Visualizing Overtwisted Discs in Open Books

by

Tetsuya Ito and Keiko Kawamuro

Abstract

We give an alternative proof of a theorem of Honda–Kazez–Matić that every non-right-veering open book supports an overtwisted contact structure. We also study two types of examples that show how overtwisted discs are embedded relative to right-veering open books.

§1. Introduction

In [15], we have introduced open book foliations and their basic machinery by using that of braid foliations [2–9] and showed applications of open book foliations including a self-linking number formula for general closed braids. In [16] we study the geometric structure of a 3-manifold by using open book foliations. In this note we study other applications of open book foliations. We will assume the readers are familiar with the definition and basic machinery of open book foliations as presented in [15].

One of the features of open book foliations is that one can visualize how surfaces are embedded with respect to general open books. In this paper we use this feature to illustrate overtwisted discs and give constructive methods to detect overtwisted contact structures.

We first give an alternative proof of a tightness criterion theorem by Honda, Kazez and Matić [14]: If an open book is not right-veering then it supports an overtwisted contact structure.

T. Ito: Research Institute for Mathematical Sciences, Kyoto University,
Kyoto, 606-8502, Japan;
e-mail: `tetitoh@kurims.kyoto-u.ac.jp`
K. Kawamuro: Department of Mathematics, The University of Iowa,
14 MacLean Hall, Iowa City, IA 52242-1419, USA;
e-mail: `keiko-kawamuro@uiowa.edu`

The converse does not hold: In fact, Honda, Kazez and Matić [14] show that if a contact structure ξ is supported by a non-right-veering open book (S, ϕ), by applying positive stabilizations to (S, ϕ) one can find a right-veering open book $(\widehat{S}, \widehat{\phi})$ that supports ξ. We concretely visualize an overtwisted disc relative to the right-veering $(\widehat{S}, \widehat{\phi})$.

Lastly, we give an infinite family of open books that are right-veering and non-destabilizable but compatible with overtwisted contact structures. This negatively answers a question of Honda, Kazez and Matić [14]. Our family generalizes the previously known examples by Lekili [19] and Lisca [20], but our proof of overtwistedness is more direct.

§2. Overtwisted discs in non-right-veering open books

Recall that an *overtwisted disc* is an embedded disc in a contact 3-manifold whose boundary is a limit cycle in the characteristic foliation of the disc. In particular, every overtwisted disc has *Legendrian* boundary. In the framework of open book foliations a *transverse overtwisted disc* plays a corresponding role:

Definition 2.1 ([15, Def. 4.1]). Let $D \subset M_{(S,\phi)}$ be an oriented disc whose boundary is positively braided (i.e., a transverse knot) with respect to the open book (S, ϕ). If the following are satisfied D is called a *transverse overtwisted disc*:

1. G_{--} is a connected tree with no fake vertices.
2. G_{++} is homeomorphic to S^1.
3. $\mathcal{F}_{\mathrm{ob}}(D)$ contains no c-circles.

(Notations and terms like G_{--}, G_{++}, fake vertices and c-circles are defined in §2.1 of [15].)

In [15] we show that the manifold $M_{(S,\phi)}$ contains a transverse overtwisted disc if and only if the contact manifold $(M_{(S,\phi)}, \xi_{(S,\phi)})$ contains an overtwisted disc. Hence from now on, we may *not* distinguish a transverse overtwisted disc and a usual overtwisted disc, and often call a transverse overtwisted disc simply an overtwisted disc.

Next we review the notion of right-veering mapping classes and then reprove Honda–Kazez–Matić's tightness criterion in Theorem 2.4.

Definition 2.2 ([14]). Let γ, γ' be oriented properly embedded arcs in the surface S that start from the same point $* \in \partial S$. Suppose that, after some isotopy relative to the endpoints, γ and γ' realize the minimal geometric intersection number. We say that γ' *lies strictly on the right side* of γ if it does so around the common starting point $*$. In that case we write $\gamma > \gamma'$.

