambda), \tag{2.33}$$

where $\widetilde{\psi}(x,\lambda) = h(x)\psi_o^{(h)}(x,\lambda+\beta)$ and $d\widetilde{\sigma}(\lambda) = d_\lambda \sigma^{(h)}(\lambda+\beta)$. By (2.4) and Lemma 5.1 of [10],

$$\widetilde{\psi}(l_1,\lambda) = -1/D_s h(l_1) \in (0,\infty), \tag{2.34}$$

$$D_s \widetilde{\psi}(l_1,\lambda) = 0. \tag{2.35}$$

Since $h(x)$ satisfies (2.28) and $\psi_o^{(h)}(\xi,\lambda+\beta)$ satisfies (2.4) with s,m,k,λ replaced by $s_h, m_h, 0, \lambda+\beta$, respectively, we have

$$D_s \widetilde{\psi}(y,\lambda) - D_s \widetilde{\psi}(x,\lambda) = \int_{(x,y]} \widetilde{\psi}(\xi,\lambda) \{-\lambda dm(\xi) + dk(\xi)\} \tag{2.36}$$

for $l_1 < x \leq y < l_2$. Noting $\int_{[0,\infty)} e^{-\lambda t} \, d\widetilde{\sigma}(\lambda) < \infty$ and putting $\psi(x,\lambda) = -D_s h(l_1) \widetilde{\psi}(x,\lambda)$ and $d\sigma(\lambda) = \{D_s h(l_1)\}^{-2} \, d\widetilde{\sigma}(\lambda)$, we obtain the conclusion from (2.33)–(2.36). □

§3. Inverse local time for $\mathbb{D}^*_{s_h,m_h,0}$

Let $\mathbb{D}_{s,m,k}$ be the diffusion process on I with generator $\mathcal{G}_{s,m,k}$ and with the end point l_i being absorbing whenever it is (s,m,k)-regular $(i=1,2)$. We denote by $p(t,x,y)$ the transition probability density with respect to dm.

Let $h \in \mathcal{H}_{s,m,k,\beta}$ and let $\mathcal{G}_{s_h,m_h,0}$ be the harmonic transform of $\mathcal{G}_{s,m,k}$ based on h. Let $\mathbb{D}_{s_h,m_h,0}$ be the diffusion process on I with generator $\mathcal{G}_{s_h,m_h,0}$ and with the end point l_i being absorbing whenever it is $(s_h,m_h,0)$-regular $(i=1,2)$. We denote by $p^{(h)}(t,x,y)$ the transition probability density with respect to dm_h.

When l_i is $(s_h,m_h,0)$-regular, it is possible to pose the reflecting boundary condition at l_i $(i=1,2)$. Let $\mathbb{D}^*_{s_h,m_h,0} = [X(t), P_x^{(h*)}]$ be the diffusion process on I whose generator is $\mathcal{G}^*_{s_h,m_h,0}$ with domain $D(\mathcal{G}^*_{s_h,m_h,0})$, that is, the end point l_i is reflecting whenever it is $(s_h,m_h,0)$-regular $(i=1,2)$.

Now we assume that l_1 is $(s_h, m_h, 0)$-regular. We consider the local time $l^{(h)}(t,\xi)$, that is,

$$\int_0^t f(X(u))\,du = \int_I l^{(h)}(t,\xi)\,dm_h(\xi), \qquad t>0,$$

for bounded continuous functions f on I. We denote by $\tau^{(h*)}(t)$ the inverse local time $(l^{(h)})^{-1}(t,l_1)$ at the end point l_1.

The following result is an immediate consequence of Proposition 2.3.

Proposition 3.1. *Assume the following conditions hold:*

(3.1) $\quad l_1$ *is* $(s_h, m_h, 0)$*-regular and reflecting.*

(3.2) $\quad s_h(l_2) = \infty, \quad$ *or* $\quad l_2$ *is* $(s_h, m_h, 0)$*-regular and reflecting.*

Then $[\tau^{(h*)}(t),\, t \ge 0]$ *is a Lévy process and there is a Lévy measure density* $n^{(h*)}(\xi)$ *such that*

$$E_{l_1}^{(h*)}[e^{-\lambda\tau^{(h*)}(t)}] = \exp\left\{-t\int_0^\infty (1-e^{-\lambda\xi})n^{(h*)}(\xi)\,d\xi\right\}, \tag{3.3}$$
$$n^{(h*)}(\xi) = \lim_{x\to l_1} q^{(h*)}(\xi,x)/\{s_h(x)-s_h(l_1)\},$$

where $E_{l_1}^{(h*)}$ *stands for the expectation with respect to* $P_{l_1}^{(h*)}$*, and*

$$\int_0^t q^{(h*)}(\xi,x)\,d\xi = P_x^{(h*)}(\sigma_{l_1} < t), \qquad t>0,\ x\in I.$$

In particular, if $s_h(l_2) = \infty$*, then*

$$n^{(h*)}(\xi) = \int_{[0,\infty)} e^{-\xi\lambda}\,d\sigma^{(h)}(\lambda) \tag{3.4}$$
$$= \lim_{x,y\to l_1} D_{s_h(x)} D_{s_h(y)} p^{(h)}(\xi,x,y), \qquad \xi>0. \tag{3.5}$$

Note that $p^{(h)}(t,x,y)$ has the representation (2.30) and $d\sigma^{(h)}(\lambda)$ satisfies (2.31).

Now we give a representation of $n^{(h*)}(\xi)$ in terms of data corresponding to the diffusion process $\mathbb{D}_{s,m,k}$. By Theorem 1.1 of [9], l_1 is $(s_h, m_h, 0)$-regular if and only if one of the following conditions is satisfied:

(3.6) $\quad l_1$ is (s,m,k)-regular and $h(l_1) \in (0,\infty)$.

(3.7) $\quad l_1$ is an (s,m,k)-entrance point, $h(l_1)=\infty$, and $|m_h(l_1)|<\infty$.

(3.8) $\quad l_1$ is (s,m,k)-natural, $h(l_1)=\infty$, and $|m_h(l_1)|<\infty$.

In the case (3.6), we have the representation (2.2) with $d\sigma(\lambda)$ satisfying (2.3). In the case (3.7), we also have the representation (2.2) with $d\sigma(\lambda)$ satisfying (2.3) and $\psi_o(x,\lambda)$ replaced by $\psi(x,\lambda)$, which is the solution of the integral equation (2.5), by Proposition 2.1, Lemma 2.6, and Remark 2.7.

Theorem 3.2. *Let $h \in \mathcal{H}_{s,m,k,\beta}$. Assume one of (3.6)–(3.8) holds. Further assume that l_1 is reflecting and $s_h(l_2) = \infty$. Then (3.3) holds true, and $n^{(h*)}(\xi)$ is given by (3.4) and (3.5). In particular, if (3.6) is satisfied, then*

$$n^{(h*)}(\xi) = h(l_1)^2 e^{-\beta\xi} \int_{[0,\infty)} e^{-\xi\lambda}\, d\sigma(\lambda) \tag{3.9}$$

$$= h(l_1)^2 e^{-\beta\xi} \lim_{x,y\to l_1} D_{s(x)} D_{s(y)} p(\xi,x,y). \tag{3.10}$$

If (3.7) is satisfied, then

$$n^{(h*)}(\xi) = D_s h(l_1)^2 e^{-\beta\xi} \int_{[0,\infty)} e^{-\xi\lambda}\, d\sigma(\lambda) \tag{3.11}$$

$$= D_s h(l_1)^2 e^{-\beta\xi} \lim_{x,y\to l_1} p(\xi,x,y). \tag{3.12}$$

Proof. Since one of (3.6)–(3.8) holds, (3.1) is satisfied. Therefore by Proposition 3.1, (3.3) holds true, and $n^{(h*)}(\xi)$ is given by (3.4) and (3.5).

Assume (3.6) holds. Then we have (2.29), and by (5.14) of [10], $d\sigma^{(h)}(\lambda)$ is a Borel measure on $[\beta,\infty)$ and $d\sigma^{(h)}(\lambda) = h(l_1)^2 d_\lambda\sigma(\lambda-\beta)$. Therefore (3.9) follows from (3.4). (3.10) follows from (2.2), (2.4), (2.8), and the dominated convergence theorem.

Assume (3.7) holds. Then we have (2.29), and by (5.18) of [10], $d\sigma^{(h)}(\lambda)$ is a Borel measure on $[\beta,\infty)$ and $d\sigma^{(h)}(\lambda) = \{D_s h(l_1)\}^2 d_\lambda\sigma(\lambda-\beta)$. Therefore (3.11) follows from (3.4). (3.12) follows from (2.2), (2.5), (2.9), and the dominated convergence theorem.

Thus the proof is complete. □

Finally we consider the conditions of Theorem 3.2. The following result shows that $h \in \mathcal{H}_{s,m,k,\beta}$ satisfying the conditions of Theorem 3.2 must be $g_2(x,\beta)$ whenever l_1 is (s,m,k)-regular or an (s,m,k)-entrance point.

Proposition 3.3. *Suppose that l_1 is (s,m,k)-regular or an (s,m,k)-entrance point, and $k \neq 0$ or $\beta > 0$. Let $h \in \mathcal{H}_{s,m,k,\beta}$. Then the following conditions are equivalent:*

(i) *(2.6) is satisfied and $h(x) = Cg_2(x,\beta)$ for some positive constant C.*

(ii) *(3.6) or (3.7) is satisfied, and $s_h(l_2) = \infty$.*

Proof. Since the statement is obvious when l_1 is regular, we only prove it when l_1 is an entrance point.

(i)⇒(ii). By Proposition 2.2, we get $s_h(l_2) = \infty$. By Lemma 2.6 and Remark 2.7, (2.6) is equivalent to $\int_{(l_1,c_o]} g_2(x,\beta)^2\, dm(x) < \infty$. Thus we obtain (3.7).

(ii)⇒(i). By Lemma 2.6 and Remark 2.7, (3.7) implies (2.6).

We note that $h(x) = C_1 g_1(x,\beta) + C_2 g_2(x,\beta)$. If $C_1 > 0$, by using Lemma 2.2 we have

$$\int_{(c_o,l_2)} h(x)^{-2}\, ds(x) \leq C_1^{-2} \int_{(c_o,l_2)} g_1(x,\beta)^{-2}\, ds(x) < \infty.$$

This contradicts $s_h(l_2) = \infty$. Therefore $C_1 = 0$ and $C_2 > 0$. □

Noting Remark 2.7, we easily obtain the following result, so we omit the proof.

Proposition 3.4. *Assume that the killing measure is null, l_1 is $(s,m,0)$-regular or an $(s,m,0)$-entrance point,* (2.6) *is satisfied, and $s(l_2) < \infty$. Take $h(x) = C\{s(l_2) - s(x)\}$ $(\in \mathcal{H}_{s,m,0,0})$, where C is a positive constant. Then* (3.6) *or* (3.7) *is satisfied, and $s_h(l_2) = \infty$.*

§4. Examples

In this section, we present some interesting diffusion operators including the $\mathcal{L}_i$'s given by (1.3)–(1.5) and (1.9). Then, using Proposition 2.3 and Theorem 3.2, we deduce the corresponding Lévy measure densities. Here we use the following functions.

The Bessel function $J_\nu(x)$:

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{n!\,\Gamma(\nu+n+1)}. \tag{4.1}$$

The modified Bessel functions $I_\nu(x)$ and $K_\nu(x)$:

$$I_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(x/2)^{2n}}{n!\,\Gamma(\nu+n+1)}, \qquad \nu > -1; \tag{4.2}$$

for n an integer,

$$\begin{aligned} K_n(x) = K_{-n}(x) = (-1)^{n+1} I_n(x)\{\gamma + \log(x/2)\} \\ + \frac{(-1)^n}{2} \sum_{k=0}^{\infty} \frac{(x/2)^{n+2k}}{k!(n+k)!} \left[\sum_{m=1}^{k} \frac{1}{m} + \sum_{m=1}^{k+n} \frac{1}{m}\right] \\ + \frac{1}{2} \sum_{r=0}^{n-1} (-1)^r \frac{(n-r-1)!}{r!} \left(\frac{x}{2}\right)^{2r-n}, \end{aligned}$$

where γ is Euler's constant, i.e. $\gamma = 0.57721\ldots$; in case ν is not an integer,

$$K_\nu(x) = \frac{\pi}{2}\,\frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi}.$$

The Whittaker functions $M_{k,l}(x)$ and $W_{k,l}(x)$:

$$\begin{aligned}
M_{k,l}(x) &= x^{l+1/2}e^{-x/2}M(l-k+1/2, 2l+1, x)\\
&= x^{l+1/2}e^{-x/2}\sum_{n=0}^{\infty}\frac{\Gamma(2l+1)\Gamma(l-k+n+1/2)}{\Gamma(2l+n+1)\Gamma(l-k+1/2)}\,\frac{x^n}{n!},\\
W_{k,l}(x) &= x^{l+1/2}e^{-x/2}U(l-k+1/2, 2l+1, x)\\
&= \frac{\Gamma(-2l)}{\Gamma(1/2-l-k)}M_{k,l}(x) + \frac{\Gamma(2l)}{\Gamma(1/2+l-k)}M_{k,-l}(x),
\end{aligned}$$

where

$$\begin{aligned}
M(a,b,x) &= 1 + \sum_{k=1}^{\infty}\frac{a(a+1)\cdots(a+k-1)x^k}{b(b+1)\cdots(b+k-1)k!},\\
U(a,b,x) &= \frac{\pi}{\sin(\pi b)}\left[\frac{M(a,b,x)}{\Gamma(1+a-b)\Gamma(b)} - x^{1-b}\frac{M(1+a-b,2-b,x)}{\Gamma(a)\Gamma(2-b)}\right].
\end{aligned}$$

Example 4.1 (Bessel process). Let us consider the following diffusion operator on $I = (0,\infty)$:

$$\mathcal{G}_1^{(\nu)} = \frac{1}{2}\,\frac{d^2}{dx^2} + \frac{2\nu+1}{2x}\,\frac{d}{dx},$$

where $-\infty < \nu < \infty$. This is the Bessel operator, and the scale function $s_1^{(\nu)}$ and the speed measure $m_1^{(\nu)}$ are given by

$$ds_1^{(\nu)}(x) = x^{-2\nu-1}\,dx, \qquad dm_1^{(\nu)}(x) = 2x^{2\nu+1}\,dx.$$

The killing measure is null. The end point 0 is:

- an $(s_1^{(\nu)}, m_1^{(\nu)}, 0)$-entrance point if $\nu \geq 0$,
- $(s_1^{(\nu)}, m_1^{(\nu)}, 0)$-regular if $-1 < \nu < 0$,
- an $(s_1^{(\nu)}, m_1^{(\nu)}, 0)$-exit point if $\nu \leq -1$.

