A Construction of Finite Index C*-algebra Inclusions from Free Actions of Compact Quantum Groups

by

Kenny De Commer and Makoto Yamashita

Abstract

Given an action of a compact quantum group on a unital C*-algebra, one can consider the associated Wassermann-type C*-algebra inclusions. One amplifies the original action with the adjoint action associated with a finite-dimensional unitary representation, and considers the induced inclusion of fixed point algebras. We show that this inclusion is a finite index inclusion of C*-algebras when the quantum group acts freely. Along the way, two natural definitions of freeness for a compact quantum group action, due respectively to D. Ellwood and M. Rieffel, are shown to be equivalent.

§1. Introduction

One of the fundamental concepts in the study of locally compact quantum groups is the notion of 'noncommutative principal bundles', or free and proper actions on 'noncommutative spaces', the noncommutative spaces being represented by various algebraic structures. In the C*-algebraic study of such principal bundles, it turned out that there can be two different ways to formulate freeness of an action.

The first is a certain density condition on the coaction map, called the *Ellwood condition*, introduced by D. A. Ellwood [6]. If the algebra is commutative, so that we are back in the classical case of a locally compact group acting continuously

K. De Commer: Department of Mathematics, University of Cergy-Pontoise, UMR CNRS 8088, F-95000 Cergy-Pontoise, France;
e-mail: `Kenny.De-Commer@u-cergy.fr`
M. Yamashita: Dipartimento di Matematica, Università degli Studi di Roma "Tor Vergata"; current address: Department of Mathematics, Ochanomizu University, Ohtsuka 2-1-1, Tokyo, Japan;
e-mail: `yamashita.makoto@ocha.ac.jp`

on a locally compact space, this condition is equivalent to freeness in the ordinary sense. In the purely algebraic setting, the Ellwood condition corresponds to the notion of a Hopf–Galois extension.

The second is the notion of a *saturated* action, which is more suited for the study of K-theory of operator algebras. It was introduced by M. Rieffel [14, 17] in the setting of actions of compact groups on C*-algebras. Since it is stated as a condition on the structure of the crossed product algebra, there is a straightforward generalization to the case of compact quantum group actions. For example, the case of *finite* quantum groups was studied by W. Szymański and C. Peligrad [23].

It has been known that these conditions are closely related to each other. For example, when $\mathbb{G}$ is a compact Lie group, C. Wahl [25, Proposition 9.8] showed that they are equivalent.

Our first main result is that the above two notions actually coincide in the setting of compact quantum group actions.

Theorem 1.1. *Let $\mathbb{G}$ be a compact quantum group acting continuously on a C*-algebra A. Then the following conditions are equivalent:*

(1) *The action satisfies the Ellwood condition.*

(2) *The action is saturated.*

Our next result relates freeness of a compact quantum group action to certain ring-theoretical properties of the associated isotypical components.

Let π be a finite-dimensional unitary representation of a compact quantum group $\mathbb{G}$, A a unital C*-algebra acted upon freely by $\mathbb{G}$, and A_π the isotypical component of A associated to π. In particular, the isotypical component for the trivial representation is the fixed point subalgebra $A^{\mathbb{G}}$. Each A_π becomes an $A^{\mathbb{G}}$-bimodule thanks to the algebra structure of A.

The spaces A_π can be interpreted as sections of a direct sum of copies of the vector bundle induced by the representation π. In the classical case of compact group actions on compact Hausdorff spaces, they are known to be finitely generated projective over the algebra of the base space. In our C*-algebraic setting, we obtain the same result from a combination of a technique used in the proof of Theorem 1.1 and Kasparov's stabilization theorem for Hilbert C*-modules.

Theorem 1.2. *Let A be a unital C*-algebra endowed with a free action of $\mathbb{G}$. Then each isotypical component A_π is finitely generated projective as a right $A^{\mathbb{G}}$-module.*

We note that, for actions of discrete group duals on general C*-algebras, this was proven by W. Szymański in an unpublished work. But even for compact groups acting on general unital C*-algebras, our result seems to be new.

Note that the conclusion in Theorem 1.2 also holds for *ergodic* actions of $\mathbb{G}$, i.e. actions for which $A^{\mathbb{G}} = \mathbb{C}$. See [7] for the case of compact groups, and [4] for the general case of compact quantum groups. Our proof of Theorem 1.2 was inspired by the short argument for this result which appears in [16], as well as by [26] and the first sections of [3]. Our arguments will cover both the free and ergodic cases at once. We stress that, since we work in the C*-algebraic setting, we need to refine some of the von Neumann algebraic techniques which appear in the above papers.

The conclusion of Theorem 1.2 is not valid for an arbitrary action: one can check for example that the isotypical components for the action of the circle group on the closed unit disc are not finitely generated over the fixed point algebra, except for the fixed point algebra itself. The problem is essentially that the field of stabilizer groups is not continuous.

Subsequent to Jones' initial subfactor paper [8], a lot of effort has gone into constructing von Neumann algebraic subfactors, starting from more classical symmetries and building up further to quantum symmetries (see e.g. [27, 29, 19, 1, 2, 12]). Motivated by this celebrated theory, Y. Watatani [28] introduced the notion of finite index inclusion of *C*-algebras*. We will show that, also in the C*-algebraic setting, the Wassermann type inclusion associated with finite-dimensional unitary representations of quantum groups provides an example of such an inclusion. This generalizes the case of finite groups in [28, Section 2.8] and finite quantum groups in [23]. The key is that the above structure theorem on A_π gives a finite quasi-basis for this inclusion.

Theorem 1.3. *Let $\mathbb{G}$ act freely on a unital C^*-algebra A, and let π be a finite-dimensional unitary representation of $\mathbb{G}$. Consider $A \otimes B(\mathscr{H}_\pi)$ with its induced action by $\mathbb{G}$. Then*
$$A^{\mathbb{G}} \subseteq (A \otimes B(\mathscr{H}_\pi))^{\mathbb{G}}$$
is a finite index inclusion of C^*-algebras. When π is irreducible, the index of the natural conditional expectation is equal to the square of the quantum dimension of $\mathscr{H}_\pi$.

The paper is organized as follows. We gather basic facts about compact quantum groups and their representations in Section 2, where we also prove a crucial Pimsner–Popa type estimate on the complete boundedness of projections onto spectral subspaces. The most technical part of this paper occupies Section 3, where we study the adjointability of Galois maps in terms of various Hilbert C*-bimodule structures of isotypical components. The first two of our main theorems are proved in Section 4, based on the results of Section 3. Finally, we study the C*-algebraic index for the Wassermann type inclusion associated with a free action and an irreducible representation in Section 5.

General notations. We denote the identity maps of various objects by ι. If X is a Banach space and $E \subseteq X$ is a subset, we denote by $[E]$ the closed linear span of E inside X. Following the convention of right Hilbert C*-modules, the scalar product of Hilbert spaces is taken to be conjugate linear in the first argument. The complex conjugate of a Hilbert space $\mathscr{H}$ will be identified with the dual of $\mathscr{H}$ by means of the inner product and denoted by $\mathscr{H}^*$. The multiplier C*-algebra of a C*-algebra C is denoted by $\mathcal{M}(C)$. When $\mathcal{E}$ is a right Hilbert C*-module over a C*-algebra A, the algebra of (adjointable) A-endomorphisms is denoted by $\mathcal{L}(\mathcal{E})_A$ and that of compact A-endomorphisms by $\mathcal{K}(\mathcal{E})_A$. If there is no risk of confusion we also write $\mathcal{L}(\mathcal{E})$ and $\mathcal{K}(\mathcal{E})$.

§2. Isotypical components of quantum group actions and associated Hilbert modules

§2.1. Compact quantum groups

In this section, we review the theory of compact quantum groups. A compact quantum group $\mathbb{G}$ is represented by a unital C*-algebra $C(\mathbb{G})$, together with a unital *-homomorphism

$$\Delta \colon C(\mathbb{G}) \to C(\mathbb{G}) \otimes_{\min} C(\mathbb{G})$$

satisfying the coassociativity and the cancellation properties [30, 11]. We will denote by $P(\mathbb{G}) \subseteq C(\mathbb{G})$ the Hopf algebra of matrix coefficients associated with $\mathbb{G}$, by S its antipode, and by $\varphi \colon C(\mathbb{G}) \to \mathbb{C}$ the invariant Haar state on $C(\mathbb{G})$.

Let π be a finite-dimensional unitary representation of $\mathbb{G}$, by which we mean a finite-dimensional Hilbert space $\mathscr{H}_\pi$ together with a left $C(\mathbb{G})$-comodule structure

$$\delta_\pi \colon \mathscr{H}_\pi \to C(\mathbb{G}) \otimes \mathscr{H}_\pi$$

satisfying $1 \otimes v^* w = \delta_\pi(v)^* \delta_\pi(w)$ for all $v, w \in \mathscr{H}_\pi$, having interpreted $\mathscr{H}_\pi$ as linear operators between the Hilbert spaces $\mathbb{C}$ and $\mathscr{H}_\pi$. Choosing an orthogonal basis $\{e_i\}$ of $\mathscr{H}_\pi$, and writing $\delta_\pi(e_i) = \sum_j u_{ij} \otimes e_j$, this means that $\sum_k u_{ik}^* u_{jk} = \delta_{ij}$. Consequently, $\sum_k u_{ki} u_{kj}^* = \delta_{ij}$, as u is invertible. We let $\operatorname{Irr} \mathbb{G}$ denote a complete representative system of irreducible finite-dimensional unitary representations of $\mathbb{G}$ up to unitary equivalence.

