

# Locally Compact Separable Abelian Group Actions on Factors with the Rokhlin Property

by

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## Abstract

We prove a classification theorem for actions with the Rokhlin property of locally compact separable abelian groups on factors. This is a generalization of the recent work due to Masuda–Tomatsu on Rokhlin flows.

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## §1. Introduction

Studying group actions is one of the most interesting topics in the theory of operator algebras. Since Connes [3], [4] completely classified single automorphisms of an approximately finite-dimensional (hereafter abbreviated by AFD) factor of type  $II_1$  up to cocycle conjugacy, the classification of group actions has been remarkably developed; discrete amenable group actions on AFD factors have been completely classified by many authors [10], [12], [17], [22], [25], [26], and there has been great progress in the classification of compact group actions on AFD factors by Jones–Takesaki [9], Kawahigashi–Takesaki [13], and Masuda–Tomatsu [19], [20], [21].

The next subject of interest is the classification of actions of non-compact continuous groups. In the study of locally compact abelian group actions, many problems are left. As a step to understanding these actions, outer actions are now intensively studied. As a candidate for outer actions for flows, the Rokhlin property was introduced by Kishimoto [15], Kawamuro [14] and Masuda–Tomatsu [18], and the last two authors have succeeded in classifying Rokhlin flows on von Neumann algebras. As mentioned in [18, Problem 8.1], the Rokhlin property can be generalized to locally compact abelian group actions. Hence it is natural to extend

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the result of Masuda–Tomatsu to locally compact abelian group actions. In this direction, Asano [2] has classified actions of  $\mathbb{R}^n$  with the Rokhlin property.

In this paper, we will generalize results in [2] and [18], and classify actions with the Rokhlin property of locally compact separable abelian groups on factors (Theorem 2). This gives an answer to [18, Problem 8.1].

This paper is organized as follows. In Section 2, we recall the definition of ultraproduct von Neumann algebras and the Rokhlin property. In Section 3, we prove the main theorem. The proof is basically modeled after that in [18]. However, there are some differences: for example, in contrast to  $\mathbb{R}$ , some locally compact abelian groups do not have enough compact quotients. In Section 4, we give some examples of Rokhlin actions.

## §2. Preliminaries

### §2.1. Notations

Let  $M$  be a von Neumann algebra. We denote the set of unitaries of  $M$  by  $U(M)$ . For  $\phi \in M_*$  and  $a \in M$ , set  $[\phi, a] := a\phi - \phi a$ . For  $\phi \in M_*^+$  and  $x \in M$ , define

$$\|x\|_\phi^\sharp := \sqrt{\frac{\phi(x^*x + xx^*)}{2}}.$$

Then  $\|\cdot\|_\phi^\sharp$  is a seminorm on  $M$ . If  $\phi$  is faithful, then this norm metrizes the strong\* topology of the unit ball of  $M$ .

### §2.2. Ultraproduct von Neumann algebras

First of all, we recall ultraproduct von Neumann algebras. Basic references are Ando–Haagerup [1] and Ocneanu [22]. Let  $\omega$  be a free ultrafilter on  $\mathbb{N}$  and  $M$  be a separable von Neumann algebra. We denote by  $l^\infty(M)$  the  $C^*$ -algebra consisting of all norm bounded sequences in  $M$ . Set

$$\begin{aligned} I_\omega &:= \left\{ (x_n) \in l^\infty(M) \mid \text{strong}^*\text{-}\lim_{n \rightarrow \omega} x_n = 0 \right\}, \\ N_\omega &:= \left\{ (x_n) \in l^\infty(M) \mid \text{for all } (y_n) \in I_\omega, \text{ we have} \right. \\ &\quad \left. (x_n y_n) \in I_\omega \text{ and } (y_n x_n) \in I_\omega \right\}, \\ C_\omega &:= \left\{ (x_n) \in l^\infty(M) \mid \text{for all } \phi \in M_*, \text{ we have } \lim_{n \rightarrow \omega} \|[\phi, x_n]\| = 0 \right\}. \end{aligned}$$

Then  $I_\omega \subset C_\omega \subset N_\omega$  and  $I_\omega$  is a closed ideal of  $N_\omega$ . Hence we can define the quotient  $C^*$ -algebra  $M^\omega := N_\omega / I_\omega$ . Denote the canonical quotient map  $N_\omega \rightarrow M^\omega$  by  $\pi$ . Set  $M_\omega := \pi(C_\omega)$ . Then  $M_\omega$  and  $M^\omega$  are von Neumann algebras as in Ocneanu [22, Proposition 5.1].

Let  $\tau^\omega : M^\omega \rightarrow M$  be the map defined by  $\tau^\omega(\pi((x_n))) = \lim_{n \rightarrow \omega} x_n$ . Here, the limit is taken in the weak topology of  $M$ . This map is a faithful normal conditional expectation (see [18, Subsection 2.4]).

Let  $\alpha$  be an automorphism of  $M$ . We define an automorphism  $\alpha^\omega$  of  $M^\omega$  by  $\alpha^\omega(\pi((x_n))) = \pi((\alpha(x_n)))$  for  $\pi((x_n)) \in M^\omega$ . Then  $\alpha^\omega(M_\omega) = M_\omega$ . By restricting  $\alpha^\omega$  to  $M_\omega$ , we obtain an automorphism  $\alpha_\omega$  of  $M_\omega$ . Hereafter we omit  $\pi$  and denote  $\alpha^\omega$  and  $\alpha_\omega$  by  $\alpha$  if no confusion arises.

### §2.3. The Rokhlin property

Next, we recall the Rokhlin property. A basic reference is [18]. In the previous subsection, we have seen that it is possible to lift automorphisms of von Neumann algebras to their ultraproducts. Hence it is natural to consider lifts of actions of locally compact abelian groups to  $M^\omega$  and  $M_\omega$ . However, lifts may not be continuous. Instead of considering  $\alpha^\omega$  on the whole  $M^\omega$ , we consider its continuous part.

Let  $G$  be a locally compact separable abelian group. In the rest of the paper, we always assume that groups and von Neumann algebras are separable, except for ultraproduct von Neumann algebras. We denote the group operation of  $G$  by  $+$ . Let  $d$  be a translation invariant metric on  $G$  (see [8, Theorem 8.3]). Choose a normal faithful state  $\varphi$  on  $M$ . For an action  $\alpha$  of  $G$  on a von Neumann algebra  $M$ , set

$$M_\alpha^\omega := \{ (x_n) \in M^\omega \mid \text{for each } \epsilon > 0, \text{ there exists } \delta > 0 \text{ such that} \\ \{ n \in \mathbb{N} \mid \|\alpha_t(x_n) - x_n\|_\varphi^\# < \epsilon \text{ for } t \in G \text{ with } d(0, t) < \delta \} \in \omega \},$$

$$M_{\omega, \alpha} := \{ (x_n) \in M_\omega \mid \text{for each } \epsilon > 0, \text{ there exists } \delta > 0 \text{ such that} \\ \{ n \in \mathbb{N} \mid \|\alpha_t(x_n) - x_n\|_\varphi^\# < \epsilon \text{ for } t \in G \text{ with } d(0, t) < \delta \} \in \omega \}.$$

Since all metrics on  $G$  are mutually equivalent, this definition does not depend on the choice of  $d$ . The condition appearing in the definition of  $M_\alpha^\omega$  means the  $\omega$ -equicontinuity of the family of maps  $\{G \ni t \mapsto \alpha_t(x_n)\}$  (see [18, Definition 3.1 and Lemma 3.2]).

