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On Nuclearity of C^*-algebras of Fell Bundles over Étale Groupoids

by

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Abstract

We show the nuclearity of the reduced C^*-algebra of a Fell bundle over an étale amenable groupoid, whose fibers over the unit space are all nuclear. We introduce the (minimal) tensor product of a Fell bundle and a C^*-algebra in order to show the uniqueness of tensor product norms.

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§1. Introduction

It is well-known that groupoid C^*-algebras can realize most of C^*-algebras related to dynamical systems—group C^*-algebras, graph algebras, uniform Roe algebras and so on. In particular, C^*-algebras related to discrete dynamical systems can be represented by étale groupoids. On the other hand, the crossed product associated with a group action on a (non-commutative) C^*-algebra is an important construction which is not included above. The Fell bundle over groupoids is a unified construction of groupoid C^*-algebras and crossed products. Hence Fell bundle C^*-algebras contain many important constructions of C^*-algebras.

There is a principle that the amenability of groups, dynamical systems or groupoids corresponds to the nuclearity of C^*-algebras. For example, the following theorem is well-known (see e.g. [2]):

Theorem 1.1. *The following hold:*

(i) *An étale locally compact Hausdorff groupoid G is amenable if and only if $C^*_r(G)$ is nuclear.*

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(ii) *If an amenable discrete group* Γ *acts on a nuclear* C^**-algebra* A*, then the crossed product* $A \rtimes_r \Gamma$ *is nuclear.*

In this paper, we show that if E is a Fell bundle over an amenable étale locally compact Hausdorff groupoid such that every fiber over the unit space is nuclear, then $C^*_r(E)$ is also nuclear (Theorem 4.1). In particular, Theorem 1.1 is included in Theorem 4.1, except for the "only if" implication of (i).

In the case of discrete groups, Quigg [13, Corollary 2.17] has already proved this theorem, and there is also an unpublished paper of Abadie-Vicens [1] related to this theorem. Abadie-Vicens uses the method of tensor products of Fell bundles (over groups), while Quigg's proof relies on the correspondence between Fell bundles and coactions, and a duality theorem for cocrossed products. The proof in this paper is fairly close to the one of Abadie-Vicens, but simpler and based on the method in Brown–Ozawa's book [2].

In connection with this topic, Sims and Williams [17] recently studied when full and reduced Fell bundle C^*-algebras coincide.

§2. Preliminaries

First, we fix some terminology and notation. A groupoid G is said to be *topological* if it has a topology which makes the source map, the range map, the multiplication map, and the inverse map continuous. Let G be a topological groupoid. An open set $S \subset G$ is called a *bisection* if the source and range maps on S are open maps which are homeomorphisms onto their images. A topological groupoid G is said to be *étale* if it has an open base consisting of bisections. Throughout this paper, groupoids are always assumed to be étale, locally compact and Hausdorff.

Let us explain the definition of amenability of étale groupoids, which is a crucial assumption of Theorem 4.1 (cf. [2, Section 5.6]).

Definition 2.1. Let G be an étale locally compact Hausdorff groupoid. A function $h\colon G \to \mathbb{C}$ is *positive definite* if for every $x \in G^{(0)}$ and any finite set $F \subset G_x$ the matrix $[h(\alpha\beta^{-1})]_{\alpha,\beta\in F}$ is positive in the matrix algebra $M_{|F|}(\mathbb{C})$.

Definition 2.2. An étale locally compact Hausdorff groupoid G is *amenable* if it has a net $\{h_i\}$ of compactly supported continuous positive definite functions which converges to 1 uniformly on compact subsets of G, and satisfies $\sup_{\gamma\in G} |h_i(\gamma)| \leq 1$ for every i.

Note that a compactly supported continuous function h on G is positive definite if and only if it is positive in the reduced C^*-algebra of G. This follows from the fact that $C_r^*(G)$ can be represented faithfully on $\bigoplus_{x\in G^{(0)}} \ell^2(G_x)$ (see [2, Section 5.6]).

For tensor products, the usual notation $\otimes$ denotes the minimal tensor product, and the algebraic tensor product is denoted by $\odot$.

We use the term "Hilbert C^*-bimodule" in the following sense:

Definition 2.3. Let A, B be C^*-algebras. An A-B-bimodule V is said to be a *Hilbert A-B-bimodule* if:

(i) V is a left Hilbert A-module and a right Hilbert B-module.

(ii) $\langle ax, y\rangle_r = \langle x, a^*y\rangle_r$ and $_l\langle x, yb\rangle = {_l\langle xb^*, y\rangle}$ for all $a \in A$, $b \in B$ and $x, y \in V$.

(iii) $_l\langle x, y\rangle z = x\langle y, z\rangle_r$ for all $x, y, z \in V$.

Here $_l\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle_r$ denote the A-valued and B-valued inner products, respectively.

Hilbert C^*-bimodules with a fullness condition are called imprimitivity bimodules. For the details about such bimodules, see the book of Raeburn and Williams [14, Chapter 3]. Note that some (but not all) properties of imprimitivity bimodules in [14] are true for Hilbert C^*-bimodules. We will tacitly use such properties. Specifically, the norms of a Hilbert C^*-bimodule induced by the left and right inner products coincide (cf. [14, Proposition 3.11]); pre-Hilbert C^*-bimodules can be completed (cf. [14, Proposition 3.12]); and the exterior tensor product of Hilbert C^*-bimodules can be defined (cf. [14, Proposition 3.36]).

§2.1. Fell bundles

If X is a topological space and $p\colon E \to X$ is a bundle, the fiber over $x \in X$ is denoted by E_x. All Banach spaces are tacitly assumed to be complex. For the definition of Banach bundles, see the book of Fell and Doran [6, Chapter II, Definition 13.4]. We assume that all Banach bundles are *continuous*, i.e., the function $e \mapsto \|e\|$ is continuous. This assumption is natural, but note that some authors assume only upper semicontinuity for Fell bundles (cf. [11],[12]).

Definition 2.4 (cf. [10]). Let G be a groupoid and $p\colon E \to G$ be a Banach bundle. Set $E^{(2)} = \{(e_1, e_2) \in E \times E \mid (p(e_1), p(e_2)) \in G^{(2)}\}$. Then $p\colon E \to G$ is a *Fell bundle* if G has the following two structure maps:

(i) An associative and continuous multiplication map $E^{(2)} \to E$, $(e_1, e_2) \mapsto e_1e_2$, which is submultiplicative with respect to the norms on the fibers, and satisfies $p(e_1e_2) = p(e_1)p(e_2)$.

(ii) A conjugate-linear and antimultiplicative continuous involution $E \to E$, $e \mapsto e^*$, which satisfies $p(e^*) = p(e)^{-1}$, the C^*-condition for norms, and $e^*e \geq 0$ in $E_{s(e)}$ for every $e \in E$.