If π is a unitary representation of $\mathbb{G}$, we denote by $\overline{\pi}$ the associated contragredient representation. It is implemented on $\mathscr{H}_\pi^*$ with the dual comodule structure, but we equip it with a new Hilbert space structure averaged out by means of φ. More precisely, choosing an orthonormal basis e_i of $\mathscr{H}_\pi$ with $\delta_\pi(e_i) = \sum_j u_{ij} \otimes e_j$,

and writing $e_i^* = \langle e_i, \, \cdot \, \rangle$, we define a new scalar product on $\mathscr{H}_\pi^*$ by

$$\langle\!\langle e_i^*, e_j^* \rangle\!\rangle = \sum_{k,l} \varphi(u_{ik}u_{jl}^*)\langle e_k^*, e_l^* \rangle = \sum_k \varphi(u_{ik}u_{jk}^*).$$

By Woronowicz's theory [30], we obtain an invertible positive matrix $Q_\pi \in B(\mathscr{H}_\pi)$ for each $\pi \in \mathrm{Irr}\,\mathbb{G}$ satisfying $\mathrm{Tr}(Q_\pi) = \mathrm{Tr}(Q_\pi^{-1})$ and

$$(1) \quad \varphi(u_{ij}^* u_{kl}) = \delta_{ik} \frac{\langle e_l, Q_\pi e_j \rangle}{\mathrm{Tr}(Q_\pi)}, \quad \varphi(u_{ij}u_{kl}^*) = \delta_{jl} \frac{\langle e_k, Q_\pi^{-1} e_i \rangle}{\mathrm{Tr}(Q_\pi^{-1})}, \quad \text{for all } i,j,k,l.$$

The number $\mathrm{Tr}(Q_\pi)$ is known as the *quantum dimension* of $\mathscr{H}_\pi$,

$$\dim_q(\mathscr{H}_\pi) = \mathrm{Tr}(Q_\pi).$$

One can then define the quantum dimension of any representation of $\mathbb{G}$ by linearity. See e.g. [18] for a detailed exposition.

§2.2. Actions of compact quantum groups

An action of $\mathbb{G}$ on a possibly non-unital C*-algebra A is given by a non-degenerate injective *-homomorphism $\alpha\colon A \to A \otimes_{\min} C(\mathbb{G})$, satisfying the coaction property $(\alpha \otimes \iota) \circ \alpha = (\iota \otimes \Delta) \circ \alpha$ and the density condition

$$[\alpha(A)(1 \otimes C(\mathbb{G}))] = A \otimes_{\min} C(\mathbb{G}).$$

We denote by $B = A^{\mathbb{G}}$ the C*-algebra of $\mathbb{G}$-invariant elements, i.e. elements $x \in A$ satisfying $\alpha(x) = x \otimes 1$. Then the map E_B defined by

$$E_B(x) = (\iota \otimes \varphi)\alpha(x), \quad x \in A,$$

is a faithful conditional expectation from A onto B, that is, a faithful c.c.p. idempotent map onto B [5, Lemma 3], [22, Section 5].

When π is a finite-dimensional unitary representation of $\mathbb{G}$, we can consider the vector space of 'equivariant functions'

$$A \,\square\, \mathscr{H}_\pi = \{z \in A \otimes \mathscr{H}_\pi \mid (\alpha \otimes \iota)z = (\iota \otimes \delta_\pi)z\}.$$

It has a natural B-bimodule structure, as well as a right B-valued Hermitian inner product which is characterized by the following identity in $B \cong B \otimes \mathbb{C}$:

$$\langle w, z \rangle_B = w^* z, \quad w, z \in A \,\square\, \mathscr{H}_\pi.$$

We further put

$$A_\pi = \text{linear span of } \{(\iota \otimes \omega)z \mid z \in A \,\square\, \mathscr{H}_\pi, \, \omega \in \mathscr{H}_\pi^*\} \subseteq A,$$

which we call the π-*isotypical component* of A. These A_π have natural B-bimodule structures where the right B-Hermitian inner product is defined by

$$\langle x, y\rangle_B = E_B(x^*y).$$

They carry an (algebraic) right $P(\mathbb{G})$-comodule structure. Note that for the trivial representation $\pi = \mathrm{triv}$, we have $A_{\mathrm{triv}} = B$. The involution on A and the conjugate operation on the unitary representations of $\mathbb{G}$ are related by

$$A_{\bar{\pi}} = \{x^* \mid x \in A_\pi\}.$$

We denote by $P(A)$ the *-algebra $\sum_{\pi \in \mathrm{Irr}\,\mathbb{G}}^{\oplus} A_\pi$.

If π_1 and π_2 are two finite-dimensional unitary representations of $\mathbb{G}$, we denote their tensor product representation on $\mathscr{H}_{\pi_1} \otimes \mathscr{H}_{\pi_2}$ by $\pi_1 \times \pi_2$. One then has the inclusion $A_{\pi_1} \cdot A_{\pi_2} \subseteq A_{\pi_1 \times \pi_2}$.

In the following lemma, we endow $\mathscr{H}_\pi^*$ with its modified Hilbert space structure $\langle\!\langle -, -\rangle\!\rangle$.

Lemma 2.1. *Let π be an irreducible representation of $\mathbb{G}$ of classical dimension n. Then the map*

$$\phi_\pi \colon (A \,\square\, \mathscr{H}_\pi) \otimes \mathscr{H}_\pi^* \to A_\pi, \qquad z \otimes \omega \mapsto \sqrt{n}(\iota \otimes \omega Q_\pi^{-1})z,$$

is an isomorphism of B-bimodules which is compatible with the B-valued inner product.

Proof. By definition of A_π, this map is surjective. Thus we only need to verify that it is compatible with the B-valued inner products of both sides.

Let $z = \sum_i a_i \otimes e_i$ and $w = \sum_i b_i \otimes e_i$ be elements of $A \,\square\, \mathscr{H}_\pi$. Then, for arbitrary k and l, we have

$$(2) \qquad \langle \phi_\pi(z \otimes e_k^*), \phi_\pi(w \otimes e_l^*)\rangle_B = n \sum_{i,j} E_B(a_i^* b_j)\langle Q_\pi^{-1}e_i, e_k\rangle\langle e_l, Q_\pi^{-1}e_j\rangle.$$

By the assumption $z, w \in A \,\square\, \mathscr{H}_\pi$, one has

$$\sum_{i,j} E_B(a_i^* b_j) \otimes e_i^* \otimes e_j = \sum_{i,j,m,n} \phi(u_{im}^* u_{jn}) a_i^* b_j \otimes e_m^* \otimes e_n.$$

Using the first half of (1), $Q_\pi^{-1}Q_\pi Q_\pi^{-1} = Q_\pi^{-1}$, and $\mathrm{Tr}(Q_\pi) = \mathrm{Tr}(Q_\pi^{-1})$, the right hand side of (2) can be computed as

$$\frac{n\langle e_l, Q_\pi^{-1}e_k\rangle}{\mathrm{Tr}(Q_\pi^{-1})} \sum a_i^* b_i.$$

By the second half of (1), this is indeed the inner product of $z \otimes e_k^*$ and $w \otimes e_l^*$. $\square$

§2.3. Hilbert module structures

The above right B-modules A_π and $A \,\square\, \mathscr{H}_\pi$ are complete with respect to their B-valued inner products. To see this, we first introduce the following special elements.

Definition 2.2. Let π be an irreducible representation of $\mathbb{G}$. We define the *quantum character* of π to be the unique element $\chi_\pi \in P(\mathbb{G})$ satisfying

$$(\varphi(\chi_\pi \cdot\,) \otimes \iota)\delta_\rho = \begin{cases} 0 & \text{if } \rho \in \operatorname{Irr}\mathbb{G} \text{ and } \rho \not\cong \pi, \\ \iota_{\mathscr{H}_\rho} & \text{if } \rho \cong \pi. \end{cases}$$

We will also use the shorthand notation

$$\omega_\pi = \hat{\chi}_\pi = \varphi(\chi_\pi \cdot\,) \in C(\mathbb{G})^*.$$

For an arbitrary representation π, we write χ_π for the sum of the quantum characters of those irreducible representations which appear in π with non-zero multiplicity.

We note that χ_π is well-defined by the Peter–Weyl theory for compact quantum groups. We record the following facts about χ_π.

Lemma 2.3. *For each representation π of $\mathbb{G}$, we have*

$$S^2(\chi_\pi) = \chi_\pi, \quad S(\chi_\pi)^* = \chi_\pi, \quad and \quad \chi_\pi^* \in C(\mathbb{G})_\pi.$$

Consider now the bounded map

$$E_\pi \colon A \to A, \quad a \mapsto (\iota \otimes \omega_\pi)\alpha(a).$$

The following lemma is a reformulation of (part of) [15, Theorem 1.5] (which holds regardless of any unitality assumption on A).

Lemma 2.4. *The map E_π is an idempotent onto A_π.*

The following Pimsner–Popa type inequality will be crucial in what follows.

Lemma 2.5. *For each finite-dimensional unitary representation π of $\mathbb{G}$, there exists $c_\pi > 0$ such that for all $n \in \mathbb{N}_0$ and $a \in M_n(\mathbb{C}) \otimes A$, we have*

$$(\iota \otimes E_\pi)(a)^*(\iota \otimes E_\pi)(a) \leq c_\pi^2 (\iota \otimes E_B)(a^*a).$$

Proof. Because of the form of the assertion, it is enough to prove it for $n = 1$. The general case can be obtained from it by replacing α with the amplified coaction $\iota \otimes \alpha \colon M_n(\mathbb{C}) \otimes A \to M_n(\mathbb{C}) \otimes A \otimes C(\mathbb{G})$, because E_π for the latter coaction is precisely equal to $\iota \otimes E_\pi$ appearing in the assertion.

Now, since $\iota \otimes \varphi$ is a c.c.p. map of $A \otimes_{\min} C(\mathbb{G})$ onto $A \otimes \mathbb{C}$, and using the inequality $\phi(x)^* \phi(x) \leq \phi(x^* x)$ for a general c.c.p. map ϕ, we find

$$E_\pi(a)^* E_\pi(a) = (\iota \otimes \varphi)((1 \otimes \chi_\pi)(\alpha(a)))^* (\iota \otimes \varphi)((1 \otimes \chi_\pi)\alpha(a))$$
$$\leq (\iota \otimes \varphi)\big(((1 \otimes \chi_\pi)\alpha(a))^*((1 \otimes \chi_\pi)\alpha(a))\big).$$

By the positivity of $\iota \otimes \phi$, the right hand side is bounded from above by

$$\|\chi_\pi\|^2 (\iota \otimes \varphi)(\alpha(a^* a)) = \|\chi_\pi\|^2 E_B(a^* a).$$

Setting $c_\pi = \|\chi_\pi\|$, we obtain the assertion. $\square$

Corollary 2.6. (1) *The right B-module A_π is complete with respect to its B-valued inner product, that is, A_π is a right Hilbert B-bimodule.*

(2) *We have $[A_\pi \cdot B] = A_\pi$ and $[A \cdot B] = A$, where the closure is with respect to the C^*-norm.*

(3) *For each representation π, the space $A \,\square\, \mathscr{H}_\pi$ is a right Hilbert B-bimodule.*

By a *right Hilbert B-bimodule*, we mean a right Hilbert B-module $\mathcal{E}$ together with a non-degenerate *-representation of B as adjointable operators on $\mathcal{E}$.