Further

$$\int_0^1 \{s_1^{(\nu)}(1) - s_1^{(\nu)}(x)\}^2\,dm_1^{(\nu)}(x) < \infty \;\Leftrightarrow\; |\nu| < 1. \tag{4.3}$$

The end point ∞ is $(s_1^{(\nu)}, m_1^{(\nu)}, 0)$-natural for all ν, and in particular,

$$s_1^{(\nu)}(\infty) = \infty \;\Leftrightarrow\; \nu \leq 0. \tag{4.4}$$

Let $\mathbb{D}_1^{(\nu)}$ be the diffusion process on I with generator $\mathcal{G}_1^{(\nu)}$, and with the end point 0 being absorbing if $-1 < \nu < 0$. We denote by $p_1^{(\nu)}(t,x,y)$ the transition probability density with respect to $dm_1^{(\nu)}$. It is known (see [1, p. 134]) that

$$p_1^{(\nu)}(t,x,y) = \frac{1}{2t}\exp\biggl\{-\frac{x^2+y^2}{2t}\biggr\}(xy)^{-\nu} I_{|\nu|}\biggl(\frac{xy}{t}\biggr),$$

which is represented as the Laplace transform of Bessel functions,

$$p_1^{(\nu)}(t,x,y) = \int_0^\infty e^{-\lambda t}\psi_1^{(\nu)}(x,\lambda)\psi_1^{(\nu)}(y,\lambda)\sigma_1^{(\nu)}(\lambda)\,d\lambda,$$

where

$$\psi_1^{(\nu)}(x,\lambda) = \begin{cases} \Gamma(\nu+1)(2/\lambda)^{\nu/2}x^{-\nu}J_\nu(\sqrt{2\lambda}\,x), & \nu \ge 0,\\ \Gamma(|\nu|)2^{-1}(2/\lambda)^{|\nu|/2}x^{|\nu|}J_{|\nu|}(\sqrt{2\lambda}\,x), & \nu < 0,\end{cases}$$

$$\sigma_1^{(\nu)}(\lambda) = \begin{cases} 2^{-\nu-1}\Gamma(\nu+1)^{-2}\lambda^{\nu}, & \nu \ge 0,\\ 2^{1-|\nu|}\Gamma(|\nu|)^{-2}\lambda^{|\nu|}, & \nu < 0,\end{cases}$$

(see [6, p. 200]). For $\alpha > 0$ we denote by $g_{1,i}^{(\nu)}(x,\alpha)$, $i = 1,2$, the functions satisfying (2.11)–(2.15) with $s = s_1^{(\nu)}$, $m = m_1^{(\nu)}$ and $k = 0$. It is known (see [1, p. 133]) that

$$g_{1,1}^{(\nu)}(x,\alpha) = x^{-\nu}I_{|\nu|}(\sqrt{2\alpha}\,x), \qquad g_{1,2}^{(\nu)}(x,\alpha) = x^{-\nu}K_{|\nu|}(\sqrt{2\alpha}\,x).$$

Now we consider the Lévy measure densities corresponding to the inverse local time at the end point 0.

(1) Let $-1 < \nu < 0$. Then the end point 0 is $(s_1^{(\nu)}, m_1^{(\nu)}, 0)$-regular. We pose the reflecting boundary condition at 0. We denote by $\mathbb{D}_1^{(\nu,*)}$ the diffusion process with generator $\mathcal{G}_1^{(\nu)}$ and with the end point 0 being reflecting. We denote by $n_1^{(\nu,*)}$ the corresponding Lévy measure density. Since $s_1^{(\nu)}(\infty) = \infty$ by (4.4), from (2.24) we deduce

$$\begin{aligned} n_1^{(\nu,*)}(\xi) &= \lim_{x,y\to 0} D_{s_1^{(\nu)}(x)} D_{s_1^{(\nu)}(y)} p_1^{(\nu)}(\xi,x,y) \\ &= \int_0^\infty e^{-\xi\lambda}\sigma_1^{(\nu)}(\lambda)\,d\lambda = 2^{-|\nu|+1}\frac{|\nu|}{\Gamma(|\nu|)}\xi^{-(|\nu|+1)}. \end{aligned} \tag{4.5}$$

(2) Let $-1 < \nu < 1$. Then the end point 0 is $(s_1^{(\nu)}, m_1^{(\nu)}, 0)$-regular or an $(s_1^{(\nu)}, m_1^{(\nu)}, 0)$-entrance point, and (2.6) is satisfied in view of (4.3). For $\beta > 0$ we put $h_1^{(\nu,\beta)}(x) = (\beta/2)^{|\nu|/2} g_{1,2}^{(\nu)}(x,\beta)$ and denote by $\mathcal{G}_1^{(\nu,\beta)}$ the harmonic transform of $\mathcal{G}_1^{(\nu)}$ based on $h_1^{(\nu,\beta)} \in \mathcal{H}_{s_1^{(\nu)},m_1^{(\nu)},0,\beta}$, that is,

$$\mathcal{G}_1^{(\nu,\beta)} = \frac{1}{2}\frac{d^2}{dx^2} + \biggl\{\frac{1}{2x} + \sqrt{2\beta}\,\frac{K_\nu'(\sqrt{2\beta}\,x)}{K_\nu(\sqrt{2\beta}\,x)}\biggr\}\frac{d}{dx},$$

which is (1.3). Note that $\mathcal{G}_1^{(\nu,\beta)} = \mathcal{G}_1^{(-\nu,\beta)}$. The scale function $s_1^{(\nu,\beta)}$ and the speed measure $m_1^{(\nu,\beta)}$ are given by

$$ds_1^{(\nu,\beta)}(x) = h_1^{(\nu,\beta)}(x)^{-2}\, ds_1^{(\nu)}(x), \qquad dm_1^{(\nu,\beta)}(x) = h_1^{(\nu,\beta)}(x)^2\, dm_2^{(\nu)}(x).$$

By Proposition 3.3, (3.6) or (3.7) is satisfied, and $s_1^{(\nu,\beta)}(\infty) = \infty$. The end point 0 is $(s_1^{(\nu,\beta)}, m_1^{(\nu,\beta)}, 0)$-regular. We consider the diffusion process $\mathbb{D}_1^{(\nu,\beta,*)}$ with generator $\mathcal{G}_1^{(\nu,\beta)}$ and with the end point 0 being reflecting. Let $n_1^{(\nu,\beta,*)}$ be the corresponding Lévy measure density.

(2-i) Assume $\nu = 0$. Since $D_{s_1^{(0)}} h_1^{(0,\beta)}(0) = -1$, (3.11) implies

$$n_1^{(0,\beta,*)}(\xi) = e^{-\beta\xi} \int_0^\infty e^{-\xi\lambda} \sigma_1^{(0)}(\lambda)\, d\lambda = \frac{1}{2\xi} e^{-\beta\xi}. \tag{4.6}$$

(2-ii) Assume $-1 < \nu < 0$. Since $h_1^{(\nu,\beta)}(0) = \Gamma(|\nu|)/2 \in (0,\infty)$, (3.9) leads to

$$\begin{aligned} n_1^{(\nu,\beta,*)}(\xi) &= h_1^{(\nu,\beta)}(0)^2 e^{-\beta\xi} \int_0^\infty e^{-\xi\lambda} \sigma_1^{(\nu)}(\lambda)\, d\lambda \\ &= 2^{-|\nu|-1}\Gamma(|\nu|+1)\xi^{-(|\nu|+1)} e^{-\beta\xi}. \end{aligned} \tag{4.7}$$

(2-iii) Assume $0 < \nu < 1$. Since $\mathcal{G}_1^{(\nu,\beta)} = \mathcal{G}_1^{(-\nu,\beta)}$, by using (4.7) we get

$$n_1^{(\nu,\beta,*)}(\xi) = n_1^{(-\nu,\beta,*)}(\xi) = 2^{-\nu-1}\Gamma(\nu+1)\xi^{-(\nu+1)} e^{-\beta\xi}. \tag{4.8}$$

(4.6)–(4.8) show (1.6). As mentioned in Section 1, (1.6) was obtained by C. Donati-Martin and M. Yor [2] (see also [11]).

(3) Let $0 < \nu < 1$. Then $s_1^{(\nu)}(\infty) < \infty$ by (4.4). Therefore Proposition 3.4 leads to a special case corresponding to $\beta = 0$.

We put $h_1^{(\nu,0)}(x) = \{s_1^{(\nu)}(\infty) - s_1^{(\nu)}(x)\}/\{s_1^{(\nu)}(\infty) - s_1^{(\nu)}(1)\} = x^{-2\nu}$. Denote by $\mathcal{G}_1^{(\nu,0)}$ the harmonic transform of $\mathcal{G}_1^{(\nu)}$ based on $h_1^{(\nu,0)} \in \mathcal{H}_{s_1^{(\nu)}, m_1^{(\nu)}, 0, 0}$, that is,

$$\mathcal{G}_1^{(\nu,0)} = \frac{1}{2}\frac{d^2}{dx^2} + \frac{-2\nu+1}{2x}\frac{d}{dx}.$$

We note that this coincides with $\mathcal{G}_1^{(-\nu)}$. The scale function $s_1^{(\nu,0)}$ and the speed measure $m_1^{(\nu,0)}$ are given by

$$\begin{aligned} ds_1^{(\nu,0)}(x) &= h_1^{(\nu,0)}(x)^{-2}\, ds_1^{(\nu)}(x) = x^{2\nu-1}\, dx, \\ dm_1^{(\nu,0)}(x) &= h_1^{(\nu,0)}(x)^2\, dm_1^{(\nu)}(x) = 2x^{-2\nu+1}\, dx. \end{aligned}$$

By Proposition 3.4, (3.7) is satisfied, and $s_1^{(\nu,0)}(\infty) = \infty$. The end point 0 is $(s_1^{(\nu,0)}, m_1^{(\nu,0)}, 0)$-regular. We consider the diffusion process $\mathbb{D}_1^{(\nu,0,*)}$ with generator

$\mathcal{G}_1^{(\nu,0)}$ and with the end point 0 being reflecting. Let $n_1^{(\nu,0,*)}$ be the corresponding Lévy measure density. Since $\mathcal{G}_1^{(\nu,0)} = \mathcal{G}_1^{(-\nu)}$, (4.5) yields

$$n_1^{(\nu,0,*)}(\xi) = 2^{-|-\nu|+1}|-\nu|\Gamma(|-\nu|)^{-1}\xi^{-(|-\nu|+1)} = 2^{-\nu+1}\frac{\nu}{\Gamma(\nu)}\xi^{-\nu-1},$$

which coincides with $n_1^{(-\nu,*)}$.

Example 4.2 (Radial Ornstein–Uhlenbeck process). Let us consider the following diffusion operator on $I = (0,\infty)$:

$$\mathcal{G}_2^{(\nu,\kappa)} = \frac{1}{2}\frac{d^2}{dx^2} + \left(\frac{2\nu+1}{2x} - \kappa x\right)\frac{d}{dx},$$

where $-\infty < \nu < \infty$ and $\kappa > 0$. This is a radial Ornstein–Uhlenbeck operator, and the scale function $s_2^{(\nu,\kappa)}$ and the speed measure $m_2^{(\nu,\kappa)}$ are given by

$$ds_2^{(\nu,\kappa)}(x) = x^{-2\nu-1}e^{\kappa x^2}\,dx, \qquad dm_2^{(\nu,\kappa)}(x) = 2x^{2\nu+1}e^{-\kappa x^2}\,dx.$$

The killing measure is null. The end point 0 is:

- an $(s_2^{(\nu,\kappa)}, m_2^{(\nu,\kappa)}, 0)$-entrance point if $\nu \geq 0$,
- $(s_2^{(\nu,\kappa)}, m_2^{(\nu,\kappa)}, 0)$-regular if $-1 < \nu < 0$,
- an $(s_2^{(\nu,\kappa)}, m_2^{(\nu,\kappa)}, 0)$-exit point if $\nu \leq -1$.

