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Positive Radial Solutions for Singular Quasilinear Elliptic Equations in a Ball

by

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Abstract

We establish the existence of positive radial solutions for the boundary value problems

$$\begin{cases} -\Delta_p u = \lambda f(u) & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p \geq 2$, B is the open unit ball $\mathbb{R}^N$, λ is a positive parameter, and $f : (0, \infty) \to \mathbb{R}$ is p-superlinear at ∞ and is allowed to be singular at 0.

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§1. Introduction

In this paper, we study the existence of positive radial solutions for the boundary value problem

$$\begin{cases} -\Delta_p u = \lambda f(u) & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u), p \geq 2$, B is the open unit ball $\mathbb{R}^N, N > 1$, λ is a positive parameter, and $f : (0, \infty) \to \mathbb{R}$.

Thus we shall consider the ODE problem

$$\begin{cases} -(r^{N-1}\phi(u'))' = \lambda r^{N-1} f(u), \quad 0 < r < 1, \\ u'(0) = 0, \quad u(1) = 0, \end{cases} \tag{1.2}$$

where $\phi(z) = |z|^{p-2}z$.

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There is a vast literature on problem (1.1) when f is nonsingular. In the semilinear case, i.e. $p = 2$, problem (1.1) on a general domain has a long history and has been studied extensively (see e.g. [Am2, Li] and the references therein). The quasilinear case, i.e. $p > 1$, has received much attention during the past two decades (see e.g. [GMS, LS1, LS2]). In the case when f is nonsingular and p-superlinear at ∞, i.e., $\lim_{u\to\infty} f(u)/u^{p-1} = \infty$, such problems have been investigated in [ANZ, Am1, AAB, DLN, GS, SW] for $p = 2$, and in [AAP, DMS, DSS, GMS, HS, HSS] for $p > 1$. We are motivated here by the results in [AAP, GMS, HS] concerning the existence of positive solutions to (1.2) when f is p-superlinear, $p > 1$. In [AAP, Theorem 4.6], assuming that $f \in C^1[0,\infty), f(0) < 0$, and there exist constants $\beta > 0$ and $\alpha \in (p, p^*)$, where $p^* = Np/\max(N-p, 0)$, such that

$$\lim_{u\to\infty} \frac{f(u)}{u^{\alpha-1}} = \beta,$$

the authors showed that (1.2) has a positive solution for $\lambda > 0$ small and there exists a connected set of positive solutions of (1.1) bifurcating from infinity at $\lambda = 0$. The result in [AAP] was extended in [HS, Theorems 2.1, 2.2] to include more general nonlinearities and to cover the case when $f(0) > 0$. We refer to [GMS] for related results in the case when $f(0) = 0$.

Problems of the type (1.1) with $p = 2$ and $f(u)$ singular at $u = 0$ arise in the theory of heat conduction in electrical conducting materials, as discussed in [FM]. The model example of this case is

$$\begin{cases} -\Delta u = A/u^\alpha + \gamma u^q & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1.3}$$

where A, γ, α, q are nonnegative constants with $\alpha \in (0,1), q > 0, A \neq 0$. Note that when $\gamma \neq 0$, this problem can be reduced to (1.1) with $f(u) = Au^{-\alpha} + u^p$ and $\lambda = \gamma^{(1+\alpha)/(q+\alpha)}$ via the transformation $v = \gamma^{1/(q+\alpha)}u$.

When $A < 0$ and $q < 1$, the existence of a positive solution to (1.3) for γ large was established in [SY, Zh]. The case when $A > 0$ was discussed in [CRT, FM, LM] for $\gamma = 0$, and in [SY, St] for $\gamma > 0$ and $p \in (0,1)$. For $A > 0, \gamma > 0$ and $q \geq 1$, it was established in [CP] that there exists a constant $\tilde{\lambda} > 0$ such that (1.3) has a positive solution for $\lambda < \tilde{\lambda}$ and no solution for $\lambda > \tilde{\lambda}$. The case when $f(u)$ is bounded away from 0 and $\lim_{u\to\infty} f(u)/u^q \in (0,\infty)$ for some $q \in (1, 2^*)$, was considered in [HKS], in which the authors showed the existence of a constant $\tilde{\lambda} > 0$ such that (1.1) with $p = 2$ has at least two positive radial solutions for $\lambda < \tilde{\lambda}$, at least one for $\lambda = \tilde{\lambda}$, and none for $\lambda > \tilde{\lambda}$.

In this paper, we are interested in positive radial solutions of the problem (1.1) for $p \geq 2$ when f is p-superlinear at ∞ and is allowed to be singular at 0. We shall consider both cases when $\lim_{u\to 0^+} f(u) > 0$ and $\lim_{u\to 0^+} f(u) < 0$. Problems of this kind appear in the the study of chemical reactions, thin films, and non-Newtonian fluids [AA, Di, DHM, DMO, HM]. Our results provide extensions of the results in [AAP, HS] to the singular case, and the results in [HKS] to the case $p \geq 2$ with more general nonlinearities $f(u)$. In particular, the existence result in Theorem 2.1 below deals with the situation when f is p-superlinear at ∞ and $\lim_{u\to 0^+} f(u) = -\infty$, which occurs in some chemical reactions (see [Di, DHM, DMO]) and has not been considered in the literature to our knowledge.

To be more precise, we shall prove in the case $\lim_{u\to 0^+} f(u) < 0$ that problem (1.2) has a positive, decreasing solution u_λ for λ small, and $u_\lambda \to \infty$ uniformly on compact subsets of $[0, 1)$ as $\lambda \to 0$.

In the case $\lim_{u\to 0^+} f(u) > 0$, we show the existence of a positive number λ^* such that (1.2) has at least two positive solutions for $\lambda < \lambda^*$, at least one for $\lambda = \lambda^*$, and none for $\lambda > \lambda^*$.

In particular, our results when applied to the model cases

$$\begin{cases} -\Delta_p u = \lambda(-1/u^\alpha + u^q(\ln(1+u))^r) & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1.4}$$

and

$$\begin{cases} -\Delta_p u = \lambda(1/u^\alpha + u^q(\ln(1+u))^r) & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1.5}$$

where $\alpha \in [0, 1)$, $r \geq 0, q \in (p-1, p^* - 1)$, give the existence of a positive radial solution to (1.4) for λ small, and the existence of a constant $\lambda^* > 0$ such that (1.5) has at least two positive radial solutions for $\lambda < \lambda^*$, at least one for $\lambda = \lambda^*$, and none for $\lambda > \lambda^*$.

Our proofs depend on degree theory and sup- and supersolutions approach as in [HS]. However, the proofs in [HS] do not carry over to the singular case since the compact operator introduced in [HS] is not defined on $C[0, 1]$ in that case. To overcome this, we come up with a modified problem whose solutions are fixed points of a compact operator in $C[0, 1]$ and then show that these solutions are in fact positive solutions of the original problem.

§2. Main results

We shall make the following assumptions:

(A.1) $f : (0, \infty) \to \mathbb{R}$ is continuous and

$$\lim_{x\to\infty} \frac{f(x)}{x^{p-1}} = \infty.$$

(A.2) $N \liminf_{x\to\infty} \dfrac{F(x)}{xf(x)} > \max\left(\dfrac{N}{p} - 1, 0\right)$, where $F(x) = \int_0^x f(t)\,dt$.