Now, we will define the Rokhlin property.

**Definition 1.** An action  $\theta$  of a locally compact abelian group  $G$  on a von Neumann algebra  $M$  is said to have the *Rokhlin property* if for each  $p \in \hat{G}$ , there exists a unitary  $u$  of  $M_{\omega, \theta}$  satisfying  $\theta_t(u) = \langle t, -p \rangle u$  for all  $t \in G$ .

The Rokhlin property can also be defined for Borel cocycle actions (see [18, Definitions 3.4 and 4.1]). For actions, by the same argument as in [18, proof of Proposition 3.5], the two definitions coincide.

§3. A classification theorem

Let  $G$  be a locally compact abelian group. Let  $\alpha^1$  and  $\alpha^2$  be two actions of  $G$  on a von Neumann algebra  $M$ . The actions  $\alpha^1$  and  $\alpha^2$  are said to be *cocycle conjugate* if there exist an  $\alpha^2$ -cocycle  $u$  and an automorphism  $\sigma$  of  $M$  satisfying  $\text{Ad } u_t \circ \alpha_t^2 = \sigma \circ \alpha_t^1 \circ \sigma^{-1}$  for all  $t \in G$ . If  $\sigma$  can be chosen to be approximately inner, then  $\alpha^1$  is said to be *strongly cocycle conjugate* to  $\alpha^2$  (see [18, Subsection 2.1]).

Our main theorem in this paper is the following.

**Main Theorem 2.** *Let  $G$  be a locally compact abelian group. Let  $\alpha$  and  $\beta$  be actions of  $G$  with the Rokhlin property on a factor  $M$ . Then  $\alpha$  and  $\beta$  are strongly cocycle conjugate if and only if  $\alpha_t \circ \beta_{-t} \in \overline{\text{Int}}(M)$  for all  $t \in G$ .*

In the rest of this section, we will present a proof of this theorem. The proof is modeled after that in [18]. However, at some points, we need to deal with different problems. One of the problems is that some locally compact abelian groups do not have enough compact quotients. Instead, we consider compact quotients of compactly generated clopen subgroups. By [8, Theorem 9.14], a compactly generated subgroup is isomorphic to  $\mathbb{R}^n \times K \times \mathbb{Z}^m$  for some compact abelian group  $K$  and non-negative integers  $n, m$ . We deal with this problem in Subsection 3.3.

§3.1. Lifts of Borel unitary paths

The first step of our proof of Theorem 2 is to find a representing unitary sequence  $\{u_t^\nu\}$  for a Borel map  $U_t: G \rightarrow U(M_\theta^\omega)$  so that the family  $\{t \mapsto u_t^\nu\}$  is “almost”  $\omega$ -equicontinuous. More precisely, we have the following.

**Lemma 3** (see [18, Lemma 3.24]). *Let  $(\theta, c)$  be a Borel cocycle action of a locally compact abelian group  $G$  on a factor  $M$ . Suppose that  $U: G \rightarrow M_\theta^\omega$  is a Borel unitary map. Let  $H$  be a compactly generated clopen subgroup of  $G$ , which is isomorphic to  $\mathbb{R}^n \times K \times \mathbb{Z}^m$  for some non-negative integers  $n, m$  and a compact abelian group  $K$ . Let  $L$  be a subset of  $H$  of the form*

$$L = [0, S_1) \times \cdots \times [0, S_n) \times K \times [0, N_1) \times \cdots \times [0, N_m)$$

*when we identify  $H$  with  $\mathbb{R}^n \times K \times \mathbb{Z}^m$ . Then for any  $\delta$  with  $0 < \delta < 1$  and a finite subset  $\Phi$  of  $M_*^+$ , there exist a compact subset  $I$  of  $L \times L$ , a compact subset  $C$  of  $L$  and a lift  $\{u_t^\nu\}$  of  $U$  satisfying the following conditions:*

- (1)  $\pi_\omega((u_t^\nu)_\nu) = U_t$  for almost every  $t \in L$ , in particular for all  $t \in C$ .
- (2)  $\mu_G(L \setminus C) < \delta$ , where  $\mu_G$  is the Haar measure on  $G$ .
- (3) For all  $\nu \in \mathbb{N}$ , the map  $L \ni t \mapsto u_t^\nu$  is Borel and its restriction to  $C$  is strongly continuous.

- (4) The family  $\{C \ni t \mapsto u_t^\nu\}_\nu$  is  $\omega$ -equicontinuous.
- (5)  $(\mu_G \times \mu_G)(I) \geq (1 - \delta)(\mu_G \times \mu_G)(L \times L)$ .
- (6) The family  $\{I \ni (t, s) \mapsto u_t^\nu \theta_t(u_s^\nu) c(t, s) (u_{t+s}^\nu)^*\}_\nu$  is  $\omega$ -equicontinuous.
- (7) The following limit is uniform on  $I$  for all  $\phi \in \Phi$ :

$$\lim_{\nu \rightarrow \omega} \|u_t^\nu \theta_t(u_s^\nu) c(t, s) (u_{t+s}^\nu)^* - 1\|_\phi^\sharp = \|U_t \theta_t(U_s) c(t, s) U_{t+s}^* - 1\|_\phi^\sharp.$$

The proof is similar to that of [18, Lemma 3.24]. Here, we only prove the following lemma, which corresponds to [18, Lemma 3.21]. The proof is by simple approximation by Borel simple step functions.

**Lemma 4** (see also [18, Lemma 3.21]). *Let  $G$  be a locally compact abelian group,  $\theta: G \rightarrow \text{Aut}(M)$  be a Borel map and  $U: G \rightarrow M_\theta^\omega$  be a Borel unitary map. Then for any Borel subset  $L$  of  $G$  with  $0 < \mu_G(L) < \infty$  and for any  $\epsilon > 0$ , there exist a compact subset  $C$  of  $L$  and a sequence  $\{u_t^\nu\}_{\nu \in \mathbb{N}}$  of unitaries of  $M$  for any  $t \in L$  which satisfy the following conditions:*

- (1)  $\pi_\omega((u_t^\nu)_\nu) = U_t$  for almost every  $t \in L$ , in particular for all  $t \in C$ .
- (2)  $\mu_G(L \setminus C) < \epsilon$ .
- (3) For all  $\nu \in \mathbb{N}$ , the map  $L \ni t \mapsto u_t^\nu$  is Borel and its restriction to  $C$  is strongly continuous.
- (4) The family  $\{C \ni t \mapsto u_t^\nu\}_\nu$  is  $\omega$ -equicontinuous.

*Proof.* By the same argument as in [18, proof of Lemma 3.21], there exists a sequence  $\{L_n\}$  of compact subsets of  $L$  satisfying:

- $L_i \cap L_j = \emptyset$  for  $i \neq j$ .
- $\mu_G(L \setminus \bigcup_{j=1}^\infty L_j) = 0$ .
- $U|_{L_i}$  is continuous for each  $i$ .

Hence we may assume that  $L$  is compact and  $U|_L$  is strongly continuous. Let  $\psi \in M_*$  be a normal faithful state. For each  $t \in L$ , take a representing unitary  $\{\tilde{U}_t^\nu\}_\nu$  of  $U_t$ . Note that  $t \mapsto \tilde{U}_t^\nu$  may not be Borel measurable. We first show the following claim.

