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Ulam Problem for the Sine Addition Formula in Hyperfunctions

by

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Abstract

We solve the Ulam problem for the sine addition formula in the spaces of Schwartz distributions and Gelfand hyperfunctions with respect to bounded distributions and bounded hyperfunctions.

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§1. Introduction

In 1950, Laurent Schwartz introduced the theory of distributions in his monograph *Théorie des distributions* [33]. In this book Schwartz systematizes the theory of generalized functions, basing it on the theory of linear topological spaces, relates all the earlier approaches, and obtains many important results. After his elegant theory appeared, many important concepts and results on the classical spaces of functions have been generalized to the space of distributions. For example, *positive functions* and *positive-definite functions* have been generalized to *positive distributions* and *positive-definite distributions*, respectively, and it was shown that every positive distribution is a positive measure [22, p. 38] and every positive-definite distribution is the Fourier transform of a positive measure μ such that $\int(1 + |x|)^{-p}\, d\mu < \infty$ for some $p \geq 0$ [21, p. 157], which is called the *Bochner–Schwartz theorem* and is a natural generalization of the famous *Bochner theorem*

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stating that every positive-definite function is the Fourier transform of a positive finite measure. For other examples, the *Paley–Wiener theorem* has been generalized to the *Paley–Wiener–Schwartz theorem* which characterizes the distributions with bounded supports [22, p. 181].

The main purpose of this paper is to prove the Hyers–Ulam type stability for the sine functional equation

$$f(x+y) - f(x)g(y) - g(x)f(y) = 0$$

in *Schwartz distributions* and *Gelfand hyperfunctions*. The Ulam problem for functional equations goes back to 1940 when S. M. Ulam proposed the following [36]:

Let f be a mapping from a group G_1 to a metric group G_2 with metric $d(\cdot,\cdot)$ such that

$$d(f(xy),\, f(x)f(y)) \le \epsilon \quad \textit{for all } x, y \in G_1.$$

Then does there exist a group homomorphism h and $\delta_\epsilon > 0$ such that

$$d(f(x),\, h(x)) \le \delta_\epsilon \quad \textit{for all } x \in G_1\textit{?}$$

This problem was solved affirmatively by D. H. Hyers under the assumption that G_2 is a Banach space (see Hyers [23], Hyers–Isac–Rassias [24]). In 1949–1951, this result was generalized by T. Aoki [2] and D. G. Bourgin [6, 7]. Since then Ulam problems for many other functional equations have been investigated [17–19, 25, 27–32]. Among the many results obtained, L. Székelyhidi developed an idea of using invariant subspaces of functions defined on a group or semigroup in connection with the Ulam problem for sine functional equations [34, 35]. As a direct consequence of the elegant results of Székelyhidi, it was shown that if $f, g : \mathbb{R}^n \to \mathbb{C}$ satisfy

$$|f(x+y) - f(x)g(y) - g(x)f(y)| \le M, \quad x, y \in \mathbb{R}^n, \tag{1.1}$$

for some $M > 0$, then either there exist $\lambda_1, \lambda_2 \in \mathbb{C}$, not both zero, and $N > 0$ such that

$$|\lambda_1 f(x) - \lambda_2 g(x)| \le N \tag{1.2}$$

for all $x \in \mathbb{R}^n$, or else

$$f(x+y) - f(x)g(y) - g(x)f(y) = 0 \tag{1.3}$$

for all $x, y \in \mathbb{R}^n$. Furthermore, the functions f and g satisfying both (1.1) and (1.2) were investigated.

As a generalization of (1.1), it is very natural to consider

$$f(x+y) - f(x)g(y) - g(x)f(y) \in L^\infty(\mathbb{R}^{2n}), \tag{1.4}$$

where f and g are Lebesgue measurable functions and $L^\infty(\mathbb{R}^{2n})$ is the space of all bounded measurable functions defined in $\mathbb{R}^{2n}$. Note that (1.4) means that the inequality (1.1) holds almost everywhere. In [8–12], some stability problems for several functional equations including the condition (1.4) were considered in various spaces of generalized functions including Schwartz distributions. In [10–12], for example, replacing f and g by distributions u and v in (1.4) we have considered the condition

$$u \circ S - u \otimes v - v \otimes u \in L^\infty(\mathbb{R}^{2n}), \tag{1.5}$$

where $S(x, y) = x + y$, $x, y \in \mathbb{R}^n$, and $\circ$ and $\otimes$ denote the pullback and the tensor product of generalized functions, respectively. The condition (1.5) is not formulated purely in the language of generalized functions because the differences are assumed to be classical bounded measurable functions; all the previous results in [10–12] have formulations as in (1.5).

Schwartz [33] generalized the space $L^\infty(\mathbb{R}^n)$ of bounded measurable functions to the space $\mathcal{D}'_{L^\infty}(\mathbb{R}^n)$ of bounded distributions. Taking this into account, it is natural to consider the following stability condition for the sine functional equation in distributions and hyperfunctions u, v with respect to bounded distributions and bounded hyperfunctions:

$$u \circ S - u \otimes v - v \otimes u \in \mathcal{D}'_{L^\infty}(\mathbb{R}^{2n}) \text{ [resp. } \mathcal{A}'_{L^\infty}(\mathbb{R}^{2n})], \tag{1.6}$$

where $\mathcal{D}'_{L^\infty}(\mathbb{R}^{2n})$ and $\mathcal{A}'_{L^\infty}(\mathbb{R}^{2n})$ are the spaces of bounded distributions and bounded hyperfunctions respectively, and S, $\circ$, $\otimes$ are as in (1.5). For some related results in Schwartz distributions, we refer the reader to [3–5, 8, 9, 22, 33]. The main tools of our proof are based on *structure theorems* for generalized functions (see Lemmas 4.3 and 4.4 below) and the *heat kernel method* initiated by T. Matsuzawa [26], which represents generalized functions as initial values of solutions of the heat equation with appropriate growth conditions [13–16, 26] (see Lemmas 4.1 and 4.2). Making use of the heat kernel method we convert (1.6) to the following classical stability statement: there exist $C, N, d > 0$ [resp. for every $\epsilon > 0$ there exist $C_\epsilon > 0$] such that

$$\begin{aligned} &|\tilde{u}(x + y, t + s) - \tilde{u}(x, t)\tilde{v}(y, s) - \tilde{v}(x, t)\tilde{u}(y, s)| \\ &\qquad \le C\left(\frac{1}{t} + \frac{1}{s}\right)^N + d \text{ [resp. } C_\epsilon e^{\epsilon(1/t+1/s)}] \end{aligned} \tag{1.7}$$

for all $x, y \in \mathbb{R}^n$ and $t, s > 0$, where $\tilde{u}, \tilde{v} : \mathbb{R}^n \times (0, \infty) \to \mathbb{C}$ are the solutions of the heat equation corresponding to u, v respectively, which are introduced in Section 4. In Section 2, we consider the stability (1.7) and combining this result with the heat kernel method we prove the stability (1.6) in Section 4. Also, as direct consequences of our result we obtain the stability (1.5) and the following L^∞-version of the stability for the sine functional equation:

$$\|f(x+y) - f(x)g(y) - g(x)f(y)\|_{L^\infty(\mathbb{R}^{2n})} \le C, \tag{1.8}$$

where $f, g : \mathbb{R}^n \to \mathbb{C}$ are Lebesgue measurable functions satisfying the following growth condition: for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$|f(x)| \le C_\epsilon e^{\epsilon |x|^2}$$

for all $x \in \mathbb{R}^n$.

§2. Stability problem in the classical sense

Let $f, g : G \times (0, \infty) \to \mathbb{C}$ with $\langle G, + \rangle$ an Abelian group. Throughout this paper N denotes a fixed nonnegative real number. We consider the following stability statements involving functional inequalities:

$$\begin{gathered}\text{There exist } C > 0 \text{ and } d > 0 \text{ such that} \\ |f(x+y, t+s) - f(x,t)g(y,s) - g(x,t)f(y,s)| \le C\left(\frac{1}{t} + \frac{1}{s}\right)^N + d \\ \text{for all } x, y \in G \text{ and } t, s > 0;\end{gathered} \tag{2.1}$$

$$\begin{gathered}\text{for every } \epsilon > 0 \text{ there exists } C_\epsilon > 0 \text{ such that} \\ |f(x+y, t+s) - f(x,t)g(y,s) - g(x,t)f(y,s)| \le C_\epsilon e^{\epsilon(1/t + 1/s)} \\ \text{for all } x, y \in G \text{ and } t, s > 0.\end{gathered} \tag{2.2}$$

From now on, a function A from a semigroup $\langle S, + \rangle$ to the field $\mathbb{C}$ of complex numbers is said to be *additive* if $A(x+y) = A(x) + A(y)$ for all $x, y \in S$, and $m : S \to \mathbb{C}$ is said to be an *exponential function* provided $m(x+y) = m(x)m(y)$ for all $x, y \in S$.