Definition 2.3 ([14, Definition 2.1]). Let C be a boundary component of S. We say that $\phi \in \mathrm{Aut}(S, \partial S)$ is *right-veering* with respect to C if $\gamma \geq \phi(\gamma)$ for any isotopy classes γ of properly embedded curves which start at a point on C. We say that the diffeomorphism ϕ (or the open book (S, ϕ)) is *right-veering* if ϕ is right-veering with respect to all the boundary components of S. In particular, the identity $\mathrm{id} \in \mathrm{Aut}(S, \partial S)$ is right-veering.

The following theorem gives a characterization of open books supporting tight contact structures.

Theorem 2.4 ([14, Theorem 1.1]). *If ϕ is not right-veering then (S, ϕ) supports an overtwisted contact structure.*

Remark 2.5. Conversely, Honda, Kazez and Matić [14, p. 444] also prove that given an overtwisted contact structure ξ there exists a non-right-veering open book (S, ϕ) supporting ξ; here Eliashberg's classification of overtwisted contact structures [10] plays an important role. By Eliashberg's classification, an overtwisted contact structure admits an open book which is a negative stabilization of some open book. Clearly such an open book has a non-right-veering monodromy. Therefore, a contact structure ξ is tight if and only if every open book supporting ξ is right-veering.

Proof of Theorem 2.4.. If ϕ is not right-veering, then there exists a properly embedded oriented arc $\alpha \subset S$ such that $\phi(\alpha) > \alpha$. By [14, Lemma 5.2], there exists a sequence of properly embedded oriented arcs $\alpha_0, \ldots, \alpha_k$ such that

 (i) $\alpha_0, \ldots, \alpha_k$ have the same initial point $n \in \partial S$,
 (ii) $\phi(\alpha) = \alpha_0 > \cdots > \alpha_k = \alpha$,
 (iii) consecutive α_i and α_{i+1} have disjoint interiors and distinct terminal points p_i, p_{i+1}.

Since $\phi(\alpha_k) = \alpha_0$ and $\phi = \mathrm{id}$ near the binding we have $p_0 = p_k$. We may assume that:

(iv) the terminal points $p_0, p_1, \ldots, p_{k-1} \in \partial S$ are mutually distinct.

Let β_i (resp. $\check{\beta}_i$) be a subarc of α_i whose endpoints are p_i and a point very close to p_i (resp. n). See Figure 1. We orient α_i *against* the parametrization, i.e., the positive direction of α_i is from p_i to n. The orientation of α_i induces those of β_i and $\check{\beta}_i$. We define sets of oriented arcs for $i = 0, \ldots, k$:

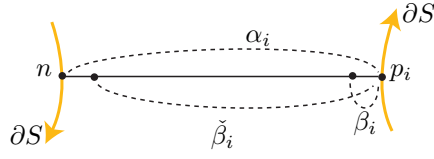$$A_i = \beta_0 \cup \cdots \cup \beta_{i-1} \cup \alpha_i \cup \beta_{i+1} \cup \cdots \cup \beta_k.$$

Figure 1. Arcs α_i, β_i, and $\check{\beta}_i$ oriented from p_i to n.

Let $t_i = i/k \in [0,1]$. In the following, we construct an oriented surface D_i properly embedded in the product region $S \times [t_i, t_{i+1}]$, where $i = 0, \ldots, k-1$, such that $D_i \cap S_{t_i} = -A_i$ and $D_i \cap S_{t_{i+1}} = A_{i+1}$.

The surface D_i consists of k connected components; one non-product region and $k-1$ product regions defined by $\beta_j \times [t_i, t_{i+1}]$ where $j \neq i, i+1$ and $0 \leq j \leq k$.

