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Positivity for Cluster Algebras of Rank 3

Dedicated to Robert Lazarsfeld on the occasion of his sixtieth birthday

by

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Abstract

We prove the positivity conjecture for skew-symmetric coefficient-free cluster algebras of rank 3.

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§1. Introduction

Cluster algebras have been introduced by Fomin and Zelevinsky in [13] in the context of total positivity and canonical bases in Lie theory. Since then cluster algebras have been shown to be related to various fields in mathematics including representation theory of finite-dimensional algebras, Teichmüller theory, Poisson geometry, combinatorics, Lie theory, tropical geometry and mathematical physics.

A cluster algebra is a subalgebra of a field of rational functions in n variables $x_1, \dots, x_n$, given by specifying a set of generators, the so-called *cluster variables*. These generators are constructed in a recursive way, starting from the initial variables $x_1, \dots, x_n$, by a procedure called *mutation*, which is determined by the choice of a skew symmetric $n \times n$ integer matrix B or, equivalently, by a quiver Q. Although each mutation is an elementary operation, it is very difficult to compute cluster variables in general, because of the recursive character of the construction.

Finding explicit computable direct formulas for the cluster variables is one of the main open problems in the theory of cluster algebras and has been studied

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by many mathematicians. In 2002, Fomin and Zelevinsky showed that every cluster variable is a Laurent polynomial in the initial variables $x_1, \ldots, x_n$, and they conjectured that this Laurent polynomial has positive coefficients [13].

This *positivity conjecture* has been proved in the following special cases

- *Acyclic cluster algebras.* These are cluster algebras given by a quiver that is mutation equivalent to a quiver without oriented cycles. In this case, positivity has been shown in [17] building on [5, 16, 23, 24] using monoidal categorifications of quantum cluster algebras and perverse sheaves over graded quiver varieties. If the initial seed itself is acyclic, the conjecture has also been shown in [9] using Donaldson–Thomas theory.
- *Cluster algebras from surfaces.* In this case, positivity has been shown in [22] building on [27, 28, 26], using the fact that each cluster variable in such a cluster algebra corresponds to a curve in an oriented Riemann surface and the Laurent expansion of the cluster variable is determined by the crossing pattern of the curve with a fixed triangulation of the surface [11, 12]. The construction and the proof of the positivity conjecture have been generalized to non-skew-symmetric cluster algebras from orbifolds in [10].

Our approach in this paper is different. We prove positivity almost exclusively by elementary algebraic computation. The advantage of this approach is that we do not need to restrict to a special type of cluster algebras but can work in the setting of an arbitrary cluster algebra. The drawback of our approach is that because of the sheer complexity of the computation, we need to restrict ourselves in this paper to rank three. Rank three is crucial since it is the smallest rank in which nonacyclic cluster algebras exist. Our main result is the following.

Theorem 1.1. *The positivity conjecture holds in every skew-symmetric coefficient-free cluster algebra of rank* 3.

Our argument provides a method for the computation of the Laurent expansions of cluster variables, and we include some examples of explicit calculation. We point out that direct formulas for the Laurent polynomials have been obtained in several special cases. The most general results are the following:

- A formula involving the Euler–Poincaré characteristic of quiver Grassmannians obtained in [15, 8] using categorification and generalizing results in [6, 7]. While this formula shows a very interesting connection between cluster algebras and geometry, it is of limited computational use, since the Euler–Poincaré characteristics of quiver Grassmannians are hard to compute. In particular, this formula does not show positivity. On the other hand, the positivity result in this paper proves the positivity of the Euler–Poincaré characteristics of the quiver Grassmannians involved.

- An elementary combinatorial formula for cluster algebras from surfaces given in [22].
- A formula for cluster variables corresponding to string modules as a product of 2×2 matrices obtained in [1], generalizing a result in [2].

The main tools of the proof are modified versions of two formulas for the rank two case, one obtained by the first author in [18] and the other obtained by both authors in [20]. These formulas allow for the computation of the Laurent expansions of a given cluster variable with respect to any seed which is close enough to the variable in the sense that there is a sequence of mutations in only two vertices which links seed and variable. The general result then follows by inductive reasoning.

If the cluster algebra is not skew-symmetric, it is shown in [25, 19] that (an adaptation of) the second rank two formula still holds. We therefore expect that our argument can be generalized to prove the positivity conjecture for non-skew-symmetric cluster algebras of rank 3.

The article is organized as follows. We start by recalling some definitions and results from the theory of cluster algebras in Section 2. In Section 3, we present several formulas for the rank 2 case when considered inside a cluster algebra of rank 3. We use each of these formulas in the proof of the positivity conjecture for rank 3 in Section 4. An example is given in Section 5.

§2. Cluster algebras

In this section, we review some notions from the theory of cluster algebras.

§2.1. Definition and Laurent phenomenon

We begin by reviewing the definition of cluster algebra, first introduced by Fomin and Zelevinsky [13]. Our definition follows the exposition in [14].

To define a cluster algebra $\mathcal{A}$ we must first fix its ground ring. Let $(\mathbb{P}, \oplus, \cdot)$ be a *semifield*, i.e., an abelian multiplicative group endowed with a binary operation of (*auxiliary*) *addition* $\oplus$ which is commutative, associative, and distributive with respect to the multiplication in $\mathbb{P}$. The group ring $\mathbb{ZP}$ will be used as a *ground ring* for $\mathcal{A}$.

As an *ambient field* for $\mathcal{A}$, we take a field $\mathcal{F}$ isomorphic to the field of rational functions in n independent variables (here n is the *rank* of $\mathcal{A}$), with coefficients in $\mathbb{QP}$. Note that the definition of $\mathcal{F}$ does not involve the auxiliary addition in $\mathbb{P}$.

Definition 2.1. A *labeled seed* in $\mathcal{F}$ is a triple $(\mathbf{x}, \mathbf{y}, B)$, where

- $\mathbf{x} = (x_1, \dots, x_n)$ is an n-tuple from $\mathcal{F}$ forming a *free generating set* over $\mathbb{QP}$,
- $\mathbf{y} = (y_1, \dots, y_n)$ is an n-tuple from $\mathbb{P}$, and
- $B = (b_{ij})$ is an $n \times n$ integer matrix which is *skew-symmetrizable*.

That is, $x_1, \dots, x_n$ are algebraically independent over $\mathbb{QP}$, and $\mathcal{F} = \mathbb{QP}(x_1, \dots, x_n)$. We refer to $\mathbf{x}$ as the (labeled) *cluster* of a labeled seed $(\mathbf{x}, \mathbf{y}, B)$, to the tuple $\mathbf{y}$ as the *coefficient tuple*, and to the matrix B as the *exchange matrix*.

We use the notation $[x]_+ = \max(x, 0)$, $[1, n] = \{1, \dots, n\}$, and

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Definition 2.2. Let $(\mathbf{x}, \mathbf{y}, B)$ be a labeled seed in $\mathcal{F}$, and let $k \in [1, n]$. The *seed mutation* μ_k in direction k transforms $(\mathbf{x}, \mathbf{y}, B)$ into the labeled seed $\mu_k(\mathbf{x}, \mathbf{y}, B) = (\mathbf{x}', \mathbf{y}', B')$ defined as follows:

- The entries of $B' = (b'_{ij})$ are given by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \operatorname{sgn}(b_{ik})\ [b_{ik} b_{kj}]_+ & \text{otherwise.} \end{cases} \tag{1}$$

- The coefficient tuple $\mathbf{y}' = (y'_1, \dots, y'_n)$ is given by

$$y'_j = \begin{cases} y_k^{-1} & \text{if } j = k, \\ y_j y_k^{[b_{kj}]_+} (y_k \oplus 1)^{-b_{kj}} & \text{if } j \neq k. \end{cases} \tag{2}$$

- The cluster $\mathbf{x}' = (x'_1, \dots, x'_n)$ is given by $x'_j = x_j$ for $j \neq k$, whereas $x'_k \in \mathcal{F}$ is determined by the *exchange relation*

$$x'_k = \frac{y_k \prod x_i^{[b_{ik}]_+} + \prod x_i^{[-b_{ik}]_+}}{(y_k \oplus 1) x_k}. \tag{3}$$

We say that two exchange matrices B and B' are *mutation-equivalent* if one can get from B to B' by a sequence of mutations.

Definition 2.3. Consider the *n-regular tree* $\mathbb{T}_n$ whose edges are labeled by the numbers $1, \dots, n$, so that the n edges emanating from each vertex receive different labels. A *cluster pattern* is an assignment of a labeled seed $\Sigma_t = (\mathbf{x}_t, \mathbf{y}_t, B_t)$ to every vertex $t \in \mathbb{T}_n$, such that the seeds assigned to the endpoints of any edge $t \overset{k}{\text{—}} t'$ are obtained from each other by the seed mutation in direction k. The components of Σ_t are written as

(4) $$\mathbf{x}_t = (x_{1;t}, \ldots, x_{n;t}), \quad \mathbf{y}_t = (y_{1;t}, \ldots, y_{n;t}), \quad B_t = (b_{ij}^t).$$

Clearly, a cluster pattern is uniquely determined by an arbitrary seed.

Definition 2.4. Given a cluster pattern, we denote

(5) $$\mathcal{X} = \bigcup_{t \in \mathbb{T}_n} \mathbf{x}_t = \{x_{i,t} : t \in \mathbb{T}_n,\, 1 \le i \le n\},$$

the union of clusters of all the seeds in the pattern. The elements $x_{i,t} \in \mathcal{X}$ are called *cluster variables*. The *cluster algebra* $\mathcal{A}$ associated with a given pattern is the $\mathbb{ZP}$-subalgebra of the ambient field $\mathcal{F}$ generated by all cluster variables: $\mathcal{A} = \mathbb{ZP}[\mathcal{X}]$. We denote $\mathcal{A} = \mathcal{A}(\mathbf{x}, \mathbf{y}, B)$, where $(\mathbf{x}, \mathbf{y}, B)$ is any seed in the underlying cluster pattern.

The cluster algebra is called *skew-symmetric* if the matrix B is skew-symmetric. In this case, it is often convenient to represent the $n \times n$ matrix B by a quiver Q_B with vertices $1, \ldots, n$ and $[b_{ij}]_+$ arrows from vertex i to vertex j.

If $\mathbb{P} = 1$ then the cluster algebra is said to be *coefficient-free*.

The main result in this paper is on coefficient-free cluster algebras. However, we need cluster algebras with coefficients in Section 3.

In [13], Fomin and Zelevinsky proved the remarkable *Laurent phenomenon* and posed the following *positivity conjecture*.

Theorem 2.5 (Laurent Phenomenon). *For any cluster algebra $\mathcal{A}$ and any seed Σ_t, each cluster variable x is a Laurent polynomial over $\mathbb{ZP}$ in the cluster variables from $\mathbf{x}_t = (x_{1;t}, \ldots, x_{n;t})$.*

Conjecture 2.6 (Positivity Conjecture). For any cluster algebra $\mathcal{A}$, any seed Σ, and any cluster variable x, the Laurent polynomial has coefficients which are nonnegative integer linear combinations of elements in $\mathbb{P}$.

§2.2. Cluster algebras with principal coefficients

One important choice for $\mathbb{P}$ is the tropical semifield; in this case we say that the corresponding cluster algebra is *of geometric type*.

Definition 2.7. Let $\mathrm{Trop}(u_1, \ldots, u_m)$ be an abelian group (written multiplicatively) freely generated by the u_j. We define $\oplus$ in $\mathrm{Trop}(u_1, \ldots, u_m)$ by

$$\prod_j u_j^{a_j} \oplus \prod_j u_j^{b_j} = \prod_j u_j^{\min(a_j, b_j)},$$

and call $(\mathrm{Trop}(u_1, \ldots, u_m), \oplus, \cdot)$ a *tropical semifield.*

Remark 2.8. In cluster algebras whose ground ring is $\mathrm{Trop}(u_1, \ldots, u_m)$ (the tropical semifield), it is convenient to replace the matrix B by an $(n+m) \times n$ matrix $\tilde{B} = (b_{ij})$ whose upper part is the $n \times n$ matrix B and whose lower part is an $m \times n$ matrix that encodes the coefficient tuple via

$$y_k = \prod_{i=1}^{m} u_i^{b_{(n+i)k}}. \tag{6}$$

Then the mutation of the coefficient tuple in (2) is determined by the mutation of the matrix $\tilde{B}$ in equation (1) and the formula (6); and the exchange relation (3) becomes

$$x'_k = x_k^{-1}\Big(\prod_{i=1}^{n} x_i^{[b_{ik}]_+} \prod_{i=1}^{m} u_i^{[b_{(n+i)k}]_+} + \prod_{i=1}^{n} x_i^{[-b_{ik}]_+} \prod_{i=1}^{m} u_i^{[-b_{(n+i)k}]_+}\Big). \tag{7}$$

Recall from [14] that a cluster algebra $\mathcal{A}$ is said to have *principal coefficients at a vertex* t if $\mathbb{P} = \mathrm{Trop}(y_1, \ldots, y_n)$ and $\mathbf{y}_t = (y_1, \ldots, y_n)$.

Definition 2.9. Let $\mathcal{A}$ be the cluster algebra with principal coefficients at t, defined by the initial seed $(\mathbf{x}_t, \mathbf{y}_t\, B_t)$ with

$$\mathbf{x}_t = (x_1, \ldots, x_n), \quad \mathbf{y}_t = (y_1, \ldots, y_n).$$

- Let $X_{\ell;t}$ be the Laurent expansion of the cluster variable $x_{\ell;t}$ in $x_1, \ldots, x_n$, $y_1, \ldots, y_n$.
- The *F-polynomial* of the cluster variable $x_{\ell,t}$ is defined as $F_{\ell;t} = X_{\ell;t}(1, \ldots, 1; y_1, \ldots, y_n)$.
- The *g-vector* $\mathbf{g}_{\ell;t}$ of the cluster variable $x_{\ell,t}$ is defined as the degree vector of the monomial $X_{\ell;t}(x_1, \ldots, x_n; 0, \ldots, 0)$.

The following theorem shows that expansion formulas in principal coefficients can be used to compute expansions in arbitrary coefficient systems.

Theorem 2.10 ([14, Theorem 3.7]). *Let $\mathcal{A}$ be a cluster algebra over an arbitrary semifield $\mathbb{P}$ with initial seed*

$$((x_1, \ldots, x_n), (\hat{y}_1, \ldots, \hat{y}_n), B).$$

Then the cluster variables in $\mathcal{A}$ can be expressed as follows:

$$x_{\ell;t} = \frac{X_{\ell;t}(x_1, \ldots, x_n; \hat{y}_1, \ldots, \hat{y}_n)}{F_{\ell;t}|_{\mathbb{P}}(\hat{y}_1, \ldots, \hat{y}_n)}. \tag{8}$$

where $F_{\ell;t}|_{\mathbb{P}}(\hat{y}_1, \ldots, \hat{y}_n)$ is the F-polynomial evaluated at $\hat{y}_1, \ldots, \hat{y}_n$ inside the semifield $\mathbb{P}$.

§3. Rank 2 considerations

In this section, we use the rank 2 formula from [20] (in the parametrization of [19]) to compute in a nonacyclic cluster algebra of rank three the Laurent expansions of those cluster variables which are obtained from the initial cluster by a mutation sequence involving only two vertices.

§3.1. Rank 2 formula

We start by recalling from [19] the formula for the Laurent expansion of an arbitrary cluster variable in the cluster algebra of rank 2 given by the initial quiver with r arrows

$$1 \xrightarrow{\quad r \quad} 2$$

where $r \geq 2$ is a positive integer. The cluster variables x_n in this cluster algebra are defined by the following recursion:

$$x_{n+1} = (x_n^r + 1)/x_{n-1} \qquad \text{for any integer } n.$$

Let $\{c_n^{[r]}\}_{n\in\mathbb{Z}}$ be the sequence defined by the recurrence relation

$$c_n^{[r]} = rc_{n-1}^{[r]} - c_{n-2}^{[r]},$$

with the initial condition $c_1^{[r]} = 0$, $c_2^{[r]} = 1$. The $c_n^{[r]}$ are Chebyshev polynomials. For example, if $r = 2$ then $c_n^{[r]} = n - 1$; if $r = 3$, the sequence $c_n^{[r]}$ takes the following values:

$$\ldots, -3, -1, 0, 1, 3, 8, 21, 55, 144, \ldots.$$

The pair of the absolute values of the integers $(c_{n-1}^{[r]}, c_{n-2}^{[r]})$ is the degree of the denominator of the cluster variable x_n.

If the value of r is clear from the context, we usually write c_n instead of $c_n^{[r]}$.

Lemma 3.1. *Let $n \geq 3$. Then $c_{n-1}c_{n+k-3} - c_{n+k-2}c_{n-2} = c_k$ for $k \in \mathbb{Z}$. In particular, $c_{n-1}^2 - c_n c_{n-2} = 1$.*

Proof. The result holds for $n = 3$. Suppose that $n \geq 4$. Then

$$\begin{aligned} c_{n+k-2}c_{n-2} &= rc_{n+k-3}c_{n-2} - c_{n+k-4}c_{n-2} \stackrel{*}{=} rc_{n+k-3}c_{n-2} - (c_k + c_{n+k-3}c_{n-3}) \\ &= c_{n+k-3}(rc_{n-2} - c_{n-3}) - c_k = c_{n+k-3}c_{n-1} - c_k, \end{aligned}$$

where $*$ holds by induction. □

Let (a_1, a_2) be a pair of nonnegative integers. A *Dyck path* of type $a_1 \times a_2$ is a lattice path from $(0, 0)$ to (a_1, a_2) that never goes above the main diagonal joining $(0, 0)$ and (a_1, a_2). Among the Dyck paths of a given type $a_1 \times a_2$, there is a (unique) *maximal* one denoted by $\mathcal{D} = \mathcal{D}^{a_1 \times a_2}$. It is defined by the property that any lattice point strictly above $\mathcal{D}$ is also strictly above the main diagonal.

Let $\mathcal{D} = \mathcal{D}^{a_1 \times a_2}$. Let $\mathcal{D}_1 = \{u_1, \dots, u_{a_1}\}$ be the set of horizontal edges of $\mathcal{D}$ indexed from left to right, and $\mathcal{D}_2 = \{v_1, \dots, v_{a_2}\}$ the set of vertical edges of $\mathcal{D}$ indexed from bottom to top. Given any points A and B on $\mathcal{D}$, let AB be the subpath starting from A, and going in the northeast direction until it reaches B (if we reach (a_1, a_2) first, we continue from $(0,0)$). By convention, if $A = B$, then AA is the subpath that starts from A, then passes (a_1, a_2) and ends at A. If we represent a subpath of $\mathcal{D}$ by its set of edges, then for $A = (i, j)$ and $B = (i', j')$, we have

$$AB = \begin{cases} \{u_k, v_\ell : i < k \le i',\ j < \ell \le j'\} & \text{if } B \text{ is to the northeast of } A, \\ \mathcal{D} - \{u_k, v_\ell : i' < k \le i,\ j' < \ell \le j\} & \text{otherwise.} \end{cases}$$

We denote by $(AB)_1$ the set of horizontal edges in AB, and by $(AB)_2$ the set of vertical edges in AB. Also let AB° denote the set of lattice points on the subpath AB excluding the endpoints A and B (here $(0,0)$ and (a_1, a_2) are regarded as the same point).

Here is an example for $(a_1, a_2) = (6, 4)$.

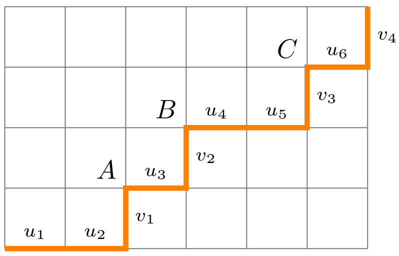


Figure 1. A maximal Dyck path.

Let $A = (2,1)$, $B = (3,2)$ and $C = (5,3)$. Then

$$(AB)_1 = \{u_3\},\ (AB)_2 = \{v_2\},\ (BA)_1 = \{u_4, u_5, u_6, u_1, u_2\},\ (BA)_2 = \{v_3, v_4, v_1\}.$$

The point C is in BA° but not in AB°. The subpath AA has length 10 (not 0).

