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Elliptic Ding–Iohara Algebra and the Free Field Realization of the Elliptic Macdonald Operator

by

Yosuke SAITO

Abstract

The Ding–Iohara algebra is a quantum algebra arising from the free field realization of the Macdonald operator. Starting from the elliptic kernel function introduced by Komori, Noumi and Shiraishi, we define an elliptic analog of the Ding–Iohara algebra. The free field realization of the elliptic Macdonald operator is also constructed.

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Contents

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Y. Saito: Mathematical Institute, Tohoku University, Sendai 980-8579, Japan;
e-mail: `yousukesaitou7@gmail.com`

Notations. In this paper, we use the following symbols:

$\mathbb{Q}(q,t)$: the field of rational functions of q, t over $\mathbb{Q}$,

$\mathbb{C}[[z,z^{-1}]]$: the set of formal power series of z, z^{-1} over $\mathbb{C}$,

The q-infinite product: $(x;q)_\infty := \prod_{n\geq 0}(1-xq^n) \quad (|q|<1)$,

$(x;q)_n := \dfrac{(x;q)_\infty}{(q^n x;q)_\infty} \quad (n\in\mathbb{Z})$,

The theta function: $\Theta_p(x) := (p;p)_\infty (x;p)_\infty (px^{-1};p)_\infty$,

The double infinite product: $(x;q,p)_\infty := \prod_{m,n\geq 0}(1-xq^m p^n)$,

The elliptic gamma function: $\Gamma_{q,p}(x) := \dfrac{(qpx^{-1};q,p)_\infty}{(x;q,p)_\infty}$.

For the theta function and the elliptic gamma function, the following relations hold:

$$\Theta_p(x) = -x\Theta_p(x^{-1}), \quad \Theta_p(px) = -x^{-1}\Theta_p(x),$$
$$\Gamma_{q,p}(qx) = \frac{\Theta_p(x)}{(p;p)_\infty}\Gamma_{q,p}(x), \quad \Gamma_{q,p}(px) = \frac{\Theta_q(x)}{(q;q)_\infty}\Gamma_{q,p}(x).$$

§1. Introduction

The aims of this paper are to introduce an elliptic analog of the Ding–Iohara algebra and to construct the free field realization of the elliptic Macdonald operator. We accomplish this by starting from the elliptic kernel function defined below. Let us explain some background and motivations.

Relations between quantum algebras and Macdonald symmetric functions have been studied by several authors. One of the most remarkable results is the construction of the q-Virasoro algebra and the q-W_N algebra by Awata, Odake, Kubo, and Shiraishi [24], [3], [4]. It is known that singular vectors of the Virasoro algebra and of the W_N algebra correspond to Jack symmetric functions [8]. On the other hand, Macdonald symmetric functions are q-analogs of Jack symmetric functions [2], [17]. Awata, Odake, Kubo, and Shiraishi constructed the q-Virasoro algebra and the q-W_N algebra whose singular vectors correspond to Macdonald symmetric functions:

$$\begin{array}{ccc} q\text{-Virasoro algebra, } q\text{-}W_N \text{ algebra} & \xrightarrow{\text{singular vectors}} & \text{Macdonald symmetric functions} \\ \uparrow q\text{-deformation} & & \uparrow q\text{-deformation} \\ \text{Virasoro algebra, } W_N \text{ algebra} & \xrightarrow{\text{singular vectors}} & \text{Jack symmetric functions} \end{array}$$

In the middle 2000's, new material emerges from the free field realization of the Macdonald operator. The *Macdonald operator* $H_N(q,t)$ $(N \in \mathbb{Z}_{>0})$ is defined by

$$H_N(q,t) := \sum_{i=1}^{N} \prod_{j\neq i} \frac{tx_i - x_j}{x_i - x_j} T_{q,x_i} \quad (T_{q,x_i} f(x_1,\dots,x_N) := f(x_1,\dots,qx_i,\dots,x_N))$$

and its free field realization tells us that we can reproduce the operator from boson operators. As we will see in Section 2, the free field realization of the Macdonald operator is based on the kernel function

$$\Pi(q,t)(x,y) := \prod_{i,j} \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty}.$$

It has been realized that from the free field realization of the Macdonald operator, a certain quantum algebra arises, the *Ding–Iohara algebra* [10], [18], [11]. Recently this algebra has been applied to several objects of mathematical physics, such as the AGT conjecture [8], [9], [1], as well as the refined topological vertex which is used to calculate amplitudes and partition functions in topological string theory [2].