In the open book foliation of D_i the point n becomes a negative elliptic point, the points $p_0, \ldots, p_{k-1}$ become positive elliptic points, and the arc $\beta_j \times \{t\}$ becomes an a-arc in the page S_t. (See [15, Prop. 2.2] for the definition of a-arcs.) The non-product component of D_i is defined by the movie presentation as sketched in Figure 2. It is a saddle shape surface with a positive hyperbolic point h_i.
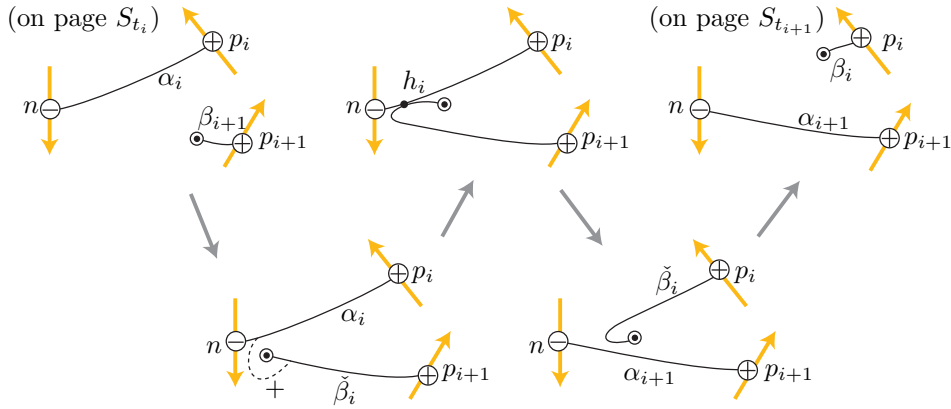

Figure 2. The non-product region of surface D_i (movie presentation).

Now we glue D_i and D_{i+1} along $A_{i+1} \subset S_{t_{i+1}}$ $(i = 0, \ldots, k-2)$ and obtain a surface $D_0 \cup \cdots \cup D_{k-1} \subset S \times [0,1]$ whose oriented boundary is $(-A_0) \cup A_k$. Since the arcs $\beta_1, \ldots, \beta_k$ are very close to ∂S and $\phi = \mathrm{id}$ near ∂S, we have $A_0 = \phi(A_k)$. So in the manifold $M_{(S,\phi)}$ we can identify A_0 and A_k and obtain a surface which we denote by D.
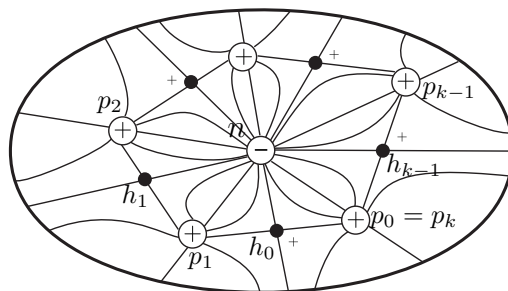
Figure 3. The transverse overtwisted disc D.

The topological type of D is the disc and its open book foliation $\mathcal{F}_{\mathrm{ob}}(D)$ is depicted in Figure 3. Clearly our D is a transverse overtwisted disc.   □

Remark 2.6. Honda, Kazez and Matić's proof and our proof are based on the same combinatorial lemma [14, Lemma 5.2] in order to use the assumption of right-veeringness. Our proof is more elementary and different from the original one that is written in the language of convex surface theory:

In [14], the authors prove the *existence* of a bypass, half of an overtwisted disc, by applying [14, Lemma 5.2] and the right-to-life principle [12, Lemma 2.9], [13, Proposition 2.2] which involves the Legendrian realization principle and Eliashberg's classification of tight contact structures on the 3-ball.

On the other hand, we use [14, Lemma 5.2] to explicitly *construct* a chain of positive elliptic and hyperbolic points surrounding the central negative elliptic point of an overtwisted disc. Hence our proof concretely visualizes an overtwisted disc.

§3. Overtwisted discs in right-veering open books

The converse of Theorem 2.4 does not hold in general. Honda, Kazez and Matić [14] show that *every* contact structure is supported by a right-veering open book.