Note that the condition $e^*e \geq 0$ makes sense since all fibers over the unit space $G^{(0)}$ are C^*-algebras by the previous assumptions. The restriction of E over $G^{(0)}$ is denoted by $E^{(0)}$. From the above remark, $E^{(0)}$ is a continuous bundle of C^*-algebras over $G^{(0)}$ (we use the term "continuous bundle of C^*-algebras" in the sense of Kirchberg–Wassermann [8]). The fiber E_γ over $\gamma \in G$ is a Hilbert $E_{r(\gamma)}$-$E_{s(\gamma)}$-bimodule. Henceforth, we omit the projection p and simply say that E is a Fell bundle.

Let E be a Fell bundle over a groupoid G. We denote by $\Gamma_c(E)$ the space of compactly supported sections, with multiplication and involution defined by

$$f * g(\gamma) = \sum_{\gamma=\alpha\beta} f(\alpha)g(\beta), \qquad f^*(\gamma) = f(\gamma^{-1})^*$$

for $f, g \in \Gamma_c(E)$. We want to complete the $*$-algebra $\Gamma_c(E)$ to produce a reduced Fell bundle C^*-algebra $C^*_r(E)$. If $G = G^{(0)}$, then just complete $\Gamma_c(E)$ in the sup-norm. In this case $C^*_r(E) = \Gamma_0(E)$, the algebra of continuous sections of E which vanish at infinity. In the general case, there is a unique C^*-completion $C^*_r(E)$ of $\Gamma_c(E)$, containing $\Gamma_0(E^{(0)})$, such that the natural restriction map $\Gamma_c(E) \to \Gamma_c(E^{(0)})$ extends to a *faithful* conditional expectation $C^*_r(E) \to \Gamma_0(E^{(0)})$.

We can construct $C^*_r(E)$ explicitly as follows. Consider the right action of $\Gamma_c(E^{(0)})$ on $\Gamma_c(E)$ and the right inner product $\langle f, g\rangle = P(f^*g)$ for $f, g \in \Gamma_c(E)$, where $P\colon \Gamma_c(E) \to \Gamma_c(E^{(0)})$ is the restriction map. Taking completion, we have the right Hilbert $\Gamma_0(E^{(0)})$-module $L^2(E)$. Left multiplication gives a faithful representation of $\Gamma_c(E)$ on $L^2(E)$, and we define $C^*_r(E)$ to be the closure of the image of this representation in $B(L^2(E))$. In this case, P defines a projection of $B(L^2(E))$, and the conditional expectation of $C^*_r(E)$ onto $\Gamma_0(E^{(0)})$ is given by $f \mapsto PfP$.

By definition, Fell bundles over (locally compact) groups are original Fell bundles in the sense of [7]. See [10] for the details on Fell bundles.

§2.2. Construction of bundles

In this section, we introduce a basic technique to construct bundles. When one constructs a Fell bundle E over a groupoid G from the given fibers $\{E_\gamma\}_{\gamma\in G}$, one can specify continuous sections of E instead of topologizing $E = \coprod_\gamma E_\gamma$ directly. The proof of the following proposition is based on [7, Chapter VIII, 2.4].

Proposition 2.5. *Let G be a groupoid. Let $E = \coprod_\gamma E_\gamma$ be an untopologized Fell bundle over G, i.e., every E_γ is a Banach space and E has multiplication and involution satisfying the axioms in Definition 2.4 except continuity. Let A_0 be a $*$-algebra of compactly supported sections of E. Assume that:*

(i) *$\{f(\gamma) \mid f \in A_0\}$ is dense in E_γ for any $\gamma \in G$.*

(ii) *$G \ni \gamma \mapsto \|f(\gamma)\|$ is continuous for any $f \in A_0$.*

Then there exists a unique topology of E which makes E a Fell bundle and all sections in A_0 continuous.

Proof. By [6, Chapter II, Theorem 13.18], E has a unique Banach bundle structure which makes all sections in A_0 continuous. We have to show the continuity of multiplication and involution of E. We will consider multiplication; the proof for involution is almost the same. Let $\{a_i\}$ and $\{b_i\}$ be two nets in E such that $(a_i, b_i) \in E^{(2)}$ and (a_i, b_i) converges to $(a, b) \in E^{(2)}$. It suffices to show that $a_i b_i$ converges to ab.

If $a = 0$, then $\|a_i b_i\| \leq \|a_i\|\|b_i\| \to 0$ and hence $a_i b_i$ converges to $0 = ab$; the same holds if $b = 0$. Thus we may assume $a, b \neq 0$. Let $\alpha_i = p(a_i)$, $\alpha = p(a)$, $\beta_i = p(b_i)$, and $\beta = p(b)$, where $p\colon E \to G$ is the canonical projection. Then $(\alpha_i, \beta_i), (\alpha, \beta) \in G^{(2)}$, and (α_i, β_i) converges to (α, β). Let $\varepsilon > 0$. By assumption (i), there exist $f, g \in A_0$ such that

$$\|f(\alpha) - a\| < \varepsilon/\|b\| \quad \text{and} \quad \|g(\beta) - b\| < \varepsilon/\|f(\alpha)\|.$$

Since A_0 is closed under the convolution product, $f * g$ is a continuous section. Let $\alpha \in S_1 \subset T_1$ and $\beta \in S_2 \subset T_2$ be bisections of open neighborhoods of α and β. Set $S = S_1 S_2$ and $T = T_1 T_2$. Take a continuous function $\varphi\colon G \to \mathbb{C}$ such that $\varphi \equiv 1$ on S and $\varphi \equiv 0$ off T. Then $h(\gamma) = \varphi(\gamma) f * g(\gamma)$ is a continuous section of E by the continuity of scalar multiplication. We have $h(\alpha\beta) = f(\alpha)g(\beta)$, and $h(\alpha_i\beta_i) = f(\alpha_i)g(\beta_i)$ for large i.

Since f and g are continuous, we have

$$\|f(\alpha_i) - a_i\| < \varepsilon/\|b_i\| \quad \text{and} \quad \|g(\beta_i) - b_i\| < \varepsilon/\|f(\alpha_i)\|$$

for large i. Then

$$\|a_i b_i - h(\alpha_i\beta_i)\| = \|a_i b_i - f(\alpha_i)g(\beta_i)\| \leq 2\varepsilon,$$

and also

$$\|ab - h(\alpha\beta)\| \leq 2\varepsilon.$$

Hence, $a_i b_i$ converges to ab by [6, Chapter II, Proposition 13.12]. □

§2.3. Tensor products of continuous bundles of C^*-algebras over locally compact spaces

In this section, we summarize some known results on the tensor product of a continuous bundle of C^*-algebras over a locally compact space with a fixed C^*-algebra.