We introduce the following conditions on $f : G \times (0, \infty) \to \mathbb{C}$ and N:

$$\begin{gathered}\text{There exist } C > 0 \text{ and } d > 0 \text{ such that} \\ |f(x,t)| \le Ct^{-N} + d, \quad \forall x \in G,\ t > 0;\end{gathered} \tag{2.3}$$

$$\begin{gathered}\text{for every } \epsilon > 0, \text{ there exists } C_\epsilon > 0 \text{ such that} \\ |f(x,t)| \le C_\epsilon e^{\epsilon/t}, \quad \forall x \in G,\ t > 0.\end{gathered} \tag{2.4}$$

Lemma 2.1 ([1, p. 212]). *All solutions* $f, g : G \times (0, \infty) \to \mathbb{C}$ *of the functional equation*

$$f(x+y, t+s) - f(x,t)g(y,s) - g(x,t)f(y,s) = 0, \qquad x, y \in G,\ t, s > 0, \tag{2.5}$$

are given by one of the following:

(i) $f = 0$ *and* g *is arbitrary,*

(ii) $f(x,t) = \frac{1}{2\lambda} m(x,t)$ *and* $g(x,t) = \frac{1}{2} m(x,t)$*, where* $\lambda \in \mathbb{C}$ *and* m *is a nonzero exponential function,*

(iii) $g(x,t) = m(x,t)$ *and* $f(x,t) = A(x,t)m(x,t)$*, where* A *is a nonzero additive function and* m *is a nonzero exponential function,*

(iv) $f(x,t) = \frac{1}{2\lambda}(m^*(x,t) - m^{**}(x,t))$ *and* $g(x,t) = \frac{1}{2}(m^*(x,t) + m^{**}(x,t))$*, where* $\lambda \in \mathbb{C}$ *and* m^**,* m^{**} *are nonzero exponential functions.*

Using the idea in [24, p. 104] we obtain the following.

Lemma 2.2. *Let* $f, g : G \times (0, \infty) \to \mathbb{C}$ *satisfy the following condition for some* $N \geq 0$*: for each* $y \in G$ *and* $s > 0$ *there exist positive constants* $C = C(y,s)$ *and* $d = d(y,s)$ [*resp. for each* $y \in G$*,* $s > 0$ *and* $\epsilon > 0$ *there exists a positive constant* $C_\epsilon = C_\epsilon(y,s)$] *such that*

$$|f(x+y, t+s) - f(x,t)g(y,s)| \leq Ct^{-N} + d \ [\textit{resp. } C_\epsilon e^{\epsilon/t}\,] \tag{2.6}$$

for all $x \in G$ *and* $t > 0$*. Then either* f *satisfies* (2.3) [*resp.* (2.4)] *or* g *is an exponential function.*

Proof. Suppose that g is not exponential. Then there exist $y_0, z_0 \in G$ and $s_0, r_0 > 0$ such that $g(y_0 + z_0, s_0 + r_0) \neq g(y_0, s_0)g(z_0, r_0)$. Now, we can write

$$\begin{aligned} &f(x + y_0 + z_0, t + s_0 + r_0) - f(x + y_0, t + s_0)g(z_0, r_0) \\ &\qquad = f(x + y_0 + z_0, t + s_0 + r_0) - f(x,t)g(y_0 + z_0, s_0 + r_0) \\ &\qquad\quad - g(z_0, r_0)\big(f(x + y_0, t + s_0) - f(x,t)g(y_0, s_0)\big) \\ &\qquad\quad + f(x,t)\big(g(y_0 + z_0, s_0 + r_0) - g(y_0, s_0)g(z_0, r_0)\big), \end{aligned}$$

and hence

$$\begin{aligned} f(x,t) = {} & \big(g(y_0 + z_0, s_0 + r_0) - g(y_0, s_0)g(z_0, r_0)\big)^{-1} \\ & \times \big(f(x + y_0 + z_0, t + s_0 + r_0) - f(x + y_0, t + s_0)g(z_0, r_0) \\ & - f(x + y_0 + z_0, t + s_0 + r_0) + f(x,t)g(y_0 + z_0, s_0 + r_0) \\ & + g(z_0, r_0)\big(f(x + y_0, t + s_0) - f(x,t)g(y_0, s_0)\big)\big). \end{aligned} \tag{2.7}$$

It follows from (2.6) and (2.7) that there exist positive constants C_1, C_2, C_3, d_1, d_2, d_3, C' and d' [resp. for every $\epsilon > 0$ there exists a positive constant C'_ϵ] such that

$$\begin{aligned}|f(x,t)| &\le C_1(t+s_0)^{-N} + d_1 + C_2 t^{-N} + d_2 + C_3 t^{-N} + d_3 \\ &\le C' t^{-N} + d' \text{ [resp. } C'_\epsilon e^{\epsilon/t}\text{]}\end{aligned}$$

for all $x \in G$ and $t > 0$. □

Lemma 2.3. *Let $g : G \times (0,\infty) \to \mathbb{C}$ be a nonzero exponential function satisfying* (2.3) [*resp.* (2.4)]. *Then g can be written in the form*

$$g(x,t) = m_1(x)m_2(t),$$

where m_1 is an exponential function on G satisfying $|m_1(x)| = 1$ for all $x \in G$ and m_2 is an exponential function on $(0,\infty)$ satisfying $0 < |m_2(t)| \le 1$ for all $t > 0$.

Proof. Assume that $g(x_0, t_0) = 0$ for some $x_0 \in G$ and $t_0 > 0$. Then, for given $x \in G$ and $t > 0$, choosing a positive integer k such that $kt > t_0$ we have

$$[g(x,t)]^k = g(kx, kt) = g(kx - x_0, kt - t_0)g(x_0, t_0) = 0.$$

Thus, $g(x,t) \neq 0$ for all $x \in G$ and $t > 0$. Let

$$m(x,t) = g(x,t)g(0,t)^{-1}.$$

Then

$$\begin{aligned}m(x,t) = g(x,t)g(0,t)^{-1} &= g(x,t+s)g(0,s)^{-1}g(0,t)^{-1} = g(x,s)g(0,s)^{-1} \\ &= m(x,s).\end{aligned}$$

Hence, m is independent of $t > 0$ and we can write $m(x,t) =: m_1(x)$. Now,

$$m_1(x+y) = g(x+y, 2t)g(0,2t)^{-1} = m(x,t)g(0,t)^{-1}g(y,t)g(0,t)^{-1} = m_1(x)m_1(y).$$

Thus, m_1 is an exponential function and we can write

$$g(x,t) = m_1(x)g(0,t) =: m_1(x)m_2(t),$$

where m_1 is an exponential function on G and m_2 is an exponential function on $(0,\infty)$. It follows from (2.3) [resp. (2.4)] that m_1 is bounded and m_2 satisfies

$$|m_2(t)| \le C t^{-N} + d \text{ [resp. } C_\epsilon e^{\epsilon/t}\text{]} \tag{2.8}$$

for all $t > 0$. If there exists $x_0 \in G$ such that $|m_1(x_0)| > 1$ or $|m_1(x_0)| < 1$, then for all integers k we have $|m_1(kx_0)| = |m_1(x_0)|^k \to \infty$ as $k \to \infty$ or $k \to -\infty$. Thus, $|m_1(x)| = 1$ for all $x \in G$. Similarly, if $|m_2(t_0)| > 1$ for some $t_0 > 0$, then for all integers k we have $|m_2(kt_0)| = |m_2(t_0)|^k \to \infty$ as $k \to \infty$, which contradicts the inequality (2.8). □

Lemma 2.4. *Let g be a nonzero exponential function satisfying* (2.3) [*resp.* (2.4)]. *Suppose that $f, g : G \times (0,\infty) \to \mathbb{C}$ satisfy* (2.1) [*resp.* (2.2)]. *Then*

$$g(x,t) = m_1(x)m_2(t),$$
$$f(x,t) = l(x)m_1(x)m_2(t) + 2f(0,t/2)g(0,t/2)m_1(x) + R(x,t),$$

where m_1 and m_2 are exponential functions on G and $(0,\infty)$ respectively such that $|m_1(x)| = 1$ for all $x \in G$ and $0 < |m_2(t)| \le 1$ for all $t > 0$, $l(x)$ is an additive function on G, and R is a function satisfying (2.3) [*resp.* (2.4)].