**Claim.** *For each  $k \in \mathbb{N}$ , there exist  $N_k \in \mathbb{N}$ ,  $F_k \in \omega$ , a finite subset  $A_k$  of  $L$ , a finite Borel partition  $P^k := \{K_l^k\}_{l=1}^{n_k}$  of  $L$  and a compact subset  $C_k$  of  $L$  satisfying:*

- (1) For  $s, t \in L$  with  $d(s, t) \leq 1/N_k$ , we have  $\|U_s - U_t\|_{\psi^\omega}^\sharp < 1/(2k)$ .
- (2)  $N_k > N_{k-1}, 2/N_k + 1/(2N_{k-1}) < 1/N_{k-1}$  for all  $k$ .
- (3)  $[k, \infty) \supset F_{k-1} \supseteq F_k$  for all  $k$ .
- (4)  $A_k \supset A_{k-1}$  for all  $k$ .

- (5)  $\bigcup_{j=1}^{\infty} A_j \subset C_k, C_{k+1} \subset C_k, \mu_G(L \setminus C_k) < \epsilon(1 - 2^{-k})$  for all  $k$ , and  $C_k \cap K_l^k$ 's are also compact for all  $k \in \mathbb{N}, l = 1, \dots, n_k$ .
- (6) For each  $k$ , the partition  $P^{k+1}$  is finer than  $P^k$  and for each  $k \in \mathbb{N}$  and  $l = 1, \dots, n_k$ , we have  $A_k \cap K_l^k = \{t_{k,l}\}$  ( $= \{\text{pt}\}$ ).
- (7) For  $s, t \in K_l^k$ , we have  $d(s, t) \leq 1/N_k$ .
- (8) For  $s, t \in A_k, \nu \in F_k$ , we have  $\|\tilde{U}_s^\nu - \tilde{U}_t^\nu\|_\psi^\sharp < \|U_s - U_t\|_{\psi^\omega}^\sharp + 1/(2k)$ .

*Proof of Claim.* First of all, choose a sequence  $\{N_k\}_{k=1}^\infty \subset \mathbb{N}$  that satisfies (1) and (2). Next, we define  $P^k$ 's. Assume that  $P^1, \dots, P^k$  are already chosen so that they satisfy (7), and that  $P^{j+1}$  is a refinement of  $P^j$  for  $j = 1, \dots, k - 1$ . By compactness of  $L$ , there exists a family  $\{B_f\}_{f \in F}$  of finite balls of radius  $1/(2N_{k+1})$  of  $L$  which covers  $L$ . This defines a partition  $\{\tilde{B}_{f'}\}_{f' \in F'}$  of  $L$ . Then  $P^{k+1} := \{K_l^l \cap \tilde{B}_{f'}\}_{f' \in F', l=1, \dots, n_k}$  is a refinement of  $P^k$  which satisfies (7). Next, we select  $C_k$ 's. Set  $C_0 := L$  and  $C_1^0 := C_0$ . By Lusin's theorem, for each  $l = 1, \dots, n_k, k \in \mathbb{N}$ , there exists a compact subset  $C_l^k$  of  $K_l^k$  which satisfies:

- $C_l^{k+1} \subset C_{l'}^k$  if  $K_l^{k+1} \subset K_{l'}^k$ .
- $\mu_G((K_l^{k+1} \cap C_{l'}^k) \setminus C_l^{k+1}) < 2^{-(k+1)}\epsilon/n_{k+1}$  if  $K_l^{k+1} \subset K_{l'}^k$ .

Set  $C_k := \bigcup_{l=1}^{n_k} C_l^k$  for each  $k \in \mathbb{N}$ . Since  $C_l^k$ 's are compact, so is  $C_k$ . On the other hand, we have

$$\begin{aligned} \mu_G(C_j \setminus C_{j+1}) &= \sum_{l=1}^{n_{j+1}} \mu_G((K_l^{j+1} \cap C_j) \setminus C_{j+1}) = \sum_{l=1}^{n_{j+1}} \mu_G((K_l^{j+1} \cap C_{l'}^j) \setminus C_{j+1}) \\ &= \sum_{l=1}^{n_{j+1}} \mu_G((K_l^{j+1} \cap C_{l'}^j) \setminus C_l^{j+1}) < n_{j+1} \frac{1}{n_{j+1}} 2^{-(j+1)}\epsilon = 2^{-(j+1)}\epsilon. \end{aligned}$$

In the above inequality, for each  $l = 1, \dots, n_{j+1}, l' \in \{1, \dots, n_j\}$  is the unique index with  $C_l^{j+1} \subset C_{l'}^j$ . Hence

$$\mu_G(L \setminus C_k) \leq \sum_{j=0}^{k-1} \mu_G(C_j \setminus C_{j+1}) < \epsilon \sum_{j=0}^{k-1} 2^{-(j+1)} = \epsilon(1 - 2^{-k}).$$

Thus  $C_k$ 's satisfy  $C_{k+1} \subset C_k$  and  $\mu_G(L \setminus C_k) < \epsilon(1 - 2^{-k})$ , and also the  $C_k \cap K_l^k$  ( $= C_l^k$ ) are compact. Next, we choose  $A_k$ 's. For each  $C_{l_1}^1 \supset C_{l_2}^2 \supset \dots$ , there exists  $t_{l_1 l_2 \dots} \in \bigcap_{k=1}^\infty C_{l_k}^k$  by compactness of  $C_l^k$ 's. By induction on  $k$ , it is possible to choose  $A_k = \{t_{k,l}\}_{l=1}^{n_k}$  so that  $A_k \subset A_{k+1}$  and  $t_{k,l} = t_{l_1 l_2 \dots l_{k+1} \dots}$ , i.e.,  $l_k = l$ . These  $A_k$ 's satisfy (4)–(6). We may choose  $F_k$ 's so that they satisfy (3) and (8). This completes the proof of the Claim.  $\square$

Now, we return to the proof of Lemma 4. For  $t \in L$ , set  $U_t^{k,\nu} := \tilde{U}_{t_{k,l}}^\nu$  if  $t \in K_l^k$  and  $u_t^\nu := U_t^{k,\nu}$  for  $\nu \in F_k \setminus F_{k+1}$ . Set  $C := \bigcap_k C_k$ . Then  $\mu_G(L \setminus C) < \epsilon$  by

condition (5) of the Claim. Since  $U_t^{k,\nu}$ 's are continuous on each  $K_l^k \cap C_k (= C_l^k)$  and  $C_1^k, \dots, C_{n_k}^k$  are compact,  $U_t^{k,\nu}$ 's are continuous on each  $C_k$ . Hence they are continuous on  $C$ . Hence by the same argument as in [18, proof of Lemma 3.21], the map  $C \ni t \mapsto u_t^\nu$  is strongly continuous for each  $\nu \in \mathbb{N}$ , and  $\{C \ni t \mapsto u_t^\nu\}_\nu$  is  $\omega$ -equicontinuous and  $\pi_\omega(u_t^\nu) = U_t$  for all  $t \in C$ . Now, we have chosen  $\{u_t^\nu\}_\nu$  and  $C$  so that they satisfy conditions (2)–(4) of Lemma 4 and the following one:

(1)'  $\pi_\omega((u_t^\nu)_\nu) = U_t$  for  $t \in C$ .