On the other hand, on the elliptic theory side it is well-known that the Macdonald operator allows the elliptic analog defined in [20],

$$H_N(q,t,p) := \sum_{i=1}^{N} \prod_{j\neq i} \frac{\Theta_p(tx_i/x_j)}{\Theta_p(x_i/x_j)} T_{q,x_i}, \tag{1.1}$$

and the kernel function for this operator was introduced by Komori, Noumi, and Shiraishi [15]:

$$\Pi(q,t,p)(x,y) := \prod_{i,j} \frac{\Gamma_{q,p}(x_iy_j)}{\Gamma_{q,p}(tx_iy_j)}.$$

Since the free field realization of the Macdonald operator is available, it can be expected that the above operator (1.1) can be derived from it. In [11], Feigin, Hashizume, Hoshino, Shiraishi and Yanagida constructed the free field realization of the elliptic Macdonald operator and an elliptic analog of the Ding–Iohara algebra based on the idea of quasi-Hopf twist. The authors of [11] noticed the crucial fact that if one wants to treat the elliptic Macdonald operator in the context of the free field realization, the Ding–Iohara algebra should become elliptic. However it is not clear whether the objects treated in [11] have connections to the elliptic kernel function. Hence the following problem remained open:

> Construct the free field realization of the elliptic Macdonald operator $H_N(q,t,p)$ and the elliptic Ding–Iohara algebra which have connections to the elliptic kernel function $\Pi(q,t,p)(x,y)$.

Our strategy to solve the above problem is the following. Since the free field realization of the Macdonald operator is based on the form of the kernel function, it is plausible that one can construct the free field realization of the elliptic Macdonald operator from the elliptic kernel function. It turns out that this leads to another elliptic analog of the Ding–Iohara algebra (FFR below stands for the free field realization):

$$\begin{array}{ccc} \text{Elliptic Macdonald operator } H_N(q,t,p) & \xrightarrow{\textbf{FFR!}} & \textbf{Elliptic Ding–Iohara algebra } \boldsymbol{\mathcal{U}(q,t,p)} \\ \uparrow \text{elliptic deformation} & & \uparrow \textbf{elliptic deformation!} \\ \text{Macdonald operator } H_N(q,t) & \xrightarrow{\text{FFR}} & \text{Ding–Iohara algebra } \mathcal{U}(q,t) \end{array}$$

Our main results are as follows.

Definition 1.1 (Elliptic Ding–Iohara algebra $\mathcal{U}(q,t,p)$). Set

$$g_p(x) := \frac{\Theta_p(qx)\Theta_p(t^{-1}x)\Theta_p(q^{-1}tx)}{\Theta_p(q^{-1}x)\Theta_p(tx)\Theta_p(qt^{-1}x)} \in \mathbb{C}[[x,x^{-1}]].$$

Here we use the notation of page 412, and assume $|q|,|p| < 1$. We define the *elliptic Ding–Iohara algebra* $\mathcal{U}(q,t,p)$ to be the associative $\mathbb{C}$-algebra generated by $\{x_n^{\pm}(p)\}_{n\in\mathbb{Z}}$, $\{\psi_n^{\pm}(p)\}_{n\in\mathbb{Z}}$ and C subject to the following relations: C is a central, invertible element and if we define $x^{\pm}(p;z) := \sum_{n\in\mathbb{Z}} x_n^{\pm}(p)z^{-n}$ and $\psi^{\pm}(p;z) := \sum_{n\in\mathbb{Z}} \psi_n^{\pm}(p)z^{-n}$ then