Proof. Dividing both sides of (2.1) [resp. (2.2)] by $g(x+y,t+s)$, setting $h(x,t) = f(x,t)g(x,t)^{-1}$ and using Lemma 2.3 we have

$$\begin{aligned}|h(x+y,t+s) - h(x,t) - h(y,s)| \\ &\le \left(C\left(\frac{1}{t}+\frac{1}{s}\right)^N + d\right)|m_1(x)m_1(y)m_2(t)m_2(s)|^{-1} \\ &\le \left(C\left(\frac{1}{t}+\frac{1}{s}\right)^N + d\right)|m_2(t)m_2(s)|^{-1} \\ &\quad [\text{resp. } C_\epsilon e^{\epsilon(1/t+1/s)}|m_2(t)m_2(s)|^{-1}\,]\end{aligned}$$

for all $x, y \in G$ and $t, s > 0$. Thus,

$$|h(x+y,t+s) - h(x,t) - h(y,s)| \le \psi(t,s) \tag{2.9}$$

for all $x, y \in G$ and $t, s > 0$, where

$$\begin{aligned}\psi(t,s) = &\left(C\left(\frac{1}{t}+\frac{1}{s}\right)^N + d\right)|m_2(t)m_2(s)|^{-1} \\ &[\text{resp. } C_\epsilon e^{\epsilon(1/t+1/s)}|m_2(t)m_2(s)|^{-1}].\end{aligned} \tag{2.10}$$

Replacing s by t and putting $y = 0$ in (2.9) we obtain

$$|h(x,2t) - h(x,t) - h(0,t)| \le \psi(t,t) \tag{2.11}$$

for all $x \in G$ and $t > 0$. Replacing t by s and putting $x = 0$ in (2.9) we have

$$|h(y,2s) - h(0,s) - h(y,s)| \le \psi(s,s) \tag{2.12}$$

for all $y \in G$ and $s > 0$. Using the triangle inequality together with (2.9), (2.11) and (2.12) we find that

$$\begin{aligned}|h(x+y,t+s) - h(x,2t) - h(y,2s) + h(0,t) + h(0,s)| \\ \le \psi(t,s) + \psi(t,t) + \psi(s,s)\end{aligned} \tag{2.13}$$

for all $x, y \in G$ and $t, s > 0$. Replacing y by x and s by t in (2.13) yields

$$|h(2x, 2t) - 2h(x, 2t) + 2h(0, t)| \le 3\psi(t, t) \tag{2.14}$$

for all $x \in G$ and $t > 0$. Fixing $t > 0$, replacing x by $2^{k-1}x$ in (2.14) and dividing the result by 2^k we have

$$|2^{-k}h(2^k x, 2t) - 2^{-k+1}h(2^{k-1}x, 2t) + 2^{-k+1}h(0, t)| \le 3 \cdot 2^{-k}\psi(t, t) \tag{2.15}$$

for all $x \in G$. For given positive integers n, m, putting $k = n, n+1, \dots, n+m$ in (2.15), summing up the results and using the triangle inequality, we can see that $A_n(x, t) := 2^{-n}h(2^n x, 2t),\ n = 1, 2, \dots,$ is a Cauchy sequence and $A(x, t) := \lim_{n\to\infty} A_n(x, t)$ exists. Replacing x, y by $2^n x$, $2^n y$ respectively in (2.13), dividing by 2^n and letting $n \to \infty$ we have

$$A(x + y, t + s) - A(x, 2t) - A(y, 2s) = 0 \tag{2.16}$$

for all $x, y \in G$ and $t, s > 0$. Letting $x = y = 0$ and replacing t, s by $t/2$ in (2.16) we get $A(0, t) = 0$ for all $t > 0$. Thus, putting $y = 0$ in (2.16) we have

$$A(x, 2t) = A(x, t + s) = A(x, s + t) = A(x, 2s) \tag{2.17}$$

for all $x \in G$ and $t, s > 0$. It follows from (2.17) that $A(x, t)$ is independent of $t > 0$ and is an additive function of $x \in G$, which we denote by $l(x)$. Using the triangle inequality together with (2.15) for $k = 1, \dots, n$ and letting $n \to \infty$ gives

$$|l(x) - h(x, 2t) + 2h(0, t)| \le 3\psi(t, t) \tag{2.18}$$

for all $x \in G$ and $t > 0$. Replacing t by $t/2$ in (2.18) and multiplying the result by $|g(x, t)|$ we obtain

$$\begin{aligned}
&|f(x, t) - l(x)g(x, t) - 2f(0, t/2)g(0, t/2)m_1(x)| \\
&\qquad \le 3C(4^N t^{-N} + d)|m_2(t/2)|^{-2}|g(x, t)| = (C't^{-N} + d')|g(0, t)|^{-1}|g(x, t)| \\
&\qquad = (C't^{-N} + d')|m_1(x)|^{-1} = C't^{-N} + d' \text{ [resp. } C'_\epsilon e^{4\epsilon/t}].
\end{aligned}$$

Letting $R(x, t) := f(x, t) - l(x)g(x, t) - 2f(0, t/2)g(0, t/2)m_1(x)$ we complete the proof. □

Lemma 2.5. *Suppose that* $f, g : G \times (0, \infty) \to \mathbb{C}$ *satisfy* (2.1) [*resp.* (2.2)]. *Then either*

$$f(x + y, t + s) - f(x, t)g(y, s) - g(x, t)f(y, s) = 0 \tag{2.19}$$

for all $x, y \in G$ *and* $t, s > 0$, *or else there exist* $\lambda_1, \lambda_2 \in \mathbb{C}$, *not both zero, such that* $\lambda_1 f(x, t) - \lambda_2 g(x, t)$ *satisfies* (2.3) [*resp.* (2.4)].

Proof. Assume that $\lambda_1 f(x,t) - \lambda_2 g(x,t)$ satisfies (2.3) [resp. (2.4)] only when $\lambda_1 = \lambda_2 = 0$. Now, it suffices to prove that f, g satisfy (2.19). Let

$$F(x,y,t,s) = f(x+y,t+s) - f(x,t)g(y,s) - g(x,t)f(y,s). \tag{2.20}$$

Choosing y_1 and s_1 with $f(y_1,s_1) \neq 0$ we have

$$g(x,t) = k_1 f(x,t) + k_2 f(x+y_1,t+s_1) - k_2 F(x,y_1,t,s_1), \tag{2.21}$$

where $k_1 = -\frac{g(y_1,s_1)}{f(y_1,s_1)}$ and $k_2 = \frac{1}{f(y_1,s_1)}$. From (2.20) and (2.21) we find that

$$\begin{aligned}
&(2.22) \qquad f\big((x+y)+z,(t+s)+r\big)\\
&= f(x+y,t+s)g(z,r) + g(x+y,t+s)f(z,r) + F(x+y,z,t+s,r)\\
&= f(x+y,t+s)g(z,r)\\
&\quad + \big(k_1 f(x+y,t+s) + k_2 f(x+y+y_1,t+s+s_1) - k_2 F(x+y,y_1,t+s,s_1)\big)f(z,r)\\
&\quad + F(x+y,z,t+s,r)\\
&= \big(f(x,t)g(y,s) + g(x,t)f(y,s) + F(x,y,t,s)\big)g(z,r)\\
&\quad + k_1\big(f(x,t)g(y,s) + g(x,t)f(y,s) + F(x,y,t,s)\big)f(z,r)\\
&\quad + k_2\big(f(x,t)g(y+y_1,s+s_1) + g(x,t)f(y+y_1,s+s_1)\\
&\quad + F(x,y+y_1,t,s+s_1) - F(x+y,y_1,t+s,s_1)\big)f(z,r)\\
&\quad + F(x+y,z,t+s,r),
\end{aligned}$$

and also

$$\begin{aligned}
&(2.23) \qquad f(x+(y+z),t+(s+r))\\
&\qquad = f(x,t)g(y+z,s+r) + g(x,t)f(y+z,s+r) + F(x,y+z,t,s+r).
\end{aligned}$$

From (2.22) and (2.23) we deduce that

$$\begin{aligned}
&(2.24) \quad f(x,t)\big(g(y,s)g(z,r) + k_1 g(y,s)f(z,r) + k_2 g(y+y_1,s+s_1)f(z,r) - g(y+z,s+r)\big)\\
&\quad + g(x,t)\big(f(y,s)g(z,r) + k_1 f(y,s)f(z,r) + k_2 f(y+y_1,s+s_1)f(z,r) - f(y+z,s+r)\big)\\
&\quad = F(x,y+z,t,s+r) - F(x+y,z,t+s,r) - F(x,y,t,s)g(z,r) - k_1 F(x,y,t,s)f(z,r)\\
&\quad - k_2\big(F(x,y+y_1,t,s+s_1) - F(x+y,y_1,t+s,s_1)\big)f(z,r).
\end{aligned}$$

Fixing y, z, s, r in (2.24), and using (2.1) and (2.20), we find that

$$\begin{aligned}
&\big|F(x,y+z,t,s+r) - F(x+y,z,t+s,r) - F(x,y,t,s)g(z,r) - k_1 F(x,y,t,s)f(z,r)\\
&\qquad - k_2\big(F(x,y+y_1,t,s+s_1) - F(x+y,y_1,t+s,s_1)\big)f(z,r)\big|\\
&\leq 2C\left(\frac{1}{t}+\frac{1}{r}\right)^N + 2d + C_1\left(\frac{1}{t}+\frac{1}{s}\right)^N + d_1 + C_2\left(\frac{1}{t}+\frac{1}{s_1}\right)^N + d_2 \leq C' t^{-N} + d',
\end{aligned}$$

where $C' = 2^N(2C + C_1 + C_2)$, $d' = 2^N(2Cr^{-N} + C_1 s^{-N} + C_2 s_1^{-N}) + 2d + d_1 + d_2$.