Definition 3.2. For $S_1 \subseteq \mathcal{D}_1$, $S_2 \subseteq \mathcal{D}_2$, we say that the pair (S_1, S_2) is *compatible* if for every $u \in S_1$ and $v \in S_2$, denoting by E the left endpoint of u and F the upper endpoint of v, there exists a lattice point $A \in EF^\circ$ such that

$$|(AF)_1| = r|(AF)_2 \cap S_2| \quad \text{or} \quad |(EA)_2| = r|(EA)_1 \cap S_1|. \tag{9}$$

With all this terminology in place we are ready to present the combinatorial expression for greedy elements. The following has been proved in [21, 19].

Theorem 3.3. *For $n \geq 3$, we have*

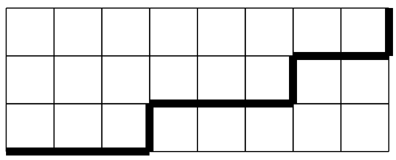
$$x_n = x_1^{-c_{n-1}} x_2^{-c_{n-2}} \sum_{(S_1,S_2)} x_1^{r|S_2|} x_2^{r|S_1|}, \tag{10}$$

$$x_{3-n} = x_2^{-c_{n-1}} x_1^{-c_{n-2}} \sum_{(S_1,S_2)} x_2^{r|S_2|} x_1^{r|S_1|}, \tag{11}$$

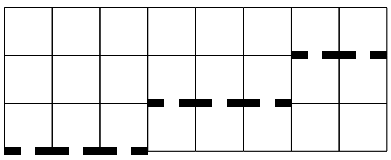
where the sums are over all compatible pairs (S_1, S_2) in $\mathcal{D}^{c_{n-1}\times c_{n-2}}$.

Remark 3.4. For $n = 2$, the formula is consistent if we impose the additional convention that $\mathcal{D}^{c_1\times c_0}$ is the empty set.

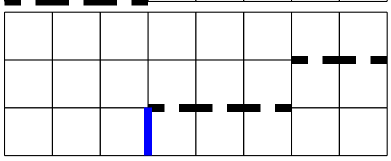
Example 3.5. Let $r = 3$ and $n = 5$. Then $\mathcal{D}^{8\times 3}$ is the following path:



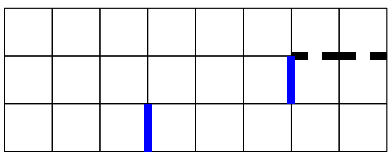
The illustrations below show the possible configurations for compatible pairs in $\mathcal{D}^{8\times 3}$. If the edge u_i is marked ▬ ▬, then u_i can occur in S_1.



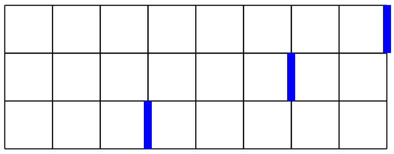
$\sum_{S_1\subset\{u_1,\ldots,u_8\},S_2=\emptyset} x_1^{r|S_2|} x_2^{r|S_1|}$
$= (1 + x_2^3)^8$



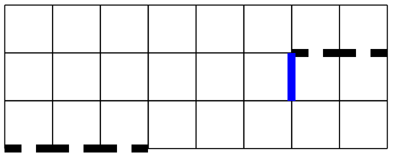
$\sum_{S_1\subset\{u_4,\ldots,u_8\},S_2=\{v_1\}} x_1^{r|S_2|} x_2^{r|S_1|}$
$= x_1^3(1 + x_2^3)^5$



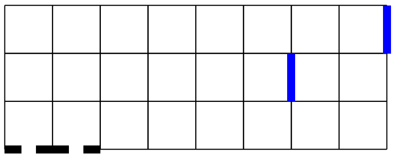
$\sum_{S_1\subset\{u_7,u_8\},S_2=\{v_1,v_2\}} x_1^{r|S_2|} x_2^{r|S_1|}$
$= x_1^6(1 + x_2^3)^2$



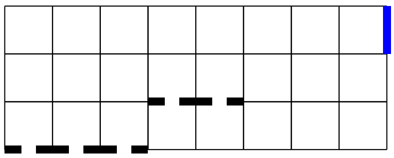
$\sum_{S_1=\emptyset, S_2=\{v_1,v_2,v_3\}} x_1^{r|S_2|} x_2^{r|S_1|}$
$= x_1^9$



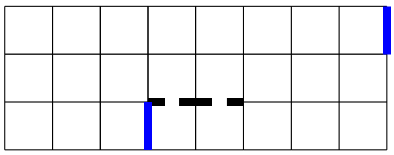
$\sum_{S_1\subset\{u_1,u_2,u_3,u_7,u_8\}, S_2=\{v_2\}} x_1^{r|S_2|} x_2^{r|S_1|}$
$= x_1^3(1+x_2^3)^5$



$\sum_{S_1\subset\{u_1,u_2\}, S_2=\{v_2,v_3\}} x_1^{r|S_2|} x_2^{r|S_1|}$
$= x_1^6(1+x_2^3)^2$



$\sum_{S_1\subset\{u_1,u_2,u_3,u_4,u_5\}, S_2=\{v_3\}} x_1^{r|S_2|} x_2^{r|S_1|}$
$= x_1^3(1+x_2^3)^5$



$\sum_{S_1\subset\{u_4,u_5\}, S_2=\{v_1,v_3\}} x_1^{r|S_2|} x_2^{r|S_1|}$
$= x_1^6(1+x_2^3)^2$

Adding the above eight polynomials together gives

$$\begin{aligned}(12)\qquad & x_2^{24} + 8x_2^{21} + 3x_1^3x_2^{15} + 28x_2^{18} + 15x_1^3x_2^{12} + 56x_2^{15} + 3x_1^6x_2^6 \\ & + 30x_1^3x_2^9 + 70x_2^{12} + x_1^9 + 6x_1^6x_2^3 + 30x_1^3x_2^6 \\ & + 56x_2^9 + 3x_1^6 + 15x_1^3x_2^3 + 28x_2^6 + 3x_1^3 + 8x_2^3 + 1.\end{aligned}$$

Then x_5 is obtained by dividing (12) by $x_1^8x_2^3$.

The following corollary can be adapted from results of [20]. Let g_ℓ be the g-vector and let F_ℓ be the F-polynomial of x_ℓ, for all integers ℓ. Then $g_3 = (-1, r)$, $g_0 = (0, -1)$, $F_3 = y_1 + 1$ and $F_0 = y_2 + 1$, and all other cases are described in the following result.

Corollary 3.6. *Let $n \geq 3$. Then*

$$g_n = (-c_{n-1}, c_n), \qquad g_{3-n} = (-c_{n-2}, c_{n-3}),$$
$$F_n = \sum_{(S_1,S_2)} y_1^{c_{n-1}-|S_1|} y_2^{|S_2|}, \qquad F_{3-n} = \sum_{(S_1,S_2)} y_1^{c_{n-2}-|S_2|} y_2^{|S_1|},$$

where the sum is over all compatible pairs (S_1, S_2) in $\mathcal{D}^{c_{n-1}\times c_{n-2}}$.

Recall from Definition 2.9 that X_n is the Laurent polynomial in x_1, x_2, y_1, y_2 corresponding to the expansion of x_n in the cluster algebra with principal coefficients in the seed $((x_1, x_2), (y_1, y_2), Q)$.

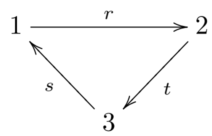
Corollary 3.7. *Let $n \geq 3$. Then*

$$X_n = x_1^{-c_{n-1}} x_2^{-c_{n-2}} \sum_{(S_1,S_2)} x_1^{r|S_2|} x_2^{r|S_1|} y_1^{c_{n-1}-|S_1|} y_2^{|S_2|}.$$

Proof. This follows directly from Theorem 3.3 and Corollary 3.6. □

§3.2. A preliminary lemma

Let $\mathcal{A}(Q)$ be a coefficient-free cluster algebra of rank 3 with initial quiver Q equal to



where r, s, t denote the numbers of arrows.

Let x_n be the cluster variable obtained from the seed $((x_1, x_2, z_3), Q)$ by the sequence of $n-2$ mutations $1, 2, 1, 2, 1, \dots$.

From Theorem 2.10, we have

$$x_n = \frac{X_n(x_1, x_2, \hat{y}_1, \hat{y}_2)}{F_n|_{\mathbb{P}}(\hat{y}_1, \hat{y}_2)},$$

where X_n is as in Corollary 3.7, $(\hat{y}_1, \hat{y}_2) = (z_3^s, z_3^{-t})$, and the denominator is the F-polynomial evaluated in $\hat{y}$ and with tropical addition. Thus, by Corollary 3.6, the denominator is $\sum_\beta^{\oplus} z_3^{s(c_{n-1}-|S_1|)-t|S_2|}$, which is equal to z_3^m where m is the smallest power occurring in this sum. Therefore we get

$$x_n = x_1^{-c_{n-1}} x_2^{-c_{n-2}} \sum_{(S_1,S_2)} x_1^{r|S_2|} x_2^{r|S_1|} z_3^{s(c_{n-1}-|S_1|)-t|S_2|-m} \tag{13}$$

where $m = \min_{(S_1,S_2)}(s(c_{n-1} - |S_1|) - t|S_2|)$.

We shall need a precise value for m. As a first step, we determine which compatible pair (S_1, S_2) in $\mathcal{D}^{c_{n-1} \times c_{n-2}}$ can realize the minimum m.

Lemma 3.8. *Let s and t be nonzero integers such that there are nonzero arrows between all pairs of the three vertices in any seed between the initial and terminal seeds inclusive. Then there is a unique compatible pair (S_1, S_2) in $\mathcal{D}^{c_{n-1} \times c_{n-2}}$ which achieves $s(c_{n-1} - |S_1|) - t|S_2| = m$. Such an (S_1, S_2) is either $(\mathcal{D}_1, \emptyset)$, $(\emptyset, \mathcal{D}_2)$, or $(\emptyset, \emptyset)$. Moreover, if s and t are positive, then (S_1, S_2) is either $(\mathcal{D}_1, \emptyset)$ or $(\emptyset, \mathcal{D}_2)$.*

Proof. We use induction on n. Consider first the case $n = 3$. Since $\mathcal{D}^{c_2\times c_1} = \mathcal{D}^{1\times 0}$, there are exactly two compatible pairs $(\emptyset, \mathcal{D}_2) = (\emptyset, \emptyset)$ and $(\mathcal{D}_1, \emptyset)$. Thus $s(c_{n-1} - |S_1|) - t|S_2| = -s|S_1|$ achieves its minimum at $(S_1, S_2) = (\mathcal{D}_1, \emptyset)$ if s is positive, and at $(S_1, S_2) = (\emptyset, \mathcal{D}_2)$ if s is negative.

Suppose now that $n \geq 3$. Consider the expression given in (13). By induction, there is a unique $(\bar{S}_1, \bar{S}_2)$ in $\mathcal{D}^{c_{n-1}\times c_{n-2}}$ such that $s(c_{n-1} - |S_1|) - t|S_2| = m$.

Suppose first that $(\bar{S}_1, \bar{S}_2) = (\mathcal{D}_1, \emptyset)$. Then the term in (13) that is not divisible by z_3 is

$$x_1^{-c_{n-1}} x_2^{-c_{n-2}} x_2^{rc_{n-1}} = x_1^{-c_{n-1}} x_2^{c_n}. \tag{14}$$

If $t > 0$ then x_{n+1} is obtained from x_n by substituting

$$x_1 \mapsto x_2, \qquad x_2 \mapsto \frac{x_2^r + z_3^t}{x_1}.$$

So (14) becomes

$$x_2^{-c_{n-1}} \left(\frac{x_2^r + z_3^t}{x_1}\right)^{c_n} = x_1^{-c_n} x_2^{-c_{n-1}} x_1^0 x_2^{rc_n} z_3^0 + \text{terms divisible by } z_3.$$

Thus the term in the expression for x_{n+1} that is not divisible by z_3 corresponds to a compatible pair (S_1, S_2) in $\mathcal{D}^{c_n\times c_{n-1}}$ with $|S_1| = c_n$, so $(S_1, S_2) = (\mathcal{D}_1, \emptyset)$.

On the other hand, if $t < 0$ then we substitute

$$x_1 \mapsto x_2, \qquad x_2 \mapsto \frac{x_2^r z_3^{-t} + 1}{x_1},$$

so (14) becomes

$$x_2^{-c_{n-1}} \left(\frac{x_2^r z_3^{-t} + 1}{x_1}\right)^{c_n} = x_1^{-c_n} x_2^{-c_{n-1}} x_1^0 x_2^0 z_3^0 + \text{terms divisible by } z_3.$$

Thus the term in the expression for x_{n+1} that is not divisible by z_3 corresponds to a compatible pair (S_1, S_2) in $\mathcal{D}^{c_n\times c_{n-1}}$ with $|S_1| = 0$ and $|S_2| = 0$, so $(S_1, S_2) = (\emptyset, \emptyset)$.

Next suppose that $(\bar{S}_1, \bar{S}_2) = (\emptyset, \emptyset)$. Then $c_{n-1} - |\bar{S}_1| = c_{n-1}$, $|\bar{S}_2| = 0$, $m = sc_{n-1}$ and $t < 0$ since $(\emptyset, \emptyset)$ realizes the minimum. Then

$$\sum_{(S_1,S_2):\,|S_1|=0} x_1^{-c_{n-1}} x_2^{-c_{n-2}} x_1^{r|S_2|} x_2^{r|S_1|} z_3^{s(c_{n-1}-|S_1|)-t|S_2|-m}$$
$$= \sum_{(S_1,S_2):\,|S_1|=0} x_1^{-c_{n-1}} x_2^{-c_{n-2}} x_1^{r|S_2|} z_3^{-t|S_2|} = x_1^{-c_{n-1}} \left(\frac{x_1^r z_3^{-t} + 1}{x_2}\right)^{c_{n-2}},$$

where the last identity holds because the condition $|S_1| = 0$ means that every subset S_2 of $\{1, \dots, c_{n-2}\}$ is compatible with S_1.

Applying the map

$$x_1 \mapsto x_2, \qquad x_2 \mapsto \frac{x_2^r z_3^{-t} + 1}{x_1}$$

yields $x_2^{-c_{n-1}} x_1^{c_{n-2}} = x_1^{-c_n} x_2^{-c_{n-1}} x_1^{rc_{n-1}} x_2^0$, which corresponds to a compatible pair (S_1, S_2) in $\mathcal{D}^{c_n \times c_{n-1}}$ with $|S_1| = 0$ and $|S_2| = c_{n-1}$, thus $(S_1, S_2) = (\emptyset, \mathcal{D}_2)$.

Finally, if $(\bar{S}_1, \bar{S}_2) = (\emptyset, \mathcal{D}_2)$, then $|\bar{S}_1| = 0$, $|\bar{S}_2| = c_{n-2}$, $m = sc_{n-1} - tc_{n-2}$ and t must be positive since $(\emptyset, \mathcal{D}_2)$ realizes the minimum. Then

$$\sum_{(S_1,S_2)\colon |S_1|=0} x_1^{-c_{n-1}} x_2^{-c_{n-2}} x_1^{r|S_2|} x_2^{r|S_1|} z_3^{s(c_{n-1}-|S_1|)-t|S_2|-m}$$
$$= \sum_{(S_1,S_2)\colon |S_1|=0} x_1^{-c_{n-1}} x_2^{-c_{n-2}} x_1^{r|S_2|} z_3^{tc_{n-2}-t|S_2|} = x_1^{-c_{n-1}} \left(\frac{x_1^r + z_3^t}{x_2}\right)^{c_{n-2}}$$

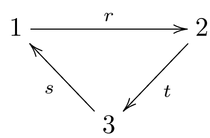
and applying the map

$$x_1 \mapsto x_2, \qquad x_2 \mapsto \frac{x_2^r + z_3^t}{x_1}$$

yields $x_2^{-c_{n-1}} x_1^{c_{n-2}} = x_1^{-c_n} x_2^{-c_{n-1}} x_1^{rc_{n-1}} x_2^0$, which corresponds to a compatible pair (S_1, S_2) in $\mathcal{D}^{c_n \times c_{n-1}}$ with $|S_1| = 0$ and $|S_2| = c_{n-1}$, thus $(S_1, S_2) = (\emptyset, \mathcal{D}_2)$. □

§3.3. Rank 2 inside rank 3: Dyck path formula

Let $\mathcal{A}(Q)$ be a *nonacyclic* coefficient-free cluster algebra of rank 3 with initial quiver Q equal to



where r, s, t denote the numbers of arrows. Now suppose that r, s and t are such that the cluster algebra $\mathcal{A}(Q)$ is nonacyclic. Then we can show that m is always zero.

Corollary 3.9. *If the cluster algebra is nonacyclic then there is a unique compatible pair* (S_1, S_2) *in* $\mathcal{D}^{c_{n-1} \times c_{n-2}}$*, which achieves* $s(c_{n-1} - |S_1|) - t|S_2| = m$*. Moreover* $(S_1, S_2) = (\mathcal{D}_1, \emptyset)$ *and* $m = 0$*.*

Proof. We use induction on n. For $n = 3$ and $n = 4$, computing x_n directly by mutation yields

$$x_3 = (x_2^r + z_3^s)/x_1, \qquad x_4 = (x_3^r + z_3^{rs-t})/x_2,$$

and hence the term in the expression for x_4 that is not divisible by z_3 is $x_2^{r^2} x_1^{-r} x_2^{-1}$, which corresponds to a compatible pair (S_1, S_2) in $\mathcal{D}^{c_{n-1} \times c_{n-2}}$ such that $|S_2| = c_{n-1} - |S_1| = 0$, thus $(S_1, S_2) = (\mathcal{D}_1, \emptyset)$.

Now the result follows by the same argument as in the proof of Lemma 3.8 using the fact that $t > 0$. □

We have proved the following theorem.

Theorem 3.10. *For $n \geq 3$,*

$$x_n = x_1^{-c_{n-1}} x_2^{-c_{n-2}} \sum_{(S_1,S_2)} x_1^{r|S_2|} x_2^{r|S_1|} z_3^{s(c_{n-1}-|S_1|)-t|S_2|},$$

where the sum is over all compatible pairs (S_1, S_2) in $\mathcal{D}^{c_{n-1}\times c_{n-2}}$.

Remark 3.11. It follows from results of [25] and [19] that an adapted version of this theorem still holds if the cluster algebra is not skew-symmetric.

Our next goal is to describe cluster monomials of the form $x_{n+1}^p x_n^q$ with $p, q \geq 0$. In order to simplify the notation we define $A_i = pc_{i+1} + qc_i$. The following lemma is a straightforward consequence of Lemma 3.1.

Lemma 3.12. *For any i, we have*

(a) $A_i = rA_{i-1} - A_{i-2}$,

(b) $A_i^2 - A_{i+1}A_{i-1} = p^2 + q^2 + rpq$. □

Theorem 3.13.

$$x_{n+1}^p x_n^q = x_1^{-A_{n-1}} x_2^{-A_{n-2}} \sum_{(S_1,S_2)} x_1^{r|S_2|} x_2^{r|S_1|} z_3^{s(c_{n-1}-|S_1|)-t|S_2|},$$

where the sum is over all $(S_1 = \bigcup_{i=1}^{p+q} S_1^i, S_2 = \bigcup_{i=1}^{p+q} S_2^i)$ such that

$$(S_1^i, S_2^i) \text{ is a compatible pair in } \begin{cases} \mathcal{D}^{c_{n-1}\times c_{n-2}} & \text{if } 1 \leq i \leq q, \\ \mathcal{D}^{c_n \times c_{n-1}} & \text{if } q+1 \leq i \leq p+q. \end{cases}$$

Proof. This follows immediately from Theorem 3.10. □

Remark 3.14. It can be shown that the summation on the right hand side in Theorem 3.13 can be taken over all compatible pairs in $\mathcal{D}^{A_{n-1}\times A_{n-2}}$ instead, without changing the sum (see [19, Theorem 1.11]).

§3.4. Rank 2 inside rank 3: Mixed formula

We keep the setup of the previous subsection. In particular, $\mathcal{A}(Q)$ is nonacyclic. We present another formula for the Laurent expansion of the cluster monomial $x_{n+1}^p x_n^q$, which is parametrized by a certain sequence of integers $\tau_0, \tau_1, \ldots, \tau_{n-2}$. This formula is a generalization of a formula given in [18, Theorem 2.1]. Combining it with the formula of Theorem 3.13 yields the mixed formula of Theorem 3.21 below, which is a key ingredient for the proof of the positivity conjecture in Section 4.