Proof. By the previous lemmas, we have the Pimsner–Popa type inequalities

$$\|\langle a, a\rangle_B\|^{1/2} \leq \|a\| \leq c_\pi \|\langle a, a\rangle_B\|^{1/2}$$

for $a \in A_\pi$. As A_π is closed in the C^*-algebra norm (being the image of a norm-bounded projection), this proves the first part of the corollary.

One also easily shows that if b_i is an approximate unit for B, then for each $a \in A$, we have $ab_i \to a$ in the Hilbert module norm, and hence also in the C^*-norm. Since the linear span $P(A)$ of all the isotypical components is norm-dense in A by [15, Theorem 1.5], which holds regardless of the unitality of A, the second part of the corollary is proven.

The third part then follows for π irreducible because $A \,\square\, \mathscr{H}_\pi$ is an orthogonally complemented summand of A_π by Lemma 2.1. The general case follows since $A \,\square\, -$ preserves finite direct sums. $\square$

Notation 2.7. To distinguish the two different norms on A_π, we will write

A_π for A_π endowed with the restriction of the C^*-norm of A,

$\mathcal{A}_\pi$ for A_π endowed with the right Hilbert B-module structure.

The natural identification map $A_\pi \to \mathcal{A}_\pi$ will be denoted by $\Lambda_{\mathcal{A}} \colon A_\pi \to \mathcal{A}_\pi$.

We will also denote by $\mathcal{A}$ the right Hilbert B-module completion of A with respect to $\langle \cdot, \cdot \rangle_B$. We then have natural inclusions $\mathcal{A}_\pi \subseteq \mathcal{A}$.

We can also put a *right* Hilbert B-$\mathcal{K}(\mathcal{A}_{\overline{\pi}})$-bimodule structure on each $\mathcal{A}_\pi$ by means of the ordinary left B-module structure and the $\mathcal{K}(\mathcal{A}_{\overline{\pi}})$-valued inner product

$$\langle x, y \rangle_{\mathcal{K}(\mathcal{A}_{\overline{\pi}})} = \Lambda_{\mathcal{A}}(x^*)\Lambda_{\mathcal{A}}(y^*)^*.$$

Then we have $\|\langle x, x \rangle_{\mathcal{K}(\mathcal{A}_{\overline{\pi}})}\| = \|E_B(xx^*)\|$, which is the norm squared obtained by considering A_π as a *left* Hilbert B-module by means of the inner product $_B\langle x, y \rangle = E_B(xy^*)$. Hence we can extend our previous notation as follows.

Notation 2.8. We write

$\quad \mathscr{A}_\pi \quad$ for A_π endowed with the right $\mathcal{K}(A_{\overline{\pi}})$-Hilbert module structure.

We denote the natural identification map $A_\pi \to \mathscr{A}_\pi$ by $\Lambda_{\mathscr{A}} \colon A_\pi \to \mathscr{A}_\pi$. In the same way, one defines the right Hilbert $\mathcal{K}(\mathcal{A})$-module completion of A by $\mathscr{A}$.

$\quad$ Note that $\|x\|_{\mathscr{A}_\pi} = \|x^*\|_{\mathcal{A}_{\overline{\pi}}}$.

Lemma 2.9. *The maps $\Lambda_{\mathcal{A}}$, $\Lambda_{\mathscr{A}}$ are completely bounded. Similarly, when ρ is a finite-dimensional unitary representation of $\mathbb{G}$, the identity map on A_ρ is completely bounded with respect to the $\mathcal{A}_\rho$-norm or $\mathscr{A}_\rho$-norm on the domain and the A_ρ-norm on the codomain.*

Proof. The boundedness for the first two maps follows from a standard calculation using the fact that E_B is c.c.p. Furthermore, Lemma 2.5 shows that ι is completely bounded for the $\mathcal{A}_\rho$-norm on the domain, with norm bounded from above by c_π. By the above remark and the general equality $\|x^*x\| = \|xx^*\|$ for C*-norms, we obtain an analogous cb-norm estimate of ι from above by $c_{\overline{\pi}}$ for the $\mathscr{A}_\pi$-norm on the domain. $\qquad\square$

§2.4. Crossed product and its corners

Let us put $\hat{h} = \phi(h \cdot) \in C(\mathbb{G})^*$ for $h \in P(\mathbb{G})$, and consider the space

$$\widehat{P(\mathbb{G})} = \{\hat{h} \mid h \in P(\mathbb{G})\} \subseteq C(\mathbb{G})^*.$$

It is a (generally non-unital) *-algebra with the convolution product and the *-operation which is determined by

$$\omega^*(x) = \overline{\omega(S(x)^*)}, \quad x \in P(\mathbb{G}).$$

This *-algebra admits a (non-unital) universal C*-envelope $C^*(\mathbb{G})$, which is a C*-algebraic direct sum of matrix algebras, the components of which are labelled by $\operatorname{Irr}\mathbb{G}$.

It is easily seen that we can extend α to a unitary comodule structure on $\mathcal{A}$,

$$\alpha_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{A} \otimes_{\min} C(\mathbb{G}),$$

where we view the right hand side as a right Hilbert $B \otimes_{\min} C(\mathbb{G})$-module in a natural way. Then to the coaction $\alpha_{\mathcal{A}}$, we can associate the (non-degenerate) *-representation

$$\widehat{\pi}_{\alpha} \colon C^*(\mathbb{G}) \to \mathcal{L}(\mathcal{A}),$$

where $\mathcal{L}(\mathcal{A})$ denotes the space of adjointable linear endomorphisms of $\mathcal{A}$. The map $\widehat{\pi}_{\alpha}$ is uniquely determined by

$$\widehat{\pi}_{\alpha}(\omega)\Lambda_{\mathcal{A}}(x) = \Lambda_{\mathcal{A}}((\iota \otimes \omega)\alpha(x)) \quad \text{for } x \in A,\, \omega \in \widehat{P(\mathbb{G})}.$$

Moreover,

$$p_{\pi} = \widehat{\pi}_{\alpha}(\widehat{\chi}_{\pi})$$

is the projection of $\mathcal{A}$ onto $\mathcal{A}_{\pi}$. In particular, $\mathcal{A}_{\pi}$ is complemented in $\mathcal{A}$.

If we perform this construction for the particular case of $(A, \alpha) = (C(\mathbb{G}), \Delta)$, the space $\mathcal{A}$ becomes a Hilbert space, which we denote by $\mathscr{L}^2(\mathbb{G})$. It is the completion of $C(\mathbb{G})$ with respect to the inner product $\langle x, y \rangle = \varphi(x^*y)$. In this case, we denote the associated GNS-map by $\Lambda_{\varphi} \colon C(\mathbb{G}) \to \mathscr{L}^2(\mathbb{G})$, and we also write $\mathscr{L}^2(\mathbb{G})_{\pi} = \Lambda_{\varphi}(C(\mathbb{G})_{\pi})$ for the finite-dimensional Hilbert space of the matrix coefficients for π. The associated representation $\widehat{\pi}_{\Delta}$ is then faithful, and we will in the following treat $\widehat{\pi}_{\Delta}$ as the identity map, so that $C^*(\mathbb{G}) \subseteq B(\mathscr{L}^2(\mathbb{G}))$.

Consider now the right Hilbert B-module $\mathcal{A} \otimes \mathscr{L}^2(\mathbb{G})$. It carries a natural non-degenerate *-representation of $A \otimes_{\min} C(\mathbb{G})$ as B-endomorphisms (cf. [10, p. 34]). By means of the homomorphism α, we obtain a representation of A on $\mathcal{A} \otimes \mathscr{L}^2(\mathbb{G})$ as well. Note that this representation might not be faithful as we do not assume $C(\mathbb{G})$ to be reduced. The space $\mathcal{A} \otimes \mathscr{L}^2(\mathbb{G})$ also carries a *-representation of $C^*(\mathbb{G})$, acting by the ordinary convolution on the second leg.

The crossed product of A by $\mathbb{G}$ for the action α is defined as

$$A \rtimes \mathbb{G} = [\alpha(A)(1 \otimes C^*(\mathbb{G}))] \subset \mathcal{L}(\mathcal{A} \otimes \mathscr{L}^2(\mathbb{G})),$$

which will be a C*-algebra. One could also use the right Hilbert A-module $A \otimes \mathscr{L}^2(\mathbb{G})$ in the above construction, and this would give the same C*-algebra. We define

$$(A \rtimes \mathbb{G})_{\pi} = [(A \rtimes \mathbb{G})(1 \otimes \widehat{\chi}_{\pi})],$$

which is a closed left ideal in $A \rtimes \mathbb{G}$. We similarly define

$$_{\pi}(A \rtimes \mathbb{G}) = (A \rtimes \mathbb{G})_{\pi}^*, \quad _{\rho}(A \rtimes \mathbb{G})_{\pi} = [(A \rtimes \mathbb{G})_{\rho}^*(A \rtimes \mathbb{G})_{\pi}].$$

Then $_{\pi}(A \rtimes \mathbb{G})_{\pi}$ is a C*-algebra.

We state a lemma which lets us realize our Hilbert modules $\mathcal{A}$ and $\mathscr{A}$ in terms of this crossed product. We omit the proof which follows from a straightforward computation. In the lemma, we will denote the unit of $C(\mathbb{G})$ by χ_{triv} for clarity.

Lemma 2.10. *There exists a unique right Hilbert B-module structure on $(A \rtimes \mathbb{G})_{\mathrm{triv}}$ such that*

$$\langle x, y \rangle_B \otimes \widehat{\chi}_{\mathrm{triv}} = x^* y \quad \text{for all } x, y \in (A \rtimes \mathbb{G})_{\mathrm{triv}},$$

and we then have a natural isomorphism

$$\mathcal{A} \to (A \rtimes \mathbb{G})_{\mathrm{triv}}, \quad \Lambda_{\mathcal{A}}(a) \mapsto \alpha(a)(1 \otimes \widehat{\chi}_{\mathrm{triv}}),$$

of right Hilbert B-bimodules.