(A.3) There exists a constant $\alpha \in [0,1)$ such that

$$\limsup_{x\to 0^+} x^\alpha |f(x)| < \infty.$$

(A.4) $f > 0$ on $(0,\infty)$ and there exist constants $B > 0$ and $\beta \in [0,1)$ such that

$$\lim_{x\to 0^+} x^\beta f(x) = B.$$

By a *positive solution* of (1.2), we mean a function $u \in C^1[0,1]$ with $u > 0$ on $[0,1)$ that satisfies (1.2).

Our main results are:

Theorem 2.1. *Let* (A.1)–(A.3) *hold. Then there exists a constant* $\lambda_0 > 0$ *such that* (1.2) *has a positive, decreasing solution* u_λ *for* $\lambda \in (0,\lambda_0)$ *with* $\|u_\lambda\|_\infty \to \infty$ *as* $\lambda \to 0$. *Furthermore, there exists a function* $L : \mathbb{R}^+ \to \mathbb{R}$ *with* $\lim_{d\to\infty} L(d) = \infty$ *such that*

$$u_\lambda(r) \geq L(\|u_\lambda\|_\infty)(1-r) \qquad \textit{for } r \in [0,1).$$

Theorem 2.2. *Let* (A.1)–(A.4) *hold. Then there exists a positive constant* λ^* *such that* (1.2) *has at least two positive solutions for* $\lambda \in (0,\lambda^*)$, *at least one for* $\lambda = \lambda^*$, *and none for* $\lambda > \lambda^*$.

Remark 2.3. (i) Theorems 2.1 and 2.2 extend Theorems 2.1 and 3.1 of [HS], and Theorem 4.6 of [AAP], to the singular case. Theorem 2.2 with $p = 2$ extends Theorem 1 of [HKS] to nonlinearities $f(u)$ that do not behave like u^q at ∞.

(ii) When f is nonsingular, condition (A.2) is satisfied under the following assumption introduced in [GMS]:

(A.2)′ There exists a constant $\theta \in (0,1)$ such that

$$N \liminf_{x\to\infty} \frac{F(\theta x)}{x f_s(x)} > \max\left(\frac{N}{p} - 1, 0\right), \quad \text{where} \quad f_s(x) = \sup_{0\le t\le x} f(t).$$

It was shown in [GMS] that when f is nondecreasing, (A.2)′ is equivalent to the following condition given in [TH]:

(A) There exists a constant $\theta \in (0,1)$ such that

$$NF(\theta x) - \frac{N-p}{p} x f(x) \geq 0 \quad \text{for } x \text{ large}.$$

§3. Preliminary results

Let $\psi(r) = 1 - r$. The following lemma is an extension of Lemma 2.2 of [HS] to the singular case.

Lemma 3.1. *Let ζ be a nonnegative number and let u be the solution of*

$$\begin{cases} -(r^{N-1}\phi(u'))' = \lambda r^{N-1}k(r), & 0 < r < 1, \\ u'(0) = 0, \quad u(1) = \zeta, \end{cases} \tag{3.1}$$

where $k \geq -m\psi^{-\alpha}$ on $(0,1)$ for some constants $m > 0$, $\alpha \in (0,1)$. Then

(i) $u' \leq \phi^{-1}(\lambda m_1)$,
(ii) $u(t) \geq u(s) - \phi^{-1}(\lambda m_1)$ *for* $0 \leq t \leq s \leq 1$,
(iii) $t^{N-1}\phi(u'(t)) \geq s^{N-1}\phi(u'(s)) - \lambda m_1$ *for* $0 \leq t \leq s \leq 1$, *where* $m_1 = m(1-\alpha)^{-1}$.

Proof. Let u be a solution of (3.1). By integrating, we obtain

$$\begin{aligned} u'(r) = -\phi^{-1}\left(\frac{\lambda}{r^{N-1}}\int_0^r \tau^{N-1}k(\tau)\,d\tau\right) &\leq \phi^{-1}\left(\frac{\lambda m}{r^{N-1}}\int_0^r \tau^{N-1}\psi^{-\alpha}\,d\tau\right) \\ \leq \phi^{-1}\left(\lambda m\int_0^r \psi^{-\alpha}d\tau\right) &\leq \phi^{-1}(\lambda m_1) \end{aligned}$$

for $r \in (0,1)$, i.e. (i) holds. Integrating this inequality on (t,s), $t < s$, gives

$$u(s) - u(t) \leq \phi^{-1}(\lambda m_1)(s-t),$$

which implies (ii). Finally, integrating the equation in (3.1) on (t,s), we obtain (iii). □

Lemma 3.2 ([HW]). *Let $q > 1$. Then there exists a constant $\nu \in (0,1)$ such that for each $g \in L^q(0,1)$, the problem*

$$\begin{cases} -(r^{N-1}\phi(u'))' = r^{N-1}g, & 0 < r < 1, \\ u'(0) = 0, \quad u(1) = 0, \end{cases}$$

has a unique solution $u \equiv Tg \in C^{1,\nu}[0,1]$. Furthermore, there exists a constant $C > 0$ independent of g such that

$$|u|_{1,\nu} \leq C\|g\|_q^{1/(p-1)},$$

and the operator $T : L^q(0,1) \to C^1[0,1]$ is compact.

Define

$$g(x) = \begin{cases} f(x) & \text{if } 0 < x \le 1, \\ f(1) & \text{if } x > 1, \end{cases} \tag{3.2}$$

$$h(x) = \begin{cases} 0 & \text{if } 0 < x \le 1, \\ f(x) - f(1) & \text{if } x > 1, \end{cases} \tag{3.3}$$

and $h(x) = 0$ if $x \le 0$. Then h is continuous, bounded below on $\mathbb{R}$ and $f = g + h$ on $(0, \infty)$. Using (A.2), it is easily seen that

$$N \liminf_{x\to\infty} \frac{H(x)}{xh(x)} > \max\left(\frac{N}{p} - 1, 0\right), \tag{3.4}$$

where $H(x) = \int_0^x h(t)\,dt$.

Lemma 3.3. (i) *There exist positive constants C, C_1, a, δ with*

$$N/p > a > N/p - 1$$

such that

$$CH(x)^{a/N} \le x, \qquad h(x) \le C_1 H(x)^{1-a/N}$$

and

$$NH(x) - axh(x) \ge \delta H(x)$$

for $x \gg 1$.

(ii) *For each $\theta \in (0,1)$, there exists a constant b_θ such that*

$$H(\theta x) \ge b_\theta H(x)$$

for $x \gg 1$. Furthermore, $b_\theta \to 1$ as $\theta \to 1$.