For arbitrary (possibly negative) integers A, B, we define the modified binomial coefficient as follows:

$$\begin{bmatrix} A \\ B \end{bmatrix} := \begin{cases} \displaystyle\prod_{i=0}^{A-B-1} \frac{A-i}{A-B-i} & \text{if } A > B, \\ 1 & \text{if } A = B, \\ 0 & \text{if } A < B. \end{cases}$$

If $A \geq 0$ then $\left[{cA \atop B} \right] = \left[{cA \atop A-B} \right]$ is just the usual binomial coefficient. In particular $\left[{cA \atop B} \right] = 0$ if $A \geq 0$ and $B < 0$.

For a sequence (τ_j) (respectively (τ'_j)) of integers, we define a sequence (s_i) (respectively (s'_i)) of weighted partial sums as follows:

$$s_0 = 0, \quad s_i = \sum_{j=0}^{i-1} c_{i-j+1}\tau_j = c_{i+1}\tau_0 + c_i\tau_1 + \cdots + c_2\tau_{i-1},$$

$$s'_0 = 0, \quad s'_i = \sum_{j=0}^{i-1} c_{i-j+1}\tau'_j = c_{i+1}\tau'_0 + c_i\tau'_1 + \cdots + c_2\tau'_{i-1}.$$

For example, $s_1 = c_2\tau_0 = \tau_0$, $s_2 = c_3\tau_0 + c_2\tau_1 = r\tau_0 + \tau_1$.

Lemma 3.15. $s_n = rs_{n-1} - s_{n-2} + \tau_{n-1}$.

Proof. Since $c_{n-j+1} = rc_{n-j} - c_{n-j-1}$, we see that s_n is equal to

$$\begin{aligned}&\sum_{j=0}^{n-1}(rc_{n-j} - c_{n-j-1})\tau_j \\ &\qquad = r\Big(\sum_{j=0}^{n-2} c_{n-j}\tau_j\Big) + rc_1\tau_{n-1} - \Big(\sum_{j=0}^{n-3} c_{n-j-1}\tau_j\Big) - c_1\tau_{n-2} - c_0\tau_{n-1} \\ &\qquad = rs_{n-1} - s_{n-2} + \tau_{n-1},\end{aligned}$$

where the last identity holds because $c_1 = 0$, $c_0 = -1$. □

Definition 3.16. Let $\mathcal{L}(\tau_0, \tau_1, \ldots, \tau_{n-2})$ denote the set of all $(\tau'_0, \tau'_1, \ldots, \tau'_{n-2}) \in \mathbb{Z}^{n-1}$ satisfying the conditions

- $0 \leq \tau'_i \leq \tau_i$ for $0 \leq i \leq n-3$,
- $s'_{n-2} = kc_{n-1}$ and $s'_{n-1} = kc_n$ for some integer $0 \leq k \leq p$.

We define a partial order on $\mathcal{L}(\tau_0, \tau_1, \ldots, \tau_{n-2})$ by

$$(\tau'_0, \tau'_1, \ldots, \tau'_{n-2}) \leq_{\mathcal{L}} (\tau''_0, \tau''_1, \ldots, \tau''_{n-2}) \quad \text{if and only if} \quad \tau'_i \leq \tau''_i \text{ for } 0 \leq i \leq n-3.$$

Let $\mathcal{L}_{\max}(\tau_0, \tau_1, \dots, \tau_{n-2})$ be the set of maximal elements of $\mathcal{L}(\tau_0, \tau_1, \dots, \tau_{n-2})$ with respect to $\leq_{\mathcal{L}}$.

We are ready to state the main result of this subsection.

Theorem 3.17. *Let $n \geq 3$. Then*

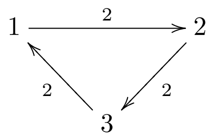
$$x_{n+1}^{p} x_n^{q} = \tag{15}$$
$$x_1^{-A_{n-1}} x_2^{-A_{n-2}} \sum_{\tau_0,\tau_1,\dots,\tau_{n-2}} \left(\prod_{i=0}^{n-2} \begin{bmatrix} A_{i+1} - rs_i \\ \tau_i \end{bmatrix} \right) x_1^{rs_{n-2}} x_2^{r(A_{n-1}-s_{n-1})} z_3^{ss_{n-1}-ts_{n-2}},$$

where the summation runs over all integers $\tau_0, \dots, \tau_{n-2}$ satisfying

$$\begin{cases} 0 \leq \tau_i \leq A_{i+1} - rs_i \ (0 \leq i \leq n-3), \\ \tau_{n-2} \leq A_{n-1} - rs_{n-2}, \\ (s_{n-1} - s'_{n-1})A_{n-2} \geq (s_{n-2} - s'_{n-2})A_{n-1} \\ \qquad\qquad \textit{for any } (\tau'_0, \dots, \tau'_{n-2}) \in \mathcal{L}_{\max}(\tau_0, \dots, \tau_{n-2}). \end{cases} \tag{16}$$

Proof. The theorem is proved in Section 3.5. □

Example 3.18. Let Q be the quiver



and let $n = 5$, $p = 1$, $q = 0$. Thus our formula computes the cluster variable obtained from the initial cluster by mutating in directions $1, 2, 1$ and 2.

First note that in this case $c_i = i - 1$, $A_i = i$ and $s_i = i\tau_0 + (i-1)\tau_1 + \cdots + 2\tau_{i-2} + \tau_{i-1}$. The first condition in (16) is $0 \leq \tau_i \leq i + 1 - 2s_i$. From this we see that τ_0 is either 0 or 1. If $\tau_0 = 1$, then $s_1 = 1$, hence $\tau_1 = 0$ by (16), whence $s_2 = 2$ and $0 \leq \tau_2 \leq (2+1) - 2(2)$, again by (16), a contradiction. Thus $\tau_0 = 0$ and the conditions on τ_i in (16) become

$$\tau_0 = 0 \qquad 0 \leq \tau_1 \leq 2 \qquad 0 \leq \tau_2 \leq 3 - 2\tau_1 \qquad \tau_3 \leq 4 - 4\tau_1 - 2\tau_2,$$

From this we conclude that there are the following 11 possibilities for $(\tau_0, \tau_1, \tau_2, \tau_3)$:

$$\begin{matrix} (0,0,0,0) & (0,0,0,1) & (0,0,0,2) & (0,0,0,3) & (0,0,0,4) & (0,0,1,0) \\ & (0,0,1,1) & (0,0,1,2) & (0,0,2,0) & (0,0,3,-2) & (0,1,0,0) \end{matrix}$$

Observe that each of these tuples satisfies the second condition in (16). Indeed, the integer $k \in \{0,1\}$ in Definition 3.16 must satisfy

$$3k = 2\tau'_1 + \tau'_2 \quad \text{and} \quad 4k = 3\tau'_1 + 2\tau'_2 + \tau'_3,$$

so for example, if $(\tau_0,\tau_1,\tau_2,\tau_3)=(0,0,1,1)$ then $\tau'_0=\tau'_1=0$, $0\le\tau'_2\le 1$, $\tau'_3\le 1$ and

$$3k=\tau'_2\le 1 \quad\text{and}\quad 4k=2\tau'_2+\tau'_3.$$

Thus $k=0$, $\mathcal{L}(0,0,1,1)=\{(0,0,0,0)\}$, and the second condition in (16) becomes

$$(s_4-s'_4)A_3\ge(s_3-s'_3)A_4 \;\Leftrightarrow\; (2+1-0)3\ge(1-0)4 \;\Leftrightarrow\; 9\ge 4.$$

On the other hand, the eleven 4-tuples above are the only ones that satisfy all conditions in (16). For example, for the tuple $(0,1,1,-2)$, we get $k=1$, $\tau'=(0,1,1,-1)\in\mathcal{L}_{\max}(0,1,1,-2)$ and the condition

$$(s_4-s'_4)A_3\ge(s_3-s'_3)A_4 \;\Leftrightarrow\; (3+2-2-3-2+1)3\ge(2+1-2-1)4 \;\Leftrightarrow\; -3\ge 0$$

is not satisfied. Therefore Theorem 3.17 yields

$$\begin{aligned}x_6=(x_2^8+\;4x_2^6z_3^2\;+6x_2^4z_3^4+4x_2^2z_3^6+\;z_3^8\;+3x_1^2x_2^4z_3^2\\ +6x_1^2x_2^2z_3^4+3x_1^2z_3^6+3x_1^4z_3^4+x_1^6z_3^2+2x_1^4x_2^2z_3^2)/x_1^4x_2^3.\end{aligned}$$

Remark 3.19. When comparing the formula of Theorem 3.17 with the Dyck path formula of Theorem 3.13, we have the following interpretation for the integer k in Definition 3.16. Let $\mathcal{D}_2$ be the set of all vertical edges in $\mathcal{D}^{c_n\times c_{n-1}}$, and fix a pair (S_1,S_2) as in Theorem 3.13. Then k in Definition 3.15 is equal to the number of times $\mathcal{D}_2$ appears in S_2. Moreover, if $(\tau'_0,\tau'_1,\dots,\tau'_{n-2})\in\mathcal{L}_{\max}$ is such that $\prod_{i=0}^{n-2}\left[{cA_{i+1}-rs_i \atop \tau_i}\right]\neq 0$ then

$$k=\min\left(\left\lfloor\frac{s_{n-2}}{c_{n-1}}\right\rfloor,p\right).$$

Corollary 3.20. *Let $x_3=(x_2^r+z_3^s)/x_1$ and let $t'=s$ and $s'=rs-t$ be the numbers of arrows from 1 to 3 and from 3 to 2 respectively, in the quiver obtained from Q by mutating at vertex 1. Then*

$$x_{n+1}^p x_n^q = \tag{17}$$

$$x_2^{-A_{n-2}}x_3^{-A_{n-3}}\sum_{\tau_0,\tau_1,\dots,\tau_{n-3}}\left(\prod_{i=0}^{n-3}\begin{bmatrix}A_{i+1}-rs_i\\ \tau_i\end{bmatrix}\right)x_2^{rs_{n-3}}x_3^{r(A_{n-2}-s_{n-2})}z_3^{s's_{n-2}-t's_{n-3}},$$

where the summation runs over all integers $\tau_0,\dots,\tau_{n-3}$ satisfying

$$\begin{cases}0\le\tau_i\le A_{i+1}-rs_i\ (0\le i\le n-4),\ \tau_{n-3}\le A_{n-2}-rs_{n-3},\\ (s_{n-2}-s'_{n-2})A_{n-3}\ge(s_{n-3}-s'_{n-3})A_{n-2}\\ \qquad\qquad \textit{for any } (\tau'_0,\dots,\tau'_{n-3})\in\mathcal{L}_{\max}(\tau_0,\dots,\tau_{n-3}).\end{cases}\tag{18}$$

Proof. This follows directly from Theorem 3.17. □

Combining Theorem 3.17 with Theorem 3.13 we get the following mixed formula.

Theorem 3.21. *Let $n \geq 3$. Then*

$$(19) \qquad x_{n+1}^p x_n^q = \sum_{\substack{\tau_0,\tau_1,\ldots,\tau_{n-3} \\ s_{n-2}\leq A_{n-1}/r}} \left(\prod_{i=0}^{n-3} \left(\begin{matrix} A_{i+1}-rs_i \\ \tau_i \end{matrix}\right)\right) {x_3}^{A_{n-1}-rs_{n-2}} x_2^{rs_{n-3}-A_{n-2}} z_3^{s's_{n-2}-t's_{n-3}} + \sum_{\substack{(S_1,S_2) \\ r|S_2|-A_{n-1}>0}} x_1^{r|S_2|-A_{n-1}} x_2^{r|S_1|-A_{n-2}} z_3^{s(c_{n-1}-|S_1|)-t|S_2|}.$$

Remark 3.22. The exponents of x_3, x_1 and z_3 are nonnegative, which is important for the proof of Theorem 4.3. The modified binomial coefficients can be replaced by the usual binomial coefficients, because the condition $s_{n-2} \leq A_{n-1}/r$ implies that $A_{n-2} - rs_{n-3}$ is nonnegative (see Lemma 3.24 below).

Proof. The first sum of the statement is obtained from Corollary 3.20 using Lemma 3.12(a) for the exponent of x_3. Observe that the new condition $s_{n-2} \leq A_{n-1}/r$ in the summation precisely means that the exponent of x_3 is nonnegative. On the other hand, the sum of all terms in which the exponent of x_3 is negative in the expression in Corollary 3.20 is equal to the second sum in the statement of Theorem 3.21. This follows from the formula of Theorem 3.13. □

In [4] the upper cluster algebra was defined as the intersection of the rings of Laurent polynomials in the $n + 1$ clusters consisting of the initial cluster and all clusters obtained from it by a single mutation. The following corollary gives a different "upper bound" for the cluster monomials in the rank 2 direction. This new upper bound is defined as the semi-ring of polynomials in the variables in the initial cluster, the first mutation of one initial variable and the inverse of another initial cluster variable.

Corollary 3.23. *Let $\widetilde{x}_1$ denote the cluster variable obtained from the initial seed by mutating in x_1. Then*

$$x_{n+1}^p x_n^q \in \mathbb{Z}_{\geq 0}[x_1, \widetilde{x}_1, z_3, x_2^{\pm 1}].$$

Proof. This follows from Theorem 3.21 because $\widetilde{x}_1 = x_3$. □

§3.5. Proof of Theorem 3.17

We use induction on n. Suppose first that $n = 3$. Since $x_3 = (x_2^r + z_3^s)/x_1$, we have

$$x_3^a = x_1^{-a} \sum_{\tau_1=0}^{a} \left(\begin{matrix} a \\ \tau_1 \end{matrix}\right) x_2^{r(a-\tau_1)} z_3^{s\tau_1},$$

and since $x_4 = (x_3^r + z_3^{rs-t})/x_2$, we have

$$x_4^p = x_2^{-p} \sum_{\tau_0=0}^{p} \binom{p}{\tau_0} x_3^{r(p-\tau_0)} z_3^{(rs-t)\tau_0}.$$

Therefore

$$\begin{aligned} x_4^p x_3^q &= x_2^{-p} \sum_{\tau_0=0}^{p} \binom{p}{\tau_0} x_3^{r(p-\tau_0)+q} z_3^{(rs-t)\tau_0} \\ &= x_2^{-p} \sum_{\tau_0=0}^{p} \binom{p}{\tau_0} \sum_{\tau_1=0}^{r(p-\tau_0)+q} \binom{r(p-\tau_0)+q}{\tau_1} x_1^{-r(p-\tau_0)-q} x_2^{r(r(p-\tau_0)+q-\tau_1)} z_3^{s\tau_1} z_3^{(rs-t)\tau_0} \\ &= x_1^{-rp-q} x_2^{-p} \sum_{\tau_0=0}^{p} \sum_{\tau_1=0}^{r(p-\tau_0)+q} \binom{p}{\tau_0} \binom{r(p-\tau_0)+q}{\tau_1} x_1^{r\tau_0} x_2^{r(r(p-\tau_0)+q-\tau_1)} z_3^{s\tau_1+(rs-t)\tau_0}, \end{aligned}$$

and the statement follows from $A_1 = p$, $A_2 = rp+q$, $s_0 = 0$, $s_1 = \tau_0$, $s_2 = r\tau_0 + \tau_1$.

Suppose now that $n \geq 4$, and assume that the statement holds for n or less. Then by the obvious shift, we have

$$\begin{aligned} &x_{n+2}^p x_{n+1}^q \\ &\quad = x_2^{-A_{n-1}} x_3^{-A_{n-2}} \sum_{\tau_0,\tau_1,\dots,\tau_{n-2}} \left[\left(\prod_{i=0}^{n-2} \begin{bmatrix} A_{i+1} - rs_i \\ \tau_i \end{bmatrix} \right) x_2^{rs_{n-2}} x_3^{r(A_{n-1}-s_{n-1})} z_3^T \right], \end{aligned}$$

where the summation runs over all integers $\tau_0, \dots, \tau_{n-2}$ satisfying (16) and

$$T = s' s_{n-1} - t' s_{n-2} = (rs-t)s_{n-1} - s s_{n-2}.$$

Using Lemma 3.12(a), we see that the exponent of x_3 is equal to $A_n - rs_{n-1}$. Then substituting $(x_2^r + z_3^s)/x_1$ for x_3, we get

$$\begin{aligned} &x_{n+2}^p x_{n+1}^q \\ &\quad = x_2^{-A_{n-1}} \sum_{\tau_0,\tau_1,\dots,\tau_{n-2}} \left[\left(\prod_{i=0}^{n-2} \begin{bmatrix} A_{i+1} - rs_i \\ \tau_i \end{bmatrix} \right) x_2^{rs_{n-2}} \left(\frac{x_2^r + z^s}{x\quad{}_1} \right)^{A_n - rs_{n-1}} z_3^T \right]. \end{aligned}$$

Expanding $(x_2^r + z^s)^{A_n - rs_{n-1}}$ yields

$$\begin{aligned} &x_1^{-A_n} x_2^{-A_{n-1}} \sum_{\tau_0,\dots,\tau_{n-2}} \left(\prod_{i=0}^{n-2} \begin{bmatrix} A_{i+1} - rs_i \\ \tau_i \end{bmatrix} \right) \\ &\qquad \times x_2^{rs_{n-2}} \sum_{\tau_{n-1}\in\mathbb{Z}} \begin{bmatrix} A_n - rs_{n-1} \\ \tau_{n-1} \end{bmatrix} x_1^{rs_{n-1}} (x_2^r)^{A_n - rs_{n-1} - \tau_{n-1}} z_3^{T + s\tau_{n-1}}. \end{aligned}$$

Note that $T + s\tau_{n-1} = ss_n - ts_{n-1}$, by Lemma 3.15. Combining the sums, we get

$$x_1^{-A_n} x_2^{-A_{n-1}} \sum_{\tau_0,\tau_1,\dots,\tau_{n-2};\,\tau_{n-1}\in\mathbb{Z}} \left(\prod_{i=0}^{n-1} \begin{bmatrix} A_{i+1} - rs_i \\ \tau_i \end{bmatrix}\right) \times x_1^{rs_{n-1}} x_2^{rs_{n-2}} (x_2^r)^{A_n - rs_{n-1} - \tau_{n-1}} z_3^{ss_n - ts_{n-1}}$$

and, by Lemma 3.15, this is equal to

(20)
$$x_1^{-A_n} x_2^{-A_{n-1}} \sum_{\tau_0,\tau_1,\dots,\tau_{n-2};\,\tau_{n-1}\in\mathbb{Z}} \left(\prod_{i=0}^{n-1} \begin{bmatrix} A_{i+1} - rs_i \\ \tau_i \end{bmatrix}\right) x_1^{rs_{n-1}} x_2^{r(A_n - s_n)} z_3^{ss_n - ts_{n-1}}.$$

Remember that $\tau_0, \dots, \tau_{n-2}$ satisfy (16).

Proposition 3.25 below implies that even if we impose the additional condition on τ_{n-2} and τ_{n-1} that

$$(s_n - s'_n)A_{n-1} - (s_{n-1} - s'_{n-1})A_n \geq 0 \quad \text{for all } (\tau'_0, \dots, \tau'_{n-1}) \in \mathcal{L}_{\max}(\tau_0, \dots, \tau_{n-2}), \tag{21}$$

the value of the expression for $x_{n+2}^p x_{n+1}^q$ in (20) does not change. From now on we impose the condition (21). On the other hand, in order to prove that Theorem 3.17 holds for $n+1$, we only need to show that τ_{n-2} can be limited to run over $0 \leq \tau_{n-2} \leq A_{n-1} - rs_{n-2}$. So we want to show that, for a fixed sequence $(\tau'_0, \dots, \tau'_{n-2}, \tau'_{n-1})$,

$$\sum_{\tau_0,\tau_1,\dots,\tau_{n-1}} \left[\left(\prod_{i=0}^{n-1} \begin{bmatrix} A_{i+1} - rs_i \\ \tau_i \end{bmatrix}\right) x_1^{rs_{n-1}} x_2^{r(A_n - s_n)}\right] = 0, \tag{22}$$

where the summation runs over all integers $\tau_0, \dots, \tau_{n-1}$ satisfying

$$\begin{cases} \text{(a)} \ \ (\tau'_0, \dots, \tau'_{n-2}) \in \mathcal{L}_{\max}(\tau_0, \dots, \tau_{n-2}), \\ \text{(b)} \ \ 0 \leq \tau_i \leq A_{i+1} - rs_i \ \ (0 \leq i \leq n-3), \\ \text{(c)} \ \ (s_{n-1} - s'_{n-1})A_{n-2} - (s_{n-2} - s'_{n-2})A_{n-1} \geq 0, \\ \text{(d)} \ \ (\tau'_0, \dots, \tau'_{n-1}) \in \mathcal{L}_{\max}(\tau_0, \dots, \tau_{n-1}), \\ \text{(e)} \ \ \tau_{n-2} \leq A_{n-1} - rs_{n-2} < 0, \ \text{and} \\ \text{(f)} \ \ (s_n - s'_n)A_{n-1} - (s_{n-1} - s'_{n-1})A_n \geq 0. \end{cases} \tag{23}$$

To do so, it is sufficient to show that $\begin{bmatrix} cA_n - rs_{n-1} \\ \tau_{n-1} \end{bmatrix} = 0$ for every τ_{n-1}, because then each summand in (22) is zero. This purely algebraic result is proved in Lemma 3.24 below.