Definition 2.6. Let E be a continuous bundle of C^*-algebras over a locally compact Hausdorff space X, and let A be a C^*-algebra. Consider the untopologized bundle $E \otimes A = \coprod_{x \in X} E_x \otimes A$ over X. Then $E \otimes A$ is said to be a *continuous bundle* if

$$x \in X \mapsto \Big\| \sum_{i=1}^{n} f_i(x) \otimes a_i \Big\|_{\min}$$

is continuous for every $f_1, \ldots, f_n \in \Gamma_c(E)$ and $a_1, \ldots, a_n \in A$.

If E, F are continuous bundles of C^*-algebras over X and Y, we can consider the notion of continuity of the bundle $E \otimes F$ over $X \times Y$ in the same way. We can see that the above definition is actually a special case, by considering Fell bundles over a single point. However, we need only tensor products with fixed C^*-algebras.

If $E \otimes A$ is a continuous bundle, then one can define a genuine continuous bundle of C^*-algebras over X by specifying that every element of $\Gamma_c(E) \odot A$ defines a continuous section. In this case, we continue to denote by $E \otimes A$ the continuous bundle of C^*-algebras topologized in the above way. Unfortunately, $E \otimes A$ is not always continuous. For example, if A is a *non-exact* C^*-algebra, then one can construct a continuous bundle of C^*-algebras E over $\hat{\mathbb{N}}$ such that $E \otimes A$ is not continuous, where $\hat{\mathbb{N}}$ is the one-point compactification of $\mathbb{N}$ (cf. [8, Section 4]). The continuity of the tensor product is a hard problem and was studied by Kirchberg and Wassermann [8]. However, this problem does not occur when every fiber is a nuclear C^*-algebra.

See [8] for the details of the proof of the following propositions:

Proposition 2.7. *If $E \otimes A$ is a continuous bundle, then $\Gamma_0(E \otimes A) = \Gamma_0(E) \otimes A$.*

Proof. By using Proposition 2.5 we can translate the definition of continuous bundles of C^*-algebras in the sense of Definition 2.4 to that of Kirchberg–Wassermann [8]. Then we can see that the cross-section algebras in the two definitions are the same by the uniqueness part of Proposition 2.5. □

Proposition 2.8 (cf. [8, Section 2]). *Let E be a continuous bundle of C^*-algebras over a locally compact Hausdorff space X such that E_x is nuclear for every $x \in X$. Then $E \otimes A$ is a continuous bundle for any C^*-algebra A.*

Proof. Let $E \otimes_{\max} A$ be the bundle of C^*-algebras whose fibers are $E_x \otimes_{\max} A$. In general, $E \otimes_{\max} A$ is an upper semicontinuous bundle and $E \otimes A$ is a lower semicontinuous bundle for any E and A. If E has nuclear fibers, then $E \otimes_{\max}$ and $E \otimes A$ coincide by definition. Hence it is a continuous bundle. □

Proposition 2.9. *If E_x is nuclear for every $x \in X$, then $\Gamma_0(E)$ is nuclear.*

Proof. If A is a C^*-algebra, then

$$\Gamma_0(E) \otimes_{\max} A = \Gamma_0(E \otimes_{\max} A) = \Gamma_0(E \otimes A) = \Gamma_0(E) \otimes A$$

since every fiber of E is nuclear. Hence $\Gamma_0(E)$ is nuclear. □

§3. Tensor products of Fell bundles with C^*-algebras

Let E be a Fell bundle over a groupoid G, and A be a C^*-algebra. Assume that $E^{(0)} \otimes A$ is a continuous bundle. We will define the tensor product $E \otimes A$.

Let $\gamma \in G$ and $x = s(\gamma), y = r(\gamma)$. Then E_γ is a Hilbert E_y-E_x-bimodule. We regard A as an imprimitive A-A-bimodule in the obvious way. Let $(E \otimes A)_\gamma = E_\gamma \otimes A$, where $E_\gamma \otimes A$ is the exterior tensor product of Hilbert C^*-bimodules (cf. [14, Chapter 3]). Then $(E \otimes A)_\gamma$ is a Hilbert $(E_y \otimes A)$-$(E_x \otimes A)$-bimodule endowed with the inner products

$$\langle e_1 \otimes a_1, e_2 \otimes a_2\rangle_r = e_1^* e_2 \otimes a_1^* a_2, \qquad {}_l\langle e_1 \otimes a_1, e_2 \otimes a_2\rangle = e_1 e_2^* \otimes a_1 a_2^*,$$

for $e_1, e_2 \in E_\gamma$ and $a_1, a_2 \in A$.

Next, we define multiplication and involution on $E \otimes A = \coprod_{\gamma \in G} (E \otimes A)_\gamma$. Let $E \odot A = \coprod_{\gamma \in G} (E \odot A)_\gamma$ and define multiplication and an involution on $E \odot A$ by

$$(e_1 \otimes a_1)(e_2 \otimes a_2) = e_1 e_2 \otimes a_1 a_2, \qquad (e \otimes a)^* = e^* \otimes a^*,$$

for $(e_1, e_2) \in E^{(2)}, e \in E$ and $a_1, a_2, a \in A$. For $s, t \in (E \odot A)_\gamma$, we have

$$\langle s, t\rangle_r = s^* t \quad \text{and} \quad {}_l\langle s, t\rangle = s t^*.$$

Lemma 3.1 (cf. [1, Proposition 3.8]). *If $s, t \in E \odot A$ are composable, then $\|st\| \leq \|s\|\,\|t\|$.*

Proof. We have $\|st\|^2 = \|t^* s^* st\| = \|\langle t, (s^*s)t\rangle_r\| \leq \|t\|\,\|(s^*s)t\| \leq \|t\|^2 \|s\|^2$. □

By Lemma 3.1, the multiplication map $(E_\alpha \odot A) \times (E_\beta \odot A) \to E_{\alpha\beta} \odot A$ for $(\alpha, \beta) \in G^{(2)}$ extends continuously to $(E_\alpha \otimes A) \times (E_\beta \otimes A) \to E_{\alpha\beta} \otimes A$. Therefore, multiplication extends to the whole $E \otimes A$. Similarly, involution extends to the whole $E \otimes A$ since involution is isometric. By the construction, $E \otimes A$ is an untopologized Fell bundle over G as in Proposition 2.5.

Lemma 3.2. *For $f_1, \ldots, f_n \in \Gamma_c(E)$ and $a_1, \ldots, a_n \in A$,*

$$\gamma \in G \mapsto \Big\|\sum_{i=1}^{n} f_i(\gamma) \otimes a_i\Big\|$$

is a continuous function.