By the previous lemma, it follows that we can realize $\mathcal{K}(\mathcal{A})$ as a closed 2-sided ideal inside $A \rtimes \mathbb{G}$ by means of a *-homomorphism

$$(3) \qquad \Pi_{\alpha} \colon \mathcal{K}(\mathcal{A}) \to A \rtimes \mathbb{G}, \quad \Lambda_{\mathcal{A}}(x) \Lambda_{\mathcal{A}}(y)^* \mapsto \alpha(x)(1 \otimes \widehat{\chi}_{\mathrm{triv}}) \alpha(y^*),$$

which is in general degenerate. We can interpret $\mathscr{A}$ as a right B-$A \rtimes \mathbb{G}$-Hilbert bimodule by composing its $\langle \cdot, \cdot \rangle_{\mathcal{K}(\mathcal{A})}$-valued inner product with the map Π_{α}. We then have an isomorphism of right Hilbert B-$A \rtimes \mathbb{G}$-bimodules

$$\mathscr{A} \to {}_{\mathrm{triv}}(A \rtimes \mathbb{G}), \quad \Lambda_{\mathscr{A}}(a) \mapsto (1 \otimes \widehat{\chi}_{\mathrm{triv}}) \alpha(a).$$

Note now that we can identify $\mathcal{A}$ with the closure of $(\Lambda_{\mathcal{A}} \otimes \Lambda_{\varphi})(\alpha(A))$ inside $\mathcal{A} \otimes \mathscr{L}^2(\mathbb{G})$. In this way $\mathcal{A}$ is an $A \rtimes \mathbb{G}$-invariant subspace of $\mathcal{A} \otimes \mathscr{L}^2(\mathbb{G})$, and we denote the resulting *-representation of $A \rtimes \mathbb{G}$ on $\mathcal{A}$ by π_{red}. Its restriction to A is given by left multiplication, while its restriction to $C^*(\mathbb{G})$ is the representation $\widehat{\pi}_{\alpha}$. It is also easy to see that $\pi_{\mathrm{red}} \circ \Pi_{\alpha}$ is the identity map on $\mathcal{K}(\mathcal{A})$.

§2.5. Galois maps

We can interpret A itself as a right Hilbert B-A-Hilbert module, the bimodule structure being given by multiplication and the A-valued inner product by $\langle x, y \rangle_A = x^* y$. Further, for any representation π, we interpret $\mathscr{A} \otimes \mathscr{L}^2(\mathbb{G})_{\pi}$ as a right $A \rtimes \mathbb{G}$-Hilbert module by means of the identification map

$$(4) \qquad \mathscr{A} \otimes \mathscr{L}^2(\mathbb{G})_{\pi} \to {}_{\pi}(A \rtimes \mathbb{G}), \quad \Lambda_{\mathscr{A}}(x) \otimes \Lambda_{\varphi}(S(h)) \mapsto (1 \otimes \widehat{h}) \alpha(x),$$

for $x \in A$ and $h \in C(\mathbb{G})_{\overline{\pi}}$. It requires a small argument to show that this Hilbert module is complete, but we will actually never use this fact.

Definition-Proposition 2.11. *The map*

(5) $A_\pi \otimes_{\mathrm{alg}} P(A) \to P(A) \otimes C(\mathbb{G})_\pi, \quad a \otimes a' \mapsto \alpha(a)(a' \otimes 1),$

induces isometric maps of right Hilbert modules

$$G_\pi \colon \mathcal{A}_\pi \otimes_B A \to A \otimes \mathscr{L}^2(\mathbb{G})_\pi, \quad \mathcal{G}_\pi \colon \mathcal{A}_\pi \otimes_B \mathcal{A} \to \mathcal{A} \otimes \mathscr{L}^2(\mathbb{G})_\pi,$$
$$\mathscr{G}_\pi \colon \mathcal{A}_\pi \otimes_B \mathscr{A} \to \mathscr{A} \otimes \mathscr{L}^2(\mathbb{G})_\pi,$$

which are uniquely determined by (5) *on the image of* $A_\pi \otimes_{\mathrm{alg}} P(A)$.

   We shall call the above maps the π-localised C*-, *Hilbert* C*- *and* crossed product-Galois map, *respectively.*

   Here $- \otimes_B -$ denotes the interior tensor product over B (see [10]). Also, for the first two maps, the tensor product on the right is simply an amplification of the corresponding right Hilbert modules with the Hilbert space $\mathscr{L}^2(\mathbb{G})_\pi$.

Proof. A trivial computation shows that the proposed formula for G_π and $\mathcal{G}_\pi$ respects the A-valued, resp. B-valued inner product on elementary tensors, so that they descend and complete to maps with domain $\mathcal{A}_\pi \otimes_B A$, resp. $\mathcal{A}_\pi \otimes_B \mathcal{A}$. The statement about $\mathscr{G}_\pi$ follows from the following easily verified identity inside $A \rtimes \mathbb{G}$:

$$(1 \otimes \widehat{S^{-1}(x_{(1)})})\alpha(x_{(0)}) = \alpha(x)(1 \otimes \widehat{\chi}_{\mathrm{triv}}), \quad x \in A_\pi,$$

where we used the Sweedler notation $\alpha(x) = x_{(0)} \otimes x_{(1)}$. $\square$

   One can show that these Galois maps add up to respective isometries $\mathcal{A} \otimes_B - \to - \otimes \mathscr{L}^2(\mathbb{G})$ which are $C^*(\mathbb{G})$-equivariant from the second to the first leg—we will use this observation only for the cases A and $\mathcal{A}$. Note that the terminology 'Galois map' comes from the corresponding Hopf algebraic theory (cf. [20]).

§3. Adjointability of Galois maps

We keep the notational conventions of the previous section. In particular, $\mathbb{G}$ is a compact quantum group acting on a not necessarily unital C*-algebra A, with fixed point C*-algebra B. We first recall a general fact about the amplification of morphisms of Hilbert C*-bimodules.

Lemma 3.1. *Let C and D be C*-algebras, and $\mathcal{E}$ and $\mathcal{E}'$ respectively a right Hilbert C*-B-C-bimodule and a right Hilbert C*-B-D-bimodule. If $T \colon \mathcal{E} \to \mathcal{E}'$ is a completely bounded B-module map, then $\iota \otimes T$ descends to a bounded map $\iota \otimes_B T$ of norm at most $\|T\|_{\mathrm{cb}}$ from $\mathcal{F} \otimes_B \mathcal{E}$ to $\mathcal{F} \otimes_B \mathcal{E}'$ for any right Hilbert C*-B-module $\mathcal{F}$.*

Proof. This is shown by a standard argument. Let $x_1, \ldots, x_n$ be elements in $\mathcal{E}$, and $y_1, \ldots, y_n$ be in $\mathcal{F}$. Then the matrix $Y = (\langle y_i, y_j \rangle_B)_{i,j}$ in $M_n(B)$ is a positive element. Hence there exists $b = (b_{i,j})_{i,j} \in M_n(B)$ satisfying $b^*b = Y$. Then

$$\left\langle \iota \otimes T\Big(\sum_i y_i \otimes x_i\Big), \iota \otimes T\Big(\sum_j y_j \otimes x_j\Big) \right\rangle_D = \sum_{i,j} \langle T(x_i), Y_{i,j} T(x_j) \rangle_D.$$

The norm of the above can be written as

$$\Big\| \sum_{i,j,k} \langle T(x_i), b_{k,i}^* b_{k,j} T(x_j) \rangle_D \Big\| = \| \iota \otimes T(\xi) \|^2,$$

where $\xi \in \mathbb{C}^n \otimes \mathcal{E}$ is a column vector whose k-th component is equal to $\sum_i b_{k,i} x_i$. By the complete boundedness of T, we obtain

$$\| \iota \otimes T(\xi) \|^2 \leq \| T \|_{\mathrm{cb}}^2 \| \xi \|^2 = \| T \|_{\mathrm{cb}}^2 \Big\| \Big\langle \sum_i y_i \otimes x_i, \sum_j y_j \otimes x_j \Big\rangle_C \Big\|^2,$$

which implies the desired estimate $\| \iota \otimes_B T \| \leq \| T \|_{\mathrm{cb}}$. $\qquad\square$

The adjointability of different Galois maps are in fact equivalent conditions.

Proposition 3.2. *Let $\mathbb{G}$ be a compact quantum group acting on a unital C^*-algebra A. Let π be a finite-dimensional unitary representation of $\mathbb{G}$. The following conditions are equivalent:*

(1) *The π-localized Hilbert-C^*-Galois map $\mathcal{G}_\pi$ has an adjoint.*
(2) *The π-localized C^*-Galois map G_π has an adjoint.*
(3) *The π-localized crossed product-Galois map $\mathscr{G}_\pi$ has an adjoint.*

Proof. Each of the implications can be argued along the same pattern: given the adjoint of one of the Galois maps, we may restrict it to isotypical components $A_\rho \otimes_{\mathrm{alg}} C(\mathbb{G})_\pi$ for $\rho \in \operatorname{Irr} \mathbb{G}$. Then Lemmas 2.9 and 3.1 imply that those restrictions are continuous with respect to the other norms. The resulting map on $P(A) \otimes_{\mathrm{alg}} C(\mathbb{G})_\pi$ is shown to be a formal adjoint of the Galois map with respect to the corresponding algebra-valued inner product. By the inner product characterization of duals for Hilbert C^*-modules, we conclude that this formal adjoint extends to the actual adjoint. As an illustration, let us prove the implication (1)$\Rightarrow$(2).

Assume that (1) holds. Take a finite-dimensional unitary representation ρ of $\mathbb{G}$, $h \in C(\mathbb{G})_\pi$, and $a \in A_\rho$. We first claim that $\mathcal{G}_\pi^*(\Lambda_\mathcal{A}(a) \otimes \Lambda_\varphi(h))$ is contained in $\mathcal{A}_\pi \otimes_B \mathcal{A}_{\overline{\pi} \times \rho}$. Observe that we can write

$$\mathcal{A}_\pi \otimes_B \mathcal{A} = \bigoplus_{\theta \in \operatorname{Irr} \mathbb{G}} (\mathcal{A}_\pi \otimes_B \mathcal{A}_\theta).$$

Take an irreducible representation θ which does not appear in $\overline{\pi} \times \rho$, and take $x \in A_\pi$, $y \in A_\theta$. By a direct calculation, we obtain

$$\langle \mathcal{G}_\pi^*(\Lambda_\mathcal{A}(a) \otimes \Lambda_\varphi(h)), \Lambda_\mathcal{A}(x) \otimes \Lambda_\mathcal{A}(y) \rangle$$
$$= (\iota \otimes \varphi \otimes \varphi)\big((\alpha(a^*) \otimes h^*)((\alpha \otimes \iota)\alpha(x))(\alpha(y) \otimes 1)\big).$$

But we see that the second factor of the evaluated element only contains matrix coefficients of the representation $\overline{\rho} \times \pi \times \theta$, which does not contain the trivial representation by the assumption on θ and Frobenius reciprocity. Hence the above expression is zero, and the claim follows.