Proof. In view of (3.4), there exist positive constants $a, \tilde{a}$ such that

$$N \liminf_{x\to\infty} \frac{H(x)}{xh(x)} > \tilde{a} > a > \max\left(\frac{N}{p} - 1, 0\right).$$

Hence

$$H(x) \ge \frac{\tilde{a}}{N} xh(x) \qquad \text{for } x \gg 1, \tag{3.5}$$

which implies

$$NH(x) - axh(x) \ge N\left(1 - \frac{a}{\tilde{a}}\right)H(x)$$

and

$$H'(x) \le \frac{N}{ax} H(x)$$

for $x \gg 1$. Solving this differential inequality gives

$$H(x) \le C_0 x^{N/a} \quad \text{for } x \gg 1,$$

and so $x \ge (H(x)/C_0)^{a/N}$ for $x \gg 1$. Note that $p < N/a$ since $\lim_{x\to\infty} H(x)/x^p = \infty$. Hence

$$h(x) \le \frac{NH(x)}{ax} \le C_1 H(x)^{1-a/N}$$

for $x \gg 1$ and (i) follows. Next, fix $\theta \in (0,1)$. By (3.5),

$$\int_{\theta x}^{x} h(t)\,dt = \int_{\theta x}^{x} \frac{th(t)}{t}\,dt \le \frac{N}{\theta a x}\int_{\theta x}^{x} H(t)\,dt \le \frac{N(1-\theta)}{\theta a} H(x)$$

for $x \gg 1$, where we have used the fact that $H(x)$ is increasing for large x. Hence

$$H(\theta x) = H(x) - \int_{\theta x}^{x} h(t)\,dt \ge b_\theta H(x)$$

for $x \gg 1$, where $b_\theta = 1 - \frac{N(1-\theta)}{\theta a}$. □

§4. Abstract setting and a priori estimates

Let $\lambda > 0$. For $v \in C[0,1]$, define $S_\lambda v = \lambda\,(g(\max(v,\psi)) + h(v))$, where g and h are defined by (3.2) and (3.3) respectively. By (A.3), there exists a constant $c_0 > 0$ such that

$$|g(x)| \le \frac{c_0}{x^\alpha} + |f(1)| \quad \text{for all } x > 0.$$

In particular,

$$|g(\max(v,\psi))| \le \frac{c_1}{\psi^\alpha}, \tag{4.1}$$

where $c_1 = c_0 + |f(1)|$. This, together with the Lebesgue Dominated Convergence Theorem, implies that $S_\lambda : C[0,1] \to L^q(0,1)$ is continuous and maps bounded sets into bounded sets, where $1 < q < 1/\alpha$.

Let $A_\lambda v = u$, where u is the solution of

$$\begin{cases} -(r^{N-1}\phi(u'))' = \lambda r^{N-1}(g(\max(v,\psi)) + h(v)), & 0 < r < 1,\\ u'(0) = 0, \quad u(1) = 0. \end{cases} \tag{4.2}$$

Since $A_\lambda = T \circ S_\lambda$, where T is defined in Lemma 3.2, it follows that $A_\lambda : C[0,1] \to C[0,1]$ is a compact operator.

Lemma 4.1. *There exists a constant $\bar\lambda > 0$ such that for each $\lambda \in (0,\bar\lambda)$, there exists a positive constant r_λ with $\lim_{\lambda\to 0} r_\lambda = \infty$ such that*

$$u = \theta A_\lambda u,\ \theta \in (0,1) \ \Rightarrow\ \|u\|_\infty \ne r_\lambda.$$

Proof. Let u satisfy $u = \theta A_\lambda u$ for some $\theta \in (0,1)$. Then

$$u(r) = \theta \int_r^1 \phi^{-1}\left(\frac{\lambda}{s^{N-1}} \int_0^s \tau^{N-1}(g(\max(u,\psi)) + h(u))\, d\tau\right) ds,$$

which, together with (4.1), implies

$$\begin{aligned}|u(r)| &\le \int_r^1 \phi^{-1}\left(\frac{\lambda}{s^{N-1}} \int_0^s \tau^{N-1}\left(\frac{c_1}{\psi^\alpha} + h_s(\|u\|_\infty)\right) d\tau\right) ds \\ &\le \phi^{-1}(\lambda c_2 + \lambda h_s(\|u\|_\infty))\end{aligned}$$

for $r \in (0,1)$, where $c_2 = c_1(1-\alpha)^{-1}$ and $h_s(t) = \sup_{x\in[0,t]} |h(x)|$.

Hence

$$\phi(\|u\|_\infty) \le \lambda(c_2 + h_s(\|u\|_\infty)). \tag{4.3}$$

Let $\bar{\lambda} = \frac{1}{2(c_2 + h_s(1))}$ and $\lambda \in (0,\bar{\lambda})$. Then

$$c_2 + h_s(1) = \frac{1}{2\bar{\lambda}} < \frac{1}{2\lambda}.$$

Since $\lim_{x\to\infty} \frac{c_2 + h_s(x)}{\phi(x)} = \infty$, there exists a constant $r_\lambda > 1$ such that

$$\frac{c_2 + h_s(r_\lambda)}{\phi(r_\lambda)} = \frac{1}{2\lambda}. \tag{4.4}$$

Clearly $\lim_{\lambda\to 0} r_\lambda = \infty$, and from (4.3) and (4.4), we see that $\|u\|_\infty \neq r_\lambda$. □

Lemma 4.2. *For each $\lambda > 0$, there exists a constant $\zeta_\lambda > 0$ such that*

$$u = A_\lambda u + \zeta,\ \zeta \ge 0 \ \Rightarrow\ \zeta \le \zeta_\lambda.$$

Proof. Let u satisfy $u = A_\lambda u + \zeta$, where $\zeta \ge 0$ and $\lambda > 0$. Then

$$\begin{cases} -(r^{N-1}|u'|^{p-2}u')' = \lambda r^{N-1}(g(\max(u,\psi)) + h(u)), & 0 < r < 1, \\ u'(0) = 0, \quad u(1) = \zeta. \end{cases}$$

Let $\lambda_1 > 0$ be the first eigenvalue of $-\Delta_p$ on the unit ball with Dirichlet boundary conditions, and let ϕ_1 be the corresponding normalized positive radial eigenfunction, i.e. $\|\phi_1\|_\infty = 1$, $\phi_1 > 0$ in $[0,1)$, and

$$\begin{cases} -(r^{N-1}|\phi_1'|^{p-2}\phi_1')' = \lambda_1 r^{N-1}\phi_1^{p-1}, & 0 < r < 1, \\ \phi_1'(0) = 0, \quad \phi_1(1) = 0. \end{cases}$$

Since there exists a constant $m > 0$ such that

$$g(\max(v,\psi)) + h(v) \ge -\frac{c_1}{\psi^\alpha} + h(v) \ge -\frac{m}{\psi^\alpha}$$

for all $v \in C[0,1]$, Lemma 3.1(ii) implies

$$u(r) \geq \zeta - \phi^{-1}(\lambda m_1), \quad \text{where} \quad m_1 = m(1-\alpha)^{-1}.$$

Choose ζ_λ so that $\zeta_\lambda > \max\{2\phi^{-1}(\lambda m_1), 2\}$ and

$$\frac{f(x)}{x^{p-1}} > \frac{2\lambda_1}{\lambda} \quad \text{for } x > \frac{\zeta_\lambda}{2}.$$

We claim that $\zeta \leq \zeta_\lambda$. Suppose $\zeta > \zeta_\lambda$ and let $\tilde{u} = u - \zeta$.