Further

$$\int_0^1 \{s_2^{(\nu,\kappa)}(1) - s_2^{(\nu,\kappa)}(x)\}^2\,dm_2^{(\nu,\kappa)}(x) < \infty \iff |\nu| < 1. \tag{4.9}$$

The end point ∞ is always $(s_2^{(\nu,\kappa)}, m_2^{(\nu,\kappa)}, 0)$-natural for all ν, and

$$s_2^{(\nu,\kappa)}(\infty) = \infty. \tag{4.10}$$

Let $\mathbb{D}_2^{(\nu,\kappa)}$ be the diffusion process on I with generator $\mathcal{G}_2^{(\nu,\kappa)}$, and with the end point 0 being absorbing if $-1 < \nu < 0$. We denote by $p_2^{(\nu,\kappa)}(t,x,y)$ the transition probability density with respect to $dm_2^{(\nu,\kappa)}$. It is known (see [1, pp. 139–140]) that

$$\begin{aligned} &p_2^{(\nu,\kappa)}(t,x,y) \\ &\quad = \frac{\kappa}{2x^\nu y^\nu \sinh(\kappa t)}\exp\left\{\kappa(\nu+1)t - \frac{\kappa e^{-\kappa t}(x^2+y^2)}{2\sinh(\kappa t)}\right\} I_{|\nu|}\left(\frac{\kappa xy}{\sinh(\kappa t)}\right). \end{aligned} \tag{4.11}$$

By (4.11), we see that

$$\lim_{\kappa\to 0} p_2^{(\nu,\kappa)}(t,x,y) = p_1^{(\nu)}(t,x,y). \tag{4.12}$$

Since $\lim_{\kappa\to 0} s_2^{(\nu,\kappa)}(x) = s_1^{(\nu)}(x)$ and $\lim_{\kappa\to 0} m_2^{(\nu,\kappa)}(x) = m_1^{(\nu)}(x)$ for $x \in I$, (4.12) also follows from Lemma 5.2 of [8]. Further we note the following: if $-1 < \nu < 0$,

$$\lim_{x,y\to 0} D_{s_2^{(\nu,\kappa)}(x)} D_{s_2^{(\nu,\kappa)}(y)} p_2^{(\nu,\kappa)}(\xi, x, y) = 2^{-|\nu|+1} \frac{|\nu|}{\Gamma(|\nu|)} \left(\frac{\kappa}{\sinh(\kappa\xi)}\right)^{|\nu|+1} e^{\kappa(\nu+1)\xi}, \tag{4.13}$$

while if $\nu \geq 0$,

$$\lim_{x,y\to 0} p_2^{(\nu,\kappa)}(\xi, x, y) = 2^{-\nu-1} \frac{1}{\Gamma(\nu+1)} \left(\frac{\kappa}{\sinh(\kappa\xi)}\right)^{\nu+1} e^{\kappa(\nu+1)\xi}. \tag{4.14}$$

For $\alpha > 0$, we denote by $g_{2,i}^{(\nu,\kappa)}(x, \alpha)$, $i = 1, 2$, the functions satisfying (2.11)–(2.15) with $s = s_2^{(\nu,\kappa)}$, $m = m_2^{(\nu,\kappa)}$ and $k = 0$. It is known (see [1, p. 139]) that

$$g_{2,1}^{(\nu,\kappa)}(x, \alpha) = \frac{\alpha^{|\nu|/2}}{2^{|\nu|/2} \kappa^{(|\nu|+1)/2} \Gamma(|\nu|+1)} x^{-\nu-1} e^{\kappa x^2/2} M_{-\frac{\alpha}{2\kappa}+\frac{\nu+1}{2}, \frac{|\nu|}{2}}(\kappa x^2),$$
$$g_{2,2}^{(\nu,\kappa)}(x, \alpha) = \frac{\kappa^{|\nu|/2-1/2}}{2^{1-|\nu|/2} \alpha^{|\nu|/2}} \Gamma\left(\frac{|\nu|}{2} - \frac{\nu}{2} + \frac{\alpha}{2\kappa}\right) x^{-\nu-1} e^{\kappa x^2/2} W_{-\frac{\alpha}{2\kappa}+\frac{\nu+1}{2}, \frac{|\nu|}{2}}(\kappa x^2).$$

Now we consider the Lévy measure densities corresponding to the inverse local time at the end point 0.

(1) Let $-1 < \nu < 0$. Then the end point 0 is $(s_2^{(\nu,\kappa)}, m_2^{(\nu,\kappa)}, 0)$-regular. We pose the reflecting boundary condition at 0. We denote by $\mathbb{D}_2^{(\nu,\kappa,*)}$ the diffusion process with generator $\mathcal{G}_2^{(\nu,\kappa)}$ and with the end point 0 being reflecting. We denote by $n_2^{(\nu,\kappa,*)}$ the corresponding Lévy measure density. Since $s_2^{(\nu,\kappa)}(\infty) = \infty$ by (4.10), from (2.24) and (4.13) we derive

$$n_2^{(\nu,\kappa,*)}(\xi) = \lim_{x,y\to 0} D_{s_2^{(\nu,\kappa)}(x)} D_{s_2^{(\nu,\kappa)}(y)} p_2^{(\nu,\kappa)}(\xi, x, y) = 2^{-|\nu|+1} \frac{|\nu|}{\Gamma(|\nu|)} \left(\frac{\kappa}{\sinh(\kappa\xi)}\right)^{|\nu|+1} e^{\kappa(\nu+1)\xi}. \tag{4.15}$$

In view of Theorem 2.4 we get

$$\lim_{\kappa\to 0} n_2^{(\nu,\kappa,*)}(\xi) = n_1^{(\nu,*)}(\xi).$$

We also note that this can be directly obtained from (4.5) and (4.15).

(2) Let $-1 < \nu < 1$. Then 0 is $(s_2^{(\nu,\kappa)}, m_2^{(\nu,\kappa)}, 0)$-regular or an $(s_2^{(\nu,\kappa)}, m_2^{(\nu,\kappa)}, 0)$-entrance point, and (2.6) is satisfied in view of (4.9). For $\beta > 0$ we put $h_2^{(\nu,\kappa,\beta)}(x) =$

$(\beta/2)^{|\nu|/2} g_{2,2}^{(\nu,\kappa)}(x,\beta)$ and denote by $\mathcal{G}_2^{(\nu,\kappa,\beta)}$ the harmonic transform of $\mathcal{G}_2^{(\nu,\kappa)}$ based on $h_2^{(\nu,\kappa,\beta)} \in \mathcal{H}_{s_2^{(\nu,\kappa)}, m_2^{(\nu,\kappa)}, 0, \beta}$, that is,

$$\mathcal{G}_2^{(\nu,\kappa,\beta)} = \frac{1}{2}\frac{d^2}{dx^2} + \left\{ -\frac{1}{2x} + 2\kappa x \frac{W'_{-\frac{\beta}{2\kappa}+\frac{\nu+1}{2}, \frac{|\nu|}{2}}(\kappa x^2)}{W_{-\frac{\beta}{2\kappa}+\frac{\nu+1}{2}, \frac{|\nu|}{2}}(\kappa x^2)} \right\} \frac{d}{dx},$$

which is (1.4). The scale function $s_2^{(\nu,\kappa,\beta)}$ and the speed measure $m_2^{(\nu,\kappa,\beta)}$ are given by

$$ds_2^{(\nu,\kappa,\beta)}(x) = h_2^{(\nu,\kappa,\beta)}(x)^{-2}\, ds_2^{(\nu,\kappa)}(x), \qquad dm_2^{(\nu,\kappa,\beta)}(x) = h_2^{(\nu,\kappa,\beta)}(x)^2\, dm_2^{(\nu,\kappa)}(x).$$

By Proposition 3.3, (3.6) or (3.7) is satisfied, and $s_2^{(\nu,\kappa,\beta)}(\infty) = \infty$. The end point 0 is $(s_2^{(\nu,\kappa,\beta)}, m_2^{(\nu,\kappa,\beta)}, 0)$-regular. We consider the diffusion process $\mathbb{D}_2^{(\nu,\kappa,\beta,*)}$ with generator $\mathcal{G}_2^{(\nu,\kappa,\beta)}$ and with the end point 0 being reflecting. Let $n_2^{(\nu,\kappa,\beta,*)}$ be the corresponding Lévy measure density.

(2-i) Assume $\nu = 0$. Since $D_{s_2^{(0,\kappa)}} h_2^{(0,\kappa,\beta)}(0) = -1$ by [7, Problem 17, p. 279] and recursion (see [6, p. 73]), (3.12) and (4.14) imply

$$n_2^{(0,\kappa,\beta,*)}(\xi) = D_{s_2^{(0,\kappa)}} h_2^{(0,\kappa,\beta)}(0)^2 e^{-\beta\xi} \lim_{x,y\to 0} p_2^{(\nu,\kappa)}(\xi,x,y) = \frac{\kappa}{2\sinh(\kappa\xi)} e^{(\kappa-\beta)\xi}. \tag{4.16}$$

(2-ii) Assume $-1 < \nu < 0$. Since $h_2^{(\nu,\kappa,\beta)}(0) = \Gamma(|\nu|)/2$, (3.10) and (4.13) imply

$$n_2^{(\nu,\kappa,\beta,*)}(\xi) = 2^{-|\nu|-1}\Gamma(|\nu|+1)\left(\frac{\kappa}{\sinh(\kappa\xi)}\right)^{|\nu|+1} e^{\{\kappa(\nu+1)-\beta\}\xi}. \tag{4.17}$$

(2-iii) Assume $0 < \nu < 1$. Since $D_{s_2^{(\nu,\kappa)}} h_2^{(\nu,\kappa,\beta)}(0) = -\Gamma(\nu+1)$ by recursion (see [6, p. 73]), (3.12) and (4.14) imply

$$n_2^{(\nu,\kappa,\beta,*)}(\xi) = D_{s_2^{(0,\kappa)}} h_2^{(0,\kappa,\beta)}(0)^2 e^{-\beta\xi} \lim_{x,y\to 0} p_2^{(\nu,\kappa)}(\xi,x,y) = 2^{-\nu-1}\Gamma(\nu+1)\left(\frac{\kappa}{\sinh(\kappa\xi)}\right)^{\nu+1} e^{\{\kappa(\nu+1)-\beta\}\xi}. \tag{4.18}$$

(4.16)–(4.18) show (1.7). As mentioned in Section 1, (1.7) was obtained by C. Donati-Martin and M. Yor [3] in the case $-1 < \nu < 0$.

By using one of the limit theorems of [1, p. 640], we see that $\lim_{\kappa\to 0} g_{2,2}^{(\nu,\kappa)}(x,\beta) = g_{1,2}^{(\nu)}(x,\beta)$, and hence $\lim_{\kappa\to 0} h_2^{(\nu,\kappa,\beta)}(x) = h_1^{(\nu,\beta)}(x)$, $\lim_{\kappa\to 0} s_2^{(\nu,\kappa,\beta)}(x) = s_1^{(\nu,\beta)}(x)$ and $\lim_{\kappa\to 0} m_2^{(\nu,\kappa,\beta)}(x) = m_1^{(\nu,\beta)}(x)$. Theorem 2.4 yields

$$\lim_{\kappa\to 0} n_2^{(\nu,\kappa,\beta,*)}(\xi) = n_1^{(\nu,\beta,*)}(\xi). \tag{4.19}$$

(4.19) also follows from (4.6)–(4.8) and (4.16)–(4.18).

We finally consider the special case $\beta = \kappa(\nu+1) > 0$. Then $\mathcal{G}_2^{(\nu,\kappa,\beta)}$ reduces to

$$\begin{aligned}\mathcal{G}_2^{(\nu,\kappa,\kappa(\nu+1))} &= \frac{1}{2}\frac{d^2}{dx^2} + \left\{-\frac{1}{2x} + 2\kappa x \frac{W'_{0,|\nu|/2}(\kappa x^2)}{W_{0,|\nu|/2}(\kappa x^2)}\right\}\frac{d}{dx} \\ &= \frac{1}{2}\frac{d^2}{dx^2} + \left\{\frac{1}{2x} + \kappa x \frac{K'_{|\nu|/2}(\kappa x^2/2)}{K_{|\nu|/2}(\kappa x^2/2)}\right\}\frac{d}{dx},\end{aligned}$$

where we used [7, Problem 19 in p. 279]. By (4.16)–(4.18), the Lévy measure density corresponding to the inverse local time at 0 for $\mathbb{D}_2^{(\nu,\kappa,\kappa(\nu+1),*)}$ is

$$n_2^{(\nu,\kappa,\kappa(\nu+1),*)}(\xi) = 2^{-|\nu|-1}\Gamma(|\nu|+1)\left(\frac{\kappa}{\sinh(\kappa\xi)}\right)^{|\nu|+1}.$$

Example 4.3 (Radial Ornstein–Uhlenbeck process). Let us consider the following diffusion operator on $I = (0,\infty)$:

$$\mathcal{G}_3^{(\nu,\kappa)} = \frac{1}{2}\frac{d^2}{dx^2} + \left(\frac{2\nu+1}{2x} + \kappa x\right)\frac{d}{dx},$$

where $-\infty < \nu < \infty$ and $\kappa > 0$. This is also a radial Ornstein–Uhlenbeck operator, and the scale function $s_3^{(\nu,\kappa)}$ and the speed measure $m_3^{(\nu,\kappa)}$ are given by

$$ds_3^{(\nu,\kappa)}(x) = x^{-2\nu-1}e^{-\kappa x^2}\,dx, \qquad dm_3^{(\nu,\kappa)}(x) = 2x^{2\nu+1}e^{\kappa x^2}\,dx.$$

The killing measure is null. The end point 0 is:

- an $(s_3^{(\nu,\kappa)}, m_3^{(\nu,\kappa)}, 0)$-entrance point if $\nu \ge 0$,
- $(s_3^{(\nu,\kappa)}, m_3^{(\nu,\kappa)}, 0)$-regular if $-1 < \nu < 0$,
- an $(s_3^{(\nu,\kappa)}, m_3^{(\nu,\kappa)}, 0)$-exit point if $\nu \le -1$.