Similarly, using (2.2) we find that for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$\big|F(x,y+z,t,s+r) - F(x+y,z,t+s,r) - F(x,y,t,s)g(z,r) - k_1F(x,y,t,s)f(z,r) \\ - k_2\big(F(x,y+y_1,t,s+s_1) - F(x+y,y_1,t+s,s_1)\big)f(z,r)\big| \\ \le 2C_\epsilon e^{\epsilon(1/t+1/r)} + C_1C_\epsilon e^{\epsilon(1/t+1/s)} + C_2C_\epsilon e^{\epsilon(1/t+1/s_1)} \le C'_\epsilon e^{\epsilon/t},$$

where $C'_\epsilon = C_\epsilon(2e^{\epsilon/r} + C_1e^{\epsilon/s} + C_2e^{\epsilon/s_1})$.

Thus, by the assumption that $\lambda_1 f(x,t) - \lambda_2 g(x,t)$ satisfies (2.3) [resp. (2.4)] only when $\lambda_1 = \lambda_2 = 0$ we have

$$g(y,s)g(z,r) + k_1g(y,s)f(z,r) + k_2g(y+y_1,s+s_1)f(z,r) - g(y+z,s+r) \\ = f(y,s)g(z,r) + k_1f(y,s)f(z,r) + k_2f(y+y_1,s+s_1)f(z,r) - f(y+z,s+r) = 0.$$

Hence

$$F(x,y+z,t,s+r) - F(x+y,z,t+s,r) \\ = \big(k_1F(x,y,t,s) + k_2F(x,y+y_1,t,s+s_1) - k_2F(x+y,y_1,t+s,s_1)\big)f(z,r) \\ + F(x,y,t,s)g(z,r). \tag{2.25}$$

Now, if we fix x, y, t, s, the left hand side of (2.25) satisfies (2.3) [resp. (2.4)] as a function of z and r. From the right hand side of (2.25), using the assumption that $\lambda_1 f(x,t) - \lambda_2 g(x,t)$ satisfies (2.3) [resp. (2.4)] only when $\lambda_1 = \lambda_2 = 0$ it follows that $F \equiv 0$. □

Theorem 2.6. *Let $f, g : G \times (0,\infty) \to \mathbb{C}$ satisfy* (2.1) [*resp.* (2.2)]. *Then (f,g) satisfies one of the following*:

(i) $f = 0$ *and g is arbitrary,*

(ii) *both f and g satisfy* (2.3) [*resp.* (2.4)],

(iii) $f(x,t) = \frac{1}{2\lambda}(m(x,t) - R(x,t))$ *and* $g(x,t) = \frac{1}{2}(m(x,t) + R(x,t))$, *where $\lambda \in \mathbb{C}$, m is a nonzero exponential function and R is a function satisfying* (2.3) [*resp.* (2.4)].

(iv) *we have*

$$g(x,t) = m_1(x)m_2(t), \\ f(x,t) = l(x)m_1(x)m_2(t) + 2f(0,t/2)g(0,t/2)m_1(x) + R(x,t),$$

where m_1 and m_2 are exponential functions on G and $(0,\infty)$ respectively such that $|m_1(x)| = 1$ for all $x \in G$ and $0 < |m_2(t)| \le 1$ for all $t > 0$, $l(x)$ is an additive function on G, and R is a function satisfying (2.3) [*resp.* (2.4)],

(v) $g(x,t) = m(x,t)$ *and* $f(x,t) = A(x,t)m(x,t)$, *where m is a nonzero exponential function and A is a nonzero additive function,*

(vi) $f(x,t) = \frac{1}{2\lambda}(m^*(x,t) - m^{**}(x,t))$ *and* $g(x,t) = \frac{1}{2}(m^*(x,t) + m^{**}(x,t))$, *where* $\lambda \in \mathbb{C}$, *and* m^* *and* m^{**} *are nonzero exponential functions.*

Proof. In view of Lemma 2.5, we first assume that the equation (2.19) holds. Then Lemma 2.1 shows that the solutions of the sine functional equation (2.19) satisfy one of the cases (i), (iii) with $R \equiv 0$, (v) or (vi). It remains to consider the case when there exist $\lambda_1, \lambda_2 \in \mathbb{C}$, not both zero, such that $\lambda_1 f(x,t) - \lambda_2 g(x,t)$ satisfies (2.3) [resp. (2.4)].

First, suppose that f $(\neq 0)$ satisfies (2.3) [resp. (2.4)]. Choosing $y_0 \in G$ and $s_0 > 0$ such that $f(y_0, s_0) \neq 0$, dividing both sides of (2.1) by $|f(y_0, s_0)|$ and using the triangle inequality we have

$$\begin{aligned}|g(x,t)| &\le \frac{1}{|f(y_0,s_0)|}\left(|f(x+y_0, t+s_0)| + |f(x,t)g(y_0,s_0)| + C\left(\frac{1}{t} + \frac{1}{s_0}\right)^N + d\right)\\ &\le C_1(t+s_0)^{-N} + d_1 + C_2 t^{-N} + d_2 + C_3\left(\frac{1}{t} + \frac{1}{s_0}\right)^N + d_3 \le C' t^{-N} + d'\end{aligned}$$

for all $x \in G$ and $t > 0$ and for some positive constants $C_1, C_2, C_3, d_1, d_2, d_3, C'$ and d'. Similarly, if f, g satisfy (2.2) we can show that for every $\epsilon > 0$ there exists $C'_\epsilon > 0$ such that

$$|g(x,t)| \le C'_\epsilon e^{\epsilon/t}$$

for all $x \in G$ and $t > 0$. Thus, we obtain case (ii).

Secondly, suppose that neither f nor g satisfies (2.3) [resp. (2.4)]. In this case we must have $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$. Thus, we can write

$$g(x,t) = \lambda f(x,t) + R(x,t) \tag{2.26}$$

for some λ $(\neq 0) \in \mathbb{C}$ and a function R satisfying (2.3) [resp. (2.4)]. Putting (2.26) in (2.1), using the triangle inequality and fixing y and s we have

$$\begin{aligned}|f(x+y,t+s) - f(x,t)(R(y,s) + 2\lambda f(y,s))| &\le |f(y,s)R(x,t)| + C\left(\frac{1}{t} + \frac{1}{s}\right)^N + d\\ &\le C' t^{-N} + d' \text{ [resp. } C'_\epsilon e^{\epsilon/t}\text{]}\end{aligned}$$

for all $x \in G$ and $t > 0$ and for some positive constants C' and d' [resp. for every $\epsilon > 0$ there exists a positive constant C'_ϵ]. Applying Lemma 2.2, we obtain

$$R(y,s) + 2\lambda f(y,s) = m(y,s) \tag{2.27}$$

for all $y \in G$ and $s > 0$, where m is an exponential function on $G \times (0, \infty)$. Thus, case (iii) follows immediately from (2.26) and (2.27).

Finally, suppose that f does not satisfy (2.3) [resp. (2.4)] and g satisfies (2.3) [resp. (2.4)]. Then we must have $g \neq 0$. In view of (2.1) [resp. (2.2)], for each $y \in G$ and $s > 0$, $f(x+y,t+s) - f(x,t)g(y,s)$ satisfies (2.3) [resp. (2.4)]. Thus, by Lemma 2.2, g is an exponential function. Using Lemma 2.4 we get case (iv). □

Corollary 2.7. *Let* $f, g : \mathbb{R}^n \times (0,\infty) \to \mathbb{C}$ *be continuous functions satisfying* (2.1) [*resp.* (2.2)]. *Then* f, g *satisfy one of the following*:

(i) $f = 0$ *and* g *is arbitrary,*

(ii) *both* f *and* g *satisfy* (2.3) [*resp.* (2.4)],

(iii) $f(x,t) = \frac{1}{2\lambda}(e^{c\cdot x+bt} - R(x,t))$ *and* $g(x,t) = \frac{1}{2}(e^{c\cdot x+bt} + R(x,t))$, *where* $\lambda, b \in \mathbb{C}$, $c \in \mathbb{C}^n$ *and* R *is a function satisfying* (2.3) [*resp.* (2.4)],

(iv) $g(x,t) = e^{ic\cdot x+bt}$ *and* $f(x,t) = a\cdot xe^{ic\cdot x+bt} + 2f(0,t/2)e^{ic\cdot x+\frac{1}{2}bt} + R(x,t)$, *where* $c \in \mathbb{R}^n$, $a \in \mathbb{C}^n$, $b \in \mathbb{C}$, *and* R *is a function satisfying* (2.3) [*resp.* (2.4)],

(v) $g(x,t) = e^{c\cdot x+bt}$ *and* $f(x,t) = (a\cdot x + dt)e^{c\cdot x+bt}$, *where* $a, c \in \mathbb{C}^n$ *and* $b, d \in \mathbb{C}$,

(vi) $f(x,t) = \frac{1}{2\lambda}(e^{c_1\cdot x+b_1 t} - e^{c_2\cdot x+b_2 t})$ *and* $g(x,t) = \frac{1}{2}(e^{c_1\cdot x+b_1 t} - e^{c_2\cdot x+b_2 t})$, *where* $\lambda, b_1, b_2 \in \mathbb{C}$ *and* $c_1, c_2 \in \mathbb{C}^n$.