Since

$$u(r) \geq \zeta_\lambda/2 > \psi \quad \text{for } r \in (0,1),$$

it follows that

$$\begin{cases} -(r^{N-1}|\tilde{u}'|^{p-2}\tilde{u}')' = \lambda r^{N-1} f(u) \geq 2\lambda_1 r^{N-1}(\tilde{u}+\zeta)^{p-1} & \text{in } (0,1), \\ \tilde{u}'(0) = 0, \quad \tilde{u}(1) = 0. \end{cases}$$

By the strong maximum principle, $\tilde{u} > 0$ in $[0,1)$ and $\tilde{u}'(1) < 0$. Let c be largest such that $\tilde{u} \geq c\phi_1$ in $[0,1)$. Then $c > 0$ and

$$-(r^{N-1}|\tilde{u}'|^{p-2}\tilde{u}')' \geq 2\lambda_1 r^{N-1}(c\phi_1)^{p-1} \quad \text{in } (0,1),$$

and the weak comparison principle implies $\tilde{u} \geq 2^{1/(p-1)}c\phi_1$ in $[0,1)$, a contradiction with the choice of c. Thus $\zeta \leq \zeta_\lambda$, as claimed. □

Lemma 4.3. *Let $\lambda < \bar{\lambda}$ and let u satisfy*

$$u = A_\lambda u + \zeta$$

for some $\zeta \geq 0$. Then there exists a positive constant $C_{\bar{\lambda}}$ such that

$$\|u\|_\infty = u(0) \quad \textit{whenever } \|u\|_\infty > C_{\bar{\lambda}}.$$

Proof. Suppose $\|u\|_\infty \equiv d = |u(r_1)|$ for some $r_1 \in (0,1)$. By Lemma 3.1(ii),

$$u(r_1) \geq -\phi^{-1}(\lambda m_1),$$

and so $u(r_1) > 0$ if $d > 2\phi^{-1}(\bar{\lambda} m_1)$. For such d,

$$u(r) \geq u(r_1) - \phi^{-1}(\bar{\lambda} m_1) \geq d/2$$

for $r \in (0, r_1)$. By integrating and using (4.1), we obtain

$$\begin{aligned} -u'(r) &= \phi^{-1}\left(\frac{\lambda}{r^{N-1}}\left(\int_0^r \tau^{N-1}(g(\max(u,\psi)) + h(u)\right) d\tau\right) \\ &\geq \phi^{-1}\left(\frac{\lambda}{r^{N-1}}\int_0^r \tau^{N-1}\left(-\frac{c_1}{\psi^\alpha} + h(u)\right) d\tau\right) \\ &\geq \phi^{-1}\left(\lambda r\left\{-\frac{c_1}{1-\alpha} + \frac{1}{N}h_i\left(\frac{d}{2}\right)\right\}\right) > 0 \end{aligned}$$

for $r \in (0, r_1)$, where $h_i(t) = \inf_{x \geq t} h(x)$, provided that $d \gg 1$. Here we have used the fact that

$$r^{-N} \int_0^r \frac{\tau^{N-1}}{(1-\tau)^\alpha} \, d\tau \leq (1-\alpha)^{-1}$$

for $r \in (0,1)$, and $h_i(t) \to \infty$ as $t \to \infty$. Thus u is decreasing on $(0, r_1)$ and so $u(0) > u(r_1)$, a contradiction. ☐

Lemma 4.4. *Let $\lambda < \bar{\lambda}$ and $\zeta_0 > 0$. Suppose u satisfies*

$$u = A_\lambda u + \zeta$$

for some $0 \leq \zeta \leq \zeta_0$. Then:

(i) *There exists a function $L : \mathbb{R}^+ \to \mathbb{R}$ depending on ζ_0 and $\bar{\lambda}$ with $\lim_{d \to \infty} L(d) = \infty$ such that*

$$u(r) \geq L(\|u\|_\infty)(1-r) \quad \text{for } r \in (0,1).$$

(ii) *There exists a constant $\bar{R} > 0$ depending on ζ_0 and $\bar{\lambda}$ such that u is decreasing on $(0,1)$ if $\|u\|_\infty > \bar{R}$.*

(iii) *If $\lambda > \underline{\lambda} > 0$ then there exists a constant $R > 0$ depending on $\underline{\lambda}, \bar{\lambda}, \zeta_0$ such that $\|u\|_\infty < R$.*

Proof. Note that

$$(4.5) \qquad \begin{cases} -(r^{N-1}|u'|^{p-2}u')' = \lambda r^{N-1}(g(\max(u,\psi)) + h(u)), & 0 < r < 1, \\ u'(0) = 0, \quad u(1) = \zeta. \end{cases}$$

Multiplying the equation in (4.5) by ru' gives

$$(4.6) \qquad \left(r^N \left(1 - \frac{1}{p}\right) |u'|^p + \lambda r^N H(u) \right)' = -\lambda r^N g(\max(u,\psi))u' + \lambda r^{N-1} N H(u) + r^{N-1}\left(1 - \frac{N}{p}\right)|u'|^p.$$

Next, multiplying the equation in (4.5) by au, where a is given by Lemma 3.3(i), we obtain

$$(4.7) \quad (ar^{N-1}|u'|^{p-2}u'u)' = -\lambda a r^{N-1} g(\max(u,\psi))u - \lambda r^{N-1} a u h(u) + r^{N-1} a |u'|^p.$$

Adding (4.6) and (4.7) yields

$$(4.8) \qquad \psi'(r) = r^{N-1}\left(a + 1 - \frac{N}{p}\right)|u'|^p + \lambda r^{N-1}(NH(u) - auh(u)) - \lambda r^N g(\max(u,\psi))u' - \lambda a r^{N-1} g(\max(u,\psi))u,$$

where $\psi(r) = r^N(1 - 1/p)|u'|^p + \lambda r^N H(u) + a r^{N-1}|u'|^{p-2}u'u$.

In what follows, we shall denote by K_i, $i = 0, 1, \dots,$ positive constants independent of u.

By Lemma 3.3(i), there exist constants $\delta, K_0 > 0$ such that

$$NH(x) - axh(x) \geq \delta H(x) - K_0 \tag{4.9}$$

for all $x \in \mathbb{R}$. Hence

$$\begin{aligned}\psi'(r) \geq \lambda\delta r^{N-1}H(u) - \lambda r^N g(\max(u,\psi))u' \\ - \lambda a r^{N-1} g(\max(u,\psi))u - \lambda K_0\end{aligned} \tag{4.10}$$

for $r \in (0,1)$. In view of Lemma 3.3(ii), there exists $\theta \in (0,1)$ such that

$$H(\theta x) \geq (1/2)H(x) \quad \text{for } x \gg 1. \tag{4.11}$$

Suppose $\|u\|_\infty = d \gg 1$. Then Lemma 4.3 implies $\|u\|_\infty = u(0)$. Let $\bar\theta \in (\theta, 1)$ and $r_0 \in (0,1)$ be such that $u(r_0) = \bar\theta d$. Note that r_0 exists since $u(0) > \bar\theta d$ and $u(1) = \zeta \leq \zeta_0 < \bar\theta d$ for large d.