Their argument is the following: Given a contact structure (M, ξ) choose a compatible open book (S, ϕ). For a boundary component C of S take two boundary-parallel arcs a_C and b_C such that the geometric intersection number $i(a_C, b_C)$ is 2. Apply positive stabilizations to (S, ϕ) along a_C and b_C for all the boundary components C on which ϕ is non-right-veering. The new open book $(\widehat{S}, \widehat{\phi})$ is now right-veering (see [14, Prop. 6.1]), and supports the same contact structure ξ.

Suppose that we start from a non-right-veering open book (S, ϕ), hence ξ is overtwisted. In the following we concretely describe how an overtwisted disc is embedded with respect to the stabilized right-veering open book $(\widehat{S}, \widehat{\phi})$.

Example 3.1 (Overtwisted discs in Honda–Kazez–Matić's stabilizations). Let (S, ϕ) be a non-right-veering open book. By the proof of Theorem 2.4 one can construct an overtwisted disc, D, in $M_{(S,\phi)}$. The open book foliation of D has a unique negative elliptic point, say n, that lies on the binding component $C \subset \partial S$, and k positive elliptic points $p_1, \ldots, p_k$ and k positive hyperbolic points $h_1, \ldots, h_k$. Let S_{t_i} be the singular fiber that contains h_i. We may assume

$$0 < t_1 < \cdots < t_k < 1/2 < 1.$$

For $t \in [0, 1)$ let $b_t \in S_t \cap D$ be the b-arc (cf. [15, Prop. 2.2]) that ends at the point n.

Now we apply Honda–Kazez–Matić's stabilization to get a right-veering open book $(\widehat{S}, \widehat{\phi})$. The monodromy $\widehat{\phi}$ satisfies $\widehat{\phi} = T_\beta \circ T_\alpha \circ \phi$, where α and β are core circles of the annuli plumbed along a_C and b_C, and T_α, T_β are positive Dehn twists along α, β.

We will construct an overtwisted disc $\widehat{D}$ by giving a movie presentation relative to the open book $(\widehat{S}, \widehat{\phi})$. For simplicity we assume that:

- ϕ is non-right-veering *only* along C.
- $p_k \in \partial S \setminus C$.

In the general case the construction of $\widehat{D}$ is similar but more complicated. It is obtained as an application of arguments in [17].

Choose stabilization arcs a_C and b_C such that $i(\alpha, b_{1/2}) = 1$ and $i(\beta, b_{1/2}) = 0$ as shown in Figure 4(a). Such arcs can always be found by the assumptions above.

To the region $\{\widehat{S}_t \mid t \in [0, 1/2]\}$ add a continuous family of b-arcs that are co-cores of the annuli plumbed along a_C (see Figure 4(a)). We denote the positive and negative elliptic points which are the endpoints of the newly added b-arcs by p' and n'. Except for this family of b-arcs, the movie presentation of $\widehat{D}$ in the interval $[0, 1/2]$ is the same as that of D. Let $\widehat{b}_t := b_t$.