Proof. It follows from the continuity of f and g that the exponential functions m, m^*, m^{**} and the additive function A in (iii)–(vi) of Theorem 2.6 are continuous. Also, in view of the proof of Lemma 2.4, the additive function $l(x)$ in (iv) of Theorem 2.6 is continuous since it is the uniform limit of a sequence of continuous functions. Now, it is well known that the continuous solutions $A : \mathbb{R}^n \times (0,\infty) \to \mathbb{C}$ of the Cauchy functional equation

$$A(x+y, t+s) = A(x,t) + A(y,s)$$

are of the form $A(x,t) = c\cdot x + bt$, and the continuous solutions $m : \mathbb{R}^n \times (0,\infty) \to \mathbb{C}$ of the exponential functional equation

$$m(x+y, t+s) = m(x,t)m(y,s)$$

are of the form $m(x,t) = e^{c\cdot x+bt}$ for some $c \in \mathbb{C}^n, b \in \mathbb{C}$. If, in particular, m satisfies (2.3) [resp. (2.4)], then $c = ia$ for some $a \in \mathbb{R}^n$ and $\Re b < 0$. Thus, cases (i)–(vi) follow immediately from (i)–(vi) of Theorem 2.6, respectively. □

§3. Bounded distributions and hyperfunctions

We first introduce the spaces $\mathcal{S}'$ of Schwartz tempered distributions and $\mathcal{G}'$ of Gelfand hyperfunctions (see [20–22, 26, 33] for more details). We use the notations $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and

$\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0$ is the set of nonnegative integers and $\partial_j = \partial/\partial x_j$.

Definition 3.1 ([33]). We denote by $\mathcal{S}$ or $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of all infinitely differentiable functions φ on $\mathbb{R}^n$ such that

$$\|\varphi\|_{\alpha,\beta} = \sup_x |x^\alpha \partial^\beta \varphi(x)| < \infty \tag{3.1}$$

for all $\alpha, \beta \in \mathbb{N}_0^n$, equipped with the topology defined by the seminorms $\|\cdot\|_{\alpha,\beta}$. The elements of $\mathcal{S}$ are called *rapidly decreasing functions* and the elements of the dual space $\mathcal{S}'$ are *tempered distributions*.

Definition 3.2 ([20, 21]). We denote by $\mathcal{G}$ or $\mathcal{G}(\mathbb{R}^n)$ the Gelfand space of all infinitely differentiable functions φ in $\mathbb{R}^n$ such that

$$\|\varphi\|_{h,k} = \sup_{x\in\mathbb{R}^n,\, \alpha,\beta\in\mathbb{N}_0^n} \frac{|x^\alpha \partial^\beta \varphi(x)|}{h^{|\alpha|} k^{|\beta|} \alpha!^{1/2} \beta!^{1/2}} < \infty$$

for some $h, k > 0$. We say that $\varphi_j \to 0$ as $j \to \infty$ if $\|\varphi_j\|_{h,k} \to 0$ as $j \to \infty$ for some h, k; we denote by $\mathcal{G}'$ the strong dual space of $\mathcal{G}$ and call its elements *Gelfand hyperfunctions*.

As a generalization of the space L^∞ of bounded measurable functions, L. Schwartz introduced the space $\mathcal{D}'_{L^\infty}$ of bounded distributions as a subspace of tempered distributions.

Definition 3.3 ([33]). We denote by $\mathcal{D}_{L^1}(\mathbb{R}^n)$ the space of smooth functions on $\mathbb{R}^n$ such that $\partial^\alpha \varphi \in L^1(\mathbb{R}^n)$ for all $\alpha \in \mathbb{N}_0^n$, equipped with the topology defined by the countable family of seminorms

$$\|\varphi\|_m = \sum_{|\alpha|\le m} \|\partial^\alpha \varphi\|_{L^1}, \qquad m \in \mathbb{N}_0.$$

We denote by $\mathcal{D}'_{L^\infty}$ the strong dual space of $\mathcal{D}_{L^1}$ and call its elements *bounded distributions*.

Generalizing bounded distributions, the space $\mathcal{A}'_{L^\infty}$ of bounded hyperfunctions has been introduced as a subspace of $\mathcal{G}'$.

Definition 3.4 ([16]). We denote by $\mathcal{A}_{L^1}$ the space of smooth functions φ on $\mathbb{R}^n$ satisfying

$$\|\varphi\|_h = \sup_\alpha \frac{\|\partial^\alpha \varphi\|_{L^1}}{h^{|\alpha|} \alpha!} < \infty$$

for some constant $h > 0$. We say that $\varphi_j \to 0$ in $\mathcal{A}_{L^1}$ as $j \to \infty$ if there is a positive constant h such that

$$\|\varphi_j\|_h \to 0 \quad \text{as } j \to \infty.$$

We denote by $\mathcal{A}'_{L^\infty}$ the strong dual space of $\mathcal{A}_{L^1}$.

It is well known that the following topological inclusions hold:

$$\mathcal{G} \hookrightarrow \mathcal{S} \hookrightarrow \mathcal{D}_{L^1}, \qquad \mathcal{D}'_{L^\infty} \hookrightarrow \mathcal{S}' \hookrightarrow \mathcal{G}',$$
$$\mathcal{G} \hookrightarrow \mathcal{A}_{L^1} \hookrightarrow \mathcal{D}_{L^1}, \quad \mathcal{D}'_{L^\infty} \hookrightarrow \mathcal{A}'_{L^\infty} \hookrightarrow \mathcal{G}'.$$

It is known that the space $\mathcal{G}(\mathbb{R}^n)$ consists of all infinitely differentiable functions φ on $\mathbb{R}^n$ which can be extended to an entire function on $\mathbb{C}^n$ satisfying

$$|\varphi(x+iy)| \le C\exp(-a|x|^2 + b|y|^2), \qquad x, y \in \mathbb{R}^n, \tag{3.2}$$

for some $a, b, C > 0$ (see [20]).

Definition 3.5. Let $u_j \in \mathcal{G}'(\mathbb{R}^{n_j})$ for $j = 1, 2$. Then the tensor product $u_1 \otimes u_2$ of u_1 and u_2, defined by

$$\langle u_1 \otimes u_2, \varphi(x_1, x_2)\rangle = \langle u_1, \langle u_2, \varphi(x_1, x_2)\rangle\rangle$$

for $\varphi(x_1, x_2) \in \mathcal{G}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, belongs to $\mathcal{G}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.

§4. Stability in distributions and hyperfunctions

In this section we consider the Hyers–Ulam stability for the sine functional equation in the space of distributions and hyperfunctions,

$$u \circ S - u \otimes v - v \otimes u \in \mathcal{D}'_{L^\infty}(\mathbb{R}^{2n}) \text{ [resp. } \mathcal{A}'_{L^\infty}(\mathbb{R}^{2n})], \tag{4.1}$$

where $\otimes$ denotes the tensor product of generalized functions, $S(x, y) = x + y$ for $x, y \in \mathbb{R}^n$, and the pullback $u \circ S$ is defined by

$$\langle u \circ S, \varphi(x, y)\rangle = \left\langle u, \int \varphi(x - y, y)\, dy \right\rangle, \qquad \varphi \in \mathcal{G}(\mathbb{R}^{2n}).$$

In view of Definition 3.2, it is easy to see that if $\varphi_j(x, y) \in \mathcal{G}(\mathbb{R}^{2n})$ is a sequence such that $\varphi_j \to 0$ in $\mathcal{G}(\mathbb{R}^n)$ as $j \to \infty$, then $\int \varphi_j(x - y, y)\, dy \to 0$ in $\mathcal{G}(\mathbb{R}^n)$ as $j \to \infty$. Thus, $u \circ S \in \mathcal{G}'(\mathbb{R}^{2n})$.

For the proof of our theorems we employ the n-dimensional heat kernel $E_t(x)$ given by

$$E_t(x) = (4\pi t)^{-n/2} \exp(-|x|^2/4t), \qquad t > 0.$$

In view of (3.2), we can see that the heat kernel E_t belongs to the Gelfand space $\mathcal{G}(\mathbb{R}^n)$ for each $t > 0$. Thus, for each $u \in \mathcal{G}'(\mathbb{R}^n)$, the convolution $(u * E_t)(x) := \langle u_y, E_t(x-y)\rangle$ is well defined. We call $(u * E_t)(x)$ the *Gauss transform* of u. From now on we denote by $\tilde{u}(x,t)$ the Gauss transform of u. It is well known that $\tilde{u}(x,t)$ is a smooth solution of the heat equation such that $\tilde{u}(\cdot,t) \to u$ in the weak* topology as $t \to 0^+$, i.e.,

$$\langle u, \varphi\rangle = \lim_{t\to 0^+} \int \tilde{u}(x,t)\varphi(x)\,dx$$

for all $\varphi \in \mathcal{G}$.