By Lemma 3.1(ii),

$$u(r) \geq u(r_0) - \phi^{-1}(\bar\lambda m_1) \geq \theta d \tag{4.12}$$

for $r < r_0$. From (4.11) and (4.12), for $r > r_0$ we obtain

$$\begin{aligned}\lambda\delta\int_0^r s^{N-1}H(u)\,ds &\geq \lambda\delta\int_0^{r_0} s^{N-1}H(u)\,ds - \lambda K_1 \\ &\geq \frac{\lambda\delta r_0^N}{N}H(\theta d) - \lambda K_1 \\ &\geq \frac{\lambda\delta r_0^N}{2N}H(d) - \lambda K_1.\end{aligned} \tag{4.13}$$

Integrating (4.10) on $(0,r)$, where $r \in (r_0, 1)$, and using (4.13), we obtain

$$\begin{aligned}\psi(r) \geq \frac{\lambda\delta r_0^N}{2N}H(d) - \lambda\int_0^r s^N g(\max(u,\psi))u'\,ds \\ - \lambda a\int_0^r s^{N-1}g(\max(u,\psi))u\,ds - \lambda K_1.\end{aligned} \tag{4.14}$$

Since $p \geq 2$, it follows from Lemma 3.1(i) that there exists a positive constant C_0 depending on $\bar\lambda$ such that

$$u' \geq \phi(u') - C_0 \tag{4.15}$$

in $(0,1)$, which together with (4.1) implies

$$(4.16)\qquad -\lambda\int_0^r s^N g(\max(u,\psi))u'\,ds$$
$$= -\lambda\int_0^r s^N\Big(g(\max(u,\psi))+\frac{c_1}{\psi^\alpha}\Big)(u'-\phi^{-1}(\lambda m_1))\,ds+\lambda c_1\int_0^r\frac{s^N u'}{\psi^\alpha}\,ds$$
$$-\lambda\phi^{-1}(\lambda m_1)\int_0^r s^N\Big(g(\max(u,\psi))+\frac{c_1}{\psi^\alpha}\Big)\,ds$$
$$\geq \lambda c_1\int_0^r\frac{s^N(\phi(u')-C_0)}{\psi^\alpha}\,ds-\lambda\phi^{-1}(\lambda m_1)\int_0^r s^N\Big(g(\max(u,\psi))+\frac{c_1}{\psi^\alpha}\Big)\,ds$$
$$\geq \lambda c_1\int_0^r\frac{s^N\phi(u')}{\psi^\alpha}\,ds-K_2.$$

By Lemma 3.1(iii),

$$\int_0^r\frac{s^N\phi(u')}{\psi^\alpha}\,ds\geq (r^{N-1}\phi(u'(r))-\lambda m_1)\Big(\int_0^r\frac{s}{\psi^\alpha}\,ds\Big).$$

From this and (4.16), we get

$$(4.17)\qquad -\lambda\int_0^r s^N g(\max(u,\psi))u'\,ds\geq\lambda c_1\Big(\int_0^r\frac{s}{\psi^\alpha}\,ds\Big)r^{N-1}\phi(u'(r))-K_3.$$

Next, using Lemma 3.1(ii), (4.1), and integration by parts, we obtain

$$(4.18)\qquad -\lambda a\int_0^r s^{N-1}g(\max(u,\psi))u\,ds$$
$$= -\lambda a\int_0^r s^{N-1}\Big(g(\max(u,\psi))+\frac{c_1}{\psi^\alpha}\Big)(u+\phi^{-1}(\lambda m_1))\,ds$$
$$+\lambda ac_1\int_0^r\frac{s^{N-1}u}{\psi^\alpha}\,ds+\lambda a\phi^{-1}(\lambda m_1)\int_0^r s^{N-1}\Big(g(\max(u,\psi))+\frac{c_1}{\psi^\alpha}\Big)\,ds$$
$$\geq -\lambda ac_1\int_0^r\frac{s^{N-1}u}{\psi^\alpha}\,ds-K_4$$
$$= -\lambda ac_1\Big(\int_0^r\frac{s^{N-1}}{\psi^\alpha}\,d\tau\Big)u(r)+\lambda ac_1\int_0^r\Big(\int_0^s\frac{\tau^{N-1}}{\psi^\alpha}\,d\tau\Big)u'\,ds-K_4.$$

From (4.15) and Lemma 3.1(i) & (iii),

$$(4.19)\qquad \int_0^r\Big(\int_0^s\frac{\tau^{N-1}}{\psi^\alpha}\,d\tau\Big)u'\,ds$$
$$= \int_0^r\Big(\int_0^s\frac{\tau^{N-1}}{\psi^\alpha}\,d\tau\Big)(u'-\phi^{-1}(\lambda m_1))\,ds+\phi^{-1}(\lambda m_1)\int_0^r\Big(\int_0^s\frac{\tau^{N-1}}{\psi^\alpha}\,d\tau\Big)\,ds$$
$$\geq \int_0^r\Big(\int_0^s\frac{d\tau}{\psi^\alpha}\Big)s^{N-1}(u'-\phi^{-1}(\lambda m_1))\,ds\geq r^{N-1}\phi(u'(r))\int_0^r\Big(\int_0^s\frac{d\tau}{\psi^\alpha}\Big)\,ds-K_5.$$

Combining (4.18) and (4.19) gives

$$-\lambda a \int_0^r s^{N-1} g(\max(u,\psi))u\,ds \ge -\lambda a c_1 \left(\int_0^r \frac{s^{N-1}}{\psi^\alpha}\,ds\right) u(r) + \lambda a c_1 \left(\int_0^r \left(\int_0^s \frac{d\tau}{\psi^\alpha}\right) ds\right) r^{N-1}\phi(u'(r)) - K_6. \tag{4.20}$$

We shall need an estimate on r_0. By Lemma 3.3(i),

$$\begin{aligned}- u'(r) &= \phi^{-1}\left(\frac{\lambda}{r^{N-1}}\left(\int_0^r \tau^{N-1}(g(\max(u,\psi)) + h(u)\Big)\,d\tau\right)\\ &\le \phi^{-1}\left(\frac{\lambda}{r^{N-1}}\left(\int_0^r \tau^{N-1}\left(\frac{c_1}{\psi^\alpha} + C_1 H(u)^{1-a/N} + K_7\right) d\tau\right)\right)\\ &\le (2\lambda C_1 H(d)^{1-a/N} r))^{1/(p-1)}\end{aligned} \tag{4.21}$$

for $r \in (0,1)$. Integrating this inequality on $(0,r_0)$ and using Lemma 3.3(i), we get
$C(1-\bar\theta)H(d)^{a/N} \le (1-\bar\theta)d \le ((p-1)/p)(2\lambda C_1)^{1/(p-1)} H(d)^{(1-a/N)1/(p-1)} r_0^{p/(p-1)}$,
which implies

$$r_0 \ge \frac{K_8}{\lambda^{1/p}} H(d)^{a/N-1/p}. \tag{4.22}$$

Next, integrating (4.21) on $(0,1)$ gives

$$d \le \zeta_0 + K_9 \lambda^{1/(p-1)} H(d)^{(1-a/N)/(p-1)},$$

and therefore, if $d \ge 2\zeta_0$,

$$C H(d)^{a/N} \le d \le 2K_9 \lambda^{1/(p-1)} H(d)^{(1-a/N)/(p-1)},$$

which implies

$$\lambda \ge K_{10} H(d)^{ap/N-1}. \tag{4.23}$$

If $N \ge p$ then it follows from (4.22) that

$$\lambda r_0^N H(d) \ge \lambda^{1-N/p} K_8^N H(d)^{a+1-N/p} \ge \bar\lambda^{1-N/p} K_8^N H(d)^{a+1-N/p}, \tag{4.24}$$

while if $N < p$, we deduce from (4.22) and (4.23) that

$$\begin{aligned}\lambda r_0^N H(d) &\ge \lambda^{1-N/p} K_8^N H(d)^{a+1-N/p}\\ &\ge (K_{10} H(d)^{ap/N-1})^{1-N/p} K_8^N H(d)^{a+1-N/p}\\ &= K_{11} H(d)^{ap/N}.\end{aligned} \tag{4.25}$$

Combining (4.14), (4.17), (4.20), (4.24), and (4.25), we get

$$\psi(r) \geq K_{12}H_1(d) + \lambda c_1\Big(\int_0^r \frac{s}{\psi^\alpha}\,ds\Big) r^{N-1}\phi(u'(r)) - \lambda a c_1\Big(\int_0^r \frac{s^{N-1}}{\psi^\alpha}\,ds\Big)u(r) + \lambda a c_1\Big(\int_0^r\Big(\int_0^s \frac{d\tau}{\psi^\alpha}\Big)\,ds\Big) r^{N-1}\phi(u'(r)) - K_{13}$$

for $r \in (r_0, 1)$, where $H_1(d) = H(d)^\gamma, \gamma = a + 1 - N/p$ if $N \geq p$, and $\gamma = ap/N$ if $N < p$.