Proof. It suffices to show that the above function is continuous on every bisection of G. Let $S \subset G$ be a bisection. Then for $f, g \in \Gamma_c(E)$,

$$x \in s(S) \mapsto f(Sx)^* g(Sx) \in E_x$$

is a continuous section of $E^{(0)}$. Therefore,

$$x \in s(S) \mapsto \Big\| \sum_{i,j=1}^{n} f_i(Sx)^* f_j(Sx) \otimes a_i^* a_j \Big\|$$

is continuous since $E^{(0)} \otimes A$ is a continuous bundle by the assumption. Now it is easy to see that

$$\gamma \in S \mapsto \Big\| \sum_{i=1}^{n} f_i(\gamma) \otimes a_i \Big\| = \Big\| \sum_{i,j=1}^{n} f_i(\gamma)^* f_j(\gamma) \otimes a_i^* a_j \Big\|^{1/2}$$

is continuous. □

Proposition 3.3. *The bundle $E \otimes A$ has a unique Fell bundle structure such that*

$$\gamma \mapsto \sum_{i=1}^{n} f_i(\gamma) \otimes a_i \in (E \otimes A)_\gamma$$

is a continuous section for every $\sum_{i=1}^{n} f_i \otimes a_i \in \Gamma_c(E) \odot A$.

Proof. It can be easily seen that the usual product of $\Gamma_c(E) \odot A$ coincides with the convolution product as sections, and the usual involution with the one as sections. Therefore, by using Lemma 3.2, we can apply Proposition 2.5 to the untopologized Fell bundle $E \otimes A$. □

Next, we investigate the reduced C^*-algebra of $E \otimes A$. From the above construction, we have $(E \otimes A)^{(0)} = E^{(0)} \otimes A$. Since $\Gamma_0(E^{(0)} \otimes A) = \Gamma_0(E^{(0)}) \otimes A$ by Proposition 2.7, $\Gamma_0(E^{(0)}) \otimes A$ is a C^*-subalgebra of $C_r^*(E \otimes A)$ with a canonical faithful conditional expectation $\tilde{P} \colon C_r^*(E \otimes A) \to \Gamma_0(E^{(0)}) \otimes A$.

Proposition 3.4. *The C^*-algebra $C_r^*(E \otimes A)$ is canonically isomorphic to $C_r^*(E) \otimes A$.*

Proof. Let $P \colon C_r^*(E) \to \Gamma_0(E^{(0)})$ be the canonical faithful conditional expectation. Then $P \otimes \mathrm{id} \colon C_r^*(E) \otimes A \to \Gamma_0(E^{(0)}) \otimes A$ is a faithful conditional expectation. We can easily see that $\tilde{P}(f) = P \otimes \mathrm{id}(f)$ for every $f \in \Gamma_c(E) \odot A$. Hence $C_r^*(E \otimes A)$ and $C_r^*(E) \otimes A$ are isomorphic, since these algebras are completions of $\Gamma_c(E) \odot A$ whose norms are defined by $\tilde{P}$ and $P \otimes \mathrm{id}$ using the faithfulness of expectations (take the completion of $\Gamma_c(E) \odot A$ as a Hilbert $\Gamma_0(E^{(0)}) \otimes A$-bimodule using this conditional expectation, and consider representations of these two algebras on it). □

§4. The main theorem

Theorem 4.1. *Let E be a Fell bundle over an étale locally compact Hausdorff groupoid G. If G is amenable, then the following conditions are equivalent:*

(i) *The C^*-algebra $C_r^*(E)$ is nuclear.*
(ii) *The fiber E_x is nuclear for every $x \in G^{(0)}$.*
(iii) *The C^*-algebra $\Gamma_0(E^{(0)})$ is nuclear.*

Since there exists a canonical faithful conditional expectation from $C_r^*(E)$ onto $\Gamma_0(E^{(0)})$, (i) implies (iii). Since E_x is a quotient of $\Gamma_0(E^{(0)})$, (iii) implies (ii). Moreover, (ii) implies (iii) by Proposition 2.9. Therefore, (ii)⇒(i) is the only non-trivial part of this theorem.

The proof we give here is based on the proof of [2, Theorem 5.6.18]. The first lemma is about a general relation between positive definite functions and contractive completely positive (c.c.p.) maps. We think it could be useful for other purposes.

Lemma 4.2. *Let E be a Fell bundle over a groupoid G, and let h be a compactly supported continuous positive definite function on G with $\sup_{\gamma\in G}|h(\gamma)| \le 1$. Then the multiplier map*

$$m_h\colon \Gamma_c(E) \to \Gamma_c(E), \qquad f \mapsto hf,$$

extends to a c.c.p. map on $C_r^*(E)$.

Proof. For $x \in G^{(0)}$, let $V_x = \bigoplus_{\gamma\in G_x} E_\gamma$ (direct sum of right Hilbert E_x-modules) and define a representation $\pi_x\colon C_r^*(E) \to B(V_x)$ by

$$\pi_x(f)\Big(\bigoplus_{\gamma\in G_x} \xi_\gamma\Big) = \bigoplus_{\gamma\in G_x}\Big(\sum_{\beta\in G_x} f(\gamma\beta^{-1})\xi_\beta\Big)$$

for $f \in \Gamma_c(E)$ and $\xi_\gamma \in E_\gamma$ with $\sum_\gamma \xi_\gamma^*\xi_\gamma$ converging in E_x. Then $\{\pi_x\}_{x\in G^{(0)}}$ is a faithful family of representations of $C_r^*(E)$ (cf. [10]). Similarly, $\lambda_x\colon C_r^*(G) \to B(\ell^2(G_x))$ is defined by

$$\lambda_x(f)\xi(\gamma) = \sum_{\beta\in G_x} f(\gamma\beta^{-1})\xi(\beta)$$

for $f \in C_c(G)$ and $\xi \in \ell^2(G_x)$.

Define $\iota\colon V_x \to \ell^2(G_x) \otimes V_x$ by

$$\bigoplus_{\gamma\in G_x} \xi_\gamma \mapsto \sum_{\gamma\in G_x} \delta_\gamma \otimes \xi_\gamma.$$

We can see that ι is adjointable and isometric. The adjoint map ι^* is given by

$$\iota^*\Big(\sum_{\alpha\in G_x}\delta_\alpha\otimes\Big(\bigoplus_{\beta\in G_x}\xi_{\alpha,\beta}\Big)\Big)=\bigoplus_{\alpha\in G_x}\xi_{\alpha,\alpha}$$

for $\xi_{\alpha,\beta}\in E_\beta$. Define $T_x\colon V_x\to\ell^2(G_x)\otimes V_x$ by

$$T_x\Big(\bigoplus_{\gamma\in G_x}\xi_\gamma\Big)=\sum_{\gamma\in G_x}\lambda_x(h)^{1/2}\delta_\gamma\otimes\xi_\gamma.$$

Then T_x is adjointable and contractive since $T_x=(\lambda_x(h)^{1/2}\otimes 1)\circ\iota$. By a simple calculation, $T_x^*(1\otimes\pi_x(f))T_x=\pi_x(m_h(f))$ for every $f\in\Gamma_c(E)$. Therefore, if one identifies $C_r^*(E)$ with $(\bigoplus_x\pi_x)(C_r^*(E))$, the restriction of the c.c.p. map

$$\Phi\colon\prod_x B(V_x)\to\prod_x B(V_x),\qquad \sum_x a_x\mapsto\sum_x T_x^*(1\otimes a_x)T_x,$$

to $C_r^*(E)$ gives the extension of m_h. □

In fact, the condition that h is compactly supported is not necessary, but we need only the case of compactly supported ones.