Hence it remains to modify  $\{u_t^\nu\}_\nu$  so that  $\pi_\omega((u_t^\nu)_\nu) = U_t$  for almost all  $t \in L$ . By repeating the same process, we can find a sequence  $\{D_n\}_{n=0}^\infty$  of compact subsets of  $L$  and a sequence  $\{D_n \ni t \mapsto u_t^{n,\nu} \in U(M)\}_{n,\nu=0}^\infty$  of strongly continuous maps which satisfy the following conditions

- $\mu_G(L \setminus (\bigcup_{n=0}^\infty D_n)) = 0$  and  $D_n$ 's are mutually disjoint.
- $\pi_\omega((u_t^{n,\nu})_\nu) = U_t$  for  $t \in D_n$ .
- $D_0 = C$  and  $u_t^{0,\nu} = u_t^\nu|_C$  for all  $\nu \in \mathbb{N}$ .

Set  $u_t^\nu := u_t^{n,\nu}$  for  $t \in D_n$ . This  $\{u_t^\nu\}_\nu$  satisfies all conditions of Lemma 4. □

### §3.2. The averaging technique

Next, we develop the “averaging technique”. For the  $\mathbb{R}$ -action case, this means that it is possible to embed  $(M \otimes L^\infty([0, S]), \theta \otimes \text{translation})$  into  $(M_\theta^\omega, \theta)$  for any  $S > 0$ . This is a key lemma for the classification theorem. For the general case, we have the following lemma.

**Lemma 5.** *Let  $G$  be a locally compact abelian group and  $\theta$  be an action with the Rokhlin property of  $G$  on a factor  $M$ . Let  $L$  be a subset of  $G$  with the following properties:*

- *There exists a compactly generated clopen subgroup  $H$  of  $G$ , which is isomorphic to  $\mathbb{R}^n \times K \times \mathbb{Z}^m$  for some compact group  $K$  and non-negative integers  $n, m$ .*
- *The set  $L$  is a subset of  $H$ . When we identify  $H$  with  $\mathbb{R}^n \times K \times \mathbb{Z}^m$ ,  $L$  is of the form  $[0, S_1] \times \dots \times [0, S_n] \times K \times [0, N_1] \times \dots \times [0, N_2]$ . Note that  $L$  can be thought of as a quotient group of  $H$ .*

*Then there exist a unitary representation  $\{u_k\}_{k \in \hat{L}}$  of  $\hat{L}$  on  $M_{\omega, \theta}$  and an injective \*-homomorphism  $\Theta : M \otimes L^\infty(L) \rightarrow M_\theta^\omega$  with the following properties:*

- $\theta_t \circ \Theta = \Theta \circ (\theta_t \otimes \gamma_t)$ , where  $\gamma : H \curvearrowright L^\infty(L)$  denotes the translation.
- $\Theta(a \otimes \langle \cdot, k \rangle) = au_k$  for  $a \in M, k \in \hat{L}$ .

- $\tau^\omega \circ \Theta = \text{id}_M \otimes \mu_L$ , where  $\mu_L$  denotes the normalized Haar measure on  $L$ , which is the normalization of the restriction of a Haar measure on  $G$ , and  $\tau^\omega$  is the normal faithful conditional expectation as in Section 2.

In order to show this, by the same argument as in [18, Lemma 5.2] (in this part, we use the fact that  $M$  is a factor), it is enough to show the following:

**Proposition 6.** *Let  $\theta : G \curvearrowright M$  be an action with the Rokhlin property of a locally compact abelian group  $G$  on a factor  $M$ , and  $L \subset H$  be subsets of  $G$  as in the above lemma. Then there exists a family of unitaries  $\{u_k\}_{k \in \hat{L}} \subset U(M_{\omega, \theta})$  with the following properties:*

- $\theta_t(u_k) = \langle t, k \rangle u_k$  for  $t \in H$ .
- The map  $k \mapsto u_k$  is an injective group homomorphism.

To show the above proposition, we need to prepare some lemmas. In the rest of this subsection,  $\theta, G, H$  and  $L$  are as in Proposition 6.

**Lemma 7.** *Let  $C$  be a subgroup of  $\hat{L}$  isomorphic to  $\mathbb{Z}/l\mathbb{Z}$ . Then there exists a family of unitaries  $\{u_k\}_{k \in C} \subset M_{\omega, \theta}$  with the following properties:*

- (1)  $\theta_t(u_k) = \langle t, k \rangle u_k$  for  $t \in H$ .
- (2) The map  $C \ni k \mapsto u_k$  is an injective group homomorphism.

*Proof.* Let  $p$  be a generator of  $C$ . Since  $\theta$  has the Rokhlin property, there exists a unitary  $w$  of  $M_{\omega, \theta}$  satisfying  $\theta_t(w) = \langle t, p \rangle w$  for  $t \in H$ . Since  $w^l \in M_{\omega, \theta}^\theta$ , there exists a unitary  $v$  of  $M_{\omega, \theta}^\theta \cap \{w\}'$  such that  $v^{-l} = w^l$ . Set  $u := vw$  and  $u_k := u^k$ . Then the family  $\{u_k\}_{k \in \mathbb{Z}/l\mathbb{Z}}$  does the job. □

By the same argument as in [18, proof of Lemma 3.16], we have the following lemma. See also Ocneanu [22, Lemma 5.3].

**Lemma 8** (Fast reindexation trick). *Let  $\theta$  be an action of  $G$  on a von Neumann algebra  $M$  and let  $F \subset M^\omega$  and  $N \subset M_\theta^\omega$  be separable von Neumann subalgebras. Suppose that the subalgebra  $N$  is globally invariant by  $\theta$ . Then there exists a faithful normal  $*$ -homomorphism  $\Phi : N \rightarrow M_\theta^\omega$  with the following properties:*

$$\begin{aligned} \Phi &= \text{id} && \text{on } F \cap M, \\ \Phi(N \cap M_{\omega, \theta}) &\subset F' \cap M_{\omega, \theta}, \\ \tau^\omega(\Phi(a)x) &= \tau^\omega(a)\tau^\omega(x) && \text{for all } a \in N, x \in F, \\ \theta_t \circ \Phi &= \Phi \circ \theta_t && \text{on } N \text{ for all } t \in L. \end{aligned}$$



**Lemma 9.** *Let  $C$  be a subgroup of  $\hat{L}$  of the form  $\mathbb{Z}^n \times F$ , where  $F := \bigoplus_{k=1}^m \mathbb{Z}/(l_k \mathbb{Z})$  is a finite abelian group. Then there exists a family of unitaries  $\{u_k\}_{k \in C} \subset M_{\omega, \theta}$  which satisfies the following conditions:*

- (1)  $\theta_t(u_k) = \langle t, k \rangle u_k$  for  $t \in H$ .
- (2) The map  $k \mapsto u_k$  is an injective group homomorphism.