Let $k > 0$ be such that $\tilde{H}(x) \equiv H(x) + kx$ is increasing on $\mathbb{R}$. Since we have $\lim_{x\to\infty} H(x)/x^p = \infty$, there exist constants k_1 and K_{14} such that

$$\psi(r) - \lambda c_1\Big(\int_0^r \frac{s}{\psi^\alpha}\,ds\Big) r^{N-1}\phi(u'(r)) + \lambda a c_1\Big(\int_0^r \frac{s^{N-1}}{\psi^\alpha}\,ds\Big)u(r) - \lambda a c_1\Big(\int_0^r\Big(\int_0^s \frac{d\tau}{\psi^\alpha}\Big)\,ds\Big) r^{N-1}\phi(u'(r)) \leq k_1\tilde{H}(|u(r)| + |u'(r)|) + K_{14}$$

for $r \in (r_0, 1)$. Consequently,

$$|u(r)| + |u'(r)| \geq \tilde{H}^{-1}\Big(\frac{K_{12}H_1(d) - K_{14}}{k_1}\Big).$$

By Lemma 3.1,

$$|u| + |u'| \leq u - u' + 4\phi^{-1}(\bar{\lambda} m_1),$$

and so

$$-u' + u \geq H_2(d) \quad \text{ on } (r_0, 1),$$

where $H_2(d) = \tilde{H}^{-1}\big(\frac{K_{12}H_1(d) - K_{14}}{k_1}\big) - 4\phi^{-1}(\bar{\lambda} m_1)$. Note that $H_2(d) \to \infty$ as $d \to \infty$. Solving the above differential inequality, we get

$$u(r) \geq e^{r-1}\zeta + e^r\Big(\int_r^1 e^{-s}\,ds\Big)H_2(d) \geq \frac{H_2(d)}{e}(1 - r)$$

for $r > r_0$ and $d \gg 1$, while (4.12) holds for $r \leq r_0$ and $d \gg 1$. On the other hand, if $d < d_0$ for some $d_0 > 0$ then it follows from the integral formula for u' that $\|u'\|_\infty < D_0$, where D_0 depends on d_0 and $\bar{\lambda}$. Hence

$$u(r) = \zeta - \int_r^1 u' \geq -D_0(1 - r) \quad \text{ for } r \in (0, 1).$$

Hence (i) follows.

(ii) Let h_0 be a positive constant such that $h(x) \geq -h_0$ for all $x \in \mathbb{R}$, and let $\tilde{R} > 2\phi^{-1}(\bar{\lambda} m_1)$ be large enough so that

$$h_i(\tilde{R}) > N2^{N+2}(c_1(1 - \alpha)^{-1} + h_0),$$

where $h_i(t) = \inf_{x\geq t} h_i(x)$, and c_1 is given by (4.1). Choose $\bar{R} > 0$ so that

$$L(z) > 4\tilde{R} \quad \text{for } z \geq \bar{R}.$$

Suppose $\|u\|_\infty > \bar{R}$. Then, by part (i),

$$\frac{u(1/2)}{2} \geq \frac{1}{4} L(\|u\|_\infty) > \tilde{R}.$$

Since

$$-\phi(u'(r)) \geq \frac{\lambda}{r^{N-1}} \int_0^r \tau^{N-1}\Big(-\frac{c_1}{\psi^\alpha} + h(u)\Big)\,d\tau \tag{4.26}$$

and

$$u(\tau) \geq u(1/2) - \phi^{-1}(\bar{\lambda} m_1) \geq \frac{u(1/2)}{2}$$

for $\tau \leq 1/2$, it follows that

$$\begin{aligned}\int_0^r \tau^{N-1}\Big(-\frac{c_1}{\psi^\alpha} + h(u)\Big)\,d\tau &\geq \frac{r^N}{N}\Big(h_i\Big(\frac{u(1/2)}{2}\Big) - Nc_1(1-\alpha)^{-1} - h_0\Big)\\ &\geq \frac{r^N}{2N} h_i\Big(\frac{u(1/2)}{2}\Big) > 0\end{aligned}$$

for $r \leq 1/2$. Hence $u' < 0$ on $(0, 1/2]$. For $r > 1/2$,

$$\begin{aligned}\int_0^r \tau^{N-1}\Big(-\frac{c_1}{\psi^\alpha} + h(u)\Big)\,d\tau &= \int_0^{1/2} \tau^{N-1}\Big(-\frac{c_1}{\psi^\alpha} + h(u)\Big)\,d\tau\\ &\quad + \int_{1/2}^r \tau^{N-1}\Big(-\frac{c_1}{\psi^\alpha} + h(u)\Big)\,d\tau\\ &\geq \frac{1}{2^{N+1}N} h_i\Big(\frac{u(1/2)}{2}\Big) - c_1(1-\alpha) - h_0\\ &> \frac{1}{2^{N+2}N} h_i\Big(\frac{u(1/2)}{2}\Big),\end{aligned} \tag{4.27}$$

and (ii) follows.

(iii) Let $R_1 > 0$ be such that

$$\frac{h_i(x)}{\phi(x)} > \frac{N2^{N+2p}}{\bar{\lambda}}$$

for $x \geq R_1$. Let $R > \bar{R}$ be such that

$$L(z) > 4R_1 \quad \text{for } z \geq R,$$

where $\bar{R}$ is defined in part (ii). We claim that $\|u\|_\infty < R$. Suppose $\|u\|_\infty \geq R$.