Combining Lemmas 2.9 and 3.1, we obtain a map $\mathcal{A}_\pi \otimes_B \mathcal{A}_{\overline{\pi} \times \rho} \to \mathcal{A}_\pi \otimes_B A_{\overline{\pi} \times \rho}$. Composing this with $\mathcal{G}_\pi^*$, we obtain a map from $A_\rho \otimes_\mathrm{alg} C(\mathbb{G})_\pi$ to $\mathcal{A}_\pi \otimes_B A_{\overline{\pi} \times \rho}$. Taking linear combinations, we obtain a map F_0 from $P(A) \otimes_\mathrm{alg} C(\mathbb{G})_\pi$ to $\mathscr{A}_\pi \otimes_B A$. We want to show that F_0 is bounded, and that its closure equals G_π^*.

Fix $a \in A_\rho$ and $h \in C(\mathbb{G})_\pi$, and take elements $(x_{n,i})_{i=1}^{N_n} \in A_\pi$ and $(y_{n,i})_{i=1}^{N_n} \in A_{\overline{\pi} \times \rho}$ for $n, N_n \in \mathbb{N}$ such that

$$\mathcal{G}_\pi^*(\Lambda_\mathcal{A}(a) \otimes \Lambda_\varphi(h)) = \lim_{n \to \infty} \sum_{i=1}^{N_n} \Lambda_\mathcal{A}(x_{n,i}) \otimes \Lambda_\mathcal{A}(y_{n,i}).$$

Take $x \in A_\pi$ and $y \in P(A)$. On the one hand, we have the convergence

$$\sum_i E_B(y_{n,i}^* E_B(x_{n,i}^* x) y) = \sum_i \langle \Lambda_\mathcal{A}(x_{n,i}) \otimes \Lambda_\mathcal{A}(y_{n,i}), \Lambda_\mathcal{A}(x) \otimes \Lambda_\mathcal{A}(y) \rangle$$
$$\xrightarrow[n \to \infty]{} \langle \Lambda_\mathcal{A}(a) \otimes \Lambda_\varphi(h), \mathcal{G}_\pi(x \otimes y) \rangle = (E_B \otimes \varphi)((a^* \otimes h^*)\alpha(x)(y \otimes 1)).$$

On the other hand, we also have the convergence

$$\langle F_0(a \otimes h), \Lambda_\mathcal{A}(x) \otimes y \rangle = \lim_{n \to \infty} \sum_{i=1}^{N_n} y_{n,i}^* E_B(x_{n,i}^* x) y.$$

Multiplying on the right with an arbitrary $z \in P(A)$, we find that

$$E_B(\langle F_0(a \otimes h), \Lambda_\mathcal{A}(x) \otimes y \rangle \cdot z) = E_B\big(\langle \Lambda_\mathcal{A}(a) \otimes \Lambda_\varphi(h), G_\pi(\Lambda_\mathcal{A}(x) \otimes y) \rangle \cdot z\big).$$

As z was arbitrary, and as $\langle \cdot, \cdot \rangle_B$ is non-degenerate on $P(A)$, we find that

$$\langle F_0(a \otimes h), \Lambda_\mathcal{A}(x) \otimes y \rangle = \langle \Lambda_\mathcal{A}(a) \otimes \Lambda_\varphi(h), G_\pi(\Lambda_\mathcal{A}(x) \otimes y) \rangle.$$

From this formula, we deduce that F_0 descends to a contractive map F from $A \otimes \mathscr{L}^2(\mathbb{G})_\pi$ to $\mathcal{A} \otimes_B A$, and that F is then precisely the adjoint of G_π, hence we obtain (2). □

Elements in the algebra B act on $\mathcal{A}_\pi$ as left multiplication operators, which are adjointable endomorphisms for the right Hilbert B-module structure. We let π_L denote the associated embedding of B into $\mathcal{L}(\mathcal{A}_\pi)_B$. The amplification of left multiplication defines an analogous action of B on $A \square \mathscr{H}_\pi$. By abuse of notation, we denote this representation also by π_L (see Lemma 2.1).

Theorem 3.3. *Let α be an action of $\mathbb{G}$ on a C^*-algebra A, and let π be a finite-dimensional unitary representation of $\mathbb{G}$. Then the following conditions are equivalent:*

(1) *The π-localized Galois maps are adjointable.*

(2) *The image of $_\pi(A \rtimes \mathbb{G})$ under π_{red} lies in $\mathcal{K}(\mathcal{A}, \mathcal{A}_\pi)$.*

(3) *The image of $\pi_L \colon B \to \mathcal{L}(\mathcal{A}_\pi)$ is contained in $\mathcal{K}(\mathcal{A}_\pi)$.*

(3′) *The image of $\pi_L \colon B \to \mathcal{L}(A \square \mathscr{H}_\pi)$ is contained in $\mathcal{K}(A \square \mathscr{H}_\pi)$.*

Furthermore, when A is unital, these conditions are equivalent to the following:

(4) *$\mathcal{A}_\pi$ is finitely generated and projective as a right B-module.*

(4′) *$A \square \mathscr{H}_\pi$ is finitely generated projective as a right B-module.*

Proof. (1)$\Rightarrow$(2) Let us identify $\mathscr{A} \otimes \mathscr{L}^2(\mathbb{G})_\pi$ with $_\pi(A \rtimes \mathbb{G})$ as explained above Proposition 2.11. Let us also identify $\mathcal{A}_\pi \otimes_B \mathscr{A}$ with the right Hilbert $\mathcal{K}(\mathcal{A})$-module $\mathcal{K}(\mathcal{A}, \mathcal{A}_\pi)$ of compact operators from $\mathcal{A}$ to $\mathcal{A}_\pi$, by means of the natural map

$$\Upsilon_\pi \colon \mathcal{A}_\pi \otimes_B \mathscr{A} \to \mathcal{K}(\mathcal{A}, \mathcal{A}_\pi), \quad \Lambda_{\mathcal{A}}(x) \otimes \Lambda_{\mathscr{A}}(y) \mapsto \Lambda_{\mathcal{A}}(x)\Lambda_{\mathcal{A}}(y^*)^*.$$

Then $\mathscr{G}_\pi$ becomes the map Π_α of (3).

As $\mathscr{G}_\pi$ is adjointable, we have, for x in $_\pi(A \rtimes \mathbb{G}) \simeq \mathscr{A} \otimes \mathscr{L}^2(\mathbb{G})_\pi$ (see (4)) and y in $\mathcal{K}(\mathcal{A}, \mathcal{A}_\pi)$,

$$\Pi_\alpha((\mathscr{G}_\pi^*(x))^* y) = x^* \Pi_\alpha(y).$$

Applying π_{red}, we conclude that $(\mathscr{G}_\pi^*(x))^* y = \pi_{\mathrm{red}}(x)^* y$ for all $y \in \mathcal{K}(\mathcal{A}, \mathcal{A}_\pi)$, and hence $\mathscr{G}_\pi^*(x) = \pi_{\mathrm{red}}(x) \in \mathcal{K}(\mathcal{A})$.

The implication (2)$\Rightarrow$(3) is of course trivial.

(3)$\Rightarrow$(1). Take $a \in P(A)$, say $a \in A_\rho$ for some representation ρ, and $h \in C(\mathbb{G})_\pi$. Since $[A_\rho B] = A_\rho$ by Corollary 2.6, the operator $\widehat{\pi}_\alpha(\widehat{S^{-1}(h)})a$ is a compact operator in $\mathcal{K}(\mathcal{A}_{\overline{\rho} \times \pi}, \mathcal{A}_\pi)$.

Let $\Upsilon_{\pi, \overline{\pi} \times \rho} \colon \mathcal{A}_\pi \otimes_B \mathscr{A}_{\overline{\pi} \times \rho} \to \mathcal{K}(\mathcal{A}_{\overline{\rho} \times \pi}, \mathcal{A}_\pi)$ be the restriction of Υ_π. It is a complete isometry. Moreover, combining Lemmas 2.9 and 3.1, we obtain an injective bounded map $\mathscr{S}_{\pi, \overline{\pi} \times \rho}$ from $\mathcal{A}_\pi \otimes_B \mathscr{A}_{\overline{\pi} \times \rho}$ to $\mathcal{A}_\pi \otimes_B \mathcal{A}_{\overline{\pi} \times \rho}$.

Define a map F_0 from $P(A) \otimes_{\mathrm{alg}} C(\mathbb{G})_\pi$ to $\mathcal{A}_\pi \otimes_B \mathcal{A}$ by

$$F_0(a \otimes h) = (\mathscr{S}_{\pi, \overline{\pi} \times \rho} \circ \Upsilon_{\pi, \overline{\pi} \times \rho}^{-1})(\widehat{\pi}_\alpha(\widehat{S^{-1}(h)})a).$$

Let us choose for each $n \in \mathbb{N}$ a finite collection of pairs $x_{n,i} \in A_\pi, y_{n,i} \in A_{\overline{\pi} \times \rho}$ such that

$$F_0(a \otimes h) = \lim_{n \to \infty} \sum_i \Lambda_{\mathcal{A}}(x_{n,i}) \otimes \Lambda_{\mathcal{A}}(y_{n,i}).$$

Then by definition of $F_0(a \otimes h)$, we have the equality

$$\lim_n \sum_i \Lambda_{\mathcal{A}}(x_{n,i}) \Lambda_{\mathcal{A}}(y_{n,i}^*)^* = \widehat{\pi}_\alpha(\widehat{S^{-1}(h)}) a$$

as operators from $\mathcal{A}_{\overline{\rho} \times \pi}$ to $\mathcal{A}_\pi$. Applying the *-operation to both sides, and applying these expressions to $\Lambda_{\mathcal{A}}(x)$ for some $x \in A_\pi$, we see that

$$\lim_n \sum_i \Lambda_{\mathcal{A}}(y_{n,i}^* E_B(x_{n,i}^* x)) = \Lambda_{\mathcal{A}}\big((\iota \otimes \varphi)((a^* \otimes h^*) \alpha(x))\big).$$

In the above formula, the vectors inside $\Lambda_{\mathcal{A}}$ belong to $\mathcal{A}_{\overline{\pi} \times \rho}$. Since the norms on $A_{\overline{\pi} \times \rho}$ and $\mathcal{A}_{\overline{\pi} \times \rho}$ are equivalent, convergence still holds in $A_{\overline{\pi} \times \rho}$. Hence if $x \in A_\pi$ and $y \in P(A)$, we have

$$\begin{aligned}
\langle F_0(a \otimes h), \Lambda_{\mathcal{A}}(x) \otimes \Lambda_{\mathcal{A}}(y) \rangle &= \lim_n \sum_i E_B(y_{n,i}^* E_B(x_{n,i}^* x) y) \\
&= E_B\big(a^*(\iota \otimes \varphi(h^* \cdot))(\alpha(x)) y\big) \\
&= \langle \Lambda_{\mathcal{A}}(a) \otimes \Lambda_\varphi(h), \mathcal{G}_\pi(\Lambda_{\mathcal{A}}(x) \otimes \Lambda_{\mathcal{A}}(y)) \rangle.
\end{aligned}$$

As in the proof of Proposition 3.2, we can conclude that F_0 extends to a bounded map F from $\mathcal{A} \otimes \mathscr{L}^2(\mathbb{G})_\pi$ to $\mathcal{A}_\pi \otimes_B \mathcal{A}$, which is then the adjoint of $\mathcal{G}_\pi$. This way we conclude that π-localized Galois maps are adjointable.