*Proof.* Let  $\{p_1, \dots, p_n, q_1, \dots, q_m\}$  be a base of  $\hat{C}$ . Then there exist unitaries  $\{u_i\}_{i=1}^n$  and  $\{v_j\}_{j=1}^m$  with  $\theta_t(u_i) = \langle t, p_i \rangle u_i$ ,  $\theta_t(v_j) = \langle t, q_j \rangle v_j$  for  $t \in H$ . By Lemma 7, we may assume that  $v_j^{l_j} = 1$ . By using the fast reindexation trick, one can choose  $\{u_i\}_{i=1}^n$  and  $\{v_j\}_{j=1}^m$  so that they pairwise commute.  $\square$

*Proof of Proposition 6.* Let  $\psi \in M_*$  be a normal faithful state and let  $\Phi = \{\phi_m\}$  be a countable dense subset of the unit ball of  $M_*$ . There exists an increasing sequence  $\{C_\nu\}$  of finitely generated subgroups of  $\hat{L}$  satisfying  $\hat{L} = \bigcup_{\nu=1}^\infty C_\nu$ . Then by the structure theorem for finitely generated abelian groups and the above lemma, for each  $\nu$ , there exists a family of unitaries  $\{u'_k\}_{k \in C_\nu} \subset U(M_{\omega, \theta})$  with  $C_\nu \ni k \mapsto u'_k$  satisfying conditions (1) and (2) of Lemma 9. For each  $k \in \hat{L}$ , define a sequence  $\{k_\nu\}$  in  $\hat{L}$  by

$$k_\nu = \begin{cases} k & \text{if } k \in C_\nu, \\ 0 & \text{if } k \notin C_\nu. \end{cases}$$

For each  $\nu \in \mathbb{N}$  and  $k \in C_\nu$ , take a representing sequence  $\{u_k^{\nu, n}\}$  of  $u'_k$ . Choose a sequence  $\{E_\nu\}$  of finite subsets of  $\hat{L}$  satisfying  $\bigcup E_\nu = \hat{L}$  and  $E_\nu \subset C_\nu$  for all  $\nu \in \mathbb{N}$ . By Lemma 3.3 of [18], the convergence

$$\lim_{n \rightarrow \omega} \|\theta_t(u_k^{\nu, n}) - \langle t, k \rangle u_k^{\nu, n}\|_\psi^\sharp = 0$$

is uniform for  $t \in L$ . Hence it is possible to choose  $F_\nu \in \omega$  ( $\nu = 1, 2, \dots$ ) so that

$$(3.1) \quad F_\nu \subsetneq F_{\nu-1} \subset [\nu - 1, \infty), \quad \nu = 2, 3, \dots,$$

$$(3.2) \quad \|u_k^{\nu, n} u_l^{\nu, n} - u_{k+l}^{\nu, n}\|_\psi^\sharp < 1/\nu, \quad k, l \in E_\nu, n \in F_\nu,$$

$$(3.3) \quad \|[\phi_m, u_k^{\nu, n}]\| < 1/\nu, \quad k \in E_\nu, m \leq \nu, n \in F_\nu,$$

$$(3.4) \quad \|\theta_t(u_k^{\nu, n}) - \langle t, k \rangle u_k^{\nu, n}\|_\psi^\sharp < 1/\nu, \quad k \in E_\nu, t \in L, n \in F_\nu.$$

Set  $(u_k)_n := u_{k_\nu}^{\nu, n}$  for  $n \in F_\nu \setminus F_{\nu+1}$ . We will show that  $u_k := \{(u_k)_n\}$  is the desired family of unitaries.

To show  $u_k \in M_\omega$ , fix  $\mu \in \mathbb{N}$  and  $k \in \hat{L}$ . Then there exists  $\nu \geq \mu$  with  $k \in E_\nu$ , and for  $n \in F_\nu$ , there exists a unique  $\lambda \geq \nu$  satisfying  $n \in F_\lambda \setminus F_{\lambda+1}$ . Thus by (3.3),

$$\|[\phi_m, (u_k)_n]\| = \|[\phi_m, (u_{k_\lambda}^{\lambda, n})]\| < 1/\lambda \leq 1/\mu$$

for  $m \leq \mu$ . Thus  $u_k \in M_\omega$ .

In a similar way, we obtain  $\theta_t(u_k) = \langle t, k \rangle u_k$ , using (3.4). One can also show that the map  $\hat{L} \ni k \mapsto u_k$  is a unitary representation by using (3.2).  $\square$

**§3.3. Cohomology vanishing**

By using Lemma 5, we show the following two propositions. See also Theorems 5.5 and 5.11 of [18], respectively.

**Proposition 10** (2-cohomology vanishing). *Let  $(\theta, c)$  be a Borel cocycle action of a locally compact abelian group  $G$  on a factor  $M$ . Suppose that  $(\theta, c)$  has the Rokhlin property. Then the 2-cocycle  $c$  is a coboundary, that is, there exists a Borel unitary map  $v : G \rightarrow U(M)$  such that*

$$v_t \theta_t(v_s) c(t, s) v_{t+s}^* = 1 \quad \text{for almost every } (t, s) \in G^2.$$

Furthermore, if  $\|c(t, s) - 1\|_\phi^\sharp$  and  $\|[c(t, s), \phi]\|$  ( $\phi \in M_*$ ) are small, then one can choose  $v_t$  so that  $\|v_t - 1\|_\phi^\sharp$  and  $\|[v_t, \phi]\|$  are small. We will explain this later.

**Proposition 11** (Approximate 1-cohomology vanishing). *Let  $\theta$  be an action with the Rokhlin property of a locally compact abelian group  $G$  on a factor  $M$ . Let  $\epsilon, \delta$  be positive numbers and  $\Phi$  be a compact subset of the unit ball of  $M_*$ . Let  $H$  be a compactly generated clopen subgroup of  $G$ , which is isomorphic to  $\mathbb{R}^n \times K \times \mathbb{Z}^m$  for some compact abelian group  $K$  and non-negative integers  $n, m$ . Let  $T, L$  be subsets of  $H$  which satisfy the following conditions:*

- When we identify  $H$  with  $\mathbb{R}^n \times K \times \mathbb{Z}^m$ ,  $L$  is of the form

$$[0, S_1) \times \cdots \times [0, S_n) \times K \times [0, N_1) \times \cdots \times [0, N_m),$$

which implies that  $L$  is a compact quotient of  $H$ .

- We have

$$\frac{\mu_G(\bigcap_{t \in T} (t + L))}{\mu_G(L)} > 1 - 4\epsilon^2.$$

Then for any  $\theta$ -cocycle  $u_t$  with

$$\frac{1}{\mu_G(L)} \int_L \|[u_t, \phi]\| d\mu_G(t) < \delta \quad \text{for all } \phi \in \Phi,$$

there exists a unitary  $w \in M$  such that

$$\begin{aligned} \|[w, \phi]\| &< 3\delta \quad \text{for all } \phi \in \Phi, \\ \|\phi \cdot (u_t \theta_t(w) w^* - 1)\| &< \epsilon, \\ \|(u_t \theta_t(w) w^* - 1) \cdot \phi\| &< \epsilon \quad \text{for all } t \in T, \phi \in \Phi. \end{aligned}$$

By carefully examining the proofs of [18, Theorems 5.5 and 5.11], we notice that we need to choose sequences  $\{L_n\}$  and  $\{T_n\}$  of subsets of  $G$  with the following properties:

- (1) There exists an increasing sequence  $\{H_k\}$  of compactly generated clopen subgroups of  $G$  with  $\bigcup_k H_k = G$  and  $L_k, T_k$  are subsets of  $H_k$ ; moreover,  $T_k$ 's are compact. When we identify  $H_k$  with  $\mathbb{R}^{n_k} \times K_k \times \mathbb{Z}^{m_k}$  for some compact abelian group  $K_k$  and nonnegative integers  $n_k, m_k$ , the subset  $L_k$  is of the form

$$[0, S_1) \times \cdots \times [0, S_{n_k}) \times K_k \times [0, N_1) \times \cdots \times [0, N_{m_k}).$$

- (2) The translation  $H_k \curvearrowright L^\infty(L_k)$  is embedded into  $(\theta, M_{\omega, \theta})$  (see Proposition 6).
- (3)  $\mu_G(L_k \setminus \bigcap_{t \in T_k + T_k} (t + L_k)) / \mu_G(L_k)$  is small.
- (4)  $L_k + T_k \subset T_{k+1}$ .
- (5)  $T_k \subset T_{k+1}$  for all  $k \in \mathbb{N}$  and  $\bigcup_{k=1}^\infty T_k = G$ .