Let E be a Fell bundle over an amenable groupoid G such that E_x is nuclear for every $x\in G^{(0)}$, and let A be an arbitrary C^*-algebra. Then $E^{(0)}\otimes A$ is continuous by Proposition 2.8.

Lemma 4.3. *Let K be a compact subset of G. Then:*

(i) *There exists a positive constant $C_K>0$ such that for $f_1,\dots,f_n\in\Gamma_c(E)$ supported in K and $a_1,\dots,a_n\in A$, we have*

$$\Big\|\sum_{i=1}^n f_i\otimes a_i\Big\|_{\max}\le C_K\sup_{\gamma\in K}\Big\|\sum_{i=1}^n f_i(\gamma)\otimes a_i\Big\|_{E_\gamma\otimes A}.$$

(ii) *For $f_1,\dots,f_n\in\Gamma_c(E)$ and $a_1,\dots,a_n\in A$, we have*

$$\sup_{\gamma\in K}\Big\|\sum_{i=1}^n f_i(\gamma)\otimes a_i\Big\|_{E_\gamma\otimes A}\le\Big\|\sum_{i=1}^n f_i\otimes a_i\Big\|_{\min}.$$

Here, $\|\cdot\|_{\max}$ and $\|\cdot\|_{\min}$ are the maximal and minimal norms of $C_r^*(E)\odot A$.

Proof. Statement (ii) follows from Proposition 3.4. Indeed, we have

$$\sup_{\gamma\in K}\Big\|\sum_{i=1}^n f_i(\gamma)\otimes a_i\Big\|_{E_\gamma\otimes A}\le\Big\|\sum_{i=1}^n f_i\otimes a_i\Big\|_\infty\le\Big\|\sum_{i=1}^n f_i\otimes a_i\Big\|_{C_r^*(E\otimes A)}$$
$$=\Big\|\sum_{i=1}^n f_i\otimes a_i\Big\|_{\min}.$$

Note that the second inequality holds for general Fell bundles (see [10]).

We now prove (i). By the usual partition of unity argument, we may assume that K is contained in some bisection. Then $f_i^* f_j$ is in $\Gamma_0(E^{(0)})$ for every i, j. Since $\Gamma_0(E^{(0)})$ is nuclear, the restriction of the maximal tensor norm to $\Gamma_0(E^{(0)}) \odot A$ coincides with the minimal tensor norm, and $\Gamma_0(E^{(0)}) \otimes A = \Gamma_0(E^{(0)} \otimes A)$. Therefore,

$$\begin{aligned}\Big\|\sum_{i=1}^{n} f_i \otimes a_i\Big\|_{\max}^2 &= \Big\|\sum_{i,j} f_i^* f_j \otimes a_i^* a_j\Big\|_{\max} = \sup_{x\in G^{(0)}} \Big\|\sum_{i,j} f_i^* f_j(x) \otimes a_i^* a_j\Big\| \\ &= \sup_{\gamma\in G} \Big\|\sum_{i,j} f_i(\gamma)^* f_j(\gamma) \otimes a_i^* a_j\Big\| = \sup_{\gamma\in G} \Big\|\sum_{i} f_i(\gamma) \otimes a_i\Big\|_{E_\gamma\otimes A}^2. \qquad \square\end{aligned}$$

Proof of Theorem 4.1. It suffices to show that the quotient map $Q\colon C_r^*(E) \otimes_{\max} A \to C_r^*(E) \otimes A$ is injective.

For every compact subset K of G, we denote by $\Gamma_K(E)$ the set of continuous sections of E whose support is contained in K. Then $\Gamma_K(E)$ is complete with respect to the sup-norm and $\Gamma_K(E) \odot A$ is a dense subspace of $\Gamma_K(E \otimes A)$ with the sup-norm topology.

Let h be a continuous positive definite function on G supported in a compact subset K of G, and assume $\sup_{\gamma\in G} |h(\gamma)| \leq 1$. Then by Lemma 4.3, the inclusion map $\Gamma_K(E) \odot A \to C_r^*(E) \odot A$ extends to bounded injective maps $\Gamma_K(E \otimes A) \to C_r^*(E) \otimes_{\max} A$ and $\Gamma_K(E \otimes A) \to C_r^*(E) \otimes A$ with closed images. Thus we can regard $\Gamma_K(E \otimes A)$ as a closed subspace of $C_r^*(E) \otimes_{\max} A$ and of $C_r^*(E) \otimes A$. Then the quotient map Q is injective on $\Gamma_K(E \otimes A)$.

By Lemma 4.2, we have the commutative diagram

$$\begin{array}{ccc} C_r^*(E) \otimes_{\max} A & \xrightarrow{\;m_h \otimes_{\max} \mathrm{id}\;} & C_r^*(E) \otimes_{\max} A \\ {\scriptstyle Q}\big\downarrow & & \big\downarrow{\scriptstyle Q} \\ C_r^*(E) \otimes A & \xrightarrow{\;m_h \otimes \mathrm{id}\;} & C_r^*(E) \otimes A \end{array}$$

Let $a \in C_r^*(E) \otimes_{\max} A$ with $Q(a) = 0$. Then we have $(Q \circ (m_h \otimes_{\max} \mathrm{id}))(a) = ((m_h \otimes \mathrm{id}) \circ Q)(a) = 0$. Since the image of $m_h \otimes_{\max} \mathrm{id}$ is contained in $\Gamma_K(E \otimes A)$ and Q is injective on $\Gamma_K(E \otimes A)$, it follows that $(m_h \otimes_{\max} \mathrm{id})(a) = 0$.

Let $h_i \in C_c(G)$ be a net of compactly supported continuous positive definite functions which converges to 1 uniformly on compact subsets of G, and satisfies $\sup_{\gamma\in G} |h_i(\gamma)| \leq 1$ for every i. Then $m_{h_i} \otimes_{\max} \mathrm{id}$ converges to $\mathrm{id}_{C_r^*(E)\otimes_{\max} A}$ in the point-norm topology. Hence, for $a \in \ker Q$, we have

$$a = \lim_i\, (m_{h_i} \otimes_{\max} \mathrm{id})(a) = 0$$

by the above argument. This proves that Q is injective. $\square$

Remark 4.4. Even when considering a non-amenable groupoid G, the C^*-algebra $C_r^*(E)$ often happens to be nuclear for some Fell bundle E over G. For example, if a (non-amenable) discrete group Γ acts amenably on a unital nuclear C^*-algebra A, then the reduced crossed product $A \rtimes_r \Gamma$ is nuclear (see [2, Section 4.3]). The notion of amenable actions on non-commutative C^*-algebras is defined as the existence of a net of functions valued in the base C^*-algebra with some technical conditions. The key point is that these functions induce multiplier maps on the crossed product.