The equivalence between (3) and (3′), and between (4) and (4′), follows from Lemma 2.1, as those conditions on π are clearly equivalent to having the corresponding ones for each irreducible subrepresentation of π.

Finally, assuming A is unital, we show that (3) and (4) are equivalent. In fact, if A is unital, so is B, and the third condition simply says that $\mathcal{K}(\mathcal{A}_\pi)$ is unital. In particular, $\mathcal{A}_\pi$ must be countably generated as a Hilbert module. By [9, Lemma 6.5] and Kasparov's stabilization theorem [10, Corollary 6.3], $\mathcal{A}_\pi \cong pB^n$ for some $n \in \mathbb{N}$ and some self-adjoint projection $p \in M_n(B)$, and is hence finitely generated projective. Conversely, if $\mathcal{A}_\pi$ is finitely generated projective, [9, Lemma 6.5] implies that $\mathcal{K}(\mathcal{A}_\pi)$ is unital. Since $\mathcal{K}(\mathcal{A}_\pi)$ is an ideal in $\mathcal{L}(\mathcal{A}_\pi)$, this completes the proof. $\square$

Remark. 1. Let us assume our action is ergodic, which means $B = \mathbb{C}1$ (and necessarily A unital). Then $\mathcal{A}$ is a Hilbert space, and as the Hilbert C*-Galois map is then an isometry between Hilbert spaces, it is necessarily adjointable. Hence the implication (1)$\Rightarrow$(5) of the previous result captures as a special case the fact that

the isotypical components of an ergodic action are finite-dimensional (cf. [4]). Of course, this special case can be proven more directly (whilst obtaining a stronger conclusion about the minimal number of generators).

2. Let $\mathbb{G}$ be a classical compact group acting on some compact space X. One can prove that the adjointability of the corresponding Galois maps is equivalent to the following purely topological condition: if $G_x \subseteq G$ is the stabilizer group of $x \in X$, then the assignment $x \mapsto G_x$ is continuous, with respect to the natural topology known as finite or Vietoris topology on the space of closed subsets of G.

We end this section with the following observation about the range projection of Galois isometries.

Proposition 3.4. *Suppose that π-localized Galois maps are adjointable. Then $\mathscr{G}_\pi \mathscr{G}_\pi^*$ is a central element inside $\mathcal{M}(_\pi(A \rtimes \mathbb{G})_\pi)$, and*

$$\mathscr{G}_\pi \mathscr{G}_\pi^* = G_\pi G_\pi^* = \mathcal{G}_\pi \mathcal{G}_\pi^*.$$

Proof. Let us write $P_\pi = \mathscr{G}_\pi \mathscr{G}_\pi^*$. Then

$$P_\pi \in \mathcal{L}(\mathscr{A} \otimes \mathscr{H}_\pi) \cong \mathcal{L}(_\pi(A \rtimes \mathbb{G})) \cong \mathcal{M}(_\pi(A \rtimes \mathbb{G})_\pi),$$

where $\mathcal{M}$ means taking the multiplier C*-algebra. Moreover, as in the proof of the previous theorem, we can interpret $\mathscr{G}$ and its adjoint as the maps Π_α and π_{red} respectively. Hence for $x \in A \rtimes \mathbb{G}$, we have $P_\pi x = \Pi_\alpha(\pi_{\mathrm{red}}(x)) = \Pi_\alpha(\pi_{\mathrm{red}}(x^*))^* = (P_\pi x^*)^* = x P_\pi$, so that P_π is central.

From the 'claim' in the proof of Proposition 3.2, we have the inclusion

$$\mathcal{G}_\pi \mathcal{G}_\pi^* \big(\Lambda_{\mathcal{A}}(P(A)) \otimes C(\mathbb{G})_\pi \big) \subseteq \Lambda_{\mathcal{A}}(P(A)) \otimes C(\mathbb{G}_\pi).$$

And from the construction of the maps G_π^* and $\mathscr{G}_\pi^*$ there, the maps $\mathcal{G}_\pi \mathcal{G}_\pi^*$, $G_\pi G_\pi^*$ and $\mathscr{G}_\pi \mathscr{G}_\pi^*$ coincide on $P(A) \otimes C(\mathbb{G})_\pi$ (after applying suitable Λ-maps).

Finally, a simple algebraic computation, coupled with a continuity argument, allows us to conclude that for $x \in {}_\pi(A \rtimes \mathbb{G})_\pi$ and $\xi \in A \otimes \mathscr{L}^2(\mathbb{G})_\pi$, one has

$$(\Lambda_{\mathcal{A}} \otimes \iota)(x\xi) = x((\Lambda_{\mathcal{A}} \otimes \iota)\xi) \quad \text{and} \quad (\Lambda_{\mathscr{A}} \otimes \iota)(x\xi) = x((\Lambda_{\mathscr{A}} \otimes \iota)\xi),$$

where the left $_\pi(A \rtimes \mathbb{G})_\pi$-module structure on $\mathscr{A} \otimes \mathscr{L}^2(\mathbb{G})_\pi$ is obtained again by making first the identification with $_\pi(A \rtimes \mathbb{G})$. It follows that $G_\pi G_\pi^* = \mathcal{G}_\pi \mathcal{G}_\pi^* = P_\pi$. $\qquad\square$

§4. Freeness of compact quantum group actions

We keep the same notation as in the previous section, thus α is an action of a compact quantum group $\mathbb{G}$ on a C*-algebra A.

The action α is said to satisfy the *Ellwood condition*, or simply be *free*, if the following cancellation property holds:

$$[\alpha(A)(A \otimes 1)] = A \otimes_{\min} C(\mathbb{G}).$$

This condition was introduced in [6] in a more general setting of actions of locally compact quantum groups. It is straightforward to check that if $A = C_0(X)$ for some locally compact space X, and $\mathbb{G}$ is an ordinary compact group G, then the above condition characterizes precisely freeness of the action of G on X.

Since the subspaces $\mathcal{A}_\pi$ of $\mathcal{A}$ for $\pi \in \operatorname{Irr}\mathbb{G}$ are orthogonal to each other, and similarly for the suspaces $(\mathscr{L}^2(\mathbb{G})_\pi)_{\pi \in \operatorname{Irr}\mathbb{G}}$ in $\mathscr{L}^2(\mathbb{G})$, the isometries G_π and $\mathcal{G}_\pi$ add up to respective isometries G and $\mathcal{G}$.

Proposition 4.1. *The following conditions are equivalent:*

(1) *The action α is free.*
(2) *The Hilbert C^*-Galois isometry $\mathcal{G}$ is unitary.*
(3) *The C^*-Galois isometry G is unitary.*

Proof. Assume the action is free. Since the natural map $(\Lambda_\mathcal{A} \otimes \Lambda_\varphi)$ from $A \otimes_{\min} C(\mathbb{G})$ to $\mathcal{A} \otimes \mathscr{L}^2(\mathbb{G})$ is contractive, we see that the image of $\mathcal{G}$ is dense in its range. As $\mathcal{G}$ is isometric, it then follows that it is bijective, hence unitary (cf. [10, Theorem 3.5]).

Let us assume that $\mathcal{G}$ is unitary. Then all $\mathcal{G}_\pi$ are unitary operators. Proposition 3.4 implies that all G_π are unitary operators as well. It follows that G is a unitary.

Finally, let us assume G is unitary. Then we have $[\alpha(A_\pi)(A \otimes 1)] = A \otimes C(\mathbb{G})_\pi$ for each representation π of $\mathbb{G}$. As $P(A)$ is dense in A, it follows that the action of $\mathbb{G}$ on A is free. $\square$

Corollary 4.2 (Theorem 1.2). *Let A be a unital C^*-algebra endowed with a free action of $\mathbb{G}$, and let π be a finite-dimensional unitary representation of $\mathbb{G}$.*

(1) *The space A_π is a finitely generated right Hilbert B-module (and hence a finitely generated and projective right B-module).*
(2) *The space $A \square \mathscr{H}_\pi$ is a finitely generated right Hilbert B-module (hence finitely generated and projective as a right B-module).*

Proof. This follows from Theorem 3.3 and the previous proposition. $\square$

Following the case of group actions, we make the following definition of saturatedness.

Definition 4.3. We say that α is *saturated* if $[(A \rtimes \mathbb{G})_{\text{triv}} \cdot (A \rtimes \mathbb{G})^*_{\text{triv}}] = A \rtimes \mathbb{G}$.

Note that if A is unital, this simply says that $1 \otimes \widehat{\chi}_{\text{triv}}$ is a full projection in $A \rtimes \mathbb{G}$. In general, this condition says that $(A \rtimes \mathbb{G})_{\text{triv}}$ is an imprimitivity bimodule between B and $A \rtimes \mathbb{G}$.

We now prove the equivalence between freeness and saturatedness.