For the  $\mathbb{R}$ -action case,  $L_k = [0, s_k)$  and  $T_k = [-t_k, t_k)$  with  $t_k \ll s_k \ll t_{k+1}$  do the job. In the following, we explain how to choose  $L_k$ 's and  $T_k$ 's for the general case. First, we show that there exists an increasing sequence  $\{H_k\}$  of clopen subgroups of  $G$  with the following properties:

- (6) For each  $k$ , the subgroup  $H_k$  is compactly generated, hence isomorphic to  $\mathbb{R}^n \times K_k \times \mathbb{Z}^{m_k}$  for some compact abelian group  $K_k$ . Note that the multiplicity  $n$  in  $\mathbb{R}$  of  $H_k$  can be chosen to be independent of  $k$  by [8, Theorem 9.14].
- (7)  $\bigcup_k H_k = G$ .

This increasing sequence is chosen in the following way. There exists an increasing sequence  $\{O_k\}$  of open subsets of  $G$  such that  $\overline{O_k}$ 's are compact,  $0 \in O_k$  for all  $k \in \mathbb{N}$  and  $\bigcup_k \overline{O_k} = G$ . For each  $k \in \mathbb{N}$ , let  $H_k$  be the subgroup of  $G$  generated by  $\overline{O_k}$ . We show that  $H_k$  is clopen. If  $t \in H_k$ , then  $t + O_k \subset H_k$ . Hence this is open. Hence by [8, Theorem 5.5],  $H_k$  is closed. By [8, Theorem 9.14],  $H_k$  is of the form  $\mathbb{R}^n \times K_k \times \mathbb{Z}^{m_k}$ .

Next, take two sequences  $\{L_k\}$  and  $\{T_k\}$  of subsets of  $G$  and a decreasing sequence  $\{\epsilon_k\} \subset \mathbb{R}_{>0}$  with the following properties:

- (8)  $L_k, T_k$  are subsets of  $H_k$ . When we identify  $H_k$  with  $\mathbb{R}^n \times K_k \times \mathbb{Z}^{m_k}$  for some compact abelian group  $K_k$  and a non-negative integer  $m_k$ , the subset  $L_k$  is of the form

$$[0, S_1) \times \cdots \times [0, S_n) \times K_k \times [0, N_1) \times \cdots \times [0, N_{m_k}).$$

Note that how we identify  $H_k$  with  $\mathbb{R}^n \times K_k \times \mathbb{Z}^{m_k}$  is not important. What is important is that  $L_k$  is a quotient of a clopen subgroup of  $G$ .

(9) We have

$$\frac{\mu_G(L_k \setminus \bigcap_{t \in T_k + T_k} (t + L_k))}{\mu_G(L_k)} > 1 - \left( \frac{\epsilon_k}{6\mu_G(T_k)^2} \right)^2.$$

(10)  $T_k + L_k \subset T_{k+1}$ ,  $\bigcup_k T_k = G$  and  $T_k$ 's are compact.

(11)  $0 < \epsilon_k < 1/k$  and

$$\sum_{k=n+1}^{\infty} \sqrt{13\mu_G(T_k)\epsilon_k} < \epsilon_n.$$

From now on, we explain how to choose  $\{L_k\}$  and  $\{T_k\}$ . For each  $k \in \mathbb{N}$ , set  $A_k := \overline{O_k}$ , where  $O_k$  is as in (7).

Assume that  $(T_l, L_l, \epsilon_l)$ ,  $l \leq k$ , are chosen. Then since  $A_{k+1} + T_k + \overline{L_k}$  is compact, we can choose  $T_{k+1} \subset H_{k+1}$  so that when we identify  $H_{k+1}$  with  $\mathbb{R}^n \times K_{k+1} \times \mathbb{Z}^{m_{k+1}}$ ,  $T_{k+1}$  is of the form

$$[-t_1, t_1] \times \cdots \times [-t_n, t_n] \times K_{k+1} \times [-M_1, M_1] \times \cdots \times [-M_{m_{k+1}}, M_{m_{k+1}}]$$

and  $A_{k+1} + T_k + L_k \subset T_{k+1}$ . Since  $\bigcup_k A_k = G$ , we also have  $\bigcup_k T_k = G$ . Choose  $\epsilon_{k+1} > 0$  so that

$$\epsilon_{k+1} < \epsilon_k, \quad \sqrt{13\mu_G(T_{k+1})\epsilon_{k+1}} < \epsilon_k/2^k.$$

Choose  $L_{k+1} \subset H_{k+1}$  so large that  $L_{k+1}$  satisfies (8) and (9). Thus we are done.

By using the above sequences  $\{L_k\}$ ,  $\{T_k\}$  instead of  $\{S_k\}$  and  $\{T_k\}$  of [18, (5.14)], Propositions 10 and 11 are shown by similar arguments to those of [18, proofs of Theorems 5.5 and 5.11]. Furthermore, one can choose  $v_t$  in Proposition 10 so that  $v_t$  satisfies the following conditions:

- If for some  $n \geq 2$  and a finite subset  $\Phi \subset (M_*)_+$ ,

$$\int_{T_{n+1}} d\mu_G(t) \int_{T_{n+1}} d\mu_G(s) \|c(t, s) - 1\|_{\phi}^{\sharp} \leq \epsilon_{n+1} \quad \text{for all } \phi \in \Phi,$$

then one can choose  $v_t$  so that

$$\int_{T_n} \|v_t - 1\|_{\phi}^{\sharp} d\mu_G(t) < \epsilon_{n-1} d(\Phi)^{1/2} \quad \text{for all } \phi \in \Phi.$$

Here,  $d(\Phi) := \max(\{1\} \cup \{\|\phi\| \mid \phi \in \Phi\})$ .

- If for some  $n \geq 2$  and a finite subset  $\Phi \subset M_*$ ,

$$\int_{T_{n+1}} d\mu_G(t) \int_{L_{n+1}} d\mu_G(s) \|[c(t, s), \phi]\| < \epsilon \quad \text{for all } \phi \in \Phi,$$

then one can choose  $v_t$  satisfying

$$\int_{T_n} \|[v_t, \phi]\| d\mu_G(t) \leq (3\epsilon_{n-1} + 3\epsilon) d(\Phi) \quad \text{for all } \phi \in \Phi.$$

In the proof, the following points are slightly different.