$$\begin{aligned} &[\psi^{\pm}(p;z),\psi^{\pm}(p;w)] = 0,\\ &\psi^{+}(p;z)\psi^{-}(p;w) = \frac{g_p(Cz/w)}{g_p(C^{-1}z/w)}\psi^{-}(p;w)\psi^{+}(p;z),\\ &\psi^{\pm}(p;z)x^{+}(p;w) = g_p\left(C^{\pm\frac{1}{2}}\frac{z}{w}\right)x^{+}(p;w)\psi^{\pm}(p;z),\\ &\psi^{\pm}(p;z)x^{-}(p;w) = g_p\left(C^{\mp\frac{1}{2}}\frac{z}{w}\right)^{-1}x^{-}(p;w)\psi^{\pm}(p;z), \end{aligned}$$

$$x^{\pm}(p;z)x^{\pm}(p;w) = g_p\left(\frac{z}{w}\right)^{\pm 1} x^{\pm}(p;w)x^{\pm}(p;z),$$

$$\begin{aligned}&[x^+(p;z), x^-(p;w)]\\&\quad = \frac{\Theta_p(q)\Theta_p(t^{-1})}{(p;p)_\infty^3 \Theta_p(qt^{-1})}\left\{\delta\left(C\frac{w}{z}\right)\psi^+(p;C^{1/2}w) - \delta\left(C^{-1}\frac{w}{z}\right)\psi^-(p;C^{-1/2}w)\right\},\end{aligned}$$

where we define the *delta function* $\delta(z)$ to be $\sum_{n\in\mathbb{Z}} z^n$.

The free field realization of the elliptic Ding–Iohara algebra $\mathcal{U}(q,t,p)$ is constructed as follows. First for the theta function $\Theta_p(x)$ one can check that

$$\Theta_p(x) \xrightarrow[p\to 0]{} 1-x.$$

On the other hand, we can rewrite $1-x$ and $\Theta_p(x)$ as follows:

$$1-x = \exp(\log(1-x)) = \exp\left(-\sum_{n>0}\frac{x^n}{n}\right) \qquad (|x|<1),$$

$$\begin{aligned}\Theta_p(x) &= (p;p)_\infty (x;p)_\infty (px^{-1};p)_\infty\\ &= (p;p)_\infty \exp(\log(x;p)_\infty (px^{-1};p)_\infty)\\ &= (p;p)_\infty \exp\left(-\sum_{n>0}\frac{p^n}{1-p^n}\frac{x^{-n}}{n}\right)\exp\left(-\sum_{n>0}\frac{1}{1-p^n}\frac{x^n}{n}\right) \qquad (|p|<|x|<1).\end{aligned}$$

From these expressions, one can derive a procedure of elliptic deformation:

$$\begin{aligned}&1-x = \exp\left(-\sum_{n>0}\frac{x^n}{n}\right)\\ &\quad\xrightarrow[\text{deformation}]{\text{elliptic}} \exp\left(-\sum_{n>0}\frac{p^n}{1-p^n}\frac{x^{-n}}{n}\right)\exp\left(-\sum_{n>0}\frac{1}{1-p^n}\frac{x^n}{n}\right) = \frac{\Theta_p(x)}{(p;p)_\infty}.\end{aligned}$$

We can also describe the above process as follows:

(1) Make the substitution

$$1-x = \exp\left(-\sum_{n>0}\frac{x^n}{n}\right) \to \exp\left(-\sum_{n>0}\frac{1}{1-p^n}\frac{x^n}{n}\right).$$

(2) Multiply the above by the negative power part

$$\begin{aligned}&\exp\left(-\sum_{n>0}\frac{1}{1-p^n}\frac{x^n}{n}\right)\\ &\qquad \to \exp\left(-\sum_{n>0}\frac{p^n}{1-p^n}\frac{x^{-n}}{n}\right)\exp\left(-\sum_{n>0}\frac{1}{1-p^n}\frac{x^n}{n}\right) = \frac{\Theta_p(x)}{(p;p)_\infty}.\end{aligned}$$

As is shown in this paper, for boson operators the procedure of elliptic deformation similar to the above process is available (for example, see Proposition 3.1 or Definition 5.1). Using two sets of boson generators, we can reproduce the theta function and the elliptic gamma function from OPE (Operator Product Expansion) of boson operators. Consequently, we have