Assuming Lemma 3.24 and Proposition 3.25, we have proved that

$$x_{n+2}^p x_{n+1}^q$$
$$= x_1^{-A_n} x_2^{-A_{n-1}} \sum_{\tau_0,\tau_1,\dots,\tau_{n-1}} \left[\left(\prod_{i=0}^{n-1} \begin{bmatrix} A_{i+1} - rs_i \\ \tau_i \end{bmatrix} \right) x_1^{rs_{n-1}} x_2^{r(A_n - s_n)} z_3^{ss_n - ts_{n-1}} \right],$$

where the summation runs over all integers $\tau_0, \dots, \tau_{n-1}$ satisfying

$$\begin{cases} 0 \le \tau_i \le A_{i+1} - rs_i \ (0 \le i \le n-2), \\ (s_{n-1} - s'_{n-1})A_{n-2} - (s_{n-2} - s'_{n-2})A_{n-1} \ge 0, \\ (s_n - s'_n)A_{n-1} - (s_{n-1} - s'_{n-1})A_n \ge 0, \end{cases} \tag{24}$$

for all $(\tau'_0, \dots, \tau'_{n-1}) \in \mathcal{L}_{\max}(\tau_0, \dots, \tau_{n-2})$. Therefore it only remains to show that we do not need to require the second condition in (24). Using Lemmas 3.15 and 3.12(a), we see that

$$\begin{aligned} s_{n-1}A_{n-2} - s_{n-2}A_{n-1} &\overset{3.15}{=} (rs_{n-2} - s_{n-3} + \tau_{n-2})A_{n-2} - s_{n-2}A_{n-1} \\ &\overset{3.12}{=} (s_{n-2}A_{n-3} - s_{n-3}A_{n-2}) + \tau_{n-2}A_{n-2}. \end{aligned}$$

Iterating this argument, we get

$$s_{n-1}A_{n-2} - s_{n-2}A_{n-1} = (s_2A_1 - s_1A_2) + \sum_{i=2}^{n-2} \tau_i A_i,$$

which means

$$\tau_1 p - \tau_0 q + \sum_{i=1}^{n-2} \tau_i A_i \ge \tau_1 p - \tau_0 q,$$

because $s_1 = \tau_0$, $s_2 = r\tau_0 + \tau_1$, $A_1 = p$ and $A_2 = rp + q$. Thus

$$s_{n-1}A_{n-2} - s_{n-2}A_{n-1} \ge \tau_1 p - \tau_0 q. \tag{25}$$

Our next goal is to estimate $-s'_{n-1}A_{n-2} + s'_{n-2}A_{n-1}$. Let k be as in Definition 3.16, so that $0 \le k \le p$ and $s'_{n-1} = kc_n$ and $s'_{n-2} = kc_{n-1}$. Then

$$\begin{aligned} -s'_{n-1}A_{n-2} + s'_{n-2}A_{n-1} &= k(-c_nA_{n-2} + c_{n-1}A_{n-1}) \\ &= k(-pc_nc_{n-1} - qc_nc_{n-2} + pc_nc_{n-1} + qc_{n-1}^2) \\ &= kq(-c_nc_{n-2} + c_{n-1}^2) = kq, \end{aligned} \tag{26}$$

where the second equality follows from the definition $A_i = pc_{i+1} + qc_i$, and the last equality holds by Lemma 3.1. On the other hand, s'_{n-2} is defined as $s'_{n-2} = c_{n-1}\tau'_0 + \sum_{j=1}^{n-3} c_{n-1-j}\tau'_j$, which implies that $kc_{n-1} = s'_{n-2} \ge c_{n-1}\tau'_0$, and thus $k \ge \tau'_0$. Moreover, $\tau_0 = \tau'_0$ by definition of $\mathcal{L}_{\max}(\tau_0, \tau_1, \dots, \tau_{n-2})$, and thus (26) implies

$$-s'_{n-1}A_{n-2} + s'_{n-2}A_{n-1} \ge \tau_0 q.$$

Adding this to (25) we get

$$(s_{n-1}-s'_{n-1})A_{n-2}-(s_{n-2}-s'_{n-2})A_{n-1}\geq 0,$$

hence the second condition in (24) is always satisfied.

This completes the proof of Theorem 3.17 modulo Lemma 3.24 and Proposition 3.25 below.

Lemma 3.24. *Assume conditions* (23). *Then*

$$\begin{bmatrix} A_n - rs_{n-1} \\ \tau_{n-1} \end{bmatrix} = 0.$$

Proof. By definition of the modified binomial coefficient, it suffices to show that $A_n - rs_{n-1} < \tau_{n-1}$.

On the one hand, we have

$$\begin{aligned}
(27)\qquad & A_{n-3}(s_{n-2}-s'_{n-2}) - A_{n-2}(A_{n-1}-s_{n-1}+s'_{n-1}) \\
&\overset{3.12(a)}{=} rA_{n-2}(s_{n-2}-s'_{n-2}) - A_{n-1}(s_{n-2}-s'_{n-2}) - A_{n-2}(A_{n-1}-s_{n-1}+s'_{n-1}) \\
&= A_{n-2}rs_{n-2} - rA_{n-2}s'_{n-2} - A_{n-1}(s_{n-2}-s'_{n-2}) - A_{n-2}(A_{n-1}-s_{n-1}+s'_{n-1}) \\
&> A_{n-2}A_{n-1} - rA_{n-2}s'_{n-2} - A_{n-1}(s_{n-2}-s'_{n-2}) - A_{n-2}(A_{n-1}-s_{n-1}+s'_{n-1}) \\
&\quad \text{(since } A_{n-1}-rs_{n-2}<0\text{)} \\
&= -rA_{n-2}s'_{n-2} + (s_{n-1}-s'_{n-1})A_{n-2} - (s_{n-2}-s'_{n-2})A_{n-1} \\
&\overset{(23)(c)}{\geq} -rA_{n-2}s'_{n-2}.
\end{aligned}$$

Thus

$$(28)\qquad s_{n-2}-s'_{n-2} > (A_{n-2}(A_{n-1}-s_{n-1}+s'_{n-1}) - rA_{n-2}s'_{n-2})/A_{n-3}.$$

Then

$$\begin{aligned}
(29)\qquad & A_{n-2}(s_{n-1}-s'_{n-1}) - A_{n-1}(s_{n-2}-s'_{n-2}) \\
&\overset{(28)}{<} A_{n-2}(s_{n-1}-s'_{n-1}) - A_{n-1}\frac{A_{n-2}(A_{n-1}-s_{n-1}+s'_{n-1}) - rA_{n-2}s'_{n-2}}{A_{n-3}} \\
&= A_{n-2}\left(s_{n-1}-s'_{n-1} - \frac{A_{n-1}}{A_{n-3}}(A_{n-1}-s_{n-1}+s'_{n-1})\right) + \frac{rA_{n-1}A_{n-2}s'_{n-2}}{A_{n-3}} \\
&= A_{n-2}\left(A_{n-1} - \left(1+\frac{A_{n-1}}{A_{n-3}}\right)(A_{n-1}-s_{n-1}+s'_{n-1})\right) + \frac{rA_{n-1}A_{n-2}s'_{n-2}}{A_{n-3}} \\
&\overset{3.12(a)}{=} A_{n-2}\left(A_{n-1} - \left(1+\frac{A_{n-1}}{A_{n-3}}\right)\frac{A_{n-2}+A_n-rs_{n-1}+rs'_{n-1}}{r}\right) + \frac{rA_{n-1}A_{n-2}s'_{n-2}}{A_{n-3}}.
\end{aligned}$$

Now, aiming for contradiction, suppose that $A_n - rs_{n-1} \geq 0$. Then

$$\begin{aligned}
(30)\qquad & \text{RHS}(29) \\
&\leq A_{n-2}\left(A_{n-1} - \frac{A_{n-3}+A_{n-1}}{A_{n-3}}\frac{A_{n-2}+rs'_{n-1}}{r}\right) + \frac{rA_{n-1}A_{n-2}s'_{n-2}}{A_{n-3}} \\
&\overset{3.12(a)}{=} A_{n-2}\left(A_{n-1} - \frac{rA_{n-2}}{A_{n-3}}\frac{A_{n-2}+rs'_{n-1}}{r}\right) + \frac{rA_{n-1}A_{n-2}s'_{n-2}}{A_{n-3}} \\
&= A_{n-2}\left(A_{n-1} - \frac{A_{n-2}^2}{A_{n-3}}\right) - r\frac{A_{n-2}^2}{A_{n-3}}s'_{n-1} + \frac{rA_{n-1}A_{n-2}s'_{n-2}}{A_{n-3}} \\
&= \frac{A_{n-2}}{A_{n-3}}\left(A_{n-1}A_{n-3} - A_{n-2}^2\right) - r\frac{A_{n-2}}{A_{n-3}}\left(A_{n-2}s'_{n-1} - A_{n-1}s'_{n-2}\right).
\end{aligned}$$

By Lemma 3.12(b) and the definition of A_i, this is equal to

$$\frac{A_{n-2}}{A_{n-3}}(-p^2 - q^2 - rpq) - r\frac{A_{n-2}}{A_{n-3}}\big((pc_{n-1} + qc_{n-2})s'_{n-1} - (pc_n + qc_{n-1})s'_{n-2}\big).$$

Let k be as in Definition 3.16; then $kc_n = s'_{n-1}$ and $kc_{n-1} = s'_{n-2}$, and using Lemma 3.1, we get

$$\frac{A_{n-2}}{A_{n-3}}(-p^2 - q^2 - rpq) + rkq\frac{A_{n-2}}{A_{n-3}},$$

and, since $k \leq p$, this is less than or equal to

$$\frac{A_{n-2}}{A_{n-3}}(-p^2 - q^2 - rpq) + rpq\frac{A_{n-2}}{A_{n-3}} \leq 0,$$

which contradicts $(s_{n-1} - s'_{n-1})A_{n-2} - (s_{n-2} - s'_{n-2})A_{n-1} \geq 0$. Hence

$$(31)\qquad A_n - rs_{n-1} < 0.$$

Next we show that $s_{n-2} > A_n - s_n$. Suppose to the contrary that $s_{n-2} \leq A_n - s_n$. Then

$$\begin{aligned}
A_{n-1} - rs_{n-2} &\geq A_{n-1} - r(A_n - s_n) \overset{(23)(f)}{\geq} A_{n-1} - r\frac{(A_{n-1} - s_{n-1} + s'_{n-1})A_n}{A_{n-1}} + rs'_n \\
&\overset{3.12(b)}{=} \frac{(p^2 + q^2 + rpq)}{A_{n-1}} + \frac{A_nA_{n-2}}{A_{n-1}} - r\frac{(A_{n-1} - s_{n-1} + s'_{n-1})A_n}{A_{n-1}} + rs'_n \\
&\overset{3.12(a)}{=} \frac{(p^2 + q^2 + rpq)}{A_{n-1}} + \frac{A_n}{A_{n-1}}\left(rs_{n-1} - A_n - rs'_{n-1} + \frac{A_{n-1}}{A_n}rs'_n\right) \\
&= \frac{p^2 + q^2}{A_{n-1}} + \frac{A_n}{A_{n-1}}(rs_{n-1} - A_n) + \frac{r}{A_{n-1}}(pq - A_ns'_{n-1} + A_{n-1}s'_n) \\
&\overset{3.16}{=} \frac{p^2 + q^2}{A_{n-1}} + \frac{A_n}{A_{n-1}}(rs_{n-1} - A_n) + \frac{r}{A_{n-1}}(pq - A_nkc_n + A_{n-1}kc_{n+1}).
\end{aligned}$$

Now $kc_nA_n - kc_{n+1}A_{n-1} = kc_n(pc_{n+1} + qc_n) - kc_{n+1}(pc_n + qc_{n-1}) = kq(c_n^2 - c_{n+1}c_{n-1}) = kq$, where the last equation holds because of Lemma 3.1. Thus

$$A_{n-1} - rs_{n-2} \geq \frac{p^2+q^2}{A_{n-1}} + \frac{A_n}{A_{n-1}}(rs_{n-1} - A_n) + \frac{r}{A_{n-1}}(q(p-k))$$
$$\overset{3.16 \text{ and } (23)(\text{d})}{\geq} \frac{p^2+q^2}{A_{n-1}} + \frac{A_n}{A_{n-1}}(rs_{n-1} - A_n) \overset{(31)}{>} 0,$$

which contradicts $A_{n-1} - rs_{n-2} < 0$ in (23). Thus $s_{n-2} > A_n - s_n$, so we have

$$A_n - rs_{n-1} < s_n + s_{n-2} - rs_{n-1} = \tau_{n-1},$$

which gives $\left[\begin{smallmatrix} cA_n - rs_{n-1} \\ \tau_{n-1} \end{smallmatrix}\right] = 0$. □

Proposition 3.25. *Let a and b be nonnegative integers satisfying*

$$\sum_{\substack{\tau_0,\tau_1,\ldots,\tau_{n-2} \\ s_{n-1}=a,\, s_{n-2}=b}} \prod_{i=0}^{n-2} \begin{bmatrix} A_{i+1} - rs_i \\ \tau_i \end{bmatrix} \neq 0.$$

Let $\tau_0, \ldots, \tau_{n-2}$ satisfy $s_{n-1} = a, s_{n-2} = b$. Then for any $(\tau_0', \ldots, \tau_{n-2}') \in \mathcal{L}_{\max}(\tau_0, \ldots, \tau_{n-2})$, we have

$$(s_{n-1} - s_{n-1}')A_{n-2} - (s_{n-2} - s_{n-2}')A_{n-1} \geq 0. \tag{32}$$

Proof. We use Theorem 3.13. Let $\mathcal{D}_{i,1}$ (respectively $\mathcal{D}_{i,2}$) be the set of horizontal (resp. vertical) edges in the i-th Dyck path in $(\mathcal{D}^{c_n \times c_{n-1}})^p \times (\mathcal{D}^{c_{n-1} \times c_{n-2}})^q$.

Choose a collection of compatible pairs

$$\begin{aligned} \beta &= (\beta_1, \ldots, \beta_{p+q}) \\ &= ((S_{1,1}, S_{1,2}), \ldots, (S_{p+q,1}, S_{p+q,2})) \text{ in } (\mathcal{D}^{c_n \times c_{n-1}})^p \times (\mathcal{D}^{c_{n-1} \times c_{n-2}})^q \end{aligned}$$

such that $\sum_{i=1}^{p+q} |S_{i,2}| = s_{n-2}$ and $pc_n + qc_{n-1} - \sum_{i=1}^{p+q} |S_{i,1}| = s_{n-1}$, and such that β has the maximal number, say w, of copies of $(\emptyset, \mathcal{D}_2)$ in $\mathcal{D}^{c_n \times c_{n-1}}$. Say $\beta_{i_1} = \cdots = \beta_{i_w} = (\emptyset, \mathcal{D}_2)$ for some $1 \leq i_1 < \cdots < i_w \leq p$. Since $|\mathcal{D}_2| = c_{n-1}$ we see that

$$w = \min\left(\left\lfloor \frac{s_{n-2}}{c_{n-1}} \right\rfloor, p\right).$$

On the other hand, by Definition 3.16 and Remark 3.19, $s_{n-2}' = kc_{n-1}$ and $s_{n-1}' = kc_n$ with $k = w$ because $(\tau_0', \tau_1', \ldots, \tau_{n-2}') \in \mathcal{L}_{\max}$. Therefore

$$\sum_{i \in \{1,\ldots,p+q\} \setminus \{i_1,\ldots,i_w\}} |\beta_i|_2 = s_{n-1} - wc_n = s_{n-1} - s_{n-1}',$$
$$\sum_{i \in \{1,\ldots,p+q\} \setminus \{i_1,\ldots,i_w\}} |\beta_i|_1 = s_{n-2} - wc_{n-1} = s_{n-2} - s_{n-2}',$$

where $|\beta_i|_2$ denotes $|\mathcal{D}_{i,1}| - |S_{i,1}|$ and $|\beta_i|_1$ denotes $|S_{i,2}|$. Hence (32) is equivalent to

$$\sum_{i\in\{1,\dots,p+q\}\setminus\{i_1,\dots,i_w\}} (A_{n-2}|\beta_i|_2 - A_{n-1}|\beta_i|_1) \geq 0. \tag{33}$$

First we show that

$$\sum_{i=p+1}^{p+q} (A_{n-2}|\beta_i|_2 - A_{n-1}|\beta_i|_1) \geq 0.$$

Due to Lemma 3.8, if $p < i \leq p+q$ then either $\beta_i = (\mathcal{D}_{i,1}, \emptyset)$ or $\beta_i = (\emptyset, \mathcal{D}_{i,2})$ gives the minimum of $A_{n-2}|\beta_i|_2 - A_{n-1}|\beta_i|_1$. If $\beta_i = (\mathcal{D}_{i,1}, \emptyset)$ then clearly $A_{n-2}|\beta_i|_2 - A_{n-1}|\beta_i|_1 = 0$. If $\beta_i = (\emptyset, \mathcal{D}_{i,2})$ then

$$\begin{aligned} A_{n-2}|\beta_i|_2 - A_{n-1}|\beta_i|_1 &= A_{n-2}c_{n-1} - A_{n-1}c_{n-2} \\ &= (pc_{n-1} + qc_{n-2})c_{n-1} - (pc_n + qc_{n-1})c_{n-2} \\ &= p(c_{n-1}^2 - c_nc_{n-2}) = p \geq 0. \end{aligned}$$

Next we show that

$$\sum_{i\in\{1,\dots,p\}\setminus\{i_1,\dots,i_w\}} (A_{n-2}|\beta_i|_2 - A_{n-1}|\beta_i|_1) \geq 0.$$

If $p = 0$ then there is nothing to show. Suppose that $p \geq 1$. Again using Lemma 3.8, we see that if $1 \leq i \leq p$ and $\beta_i \neq (\emptyset, \mathcal{D}_{i,2})$ then

$$c_{n-2}|\beta_i|_2 - c_{n-1}|\beta_i|_1 > c_{n-2}c_n - c_{n-1}^2 = -1,$$

so $c_{n-2}|\beta_i|_2 - c_{n-1}|\beta_i|_1 \geq 0$. Also $c_{n-1}|\beta_i|_2 - c_n|\beta_i|_1 \geq c_{n-1}c_n - c_nc_{n-1} = 0$. Thus

$$\begin{aligned} A_{n-2}|\beta_i|_2 - A_{n-1}|\beta_i|_1 &= (pc_{n-1} + qc_{n-2})|\beta_i|_2 - (pc_n + qc_{n-1})|\beta_i|_1 \\ &= p(c_{n-1}|\beta_i|_2 - c_n|\beta_i|_1) + q(c_{n-2}|\beta_i|_2 - c_{n-1}|\beta_i|_1) \geq 0. \quad \square \end{aligned}$$

§3.6. Divisibility in rank 2

We end this section with the following rank 2 result which we will need in Section 4 for the rank 3 case.

Theorem 3.26. *Let $a \geq A_n/r$ be an integer. Then*

$$\sum_{\substack{\tau_0,\tau_1,\dots,\tau_{n-2} \\ s_{n-1}=a}} \prod_{i=0}^{n-2} \begin{bmatrix} A_{i+1} - rs_i \\ \tau_i \end{bmatrix} x_1^{rs_{n-2}-A_{n-1}} x_2^{r(A_{n-1}-a)-A_{n-2}}$$

is divisible by $(1+x_1^r)^{ra-A_n}$ and the resulting quotient has nonnegative coefficients.