Example. Let $u(x) = x^\alpha$, $\alpha \in \mathbb{N}_0^n$, $v(x) = e^{c\cdot x}$, $w(x) = a \cdot x e^{c\cdot x}$, $a = (a_1, \dots, a_n)$, $c = (c_1, \dots, c_n) \in \mathbb{C}^n$. Then $u, v, w \in \mathcal{G}'(\mathbb{R}^n)$ and simple calculations show that

$$\begin{aligned}
\tilde{u}(x,t) &= [\xi^\alpha * E_t(\xi)](x) = \alpha! \sum_{0\le 2\gamma\le\alpha} \frac{t^{|\gamma|} x^{\alpha-2\gamma}}{\gamma!(\alpha-2\gamma)!},\\
\tilde{v}(x,t) &= [e^{c\cdot\xi} * E_t(\xi)](x) = e^{c\cdot x + (c_1^2+\cdots+c_n^2)t},\\
\tilde{w}(x,t) &= [a\cdot\xi\, e^{c\cdot\xi} * E_t(\xi)](x) = (a\cdot x + 2a\cdot ct)e^{c\cdot x+(c_1^2+\cdots+c_n^2)t}.
\end{aligned}$$

The proof of Theorem 2.3 of [15] works even when $p = \infty$, i.e., we obtain the following.

Lemma 4.1 ([15]). *The Gauss transform $\tilde{u}(x,t) := (u * E)(x,t)$ of $u \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$ is a smooth solution of the heat equation $(\Delta - \partial/\partial_t)\tilde{u} = 0$ satisfying:*

(i) *There exist constants $C > 0$ and $N \ge 0$ such that*

$$|\tilde{u}(x,t)| \le Ct^{-N} \quad \text{for all } x \in \mathbb{R}^n \text{ and } t > 0. \tag{4.2}$$

(ii) *$\tilde{u}(\cdot,t) \to u$ as $t \to 0^+$ in the sense that for every $\varphi \in \mathcal{D}_{L^1}$,*

$$\langle u, \varphi\rangle = \lim_{t\to 0^+} \int \tilde{u}(x,t)\varphi(x)\,dx.$$

Conversely, every smooth solution $\tilde{u}(x,t)$ of the heat equation satisfying the estimate (4.2) can be uniquely expressed as $\tilde{u}(x,t) = (u * E)(x,t)$ for some $u \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$.

The following lemma is a special case of Theorem 3.5 of [16] when $p = \infty$ (in [16], the space $\mathcal{A}'_{L^\infty}(\mathbb{R}^n)$ is denoted by $\mathcal{B}_{L^\infty}(\mathbb{R}^n)$).

Lemma 4.2 ([16]). *The Gauss transform $\tilde{u}(x,t) := (u * E)(x,t)$ of $u \in \mathcal{A}'_{L^\infty}(\mathbb{R}^n)$ is a smooth solution of the heat equation $(\Delta - \partial/\partial_t)\tilde{u} = 0$ satisfying:*

(i) *For every* $\epsilon > 0$ *there exists a constant* $C_\epsilon > 0$ *such that*

$$|\tilde{u}(x,t)| \le C_\epsilon e^{\epsilon/t} \quad \text{for all } x \in \mathbb{R}^n,\ t > 0. \tag{4.3}$$

(ii) $\tilde{u}(\cdot,t) \to u$ *as* $t \to 0^+$ *in the sense that for every* $\varphi \in \mathcal{A}_{L^1}$,

$$\langle u, \varphi \rangle = \lim_{t\to 0^+} \int \tilde{u}(x,t)\varphi(x)\,dx.$$

Conversely, every smooth solution $\tilde{u}$ *of the heat equation satisfying* (4.3) *can be uniquely expressed as* $\tilde{u}(x,t) = (u * E)(x,t)$ *for some* $u \in \mathcal{A}'_{L^\infty}(\mathbb{R}^n)$.

The following structure theorem for bounded distributions is well known. We refer the reader to [33, Theorem 25 in Chapter 6].

Lemma 4.3. *Every* $u \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$ *can be expressed as*

$$u = \sum_{|\alpha|\le p} \partial^\alpha f_\alpha \tag{4.4}$$

for some $p \in \mathbb{N}_0$ *where* $f_\alpha \in L^\infty(\mathbb{R}^n)$ *for all* $|\alpha| \le p$. *The equality* (4.4) *implies that*

$$\langle u, \varphi \rangle = \sum_{|\alpha|\le p} (-1)^{|\alpha|} \int f_\alpha(x)\partial^\alpha \varphi(x)\,dx$$

for all $\varphi \in \mathcal{D}_{L^1}$.

As a special case of Theorem 3.4 of [16] when $p = \infty$ we obtain the following.

Lemma 4.4 ([16]). *Every* $u \in \mathcal{A}'_{L^\infty}(\mathbb{R}^n)$ *can be expressed as*

$$u = \Big(\sum_{k=0}^{\infty} a_k \Delta^k\Big) g + h \tag{4.5}$$

where Δ *denotes the Laplacian,* g, h *are bounded continuous functions on* $\mathbb{R}^n$ *and* a_k, $k = 0,1,2,\ldots$, *satisfy the following estimates: for every* $L > 0$ *there exists* $C > 0$ *such that*

$$|a_k| \le C L^k / k!^2$$

for all $k = 0,1,2,\ldots$.

The following properties of the heat kernel will be useful; they can be found in [26]. Here we give a slightly modified proof.

Proposition 4.5 ([26]). *For each* $t > 0$, $E_t(\cdot)$ *is an entire function and the following estimate holds: there exists* $C > 0$ *such that for all* $x \in \mathbb{R}^n$ *and* $t > 0$,

$$|\partial_x^\alpha E_t(x)| \le C^{|\alpha|} t^{-(n+|\alpha|)/2} \alpha!^{1/2} \exp(-|x|^2/8t). \tag{4.6}$$

Also for all $x \in \mathbb{R}^n$ *and* $t, s > 0$,

$$(E_t * E_s)(x) := \int E_t(x-y)E_s(y)dy = E_{t+s}(x). \tag{4.7}$$

Proof. The equality (4.7) is proved by the well-known calculation which we omit. We prove (4.6) for $n = 1$. By the Cauchy integral formula we have

$$\frac{d^k}{dx^k}E_t(x) = \frac{k!}{2\pi i}\int_{C_r}\frac{E_t(z)}{(z-x)^{k+1}}\,dz, \tag{4.8}$$

where C_r is the circle of radius r with center at $z = x$. Using (4.8) and the triangle inequality we obtain

$$\begin{aligned} |\partial^k E_t(x)| &\le \frac{k!}{\sqrt{4\pi t}\,r^k}\sup_{z\in C_r}|\exp(-z^2/4t)| \\ &\le \frac{k!}{\sqrt{4\pi t}\,r^k}\sup_{0\le\theta\le 2\pi}\exp\left(\frac{-(x+r\cos\theta)^2 + r^2\sin^2\theta}{4t}\right) \\ &\le \frac{k!}{\sqrt{4\pi t}\,r^k}\exp\left(\frac{-\frac{1}{2}x^2 + r^2}{4t}\right) = \frac{k!}{\sqrt{4\pi t}\,r^k}\exp\left(\frac{r^2}{4t}\right)\exp\left(-\frac{x^2}{8t}\right). \end{aligned} \tag{4.9}$$

The right hand side of (4.9) attains its minimum at $r = \sqrt{2kt}$. Thus, (4.9) reduces to

$$|\partial^k E_t(x)| \le \frac{(e/2)^{k/2}}{\sqrt{4\pi}}k!^{1/2}t^{-(1+k)/2}\exp\left(-\frac{x^2}{8t}\right).$$

The general case is proved in the same manner. □

Now, we state and prove the main theorems.

Theorem 4.6. *Let* $u, v \in \mathcal{G}'(\mathbb{R}^n)$. *Then* (u, v) *satisfies* (4.1) *if and only if* (u, v) *satisfies one of the following*:

(i) $u = 0$ *and* v *is arbitrary,*

(ii) u *and* v *are bounded distributions* [*resp. bounded hyperfunctions*],

(iii) $u = \frac{1}{2\lambda}(e^{c\cdot x} - w_0)$ *and* $v = \frac{1}{2}(e^{c\cdot x} + w_0)$ *for some* $c \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$ *and* $w_0 \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$ [*resp.* $\mathcal{A}'_{L^\infty}(\mathbb{R}^n)$],

(iv) $u = a \cdot x\, e^{ic\cdot x} + w_0$ *and* $v = e^{ic\cdot x}$ *for some* $a \in \mathbb{C}^n$, $c \in \mathbb{R}^n$ *and* $w_0 \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$ [*resp.* $\mathcal{A}'_{L^\infty}(\mathbb{R}^n)$],

(v) $u = a \cdot x\, e^{c\cdot x}$ *and* $v = e^{c\cdot x}$ *for some* $a, c \in \mathbb{C}^n$,

(vi) $u = \frac{1}{2\lambda}(e^{c_1\cdot x} - e^{c_2\cdot x})$ *and* $v = \frac{1}{2}(e^{c_1\cdot x} + e^{c_2\cdot x})$ *for some* $c_1, c_2 \in \mathbb{C}^n$ *and* $\lambda \in \mathbb{C}$.