- (1) The inequality corresponding to [18, (5.12)] is

$$\frac{2\mu_G(L \setminus (\bigcap_{t \in T+T} t + L))^{1/2}}{\mu_G(L)^{1/2}} < \frac{\delta}{6\mu_G(T)^2}.$$

- (2) We need to show a lemma which corresponds to [18, Lemma 5.4]. In the proof, the inequality corresponding to [18, (5.13)] is

$$\begin{aligned} \|U_t \alpha_t(U_s) c(t, s) U_{s+t}^* - 1\|_{\phi}^{\sharp} &\leq \| \chi_{\bigcap_{t \in T+T} t+L} - 1 \|_{\phi \otimes \mu_L}^{\sharp} \\ &\quad + \| \chi_{L \setminus (\bigcap_{t \in T+T} t+L)} ((\text{a unitary valued function}) - 1) \|_{\phi \otimes \mu_L}^{\sharp} \\ &\leq 0 + 2 \| \chi_{L \setminus (\bigcap_{t \in T+T} t+L)} \|_{\phi \otimes \mu_L}^{\sharp} \\ &\leq 2 \| \phi \|^{1/2} \frac{\mu_G(L \setminus (\bigcap_{t \in T+T} t + L))^{1/2}}{\mu_G(L)^{1/2}} < \frac{\delta}{6\mu_G(T)^2} \end{aligned}$$

for all  $t, s \in T$ ,  $\phi \in \Phi$ . The other parts of the proof are completely the same.

- (3) In [18, proof of Theorem 5.5], the inequality

$$\int_{T_n}^{T_n} \|W^* u_t \alpha_t^n(W) - 1\|_2^2 dt < 18\epsilon_n$$

is shown. Instead, in the proof of Proposition 10, we show

$$\begin{aligned} &\int_{T_n} \|W^* u_t \alpha_t^n(W) - 1\|_2^2 d\mu_G(t) \\ &\leq \frac{2}{\mu_G(L_n)} \int_{T_n} d\mu_G(t) \left( \int_{\bigcap_{t \in T_n} t+L_n} d\mu_G(s) \| \tilde{u}_s^* u_t \alpha_t^n(\tilde{u}_{s-t}) - 1 \|_2^2 \right. \\ &\quad \left. + \int_{L_n \setminus \bigcap_{t \in T_n} t+L_n} d\mu_G(s) \| \tilde{u}_s^* u_t \alpha_t^n(\tilde{u}_{s-t}) - 1 \|_2^2 \right) \\ &\leq \frac{2}{\mu_G(L_n)} \int_{T_n} d\mu_G(t) \int_{\bigcap_{t \in T_n} t+L_n} d\mu_G(s) \| \tilde{u}_s^* u_t \alpha_t^n(\tilde{u}_{s-t}) - 1 \|_2^2 \\ &\quad + \frac{8}{\mu_G(L_n)} \mu_G(T_n) \mu_G\left(L_n \setminus \bigcap_{t \in T_n} t + L_n\right) \\ &< \frac{2}{\mu_G(L_n)} \int_{T_{n+1} \times T_{n+1}} d\mu_G(t) d\mu_G(s) \| \tilde{u}_s^* u_t \alpha_t^n(\tilde{u}_{s-t}) - 1 \|_2^2 \\ &\quad + \mu_G(T_n) \frac{\epsilon_n^2}{18\mu_G(T_n)^4} \\ &< 9\epsilon_n. \end{aligned}$$

The other parts of the proof of Proposition 10 are the same as the corresponding parts of the proof of [18, Theorem 5.5].

(4) In the proof of Proposition 11, we need to show that

$$\|u_t \alpha_t(W)W^* - 1\|_{|\phi|^\omega}^\# \leq 2\|\chi_{L \setminus (\cap_{t \in T} t+L)}\|_{|\phi| \otimes \mu_L}^\#,$$

which corresponds to

$$\|u_t \alpha_t(W)W^* - 1\|_{|\phi|^\omega}^\# \leq 2 \frac{t^{1/2} \|\phi\|^{1/2}}{S^{1/2}}$$

in [18, proof of Theorem 5.11]. This is obtained by a similar computation to (3) above.

By using Proposition 10, one can prove the following lemma, which corresponds to [18, Lemma 5.8].

**Lemma 12.** *Let  $\alpha, \beta$  be actions with the Rokhlin property of a locally compact abelian group  $G$  on a factor  $M$ . Suppose that  $\alpha_t \circ \beta_{-t} \in \overline{\text{Int}}(M)$  for all  $t \in G$ . Let  $H$  be a compactly generated clopen subgroup of  $G$  and  $T$  be a subset of  $H$  such that when we identify  $H$  with  $\mathbb{R}^n \times K \times \mathbb{Z}^m$  for some compact abelian group  $K$  and non-negative integers  $n, m$ , then  $T$  is of the form*

$$[-t_1, t_1] \times \cdots \times [-t_n, t_n] \times K \times [-M_1, M_1] \times \cdots \times [-M_m, M_m].$$

Then for any  $\epsilon > 0$  and a finite set  $\Phi \subset M_*$ , there exists an  $\alpha$ -cocycle  $u$  such that

$$\int_T \|\text{Ad } u_t \circ \alpha_t(\phi) - \beta_t(\phi)\| d\mu_G(t) < \epsilon \quad \text{for all } \phi \in \Phi.$$

In the proof of this lemma, the set corresponding to [18, (5.18)] is obtained in the following way. For a small  $\eta > 0$ , take a small  $r > 0$  so that

$$\|\alpha_t(\phi) - \phi\| < \eta, \quad \|\beta_t(\phi) - \phi\| < \frac{2\eta}{\mu_G(T)}$$

for all  $\phi \in \Phi$  and  $t \in G$  with  $d(t, 0) < r$ . Choose  $A(r, T) := \{t_j\}_{j=1}^N$  so that for any  $t \in T$ , there exists  $t_j \in A(r, T)$  with  $d(t, t_j) < r$ . This is possible because  $T$  is compact.

Now, we return to the proof of Theorem 2. The proof is basically the same as that of [18, Case 2 of Lemma 5.12]. Here, we only explain the outline. We apply Proposition 11 and Lemma 12 alternately (the Bratteli–Elliott–Evans–Kishimoto type argument). However, we need to change the following part. In the proof of [18, Case 2 of Lemma 5.12], one takes  $\{M_n\} \subset \mathbb{N}$  and  $\{A(M_n, T_n)\}$ , which appear in conditions (n.1) and (n.8) there. Instead, in (n.8), take  $r_n \in \mathbb{R}_{>0}$  so that

$$\|(v^{\hat{n}}(t) - v^{\hat{n}}(s)) \cdot \phi\| < \epsilon_n, \quad \|\phi \cdot (v^{\hat{n}}(t) - v^{\hat{n}}(s))\| < \epsilon_n$$

for  $t, s \in T_n$  with  $d(t, s) < r_n$ , and  $\phi \in \hat{\Phi}_{n-1}$ . Choose a finite subset  $A(r_k, T_k)$  of  $T_k$  so that for each  $t \in T_k$ , there exists  $t_0 \in A(r_k, T_k)$  with  $d(t, t_0) < r_k$ . This is possible because  $T_k$  is compact.

**§4. Examples**

Here, we will give some examples of Rokhlin actions. First, we consider actions which fix Cartan subalgebras. This type of examples have been classified by Kawahigashi [11]. One of the most important examples of actions of this form is an infinite tensor product action.

Let  $\{p_n\}$  be a sequence in the dual group  $\hat{G}$  of  $G$ . Set

$$M := \bigotimes_{n=1}^{\infty} (M_2(\mathbb{C}), \text{tr}),$$

We define an action  $\theta$  of  $G$  on  $M$  by

$$\theta_t := \bigotimes \text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & \langle t, p_n \rangle \end{pmatrix}.$$

Then  $\theta$  has the Rokhlin property if and only if the set

$$A := \{p \in \hat{G} : \text{there exists a subsequence of } \{p_n\} \text{ which converges to } p\}$$

generates a dense subgroup  $\Gamma$  in  $\hat{G}$ . We prove the “if” part. Here, we show this implication in the case where for each  $p \in A$ , a subsequence of  $\{p_n\}$  which converges to  $p$  can be chosen to be a constant sequence. This case is needed for the proof of Example 13. The general case of this implication will follow from Example 13.