Theorem 1.2 (Free field realization of the elliptic Ding–Iohara algebra). *Define an algebra $\mathcal{B}_{a,b}$ of bosons to be generated by $a=\{a_n\}_{n\in\mathbb{Z}\setminus\{0\}}$, $b=\{b_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ with the relations*

$$[a_m,a_n]=m(1-p^{|m|})\frac{1-q^{|m|}}{1-t^{|m|}}\delta_{m+n,0},\qquad [b_m,b_n]=m\frac{1-p^{|m|}}{(qt^{-1}p)^{|m|}}\frac{1-q^{|m|}}{1-t^{|m|}}\delta_{m+n,0},$$

$$[a_m,b_n]=0.$$

We define the boson Fock space $\mathcal{F}$ to be the left $\mathcal{B}_{a,b}$-module generated by the vacuum vector $|0\rangle$ which satisfies $a_n|0\rangle=b_n|0\rangle=0$ $(n>0)$:

$$\mathcal{F}=\mathrm{span}\{a_{-\lambda}b_{-\mu}|0\rangle:\lambda,\mu\in\mathcal{P}\},$$

where $\mathcal{P}$ denotes the set of partitions and $a_{-\lambda}:=a_{-\lambda_1}\cdots a_{-\lambda_{\ell(\lambda)}}$ $(\lambda\in\mathcal{P})$. Set $\gamma:=(qt^{-1})^{-1/2}$ and define operators $\eta(p;z)$, $\xi(p;z)$, $\varphi^{\pm}(p;z):\mathcal{F}\to\mathcal{F}\otimes\mathbb{C}[[z,z^{-1}]]$ as follows:

$$\eta(p;z):=\boldsymbol{:}\exp\Bigg(-\sum_{n\neq 0}\frac{1-t^{-n}}{1-p^{|n|}}p^{|n|}b_n\frac{z^n}{n}\Bigg)\exp\Bigg(-\sum_{n\neq 0}\frac{1-t^{n}}{1-p^{|n|}}a_n\frac{z^{-n}}{n}\Bigg)\boldsymbol{:},$$

$$\xi(p;z):=\boldsymbol{:}\exp\Bigg(\sum_{n\neq 0}\frac{1-t^{-n}}{1-p^{|n|}}\gamma^{-|n|}p^{|n|}b_n\frac{z^n}{n}\Bigg)\exp\Bigg(\sum_{n\neq 0}\frac{1-t^{n}}{1-p^{|n|}}\gamma^{|n|}a_n\frac{z^{-n}}{n}\Bigg)\boldsymbol{:},$$

$$\varphi^{+}(p;z):=\boldsymbol{:}\eta(p;\gamma^{1/2}z)\xi(p;\gamma^{-1/2}z)\boldsymbol{:},\qquad \varphi^{-}(p;z):=\boldsymbol{:}\eta(p;\gamma^{-1/2}z)\xi(p;\gamma^{1/2}z)\boldsymbol{:}.$$

Then the map defined by

$$C\mapsto\gamma,\qquad x^{+}(p;z)\mapsto\eta(p;z),\qquad x^{-}(p;z)\mapsto\xi(p;z),\qquad \psi^{\pm}(p;z)\mapsto\varphi^{\pm}(p;z)$$

gives a representation of the elliptic Ding–Iohara algebra $\mathcal{U}(q,t,p)$.

Theorem 1.3 (Free field realization of the elliptic Macdonald operator). *Let $\phi(p;z):\mathcal{F}\to\mathcal{F}\otimes\mathbb{C}[[z,z^{-1}]]$ be an operator defined as*

$$\phi(p;z):=\exp\Bigg(\sum_{n>0}\frac{(1-t^n)(qt^{-1}p)^n}{(1-q^n)(1-p^n)}b_{-n}\frac{z^{-n}}{n}\Bigg)\exp\Bigg(\sum_{n>0}\frac{1-t^n}{(1-q^n)(1-p^n)}a_{-n}\frac{z^{n}}{n}\Bigg).$$