Proof. Using Theorem 3.17 with $z_3 = 1$, we see that $x_{n+1}^p x_n^q$ is equal to

$$\sum_{\tau_0,\tau_1,\dots,\tau_{n-2}} \left(\prod_{i=0}^{n-2} \left[\begin{matrix} A_{i+1} - rs_i \\ \tau_i \end{matrix} \right] \right) x_1^{rs_{n-2}-A_{n-1}} x_2^{A_n - rs_{n-1}}.$$

On the other hand, using Theorem 3.17 to express $x_{n+1}^p x_n^q$ in the cluster $(x_0 = (x_1^r + 1)/x_2, x_1)$, we get

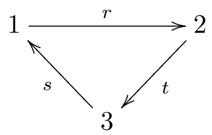
$$\sum_{\tau_0,\tau_1,\dots,\tau_{n-2},\tau_{n-1}} \left(\prod_{i=0}^{n-1} \left[\begin{matrix} A_{i+1} - rs_i \\ \tau_i \end{matrix} \right] \right) \left(\frac{x_1^r + 1}{x_2} \right)^{rs_{n-1}-A_n} x_1^{A_{n+1}-rs_n}.$$

Since the positivity conjecture is known to hold for rank 2, it follows that all the sums of products of modified binomial coefficients in this expression are positive. Now the result follows by fixing $s_{n-1} = a$. □

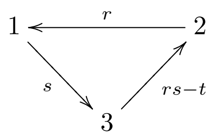
§4. Rank 3

§4.1. Nonacyclic mutation classes of rank 3

In this subsection we collect some basic results on quivers of rank 3 which are not mutation equivalent to an acyclic quiver. First let us recall how mutations act on a rank 3 quiver. Given a quiver

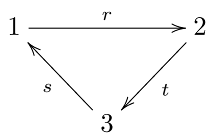


where $r, s, t \geq 1$ denote the numbers of arrows, its mutation in 1 is the quiver



where we agree that if $rs - t < 0$ then there are $|rs - t|$ arrows from 2 to 3.

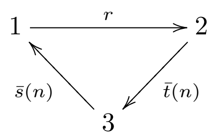
Lemma 4.1. *Let $\mathcal{A}$ be a nonacyclic cluster algebra of rank 3 with initial quiver Q equal to*



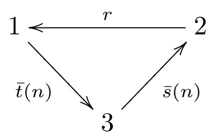
where $r, s, t \geq 1$ denote the numbers of arrows. Then

(i) *$r, s, t \geq 2$;*

(ii) *applying to Q a mutation sequence at the vertices $1, 2, 1, 2, 1, 2, \ldots$ consisting of n mutations yields the quiver*



if n is even,

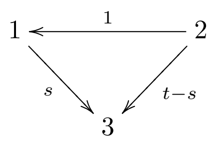


if n is odd, where

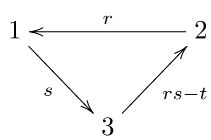
$$\bar{s}(n) = c_{n+2}^{[r]} s - c_{n+1}^{[r]} t \quad \textit{and} \quad \bar{t}(n) = c_{n+1}^{[r]} s - c_n^{[r]} t.$$

(iii) *$c_{n+1}^{[r]} s - c_n^{[r]} t \geq 2$ for all $n \geq 1$.*

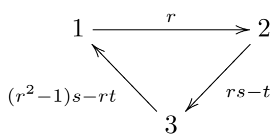
Proof. (i) This has been shown in [3][1], but we include a proof for convenience. Suppose that one of r, s, t is less than 2. Without loss of generality, we may suppose $r < 2$ and $s \leq t$. Since Q is not acyclic, r cannot be zero, whence $r = 1$. But then mutating Q at vertex 1 would produce the following acyclic quiver:



(ii) We proceed by induction on n. For $n = 1$, the quiver obtained from Q by mutation at 1 is the following:



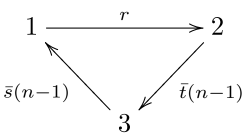
and for $n = 2$, the quiver obtained from Q by mutation at 1 and 2 is the following:



In both cases, the result follows from $c_1^{[r]} = 0, c_2^{[r]} = 1, c_3^{[r]} = r, c_4^{[r]} = r^2 - 1$.

[1] It is shown there that A is nonacyclic if and only if $r, s, t \geq 2$ and $r^2 + s^2 + t^2 - rst \leq 4$.

Suppose that $n > 2$. If n is odd then by induction we know that the quiver we are considering is obtained by mutating the following quiver at vertex 1:



and the result follows from $\bar{t}(n) = \bar{s}(n-1)$ and

$$r\bar{s}(n-1) - \bar{t}(n-1) = rc_{n+1}^{[r]}s - rc_n^{[r]}t - c_n^{[r]}s + c_{n-1}^{[r]}t = c_{n+2}^{[r]}s - c_{n+1}^{[r]}t = \bar{s}(n).$$

If n is even then the proof is similar. (iii) follows from (i) and (ii). □

§4.2. Positivity

In this section, we prove the positivity conjecture in rank 3.

Theorem 4.2. *Let $\mathcal{A}(Q)$ be a skew-symmetric coefficient-free cluster algebra of rank 3 with initial cluster $\mathbf{x}$ and let $\mathbf{x}_{t_0}$ be any cluster. Then the Laurent expansion of any cluster variable in $\mathbf{x}_{t_0}$ with respect to the cluster $\mathbf{x}$ has nonnegative coefficients.*

The remainder of this section is devoted to the proof of this theorem.

If $\mathcal{A}$ is acyclic then the theorem has been proved by Kimura and Qin [17]. We therefore assume that $\mathcal{A}$ is nonacyclic, but we point out that this is not a necessary assumption for our argument. Our methods would work also in the acyclic case, but restricting to the nonacyclic case considerably simplifies the exposition.

Since $\mathcal{A}$ is nonacyclic, every quiver Q' which is mutation equivalent to the initial quiver Q has at least one oriented cycle. Since Q' has three vertices and no 2-cycles, this implies that every arrow in Q' lies in at least one oriented cycle.

Let $\mathbf{x}_{t_0}$ be an arbitrary cluster. Choose a finite sequence μ of mutations which transforms $\mathbf{x}_{t_0}$ into the initial cluster $\mathbf{x}$ and which is of minimal length among all such sequences. Each mutation in the sequence μ is a mutation at one of the three vertices 1, 2 or 3 of the quiver, thus μ induces a finite sequence of vertices $(V_1, V_2, \ldots)$, where each V_i is 1, 2 or 3. Since μ is of minimal length, it follows that $V_i \neq V_{i+1}$.

Let $e_{0,1} = V_1$ and $e_{0,2} = V_2$. Let $k_1 > 1$ be the least integer such that $V_{k_1} \neq e_{0,1}, e_{0,2}$, and denote by t_1 the seed obtained from t_0 after mutating at $V_1, \ldots, V_{k_1-1}$. Then let $e_{1,1} = V_{k_1-1}$ and $e_{1,2} = V_{k_1}$. Let $k_2 > k_1$ be the least integer such that $V_{k_2} \neq e_{1,1}, e_{1,2}$, and denote by t_2 the seed obtained from t_0 after mutating at $V_1, \dots, V_{k_2-1}$. Recursively, we define a sequence of seeds $\Sigma_{t_0}, \Sigma_{t_1}, \Sigma_{t_2}, \dots, \Sigma_{t_m}$ along our sequence μ such that $\mathbf{x}_{t_m} = \mathbf{x}$ is the initial cluster and, for each i, the subsequence of μ between t_i and t_{i+1} is a sequence of mutations at two vertices.

The sequence of mutations can be visualized in the following diagram:

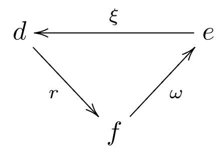
$$t_0 \overset{V_1}{\rightsquigarrow} \bullet \cdots\cdots \bullet \overset{V_{k_1-1}}{\rightsquigarrow} t_1 \overset{V_{k_1}}{\rightsquigarrow} \bullet \cdots\cdots \bullet \overset{V_{k_i-1}}{\rightsquigarrow} t_i \overset{V_{k_i}}{\rightsquigarrow} \bullet \cdots\cdots \bullet \overset{V_{k_m-1}}{\rightsquigarrow} t_m$$

The main idea of our proof is to use our rank 2 formula from the previous section to compute the Laurent expansion of a cluster variable of t_0 in the cluster at t_1, then replace the cluster variables of t_1 in this expression by their Laurent expansions in the cluster t_2, which we can compute again because of our rank 2 formulas. Continuing this procedure we obtain, at least theoretically, a Laurent expansion in the initial cluster.

Fix an arbitrary $j \geq 0$, let $d = e_{j,1}$, $e = e_{j,2}$, and let f be the integer such that $\{d, e, f\} = \{1, 2, 3\}$. Thus the sequence μ of mutations around the node t_j is of the form

$$\cdots\cdots \bullet \overset{f}{\rightsquigarrow} \bullet \overset{d}{\rightsquigarrow} t_j \overset{e}{\rightsquigarrow} \bullet \overset{d}{\rightsquigarrow} \bullet \overset{e}{\rightsquigarrow} \bullet \cdots\cdots$$

Let



be the quiver at t_j, and use the notation $\{x_{1;t_j}, x_{2;t_j}, x_{3;t_j}\}$ for the cluster $\mathbf{x}_{t_j}$ at t_j. Let

$$\widetilde{x_{d;t_j}} = \frac{x_{e;t_j}^{\xi} + x_{f;t_j}^{r}}{x_{d;t_j}}$$

and

$$\widetilde{\widetilde{x_{e;t_j}}} = \frac{\widetilde{x_{d;t_j}}^{\,\xi} + x_{f;t_j}^{r\xi-\omega}}{x_{e;t_j}}.$$

Thus $\widetilde{x_{d;t_j}} = \mu_d(x_{d;t_j})$ is the new cluster variable obtained by mutation of the cluster at t_j in direction d and $\widetilde{\widetilde{x_{e;t_j}}} = \mu_e\mu_d(x_{e;t_j})$ is the new cluster variable obtained by mutation of the cluster at t_j in direction d and then e.

Theorem 4.2 follows easily from the following result.

Theorem 4.3. *For any t_i, t_j with $i < j$, the Laurent expansion of the cluster monomial $x^p_{e_{i,1};t_i} x^q_{e_{i,2};t_i}$ in the cluster $\mathbf{x}_{t_j}$ is of the form*

$$\sum_{u_1\in\mathbb{Z},\, p_1,q_1\geq 0} C_{1;\, i\to j}\, x^{u_1}_{f;t_j} \widetilde{x_{d;t_j}}^{\,p_1} \widetilde{\widetilde{x_{e;t_j}}}^{\,q_1} + \sum_{u_2\in\mathbb{Z},\, p_2,q_2\geq 0} C_{2;\, i\to j}\, x^{u_2}_{f;t_j} \widetilde{x_{d;t_j}}^{\,p_2} x^{q_2}_{e;t_j} + \sum_{u_3\in\mathbb{Z},\, p_3,q_3\geq 0} C_{3;\, i\to j}\, x^{u_3}_{f;t_j} x^{p_3}_{d;t_j} x^{q_3}_{e;t_j},$$

where

$$C_{1;\, i\to j} = C_1(Q_{t_j}, V_{k_i}, \ldots, V_{k_j-1}, p, q; u_1, p_1, q_1),$$
$$C_{2;\, i\to j} = C_2(Q_{t_j}, V_{k_i}, \ldots, V_{k_j-1}, p, q; u_2, p_2, q_2),$$
$$C_{3;\, i\to j} = C_3(Q_{t_j}, V_{k_i}, \ldots, V_{k_j-1}, p, q; u_3, p_3, q_3)$$

are nonnegative integers which depend on $Q_{t_j}, V_{k_i}, \ldots, V_{k_j-1}, p, q, u_1, u_2, u_3, p_1, p_2, p_3, q_1, q_2, q_3$.

§4.3. Proof of positivity

Before proving Theorem 4.3, let us show that it implies Theorem 4.2.

Let x be a cluster variable in $\mathbf{x}_{t_0}$. Using Theorem 4.3 with $i = 0$ and $j = m-1$, we get an expression $LP(x, t_j)$ for x as a Laurent polynomial with nonnegative coefficients in which each summand is a quotient of a certain monomial in the variables $x_{f;t_j}, \widetilde{x_{d;t_j}}, \widetilde{\widetilde{x_{e;t_j}}}, x_{e;t_j}$ by a power of $x_{f;t_j}$. In other words, only the variable $x_{f;t_j}$ appears in the denominator of $LP(x, t_j)$.

Since the seed t_j is obtained from the seed t_m by a sequence of mutations at the vertices d and e, we see that $x_{f;t_j} = x_{f;t_m}$ is one of the initial cluster variables and that the variables $x_{d;t_j}, x_{e;t_j}\widetilde{x_{d;t_j}}, \widetilde{\widetilde{x_{e;t_j}}}$ are obtained from the initial cluster $\mathbf{x}_{t_m}$ by a sequence of mutations at d and e. Therefore, in order to write x as a Laurent polynomial in the initial cluster $\mathbf{x}_{t_m}$, we only need to compute the expansions for the variables $x_{d;t_j}, x_{e;t_j}, \widetilde{x_{d;t_j}}, \widetilde{\widetilde{x_{e;t_j}}}$ in the cluster $\mathbf{x}_{t_m}$ and substitute these expansions into $LP(x, t_j)$. But since the exponents p_i, q_i of these variables are nonnegative, we know from Theorem 3.10 that these expansions are Laurent polynomials with nonnegative coefficients, and hence, after substitution into $LP(x, t_j)$, we get an expansion for x as a Laurent polynomial with nonnegative coefficients in the initial cluster $\mathbf{x}_{t_m}$. □

§4.4. Proof of Theorem 4.3

We use induction on j. Suppose that the statement holds true for j, and we prove it for $j + 1$. Note that $f = e_{j+1,2}$, since the rank is 3. Without loss of generality, let $d=1$, so the sequence of mutations μ around the node t_{j+1} is of the form

$$\cdots\cdots \bullet \overset{e}{\rightsquigarrow} \bullet \overset{1}{\rightsquigarrow} t_{j+1} \overset{f}{\rightsquigarrow} \bullet \overset{1}{\rightsquigarrow} \bullet \overset{f}{\rightsquigarrow} \bullet \cdots\cdots$$

We analyze the three sums in the statement of Theorem 4.3 separately. For the first sum, thanks to Theorem 3.21 with $x_1 = x_{1;t_{j+1}}$, $x_2 = x_{e;t_{j+1}}$, $x_3 = \widetilde{x_{1;t_{j+1}}}$, $z_3 = x_{f;t_{j+1}}$, there exist nonnegative coefficients $\overline{\overline{C_{2;\,j\to j+1}}}$ and $\overline{\overline{C_{3;\,j\to j+1}}}$ such that

$$\sum_{u_1\in\mathbb{Z},\,p_1,q_1\geq 0} C_{1;\,i\to j}\, x_{f;t_j}^{u_1}\widetilde{x_{1;t_j}}^{p_1}\widetilde{\widetilde{x_{e;t_j}}}^{q_1} = \sum_{u_1\in\mathbb{Z},\,p_1,q_1\geq 0} C_{1;\,i\to j}\, x_{f;t_j}^{u_1} \times\left(\begin{array}{l}\displaystyle\sum_{v_e^{(1)}\in\mathbb{Z},\,p_2^{(1)},q_2^{(1)}\geq 0}\overline{\overline{C_{2;\,j\to j+1}}}\, x_{e;t_{j+1}}^{v_e^{(1)}}\widetilde{x_{1;t_{j+1}}}^{p_2^{(1)}}x_{f;t_{j+1}}^{q_2^{(1)}}\\ \displaystyle +\sum_{w_e^{(1)}\in\mathbb{Z},\,p_3^{(1)},q_3^{(1)}\geq 0}\overline{\overline{C_{3;\,j\to j+1}}}\, x_{e;t_{j+1}}^{w_e^{(1)}}x_{1;t_{j+1}}^{p_3^{(1)}}x_{f;t_{j+1}}^{q_3^{(1)}}\end{array}\right), \tag{34}$$

where $\overline{\overline{C_{2;\,j\to j+1}}}$ depends on $v_e^{(1)}, p_2^{(1)}, q_2^{(1)}$, while $\overline{\overline{C_{3;\,j\to j+1}}}$ depends on $w_e^{(1)}$, $p_3^{(1)}, q_3^{(1)}$.

We can rewrite (34) as

$$\sum C_{1;\,i\to j}\overline{\overline{C_{2;\,j\to j+1}}}\, x_{e;t_{j+1}}^{v_e^{(1)}}\widetilde{x_{1;t_{j+1}}}^{p_2^{(1)}}x_{f;t_{j+1}}^{u_1+q_2^{(1)}} + \sum C_{1;\,i\to j}\overline{\overline{C_{3;\,j\to j+1}}}\, x_{e;t_{j+1}}^{w_e^{(1)}}x_{1;t_{j+1}}^{p_3^{(1)}}x_{f;t_{j+1}}^{u_1+q_3^{(1)}}. \tag{35}$$

For the second sum, we have

$$\sum_{u_2\in\mathbb{Z},\,p_2,q_2\geq 0} C_{2;\,i\to j}\, x_{f;t_j}^{u_2}\widetilde{x_{1;t_j}}^{p_2}x_{e;t_j}^{q_2} = \sum_{u_2\in\mathbb{Z},\,p_2,q_2\geq 0} C_{2;\,i\to j}\, x_{f;t_j}^{u_2} \times\left(\begin{array}{l}\displaystyle\sum_{v_e^{(2)}\in\mathbb{Z},\,p_2^{(2)},q_2^{(2)}\geq 0}\overline{C_{2;\,j\to j+1}}\, x_{e;t_{j+1}}^{v_e^{(2)}}\widetilde{x_{1;t_{j+1}}}^{p_2^{(2)}}x_{f;t_{j+1}}^{q_2^{(2)}}\\ \displaystyle +\sum_{w_e^{(2)}\in\mathbb{Z},\,p_3^{(2)},q_3^{(2)}\geq 0}\overline{C_{3;\,j\to j+1}}\, x_{e;t_{j+1}}^{w_e^{(2)}}x_{1;t_{j+1}}^{p_3^{(2)}}x_{f;t_{j+1}}^{q_3^{(2)}}\end{array}\right), \tag{36}$$

where $\overline{C_{2;\,j\to j+1}}$ depends on $v_e^{(2)}, p_2^{(2)}, q_2^{(2)}$, while $\overline{C_{3;\,j\to j+1}}$ depends on $w_e^{(2)}$, $p_3^{(2)}, q_3^{(2)}$.

We can rewrite (36) as

$$\sum C_{2;\,i\to j}\overline{C_{2;\,j\to j+1}}\, x_{e;t_{j+1}}^{v_e^{(2)}}\widetilde{x_{1;t_{j+1}}}^{p_2^{(2)}}x_{f;t_{j+1}}^{u_2+q_2^{(2)}} + \sum C_{2;\,i\to j}\overline{C_{3;\,j\to j+1}}\, x_{e;t_{j+1}}^{w_e^{(2)}}x_{1;t_{j+1}}^{p_3^{(2)}}x_{f;t_{j+1}}^{u_2+q_3^{(2)}}. \tag{37}$$

Similarly, for the third sum

$$\begin{aligned}(38)\qquad &\sum_{u_3\in\mathbb{Z},\, p_3,q_3\geq 0} C_{3;\,i\to j}\, x_{f;t_j}^{u_3} x_{1;t_j}^{p_3} x_{e;t_j}^{q_3} \\ &= \sum_{u_3\in\mathbb{Z},\, p_3,q_3\geq 0} C_{3;\,i\to j}\, x_{f;t_j}^{u_3} \\ &\quad\times\left(\begin{array}{l}\displaystyle\sum_{v_e^{(3)}\in\mathbb{Z},\, p_2^{(3)},q_2^{(3)}\geq 0} C_{2;\,j\to j+1}\, x_{e;t_{j+1}}^{v_e^{(3)}} \widetilde{x_{1;t_{j+1}}}^{p_2^{(3)}} x_{f;t_{j+1}}^{q_2^{(3)}} \\ \displaystyle +\sum_{w_e^{(3)}\in\mathbb{Z},\, p_3^{(3)},q_3^{(3)}\geq 0} C_{3;\,j\to j+1}\, x_{e;t_{j+1}}^{w_e^{(3)}} x_{1;t_{j+1}}^{p_3^{(3)}} x_{f;t_{j+1}}^{q_3^{(3)}}\end{array}\right),\end{aligned}$$

where $C_{2;\,j\to j+1}$ depends on $v_e^{(3)}, p_2^{(3)}, q_2^{(3)}$, while $C_{3;\,j\to j+1}$ depends on $w_e^{(3)}$, $p_3^{(3)}, q_3^{(3)}$.