Theorem 4.4 (Theorem 1.1). *A compact quantum group action of $\mathbb{G}$ on a (not necessarily unital) C^*-algebra A is saturated if and only if it is free.*

Proof. Observe first that the saturatedness condition is equivalent to saying that $\Pi_\alpha(\mathcal{K}(\mathcal{A}, \mathcal{A}_\pi))$ is equal to $_\pi(A \rtimes \mathbb{G})$ for each representation π of $\mathbb{G}$. This in turn is equivalent to all maps $\mathscr{G}_\pi$ having dense range, i.e. being unitaries. The theorem then follows from Propositions 3.4 and 4.1. $\qquad\square$

§5. Finite index inclusions of C*-algebras

Let $B \subseteq C$ be a unital inclusion of unital C*-algebras. Following [28], we call this an *inclusion of finite-index type* if there exists a conditional expectation $E \colon C \to B$ and a finite set of pairs $(v_i, w_i) \in C \times C$, called a *quasi-basis for E*, such that

$$\sum_i v_i E(w_i x) = x = \sum_i E(x v_i) w_i \quad (x \in C).$$

Such a conditional expectation is itself called a conditional expectation of finite-index type. Then, *the index of E* is defined as

(6) $$\text{Index}(E) = \sum_i v_i w_i.$$

This does not depend on the choice of quasi-basis, and belongs to the centre of C.

An equivalent characterization of a conditional expectation of finite-index type is the following.

Lemma 5.1. *Let $B \subseteq C$ be as above, and $E \colon C \to B$ a conditional expectation. Then E is of finite-index type if and only if the right B-module C, together with the B-valued inner product $\langle x, y \rangle_B = E(x^* y)$, is a finitely generated right Hilbert B-module.*

Proof. If C is a finitely generated right Hilbert B-module in the way prescribed above, it follows from [28, Corollary 3.1.4] and the remark following it that E is of finite-index type.

Conversely, assume that E is of finite-index type. We know that C is finitely generated as a right B-module, and, by [28, Proposition 2.1.5], that there exists a

constant $c > 0$ such that

$$\|E(x^*x)\| \leq \|x^*x\| \leq c\|E(x^*x)\|.$$

Hence C with its B-valued inner product is complete, and becomes a finitely generated right Hilbert B-module. This concludes the proof. $\square$

Assume now again that A is a unital C*-algebra with an action of a compact quantum group $\mathbb{G}$, with B denoting the C*-algebra of $\mathbb{G}$-fixed elements. Let us fix a representation π of $\mathbb{G}$ with a fixed orthogonal basis $\{e_i\}$ for $\mathscr{H}_\pi$, and write $\delta_\pi(e_i) = \sum_j u_{ij} \otimes e_j$ with $u_{ij} \in C(\mathbb{G})$.

On the one hand, we can twist the coaction α with the representation π to obtain the coaction

$$\alpha_\pi \colon A \otimes B(\mathscr{H}_\pi) \to (A \otimes B(\mathscr{H}_\pi)) \otimes_{\min} C(\mathbb{G}),$$

which, using the Sweedler notation $\alpha(a) = a_{(0)} \otimes a_{(1)}$, is given by the formula

$$\alpha_\pi(a \otimes e_i e_j^*) = \sum_{k,l} a_{(0)} \otimes e_k e_l^* \otimes u_{ki}^* a_{(1)} u_{lj}.$$

On the other hand, we can also consider the following *left* coaction of $C(\mathbb{G})$ on $B(\mathscr{H}_\pi)$:

$$\mathrm{Ad}_\pi \colon B(\mathscr{H}_\pi) \to C(\mathbb{G}) \otimes B(\mathscr{H}_\pi), \quad e_i e_j^* \mapsto \sum_{k,l} u_{ik} u_{jl}^* \otimes e_k e_l^*.$$

We let $A \square B(\mathscr{H}_\pi)$ denote the space $\{x \in A \otimes B(\mathscr{H}_\pi) \colon (\alpha \otimes \iota)(x) = (\iota \otimes \mathrm{Ad}_\pi)(x)\}$.

The following lemma follows from a straightforward computation, which we omit.

Lemma 5.2. *The C*-subalgebras $(A \otimes B(\mathscr{H}_\pi))^{\mathbb{G}}$ and $A \square B(\mathscr{H}_\pi)$ of $A \otimes B(\mathscr{H}_\pi)$ coincide.*

Theorem 5.3 (Theorem 1.3, first part). *Let A be a unital C*-algebra endowed with a free action of $\mathbb{G}$. Then the inclusion $B = A^{\mathbb{G}} \subseteq (A \otimes B(\mathscr{H}_\pi))^{\mathbb{G}}$ is a finite-index type inclusion.*

Proof. By the previous lemma, it is equivalent to show that $B \subseteq A \square B(\mathscr{H}_\pi)$ is an inclusion of finite-index type. Let us choose a faithful Ad_π-invariant state θ_π on $B(\mathscr{H}_\pi)$. Then we can view $B(\mathscr{H}_\pi)$ as a Hilbert space with the inner product $\langle x, y \rangle = \theta_\pi(x^*y)$, and it is immediate that this turns Ad_π into a representation of $\mathbb{G}$.

Consider the map

$$E \colon A \square B(\mathscr{H}_\pi) \to A, \quad x \mapsto (\iota \otimes \theta_\pi)x.$$

First, it is faithful, being a restriction of the faithful map $\iota \otimes \theta_\pi$. Next, its image is the intersection $A \otimes \mathbb{C} \cap A \,\square\, B(\mathscr{H}_\pi) = B$. Moreover, the E-induced inner product $\langle x, y \rangle_B = E(x^*y)$ on $A \,\square\, B(\mathscr{H}_\pi)$ coincides precisely with the one we defined in Section 2 (viewing $B(\mathscr{H}_\pi)$ as a Hilbert space as above). Hence, by Corollary 4.2, $A \,\square\, B(\mathscr{H}_\pi)$ is a finitely generated right Hilbert B-module. The theorem now follows from Lemma 5.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

We remark that, by the same proof, the theorem holds more generally for any coaction whose associated C*-Galois map is adjointable (or indeed just its $\pi \otimes \bar{\pi}$-localization is adjointable).

To end, let us show that, when π is irreducible, the index of the above inclusion is in fact a scalar, equal to the square of the quantum dimension of $\mathscr{H}_\pi$. This should not be surprising: it is the quantum analogue of the fact that if V is a representation of a compact group G, and X a compact space with a free action of G, the pullback of V to X by means of the action gives a vector bundle of constant rank the dimension of V. For more on this in a von Neumann algebraic context, see e.g. [27], [2] and [24].

Note that the formula (6) still makes sense in case E is just a B-bimodular map from C to B and the 'inclusion map' map $B \to C$ is not injective. We will make use of the following lemma, whose proof is very similar to the one for the statement that finite index is stable under the Jones tower construction (cf. [28, Proposition 1.6.6]). It will later on allow us to tune down half of an argument of [27].

Lemma 5.4. *Let B be a unital C*-algebra, and $\mathcal{E}$ a right Hilbert B-bimodule which is finitely generated as a left and as a right B-module. Assume that $\mathcal{E}$ has a left B-valued inner product ${}_B\langle\,\cdot\,,\,\cdot\,\rangle$ such that $\mathcal{E}$ becomes also a left Hilbert B-bimodule (with respect to the given B-bimodule structure on $\mathcal{E}$). Then the map*

$$E \colon \mathcal{K}(\mathcal{E})_B \to B, \quad \xi\eta^* \mapsto {}_B\langle \xi, \eta \rangle,$$

is well-defined and B-bimodular (with respect to the natural map $B \to \mathcal{K}(\mathcal{E})_B$). Choosing further a finite set of elements $\xi_j, \eta_j, \widetilde{\xi}_i, \widetilde{\eta}_i \in \mathcal{E}$ such that

$$\sum_j \xi_j \langle \eta_j, \xi \rangle_B = \xi = \sum_i {}_B\langle \xi, \widetilde{\xi}_i \rangle \widetilde{\eta}_i \quad \text{for all } \xi \in \mathcal{E},$$

the elements

$$v_{ij} = \xi_j \widetilde{\eta}_i^*, \quad w_{ij} = \widetilde{\xi}_i \eta_j^*$$

form a quasi-basis for E, and hence the index of E is given by

$$\mathrm{Index}(E) = \sum_j \xi_j \Big(\sum_i \langle \widetilde{\eta}_i, \widetilde{\xi}_i \rangle_B \Big) \eta_j^*.$$

To be clear, the adjoint operation is taken with respect to the right Hilbert module structure. Note that, if the sum $\sum_i \langle \widetilde{\eta}_i, \widetilde{\xi}_i \rangle_B$ equals $c1$ for some scalar c, then $\mathrm{Index}(E) = c1$.

Proof. We will only verify that half of the quasi-basis property with respect to E is satisfied for the pairs (v_{ij}, w_{ij}). Note that we can restrict ourselves to the verification of the identity $T = \sum_{i,j} v_{i,j} E(w_{i,j} T)$ for rank 1 endomorphisms T of $\mathcal{E}$, as we can obtain the same formula for arbitrary $T \in \mathcal{K}(\mathcal{E})$ by linearity.

Suppose $T = \xi \eta^*$ for some vectors ξ, η in $\mathcal{E}$. Then $w_{ij} T = \widetilde{\xi}_i \langle \eta_j, \xi \rangle_B \eta^*$. Hence we can compute

$$\sum_{i,j} v_{ij} E(w_{ij} T) = \sum_{i,j} \xi_j \widetilde{\eta}_i^* {}_B \langle \widetilde{\xi}_i \langle \eta_j, \xi \rangle_B, \eta \rangle = \sum_{i,j} \xi_j \left({}_B \langle \widetilde{\xi}_i \langle \eta_j, \xi \rangle_B, \eta \rangle^* \widetilde{\eta}_i \right)^*.$$

Note that the inner product satisfies the symmetry ${}_B \langle x, y \rangle^* = {}_B \langle y, x \rangle$, and the compatibility ${}_B \langle x, ya \rangle = {}_B \langle xa^*, y \rangle$ with the bimodule structure. Using these, we may further transform the right hand side above to

$$\sum_{i,j} \xi_j \left({}_B \langle \eta \langle \eta_j, \xi \rangle_B^*, \widetilde{\xi}_i \rangle \widetilde{\eta}_i \right)^* = \sum_j \xi_j \left(\eta \langle \eta_j, \xi \rangle_B^* \right)^* = \sum_j \xi_j \langle \eta_j, \xi \rangle_B \eta^* = \xi \eta^*,$$

which proves $\sum_{i,j} v_{ij} E(w_{ij} T) = T$. $\qquad\square$

Let us now fix a free action of $\mathbb{G}$ on a unital C*-algebra A. We also fix an irreducible finite-dimensional unitary representation π of $\mathbb{G}$. Let us also recall here the following *strong left invariance* property for φ: if $g, h \in P(\mathbb{G})$, then

$$\varphi(g h_{(2)}) h_{(1)} = \varphi(g_{(2)} h) S(g_{(1)}).$$

Lemma 5.5. *The natural representation of $A \square B(\mathscr{H}_\pi)$ on $A \square \mathscr{H}_\pi$ is faithful.*

Proof. Note first that we can realize $\mathcal{K}(A \square \mathscr{H}_\pi)$ as an ideal inside $A \square B(\mathscr{H}_\pi)$, formed by the span of the elements of the form $\sum_{i,j} x_i y_j^* \otimes e_i e_j^*$ with $\sum x_i \otimes e_i$ and $\sum y_i \otimes e_i$ inside $A \square \mathscr{H}_\pi$. The faithfulness of our representation is then equivalent to this ideal being the whole of $A \square B(\mathscr{H}_\pi)$.