Set $\phi_N(p;x):=\prod_{j=1}^N\phi(p;x_j)$ $(N\in\mathbb{Z}_{>0})$. Then the operator $\eta(p;z)$ in Theorem 1.2 and the operator $\phi(p;z)$ reproduce the elliptic Macdonald operator $H_N(q,t,p)$:

$$[\eta(p;z) - t^{-N}(\eta(p;z))_-(\eta(p;p^{-1}z))_+]_1\phi_N(p;x)|0\rangle = \frac{t^{-N+1}\Theta_p(t^{-1})}{(p;p)_\infty^3}H_N(q,t,p)\phi_N(p;x)|0\rangle,$$

where $(\eta(p;z))_\pm$ *stand for the plus and minus parts of* $\eta(p;z)$ *defined as*

$$(\eta(p;z))_\pm = \exp\biggl(-\sum_{\pm n>0}\frac{1-t^{-n}}{1-p^{|n|}}p^{|n|}b_n\frac{z^n}{n}\biggr)\exp\biggl(-\sum_{\pm n>0}\frac{1-t^n}{1-p^{|n|}}a_n\frac{z^{-n}}{n}\biggr),$$

and $[f(z)]_1$ *denotes the constant term of* $f(z)$ *in* z.

Organization of the paper. This paper is organized as follows. In Section 2, we give a review of the trigonometric case. In Section 3, we show how we can obtain the elliptic Ding–Iohara algebra. First, we define the elliptic kernel function introduced by Komori, Noumi and Shiraishi [15]. This function is important in constructing an elliptic analog of Macdonald symmetric functions. Second, from the elliptic kernel function, we define elliptic currents $\eta(p;z)$, $\xi(p;z)$, and $\varphi^\pm(p;z)$ which satisfy elliptic deformed relations of the Ding–Iohara algebra. Consequently, we can define the elliptic Ding–Iohara algebra $\mathcal{U}(q,t,p)$.

In Section 4, to clarify whether the elliptic Macdonald operator can be represented by $\eta(p;z)$, we study relations between the elliptic current $\eta(p;z)$ and the elliptic Macdonald operator. We derive the free field realization for the elliptic Macdonald operator in the form of Theorem 1.3.

In Section 5, some observations and remarks are given, and Section 6 is an appendix which contains the proofs of Wick's theorem, Ramanujan's summation formula, and a partial fraction expansion involving the theta functions.

§2. A review of the trigonometric case

In this section, before considering the elliptic case we review some background material: Macdonald symmetric functions, the free field realization of the Macdonald operator, and the Ding–Iohara algebra.

§2.1. Macdonald symmetric functions

First, we give some notations for symmetric polynomials and symmetric functions [16], [17], [23]. Let $q, t \in \mathbb{C}$ be parameters and assume $|q| < 1$. We denote the symmetric group of degree N by $\mathfrak{S}_N$ and write $\Lambda_N(q,t) := \mathbb{Q}(q,t)[x_1,\dots,x_N]^{\mathfrak{S}_N}$ for the space of N-variable symmetric polynomials over $\mathbb{Q}(q,t)$. If $\lambda = (\lambda_1,\dots,\lambda_N) \in (\mathbb{Z}_{\geq 0})^N$ satisfies the condition $\lambda_i \geq \lambda_{i+1}$ $(1 \leq i \leq N-1)$, then λ is called a *partition*. We denote the set of partitions by $\mathcal{P}$. For a partition λ, $\ell(\lambda) := \sharp\{i : \lambda_i \neq 0\}$ is the *length* of λ, and $|\lambda| := \sum_{i=1}^{\ell(\lambda)}\lambda_i$ is the *size* of λ.

For $\alpha = (\alpha_1, \dots, \alpha_N) \in (\mathbb{Z}_{\geq 0})^N$, we set $x^\alpha := x_1^{\alpha_1} \cdots x_N^{\alpha_N}$. For a partition λ, we define the monomial symmetric polynomial $m_\lambda(x)$ as follows:

$$m_\lambda(x) := \sum_{\alpha\,:\,\alpha \text{ is a permutation of } \lambda} x^\alpha.$$

As is well-known, $\{m_\lambda(x)\}_{\lambda \in \mathcal{P}}$ is a basis of $\Lambda_N(q,t)$. Let $p_n(x) := \sum_{i=1}^N x_i^n$ ($n \in \mathbb{Z}_{>0}$) be the power sum, and for a partition λ, set $p_\lambda(x) := \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}(x)$.