Further

$$\int_0^1 \{s_3^{(\nu,\kappa)}(1) - s_3^{(\nu,\kappa)}(x)\}^2\,dm_3^{(\nu,\kappa)}(x) < \infty \;\Leftrightarrow\; |\nu| < 1. \tag{4.20}$$

The end point ∞ is always $(s_3^{(\nu,\kappa)}, m_3^{(\nu,\kappa)}, 0)$-natural for all ν, and

$$s_3^{(\nu,\kappa)}(\infty) < \infty. \tag{4.21}$$

Let $\mathbb{D}_3^{(\nu,\kappa)}$ be the diffusion process on I with generator $\mathcal{G}_3^{(\nu,\kappa)}$, and with the end point 0 being absorbing if $-1 < \nu < 0$. We denote by $p_3^{(\nu,\kappa)}(t,x,y)$ the transition probability density with respect to $dm_3^{(\nu,\kappa)}$. It is known (see [1, pp. 139–140]) that

$$\begin{aligned}&p_3^{(\nu,\kappa)}(t,x,y) \\ &\quad= \frac{\kappa}{2x^\nu y^\nu \sinh(\kappa t)}\exp\left\{-\kappa(\nu+1)t - \frac{\kappa e^{\kappa t}(x^2+y^2)}{2\sinh(\kappa t)}\right\} I_{|\nu|}\left(\frac{\kappa xy}{\sinh(\kappa t)}\right).\end{aligned} \tag{4.22}$$

By (4.22),

$$\lim_{\kappa\to 0} p_3^{(\nu,\kappa)}(t,x,y) = p_1^{(\nu)}(t,x,y). \tag{4.23}$$

Since $\lim_{\kappa\to 0} s_3^{(\nu,\kappa)}(x) = s_1^{(\nu)}(x)$ and $\lim_{\kappa\to 0} m_3^{(\nu,\kappa)}(x) = m_1^{(\nu)}(x)$ for $x \in I$, (4.23) also follows from Lemma 5.2 of [8]. In the same way as for (4.13) and (4.14), we find that if $-1 < \nu < 0$,

$$\begin{aligned}\lim_{x,y\to 0} D_{s_3^{(\nu,\kappa)}(x)} D_{s_3^{(\nu,\kappa)}(y)} p_3^{(\nu,\kappa)}(\xi,x,y) \\ = 2^{-|\nu|+1}\frac{|\nu|}{\Gamma(|\nu|)}\left(\frac{\kappa}{\sinh(\kappa\xi)}\right)^{|\nu|+1} e^{-\kappa(\nu+1)\xi},\end{aligned} \tag{4.24}$$

while if $\nu \geq 0$,

$$\lim_{x,y\to 0} p_3^{(\nu,\kappa)}(\xi,x,y) = 2^{-\nu-1}\frac{1}{\Gamma(\nu+1)}\left(\frac{\kappa}{\sinh(\kappa\xi)}\right)^{\nu+1} e^{-\kappa(\nu+1)\xi}. \tag{4.25}$$

For $\alpha > 0$ we denote by $g_{3,i}^{(\nu,\kappa)}(x,\alpha)$, $i = 1, 2$, the functions satisfying (2.11)–(2.15) with $s = s_3^{(\nu,\kappa)}$, $m = m_3^{(\nu,\kappa)}$ and $k = 0$. It is known (see [1, p. 140][1]) that

$$\begin{aligned}g_{3,1}^{(\nu,\kappa)}(x,\alpha) &= \frac{\alpha^{|\nu|/2}}{2^{|\nu|/2}\kappa^{(|\nu|+1)/2}\Gamma(|\nu|+1)} x^{-\nu-1} e^{-\kappa x^2/2} M_{-\frac{\alpha}{2\kappa}-\frac{\nu+1}{2},\frac{|\nu|}{2}}(\kappa x^2),\\ g_{3,2}^{(\nu,\kappa)}(x,\alpha) &\\ &= \frac{\kappa^{|\nu|/2-1/2}}{2^{1-|\nu|/2}\alpha^{|\nu|/2}}\Gamma\left(\frac{|\nu|}{2}+\frac{\nu}{2}+\frac{\alpha}{2\kappa}+1\right) x^{-\nu-1} e^{-\kappa x^2/2} W_{-\frac{\alpha}{2\kappa}-\frac{\nu+1}{2},\frac{|\nu|}{2}}(\kappa x^2).\end{aligned}$$

Now we consider the Lévy measure densities corresponding to the inverse local time at the end point 0. We assume $-1 < \nu < 1$. Then the end point 0 is $(s_3^{(\nu,\kappa)}, m_3^{(\nu,\kappa)}, 0)$-regular or an $(s_3^{(\nu,\kappa)}, m_3^{(\nu,\kappa)}, 0)$–entrance point, and (2.6) is satisfied by (4.20).

(1) For $\beta > 0$ we put $h_3^{(\nu,\kappa,\beta)}(x) = (\beta/2)^{|\nu|/2} g_{3,2}^{(\nu,\kappa)}(x,\beta)$ and denote by $\mathcal{G}_3^{(\nu,\kappa,\beta)}$ the harmonic transform of $\mathcal{G}_3^{(\nu,\kappa)}$ based on $h_3^{(\nu,\kappa,\beta)} \in \mathcal{H}_{s_3^{(\nu,\kappa)},m_3^{(\nu,\kappa)},0,\beta}$, that is,

$$\mathcal{G}_3^{(\nu,\kappa,\beta)} = \frac{1}{2}\frac{d^2}{dx^2} + \left\{-\frac{1}{2x} + 2\kappa x \frac{W'_{-\frac{\beta}{2\kappa}-\frac{\nu+1}{2},\frac{|\nu|}{2}}(\kappa x^2)}{W_{-\frac{\beta}{2\kappa}-\frac{\nu+1}{2},\frac{|\nu|}{2}}(\kappa x^2)}\right\}\frac{d}{dx},$$

which is (1.5). The scale function $s_3^{(\nu,\kappa,\beta)}$ and the speed measure $m_3^{(\nu,\kappa,\beta)}$ are given by

$$ds_3^{(\nu,\kappa,\beta)}(x) = h_3^{(\nu,\kappa,\beta)}(x)^{-2}\, ds_3^{(\nu,\kappa)}(x), \qquad dm_3^{(\nu,\kappa,\beta)}(x) = h_3^{(\nu,\kappa,\beta)}(x)^2\, dm_3^{(\nu,\kappa)}(x).$$

[1]Misprints in [1, p. 140]: minus signs are missing from exponents of Green functions in the case $\gamma < 0$.

By Proposition 3.3, (3.6) or (3.7) is satisfied, and $s_3^{(\nu,\kappa,\beta)}(\infty) = \infty$. The end point 0 is $(s_3^{(\nu,\kappa,\beta)}, m_3^{(\nu,\kappa,\beta)}, 0)$-regular. We consider the diffusion process $\mathbb{D}_3^{(\nu,\kappa,\beta,*)}$ with generator $\mathcal{G}_3^{(\nu,\kappa,\beta)}$ and with the end point 0 being reflecting. Let $n_3^{(\nu,\kappa,\beta,*)}$ be the corresponding Lévy measure density.

(1-i) Assume $\nu = 0$. Since $D_{s_3^{(0,\kappa)}} h_3^{(0,\kappa,\beta)}(0) = -1$ by [7, Problem 17, p. 279] and recursion (see [6, p. 73]), (3.12) and (4.25) imply

$$\begin{aligned} n_3^{(0,\kappa,\beta,*)}(\xi) &= D_{s_3^{(0,\kappa)}} h_3^{(0,\kappa,\beta)}(0)^2 e^{-\beta\xi} \lim_{x,y\to 0} p_3^{(\nu,\kappa)}(\xi,x,y) \\ &= \frac{\kappa}{2\sinh(\kappa\xi)} e^{(-\kappa-\beta)\xi}. \end{aligned} \tag{4.26}$$

(1-ii) Assume $-1 < \nu < 0$. Since $h_3^{(\nu,\kappa,\beta)}(0) = \Gamma(|\nu|)/2$, (3.10) and (4.24) imply

$$n_3^{(\nu,\kappa,\beta,*)}(\xi) = 2^{-|\nu|-1}\Gamma(|\nu|+1)\left(\frac{\kappa}{\sinh(\kappa\xi)}\right)^{|\nu|+1} e^{\{-\kappa(\nu+1)-\beta\}\xi}. \tag{4.27}$$

(1-iii) Assume $0 < \nu < 1$. Since $D_{s_3^{(\nu,\kappa)}} h_3^{(\nu,\kappa,\beta)}(0) = -\Gamma(\nu+1)$ by recursion (see [6, p. 73]), (3.12) and (4.25) imply

$$\begin{aligned} n_3^{(\nu,\kappa,\beta,*)}(\xi) &= D_{s_3^{(0,\kappa)}} h_3^{(0,\kappa,\beta)}(0)^2 e^{-\beta\xi} \lim_{x,y\to 0} p_3^{(\nu,\kappa)}(\xi,x,y) \\ &= 2^{-\nu-1}\Gamma(\nu+1)\left(\frac{\kappa}{\sinh(\kappa\xi)}\right)^{\nu+1} e^{\{-\kappa(\nu+1)-\beta\}\xi}. \end{aligned} \tag{4.28}$$

By one of the limit theorems of [1, p. 640], we get $\lim_{\kappa\to 0} g_{3,2}^{(\nu,\kappa)}(x,\beta) = g_{1,2}^{(\nu)}(x,\beta)$, and hence $\lim_{\kappa\to 0} s_3^{(\nu,\kappa,\beta)}(x) = s_1^{(\nu,\beta)}(x)$ and $\lim_{\kappa\to 0} m_3^{(\nu,\kappa,\beta)}(x) = m_1^{(\nu,\beta)}(x)$. Theorem 2.4 yields

$$\lim_{\kappa\to 0} n_3^{(\nu,\kappa,\beta,*)}(\xi) = n_1^{(\nu,\beta,*)}(\xi).$$

This also follows from (4.6)–(4.8) and (4.26)–(4.28).

(2) Since $s_3^{(\nu,\kappa)}(\infty) < \infty$ by (4.21), Proposition 3.4 leads to a special case corresponding to $\beta = 0$. Put $h_3^{(\nu,\kappa,0)}(x) = \{s_3^{(\nu,\kappa)}(\infty) - s_3^{(\nu,\kappa)}(x)\}/\{s_3^{(\nu,\kappa)}(\infty) - s_3^{(\nu,\kappa)}(1)\}$. Denote by $\mathcal{G}_3^{(\nu,\kappa,0)}$ the harmonic transform of $\mathcal{G}_3^{(\nu,\kappa)}$ based on $h_3^{(\nu,\kappa,0)} \in \mathcal{H}_{s_3^{(\nu,\kappa)}, m_3^{(\nu,\kappa)}, 0, 0}$, that is,

$$\mathcal{G}_3^{(\nu,\kappa,0)} = \frac{1}{2}\frac{d^2}{dx^2} + \left\{\frac{2\nu+1}{2x} + \kappa x - \frac{x^{-(2\nu+1)}e^{-\kappa x^2}}{\int_x^\infty y^{-(2\nu+1)}e^{-\kappa y^2}\,dy}\right\}\frac{d}{dx}.$$

The scale function $s_3^{(\nu,\kappa,0)}$ and the speed measure $m_3^{(\nu,\kappa,0)}$ are given by

$$ds_3^{(\nu,\kappa,0)}(x) = h_3^{(\nu,\kappa,0)}(x)^{-2}\, ds_3^{(\nu,\kappa)}(x), \qquad dm_3^{(\nu,\kappa,0)}(x) = h_3^{(\nu,\kappa,0)}(x)^2\, dm_3^{(\nu,\kappa)}(x).$$

By Proposition 3.4, (3.6) or (3.7) is satisfied, and $s_3^{(\nu,\kappa,0)}(\infty) = \infty$. The end point 0 is $(s_3^{(\nu,\kappa,0)}, m_3^{(\nu,\kappa,0)}, 0)$-regular. We consider the diffusion process $\mathbb{D}_3^{(\nu,\kappa,0,*)}$ with generator $\mathcal{G}_3^{(\nu,\kappa,0)}$ and with the end point 0 being reflecting. Let $n_3^{(\nu,\kappa,0,*)}$ be the corresponding Lévy measure density.