In the interval $[1/2, 1)$, $\widehat{D}$ is described as in the passage (b)→(c)→(d) of Figure 4. We form one negative and one positive hyperbolic point, h_- and h_+, as in (b) and (c), respectively. The describing arcs of h_- and h_+ are parallel to α. We have $\widehat{b}_1 = T_\alpha^{-1}(b_1)$. Note that $\widehat{\phi}(\widehat{b}_1) = (T_\beta \circ T_\alpha \circ \phi)(\widehat{b}_1) = T_\beta \circ T_\alpha(T_\alpha^{-1}b_0) = T_\beta(b_0) = b_0 = \widehat{b}_0$, moreover $\widehat{\phi}(\widehat{D} \cap S_1) = \widehat{D} \cap S_0$ so this movie presentation indeed defines an overtwisted disc in $M_{(\widehat{S}, \widehat{\phi})}$.
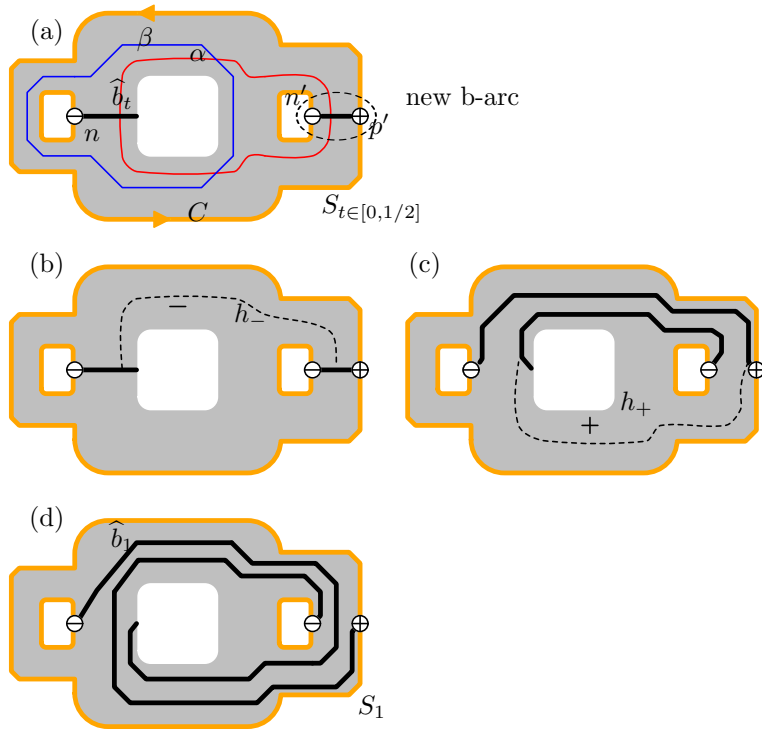
Figure 4. A movie presentation of the overtwisted disc $\widehat{D}$. The shaded region is the union of a collar neighborhood of C and the two plumbed annuli. All the a-arcs are omitted for simplicity.
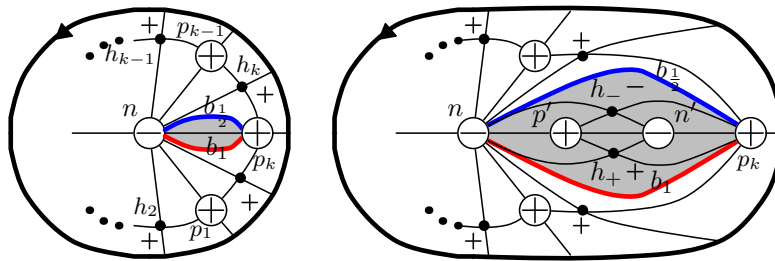

Figure 5. $\mathcal{F}_{\mathrm{ob}}(D)$ and $\mathcal{F}_{\mathrm{ob}}(\widehat{D})$.

The open book foliations of the overtwisted discs D and $\widehat{D}$ are depicted in Figure 5, where $\mathcal{F}_{\mathrm{ob}}(\widehat{D})$ is obtained by inserting two bb-tiles of opposite signs into the shaded region of $\mathcal{F}_{\mathrm{ob}}(D)$ that is bounded by $b_{1/2}$ and b_1.

Strictly speaking, the disc $\widehat{D}$ is not a transverse overtwisted disc since condition (3) of Definition 2.1 is not satisfied. However, applying the same technique as in the alternative proof in [15] of the Bennequin–Eliashberg inequality [1, 11], condition (3) will be satisfied. Thus we can regard $\widehat{D}$ as an overtwisted disc.

§4. Generalization of Lekili's and Lisca's examples

In [14, Question 6.2] Honda, Kazez and Matić ask whether a right-veering and non-destabilizable open book always supports a tight contact structure. Lekili [19] and Lisca [20] negatively answer the question by constructing examples. They study open book decompositions of 3-manifolds whose tight contact structures are well-studied and classified (in [19] Poincaré homology 3-spheres, and in [20] lens spaces). In both constructions the most technical points are showing that their open books indeed support overtwisted contact structures. Advanced tools such as Ozsváth–Szábo's Heegaard Floer invariants and properties of planar open books enable them to overcome the difficulty.