Choose  $p \in A$ . By ignoring other tensor components, we may assume that  $p_n = p$  for all  $n$ . For each  $m \in \mathbb{N}$ , set

$$S^m := \{\sigma : \{1, \dots, 2m - 1\} \rightarrow \{1, 2\} \mid \#\sigma^{-1}(1) = m - 1, \#\sigma^{-1}(2) = m\}.$$

For  $\sigma \in S^m$ ,  $m \in \mathbb{N}$  and  $k \in \{1, \dots, 2m - 1\}$ , set  $\tau(k) := 3 - \sigma(k)$  and

$$v_\sigma := e_{\tau(1)\sigma(1)} \otimes \cdots \otimes e_{\tau(2m-1)\sigma(2m-1)} \otimes 1 \otimes \cdots .$$

Then we have

$$\begin{aligned} e_\sigma &:= v_\sigma^* v_\sigma = e_{\sigma(1)\sigma(1)} \otimes \cdots \otimes e_{\sigma(2m-1)\sigma(2m-1)} \otimes 1 \otimes \cdots , \\ f_\sigma &:= v_\sigma v_\sigma^* = e_{\tau(1)\tau(1)} \otimes \cdots \otimes e_{\tau(2m-1)\tau(2m-1)} \otimes 1 \otimes \cdots , \\ \theta_t(v_\sigma) &= \langle t, p \rangle v_\sigma \end{aligned}$$

for  $t \in G$ . Hence if we set

$$T := \bigcup_{m=1}^{\infty} \{ \sigma \in S^m \mid \#(\sigma^{-1}(1) \cap \{1, \dots, k\}) \geq \#(\sigma^{-1}(2) \cap \{1, \dots, k\}) \}$$

for  $k = 1, \dots, 2m - 2$ ,

then  $\{e_\sigma\}_{\sigma \in T}$  and  $\{f_\sigma\}_{\sigma \in T}$  are orthogonal families. We will show that  $\sum_{\sigma \in T} e_\sigma = 1$ , which implies that  $\sum_{\sigma \in T} v_\sigma$  is a unitary. Consider the gambler's ruin problem when one player has infinite money, the other has no money and they have equal chances to win. Then  $\|\sum_{\sigma \in T} e_\sigma\|_1$  is equal to the probability of the poor's ruin. This is 1. Set

$$u_n := 1 \otimes \cdots \otimes 1 \otimes \sum_{\sigma \in T} v_\sigma \in M_2(\mathbb{C})^{\otimes n} \otimes M.$$

Then  $\{u_n\} \in M_{\omega, \theta}$  and  $\theta_t((u_n)_\omega) = \langle t, p \rangle (u_n)_\omega$  for  $t \in G$ . By assumption, the set  $A$  generates a dense subgroup of  $\hat{G}$ . Hence  $\theta$  has the Rokhlin property.

Conversely, assume that the subgroup  $\Gamma$  is not dense in  $\hat{G}$ . Then there exists a non-empty open subset  $U$  of  $\hat{G}$  with  $U \cap \Gamma = \emptyset$ . By a similar argument to that in [11, proof of Proposition 1.2], it is shown that the Connes spectrum of  $\theta$  and  $U$  do not intersect, which implies that  $\theta$  does not have the Rokhlin property.

**Example 13** (see also Kawahigashi [11, Corollary 1.9]). Let  $\alpha$  be an action of a locally compact abelian group  $G$  on an AFD factor  $R$  of type  $\text{II}_1$ . Assume that  $\alpha$  fixes a Cartan subalgebra of  $R$ . Then  $\alpha$  has the Rokhlin property if and only if its Connes spectrum is  $\hat{G}$ .

The proof is just a combination of an analogue of [11, Corollary 5.17], which follows from the above example of an infinite tensor product action and Theorem 2, and [18, Lemma 6.2]. In the proof, the crucial fact is that invariantly approximate innerness (see [18, Definition 4.5]) is the dual of the Rokhlin property. This fact is shown by completely the same argument as in [18, proof of Theorem 4.11].

By this example and the main theorem, all the actions fixing Cartan subalgebras with full Connes spectrum are cocycle conjugate to an infinite tensor product action with full Connes spectrum.

**Example 14** (see [18, Theorem 6.12]). Let  $\theta$  be an almost periodic minimal action of a locally compact abelian group  $G$  on an AFD factor of type  $\text{II}_1$ . Then  $\theta$  has the Rokhlin property.

*Proof.* An almost periodic action is the restriction of a compact abelian group action to the dense subgroup (see Thomsen [27, Proposition 7.3]). If  $\theta$  is minimal,



then the original compact group action is also minimal, which is unique up to cocycle conjugacy by Jones–Takesaki [9]. This has the Rokhlin property.  $\square$

There exist “many” Rokhlin actions on AFD factors of type  $\text{III}_0$ . The following is a generalization of a part of [24].

**Example 15** (see [24, Theorem 5]). Let  $\alpha$  be an action of a locally compact abelian group  $G$  on an AFD factor  $M$  and  $(\theta, Z)$  be the flow of weights of  $M$ . Assume that an action  $\{\text{mod}(\alpha_g) \circ \theta_t\}_{(g,t) \in G \times \mathbb{R}}$  of  $G \times \mathbb{R}$  is faithful. Then  $\alpha$  has the Rokhlin property. Here,  $\text{mod}(\alpha)$  denotes the Connes–Takesaki module of  $\alpha$  (see Connes–Takesaki [5], Haagerup–Størmer [7]).

*Proof.* The proof is the same as that of [24, case  $\ker = 0$  of Lemma 6]. In the proof in [24], we use Rokhlin’s lemma for actions of  $\mathbb{R}^2$ . Rokhlin’s lemma for actions of  $G \times \mathbb{R}$  also holds, which is shown by the same argument as in Lind [16, Lemma] and [6, Theorem 1].  $\square$

The classification theorem is also applicable to actions of locally compact abelian groups on non-McDuff factors.

**Example 16.** Let  $(X_0, \mu_0)$  be a probability measured space and let  $\theta_0: G \curvearrowright L^\infty(X_0, \mu_0)$  be a faithful  $\mu_0$ -preserving action (if  $G$  is compactly generated, an example of such an action can be constructed by taking a direct product of increasing compact quotients). Set  $D := \bigotimes_{\mathbb{Z}} L^\infty(X_0, \mu_0)$ . Let  $\alpha: \mathbb{Z} \curvearrowright D$  be the Bernoulli shift and let  $\theta: G \curvearrowright D$  be the diagonal action of  $\theta_0$ . Then  $\theta$  canonically extends to  $M := D \rtimes_{\alpha * \alpha} \mathbf{F}_2$ , which is a non-McDuff factor of type  $\text{II}_1$  (see, for example, Ueda [28, Theorem 10]). Then this action has the Rokhlin property.

*Proof.* The proof is the same as that of [23, Theorem 3.3].  $\square$

Although there are Rokhlin actions on non-McDuff factors, as mentioned in [23], the effect of our classification theorem is limited because  $\overline{\text{Int}}(M)$  is too small in many cases.

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