(2-i) Assume $0 \le \nu < 1$. Since $D_{s_3^{(\nu,\kappa)}} h_3^{(\nu,\kappa,0)}(0) = -\{s_3^{(\nu,\kappa)}(\infty) - s_3^{(\nu,\kappa)}(1)\}^{-1}$, (3.12) implies

$$n_3^{(\nu,\kappa,0,*)}(\xi) = \{s_3^{(\nu,\kappa)}(\infty) - s_3^{(\nu,\kappa)}(1)\}^{-2} 2^{-\nu-1} \frac{1}{\Gamma(\nu+1)} \left(\frac{\kappa}{\sinh(\kappa\xi)}\right)^{\nu+1} e^{-\kappa(\nu+1)\xi},$$

where we used (4.25). We note that

$$\lim_{\kappa\to 0} n_3^{(\nu,\kappa,0,*)}(\xi) = 2^{-\nu+1} \frac{\nu^2}{\Gamma(\nu+1)} \xi^{-\nu-1}. \tag{4.29}$$

On the other hand,

$$\lim_{\kappa\to 0} h_3^{(\nu,\kappa,0)}(x) = x^{-2\nu},$$

and hence $\lim_{\kappa\to 0} s_3^{(\nu,\kappa,0)}(x) = s_1^{(-\nu)}(x)$ and $\lim_{\kappa\to 0} m_3^{(\nu,\kappa,0)}(x) = m_1^{(-\nu)}(x)$.

In the case $0 < \nu < 1$, the end point 0 is $(s_1^{(-\nu)}, m_1^{(-\nu)}, 0)$-regular, and hence by Theorem 2.4,

$$\lim_{\kappa\to 0} n_3^{(\nu,\kappa,0,*)}(\xi) = n_1^{(-\nu,*)} = 2^{-\nu+1} \frac{\nu}{\Gamma(\nu)} \xi^{-\nu-1}.$$

This shows (4.29) with $0 < \nu < 1$.

In the case $\nu = 0$, the end point 0 is an $(s_1^{(0)}, m_1^{(0)}, 0)$-entrance point and hence the local time at 0 does not exist.

(2-ii) Assume $-1 < \nu < 0$. Since $h_3^{(\nu,\kappa,0)}(0) \in (0, \infty)$, by (3.10) and (4.24) we get

$$n_3^{(\nu,\kappa,0,*)}(\xi) = h_3^{(\nu,\kappa,0)}(0)^2 2^{-|\nu|+1} \frac{|\nu|}{\Gamma(|\nu|)} \left(\frac{\kappa}{\sinh(\kappa\xi)}\right)^{|\nu|+1} e^{-\kappa(\nu+1)\xi}. \tag{4.30}$$

Since $\lim_{\kappa\to 0} h_3^{(\nu,\kappa,0)}(0) = 1$, we have $\lim_{\kappa\to 0} s_3^{(\nu,\kappa,0)}(x) = s_1^{(\nu)}(x)$ and $\lim_{\kappa\to 0} m_3^{(\nu,\kappa,0)}(x) = m_1^{(\nu)}(x)$. By Theorem 2.4 and (4.5),

$$\lim_{\kappa\to 0} n_3^{(\nu,\kappa,0,*)}(\xi) = n_1^{(\nu,*)}(\xi) = 2^{-|\nu|+1} \frac{|\nu|}{\Gamma(|\nu|)} \xi^{-|\nu|-1}.$$

This also follows from (4.30).

Example 4.4 (Squared Bessel process). Let us consider the following diffusion operator on $I = (0, \infty)$:

$$\mathcal{G}_4^{(\nu)} = 2x \frac{d^2}{dx^2} + (2\nu + 2) \frac{d}{dx},$$

where $-\infty < \nu < \infty$. This is the squared Bessel operator, and the scale function $s_4^{(\nu)}$ and the speed measure $m_4^{(\nu)}$ are given by

$$ds_4^{(\nu)}(x) = x^{-\nu-1}\,dx, \qquad dm_4^{(\nu)}(x) = 2^{-1}x^{\nu}\,dx.$$

The killing measure is null. The end point 0 is:

- an $(s_4^{(\nu)}, m_4^{(\nu)}, 0)$-entrance point if $\nu \geq 0$,
- $(s_4^{(\nu)}, m_4^{(\nu)}, 0)$-regular if $-1 < \nu < 0$,
- an $(s_4^{(\nu)}, m_4^{(\nu)}, 0)$-exit point if $\nu \leq -1$.

Further

$$\int_0^1 \{s_4^{(\nu)}(1) - s_4^{(\nu)}(x)\}^2\,dm_4^{(\nu)}(x) < \infty \;\Leftrightarrow\; |\nu| < 1. \tag{4.31}$$

The end point ∞ is $(s_4^{(\nu)}, m_4^{(\nu)}, 0)$-natural for all ν, and

$$s_4^{(\nu)}(\infty) = \infty \;\Leftrightarrow\; \nu \leq 0. \tag{4.32}$$

Let $\mathbb{D}_4^{(\nu)}$ be the diffusion process on I with generator $\mathcal{G}_4^{(\nu)}$, and with the end point 0 being absorbing if $-1 < \nu < 0$. We denote by $p_4^{(\nu)}(t, x, y)$ the transition probability density with respect to $dm_4^{(\nu)}$. It is known (see [1, p. 136]) that

$$p_4^{(\nu)}(t, x, y) = \frac{1}{t}\exp\Big\{-\frac{x+y}{2t}\Big\}(xy)^{-\nu/2} I_{|\nu|}\Big(\frac{\sqrt{xy}}{t}\Big).$$

In the same way as for the spectral representation of $p_1^{(\nu)}(t, x, y)$ in Example 4.1, we get

$$p_4^{(\nu)}(t, x, y) = \int_0^\infty e^{-\lambda t}\psi_4^{(\nu)}(x, \lambda)\psi_4^{(\nu)}(y, \lambda)\sigma_4^{(\nu)}(\lambda)\,d\lambda,$$

where

$$\psi_4^{(\nu)}(x, \lambda) = \begin{cases} \Gamma(\nu+1)(2/\lambda)^{\nu/2}x^{-\nu/2}J_\nu(\sqrt{2\lambda x}), & \nu \geq 0, \\ \Gamma(|\nu|)(2/\lambda)^{|\nu|/2}x^{|\nu|/2}J_{|\nu|}(\sqrt{2\lambda x}), & \nu < 0, \end{cases}$$

$$\sigma_4^{(\nu)}(\lambda) = \begin{cases} 2^{-\nu}\Gamma(\nu+1)^{-2}\lambda^{\nu}, & \nu \geq 0, \\ 2^{-|\nu|}\Gamma(|\nu|)^{-2}\lambda^{|\nu|}, & \nu < 0. \end{cases}$$

For $\alpha > 0$ we denote by $g_{4,i}^{(\nu)}(x, \alpha)$, $i = 1, 2$, the functions satisfying (2.11)–(2.15) with $s = s_4^{(\nu)}$, $m = m_4^{(\nu)}$ and $k = 0$. It is known (see [1, p. 135]) that

$$g_{4,1}^{(\nu)}(x, \alpha) = x^{-\nu/2}I_{|\nu|}(\sqrt{2\alpha x}), \qquad g_{4,2}^{(\nu)}(x, \alpha) = x^{-\nu/2}K_{|\nu|}(\sqrt{2\alpha x}).$$

Now we consider the Lévy measure densities corresponding to the inverse local time at the end point 0.

(1) Let $-1 < \nu < 0$. Then the end point 0 is $(s_4^{(\nu)}, m_4^{(\nu)}, 0)$-regular. We pose the reflecting boundary condition at 0. We denote by $\mathbb{D}_4^{(\nu,*)}$ the diffusion process with generator $\mathcal{G}_4^{(\nu)}$ and with the end point 0 being reflecting. We denote by $n_4^{(\nu,*)}$ the corresponding Lévy measure density. Since $s_4^{(\nu)}(\infty) = \infty$ by (4.32), from (2.24) we derive

$$(4.33)\qquad \begin{aligned} n_4^{(\nu,*)}(\xi) &= \lim_{x,y\to 0} D_{s_4^{(\nu)}(x)} D_{s_4^{(\nu)}(y)} p_4^{(\nu)}(\xi, x, y) \\ &= \int_0^\infty e^{-\xi\lambda}\sigma_4^{(\nu)}(\lambda)\, d\lambda = 2^{-|\nu|}\frac{|\nu|}{\Gamma(|\nu|)}\xi^{-(|\nu|+1)}. \end{aligned}$$

(2) Let $-1 < \nu < 1$. Then 0 is $(s_4^{(\nu)}, m_4^{(\nu)}, 0)$-regular or an $(s_4^{(\nu)}, m_4^{(\nu)}, 0)$–entrance point, and (2.6) is satisfied in view of (4.31). For $\beta > 0$ we put $h_4^{(\nu,\beta)}(x) = (\beta/2)^{|\nu|/2} g_{4,2}^{(\nu)}(x, \beta)$ and denote by $\mathcal{G}_4^{(\nu,\beta)}$ the harmonic transform of $\mathcal{G}_4^{(\nu)}$ based on $h_4^{(\nu,\beta)} \in \mathcal{H}_{s_4^{(\nu)}, m_4^{(\nu)}, 0, \beta}$, that is,

$$\mathcal{G}_4^{(\nu,\beta)} = 2x\frac{d^2}{dx^2} + 2\left\{1 + \sqrt{2\beta x}\,\frac{K_\nu'(\sqrt{2\beta x})}{K_\nu(\sqrt{2\beta x})}\right\}\frac{d}{dx}.$$

The scale function $s_4^{(\nu,\beta)}$ and the speed measure $m_4^{(\nu,\beta)}$ are given by

$$ds_4^{(\nu,\beta)}(x) = h_4^{(\nu,\beta)}(x)^{-2}\, ds_4^{(\nu)}(x), \qquad dm_4^{(\nu,\beta)}(x) = h_4^{(\nu,\beta)}(x)^2\, dm_4^{(\nu)}(x).$$

By Proposition 3.3, (3.6) or (3.7) is satisfied, and $s_4^{(\nu,\beta)}(\infty) = \infty$. The end point 0 is $(s_4^{(\nu,\beta)}, m_4^{(\nu,\beta)}, 0)$-regular. We consider the diffusion process $\mathbb{D}_4^{(\nu,\beta,*)}$ with generator $\mathcal{G}_4^{(\nu,\beta)}$ and with the end point 0 being reflecting. Let $n_4^{(\nu,\beta,*)}$ be the corresponding Lévy measure density.

(2-i) Assume $\nu = 0$. Since $D_{s_4^{(0)}} h_4^{(0,\beta)}(0) = -1/2$, (3.11) implies

$$(4.34)\qquad n_4^{(0,\beta,*)}(\xi) = \frac{1}{4}e^{-\beta\xi}\int_{[0,\infty)} e^{-\xi\lambda}\sigma_4^{(0)}(\lambda)\, d\lambda = \frac{1}{4\xi}e^{-\beta\xi}.$$

(2-ii) Assume $-1 < \nu < 0$. Since $h_4^{(\nu,\beta)}(0) = \Gamma(-\nu)/2$, (3.9) leads to

$$(4.35)\qquad \begin{aligned} n_4^{(\nu,\beta,*)}(\xi) &= h_4^{(\nu,\beta)}(0)^2 e^{-\beta\xi}\int_0^\infty e^{-\xi\lambda}\sigma_4^{(\nu)}(\lambda)\, d\lambda \\ &= 2^{\nu-2}\Gamma(|\nu|+1)\xi^{\nu-1}e^{-\beta\xi}. \end{aligned}$$

(2-iii) Assume $0 < \nu < 1$. Since $\mathcal{G}_4^{(\nu,\beta)} = \mathcal{G}_4^{(-\nu,\beta)}$, by using (4.35) we get

$$(4.36)\qquad n_4^{(\nu,\beta,*)}(\xi) = n_4^{(-\nu,\beta,*)}(\xi) = 2^{-\nu-2}\Gamma(\nu+1)\xi^{-\nu-1}e^{-\beta\xi}.$$

(3) Let $0 < \nu < 1$. Then $s_4^{(\nu)}(\infty) < \infty$ by (4.32). Therefore Proposition 3.4 leads to a special case corresponding to $\beta = 0$. We put $h_4^{(\nu,0)}(x) = \{s_4^{(\nu)}(\infty) -$

$s_4^{(\nu)}(x)\}/\{s_4^{(\nu)}(\infty)-s_4^{(\nu)}(1)\}=x^{-\nu}$. Denote by $\mathcal{G}_4^{(\nu,0)}$ the harmonic transform of $\mathcal{G}_4^{(\nu)}$ based on $h_4^{(\nu,0)}\in\mathcal{H}_{s_4^{(\nu)},m_4^{(\nu)},0,0}$, that is,

$$\mathcal{G}_4^{(\nu,0)}=2x\frac{d^2}{dx^2}+(-2\nu+2)\frac{d}{dx}.$$

We note that this coincides with $\mathcal{G}_4^{(-\nu)}$. The scale function $s_4^{(\nu,0)}$ and the speed measure $m_4^{(\nu,0)}$ are given by

$$ds_4^{(\nu,0)}(x)=h_4^{(\nu,0)}(x)^{-2}\,ds_4^{(\nu)}(x)=x^{\nu-1}\,dx=ds_4^{(-\nu)}(x),$$
$$dm_4^{(\nu,0)}(x)=h_4^{(\nu,0)}(x)^{2}\,dm_4^{(\nu)}(x)=2^{-1}x^{-\nu}\,dx=dm_4^{(-\nu)}(x).$$