Then, by integrating on $(1/2, 1)$ the inequality

$$-u' \geq \phi^{-1}\left(\frac{\lambda}{2^{N+2}N} h_i\left(\frac{u(1/2)}{2}\right)\right),$$

obtained from (4.26) and (4.27), we get

$$2u(1/2) \geq \phi^{-1}\left(\frac{\lambda}{2^{N+2}N} h_i\left(\frac{u(1/2)}{2}\right)\right),$$

or, equivalently,

$$\frac{h_i\left(\frac{u(1/2)}{2}\right)}{\phi\left(\frac{u(1/2)}{2}\right)} \leq \frac{N2^{N+2p}}{\lambda} < \frac{N2^{N+2p}}{\underline{\lambda}}.$$

This implies $u(1/2)/2 < R_1$, and since

$$L(\|u\|_\infty) \leq 2u(1/2) < 4R_1,$$

it follows that $\|u\|_\infty < R$, a contradiction which proves the claim. This completes the proof of Lemma 4.4. □

§5. Proofs of the main results

Proof of Theorem 2.1. Suppose $\lambda < \bar{\lambda}$, where $\bar{\lambda}$ is defined by Lemma 4.1. In view of Lemmas 4.1, 4.2, and 4.4(iii), it follows that

$$\deg(I - A_\lambda, B(0, r_\lambda), 0) = 1, \qquad \deg(I - A_\lambda, B(0, R), 0) = 0,$$

and the excision property of the Leray–Schauder degree gives the existence of a fixed point u_λ of A_λ such that

$$\|u_\lambda\|_\infty > r_\lambda.$$

Since $r_\lambda \to \infty$ as $\lambda \to 0$, it follows from Lemma 4.4(i) & (ii) with $\zeta_0 = 0$ that, for λ small, u_λ is decreasing and

$$u_\lambda(r) \geq L(\|u_\lambda\|_\infty)(1 - r) \geq \psi(r)$$

for $r \in [0, 1]$. In particular, u_λ is a positive solution of (1.2) for $\lambda > 0$ small and $u_\lambda \to \infty$ uniformly on compact subsets of $[0, 1)$. This completes the proof of Theorem 2.1. □

We now turn our attention to the positone case. By (A.1) and (A.4), there exists a positive number κ such that

$$f(x) \geq \kappa \quad \text{for all } x > 0.$$

Let $\psi_\lambda = c_\lambda \psi$, where $c_\lambda = (\lambda\kappa/N)^{1/(p-1)}(p - 1)/p$.

For $\lambda > 0$ and $v \in C[0,1]$, let $u = \tilde{A}_\lambda v$ be the solution of

$$\begin{cases} -(r^{N-1}\phi(u'))' = \lambda r^{N-1}(g\max(v,\psi_\lambda)) + h(v)), & 0 < r < 1, \\ u'(0) = 0, \quad u(1) = 0. \end{cases} \tag{5.1}$$

Then $\tilde{A}_\lambda : C[0,1] \to C[0,1]$ is a compact operator and using the same arguments as above, we obtain the following results for $\tilde{A}_\lambda$.

Lemma 5.1. (i) *Let $0 < \underline{\lambda} < \lambda < \bar{\lambda}$. Then there exists a positive number $R_0 > 0$ depending on $\underline{\lambda}$ and $\bar{\lambda}$ such that any solution u_λ of*

$$u = \tilde{A}_\lambda u$$

satisfies $\|u\|_\infty < R_0$. Furthermore

$$\deg(I - \tilde{A}_\lambda, B(0,R_0), 0) = 0.$$

(ii) *$\tilde{A}_\lambda$ has a fixed point for λ small.*

Lemma 5.2. (i) *Let u satisfy*

$$\begin{cases} -(r^{N-1}\phi(u'))' \geq \lambda r^{N-1}\kappa, & 0 < r < 1, \\ u'(0) = 0, \quad u(1) = 0. \end{cases} \tag{5.2}$$

Then $u \geq \psi_\lambda$ in $(0,1)$. In particular, u is a fixed point of $\tilde{A}_\lambda$ if and only if u is a solution of (1.2).

(ii) *There exists a positive number $\tilde{\lambda}$ such that (1.2) has no solution for $\lambda \geq \tilde{\lambda}$.*

Proof. (i) Using the integral formula for u, we see that

$$\begin{aligned} u(r) &\geq \int_r^1 \phi^{-1}\left(\frac{\lambda}{s^{N-1}}\int_0^s \tau^{N-1}\kappa\, d\tau\right) ds = \int_r^1 (\lambda\kappa s/N)^{1/(p-1)}\, ds \\ &\geq (\lambda\kappa/N)^{1/(p-1)}((p-1)/p)(1-r) \end{aligned} \tag{5.3}$$

for $r \in (0,1)$. Consequently, if u is a solution of (1.2) then $u = \max(u,\psi_\lambda)$ and so u is a fixed point of $\tilde{A}_\lambda$. Conversely, suppose $u = \tilde{A}_\lambda u$. Since

$$g(\max(u,\psi_\lambda)) + h(u) = f(\max(u,\psi_\lambda))$$

if $\max(u,\psi_\lambda) \leq 1$, and

$$g(\max(u,\psi_\lambda)) + h(u) = f(\max(u,1))$$

if $\max(u,\psi_\lambda) > 1$, it follows that $u \geq \psi_\lambda$ in $(0,1)$, and so u is a positive solution of (1.2).

(ii) Let u be a solution of (1.2). Then u is decreasing and satisfies

$$\begin{aligned} u(1/2) &\geq \int_{1/2}^{1} \phi^{-1}\left(\frac{\lambda}{s^{N-1}} \int_0^{1/2} \tau^{N-1} f(u)\, d\tau\right) ds \\ &\geq \frac{1}{2}\phi^{-1}\left(\frac{\lambda}{N2^N} f_i(u(1/2))\right), \end{aligned}$$

or

$$\frac{f_i(u(1/2))}{\phi(u(1/2))} \leq \frac{N2^{N+p-1}}{\lambda},$$

which is a contradiction to (5.3) and the fact that $\lim_{x\to\infty} f_i(x)/\phi(x) = \infty$ if λ is sufficiently large. □

Let $\Lambda = \{\lambda > 0 : (1.2) \text{ has a solution}\}$ and let $\lambda^* = \sup \Lambda$.

Lemma 5.3. *$\lambda^* \in (0, \infty)$ and $\lambda^* \in \Lambda$.*

Proof. Using Lemmas 5.1(ii) and 5.2(ii), we see that $\lambda^* \in (0, \infty)$. Let (λ_n) be a sequence in Λ such that $\lambda_n \to \lambda^*$ and let (u_n) be the corresponding solutions of (1.2). By Lemma 5.1(i), $(\|u_n\|_\infty)$ is bounded, and so there exists a constant $C > 0$ such that

$$f(u_n) \leq \frac{C}{u_n^\alpha} \leq \frac{C}{c_{\lambda_n}^\alpha \psi^\alpha} \quad \text{in } (0,1)$$

for all n. From this and the formula

$$u_n'(r) = -\phi^{-1}\left(\frac{\lambda_n}{r^{N-1}} \int_0^r \tau^{N-1} f(u_n)\, d\tau\right),$$

we deduce that

$$|u_n'(r)| \leq \phi^{-1}\left(\frac{\lambda_n C}{c_{\lambda_n}^\alpha r^{N-1}} \int_0^r \frac{\tau^{N-1}}{\psi^\alpha}\, d\tau\right) \leq \phi^{-1}\left(\frac{\lambda_n C}{c_{\lambda_n}^\alpha (1-\alpha)}\right) \leq C_1$$

for all $r \in (0,1)$ and n, where C_1 is a constant depending on $\lambda^*, C, \alpha, N, p$.