In this connection, an approximation property for Fell bundles over discrete groups is defined and studied by Exel [3]. This approximation property is a sufficient condition for the coincidence of full and reduced Fell bundle C^*-algebras (over groups). Exel's approximation property also requires the existence of a net of functions with some conditions without relevance to multiplier maps. If a group is acting amenably on a C^*-algebra, then the semidirect product bundle has the approximation property.

§5. Examples and applications

§5.1. Crossed products

Definition 5.1 (cf. [10]). Let $q\colon A \to G^{(0)}$ be a continuous bundle of C^*-algebras over $G^{(0)}$. Define $G*A = \{(\gamma, a) \mid s(\gamma) = q(a)\}$. An *action* of G on A is a continuous map $\alpha\colon G * A \to A$, where $\alpha(\gamma, a)$ is denoted by $\alpha_\gamma(a)$, satisfying the following conditions:

(i) $q(\alpha_\gamma(a)) = r(\gamma)$.
(ii) $\alpha_\gamma\colon A_{s(\gamma)} \to A_{r(\gamma)}$ is an isomorphism of C^*-algebras.
(iii) $\alpha_x(a) = a$ for $x \in G^{(0)}$ and $a \in A_x$.
(iv) $\alpha_{\gamma_1\gamma_2}(a) = \alpha_{\gamma_1}(\alpha_{\gamma_2}(a))$.

If there is an action α of a groupoid G on a continuous bundle A of C^*-algebras over $G^{(0)}$, we can define a Fell bundle structure on $G * A$. The topology of $G * A$ is the relative topology of the usual product topology, and the projection $p\colon G * A \to G$ is the projection onto the first coordinate. The norm on $G * A$ is defined by $\|(\gamma, a)\| = \|a\|$. Multiplication and involution are defined by

$$(\gamma_1, a_1)(\gamma_2, a_2) = (\gamma_1\gamma_2, \alpha_{\gamma_2^{-1}}(a_1)a_2), \qquad (\gamma, a)^* = (\gamma^{-1}, \alpha_\gamma(a^*)).$$

$G * A$ is called a *semidirect product bundle*.

A semidirect product bundle is a generalization of usual crossed products by groups to groupoids. If $G = \Gamma$ is a discrete group, then $\Gamma^{(0)} = \{e\}$ and a continuous bundle of C^*-algebras over $\Gamma^{(0)}$ is just a C^*-algebra. In this case, $\Gamma_c(\Gamma * A)$ is identified with the finitely supported A-valued functions on Γ, and multiplication and involution on $\Gamma_c(\Gamma * A)$ coincide with those of crossed products. Therefore, $C_r(\Gamma * A) \cong A \rtimes_r \Gamma$ since both sides are completions of $\Gamma_c(\Gamma * A)$ characterized by the canonical faithful expectation onto A. In the case of crossed products by groups, Theorem 4.1 reduces to Theorem 1.1(ii).

The semidirect product bundle is certainly a proper generalization of crossed products, but it may seem less natural that actions on bundles are used instead of actions of C^*-algebras. Indeed, we can define crossed products by groupoids from actions on a certain class of C^*-algebras—$C_0(X)$-*algebras* (where X is the unit space of the groupoid). Actually, we can describe actions on $C_0(X)$-algebras in the form of bundles, but we need *upper semicontinuous Fell bundles* to handle the general case, which requires only the upper semicontinuity instead of continuity of norms in the axioms of Banach bundles. See [11], [12] for general crossed products and the treatment of upper semicontinuous bundles.

§5.2. Line bundles

A Fell bundle E over a groupoid is called a *Fell line bundle* if each fiber of E is one-dimensional. If X is a locally compact space, then there is only one Fell line bundle over X—the trivial one. For a groupoid G, there is a canonical one-to-one correspondence between Fell line bundles over G and *twists* over G—an extended notion of the second cohomology of groupoids. See [9], [16] for twists and [15] for cohomologies.

Fell line bundles are related to the theory of Cartan subalgebras for C^*-algebras. In the well-known theorem of Feldman–Moore, a von Neumann algebra which has a Cartan subalgebra is described by an equivalence relation and a 2-cocycle (cf. [4], [5]). In the case of C^*-algebras, equivalence relations are replaced by *topologically principal* étale groupoids and 2-cocycles are replaced by twists.

Definition 5.2 (Renault, [16]). An abelian C^*-subalgebra B of a C^*-algebra A is said to be a *Cartan subalgebra* if we have the following conditions:

(i) B contains an approximate unit of A.
(ii) B is a maximal abelian subalgebra of A.
(iii) The normalizer $N(B)$ of B generates A.
(iv) There exists a faithful conditional expectation of A onto B.

Here, $N(B)$ is the set $\{a \in A \mid aBa^*, a^*Ba \subset B\}$.

Renault [16] defined the notion of Cartan subalgebras for C^*-algebras and proved a theorem analogous to the theorem of Feldman–Moore.

Theorem 5.3 (Renault, [16]). *If G is a groupoid which is locally compact Hausdorff, étale, topologically principal and second countable, and E is a Fell line bundle over G, then $\Gamma_0(E^{(0)}) = C_0(G^{(0)})$ is a Cartan subalgebra of $C^*_r(E)$. Conversely, if A is a separable C^*-algebra and B is a Cartan subalgebra of A, then there exists a locally compact Hausdorff, étale, topologically principal, second countable groupoid G, a Fell line bundle E over G, and an isomorphism of $C^*_r(E)$ onto A which carries $C_0(G^{(0)})$ onto B. The groupoid G and the Fell line bundle E are unique up to isomorphism.*

In the above theorem, Fell line bundles are used to fetch some extra information lost in the construction of groupoids. Therefore, for a groupoid G, the reduced C^*-algebras associated to all Fell line bundles over G are expected to share some properties. For example, the nuclearity is one of such properties (we do not have other examples, but there should be a great deal of such properties).

Theorem 5.4. *For a locally compact Hausdorff étale groupoid G, the following conditions are equivalent:*

(i) *G is amenable.*
(ii) *$C^*_r(E)$ is nuclear for all Fell line bundles E over G.*
(iii) *$C^*_r(E)$ is nuclear for some Fell line bundle E over G.*

Proof. Condition (i) implies (ii) by Theorem 4.1, and obviously (ii) implies (iii). We will prove that (iii) implies (i). The proof is based on [2, Theorem 5.6.18].