By Proposition 3.4, (3.7) is satisfied, and $s_4^{(\nu,0)}(\infty)=\infty$. The end point 0 is $(s_4^{(\nu,0)},m_4^{(\nu,0)},0)$-regular. We consider the diffusion process $\mathbb{D}_4^{(\nu,0,*)}$ with generator $\mathcal{G}_4^{(\nu,0)}$ and with the end point 0 being reflecting. Let $n_4^{(\nu,0,*)}$ be the corresponding Lévy measure density. Since $\mathcal{G}_4^{(\nu,0)}=\mathcal{G}_4^{(-\nu)}$, (4.33) yields

$$n_4^{(\nu,0,*)}(\xi)=2^{-\nu}\frac{\nu}{\Gamma(\nu)}\xi^{-(\nu+1)}=n_4^{(-\nu,*)}(\xi).$$

Example 4.5 (Squared radial Ornstein–Uhlenbeck process). Let us consider the following diffusion operator on $I=(0,\infty)$:

$$\mathcal{G}_5^{(\nu,\kappa)}=2x\frac{d^2}{dx^2}+(2\nu+2-2\kappa x)\frac{d}{dx},$$

where $-\infty<\nu<\infty$ and $\kappa>0$. This is a squared radial Ornstein–Uhlenbeck operator, and the scale function $s_5^{(\nu,\kappa)}$ and the speed measure $m_5^{(\nu,\kappa)}$ are

$$ds_5^{(\nu,\kappa)}(x)=x^{-\nu-1}e^{\kappa x}\,dx,\qquad dm_5^{(\nu,\kappa)}(x)=2^{-1}x^{\nu}e^{-\kappa x}\,dx.$$

The killing measure is null. The end point 0 is:

- an $(s_5^{(\nu,\kappa)},m_5^{(\nu,\kappa)},0)$-entrance point if $\nu\ge 0$,
- $(s_5^{(\nu,\kappa)},m_5^{(\nu,\kappa)},0)$-regular if $-1<\nu<0$,
- an $(s_5^{(\nu,\kappa)},m_5^{(\nu,\kappa)},0)$-exit point if $\nu\le -1$.

Further

$$\int_0^1\{s_5^{(\nu,\kappa)}(1)-s_5^{(\nu,\kappa)}(x)\}^2\,dm_5^{(\nu,\kappa)}(x)<\infty\ \Leftrightarrow\ |\nu|<1. \tag{4.37}$$

The end point ∞ is $(s_5^{(\nu,\kappa)},m_5^{(\nu,\kappa)},0)$-natural for all ν and

$$s_5^{(\nu,\kappa)}(\infty)=\infty. \tag{4.38}$$

Let $\mathbb{D}_5^{(\nu,\kappa)}$ be the diffusion process on I with generator $\mathcal{G}_5^{(\nu,\kappa)}$, and with the end point 0 being absorbing if $-1 < \nu < 0$. We denote by $p_5^{(\nu,\kappa)}(t,x,y)$ the transition probability density with respect to $dm_5^{(\nu,\kappa)}$. It is known (see [1, p. 142]) that

$$p_5^{(\nu,\kappa)}(t,x,y) = \frac{\kappa}{\sinh(\kappa t)}(xy)^{-\nu/2}\exp\biggl\{\kappa(\nu+1)t - \frac{\kappa e^{-\kappa t}(x+y)}{2\sinh(\kappa t)}\biggr\} I_{|\nu|}\biggl(\frac{\kappa\sqrt{xy}}{\sinh(\kappa t)}\biggr). \tag{4.39}$$

By (4.39),

$$\lim_{\kappa\to 0} p_5^{(\nu,\kappa)}(t,x,y) = p_4^{(\nu)}(t,x,y). \tag{4.40}$$

Since $\lim_{\kappa\to 0} s_5^{(\nu,\kappa)}(x) = s_4^{(\nu)}(x)$ and $\lim_{\kappa\to 0} m_5^{(\nu,\kappa)}(x) = m_4^{(\nu)}(x)$ for $x \in I$, (4.40) also follows from Lemma 5.2 of [8]. Further we note that if $-1 < \nu < 0$,

$$\lim_{x,y\to 0} D_{s_5^{(\nu,\kappa)}(x)} D_{s_5^{(\nu,\kappa)}(y)} p_5^{(\nu,\kappa)}(\xi,x,y) = 2^{-|\nu|}\frac{|\nu|}{\Gamma(|\nu|)}\biggl(\frac{\kappa}{\sinh(\kappa\xi)}\biggr)^{|\nu|+1} e^{\kappa(\nu+1)\xi}, \tag{4.41}$$

while if $\nu \ge 0$,

$$\lim_{x,y\to 0} p_6^{(\nu,\kappa)}(\xi,x,y) = 2^{-\nu}\frac{1}{\Gamma(\nu+1)}\biggl(\frac{\kappa}{\sinh(\kappa\xi)}\biggr)^{\nu+1} e^{\kappa(\nu+1)\xi}. \tag{4.42}$$

For $\alpha > 0$ we denote by $g_{5,i}^{(\nu,\kappa)}(x,\alpha)$, $i = 1,\ 2$, the functions satisfying (2.11)–(2.15) with $s = s_5^{\nu,\kappa}$, $m = m_5^{\nu,\kappa}$ and $k = 0$. It is known (see [1, pp. 141–142]) that

$$g_{5,1}^{(\nu,\kappa)}(x,\alpha) = \frac{\alpha^{|\nu|/2}}{2^{|\nu|/2}\Gamma(|\nu|+1)\kappa^{(|\nu|+1)/2}} x^{-\frac{\nu+1}{2}} e^{\kappa x/2} M_{-\frac{\alpha}{2\kappa}+\frac{\nu+1}{2},\frac{|\nu|}{2}}(\kappa x),$$
$$g_{5,2}^{(\nu,\kappa)}(x,\alpha) = \frac{2^{|\nu|/2-1}\kappa^{(|\nu|-1)/2}}{\alpha^{|\nu|/2}}\Gamma\biggl(\frac{\alpha}{2\kappa}+\frac{|\nu|}{2}-\frac{\nu}{2}\biggr) x^{-\frac{\nu+1}{2}} e^{\kappa x/2} W_{-\frac{\alpha}{2\kappa}+\frac{\nu+1}{2},\frac{|\nu|}{2}}(\kappa x).$$

Now we consider the Lévy measure densities corresponding to the inverse local time at the end point 0.

(1) Let $-1 < \nu < 0$. Then the end point 0 is $(s_5^{(\nu,\kappa)}, m_5^{(\nu,\kappa)}, 0)$-regular. We pose the reflecting boundary condition at 0. We denote by $\mathbb{D}_5^{(\nu,\kappa,*)}$ the diffusion process with generator $\mathcal{G}_5^{(\nu,\kappa)}$ and with the end point 0 being reflecting. We denote by $n_5^{(\nu,\kappa,*)}$ the corresponding Lévy measure density. Since $s_5^{(\nu,\kappa)}(\infty) = \infty$ by (4.38), from (2.24) and (4.41) we derive

$$\begin{aligned} n_5^{(\nu,\kappa,*)}(\xi) &= \lim_{x,y\to 0} D_{s_5^{(\nu,\kappa)}(x)} D_{s_5^{(\nu,\kappa)}(y)} p_5^{(\nu,\kappa)}(\xi,x,y) \\ &= \frac{2^{-|\nu|}|\nu|}{\Gamma(|\nu|)}\biggl(\frac{\kappa}{\sinh(\kappa\xi)}\biggr)^{|\nu|+1} e^{\kappa(\nu+1)\xi}. \end{aligned}$$

(2) Let $-1 < \nu < 1$. Then 0 is $(s_5^{(\nu,\kappa)}, m_5^{(\nu,\kappa)}, 0)$-regular or an $(s_5^{(\nu,\kappa)}, m_5^{(\nu,\kappa)}, 0)$-entrance point, and (2.6) is satisfied in view of (4.37). For $\beta > 0$ we put $h_5^{(\nu,\kappa,\beta)}(x) = (\beta/2)^{|\nu|/2} g_{5,2}^{(\nu,\kappa)}(x,\beta)$ and denote by $\mathcal{G}_5^{(\nu,\kappa,\beta)}$ the harmonic transform of $\mathcal{G}_5^{(\nu,\kappa)}$ based on $h_5^{(\nu,\kappa,\beta)} \in \mathcal{H}_{s_5^{(\nu,\kappa)}, m_5^{(\nu,\kappa)}, 0, \beta}$, that is,

$$\mathcal{G}_5^{(\nu,\kappa,\beta)} = 2x\frac{d^2}{dx^2} + 4\kappa x \frac{W'_{-\frac{\beta}{2\kappa}+\frac{\nu+1}{2}, \frac{|\nu|}{2}}(\kappa x)}{W_{-\frac{\beta}{2\kappa}+\frac{\nu+1}{2}, \frac{|\nu|}{2}}(\kappa x)} \frac{d}{dx}.$$

The scale function $s_5^{(\nu,\kappa,\beta)}$ and the speed measure $m_5^{(\nu,\kappa,\beta)}$ are given by

$$ds_5^{(\nu,\kappa,\beta)}(x) = h_5^{(\nu,\kappa,\beta)}(x)^{-2}\, ds_5^{(\nu,\kappa)}(x), \qquad dm_5^{(\nu,\kappa,\beta)}(x) = h_5^{(\nu,\kappa,\beta)}(x)^2\, dm_5^{(\nu,\kappa)}(x).$$

By Proposition 3.3, (3.6) or (3.7) is satisfied, and $s_5^{(\nu,\kappa,\beta)}(\infty) = \infty$. Therefore the end point 0 is $(s_5^{(\nu,\kappa,\beta)}, m_5^{(\nu,\kappa,\beta)}, 0)$-regular. We consider the diffusion process $\mathbb{D}_5^{(\nu,\kappa,\beta,*)}$ with generator $\mathcal{G}_5^{(\nu,\kappa,\beta)}$ and with the end point 0 being reflecting. Let $n_5^{(\nu,\kappa,\beta,*)}$ be the corresponding Lévy measure density.

(2-i) Assume $\nu = 0$. Since $D_{s_5^{(0,\kappa)}} h_5^{(0,\kappa,\beta)}(0) = -1/2$ by [7, Problem 17, p. 279] and recursion (see [6, p. 73]), (3.12) and (4.42) imply

$$n_5^{(0,\kappa,\beta,*)}(\xi) = \frac{\kappa}{4\sinh(\kappa\xi)} e^{(\kappa-\beta)\xi}. \tag{4.43}$$

(2-ii) Assume $-1 < \nu < 0$. Since $h_5^{(\nu,\kappa,\beta)}(0) = \Gamma(|\nu|)/2$, (3.10) and (4.41) lead to

$$n_5^{(\nu,\kappa,\beta,*)}(\xi) = 2^{-|\nu|-2}\Gamma(|\nu|+1)\left(\frac{\kappa}{\sinh(\kappa\xi)}\right)^{|\nu|+1} e^{(\kappa(\nu+1)-\beta)\xi}. \tag{4.44}$$

(2-iii) Assume $0 < \nu < 1$. Since $D_{s_5^{(\nu,\kappa)}} h_5^{(\nu,\kappa,\beta)}(0) = -\Gamma(1+\nu)/2$ by recursion (see [6, p. 73]), (3.12) and (4.42) imply

$$n_5^{(\nu,\kappa,\beta,*)}(\xi) = 2^{-\nu-2}\Gamma(\nu+1)\left(\frac{\kappa}{\sinh(\kappa\xi)}\right)^{\nu+1} e^{(\kappa(\nu+1)-\beta)\xi}. \tag{4.45}$$

By using one of the limit theorems of [1, p. 640], we see that $\lim_{\kappa\to 0} g_{5,2}^{(\nu,\kappa)}(x,\beta) = g_{4,2}^{(\nu)}(x,\beta)$, and hence $\lim_{\kappa\to 0} h_5^{(\nu,\kappa,\beta)}(x) = h_4^{(\nu,\beta)}(x)$, $\lim_{\kappa\to 0} s_5^{(\nu,\kappa,\beta)}(x) = s_4^{(\nu,\beta)}(x)$ and $\lim_{\kappa\to 0} m_5^{(\nu,\kappa,\beta)}(x) = m_4^{(\nu,\beta)}(x)$. Theorem 2.4 yields

$$\lim_{\kappa\to 0} n_5^{(\nu,\kappa,\beta,*)}(\xi) = n_4^{(\nu,\beta,*)}(\xi). \tag{4.46}$$

(4.46) also follows from (4.34)–(4.36) and (4.43)–(4.45).