Let $\rho_N^{N+1} : \Lambda_{N+1}(q,t) \to \Lambda_N(q,t)$ be the homomorphism defined by

$$(\rho_N^{N+1} f)(x_1, \dots, x_N) := f(x_1, \dots, x_N, 0) \quad (f \in \Lambda_{N+1}(q,t)).$$

We define the ring $\Lambda(q,t)$ of symmetric functions as the projective limit with respect to $\{\rho_N^{N+1}\}_{N \geq 1}$:

$$\Lambda(q,t) := \varprojlim \Lambda_N(q,t).$$

It is known that $\{p_\lambda(x)\}_{\lambda \in \mathcal{P}}$ is a basis of $\Lambda(q,t)$. Define $n_\lambda(a) := \sharp\{i : \lambda_i = a\}$ and

$$z_\lambda := \prod_{a \geq 1} a^{n_\lambda(a)} n_\lambda(a)!, \qquad z_\lambda(q,t) := z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}.$$

Then we define an inner product $\langle\, , \rangle_{q,t}$ in $\Lambda(q,t)$ as follows:

$$\langle p_\lambda(x), p_\mu(x) \rangle_{q,t} = \delta_{\lambda\mu} z_\lambda(q,t). \tag{2.1}$$

We define the *kernel function* $\Pi(q,t)(x,y)$ by

$$\Pi(q,t)(x,y) := \prod_{i,j} \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty}.$$

Then we have

$$\sum_{\lambda \in \mathcal{P}} \frac{1}{z_\lambda(q,t)} p_\lambda(x) p_\lambda(y) = \Pi(q,t)(x,y).$$

Remark 2.1. Assume that $u_\lambda(x)$, $v_\lambda(x)$ ($\lambda \in \mathcal{P}$) are homogeneous symmetric functions of degree $|\lambda|$, and suppose that $\{u_\lambda(x)\}_{\lambda \in \mathcal{P}}$ and $\{v_\lambda(x)\}_{\lambda \in \mathcal{P}}$ are two bases of $\Lambda(q,t)$. Then

$\{u_\lambda(x)\}_{\lambda \in \mathcal{P}}$ and $\{v_\lambda(x)\}_{\lambda \in \mathcal{P}}$ are dual bases under the inner product $\langle\, , \rangle_{q,t}$

$$\Leftrightarrow \sum_{\lambda \in \mathcal{P}} u_\lambda(x) v_\lambda(y) = \Pi(q,t)(x,y).$$

Due to this fact, the inner product $\langle\, , \rangle_{q,t}$ is determined by the kernel function $\Pi(q,t)(x,y)$.

Macdonald symmetric functions are q-analogs of Schur symmetric functions and Jack symmetric functions. The existence of Macdonald symmetric functions, due to Macdonald, is stated below [16], [17], [23]. We define an order in $\mathcal{P}$ as follows:

$$\lambda \geq \mu \ \Leftrightarrow\ |\lambda| = |\mu| \text{ and } \lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i \text{ for all } i.$$

Theorem 2.2 (Existence of Macdonald symmetric functions). *For each partition λ, there is a unique symmetric function $P_\lambda(x) \in \Lambda(q,t)$ satisfying*

$$P_\lambda(x) = \sum_{\mu \leq \lambda} u_{\lambda\mu} m_\mu(x) \qquad (u_{\lambda\mu} \in \mathbb{Q}(q,t)),$$

$$\lambda \neq \mu \ \Rightarrow\ \langle P_\lambda(x), P_\mu(x)\rangle_{q,t} = 0.$$

Remark 2.3. Set $\langle \lambda \rangle_{q,t} := \langle P_\lambda(x), P_\lambda(x)\rangle_{q,t}$. Then

$$\sum_{\lambda \in \mathcal{P}} \frac{1}{\langle \lambda \rangle_{q,t}} P_\lambda(x) P_\lambda(y) = \Pi(q,t)(x,y).$$

This means that $\{P_\lambda(x)\}_{\lambda\in\mathcal{P}}$ is a basis of $\Lambda(q,t)$.