We generalize Lekili's and Lisca's examples in Theorem 4.1 below. Our proof of overtwistedness is direct and does not require any knowledge of classification of tight contact structures of ambient manifolds or Ozsváth–Szábo's invariants.

Theorem 4.1. *Let S be a 2-sphere with four holes. Let a, b, c, d, e be simple closed curves on S as shown in Figure 6. Let $\Phi_{h,i,k} = T_a^h T_b^i T_c T_d T_e^{-k-1}$ where T_x ($x = a, b, c, d, e$) denotes the right-handed Dehn twist along x. Then for all $h, i, k \geq 1$, $\Phi_{h,i,k}$ is right-veering and the open book $(S, \Phi_{h,i,k})$ is non-destabilizable and supports an overtwisted contact structure.*
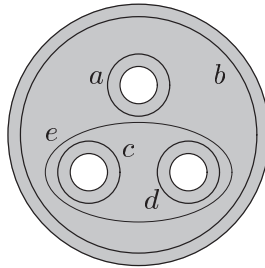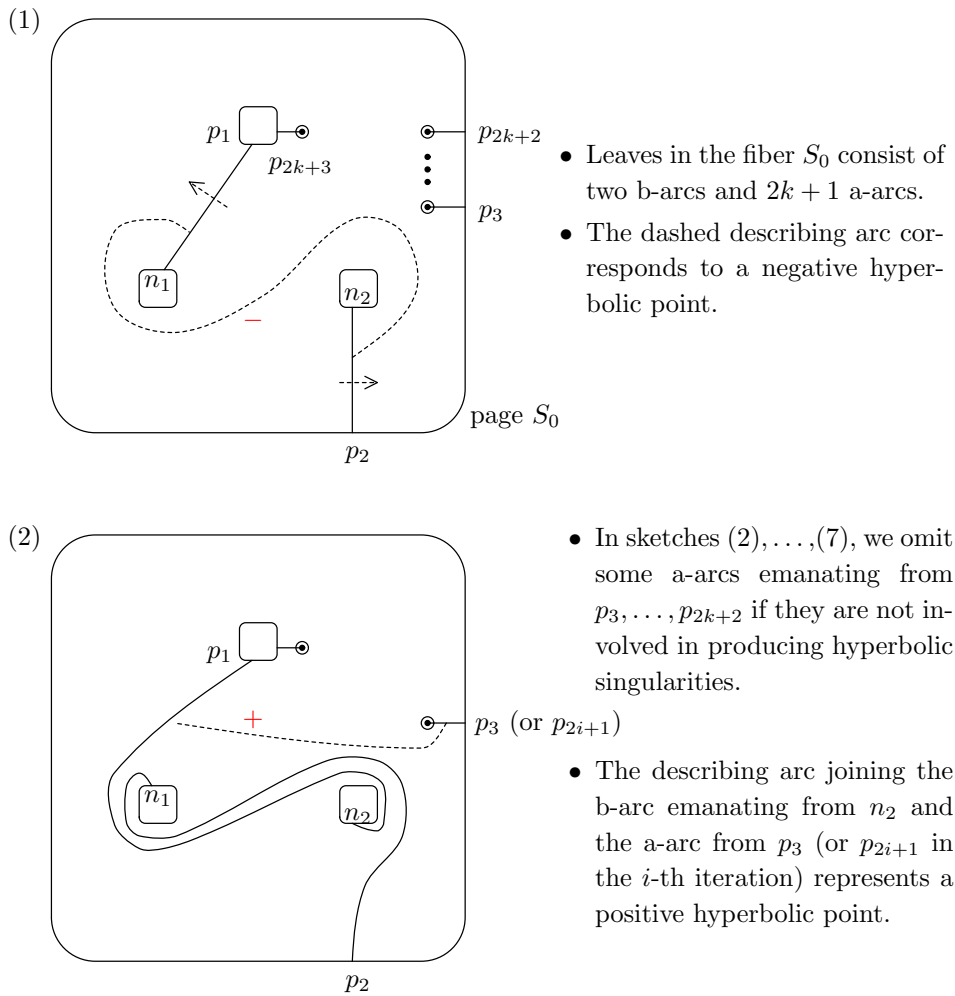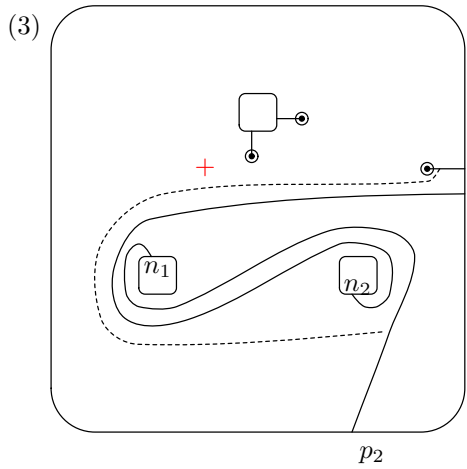