Now choose an orthonormal basis $e_i \in \mathscr{H}_\pi$, and write $\delta_\pi(e_i) = \sum_j u_{ij} \otimes e_j$. Take $x \in P(A)$. Then, using that $S(u_{ij}^*) = u_{ji}$ together with strong left invariance, one finds that $\sum_j \varphi(u_{ji}^* x_{(1)}) x_{(0)} \otimes e_j$ lies in $A \square \mathscr{H}_\pi$, where we have used again the Sweedler notation for the coaction α. Hence $\mathcal{K}(A \square \mathscr{H}_\pi)$ contains all elements of the form

$$(7) \qquad \sum_{k,l} \varphi(u_{ki}^* x_{(1)}) \varphi(y_{(1)}^* u_{lj}) x_{(0)} y_{(0)}^* \otimes e_k e_l^*.$$

But we can write this in the form

$$\sum_{k,l} \varphi(u_{ki}^* x_{(2)})\varphi\big(S^{-1}(x_{(1)})(x_{(0)}y^*)_{(1)}u_{lj}\big)(x_{(0)}y^*)_{(0)} \otimes e_k e_l^*.$$

Now from the proof of Propositions 3.2 and 4.1, it follows that we can express any $a \otimes h$ with $a \in P(A)$ and $h \in P(\mathbb{G})$ as a linear combination of elements of the form $\alpha(x)(y^* \otimes 1)$. Hence, taking the particular case $a = 1$, we see that $\mathcal{K}(A \,\square\, \mathscr{H}_\pi)$ contains all elements of the form

$$\sum_{k,l} \varphi(u_{ki}^* h_{(2)})\varphi(S^{-1}(h_{(1)})u_{lj})\, 1 \otimes e_k e_l^*.$$

Rewriting this slightly by means of strong left invariance, this becomes

$$(8) \qquad \sum_{k,l,p} \varphi(u_{pi}^* h)\varphi(u_{kp}^* u_{lj})\, 1 \otimes e_k e_l^*.$$

Using the matrices Q_π from Section 2.1, the above simplifies to

$$\dim_q(\mathscr{H}_\pi)^{-1}\Big(\sum_p \langle e_j, Q_\pi e_p \rangle\, \varphi(u_{pi}^* h) \Big)\, 1 \otimes 1.$$

It is clear that the sum does not vanish for some combination of i, j and h, proving that $\mathcal{K}(A \,\square\, \mathscr{H}_\pi)$ contains the unit of $A \,\square\, B(\mathscr{H}_\pi)$, and is thus equal to the latter algebra. $\qquad\square$

Note that the above proof, with a slightly modified last step, works just as well for representations which are not irreducible. Although we will not really need it, let us make at this point the link with the theory of eigenmatrices.

Let π be a representation of $\mathbb{G}$, and choose an orthonormal basis (e_i) for $\mathscr{H}_\pi$. As before, put $\delta(e_i) = \sum_j u_{ij} \otimes e_j$, and consider the space

$$A(\pi) = \Big\{ x = \sum_{ij} x_{ij} \otimes e_{ji} \in M_n(A) \,\Big|\, \alpha(x_{ij}) = \sum_k x_{ik} \otimes u_{kj} \Big\} \subseteq A \otimes B(\mathscr{H}_\pi),$$

which we call the space of π-*eigenmatrices*.

Corollary 5.6. *In the above notation, we have*

$$(9) \qquad A(\pi)A(\pi)^* = (A \otimes B(\mathscr{H}_\pi))^{\mathbb{G}}.$$

Proof. It is easily verified that we have an isomorphism of vector spaces

$$(A \,\square\, \mathscr{H}_\pi) \otimes \mathscr{H}_\pi^* \to A(\pi), \qquad \Big(\sum_i a_i \otimes e_i \Big) \otimes e_j^* \mapsto \sum_i a_i \otimes e_i e_j^*.$$

It follows immediately that

$$A(\pi)A(\pi)^* = (A \,\square\, \mathscr{H}_\pi)(A \,\square\, \mathscr{H}_\pi)^*,$$

from which the corollary follows by the previous lemma. $\qquad\square$

Peligrad [13, Corollary 3.5] showed that, for compact group actions, the condition (9) is equivalent to the saturatedness of the action. These results are also closely related to the *strong monoidality* of the operation $A \,\square\, -$.

Assume now again π irreducible, and consider on $A \,\square\, \mathscr{H}_\pi$ the left Hilbert B-module structure via

$$_B\langle \xi, \eta \rangle = (\iota \otimes \theta_\pi)\xi\eta^*,$$

where θ_π is the unique Ad_π-invariant state on $B(\mathscr{H}_\pi)$. It is easy to see that the conditional expectation $E \colon A \,\square\, B(\mathscr{H}_\pi) \cong \mathcal{K}(A \,\square\, \mathscr{H}_\pi) \to B$ from Theorem 5.3 corresponds precisely to the one from Lemma 5.4.

Theorem 5.7 (Theorem 1.3, second part). *The index of the conditional expectation $(\iota \otimes \theta_\pi)$ for $B \subseteq (A \otimes B(\mathscr{H}_\pi))^{\mathbb{G}}$ equals $\dim_q(\mathscr{H}_\pi)^2$, the square of the quantum dimension of $\mathscr{H}_\pi$.*

Proof. As in Section 2.1, we identify $\mathscr{H}_{\overline{\pi}}$ with $\mathscr{H}_\pi^*$ endowed with the modified inner product $\langle\!\langle e_i^*, e_j^* \rangle\!\rangle = \theta_\pi(e_i e_j^*)$.

Choose an orthonormal basis $(e_i)_i$ (resp. $(f_i)_i$) of $\mathscr{H}_\pi$ (resp. of $\mathscr{H}_{\overline{\pi}}$), and write $\delta(e_i) = \sum_j u_{ij} \otimes e_j$ (resp. $\delta_{\overline{\pi}}(f_i) = \sum_j v_{ij} \otimes f_j$). Let h and g be the elements of $P(\mathbb{G})$ satisfying

$$(10) \qquad\qquad \varphi(u_{ij}^* h) = \delta_{ij}, \qquad \varphi(v_{ij}^* g) = \delta_{ij}.$$

Finally, let us choose a finite collection of $x_k, y_k, w_l, z_l \in P(A)$ such that

$$\sum_k \alpha(x_k)(y_k^* \otimes 1) = 1 \otimes h, \qquad \sum_l \alpha(w_l)(z_l^* \otimes 1) = 1 \otimes g.$$

Then $x_k, y_k \in A_\pi$ and $z_l, w_l \in A_{\overline{\pi}} A_\pi^*$.

Consider the vectors of $A \,\square\, \mathscr{H}_\pi$ (see the proof of Lemma 5.5) defined by

$$\xi_{i,k} = \sum_p \varphi(u_{pi}^* x_{k(1)})\, x_{k(0)} \otimes e_p, \qquad \eta_{i,k} = \sum_p \varphi(u_{pi}^* y_{k(1)})\, y_{k(0)} \otimes e_p.$$

Going through the argument from (7) to (8) and then applying (10), we deduce that $\sum_{i,k} \xi_{i,k} \eta_{i,k}^*$ represents the identity operator on $A \,\square\, \mathscr{H}_\pi$.

When $\sum_i x_i \otimes e_i$, $\sum_j y_j \otimes e_j$, and $\sum_k z_k \otimes e_k$ in $A \,\square\, \mathscr{H}_\pi$, we calculate

$$\left({}_B\!\left\langle \sum_i x_i \otimes e_i, \sum_j y_j \otimes e_j \right\rangle \sum_k z_k \otimes e_k \right)^* = \sum_k z_k^* \otimes e_k^* \left\langle \sum_j y_j^* \otimes e_j^*, \sum_i x_i^* \otimes e_i^* \right\rangle_B.$$

Here, the B-valued inner product on the right hand side is defined on the right B-module $A \,\square\, \mathscr{H}_{\overline{\pi}}$, and it makes sense because $\sum_i x_i^* \otimes e_i^*$ is in $A \,\square\, \mathscr{H}_{\overline{\pi}}$, etc. From this, we find that the elements $\xi_{i,k}$, $\eta_{i,k}$ and

$$\widetilde{\xi}_{i,l} = \sum_p \varphi(z_{l(1)}^* v_{pi}) \, z_{l(0)}^* \otimes f_p^*, \qquad \widetilde{\eta}_{i,l} = \sum_p \varphi(w_{l(1)}^* v_{pi}) w_{l(0)}^* \otimes f_p^*$$

satisfy the hypotheses of Lemma 5.4. More precisely, when $v = \sum_i c_i e_i^* \in \mathscr{H}_{\overline{\pi}} = \mathscr{H}_\pi^*$, then v^* denotes the vector $\sum_i \overline{c_i} e_i$ in $\mathscr{H}_\pi$.

Now the same computation as in Lemma 5.5 shows that

$$\sum_{i,k} \big\langle \widetilde{\eta}_{i,k}, \widetilde{\xi}_{i,k} \big\rangle_B = \Big(\sum_i f_i f_i^* \Big) 1.$$

Therefore $\text{Index}(E) = \big(\sum_i f_i f_i^* \big) 1$. From the way $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ was defined and (1), we see that a possible choice of $(f_i)_i$ is

$$f_i = \dim_q(\mathscr{H}_\pi)^{1/2} \sum_k \langle e_i, Q_\pi^{1/2} e_k \rangle e_k^*.$$

For this choice we can compute $\sum_i f_i f_i^* = \dim_q(\mathscr{H}_\pi) \text{Tr}(Q_\pi) = \dim_q(\mathscr{H}_\pi)^2$, finishing the proof. $\qquad\square$

Acknowledgements

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