Proof. We first obtain the following inequality from (4.1): there exist $C, d > 0$ [resp. for every $\epsilon > 0$ there exists $C_\epsilon > 0$] such that

$$|\tilde{u}(x+y,t+s) - \tilde{u}(x,t)\tilde{v}(y,s) - \tilde{v}(x,t)\tilde{u}(y,s)| \le C\left(\frac{1}{t}+\frac{1}{s}\right)^N + d \text{ [resp. } C_\epsilon e^{\epsilon(1/t+1/s)}], \tag{4.10}$$

where $\tilde{u}$, $\tilde{v}$ are the Gauss transforms of u, v, respectively, given in Lemma 4.1.

Convolving with the tensor product $E_t(x)E_s(y)$ of the n-dimensional heat kernels on the left hand side of (4.1), in view of the semigroup property $(E_t * E_s)(x) = E_{t+s}(x)$ of the heat kernel we have

$$\begin{aligned}[(u\circ S) * (E_t(\xi)E_s(\eta))](x,y) &= \left\langle u_\xi, \int E_t(x-\xi+\eta)E_s(y-\eta)\,d\eta \right\rangle \\ &= \langle u_\xi, (E_t * E_s)(x+y-\xi)\rangle = \tilde{u}(x+y,t+s).\end{aligned} \tag{4.11}$$

Similarly,

$$\begin{aligned}[(u\otimes v) * (E_t(\xi)E_s(\eta))](x,y) &= \tilde{u}(x,t)\tilde{v}(y,s), \\ [(v\otimes u) * (E_t(\xi)E_s(\eta))](x,y) &= \tilde{v}(x,t)\tilde{u}(y,s),\end{aligned} \tag{4.12}$$

where $\tilde{u}(x,t), \tilde{v}(x,t)$ are the Gauss transforms of u, v, respectively.

Let $w := u\circ S - u\otimes v - v\otimes u$. Then $w \in \mathcal{D}'_{L^\infty}(\mathbb{R}^{2n})$ [resp. $\mathcal{A}'_{L^\infty}(\mathbb{R}^{2n})$]. First, we suppose that $w \in \mathcal{D}'_{L^\infty}(\mathbb{R}^{2n})$. Using (4.4) and (4.6) we have

$$\begin{aligned}|[w * (E_t(\xi)E_s(\eta))](x,y)| &\le \sum_{|\alpha|\le p} |[\partial^\alpha f_\alpha * (E_t(\xi)E_s(\eta))](x,y)| \\ &\le \sum_{|\alpha|\le p} |[f_\alpha * \partial^\alpha_{\xi,\eta}(E_t(\xi)E_s(\eta))](x,y)| \le \sum_{|\alpha|\le p} \|f_\alpha\|_{L^\infty}\|\partial^\alpha_{\xi,\eta}(E_t(\xi)E_s(\eta))\|_{L^1} \\ &\le C_1 \sum_{|\beta|+|\gamma|\le p} \|\partial^\beta_\xi E_t(\xi)\|_{L^1}\|\partial^\gamma_\eta E_s(\eta)\|_{L^1} \\ &\le C_2 \sum_{|\beta|+|\gamma|\le p} t^{-(n+|\beta|)/2}s^{-(n+|\gamma|)/2} \le C\left(\frac{1}{t}+\frac{1}{s}\right)^N + d,\end{aligned}$$

where $N = n + p/2$ and the constants C and d depend only on p.

Secondly, we suppose that $w \in \mathcal{A}'_{L^\infty}(\mathbb{R}^{2n})$. Then using (4.6) we have

$$\begin{aligned}\|\Delta^k(E_t(\xi)E_s(\eta))\|_{L^1} &\le \sum_{|\alpha|=k} \frac{k!}{\alpha!}\|\partial^{2\alpha}(E_t(\xi)E_s(\eta))\|_{L^1} \\ &\le \sum_{|\beta|+|\gamma|=k} \frac{k!}{\beta!\gamma!}\|\partial^{2\beta}_\xi E_t(\xi)\|_{L^1}\|\partial^{2\gamma}_\eta E_s(\eta)\|_{L^1}\end{aligned}$$

$$\leq \sum_{|\beta|+|\gamma|=k} \frac{k!(2\beta)!^{1/2}(2\gamma)!^{1/2}M^{2k}}{\beta!\,\gamma!} t^{-n/2-|\beta|} s^{-n/2-|\gamma|}$$
$$\leq \sum_{|\beta|+|\gamma|=k} k!(2M)^{2k} t^{-n/2-|\beta|} s^{-n/2-|\gamma|} \leq k!(2\sqrt{n}\,M)^{2k}\,(1/t+1/s)^{n+k}\,.$$

Now, by the structure (4.5) of bounded hyperfunctions together with the growth condition on a_k, $k=0,1,2,\ldots$, we have

$$\begin{aligned}
&|[w*(E_t(\xi)E_s(\eta))](x,y)| \\
&\qquad \leq \sum_{k=0}^{\infty} \|a_k(\Delta^k g)*(E_t(\xi)E_s(\eta))\|_{L^\infty} + \|h*(E_t(\xi)E_s(\eta))\|_{L^\infty} \\
&\qquad \leq \|g\|_{L^\infty} \sum_{k=0}^{\infty} \|a_k \Delta^k (E_t(\xi)E_s(\eta))\|_{L^1} + \|h\|_{L^\infty}\|E_t(\xi)E_s(\eta)\|_{L^1} \\
&\qquad \leq C_1 \sum_{k=0}^{\infty} \frac{1}{k!}(4nM^2L)^k\,(1/t+1/s)^{n+k} + \|h\|_{L^\infty} \\
&\qquad \leq C_2 \sum_{k=0}^{\infty} \frac{1}{k!}\epsilon^k\,(1/t+1/s)^{n+k} + \|h\|_{L^\infty} \leq C_\epsilon\, e^{\epsilon(1/t+1/s)},
\end{aligned}$$

where L is taken so that $4nM^2L<\epsilon$ and the constant C_ϵ depends only on w and ϵ. Thus, we have the inequality (4.10). Replacing f by $\tilde{u}$, g by $\tilde{v}$ and using Corollary 2.7, we obtain one of the following:

(I) $\tilde{u}=0$ and $\tilde{v}$ is arbitrary,

(II) both $\tilde{u}$ and $\tilde{v}$ satisfy (2.3) [resp. (2.4)],

(III) $\tilde{u}(x,t) = \frac{1}{2\lambda}(e^{c\cdot x+bt} - R(x,t))$ and $\tilde{v}(x,t) = \frac{1}{2}(e^{c\cdot x+bt} + R(x,t))$, where $\lambda, b\in\mathbb{C}$, $c\in\mathbb{C}^n$ and R is a function satisfying (2.3) [resp. (2.4)],

(IV) $\tilde{v}(x,t) = e^{ic\cdot x+bt}$ and $\tilde{u}(x,t) = a\cdot x e^{ic\cdot x+bt} + 2\tilde{u}(0,t/2)e^{ic\cdot x+\frac{1}{2}bt} + R(x,t)$ where $c\in\mathbb{R}^n$, $a\in\mathbb{C}^n$, $d\in\mathbb{C}$, and R is a function satisfying (2.3) [resp. (2.4)],

(V) $\tilde{v}(x,t) = e^{c\cdot x+bt}$ and $\tilde{u}(x,t) = (a\cdot x+dt)e^{c\cdot x+bt}$ where $a,c\in\mathbb{C}^n$ and $b,d\in\mathbb{C}$,

(VI) $\tilde{u}(x,t) = \frac{1}{2\lambda}(e^{c_1\cdot x+b_1t} - e^{c_2\cdot x+b_2t})$ and $\tilde{v}(x,t) = \frac{1}{2}(e^{c_1\cdot x+b_1t} - e^{c_2\cdot x+b_2t})$, where $\lambda, b_1, b_2\in\mathbb{C}$ and $c_1, c_2\in\mathbb{C}^n$.

Case (i) is obvious. By Lemma 4.1, case (II) implies (ii). From (III) we have

$$\tilde{v}(x,t) - \lambda\tilde{u}(x,t) = R(x,t). \tag{4.13}$$

Thus, $R(x,t)$ is a solution of the heat equation. Letting $t\to 0^+$ in (III) and using Lemma 4.1 we have $R(x,t)\to w_0$ in the weak* topology for some w_0 in

$\mathcal{D}'_{L^\infty}(\mathbb{R}^n)$ [resp. $\mathcal{A}'_{L^\infty}(\mathbb{R}^n)$], which gives case (iii). Letting $t \to 0^+$ in (V) and (VI) we get (v) and (vi), respectively.