We can rewrite (38) as

$$\begin{aligned}(39)\qquad &\sum C_{3;\,i\to j} C_{2;\,j\to j+1}\, x_{e;t_{j+1}}^{v_e^{(3)}} \widetilde{x_{1;t_{j+1}}}^{p_2^{(3)}} x_{f;t_{j+1}}^{u_3+q_2^{(3)}} \\ &\qquad + \sum C_{3;\,i\to j} C_{3;\,j\to j+1}\, x_{e;t_{j+1}}^{w_e^{(3)}} x_{1;t_{j+1}}^{p_3^{(3)}} x_{f;t_{j+1}}^{u_3+q_3^{(3)}}.\end{aligned}$$

So by induction we have an expression of $x_{e_{i,1};t_i}^{p} x_{e_{i,2};t_i}^{q}$ with respect to the cluster t_{j+1}, that is,

$$(40)\qquad x_{1;t_i}^{p} x_{e_{i+1};t_i}^{q} = (35) + (37) + (39).$$

This expression allows us to compute the new coefficients $C_{2;\,i\to j+1}$ and $C_{3;\,i\to j+1}$ by collecting terms with nonnegative exponents on $x_{f;t_{j+1}}$; we have

$$C_{2;\,i\to j+1} = \sum C_{1;\,i\to j}\overline{\overline{C_{2;\,j\to j+1}}} + \sum C_{2;\,i\to j}\overline{C_{2;\,j\to j+1}} + \sum C_{3;\,i\to j} C_{2;\,j\to j+1},$$

where the three sums are over all possible variables satisfying $u_1 + q_2^{(1)} \geq 0$, $u_2 + q_2^{(2)} \geq 0$, and $u_3 + q_2^{(3)} \geq 0$ respectively, so that the exponent of $x_{f;t_{j+1}}$ is nonnegative.

Similarly,

$$C_{3;\,i\to j+1} = \sum C_{1;\,i\to j}\overline{\overline{C_{3;\,j\to j+1}}} + \sum C_{2;\,i\to j}\overline{C_{3;\,j\to j+1}} + \sum C_{3;\,i\to j} C_{3;\,j\to j+1},$$

where the three sums are over all possible variables satisfying $u_1 + q_3^{(1)} \geq 0$, $u_2 + q_3^{(2)} \geq 0$, and $u_3 + q_3^{(3)} \geq 0$ respectively. In particular, this shows that $C_{2;\,i\to j+1}$ and $C_{3;\,i\to j+1}$ are nonnegative integers.

Since u_1, u_2 or u_3 can be negative, $x_{f;t_{j+1}}$ can have negative exponents. Now we analyze the terms in which $x_{f;t_{j+1}}$ appears with a negative exponent. For every positive integer θ, let $\mathcal{P}_\theta$ be the sum of all terms in (40) with exponent of $x_{f,t_{j+1}}$ equal to $-\theta$, that is, $\mathcal{P}_\theta = \mathcal{P}_{\theta,2} + \mathcal{P}_{\theta,3}$, where

$$\begin{aligned}
\mathcal{P}_{\theta,2} = &\sum C_{1;\, i\to j}\overline{\overline{C_{2;\, j\to j+1}}}\, x_{e;t_{j+1}}^{v_e^{(1)}}\, \widetilde{x_{1;t_{j+1}}}^{\,p_2^{(1)}}\, x_{f;t_{j+1}}^{u_1+q_2^{(1)}} \\
&+ \sum C_{2;\, i\to j}\overline{C_{2;\, j\to j+1}}\, x_{e;t_{j+1}}^{v_e^{(2)}}\, \widetilde{x_{1;t_{j+1}}}^{\,p_2^{(2)}}\, x_{f;t_{j+1}}^{u_2+q_2^{(2)}} \\
&+ \sum C_{3;\, i\to j}C_{2;\, j\to j+1}\, x_{e;t_{j+1}}^{v_e^{(3)}}\, \widetilde{x_{1;t_{j+1}}}^{\,p_2^{(3)}}\, x_{f;t_{j+1}}^{u_3+q_2^{(3)}}, \\
\mathcal{P}_{\theta,3} = &\sum C_{1;\, i\to j}\overline{\overline{C_{3;\, j\to j+1}}}\, x_{e;t_{j+1}}^{w_e^{(1)}} x_{1;t_{j+1}}^{p_3^{(1)}} x_{f;t_{j+1}}^{u_1+q_3^{(1)}} \\
&+ \sum C_{2;\, i\to j}\overline{C_{3;\, j\to j+1}}\, x_{e;t_{j+1}}^{w_e^{(2)}} x_{1;t_{j+1}}^{p_3^{(2)}} x_{f;t_{j+1}}^{u_2+q_3^{(2)}} \\
&+ \sum C_{3;\, i\to j}C_{3;\, j\to j+1}\, x_{e;t_{j+1}}^{w_e^{(3)}} x_{1;t_{j+1}}^{p_3^{(3)}} x_{f;t_{j+1}}^{u_3+q_3^{(3)}},
\end{aligned}$$

where the first sum in the expression for $\mathcal{P}_{\theta,h}$ $(h = 2, 3)$ is over all

$$u_1, v_e^{(1)}, w_e^{(1)} \in \mathbb{Z},\ p_1, q_1, p_h^{(1)}, q_h^{(1)} \geq 0 \quad \text{satisfying} \quad u_1 + q_h^{(1)} = -\theta,$$

the second sum is over all

$$u_2, v_e^{(2)}, w_e^{(2)} \in \mathbb{Z},\ p_2, q_2, p_h^{(2)}, q_h^{(2)} \geq 0 \quad \text{satisfying} \quad u_2 + q_h^{(2)} = -\theta,$$

and the third sum is over all

$$u_3, v_e^{(3)}, w_e^{(3)} \in \mathbb{Z},\ p_3, q_3, p_h^{(3)}, q_h^{(3)} \geq 0 \quad \text{satisfying} \quad u_3 + q_h^{(3)} = -\theta.$$

To complete the proof, we shall compute $\mathcal{P}_{\theta,2}$ and $\mathcal{P}_{\theta,3}$ separately. We show that $\mathcal{P}_{\theta,3} = 0$ in Lemma 4.10, and thus to complete the proof it suffices to show the following result on $\mathcal{P}_{\theta,2}$.

Lemma 4.4. *$\mathcal{P}_{\theta,2}$ is of the form*

$$\widetilde{\widetilde{x_{f;t_{j+1}}}}^{\,\theta} \sum_{u_e \in \mathbb{Z}, p_1 \geq 0} C_{1;i\to j+1}\, x_{e;t_{j+1}}^{u_e}\, \widetilde{x_{1;t_{j+1}}}^{\,p_1},$$

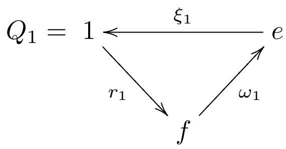
where

$$C_{1;i\to j+1} = C_1(B_{t_{j+1}}, V_{k_i}, \ldots, V_{k_{j+1}-1}, p, q; u_e, p_1, \theta)$$

are nonnegative integers.

Proof. P_θ is the sum of all terms in the Laurent expansion of $x_{1;t_i}^p x_{f;t_i}^q$ in the cluster $\mathbf{x}_{t_{j+1}}$ with exponent of $x_{f;t_{j+1}}$ equal to $-\theta$. Clearly $P_{\theta,2} = 0$, for $j = i$. We shall start over and compute P_θ using Corollary 3.20. To keep the notation simple, we give a detailed proof for $j = i + 1$. The case $j > i + 1$ uses the same argument.

Let

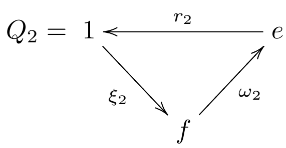


be the quiver at the seed $\mu_1(t_{i+1})$, where r_1, ω_1 and ξ_1 are the numbers of arrows.

Applying Corollary 3.20 to $x^p_{1;t_i} x^q_{f;t_i}$, with $x_2 = x_{f;t_{i+1}}$, $x_3 = \widetilde{x_{1;t_{i+1}}}$, and $z_3 = x_{e;t_{i+1}}$, we obtain

$$\sum_{\tau_{0;1},\tau_{1;1},\dots,\tau_{n_1-3;1}} \left(\prod_{w=0}^{n_1-3} \begin{bmatrix} A_{w+1;1} - r_1 s_{w;1} \\ \tau_{w;1} \end{bmatrix}\right) \times \widetilde{x_{1;t_{i+1}}}^{A_{n_1-1;1} - r_1 s_{n_1-2;1}} x_{f;t_{i+1}}^{r_1 s_{n_1-3;1} - A_{n_1-2;1}} x_{e;t_{i+1}}^{\omega_1 s_{n_1-2;1} - \xi_1 s_{n_1-3;1}}. \tag{41}$$

Let n_2 be the number of seeds between $\mu_1(t_{i+1})$ and t_{i+2} inclusive. Suppose that n_2 is an even integer. The case of n_2 odd is similar, except that the roles of $x_{1;t_i+1}$ and $x_{e;t+1}$ are interchanged. Let



be the quiver at the seed $\mu_1(t_{i+2})$, where r_2, ω_2 and ξ_2 are the numbers of arrows. Since the mutation sequence relating the quivers Q_1 and Q_2 consists of mutations at the vertices 1 and e, we see from Lemma 4.1 that

$$\begin{aligned} r_2 &= \xi_1, \\ \omega_2 &= c^{[r_2]}_{n_2-1} r_1 - c^{[r_2]}_{n_2-2} \omega_1, \\ \xi_2 &= c^{[r_2]}_{n_2} r_1 - c^{[r_2]}_{n_2-1} \omega_1. \end{aligned} \tag{42}$$

Now let $p_2 = A_{n_1-1;1} - r_1 s_{n_1-2;1}$, $q_2 = \omega_1 s_{n_1-2;1} - \xi_1 s_{n_1-3;1}$ be the exponents of $\widetilde{x_{1;t_{i+1}}}$ and $x_{e;t_{i+1}}$ in (41) respectively. Applying Corollary 3.20 to $\widetilde{x_{1;t_{i+1}}}^{p_2} x^{q_2}_{e;t_{i+1}}$ we see that (41) is equal to

$$\begin{aligned} &\sum_{\tau_{0;1},\tau_{1;1},\dots,\tau_{n_1-3;1}} \left(\prod_{w=0}^{n_1-3} \begin{bmatrix} A_{w+1;1} - r_1 s_{w;1} \\ \tau_{w;1} \end{bmatrix}\right) x_{f;t_{i+2}}^{r_1 s_{n_1-3;1} - A_{n_1-2;1}} \\ &\qquad \times \sum_{\tau_{0;2},\tau_{1;2},\dots,\tau_{n_2-3;2}} \left(\prod_{w=0}^{n_2-3} \begin{bmatrix} A_{w+1;2} - r_2 s_{w;2} \\ \tau_{w;2} \end{bmatrix}\right) \\ &\qquad \times \widetilde{x_{1;t_{i+2}}}^{A_{n_2-1;2} - r_2 s_{n_2-2;2}} x_{e;t_{i+2}}^{r_2 s_{n_2-3;2} - A_{n_2-2;2}} x_{f;t_{i+2}}^{\omega_2 s_{n_2-2;2} - \xi_2 s_{n_2-3;2}} \end{aligned} \tag{43}$$

$$= \sum_{\tau_{0;1},\tau_{1;1},\dots,\tau_{n_1-3;1}} \left(\prod_{w=0}^{n_1-3} \begin{bmatrix} A_{w+1;1} - r_1 s_{w;1} \\ \tau_{w;1} \end{bmatrix}\right)$$
$$\times \sum_{\tau_{0;2},\tau_{1;2},\dots,\tau_{n_2-3;2}} \left(\prod_{w=0}^{n_2-3} \begin{bmatrix} A_{w+1;2} - r_2 s_{w;2} \\ \tau_{w;2} \end{bmatrix}\right)$$
$$\times \widetilde{x_{1;t_{i+2}}}^{A_{n_2-1;2}-r_2 s_{n_2-2;2}} x_{e;t_{i+2}}^{r_2 s_{n_2-3;2}-A_{n_2-2;2}} x_{f;t_{i+2}}^{\omega_2 s_{n_2-2;2}-\xi_2 s_{n_2-3;2}+r_1 s_{n_1-3;1}-A_{n_1-2;1}}$$

where $A_{i;2}$ and $s_{i;2}$ are as defined before Lemma 3.12 and Lemma 3.15 but in terms of p_2, q_2, and r_2, thus $A_{i;2} = p_2 c_{i+1}^{[r_2]} + q_2 c_i^{[r_2]}$ and $s_{i;2} = \sum_{j=0}^{i-1} c_{i-j+1}^{[r_2]} \tau_{j;2}$.

Let θ be a positive integer. We want to compute P_θ, which is the sum of all terms in the sum above for which the exponent of $x_{f;t_{i+2}}$ is equal to $-\theta$, and show that it is divisible by $\widetilde{\widetilde{x_{f;t_{j+1}}}}^{\theta}$. Thus $-\theta$ is equal to

$$\omega_2 s_{n_2-2;2} - \xi_2 s_{n_2-3;2} + r_1 s_{n_1-3;1} - A_{n_1-2;1}.$$

It is convenient to introduce ς such that $\tau_{0;2} = \varsigma - s_{n_1-3;1}$. Then

$$s_{n_2-2;2} = c_{n_2-1}^{[r_2]}(\varsigma - s_{n_1-3;1}) + \sum_{j=1}^{n_2-3} c_{n_2-1-j}^{[r_2]} \tau_{j;2},$$
$$s_{n_2-3;2} = c_{n_2-2}^{[r_2]}(\varsigma - s_{n_1-3;1}) + \sum_{j=1}^{n_2-4} c_{n_2-2-j}^{[r_2]} \tau_{j;2}.$$

Using (42), the expressions for $s_{n_2-2;2}$ and $s_{n_2-3;2}$ and the fact that $c_1^{[\xi]} = 0$, we have

$$\omega_2 s_{n_2-2;2} - \xi_2 s_{n_2-3;2}$$
$$= (c_{n_2-1}^{[\xi_1]} r_1 - c_{n_2-2}^{[\xi_1]} \omega_1)\Big[c_{n_2-1}^{[\xi_1]}(\varsigma - s_{n_1-3;1}) + \sum_{j=1}^{n_2-3} c_{n_2-1-j}^{[\xi_1]} \tau_{j;2}\Big]$$
$$- (c_{n_2}^{[\xi_1]} r_1 - c_{n_2-1}^{[\xi_1]} \omega_1)\Big[c_{n_2-2}^{[\xi_1]}(\varsigma - s_{n_1-3;1}) + \Big(\sum_{j=1}^{n_2-3} c_{n_2-2-j}^{[\xi_1]} \tau_{j;2}\Big) - c_1^{[\xi_1]} \tau_{n_2-3;2}\Big]$$
$$= (\varsigma - s_{n_1-3;1})\, r_1 ((c_{n_2-1}^{[\xi_1]})^2 - c_{n_2}^{[\xi_1]} c_{n_2-2}^{[\xi_1]})$$
$$+ \sum_{j=1}^{n_2-3} \tau_{j;2} [r_1 (c_{n_2-1}^{[\xi_1]} c_{n_2-1-j}^{[\xi_1]} - c_{n_2}^{[\xi_1]} c_{n_2-2-j}^{[\xi_1]}) + \omega_1 (-c_{n_2-2}^{[\xi_1]} c_{n_2-1-j}^{[\xi_1]} + c_{n_2-1}^{[\xi_1]} c_{n_2-2-j}^{[\xi_1]})]$$
$$\overset{3.1}{=} (\varsigma - s_{n_1-3;1})\, r_1 + \sum_{j=1}^{n_2-3} \tau_{j;2} [r_1(-c_{-j}^{[\xi_1]}) + \omega_1 c_{1-j}^{[\xi_1]}].$$

And since $-c^{[\xi_1]}_{-j} = c^{[\xi_1]}_{j+2}$, we get

$$\begin{aligned}(44)\qquad -\theta &= r_1(\varsigma - s_{n_1-3;1}) + \sum_{j=1}^{n_2-3} \tau_{j;2}(c^{[\xi_1]}_{j+2} r_1 - c^{[\xi_1]}_{j+1}\omega_1) + r_1 s_{n_1-3;1} - A_{n_1-2;1}\\ &= -A_{n_1-2;1} + r_1\varsigma + \sum_{j=1}^{n_2-3} \tau_{j;2}(c^{[\xi_1]}_{j+2} r_1 - c^{[\xi_1]}_{j+1}\omega_1).\end{aligned}$$

Also, the exponents of $\widetilde{x_{1;t_{i+2}}}$ and $x_{e;t_{i+2}}$ in (43) can be expressed as follows:

$$\begin{aligned}A_{n_2-1;2} - r_2 s_{n_2-2;2} &= c^{[\xi_1]}_{n_2} p_2 + c^{[\xi_1]}_{n_2-1} q_2 - \xi_1\Big(c^{[\xi_1]}_{n_2-1}(\varsigma - s_{n_1-3;1}) + \sum_{j=1}^{n_2-3} c^{[\xi_1]}_{n_2-1-j}\tau_{j;2}\Big)\\ &= c^{[\xi_1]}_{n_2}(A_{n_1-1;1} - r_1 s_{n_1-2;1}) + c^{[\xi_1]}_{n_2-1}(\omega_1 s_{n_1-2;1} - \xi_1 s_{n_1-3;1})\\ &\quad - \xi_1\Big(c^{[\xi_1]}_{n_2-1}(\varsigma - s_{n_1-3;1}) + \sum_{j=1}^{n_2-3} c^{[\xi_1]}_{n_2-1-j}\tau_{j;2}\Big)\\ &= c^{[\xi_1]}_{n_2}(A_{n_1-1;1} - r_1 s_{n_1-2;1}) + c^{[\xi_1]}_{n_2-1}\omega_1 s_{n_1-2;1}\\ &\quad - \xi_1\Big(c^{[\xi_1]}_{n_2-1}\varsigma + \sum_{j=1}^{n_2-3} c^{[\xi_1]}_{n_2-1-j}\tau_{j;2}\Big),\end{aligned}$$

and similarly

$$\begin{aligned}\xi_1 s_{n_2-3;2} - A_{n_2-2;2} &= \xi_1\Big(c^{[\xi_1]}_{n_2-2}\varsigma + \sum_{j=1}^{n_2-4} c^{[\xi_1]}_{n_2-2-j}\tau_{j;2}\Big)\\ &\qquad - \big(c^{[\xi_1]}_{n_2-1}(A_{n_1-1;1} - r_1 s_{n_1-2;1}) + c^{[\xi_1]}_{n_2-2}\omega_1 s_{n_1-2;1}\big).\end{aligned}$$

By fixing $\varsigma, \tau_{1;2}, \dots, \tau_{n_2-3;2}$ in (43), we have

$$\begin{aligned}&\sum_{\tau_{0;1},\tau_{1;1},\dots,\tau_{n_1-3;1}} \left(\prod_{w=0}^{n_1-3}\begin{bmatrix} A_{w+1;1} - r_1 s_{w;1}\\ \tau_{w;1}\end{bmatrix}\right)\left(\prod_{w=0}^{n_2-3}\begin{bmatrix} A_{w+1;2} - r_2 s_{w;2}\\ \tau_{w;2}\end{bmatrix}\right)\\ &\quad\times \widetilde{x_{1;t_{i+2}}}^{\,c^{[\xi_1]}_{n_2}(A_{n_1-1;1}-r_1 s_{n_1-2;1}) + c^{[\xi_1]}_{n_2-1}\omega_1 s_{n_1-2;1} - \xi_1(c^{[\xi_1]}_{n_2-1}\varsigma + \sum_{j=1}^{n_2-3} c^{[\xi_1]}_{n_2-1-j}\tau_{j;2})}\\ &\quad\times x_{e;t_{i+2}}^{\xi_1(c^{[\xi_1]}_{n_2-2}\varsigma + \sum_{j=1}^{n_2-4} c^{[\xi_1]}_{n_2-2-j}\tau_{j;2}) - (c^{[\xi_1]}_{n_2-1}(A_{n_1-1;1}-r_1 s_{n_1-2;1}) + c^{[\xi_1]}_{n_2-2}\omega_1 s_{n_1-2;1})}\\ &\quad\times x_{f;t_{i+2}}^{-A_{n_1-2;1} + r_1\varsigma + \sum_{j=1}^{n_2-3}(c^{[t_1]}_{j+2} r_1 - c^{[t_1]}_{j+1}\omega_1)\tau_{j;2}}\end{aligned}$$

and this is equal to a product $\phi\varphi$ where ϕ is a Laurent monomial in $\widetilde{x_{1;t_{i+2}}}, x_{e;t_{i+2}}$,