Let E be a Fell bundle over G such that $C^*_r(E)$ is nuclear. By the definition of amenability, it suffices to show that there exists a net $\{h_i\}$ of compactly supported positive definite functions on G with $\sup_{x\in G^{(0)}} |h(x)| \leq 1$ which converges to 1 uniformly on compact subsets of G. We will construct h_i of the form $h_i = \zeta_i^* * \zeta_i$ with $\|\zeta_i\|_{L^2(G)} \leq 1$, where

$$\|\zeta_i\|_{L^2(G)} = \sup_{x\in G^{(0)}} \Big(\sum_{\gamma\in G_x} |\zeta_i(\gamma)|^2 \Big)^{1/2}.$$

Fix $\varepsilon > 0$ and a compact subset K of G such that $K^{-1} = K$. Note that Banach line bundles are always "locally trivial" by [6, Remark 13.19]. Take relatively compact bisections $U_1, \ldots, U_n, V_1, \ldots, V_n$ of G satisfying $K \subset U_1 \cup \cdots \cup U_n$, $\overline{U_l} \subset V_l$ and such that E is trivial over each V_l. Let $\Phi_l\colon E|_{V_l} \to V_l \times \mathbb{C}$ be an isomorphism of Banach bundles over V_l. Then Φ_l is an isomorphism as complex vector bundles and isometric on each fiber. Let $\tilde{f}_l$ be a continuous function on G such that $\tilde{f}_l$ is identical to 1 on $\overline{U_l}$, $0 \leq \tilde{f}_l \leq 1$ and the support of $\tilde{f}_l$ is contained in V_l. Define a continuous section f_l of E by

$$f_l(\gamma) = \begin{cases} \Phi_l^{-1}(\gamma, \tilde{f}_l(\gamma)) & (\gamma \in V_l), \\ 0_\gamma & (\gamma \notin V_l), \end{cases}$$

where 0_γ is the zero element of E_γ.

Since $C_r^*(E)$ is nuclear by the assumption, there exist c.c.p. maps $\psi\colon C_r^*(E) \to M_n(\mathbb{C})$ and $\varphi\colon M_n(\mathbb{C}) \to C_r^*(E)$ satisfying

$$\|\varphi \circ \psi(f_l) - f_l\| \leq \varepsilon, \quad \|\varphi \circ \psi(f_l^* * f_l) - f_l^* * f_l\| \leq \varepsilon$$

for $l = 1, \dots, n$. Note that $f_l^* * f_l \in \Gamma_c(E^{(0)})$ and $\|f_l^* * f_l\| = \|f_l\|_\infty^{1/2} = 1$.

Let $\{e_{ij}\}_{i,j}$ be the matrix units of $M_n(\mathbb{C})$ and let $b = [\varphi(e_{ij})]_{i,j}^{1/2} \in M_n(C_r^*(E))$. Note that $[\varphi(e_{ij})]_{ij} \geq 0$ as $[e_{ij}]_{ij} \geq 0$ in $M_n(M_n(\mathbb{C}))$. Put $\eta_\varphi = \sum_{j,k} \xi_j \otimes \xi_k \otimes b_{kj} \in \ell_n^2 \otimes \ell_n^2 \otimes C_r^*(E)$, where $\{\xi_j\}_{j=1}^n$ is an orthonormal basis of the n-dimensional Hilbert space ℓ_n^2 (the inner product is assumed to be linear in the second variable). We regard $\ell_n^2 \otimes \ell_n^2 \otimes C_r^*(E)$ as a right Hilbert $C_r^*(E)$-module in the natural way. Then $\langle \eta_\varphi, (a \otimes 1 \otimes 1)\eta_\varphi \rangle = \varphi(a)$ for $a \in M_n(\mathbb{C})$ since

$$\langle \eta_\varphi, (a \otimes 1 \otimes 1)\eta_\varphi \rangle = \sum_{j,k,l} \langle \xi_j, a\xi_l \rangle b_{kj}^* b_{kl} = \sum_{j,l} a_{jl} \varphi(e_{jl}) = \varphi(a).$$

In particular, $\|\eta_\varphi\|^2 = \|\varphi(1)\| \leq 1$.

Put $a_l = \psi(f_l)$. Then

$$\begin{aligned} &|(a_l \otimes 1 \otimes 1)\eta_\varphi - \eta_\varphi f_l|^2 \\ &\quad = |(a_l \otimes 1 \otimes 1)\eta_\varphi|^2 - \langle \eta_\varphi f_l, (a_l \otimes 1 \otimes 1)\eta_\varphi \rangle - \langle (a_l \otimes 1 \otimes 1)\eta_\varphi, \eta_\varphi f_l \rangle + |\eta_\varphi f_l|^2 \\ &\quad = \varphi(a_l^* a_l) - f_l^* \varphi(a_l) - \varphi(a_l^*) f_l + f_l^* f_l \\ &\quad = (\varphi(a_l^* a_l) - f_l^* f_l) + f_l^*(f_l - \varphi(a_l)) + (f_l - \varphi(a_l))^* f_l \leq 3\varepsilon. \end{aligned}$$

Let us take $\eta'_\varphi \in \ell_n^2 \odot \ell_n^2 \odot \Gamma_c(E)$ satisfying $\|\eta_\varphi - \eta'_\varphi\| \leq \varepsilon$ and $\|\eta'_\varphi\| \leq 1$. We write $\eta'_\varphi = \sum_{j,k} \xi_j \otimes \xi_k \otimes \zeta_{kj}$ for some $\zeta_{kj} \in \Gamma_c(E)$. For $\gamma \in G$, define

$$\eta'_\varphi(\gamma) = \sum_{j,k} \xi_j \otimes \xi_k \otimes \zeta_{kj}(\gamma) \in \ell_n^2 \otimes \ell_n^2 \otimes E_\gamma,$$

where $\ell_n^2 \otimes \ell_n^2 \otimes E_\gamma$ is considered as a right Hilbert $E_{s(\gamma)}$-module. Since $E_{s(\gamma)}$ is isomorphic to $\mathbb{C}$, this is just an n^2-dimensional Hilbert space. Put $\zeta(\gamma) = \|\eta'_\varphi(\gamma)\|$. Then ζ is a compactly supported continuous function on G with

$$\begin{aligned} \|\zeta\|_{L^2(G)}^2 &\leq \sup_{x \in G^{(0)}} \sum_{\gamma \in G_x} \sum_{k,j} \|\zeta_{kj}(\gamma)^* \zeta_{kj}(\gamma)\| = \sup_{x \in G^{(0)}} \Big\| \sum_{k,j} \sum_{\gamma \in G_x} \zeta_{kj}(\gamma)^* \zeta_{kj}(\gamma) \Big\| \\ &= \sup_{x \in G^{(0)}} \|\langle \eta'_\varphi, \eta'_\varphi \rangle(x)\| \leq 1. \end{aligned}$$

Note that the equality on the first line is valid since E_x is one-dimensional.