Hence (u_n) is bounded in $C^1[0,1]$, and, by passing to a subsequence, we can assume that $u_n \to u_{\lambda^*}$ in $C[0,1]$. Letting $n \to \infty$ in

$$u_n(r) = \int_r^1 \phi^{-1}\left(\frac{\lambda_n}{s^{N-1}} \int_0^s \tau^{N-1} f(u_n)\, d\tau\right) ds$$

we obtain

$$u_{\lambda^*}(r) = \int_r^1 \phi^{-1}\left(\frac{\lambda^*}{s^{N-1}} \int_0^s \tau^{N-1} f(u_{\lambda^*})\, d\tau\right) ds,$$

i.e. u_{λ^*} is a solution of (1.2) with $\lambda = \lambda^*$. □

Lemma 5.4. *Let $\lambda \in (0, \lambda^*)$ and let u_{λ^*} be a solution of* (1.2)$_{\lambda^*}$. *Then there exists a positive number ε such that $u_{\lambda^*} + \varepsilon$ is a supersolution of* (1.2)$_\lambda$.

Proof. Let $p(x) = x^\beta f(x)$ and $\varepsilon_0 = 1 - \lambda/\lambda^*$. Then there exists a positive number κ_0 such that $p \geq \kappa_0$ in $(0, \infty)$. Since p is uniformly continuous on $(0, \|u_{\lambda^*}\|_\infty + 1]$, there exists a number $\varepsilon \in (0,1)$ such that for all $x \in (0, \|u_{\lambda^*}\|_\infty]$,

$$|p(x) - p(x+\varepsilon)| < \varepsilon_0 \kappa_0,$$

hence

$$\left| \frac{p(x)}{p(x+\varepsilon)} - 1 \right| < \varepsilon_0,$$

which implies

$$\frac{f(x)}{f(x+\varepsilon)} = \frac{p(x)}{p(x+\varepsilon)} \left(1 + \frac{\varepsilon}{x}\right)^\beta > 1 - \varepsilon_0 = \frac{\lambda}{\lambda^*}. \tag{5.4}$$

Consequently,

$$-(r^{N-1}\phi(u'_{\lambda^*}))' = \lambda^* r^{N-1} f(u_{\lambda^*}) > \lambda r^{N-1} f(u_{\lambda^*} + \varepsilon) \quad \text{in } (0,1),$$

i.e., $u_{\lambda^*} + \varepsilon$ is a supersolution of (1.2)$_\lambda$.

Next, for each $v \in C[0,1]$, let $u = T_\lambda v$ be the solution of

$$\begin{cases} -(r^{N-1}\phi(u'))' = \lambda r^{N-1}(g(\min(\max(v, \psi_\lambda), u_{\lambda^*} + \varepsilon)) + h(\min(v, u_{\lambda^*} + \varepsilon)), \\ u'(0) = 0, \quad u(1) = 0, \end{cases}$$

where ε is defined in Lemma 5.4. Then $T_\lambda : C[0,1] \to C[0,1]$ is a compact operator and since

$$\psi_\lambda \leq \min(\max(v, \psi_\lambda), u_{\lambda^*} + \varepsilon) \leq u_{\lambda^*} + \varepsilon,$$

it follows from (A.3) that T_λ is bounded. □

Lemma 5.5. *Every fixed point u of T_λ is a solution of* (1.2) *and satisfies*

$$\psi_\lambda \leq u \leq u_{\lambda^*} + \varepsilon \quad \text{in } [0,1].$$

Proof. Let u be a fixed point of T_λ. Since $u_{\lambda^*} \geq \psi_{\lambda^*} > \psi_\lambda$, we have

$$g(\min(\max(u, \psi_\lambda), u_{\lambda^*} + \varepsilon))) + h(\min(u, u_{\lambda^*} + \varepsilon)) = g(\max(u, \psi_\lambda)) + h(u) \geq \kappa$$

if $u \leq u_{\lambda^*} + \varepsilon$, and

$$g(\min(\max(u, \psi_\lambda), u_{\lambda^*} + \varepsilon))) + h(\min(u, u_{\lambda^*} + \varepsilon)) = f(u_{\lambda^*} + \varepsilon) \geq \kappa$$

if $u \geq u_{\lambda^*} + \varepsilon$. This implies $u \geq \psi_\lambda$ in $(0,1)$, by Lemma 5.2(i). Suppose there exists $r_0 \in (0,1)$ such that $u(r_0) > u_{\lambda^*}(r_0) + \varepsilon$. Then there exist numbers r_1, r_2

with $0 \le r_1 < r < r_2 < 1$ such that $u(r_2) = u_{\lambda^*}(r_2) + \varepsilon$, $u'(r_1) = u'_{\lambda^*}(r_1)$ or $u(r_1) = u_{\lambda^*}(r_1) + \varepsilon$, and $u > u_{\lambda^*} + \varepsilon$ on (r_1, r_2).

Hence, by Lemma 5.4,

$$-(r^{N-1}\phi(u'))' = \lambda r^{N-1} f(u_{\lambda^*} + \varepsilon) < \lambda^* r^{N-1} f(u_{\lambda^*}) = -(r^{N-1}\phi(u'_{\lambda^*}))'$$

in (r_1, r_2). Consequently,

$$\begin{aligned} 0 &< \int_{r_1}^{r_2} (r^{N-1}(\phi(u') - \phi(u'_{\lambda^*})))'(u - (u_{\lambda^*} + \varepsilon))\, dr \\ &= -\int_{r_1}^{r_2} r^{N-1}(\phi(u') - \phi(u'_{\lambda^*}))(u' - u'_{\lambda^*})\, dr \le 0, \end{aligned}$$

a contradiction. Thus $u \le u_{\lambda^*} + \varepsilon$ in $(0,1)$, which completes the proof. □

Proof of Theorem 2.2. Let $\lambda \in (0, \lambda^*)$. Since $\nu\phi_1$ is a subsolution of $(1.2)_\lambda$ if $\nu > 0$ is sufficiently small and u_{λ^*} is a supersolution of $(1.2)_\lambda$, it follows that (1.2) has a solution u_λ such that $\nu\phi_1 \le u_\lambda \le u_{\lambda^*}$. We shall show that $(1.2)_\lambda$ has a second solution. Define

$$D = \{u \in C[0,1] : -\varepsilon < u < u_{\lambda^*} + \varepsilon \text{ in } [0,1]\}.$$

Then D is an open set and $u_\lambda \in D$. By Lemma 5.5, all fixed points of T_λ are in $\bar{D}$. Since T_λ is bounded,

$$\deg(I - T_\lambda, B(u_\lambda, R), 0) = 1 \quad \text{for } R \gg 1.$$

If there exists $u \in \partial D$ such that $u = T_\lambda u$ then u is a second solution of $(1.2)_\lambda$. Suppose that $u \ne T_\lambda u$ for all $u \in \partial D$. Then $\deg(I - T_\lambda, D, 0)$ is defined and since T_λ has no fixed point in $B(u_\lambda, R) \setminus D$, it follows that

$$\deg(I - T_\lambda, D, 0) = \deg(I - T_\lambda, B(u_\lambda, R), 0) = 1.$$

Since $\tilde{A}_\lambda = T_\lambda$ on D, we have

$$\deg(I - \tilde{A}_\lambda, D, 0) = 1,$$

and since by Lemma 5.2(i),

$$\deg(I - \tilde{A}_\lambda, B(0, R_0), 0) = 0$$

for some $R_0 \gg 1$, we arrive at

$$\deg(I - \tilde{A}_\lambda, B(0, R_0) \setminus D, 0) = -1.$$

Thus there exists a fixed point u of $\tilde{A}_\lambda$ in $B(0, R_0) \setminus D$, which is a second positive solution of (1.2). This completes the proof of Theorem 2.2. □

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