We finally consider the special case $\beta = \kappa(\nu+1) > 0$. Then $\mathcal{G}_5^{(\nu,\kappa,\beta)}$ reduces to

$$\begin{aligned}\mathcal{G}_5^{(\nu,\kappa,\kappa(\nu+1))} &= 2x\frac{d^2}{dx^2} + 4\kappa x \frac{W'_{0,|\nu|/2}(\kappa x)}{W_{0,|\nu|/2}(\kappa x)}\frac{d}{dx} \\ &= 2x\frac{d^2}{dx^2} + \left\{1 + \kappa x \frac{K'_{|\nu|/2}(\kappa x/2)}{K_{|\nu|/2}(\kappa x/2)}\right\}\frac{d}{dx},\end{aligned}$$

where we used [7, Problem 19 in p. 279]. By (4.43), (4.44), and (4.45), the Lévy measure density corresponding to the inverse local time at 0 for $\mathbb{D}_5^{(\nu,\kappa,\kappa(\nu+1),*)}$ is

$$n_5^{(\nu,\kappa,\kappa(\nu+1),*)}(\xi) = 2^{-|\nu|-2}\Gamma(|\nu|+1)\left(\frac{\kappa}{\sinh(\kappa\xi)}\right)^{|\nu|+1}. \tag{4.47}$$

Example 4.6 (Squared radial Ornstein–Uhlenbeck process). Let us consider the following diffusion operator on $I = (0,\infty)$:

$$\mathcal{G}_6^{(\nu,\kappa)} = 2x\frac{d^2}{dx^2} + (2\nu + 2 + 2\kappa x)\frac{d}{dx},$$

where $-\infty < \nu < \infty$ and $\kappa > 0$. This is also a squared radial Ornstein–Uhlenbeck operator, and the scale function $s_6^{(\nu,\kappa)}$ and the speed measure $m_6^{(\nu,\kappa)}$ are given by

$$ds_6^{(\nu,\kappa)}(x) = x^{-\nu-1}e^{-\kappa x}\,dx, \qquad dm_6^{(\nu,\kappa)}(x) = 2^{-1}x^{\nu}e^{\kappa x}\,dx.$$

The killing measure is null. The end point 0 is:

- an $(s_6^{(\nu,\kappa)}, m_6^{(\nu,\kappa)}, 0)$-entrance point if $\nu \ge 0$,
- $(s_6^{(\nu,\kappa)}, m_6^{(\nu,\kappa)}, 0)$-regular if $-1 < \nu < 0$,
- an $(s_6^{(\nu,\kappa)}, m_6^{(\nu,\kappa)}, 0)$-exit point if $\nu \le -1$.

Further

$$\int_0^1 \{s_6^{(\nu,\kappa)}(1) - s_6^{(\nu,\kappa)}(x)\}^2\,dm_6^{(\nu,\kappa)}(x) < \infty \;\Leftrightarrow\; |\nu| < 1. \tag{4.48}$$

The end point ∞ is $(s_6^{(\nu,\kappa)}, m_6^{(\nu,\kappa)}, 0)$-natural for all ν and

$$s_6^{(\nu,\kappa)}(\infty) < \infty. \tag{4.49}$$

Let $\mathbb{D}_6^{(\nu,\kappa)}$ be the diffusion process on I with generator $\mathcal{G}_6^{(\nu,\kappa)}$, and with the end point 0 being absorbing if $-1 < \nu < 0$. We denote by $p_6^{(\nu,\kappa)}(t,x,y)$ the transition probability density with respect to $dm_6^{(\nu,\kappa)}$. It is known (see [1, p. 142]) that

$$\begin{aligned}&p_6^{(\nu,\kappa)}(t,x,y) \\ &= \frac{\kappa}{\sinh(\kappa t)}(xy)^{-\nu/2}\exp\left\{-\kappa(\nu+1)t - \frac{\kappa e^{\kappa t}(x+y)}{2\sinh(\kappa t)}\right\}I_{|\nu|}\left(\frac{\kappa\sqrt{xy}}{\sinh(\kappa t)}\right).\end{aligned} \tag{4.50}$$

By (4.50),

$$\lim_{\kappa\to 0} p_6^{(\nu,\kappa)}(t,x,y) = p_4^{(\nu)}(t,x,y). \tag{4.51}$$

Since $\lim_{\kappa\to 0} s_6^{(\nu,\kappa)}(x) = s_4^{(\nu)}(x)$ and $\lim_{\kappa\to 0} m_6^{(\nu,\kappa)}(x) = m_4^{(\nu)}(x)$ for $x \in I$, (4.51) also follows from Lemma 5.2 of [8].

In the same way as for (4.41) and (4.42), we find that if $-1 < \nu < 0$,

$$\begin{aligned}\lim_{x,y\to 0} D_{s_6^{(\nu,\kappa)}(x)} D_{s_6^{(\nu,\kappa)}(y)} p_6^{(\nu,\kappa)}(\xi,x,y) \\ = 2^{-|\nu|}\frac{|\nu|}{\Gamma(|\nu|)}\left(\frac{\kappa}{\sinh(\kappa\xi)}\right)^{|\nu|+1} e^{-\kappa(\nu+1)\xi},\end{aligned} \tag{4.52}$$

while if $\nu \geq 0$,

$$\lim_{x,y\to 0} p_6^{(\nu,\kappa)}(\xi,x,y) = 2^{-\nu}\frac{1}{\Gamma(\nu+1)}\left(\frac{\kappa}{\sinh(\kappa\xi)}\right)^{\nu+1} e^{-\kappa(\nu+1)\xi}. \tag{4.53}$$

For $\alpha > 0$ we denote by $g_{6,i}^{(\nu,\kappa)}(x,\alpha)$, $i = 1,\ 2$, the functions satisfying (2.11)–(2.15) with $s = s_6^{(\nu,\kappa)}$, $m = m_6^{(\nu,\kappa)}$ and $k = 0$. It is known (see [1, p. 142][2]) that

$$g_{6,1}^{(\nu,\kappa)}(x,\alpha) = \frac{\alpha^{|\nu|/2}}{2^{|\nu|/2}\kappa^{(|\nu|+1)/2}\Gamma(|\nu|+1)} x^{-\frac{\nu+1}{2}} e^{-\kappa x/2} M_{-\frac{\alpha}{2\kappa}-\frac{\nu+1}{2},\frac{|\nu|}{2}}(\kappa x),$$

$$g_{6,2}^{(\nu,\kappa)}(x,\alpha) = \frac{\kappa^{\frac{|\nu|-1}{2}}}{2^{1-\frac{|\nu|}{2}}\alpha^{\frac{|\nu|}{2}}}\Gamma\left(\frac{|\nu|}{2}+\frac{\nu}{2}+\frac{\alpha}{2\kappa}+1\right) x^{-\frac{\nu+1}{2}} e^{-\kappa x/2} W_{-\frac{\alpha}{2\kappa}-\frac{\nu+1}{2},\frac{|\nu|}{2}}(\kappa x).$$

Now we consider the Lévy measure densities corresponding to the inverse local time at the end point 0.

In the following we assume $-1 < \nu < 1$. Then the end point 0 is $(s_6^{(\nu,\kappa)}, m_6^{(\nu,\kappa)}, 0)$-regular or an $(s_6^{(\nu,\kappa)}, m_6^{(\nu,\kappa)}, 0)$-entrance point, and (2.6) is satisfied in view of (4.48).

(1) For $\beta > 0$, we put $h_6^{(\nu,\kappa,\beta)}(x) = (\beta/2)^{|\nu|/2} g_{6,2}^{(\nu,\kappa)}(x,\beta)$ and denote by $\mathcal{G}_6^{(\nu,\kappa,\beta)}$ the harmonic transform of $\mathcal{G}_6^{(\nu,\kappa)}$ based on $h_6^{(\nu,\kappa,\beta)} \in \mathcal{H}_{s_6^{(\nu,\kappa)},m_6^{(\nu,\kappa)},0,\beta}$, that is,

$$\mathcal{G}_6^{(\nu,\kappa,\beta)} = 2x\frac{d^2}{dx^2} + 4\kappa x\frac{W'_{-\frac{\beta}{2\kappa}-\frac{\nu+1}{2},\frac{|\nu|}{2}}(\kappa x)}{W_{-\frac{\beta}{2\kappa}-\frac{\nu+1}{2},\frac{|\nu|}{2}}(\kappa x)}\frac{d}{dx}.$$

The scale function $s_6^{(\nu,\kappa,\beta)}$ and the speed measure $m_6^{(\nu,\kappa,\beta)}$ are given by

$$ds_6^{(\nu,\kappa,\beta)}(x) = h_6^{(\nu,\kappa,\beta)}(x)^{-2}\, ds_6^{(\nu,\kappa)}(x), \qquad dm_6^{(\nu,\kappa,\beta)}(x) = h_6^{(\nu,\kappa,\beta)}(x)^2\, dm_6^{(\nu,\kappa)}(x).$$

[2]Misprints in [1, p. 142]: there are unnecessary minus signs in exponents of Green functions in the case $\gamma < 0$.

By Proposition 3.3, (3.6) or (3.7) is satisfied, and $s_6^{(\nu,\kappa,\beta)}(\infty) = \infty$. The end point 0 is $(s_6^{(\nu,\kappa,\beta)}, m_6^{(\nu,\kappa,\beta)}, 0)$-regular. We consider the diffusion process $\mathbb{D}_6^{(\nu,\kappa,\beta,*)}$ with generator $\mathcal{G}_6^{(\nu,\kappa,\beta)}$ and with the end point 0 being reflecting. Let $n_6^{(\nu,\kappa,\beta,*)}$ be the corresponding Lévy measure density.

(1-i) Assume $\nu = 0$. Since $D_{s_6^{(0,\kappa)}} h_6^{(0,\kappa,\beta)}(0) = -1/2$ by [7, Problem 17, p. 279] and recursion (see [6, p. 73]), (3.12) and (4.53) imply

$$n_6^{(0,\kappa,\beta,*)}(\xi) = \frac{\kappa}{4\sinh(\kappa\xi)} e^{(-\kappa-\beta)\xi}. \tag{4.54}$$

(1-ii) Assume $-1 < \nu < 0$. Since $h_6^{(\nu,\kappa,\beta)}(0) = \Gamma(|\nu|)/2$, (3.10) and (4.52) lead to

$$n_6^{(\nu,\kappa,\beta,*)}(\xi) = 2^{-|\nu|-2}\Gamma(|\nu|+1)\left(\frac{\kappa}{\sinh(\kappa\xi)}\right)^{|\nu|+1} e^{(-\kappa(\nu+1)-\beta)\xi}. \tag{4.55}$$

(1-iii) Assume $0 < \nu < 1$. Since $D_{s_6^{(\nu,\kappa)}} h_6^{(\nu,\kappa,\beta)}(0) = -\Gamma(\nu+1)/2$ by recursion (see [6, p. 73]), (3.12) and (4.53) imply

$$n_6^{(\nu,\kappa,\beta,*)}(\xi) = 2^{-\nu-2}\Gamma(\nu+1)\left(\frac{\kappa}{\sinh(\kappa\xi)}\right)^{\nu+1} e^{(-\kappa(\nu+1)-\beta)\xi}. \tag{4.56}$$

By one of the limit theorems of [1, p. 640], we get $\lim_{\kappa\to 0} g_{6,2}^{(\nu,\kappa)}(x,\beta) = g_{4,2}^{(\nu)}(x,\beta)$, and hence $\lim_{\kappa\to 0} s_6^{(\nu,\kappa,\beta)}(x) = s_4^{(\nu,\beta)}(x)$ and $\lim_{\kappa\to 0} m_6^{(\nu,\kappa,\beta)}(x) = m_4^{(\nu,\beta)}(x)$. By Theorem 2.4,

$$\lim_{\kappa\to 0} n_6^{(\nu,\kappa,\beta,*)}(\xi) = n_4^{(\nu,\beta,*)}(\xi).$$

This also follows from (4.34)–(4.36) and (4.54)–(4.56).

(2) Since $s_6^{(\nu,\kappa)}(\infty) < \infty$ by (4.49), Proposition 3.4 leads to a special case corresponding to $\beta = 0$. Put $h_6^{(\nu,\kappa,0)}(x) = \{s_6^{(\nu,\kappa)}(\infty) - s_6^{(\nu,\kappa)}(x)\}/\{s_6^{(\nu,\kappa)}(\infty) - s_6^{(\nu,\kappa)}(1)\}$. Denote by $\mathcal{G}_6^{(\nu,\kappa,0)}$ the harmonic transform of $\mathcal{G}_6^{(\nu,\kappa)}$ based on $h_6^{(\nu,\kappa,0)} \in \mathcal{H}_{s_6^{(\nu,\kappa)}, m_6^{(\nu,\kappa)}, 0, 0}$, that is,

$$\mathcal{G}_6^{(\nu,\kappa,0)} = 2x\frac{d^2}{dx^2} + \left\{2\nu + 2 + 2\kappa x - \frac{4x^{-\nu}e^{-\kappa x}}{\int_x^\infty y^{-(\nu+1)}e^{-\kappa y}\,dy}\right\}\frac{d}{dx}.$$

The scale function $s_6^{(\nu,\kappa,0)}$ and the speed measure $m_6^{(\nu,\kappa,0)}$ are given by

$$ds_6^{(\nu,\kappa,0)}(x) = h_6^{(\nu,\kappa,0)}(x)^{-2}\,ds_6^{(\nu,\kappa)}(x), \qquad dm_6^{(\nu,\kappa,0)}(x) = h_6^{(\nu,\kappa,0)}(x)^2\,dm_6^{(\nu,\kappa)}(x).$$

By Proposition 3.4, (3.6) or (3.7) is satisfied, and $s_6^{(\nu,\kappa,0)}(\infty) = \infty$. The end point 0 is $(s_6^{(\nu,\kappa,0)}, m_6^{(\nu,\kappa,0)}, 0)$-regular. We consider the diffusion process $\mathbb{D}_6^{(\nu,\kappa,0,*)}$ with generator $\mathcal{G}_6^{(\nu,\kappa,0)}$ and with the end point 0 being reflecting. Let $n_6^{(\nu,\kappa,0,*)}$ be the corresponding Lévy measure density.