Figure 6. The surface S.

Remark 4.2. Lekili's examples [19, Theorem 1.2 and Remark 4.1] are $\Phi_{2,i,1}$ ($i \leq 5$), and Lisca's examples [20, Theorem 1.1] are $\Phi_{h,1,l}$ ($h, l > 0$).

Proof of Theorem 4.1. We owe to [19], [20] the proof that $\Phi_{h,i,k}$ is right-veering and non-destabilizable. Hence we only show that $(S, \Phi_{h,i,k})$ supports an overtwisted contact structure. We define a transverse overtwisted disc D in the open book $(S, \Phi_{h,i,k})$ by the movie presentation as in sketches (1)–(7) of Figure 7. For example, sketch (1) depicts the page S_0 with the set of arcs $D \cap S_0$. On ∂S there are two negative elliptic points n_1, n_2, and $2k+3$ positive elliptic points $p_1, \ldots, p_{2k+3}$. The movie presentation shows that $\mathcal{F}_{ob}(D)$ contains two negative hyperbolic points and $2k+3$ positive hyperbolic points. Note that $\Phi_{h,i,k}(D \cap S_1) = D \cap S_0$.

The corresponding open book foliation $\mathcal{F}_{ob}(D)$ is depicted in Figure 8. We can verify that D meets the conditions in Definition 2.1 of transverse overtwisted discs. $\qquad \square$

(1)

- Leaves in the fiber S_0 consist of two b-arcs and $2k+1$ a-arcs.
- The dashed describing arc corresponds to a negative hyperbolic point.

(2)

- In sketches $(2), \ldots, (7)$, we omit some a-arcs emanating from $p_3, \ldots, p_{2k+2}$ if they are not involved in producing hyperbolic singularities.

- The describing arc joining the b-arc emanating from n_2 and the a-arc from p_3 (or p_{2i+1} in the i-th iteration) represents a positive hyperbolic point.

(3)

p_4 (or p_{2i+2})
p_3 (or p_{2i+1})

- The describing arc joining the b-arc emanating from n_1 and the a-arc from p_4 (or p_{2i+2} in the i-th iteration) represents a positive hyperbolic point.

- Iterate steps (2) and (3) k times.
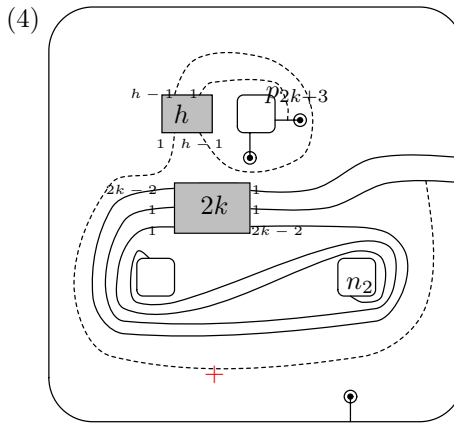
- The shaded boxes labeled $h, 2k$ contain parallel h and $2k$ arcs.