$x_{f;t_{i+2}}$, while φ is equal to

$$\sum_{\tau_{0;1},\tau_{1;1},\ldots,\tau_{n_1-3;1}} \left(\prod_{w=0}^{n_1-3} \begin{bmatrix} A_{w+1;1} - r_1 s_{w;1} \\ \tau_{w;1} \end{bmatrix}\right)\left(\prod_{w=0}^{n_2-3} \begin{bmatrix} A_{w+1;2} - r_2 s_{w;2} \\ \tau_{w;2} \end{bmatrix}\right) \times \left(\frac{\widetilde{x_{1;t_{i+2}}}^{c^{[\xi_1]}_{n_2} r_1 - c^{[\xi_1]}_{n_2-1}\omega_1}}{x_{e;t_{i+2}}^{c^{[\xi_1]}_{n_2-1} r_1 - c^{[\xi_1]}_{n_2-2}\omega_1}}\right)^{\lfloor (A_{n_1-1;1}-\varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}}\rfloor - s_{n_1-2;1}}$$

and transferring the 0-th term of the second product to an (n_1-2)-nd term in the first product, we get

$$\sum_{\tau_{0;1},\tau_{1;1},\ldots,\tau_{n_1-2;1}} \left(\prod_{w=0}^{n_1-2} \begin{bmatrix} A_{w+1;1} - r_1 s_{w;1} \\ \tau_{w;1} \end{bmatrix}\right)\left(\prod_{w=1}^{n_2-3} \begin{pmatrix} A_{w+1;2} - r_2 s_{w;2} \\ \tau_{w;2} \end{pmatrix}\right) \times \left(\frac{\widetilde{x_{1;t_{i+2}}}^{c^{[\xi_1]}_{n_2} r_1 - c^{[\xi_1]}_{n_2-1}\omega_1}}{x_{e;t_{i+2}}^{c^{[\xi_1]}_{n_2-1} r_1 - c^{[\xi_1]}_{n_2-2}\omega_1}}\right)^{\lfloor (A_{n_1-1;1}-\varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}}\rfloor - s_{n_1-2;1}}, \tag{45}$$

where $\tau_{n_1-2;1} = A_{n_1-1;1} - r_1 s_{n_1-2;1} - \tau_{0;2} = A_{n_1-1;1} - r_1 s_{n_1-2;1} - \varsigma + s_{n_1-3;1}$. Moreover, using Lemma 3.15, we observe that

$$s_{n_1-1;1} = r_1 s_{n_1-2;1} - s_{n_1-3;1} + \tau_{n_1-2;1} = A_{n_1-1;1} - \varsigma, \tag{46}$$

and by Theorem 3.26,

$$\sum_{\substack{\tau_{0;1},\tau_{1;1},\ldots,\tau_{n_1-2;1} \\ s_{n_1-1;1}=A_{n_1-1;1}-\varsigma}} \left(\prod_{w=0}^{n_1-2} \begin{bmatrix} A_{w+1;1} - r_1 s_{w;1} \\ \tau_{w;1} \end{bmatrix}\right) x_{1;t_{i+1}}^{A_{n_1-1;1} - r_1 s_{n_1-2;1}}$$

is divisible by $(1 + {x_{1;t_{i+1}}}^{r_1})^{r_1(A_{n_1-1;1}-\varsigma)-A_{n_1;1}}$ in $\mathbb{Z}[x^{\pm1}_{1;t_{i+1}}]$, and the resulting quotient has nonnegative coefficients. Multiplying the sum with $x_{1;t_{i+1}}^{r_1\lfloor (A_{n_1-1;1}-\varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}}\rfloor - A_{n_1-1;1}}$ shows that

$$\sum_{\substack{\tau_{0;1},\tau_{1;1},\ldots,\tau_{n_1-2;1} \\ s_{n_1-1;1}=A_{n_1-1;1}-\varsigma}} \left(\prod_{w=0}^{n_1-2} \begin{bmatrix} A_{w+1;1} - r_1 s_{w;1} \\ \tau_{w;1} \end{bmatrix}\right) ({x_{1;t_{i+1}}}^{r_1})^{\lfloor (A_{n_1-1;1}-\varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}}\rfloor - s_{n_1-2;1}} \tag{47}$$

is also divisible by $(1 + x^{r_1}_{1;t_{i+1}})^{r_1(A_{n_1-1;1}-\varsigma)-A_{n_1;1}}$, and the resulting quotient has nonnegative coefficients. Moreover, we shall show in Lemma 4.5 below that the exponents in (47) are nonnegative, which implies that the quotient is a polynomial.

Note that the statement about the divisibility of (47) also holds when we replace $x^r_{1;t_{i+1}}$ with any other expression X. We can write (45) as follows:

$$(45) = \sum\sum_m q(m)p(m)X^{b-m},$$

where

$$q(m) = \prod_{w=0}^{n_1-2}\left[\begin{matrix}A_{w+1;1} - r_1 s_{w;1}\\ \tau_{w;1}\end{matrix}\right], \qquad p(m) = \prod_{w=1}^{n_2-3}\binom{A_{w+1;2} - r_2 s_{w;2}}{\tau_{w;2}},$$

$$b = \left\lfloor (A_{n_1-1;1} - \varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}}\right\rfloor, \qquad m = s_{n_1-2;1},$$

$$X = \frac{\widetilde{x_{1;t_{i+2}}}^{c^{[\xi_1]}_{n_2} r_1 - c^{[\xi_1]}_{n_2-1}\omega_1}}{x_{e;t_{i+2}}^{c^{[\xi_1]}_{n_2-1} r_1 - c^{[\xi_1]}_{n_2-2}\omega_1}}.$$

Then Lemma 4.7 below yields

$$p(m) = \sum_{i=0}^{\sum_{w=1}^{n_2-3}\tau_{w;2}} d_i \binom{\lfloor (A_{n_1-1;1} - \varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}}\rfloor - s_{n_1-2;1}}{i}$$

and using Lemma 4.6 with $g = r_1(A_{n_1-1;1} - \varsigma) - A_{n_1;1}$ and $h = \sum_{j=1}^{n_2-3}\tau_{j;2}$, we find that the expression in (45) is divisible by

$$(48) \qquad \left(1 + \frac{\widetilde{x_{1;t_{i+2}}}^{c^{[\xi_1]}_{n_2} r_1 - c^{[\xi_1]}_{n_2-1}\omega_1}}{x_{e;t_{i+2}}^{c^{[\xi_1]}_{n_2-1} r_1 - c^{[\xi_1]}_{n_2-2}\omega_1}}\right)^{r_1(A_{n_1-1;1}-\varsigma) - A_{n_1;1} - \sum_{j=1}^{n_2-3}\tau_{j;2}}$$
$$= \left(1 + \frac{\widetilde{x_{1;t_{i+2}}}^{\xi_2}}{x_{e;t_{i+2}}^{\omega_2}}\right)^{A_{n_1-2;1} - r_1\varsigma - \sum_{j=1}^{n_2-3}\tau_{j;2}},$$

and the resulting quotient has nonnegative coefficients. Finally, dividing (48) by $x^\theta_{f;t_{i+2}}$ and using the fact that

$$\widetilde{\widetilde{x_{f;t_{i+2}}}} = (x^{\omega_2}_{e;t_{i+2}} + \widetilde{x_{1;t_{i+2}}}^{\xi_2})/x_{f;t_{i+2}}$$

we see that $\mathcal{P}_{\theta,2}$ is divisible by $\widetilde{\widetilde{x_{f;t_{i+2}}}}^{\theta}$. □

Lemma 4.5.
$$\left\lfloor (A_{n_1-1;1} - \varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}}\right\rfloor - s_{n_1-2;1} \geq 0.$$

Proof. We have

$$s_{n_1-1;1} - s'_{n_1-1;1} \geq (s_{n_1-2;1} - s'_{n_1-2;1})\frac{A_{n_1-1;1}}{A_{n_1-2;1}} \geq (s_{n_1-2;1} - s'_{n_1-2;1})\frac{A_{n_1;1}}{A_{n_1-1;1}},$$

where the first inequality follows from (16) and the second from Lemma 3.12(b).

Hence

$$(s_{n_1-1;1} - s'_{n_1-1;1})A_{n_1-1;1} \geq (s_{n_1-2;1} - s'_{n_1-2;1})A_{n_1;1}. \tag{49}$$

On the other hand, since $s'_{n_1-2;1} = kc^{[r_1]}_{n_1-1;1}$, $s'_{n_1-1;1} = kc^{[r_1]}_{n;1}$, by Definition 3.16, and $A_{i;1} = pc^{[r_1]}_{i+1;1} + qc^{[r_1]}_{i;1}$, we have

$$\begin{aligned} s'_{n_1-1;1}A_{n_1-1;1} &= kc^{[r_1]}_{n;1}(pc^{[r_1]}_{n;1} + qc^{[r_1]}_{n_1-1;1}) \\ &\geq kc^{[r_1]}_{n_1-1;1}(pc^{[r_1]}_{n+1;1} + qc^{[r_1]}_{n;1}) = s'_{n_1-2;1}A_{n_1;1}, \end{aligned} \tag{50}$$

where the inequality follows from Lemma 3.1. Adding (49) and (50) we get

$$s_{n_1-1;1}A_{n_1-1;1} \geq s_{n_1-2;1}A_{n_1;1}.$$

Therefore (46) yields

$$(A_{n_1-1;1} - \varsigma)A_{n_1-1;1} \geq s_{n_1-2;1}A_{n_1;1},$$

and we get

$$(A_{n_1-1;1} - \varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}} - s_{n_1-2;1} \geq s_{n_1-2;1} - s_{n_1-2;1} = 0. \qquad \square$$

Lemma 4.6. *Suppose that a polynomial in x of the form*

$$\sum_{m\in I} q(m)x^{b-m}$$

is divisible by $(1+x)^g$ and the quotient has nonnegative coefficients. Let

$$p(m) = \sum_{i=0}^{h} d_i \binom{b-m}{i}$$

be a polynomial in m with $d_i \geq 0$. Then

$$\sum_{m\in I} p(m)q(m)x^{b-m}$$

is divisible by $(1+x)^{g-h}$ and the quotient has nonnegative coefficients.

Proof. This is because $x^i \frac{d^i}{dx^i}\sum_{m\in I} q(m)x^{b-m}$ is divisible by $(1+x)^{g-i}$ and the quotient has nonnegative coefficients. $\square$

Lemma 4.7. *With assumptions in the proof of Lemma 4.4, we have*

$$\prod_{w=1}^{n_2-3}\binom{A_{w+1;2} - r_2 s_{w;2}}{\tau_{w;2}} = \sum_{i=0}^{\sum_{w=1}^{n_2-3}\tau_{w;2}} d_i \binom{\lfloor (A_{n_1-1;1} - \varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}}\rfloor - s_{n_1-2;1}}{i}$$

for some $d_i \in \mathbb{N}$, which are independent of $s_{n_1-2;1}$.

Proof. Once we know that there are nonnegative integers a and b such that $A_{w+1;2}-r_2 s_{w;2} = a\big(\lfloor (A_{n_1-1;1}-\varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}}\rfloor - s_{n_1-2;1}\big)+b$, it is clear, by Lemma 4.8, that

$$\binom{A_{w+1;2}-r_2 s_{w;2}}{\tau_{w;2}} = \sum_{i=0}^{\tau_{w;2}} d'_i \binom{\lfloor (A_{n_1-1;1}-\varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}}\rfloor - s_{n_1-2;1}}{i}$$

for some $d'_i \in \mathbb{N}$, and by Lemma 4.9, for any nonnegative integers j and k,

$$\binom{\lfloor (A_{n_1-1;1}-\varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}}\rfloor - s_{n_1-2;1}}{j}\binom{\lfloor (A_{n_1-1;1}-\varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}}\rfloor - s_{n_1-2;1}}{k} = \sum_{i=0}^{j+k} d''_i \binom{\lfloor (A_{n_1-1;1}-\varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}}\rfloor - s_{n_1-2;1}}{i}$$

for some $d''_i \in \mathbb{N}$. Then the desired statement easily follows.

Thus we need to show the existence of the nonnegative integers a and b. Using the definitions of $A_{w+1;2}$ and ς as well as the fact that $r_2 = \xi_1$, we get

$$A_{w+1;2} - r_2 s_{w;2} = c^{[\xi_1]}_{w+2}(A_{n_1-1;1} - r_1 s_{n_1-2;1}) + c^{[\xi_1]}_{w+1}\omega_1 s_{n_1-2;1} - \xi_1\Big(c^{[\xi_1]}_{w+1}\varsigma + \sum_{j=1}^{w-1} c^{[\xi_1]}_{w+1-j}\tau_{j;2}\Big),$$

which can be written as

$$A_{w+1;2}-r_2 s_{w;2} = (c^{[\xi_1]}_{w+2} r_1 - c^{[\xi_1]}_{w+1}\omega_1)\left(\left\lfloor (A_{n_1-1;1}-\varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}}\right\rfloor - s_{n_1-2;1}\right) + C(w),$$

where $C(w)$ is some function of w, which is independent of $s_{n_1-2;1}$. Note that

$$c^{[\xi_1]}_{w+2} r_1 - c^{[\xi_1]}_{w+1}\omega_1 > 0,$$

because, by Lemma 4.1, this is the number of arrows between some pair of vertices in some seed between t_{i+1} and t_{i+2}. Thus it suffices to show that $C(w)$ is nonnegative. Indeed,

$$C(w) = (c^{[\xi_1]}_{w+2} r_1 - c^{[\xi_1]}_{w+1}\omega_1)\left((A_{n_1-1;1}-\varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}} - \left\lfloor (A_{n_1-1;1}-\varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}}\right\rfloor\right) + \tilde{C}(w)\theta(w),$$

where

$$\tilde{C}(w) = c^{[\xi_1]}_{w+2} - (c^{[\xi_1]}_{w+2} r_1 - c^{[\xi_1]}_{w+1}\omega_1)\frac{A_{n_1-1;1}}{A_{n_1;1}},$$

$$\theta(w) = A_{n_1-1;1} - \frac{\xi_1 c^{[\xi_1]}_{w+1} - (c^{[\xi_1]}_{w+2} r_1 - c^{[\xi_1]}_{w+1}\omega_1)\frac{A_{n_1-1;1}}{A_{n_1;1}}}{c^{[\xi_1]}_{w+2} - (c^{[\xi_1]}_{w+2} r_1 - c^{[\xi_1]}_{w+1}\omega_1)\frac{A_{n_1-1;1}}{A_{n_1;1}}}\varsigma$$
$$- \sum_{j=1}^{w-1} \frac{\xi_1 c^{[\xi_1]}_{w+1-j}}{c^{[\xi_1]}_{w+2} - (c^{[\xi_1]}_{w+2} r_1 - c^{[\xi_1]}_{w+1}\omega_1)\frac{A_{n_1-1;1}}{A_{n_1;1}}}\tau_{j;2}.$$

We want to show that $C(w)$ is nonnegative for $w \geq 1$, for which it suffices to show that $\tilde{C}(w)$ and $\theta(w)$ are nonnegative for $w \geq 1$.

First we show that $\tilde{C}(w)$ are nonnegative for $w \geq 1$. Note that $\tilde{C}(w) = \xi_1\tilde{C}(w-1) - \tilde{C}(w-2)$. Moreover $\xi_1 \geq 2$, by Lemma 4.1. Hence if we show $\tilde{C}(1) > 0 \geq \tilde{C}(0)$ then induction on w will show that $\tilde{C}(w)$ is increasing in w. It is easy to see that $\tilde{C}(0) = 1 - r_1\frac{A_{n_1-1;1}}{A_{n_1;1}} \leq 0$. On the other hand,

$$\tilde{C}(1) = \xi_1 - (\xi_1 r_1 - \omega_1)\frac{A_{n_1-1;1}}{A_{n_1;1}} = \xi_1\frac{A_{n_1;1} - r_1 A_{n_1-1;1}}{A_{n_1;1}} + \omega_1\frac{A_{n_1-1;1}}{A_{n_1;1}}$$
$$= \xi_1\frac{-A_{n_1-2;1}}{A_{n_1;1}} + \omega_1\frac{A_{n_1-1;1}}{A_{n_1;1}},$$

which is positive because

$$\begin{aligned}(51)\quad \omega_1 A_{n_1-1;1} - \xi_1 A_{n_1-2;1} &= \omega_1(pc_{n_1} + qc_{n_1-1}) - \xi_1(pc_{n_1-1} + qc_{n_1-2})\\ &= p(\omega_1 c_{n_1} - \xi_1 c_{n_1-1}) + q(\omega_1 c_{n_1-1} - \xi_1 c_{n_1-2})) > 0,\end{aligned}$$

where the last inequality follows from

$$\begin{aligned}&\{\omega_1 c_{n_1} - \xi_1 c_{n_1-1}, \omega_1 c_{n_1-1} - \xi_1 c_{n_1-2}\}\\ &\quad = \{\text{the number of arrows between } e \text{ and } 1 \text{ in the seed } t_i,\\ &\qquad \text{the number of arrows between } e \text{ and } f \text{ in the seed } t_i\}.\end{aligned}$$

Next we show that $\theta(w)$ is nonnegative for all w such that $1 \leq w \leq n_2 - 3$. Recall from (44) that

$$\theta = A_{n_1-2;1} - r_1\varsigma - \sum_{j=1}^{n_2-3}(c^{[t_1]}_{j+2} r_1 - c^{[t_1]}_{j+1}\omega_1)\tau_{j;2} > 0,$$

which implies that

$$A_{n_1-1;1} - \frac{r_1 A_{n_1-1;1}}{A_{n_1-2;1}}\varsigma - \sum_{j=1}^{w-1}\frac{(c^{[t_1]}_{j+2} r_1 - c^{[t_1]}_{j+1}\omega_1)A_{n_1-1;1}}{A_{n_1-2;1}}\tau_{j;2} > 0.$$

So it is enough to show

$$\frac{r_1 A_{n_1-1;1}}{A_{n_1-2;1}} > \frac{\xi_1 c^{[\xi_1]}_{w+1} - (c^{[\xi_1]}_{w+2} r_1 - c^{[\xi_1]}_{w+1}\omega_1)\frac{A_{n_1-1;1}}{A_{n_1;1}}}{c^{[\xi_1]}_{w+2} - (c^{[\xi_1]}_{w+2} r_1 - c^{[\xi_1]}_{w+1}\omega_1)\frac{A_{n_1-1;1}}{A_{n_1;1}}}$$

and

$$\frac{(c^{[t_1]}_{j+2} r_1 - c^{[t_1]}_{j+1}\omega_1) A_{n_1-1;1}}{A_{n_1-2;1}} > \frac{\xi_1 c^{[\xi_1]}_{w+1-j}}{c^{[\xi_1]}_{w+2} - (c^{[\xi_1]}_{w+2} r_1 - c^{[\xi_1]}_{w+1}\omega_1)\frac{A_{n_1-1;1}}{A_{n_1;1}}};$$

but these inequalities can be proved by induction on w. □

Lemma 4.8. *Let a, b, c be any nonnegative integers. Then there are nonnegative integers $d_0, \ldots, d_c$ such that*

$$\binom{aX+b}{c} = \sum_{i=0}^{c} d_i \binom{X}{i} \qquad \textit{for all nonnegative integers } X.$$

Proof. The Vandermonde identity shows

$$\binom{aX+b}{c} = \sum_{w_0+w_1+\cdots+w_a=c,\; w_0,\ldots,w_a\in\mathbb{N}} \binom{b}{w_0} \prod_{v=1}^{a} \binom{X}{w_v},$$

and then the statement follows from Lemma 4.9. □

Lemma 4.9. *Let a, b be any nonnegative integers. Then there are nonnegative integers $e_0, \ldots, e_{a+b}$ such that*

$$\binom{X}{a}\binom{X}{b} = \sum_{i=0}^{a+b} e_i \binom{X}{i} \qquad \textit{for all nonnegative integers } X.$$

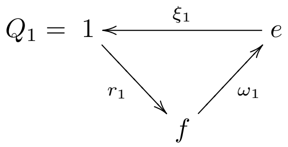
Proof. There are many proofs. This proof is due to Qiaochu Yuan and Brendan McKay. It is enough to prove the result for large enough integers X, because both sides can be regarded as polynomials in X. Now $\binom{X}{a}\binom{X}{b}$ is the number of ways to choose a subset of size a and a subset of size b from a set of size X. The union of these two subsets is a subset of size anywhere from $\max(a, b)$ to $a + b$, so e_i is the number of different ways a subset of size i can be realized as the union of a subset of size a and a subset of size b. □

This completes the proof of Lemma 4.4.