Finally, we prove that (IV) implies (iv). Since $\tilde{v}(x,t) = e^{ic\cdot x+bt}$ in (IV) is a solution of the heat equation, we have $b = -(c_1^2 + \cdots + c_n^2) := -|c|^2$, where $c = (c_1, \ldots, c_n)$. Let

$$R_1(x,t) = \tilde{u}(x,t) - (a\cdot x + 2ia\cdot ct)e^{ic\cdot x-|c|^2 t}. \tag{4.14}$$

Then R_1 is a solution of the heat equation. Also from (IV) we have

$$R_1(x,t) = -2ia\cdot cte^{ic\cdot x-|c|^2 t} + 2\tilde{u}(0,t/2)e^{ic\cdot x+\frac{1}{2}bt} + R(x,t). \tag{4.15}$$

By the continuity of $\tilde{u}$, there exists $M > 0$ such that $|\tilde{u}(0,t+1)| \le M$ for all $t \in [0,1]$. Putting $x = y = 0, s = 1$ in (4.10), dividing the result by $|\tilde{v}(0,1)|$, and using the triangle inequality we obtain

$$\begin{aligned} |\tilde{u}(0,t)| &\le \frac{|\tilde{u}(0,t+1) - \tilde{u}(0,1)e^{-|c|^2 t}| + C(1/t+1)^N + d}{|\tilde{v}(0,1)|} \\ &\le \frac{2M + C(1/t+1)^N + d}{|\tilde{v}(0,1)|} \le C' t^{-N} \text{ [resp. } C_\epsilon e^{\epsilon/t}] \end{aligned} \tag{4.16}$$

for all $t \in (0,1)$. From (4.14)–(4.16) we can see that R_1 is a solution of the heat equation satisfying (4.2) [resp. (4.3)]. By Lemma 4.1 [resp. Lemma 4.2], there exists $w_0 \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$ [resp. $\mathcal{A}'_{L^\infty}(\mathbb{R}^n)$] such that $R_1 \to w_0$ as $t \to 0^+$. Thus, letting $t \to 0^+$ in (IV) and using (4.15) we get (iv). □

Now, we consider the stability condition (4.1) in the space of Schwartz tempered distributions. Recall that the following topological inclusions hold:

$$\begin{aligned} &\mathcal{G} \hookrightarrow \mathcal{S} \hookrightarrow \mathcal{D}_{L^1}, \qquad \mathcal{D}'_{L^\infty} \hookrightarrow \mathcal{S}' \hookrightarrow \mathcal{G}', \\ &\mathcal{G} \hookrightarrow \mathcal{A}_{L^1} \hookrightarrow \mathcal{D}_{L^1}, \quad \mathcal{D}'_{L^\infty} \hookrightarrow \mathcal{A}'_{L^\infty} \hookrightarrow \mathcal{G}'. \end{aligned}$$

In view of these inclusions, if $u, v \in \mathcal{S}'(\mathbb{R}^n)$ it is natural to consider the condition

$$u \circ S - u \otimes v - v \otimes u \in \mathcal{D}'_{L^\infty}(\mathbb{R}^{2n}). \tag{4.17}$$

Theorem 4.7. *Let $u, v \in \mathcal{S}'(\mathbb{R}^n)$. Then (u, v) satisfies (4.17) if and only if (u, v) satisfies one of the following:*

(i) *$u = 0$ and v is arbitrary,*

(ii) *u and v are bounded distributions,*

(iii) *$u = a\cdot x\, e^{ic\cdot x} + w_0$ and $v = e^{ic\cdot x}$ for some $a \in \mathbb{C}^n$, $c \in \mathbb{R}^n$ and $w_0 \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$.*

Proof. It is easy to see that $e^{c\cdot x} \in \mathcal{S}'(\mathbb{R}^n)$ only when $c = ia$ for some $a \in \mathbb{R}^n$. Thus, if $u, v \in \mathcal{S}'(\mathbb{R}^n)$, cases (iii), (vi) in Theorem 4.6 reduce to case (ii), and case (v) is contained in (iv). □

Finally, we discuss the following stability (see [11, 12] for related results):

$$u \circ S - u \otimes v - v \otimes u \in L^\infty(\mathbb{R}^{2n}), \tag{4.18}$$

where $L^\infty(\mathbb{R}^{2n})$ denotes the space of bounded measurable functions on $\mathbb{R}^{2n}$. For the proof we use the following lemma instead of Lemma 4.1.

Lemma 4.8 ([37, p. 122]). *Let $f(x,t)$ be a solution of the heat equation. Then $f(x,t)$ satisfies*

$$|f(x,t)| \le M, \qquad x \in \mathbb{R}^n,\ t \in (0,1),$$

for some $M > 0$, if and only if

$$f(x,t) = (f_0 * E_t)(x) = \int f_0(y) E_t(x-y)\, dy$$

for some bounded measurable function f_0 defined in $\mathbb{R}^n$. In particular, $f(x,t) \to f_0(x)$ for almost every $x \in \mathbb{R}^n$ as $t \to 0^+$.

Following the approach in the proof of Theorem 4.6 we have

$$|\tilde{u}(x+y, t+s) - \tilde{u}(x,t)\tilde{v}(y,s) - \tilde{v}(x,t)\tilde{u}(y,s)| \le C, \tag{4.19}$$

where $\tilde{u}$, $\tilde{v}$ are the Gauss transforms of u, v. Now, using Corollary 2.7 for $N = 0$ and Theorem 4.6 we obtain the following.

Theorem 4.9. *Let $u, v \in \mathcal{G}'(\mathbb{R}^n)$. Then (u,v) satisfies* (4.18) *if and only if (u,v) satisfies one of the following:*

(i) *$u = 0$ and v is arbitrary,*
(ii) *u and v are bounded measurable functions,*
(iii) *$u = \frac{1}{2\lambda}(e^{c\cdot x} - B(x))$ and $v = \frac{1}{2}(e^{c\cdot x} + B(x))$ for some $c \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$ and $B \in L^\infty(\mathbb{R}^n)$,*
(iv) *$u = a\cdot x\, e^{ic\cdot x} + B(x)$ and $v = e^{ic\cdot x}$ for some $a \in \mathbb{C}^n$, $c \in \mathbb{R}^n$ and $B \in L^\infty(\mathbb{R}^n)$,*
(v) *$u = a \cdot x\, e^{c\cdot x}$ and $v = e^{c\cdot x}$ for some $a, c \in \mathbb{C}^n$,*
(vi) *$u = \frac{1}{2\lambda}(e^{c_1\cdot x} - e^{c_2\cdot x})$ and $v = \frac{1}{2}(e^{c_1\cdot x} + e^{c_2\cdot x})$ for some $c_1, c_2 \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$.*

Let f be a Lebesgue measurable function on $\mathbb{R}^n$ satisfying the following condition: for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$|f(x)| \le C_\epsilon e^{\epsilon |x|^2} \tag{4.20}$$

for all $x \in \mathbb{R}^n$. The function satisfying (4.20) is said to be an *infra-exponential function of order* 2. It is easy to see that every infra-exponential function f of order 2 defines an element of $\mathcal{G}'(\mathbb{R}^n)$ via the correspondence

$$\langle f, \varphi \rangle = \int f(x)\varphi(x)\,dx$$

for $\varphi \in \mathcal{G}$. Thus, as a direct consequence of Theorem 4.7 we obtain the following.

Theorem 4.10. *Let f, g be infra-exponential functions of order 2. There exists $C > 0$ such that*

$$\|f(x+y) - f(x)g(y) - g(x)f(y)\|_{L^\infty(\mathbb{R}^{2n})} \le C \tag{4.21}$$

if and only if f, g satisfy one of the following in the almost everywhere sense:

(i) $f = 0$ *and g is arbitrary,*

(ii) f *and g are bounded measurable functions,*

(iii) $f(x) = \frac{1}{2\lambda}(e^{c\cdot x} - B(x))$ *and* $g(x) = \frac{1}{2}(e^{c\cdot x} + B(x))$ *for some* $c \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$ *and* $B \in L^\infty(\mathbb{R}^n)$,

(iv) $f(x) = a \cdot x\, e^{ic\cdot x} + B(x)$, $g(x) = e^{ic\cdot x}$ *for some* $a \in \mathbb{C}^n$, $c \in \mathbb{R}^n$ *and* $B \in L^\infty(\mathbb{R}^n)$,

(v) $f(x) = a \cdot x\, e^{c\cdot x}$ *and* $g(x) = e^{c\cdot x}$ *for some* $a, c \in \mathbb{C}^n$,

(vi) $f(x) = \frac{1}{2\lambda}(e^{c_1\cdot x} - e^{c_2\cdot x})$ *and* $g(x) = \frac{1}{2}(e^{c_1\cdot x} + e^{c_2\cdot x})$ *for some* $c_1, c_2 \in \mathbb{C}^n$ *and* $\lambda \in \mathbb{C}$.

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