(2-i) Assume $0 \le \nu < 1$. Since $D_{s_6^{(\nu,\kappa)}} h_6^{(\nu,\kappa,0)}(0) = -\{s_6^{(\nu,\kappa)}(\infty) - s_6^{(\nu,\kappa)}(1)\}^{-1}$, (3.12) implies

$$n_6^{(\nu,\kappa,0,*)}(\xi) = \{s_6^{(\nu,\kappa)}(\infty) - s_6^{(\nu,\kappa)}(1)\}^{-2} 2^{-\nu} \frac{1}{\Gamma(\nu+1)} \left(\frac{\kappa}{\sinh(\kappa\xi)}\right)^{\nu+1} e^{-\kappa(\nu+1)\xi},$$

where we used (4.52). We note that

$$\lim_{\kappa\to 0} n_6^{(\nu,\kappa,0,*)}(\xi) = 2^{-\nu} \frac{\nu^2}{\Gamma(\nu+1)} \xi^{-\nu-1}. \tag{4.57}$$

On the other hand,

$$\lim_{\kappa\to 0} h_6^{(\nu,\kappa,0)}(x) = x^{-\nu},$$

and hence $\lim_{\kappa\to 0} s_6^{(\nu,\kappa,0)} = s_4^{(-\nu)}$ and $\lim_{\kappa\to 0} m_6^{(\nu,\kappa,0)} = m_4^{(-\nu)}$.

In the case $0 < \nu < 1$, the end point 0 is $(s_4^{(-\nu)}, m_4^{(-\nu)}, 0)$-regular, and hence

$$\lim_{\kappa\to 0} n_6^{(\nu,\kappa,0,*)}(\xi) = n_4^{(-\nu,*)}(\xi) = 2^{-\nu} \frac{\nu^2}{\Gamma(\nu+1)} \xi^{-\nu-1}.$$

This shows (4.57) with $0 < \nu < 1$.

In the case $\nu = 0$, the end point 0 is an $(s_4^{(0)}, m_4^{(0)}, 0)$-entrance point and hence the local time at the end point 0 does not exist.

(2-ii) Assume $-1 < \nu < 0$. Since $h_6^{(\nu,\kappa,0)}(0) \in (0,\infty)$, by (3.10) and (4.52) we get

$$n_6^{(\nu,\kappa,0,*)}(\xi) = h_6^{(\nu,\kappa,0)}(0)^2 2^{-|\nu|} \frac{|\nu|}{\Gamma(|\nu|)} \left(\frac{\kappa}{\sinh(\kappa\xi)}\right)^{|\nu|+1} e^{-\kappa(\nu+1)\xi}. \tag{4.58}$$

As $\lim_{\kappa\to 0} h_6^{(\nu,\kappa,0)}(x) = 1$, we have $\lim_{\kappa\to 0} s_6^{(\nu,\kappa,0)}(x) = s_4^{(\nu)}$ and $\lim_{\kappa\to 0} m_6^{(\nu,\kappa,0)}(x) = m_4^{(\nu)}$. By Theorem 2.4 and (4.33),

$$\lim_{\kappa\to 0} n_6^{(\nu,\kappa,0,*)}(\xi) = n_4^{(\nu,*)}(\xi) = 2^{-|\nu|} \frac{|\nu|}{\Gamma(|\nu|)} \xi^{-|\nu|-1}.$$

This also follows from (4.58).

Example 4.7 (Brownian motion). Let us consider the following diffusion operator on $I = (0,a)$, where $0 < a < \infty$:

$$\mathcal{G}_7^{(a)} = \frac{1}{2} \frac{d^2}{dx^2}.$$

This is the generator of Brownian motion on $(0,a)$, and the scale function $s_7^{(a)}$ and the speed measure $m_7^{(a)}$ are given by

$$ds_7^{(a)}(x) = dx, \qquad dm_7^{(a)}(x) = 2\,dx.$$

The killing measure is null. The end points 0 and a are both $(s_7^{(a)}, m_7^{(a)}, 0)$-regular.

Let $\mathbb{D}_7^{(a)}$ be the Brownian motion on I with generator $\mathcal{G}_7^{(a)}$, and with the end points 0 and a being both absorbing. We denote by $p_7^{(a)}(t,x,y)$ the transition probability density with respect to $dm_7^{(a)}$. It is known (see [1, p. 122]) that

$$\begin{aligned} p_7^{(a)}(t,x,y) &= \frac{1}{a}\sum_{n=1}^{\infty} e^{-n^2\pi^2 t/(2a^2)} \sin(n\pi x/a)\sin(n\pi y/a) \\ &= \int_{(0,\infty)} e^{-\lambda t}\psi_7^{(a)}(x,\lambda)\psi_7^{(a)}(y,\lambda)\, d\sigma_7^{(a)}(\lambda), \end{aligned}$$

where

$$\psi_7^{(a)}(x,\lambda_n) = \frac{a}{n\pi}\sin(n\pi x/a), \qquad d\sigma_7^{(a)}(\lambda) = \sum_{n=1}^{\infty}\frac{n^2\pi^2}{a^3}\delta_{\lambda_n}(d\lambda),$$

$\lambda_n = n^2\pi^2/(2a^2)$, and $\delta_p(d\lambda)$ is the unit measure concentrated at p. For $\alpha > 0$, we denote by $g_{7,i}^{(a)}(x,\alpha)$, $i = 1,2$, the function satisfying (2.11)–(2.15) with $s = s_7^{(a)}$, $m = m_7^{(a)}$ and $k = 0$. It is easy to see that

$$g_{7,1}^{(a)}(x,\alpha) = e^{\sqrt{2\alpha}x} - e^{-\sqrt{2\alpha}x}, \qquad g_{7,2}^{(a)}(x,\alpha) = e^{\sqrt{2\alpha}(a-x)} - e^{-\sqrt{2\alpha}(a-x)}.$$

Now we consider the Lévy measure densities corresponding to the inverse local time at the end point 0. For $\beta > 0$ we put $h_7^{(a)}(x) = Cg_{7,2}^{(a)}(x,\beta)$, where $C = g_{7,2}^{(a)}(0,\beta)^{-1}$ and denote by $\mathcal{G}_7^{(a,\beta)}$ the harmonic transform of $\mathcal{G}_7^{(a)}$ based on $h_7^{(a,\beta)} \in \mathcal{H}_{s_7^{(a)},m_7^{(a)},0,\beta}$, that is,

$$\mathcal{G}_7^{(a,\beta)} = \frac{1}{2}\frac{d^2}{dx^2} - \sqrt{2\beta}\,\frac{e^{2\sqrt{2\beta}(a-x)}+1}{e^{2\sqrt{2\beta}(a-x)}-1}\frac{d}{dx},$$

which is (1.9). The scale function $s_7^{(a,\beta)}$ and the speed measure $m_7^{(a,\beta)}$ are given by

$$ds_7^{(a,\beta)}(x) = h_7^{(a,\beta)}(x)^{-2}\, ds_7^{(a)}(x), \qquad dm_7^{(a,\beta)}(x) = h_7^{(a,\beta)}(x)^2\, dm_7^{(a)}(x).$$

Since $h_7^{(a)}(0) = 1$, by Proposition 3.3, (3.6) is satisfied, and $s_7^{(a,\beta)}(a) = \infty$. The end point 0 is $(s_7^{(a,\beta)}, m_7^{(a,\beta)}, 0)$-regular. We consider the diffusion process $\mathbb{D}_7^{(a,\beta,*)}$ with generator $\mathcal{G}_7^{(a,\beta)}$ and with the end point 0 being reflecting. Let $n_7^{(a,\beta,*)}$ be the corresponding Lévy measure density. By (3.9),

$$n_7^{(a,\beta,*)}(\xi) = e^{-\beta\xi}\int_{(0,\infty)} e^{-\xi\lambda}\, d\sigma_7^{(a)}(\lambda) = e^{-\beta\xi}\sum_{n=1}^{\infty} e^{-\xi\lambda_n}\frac{n^2\pi^2}{a^3}.$$

This shows (1.10).

We note that

$$\lim_{a\to\infty} n_7^{(a,\beta,*)}(\xi) = \sqrt{\pi/2}\, e^{-\beta\xi}\xi^{-3/2}. \tag{4.59}$$

Indeed, for $\Lambda > 0$,

$$\int_{(0,\Lambda]} d\sigma_7^{(a)}(\lambda) = \frac{\pi^2}{a^3}\sum_{n:\lambda_n\le\Lambda} n^2 = \frac{\pi^2}{a^3}\sum_{n=1}^{\Lambda_0} n^2 = \frac{\pi^2}{a^3}\left(\frac{1}{3}\Lambda_0^3 + \frac{1}{2}\Lambda_0^2 + \frac{1}{6}\Lambda_0\right)$$
$$\to \frac{2\sqrt{2}}{3\pi}\Lambda^{3/2} \quad \text{as } a\to\infty,$$

where Λ_0 indicates the integral part of $\sqrt{2\Lambda}\, a/\pi$. Thus we have (4.59). We can also show (4.59) by using Theorem 2.4. Here is a proof. Let us consider the diffusion operator

$$\mathcal{G}_7^{(\uparrow,\beta)} = \frac{1}{2}\frac{d^2}{dx^2} - \sqrt{2\beta}\frac{d}{dx}.$$

The scale function $s_7^{(\uparrow,\beta)}$ and the speed measure $m_7^{(\uparrow,\beta)}$ are given by

$$ds_7^{(\uparrow,\beta)}(x) = e^{2\sqrt{2\beta}x}\,dx, \qquad dm_7^{(\uparrow,\beta)}(x) = 2e^{-2\sqrt{2\beta}x}\,dx.$$

The killing measure is null. The end point 0 is $(s_7^{(\uparrow,\beta)}, m_7^{(\uparrow,\beta)}, 0)$-regular. Let $\mathbb{D}_7^{(\uparrow,\beta,*)}$ be the diffusion process on $(0,\infty)$ with generator $\mathcal{G}_7^{(\uparrow,\beta)}$, and with the end point 0 being reflecting. We denote by $n_7^{(\uparrow,\beta,*)}(\xi)$ the corresponding Lévy measure density. By using [10, Example 6.1] and (2.24), we have

$$n_7^{(\uparrow,\beta,*)}(\xi) = \int_\beta^\infty e^{-\lambda\xi}\sigma_7^{(\uparrow,\beta)}(\lambda)\,d\lambda = \sqrt{\pi/2}\, e^{-\beta\xi}\xi^{-3/2}.$$

As $\lim_{a\to\infty} h_7^{(a,\beta)}(x) = e^{-\sqrt{2\beta}x}$, $\lim_{a\to\infty} s_7^{(a,\beta)}(x) = s_7^{(\uparrow,\beta)}(x)$, and $\lim_{a\to\infty} m_7^{(a,\beta)}(x) = m_7^{(\uparrow,\beta)}(x)$, by Theorem 2.4 we obtain

$$\lim_{a\to\infty} n_7^{(a,\beta,*)}(\xi) = n_7^{(\uparrow,\beta,*)}(\xi).$$

This shows (4.59).

Acknowledgements

The second author was partially supported by Grant-in-Aid for Scientific Research (C) 22540132.

References

[1] A. N. Borodin and P. Salminen, *Handbook of Brownian motion—facts and formulae*, Birkhäuser, Basel, 2002. Zbl 1012.60003 MR 1912205

[2] C. Donati-Martin and M. Yor, Some explicit Krein representations of certain subordinators, including the Gamma processes, Publ. RIMS Kyoto Univ. **42** (2006), 879–895. Zbl 1123.60028 MR 2289152

[3] C. Donati-Martin and M. Yor, Further examples of explicit Krein representations of certain subordinators. Publ. RIMS Kyoto Univ. **43** (2007), 315–328. Zbl 1129.60042 MR 2341013

[4] W. Feller, The parabolic differential equations and the associated semi-groups of transformations, Ann. of Math. **55** (1952), 468–519. Zbl 0047.09303 MR 0047886

[5] K. Itô and H. P. McKean, Jr., *Diffusion processes and their sample paths*, Springer, New York, 1974. Zbl 0285.60063 MR 0345224

[6] S. Moriguchi, K. Udagawa and S. Hitotsumatsu, *Mathematical formulas. III*, Iwanami, 1960 (in Japanese).

[7] N. N. Lebedev, *Spectral functions and their applications*, Dover Publ., New York, 1972.

[8] Y. Ogura, One-dimensional bi-generalized diffusion processes, J. Math. Soc. Japan **41** (1989), 213–242. Zbl 0701.60078 MR 0984748

[9] T. Takemura, State of boundaries for harmonic transforms of one-dimensional generalized diffusion processes, Annual Reports Grad. School Humanities Sci. Nara Women's Univ. **25** (2019), 285–294.

[10] T. Takemura and M. Tomisaki, h transform of one-dimensional generalized diffusion operators, Kyushu J. Math. **66** (2012), 171–191. Zbl pre06135998 MR 2962397

[11] M. Tomisaki, Intrinsic ultracontractivity and small perturbation for one-dimensional generalized diffusion operators, J. Funct. Anal. **251** (2007), 289–324. Zbl 1135.47044 MR 2353708

[12] M. Tomisaki, Inverse local times of harmonic transformed Bessel processes, Annual Reports Grad. School Humanities Sci. Nara Women's Univ. **22** (2011), 269–279.