Hence, all we have to show is that $\sup_{\gamma\in K}|1-\zeta^* * \zeta(\gamma)|$ is small. First, we have $\|f_l^* * f_l - \varphi(\psi(f_l)^*\psi(f_l))\| \le 4\varepsilon$ since

$$f_l^* * f_l \approx_{2\varepsilon} \varphi\circ\psi(f_l)^*\varphi\circ\psi(f_l) \le \varphi(\psi(f_l)^*\psi(f_l)) \le \varphi\circ\psi(f_l^* * f_l) \approx_\varepsilon f_l^* * f_l.$$

Hence

$$\begin{aligned} f_l^* * f_l &\approx_{4\varepsilon} \varphi(a_l^* a_l) = \langle (a_l\otimes 1\otimes 1)\eta_\varphi, (a_l\otimes 1\otimes 1)\eta_\varphi\rangle \\ &\approx_{\sqrt{3\varepsilon}} \langle (a_l\otimes 1\otimes 1)\eta_\varphi, \eta_\varphi f_l\rangle \approx_{2\varepsilon} \langle (a_l\otimes 1\otimes 1)\eta'_\varphi, \eta'_\varphi f_l\rangle. \end{aligned}$$

Fix $1\le l\le n$ and $\gamma\in U_l$. Put $x=s(\gamma)$. Since $|f_l(\gamma)|^2$ is a positive element of norm 1, we have $f_l^* * f_l(x) = |f_l(\gamma)|^2 = 1$, where 1 is the unit of the C^*-algebra E_x (which is isomorphic to $\mathbb{C}$). In addition, for $\beta\in G_x$,

$$\eta'_\varphi f_l(\beta) = \sum_{\alpha\in G_x} \eta'_\varphi(\beta\alpha^{-1}) f_l(\alpha) = \eta'_\varphi(\beta\gamma^{-1}) f_l(\gamma).$$

Hence, $\|\eta'_\varphi f_l(\beta)\| \le \|\eta'_\varphi(\beta\gamma^{-1})\|$.

Finally, we have

$$\begin{aligned} 1 = \|f_l^* * f_l(x)\| &\approx \|\langle (a_l\otimes 1\otimes 1)\eta'_\varphi, \eta'_\varphi f_l\rangle(x)\| \\ &= \Big\| \sum_{\beta\in G_x} \langle (a_l\otimes 1\otimes 1)\eta'_\varphi(\beta), \eta'_\varphi f_l(\beta)\rangle \Big\| \le \sum_{\beta\in G_x} \|\eta'_\varphi(\beta)\|\,\|\eta'_\varphi f_l(\beta)\| \\ &\le \sum_{\beta\in G_x} \|\eta'_\varphi(\beta)\|\,\|\eta'_\varphi(\beta\gamma^{-1})\| = \zeta^* * \zeta(\gamma^{-1}) \le 1, \end{aligned}$$

and therefore $1\approx \zeta^* * \zeta(\gamma^{-1})$. This leads to the conclusion, since $K^{-1}=K$ and the error in the above does not depend on γ. □

For the case of discrete groups, exactness is also shared by C^*-algebras of Fell line bundles.

Proposition 5.5. *For a discrete group* Γ*, the following conditions are equivalent:*

(i) Γ *is exact.*
(ii) $C_r^*(E)$ *is exact for all Fell line bundles* E *over* Γ*.*
(iii) $C_r^*(E)$ *is exact for some Fell line bundle* E *over* Γ*.*

Proof. First, we will prove (i) implies (ii). Assume Γ is exact and take an amenable action on a compact space X. Let $G=\Gamma * X$ be the associated transformation groupoid. If $q: E\to\Gamma$ is a Fell bundle, then we can define a Fell bundle over G, denoted by $E*X$, as follows. The space $E*X$ equals $E\times X$ as a topological space, and the projection $p: E*X\to G$ is defined by $p(e,x)=(q(e),x)$ for $(e,x)\in E*X$.

Multiplication and involution are defined by $(e_1, q(e_2)x)(e_2, x) = (e_1e_2, x)$ and $(e, x)^* = (e^*, q(e)x)$. The norm on E is defined by $\|(e, x)\| = \|e\|$. We can easily check that $E * X$ is actually a Fell bundle.

Let E be a line bundle over Γ. Then $E * X$ is a line bundle over the amenable étale groupoid G, and hence $C^*_r(E * X)$ is nuclear by Theorem 5.4. The exactness of $C^*_r(E)$ follows since we can embed $C^*_r(E)$ into $C^*_r(E*X)$. Indeed, for $f \in \Gamma_c(E)$ define a section $\tilde{f}$ of $E * X$ by $\tilde{f}(g, x) = (f(g), x)$ for $(g, x) \in \Gamma * X$. Clearly $\tilde{f}$ is a compactly supported continuous section of $E * X$, and we can see that the map $f \mapsto \tilde{f}$ defines an isometric $*$-homomorphism $C^*_r(E) \to C^*_r(E * X)$.

Next, we will show (iii) implies (i). Let $q : E \to \Gamma$ be a Fell line bundle such that $C^*_r(E)$ is exact. Let $\bar{E}$ be the conjugate bundle of E, i.e., $\bar{E}_g = E_{g^{-1}}$ for $g \in \Gamma$. The element of $\bar{E}_g$ corresponding to $e \in E_{g^{-1}}$ will be denoted by $\bar{e}$. Then $C^*_r(\bar{E})$ equals $C^*_r(E)^{\mathrm{op}}$ and hence it is exact. We do not give the details, but we can define the tensor product bundle $E \otimes \bar{E}$ on $\Gamma \times \Gamma$ as in Section 3. The fiber of the bundle $E \otimes \bar{E}$ over $(g, h) \in \Gamma \times \Gamma$ is $E_g \otimes E_h$, and $C^*_r(E \otimes \bar{E})$ is isomorphic to $C^*_r(E) \otimes C^*_r(E)^{\mathrm{op}}$. Let Δ be the diagonal subgroup of $\Gamma \times \Gamma$, which is isomorphic to Γ. Then the restriction of $E \otimes \bar{E}$ over Δ is isomorphic to the trivial bundle $\Gamma \times \mathbb{C}$ over Γ. The isomorphism is given by

$$(E \otimes \bar{E})|_\Delta \to \Delta \times \mathbb{C}, \qquad e \otimes \bar{f} \mapsto (q(e), ef),$$

where the fiber of E over the unit is identified with $\mathbb{C}$. Hence the subalgebra $C^*_r((E \otimes \bar{E})|_\Delta)$ of $C^*_r(E \otimes \bar{E})$ is isomorphic to $C^*_r(\Gamma)$. Therefore, $C^*_r(\Gamma)$ is exact. $\square$

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