Lemma 4.10. $\mathcal{P}_{\theta,3} = 0$.

Proof. In trying to keep the notation simple, we give a detailed proof for $j = i+1$. The case $j > i + 1$ uses the same argument.

Let



be the quiver at the seed $\mu_1(t_{i+1})$, where r_1, ω_1 and ξ_1 are the numbers of arrows. For a compatible pair $\beta = (S_1, S_2)$, let $|\beta|_2$ denote $|\mathcal{D}_1| - |S_1|$ and $|\beta|_1$ denote $|S_2|$.

Applying Theorem 3.21 to $x^p_{1;t_i} x^q_{f;t_i}$, we obtain

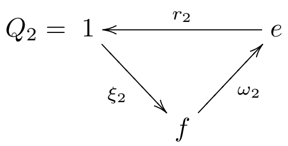
$$(52) \quad \sum_{\substack{\tau_{0;1},\dots,\tau_{n_1-3;1} \\ A_{n_1-1;1}-r_1 s_{n_1-2;1}\geq 0}} \left(\prod_{w=0}^{n_1-3} \begin{bmatrix} A_{w+1;1} - r_1 s_{w;1} \\ \tau_{w;1} \end{bmatrix} \right) \times \widetilde{x_{1;t_{i+1}}}^{A_{n_1-1;1}-r_1 s_{n_1-2;1}} x_{f;t_{i+1}}^{r_1 s_{n_1-3;1}-A_{n_1-2;1}} x_{e;t_{i+1}}^{\omega_1 s_{n_1-2;1}-\xi_1 s_{n_1-3;1}}$$

$$(53) \quad + x_{1;t_{i+1}}^{-A_{n_1-1;1}} x_{f;t_{i+1}}^{-A_{n_1-2;1}} \sum_{\beta:\, r_1|\beta|_1 - A_{n_1-1;1}>0} x_{1;t_{i+1}}^{r_1|\beta|_1} x_{f;t_{i+1}}^{r_1(A_{n_1-1;1}-|\beta|_2)} x_{e;t_{i+1}}^{\xi_1|\beta|_2-(\xi_1 r_1-\omega_1)|\beta|_1},$$

where the second sum is over all $\beta = (S_1 = \bigcup_{i=1}^{p+q} S_1^i, S_2 = \bigcup_{i=1}^{p+q} S_2^i)$ such that

$$(S_1^i, S_2^i) \text{ is a compatible pair in } \begin{cases} \mathcal{D}^{c_{n_1-1;1}\times c_{n_1-2;1}} & \text{if } 1 \leq i \leq q, \\ \mathcal{D}^{c_{n_1;1}\times c_{n_1-1;1}} & \text{if } q+1 \leq i \leq p+q. \end{cases}$$

Let n_2 be the number of seeds between $\mu_1(t_{i+1})$ and t_{i+2} inclusive. Suppose that n_2 is an even integer. The case of n_2 odd is similar, except that the roles of $x_{1;t_i+1}$ and $x_{e;t+1}$ are interchanged. Let



be the quiver at the seed $\mu_1(t_{i+2})$, where r_2, ω_2 and ξ_2 are the numbers of arrows.

Here we show that if $C_{2;\, i\to j}\overline{C_{3;\, j\to j+1}} \neq 0$ then $u_2 + q_3^{(2)}$ (the exponent of x_f) can never be negative, so that the second sum in $\mathcal{P}_{\theta,3}$ is equal to 0. A similar argument can be applied to show that the other sums are 0.

Let $p_2 = A_{n_1-1;1} - r_1 s_{n_1-2;1}$ and $q_2 = \omega_1 s_{n_1-2;1} - \xi_1 s_{n_1-3;1}$ be the exponents of $\widetilde{x_{1;t_{i+1}}}$ and $x_{e;t_{i+1}}$ in (52), respectively. Applying Theorem 3.13 to $\widetilde{x_{1;t_{i+1}}}^{p_2} x^{q_2}_{e;t_{i+1}}$ in (52), we have

$$\sum_{\substack{\tau_{0;1},\dots,\tau_{n_1-3;1}\\ A_{n_1-1;1}-r_1s_{n_1-2;1}\geq 0}} \left(\prod_{w=0}^{n_1-3}\begin{bmatrix} A_{w+1;1}-r_1s_{w;1}\\ \tau_{w;1}\end{bmatrix}\right) x_{f;t_{i+1}}^{r_1s_{n_1-3;1}-A_{n_1-2;1}}$$

$$\times\sum_{\beta} x_{1;t_{i+2}}^{r_2|\beta|_1-A_{n_2-1;2}} x_{e;t_{i+2}}^{A_{n_2;2}-r_2|\beta|_2} x_{f;t_{i+2}}^{\xi_2|\beta|_2-(\xi_2r_2-\omega_2)|\beta|_1}$$

$$= \sum_{\substack{\tau_{0;1},\dots,\tau_{n_1-3;1}\\ A_{n_1-1;1}-r_1s_{n_1-2;1}\geq 0}} \left(\prod_{w=0}^{n_1-3}\begin{bmatrix} A_{w+1;1}-r_1s_{w;1}\\ \tau_{w;1}\end{bmatrix}\right)$$

$$\times\sum_{\beta} x_{1;t_{i+2}}^{r_2|\beta|_1-A_{n_2-1;2}} x_{e;t_{i+2}}^{A_{n_2;2}-r_2|\beta|_2} x_{f;t_{i+2}}^{\xi_2|\beta|_2-(\xi_2r_2-\omega_2)|\beta|_1+r_1s_{n_1-3;1}-A_{n_1-2;1}},$$

where each second sum is over all $\beta = (S_1 = \bigcup_{i=1}^{p_2+q_2} S_1^i, S_2 = \bigcup_{i=1}^{p_2+q_2} S_2^i)$ such that

$$(S_1^i, S_2^i) \text{ is a compatible pair in } \begin{cases} \mathcal{D}^{c_{n_2-1;2}\times c_{n_2-2;2}} & \text{if } 1\leq i\leq q_2,\\ \mathcal{D}^{c_{n_2;2}\times c_{n_2-1;2}} & \text{if } q_2+1\leq i\leq p_2+q_2.\end{cases}$$

The exponent of x_1 is positive by definition of $\mathcal{P}_{\theta,3}$. Therefore $A_{n_2-1;2} < r_2|\beta|_1$, hence $A_{n_2-1;2}/(r_2|\beta|_1) < 1$ and thus

$$\frac{r_1A_{n_2-1;2}}{c_{n_2-1}^{[r_2]}\,r_2} = \frac{r_1|\beta|_1}{c_{n_2-1}^{[r_2]}}\,\frac{A_{n_2-1;2}}{r_2|\beta|_1} < \frac{r_1|\beta|_1}{c_{n_2-1}^{[r_2]}} \leq \xi_2|\beta|_2-(\xi_2r_2-\omega_2)|\beta|_1,$$

where the last inequality is proved in [21, Lemma 4.10] and [19, Proposition 4.1]. Using $r_2=\xi_1$ and the definition of $A_{n_2-1;2}$, we get

$$\begin{aligned}
&r_1s_{n_1-3;1}-A_{n_1-2;1}+\xi_2|\beta|_2-(\xi_2r_2-\omega_2)|\beta|_1\\
&\quad\geq r_1s_{n_1-3;1}-A_{n_1-2;1}\\
&\qquad+\frac{r_1\big(c_{n_2}^{[\xi_1]}(A_{n_1-1;1}-r_1s_{n_1-2;1})+c_{n_2-1}^{[\xi_1]}(\omega_1s_{n_1-2;1}-\xi_1s_{n_1-3;1})\big)}{c_{n_2-1}^{[\xi_1]}\xi_1}\\
&\quad= -A_{n_1-2;1}+\frac{r_1\big(c_{n_2}^{[\xi_1]}(A_{n_1-1;1}-r_1s_{n_1-2;1})+c_{n_2-1}^{[\xi_1]}\omega_1s_{n_1-2;1}\big)}{c_{n_2-1}^{[\xi_1]}\xi_1}\\
&\quad= -A_{n_1-2;1}+\frac{r_1\big(c_{n_2}^{[\xi_1]}A_{n_1-1;1}-(c_{n_2}^{[\xi_1]}r_1-c_{n_2-1}^{[\xi_1]}\omega_1)s_{n_1-2;1}\big)}{c_{n_2-1}^{[\xi_1]}\xi_1}\\
&\quad\overset{(A_{n_1-1;1}-r_1s_{n_1-2;1}\geq 0)}{\geq} -A_{n_1-2;1}+\frac{r_1\big(c_{n_2}^{[\xi_1]}A_{n_1-1;1}-(c_{n_2}^{[\xi_1]}r_1-c_{n_2-1}^{[\xi_1]}\omega_1)\frac{A_{n_1-1;1}}{r_1}\big)}{c_{n_2-1}^{[\xi_1]}\xi_1}\\
&\quad= \frac{1}{\xi_1}(\omega_1A_{n_1-1;1}-\xi_1A_{n_1-2;1}) \overset{(51)}{>} 0.
\end{aligned}$$

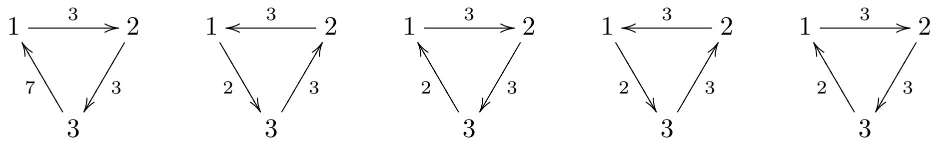
Thus the exponent of $x_{f;t_{i+2}}$ in the expansion of (52) is positive. The proof for the expansion of (53) uses a similar argument. □

§5. Example

Example 5.1. Let Σ_{t_0} be a seed connected to the initial seed Σ_{t_3} by the following sequence of mutations:

$$t_0 \overset{2}{\rightsquigarrow} t'_1 \overset{1}{\rightsquigarrow} t_1 \overset{3}{\rightsquigarrow} t'_2 \overset{1}{\rightsquigarrow} t_2 \overset{2}{\rightsquigarrow} \bullet \overset{1}{\rightsquigarrow} \bullet \cdots\cdots \bullet \overset{2}{\rightsquigarrow} \bullet \overset{1}{\rightsquigarrow} t_3$$

Suppose that the quivers corresponding to the first 5 seeds are as follows



$$Q_{t_0} \overset{2}{\rightsquigarrow} Q_{t'_1} \overset{1}{\rightsquigarrow} Q_{t_1} \overset{3}{\rightsquigarrow} Q_{t'_2} \overset{1}{\rightsquigarrow} Q_{t_2}$$

We want to illustrate the proof of positivity for $x_{2;t_0}$.

First, we compute its expansion in the cluster $\mathbf{x}_{t_1}$ using Theorem 3.21. The mutation sequence from t_0 to t_1 is at the vertices 1 and 2. Thus we have $r = 3$, $n = 3$, $p = 1$, $q = 0$. Moreover,

$$c_1 = 0,\ c_2 = 1,\ c_3 = 3, \qquad A_1 = 1,\ A_2 = 3, \qquad s_1 = \tau_0,$$

and the summation is on $\tau_0 = 0, 1$. The condition $s_{n-2} \le A_{n-1}/r$ in the first sum of Theorem 3.21 becomes $\tau_0 \le 1$, which is always satisfied, so the second sum in the theorem is empty. Finally, the variables in the theorem are $x_2 = x_{2;t_1}$, $z_3 = x_{3;t_1}$ and $x_3 = x_{1;t'_1} = (x_{3;t_1}^2 + x_{2;t_1}^3)/x_{1;t_1}$. Thus $x_{2;t_0}$ is equal to

$$x_{2;t_1}^{-1}\binom{1}{0}\left(\frac{x_{3;t_1}^2 + x_{2;t_1}^3}{x_{1;t_1}}\right)^3 = x_{2;t_1}^{-1}\binom{1}{0}x_{1;t'_1}^3 \tag{54}$$

$$+\, x_{2;t_1}^{-1}\binom{1}{1}x_{3;t_1}^3. \tag{55}$$

Now we compute the expansion of this expression in the cluster $\mathbf{x}_{t_2}$ again using Theorem 3.21. We treat the two terms (54) and (55) separately. For the first term, we need to expand $x_{1;t'_1}^3$ which lies in the cluster $\mathbf{x}_{t'_1}$. The mutation sequence from t'_1 to t_2 is at the vertices 1 and 3. Thus we have $r = 2$, $n = 4$, $p = 3$, $q = 0$.

Moreover,

$$c_1 = 0,\ c_2 = 1,\ c_3 = 2,\ c_4 = 3, \qquad A_1 = 3,\ A_2 = 6,\ A_3 = 9,$$
$$s_1 = \tau_0, \qquad s_2 = 2\tau_0 + \tau_1, \qquad s_3 = 3\tau_0 + 2\tau_1 + \tau_2.$$

The two binomial coefficients in the first sum are $\binom{3}{\tau_0}$ and $\binom{6-2\tau_0}{\tau_1}$, so their product is zero unless

$$0 \le \tau_0 \le 3, \qquad 0 \le \tau_1 \le 6 - 2\tau_0.$$

Finally, the condition $s_{n-2} \le A_{n-1}/r$ in the first sum of Theorem 3.21 implies that $\tau_0 < 3$, and that $\tau_1 \le 4$ if $\tau_0 = 0$, $\tau_1 \le 2$ if $\tau_0 = 1$, and $\tau_1 = 0$ if $\tau_0 = 2$. Therefore, the first sum is over the following pairs (τ_0, τ_1)

$$(0,0)\ (0,1)\ (0,2)\ (0,3)\ (0,4)\ (1,0)\ (1,1)\ (1,2)\ (2,0),$$

and the corresponding terms are

$$x_{2;t_2}^{-1}\binom{1}{0}x_{3;t_2}^{-6}\binom{3}{0}\binom{6}{0}x_{1;t_2'}^{9} \tag{56}$$

$$+\, x_{2;t_2}^{-1}\binom{1}{0}x_{3;t_2}^{-6}\binom{3}{0}\binom{6}{1}x_{1;t_2'}^{7}x_{2;t_2}^{3} \tag{57}$$

$$+\, x_{2;t_2}^{-1}\binom{1}{0}x_{3;t_2}^{-6}\binom{3}{0}\binom{6}{2}x_{1;t_2'}^{5}x_{2;t_2}^{6} \tag{58}$$

$$+\, x_{2;t_2}^{-1}\binom{1}{0}x_{3;t_2}^{-6}\binom{3}{0}\binom{6}{3}x_{1;t_2'}^{3}x_{2;t_2}^{9} \tag{59}$$

$$+\, x_{2;t_2}^{-1}\binom{1}{0}x_{3;t_2}^{-6}\binom{3}{0}\binom{6}{4}x_{1;t_2'}^{1}x_{2;t_2}^{12} \tag{60}$$

$$+\, x_{2;t_2}^{-1}\binom{1}{0}x_{3;t_2}^{-4}\binom{3}{1}\binom{4}{0}x_{1;t_2'}^{5}x_{2;t_2}^{3} \tag{61}$$

$$+\, x_{2;t_2}^{-1}\binom{1}{0}x_{3;t_2}^{-4}\binom{3}{1}\binom{4}{1}x_{1;t_2'}^{3}x_{2;t_2}^{6} \tag{62}$$

$$+\, x_{2;t_2}^{-1}\binom{1}{0}x_{3;t_2}^{-4}\binom{3}{1}\binom{4}{2}x_{1;t_2'}^{1}x_{2;t_2}^{9} \tag{63}$$

$$+\, x_{2;t_2}^{-1}\binom{1}{0}x_{3;t_2}^{-2}\binom{3}{2}\binom{2}{0}x_{1;t_2'}^{1}x_{2;t_2}^{6}. \tag{64}$$

The second sum in Theorem 3.21 is over all compatible pairs (S_1, S_2) in $\mathcal{D}^{9\times 6}$ such that $|S_2| > 9/2$. The condition $|S_2| > 9/2$ implies that S_2 must be equal to $\mathcal{D}_2$ or $\mathcal{D}_2 \setminus \{\text{any single vertical edge}\}$. If $S_2 = \mathcal{D}_2$ then S_1 must be the empty set. If $S_2 = \mathcal{D}_2 \setminus \{v_{2i-1}\}$ for $i = 1, 2, 3$, then $S_1 = \{u_{3i-2}\}$ or $\emptyset$. If $S_2 = \mathcal{D}_2 \setminus \{v_{2i}\}$ for $i = 1, 2, 3$, then $S_1 = \{u_{3i}\}$ or $\emptyset$. Therefore the second sum in Theorem 3.21 is equal to

$$x_{2;t_2}^{-1}\binom{1}{0}6x_{3;t_2}^{-6}x_{1;t_2}x_{2;t_2}^{12} \tag{65}$$

$$+\, x_{2;t_2}^{-1}\binom{1}{0}x_{3;t_2}^{-6}x_{1;t_2}^{3}x_{2;t_2}^{9} \tag{66}$$

$$+\, x_{2;t_2}^{-1}\binom{1}{0}6x_{3;t_2}^{-4}x_{1;t_2}x_{2;t_2}^{9}. \tag{67}$$

This shows that (54) is equal to the sum of all terms (56)–(67).

Applying Theorem 3.21 to the expression (55) and using a similar analysis, we see that (55) is equal to

$$x_{2;t_2}^{-1}\binom{1}{1}x_{3;t_2}^{-3}\binom{3}{0}x_{1;t_2'}^{6} \tag{68}$$

$$+\, x_{2;t_2}^{-1}\binom{1}{1}x_{3;t_2}^{-3}\binom{3}{1}x_{1;t_2'}^{4}x_{2;t_2}^{3} \tag{69}$$

$$+\, x_{2;t_2}^{-1}\binom{1}{1}x_{3;t_2}^{-3}\binom{3}{2}x_{1;t_2'}^{2}x_{2;t_2}^{6} \tag{70}$$

$$+\, x_{2;t_2}^{-1}\binom{1}{1}x_{3;t_2}^{-3}\binom{3}{3}x_{2;t_2}^{9}. \tag{71}$$

So $x_{2;t_0}$ is equal to the sum of all terms (56)–(71). Observe that the powers of the variables $x_{1;t_2'}, x_{1;t_2}$ in all terms are positive and that the powers of the variable $x_{2;t_2}$ are positive in all terms except for (56) and (68).

On the other hand,

$$(56) + (68) = x_{3;t_2}^{-6}x_{1;t_2'}^{6}\left(\frac{x_{1;t_2'}^{3} + x_{3;t_2}^{3}}{x_{2;t_2}}\right) = x_{3;t_2}^{-6}x_{1;t_2'}^{6}x_{2;t_2''}$$

where $\mathbf{x}_{t_2''} = \mu_2(\mathbf{x}_{t_2'})$ denotes the cluster obtained from $\mathbf{x}_{t_2'}$ by mutation at 2.

Thus we obtain an expression for $x_{2;t_0}$ as a Laurent polynomial in the variables $x_{2;t_2''}$, $x_{1;t_2'}$, $x_{1;t_2}$, $x_{2;t_2}$, $x_{3;t_2}$ with nonnegative coefficients and in which only the variable $x_{3;t_2}$ appears with negative powers. Note that $x_{2;t_2''} = \widetilde{\widetilde{x_{2;t_2}}}$ and $x_{1;t_2'} = \widetilde{x_{1;t_2}}$, thus the sum (56)+(68) is of the form of the first sum in Theorem 4.3, the sum of (57)–(64) and (69)–(71) is of the form of the second sum, and the sum of (65)–(67) is of the form of the third sum in Theorem 4.3.

Since the mutation sequence linking the seeds $\Sigma_{t_2''}, \Sigma_{t_2'}$ and Σ_{t_2} to the seed Σ_{t_3} consists of mutations at the vertices 1 and 2 only, we see that $x_{3;t_2} = x_{3;t_3}$ and replacing the other variables with their expansions in the seed Σ_{t_3} (which have nonnegative coefficients by the rank 2 case) produces again a Laurent polynomial with nonnegative coefficients in the cluster $\mathbf{x}_{t_3}$.

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