- The edges out of the shaded boxes are weighted as indicated.

- The describing arc joins the b-arc from n_2 to p_{2k+1} and the a-arc emanating from p_{2k+3} and it represents a positive hyperbolic point.
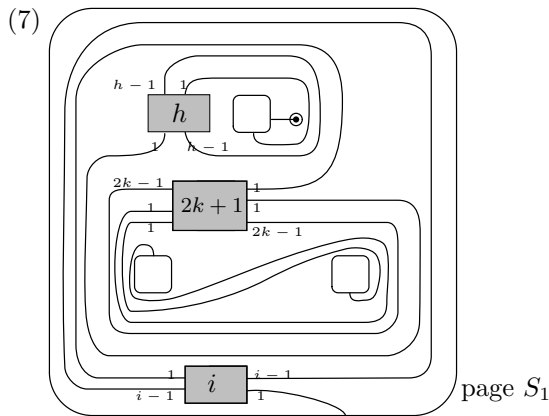
(4)

p_{2k+3}
p_{2k+2}
p_{2k+1}

(5)
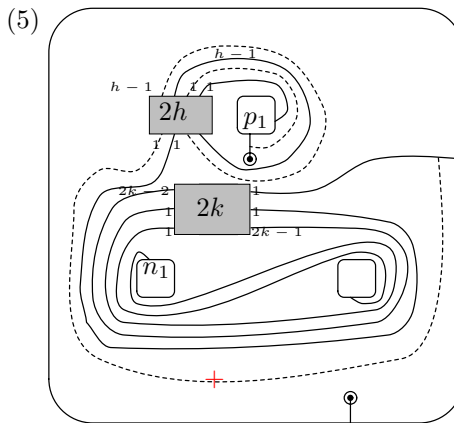
p_{2k+2}

- The describing arc joins the b-arc from n_1 to p_{2k+2} and the a-arc emanating from p_1 and it represents a positive hyperbolic point.

(6)

(7)
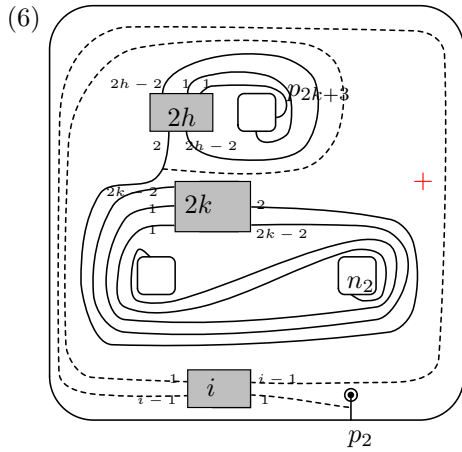
- The describing arc joins the b-arc from n_2 to p_{2k+3} and the a-arc emanating from p_2 and it represents a positive hyperbolic point.

- The leaves in the page S_1. It satisfies $\Phi_{h,i,k}(D \cap S_1) = D \cap S_0$.

page S_1

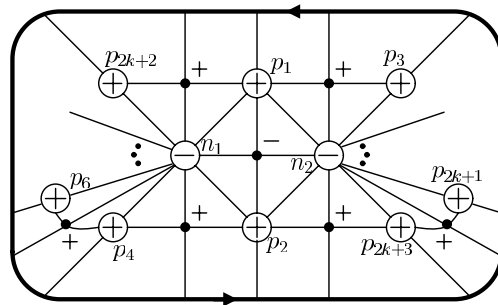Figure 7. Movie presentation of a transverse overtwisted disc in $(S, \Phi_{h,i,k})$.

Figure 8. The open book foliation $\mathcal{F}_{\mathrm{ob}}(D)$.

Remark 4.3. After the announcement of this example (on December 26, 2011), Kazez and Roberts [18] found more counterexamples to the conjecture of Honda, Kazez and Matić.

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