For the Macdonald symmetric function $P_\lambda(x)$, we define the N-variable symmetric polynomial $P_\lambda(x_1,\dots,x_N)$ by

$$P_\lambda(x_1,\dots,x_N) := P_\lambda(x_1,\dots,x_N,0,0,\dots) \qquad (\ell(\lambda) \leq N).$$

We call it the *N-variable Macdonald polynomial*. We define the *q-shift operator* by

$$T_{q,x_i} f(x_1,\dots,x_N) := f(x_1,\dots,qx_i,\dots,x_N)$$

and define the *Macdonald operator* $H_N(q,t) : \Lambda_N(q,t) \to \Lambda_N(q,t)$ as follows:

$$H_N(q,t) := \sum_{i=1}^{N} \prod_{j\neq i} \frac{tx_i - x_j}{x_i - x_j} T_{q,x_i}. \tag{2.2}$$

Proposition 2.4. (1) *For each partition λ with $\ell(\lambda) \leq N$, the Macdonald polynomial $P_\lambda(x_1,\dots,x_N)$ is an eigenfunction of the Macdonald operator:*

$$H_N(q,t)P_\lambda(x_1,\dots,x_N) = \varepsilon_N(\lambda)P_\lambda(x_1,\dots,x_N), \qquad \varepsilon_N(\lambda) := \sum_{i=1}^{N} q^{\lambda_i} t^{N-i}.$$

(2) *We have*

$$H_N(q,t)_x \Pi(q,t)(x,y) = H_N(q,t)_y \Pi(q,t)(x,y),$$

where

$$H_N(q,t)_x := \sum_{i=1}^{N} \prod_{j\neq i} \frac{tx_i - x_j}{x_i - x_j} T_{q,x_i}, \qquad H_N(q,t)_y := \sum_{i=1}^{N} \prod_{j\neq i} \frac{ty_i - y_j}{y_i - y_j} T_{q,y_i}.$$

§2.2. Free field realization of the Macdonald operator

In this subsection, we give the free field realization of the Macdonald operator [23]. Let q, $t \in \mathbb{C}$ be parameters and assume $|q| < 1$. First we define the algebra $\mathcal{B}$ of bosons to be the algebra generated by $\{a_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ with the relations

$$[a_m, a_n] = m\frac{1-q^{|m|}}{1-t^{|m|}}\delta_{m+n,0}. \tag{2.3}$$

We define the normal ordering $:\cdot:$ as

$$:a_m a_n: = \begin{cases} a_m a_n & (m < n), \\ a_n a_m & (m \geq n). \end{cases}$$

Let $|0\rangle$ be the vacuum vector which satisfies $a_n|0\rangle = 0$ $(n > 0)$. For a partition λ, we set $a_{-\lambda} := a_{-\lambda_1} \cdots a_{-\lambda_{\ell(\lambda)}}$ and define the boson Fock space $\mathcal{F}$ as a left $\mathcal{B}$-module:

$$\mathcal{F} := \mathrm{span}\{a_{-\lambda}|0\rangle : \lambda \in \mathcal{P}\}.$$

We introduce the dual vacuum vector $\langle 0|$ which satisfies $\langle 0|a_n = 0$ $(n < 0)$. Similarly to the definition of $\mathcal{F}$, we define the dual boson Fock space $\mathcal{F}^*$ as a right $\mathcal{B}$-module:

$$\mathcal{F}^* := \mathrm{span}\{\langle 0|a_\lambda : \lambda \in \mathcal{P}\} \qquad (a_\lambda := a_{\lambda_1} \cdots a_{\lambda_{\ell(\lambda)}}).$$

Let us define a bilinear form $\langle\cdot|\cdot\rangle : \mathcal{F}^* \times \mathcal{F} \to \mathbb{C}$ by the conditions

$$\langle 0|0\rangle = 1, \qquad \langle 0|a_\lambda a_{-\mu}|0\rangle = \delta_{\lambda\mu} z_\lambda(q,t).$$

Remark 2.5. It is clear that the bilinear form defined above corresponds to the inner product $\langle\,,\rangle_{q,t}$ in (2.1). Therefore the relation (2.3) is determined by the inner product $\langle\,,\rangle_{q,t}$, or equivalently, by the kernel function $\Pi(q,t)(x,y)$.

To recover the Macdonald operator from a boson operator, let us define operators $\eta(z)$, $\xi(z) : \mathcal{F} \to \mathcal{F} \otimes \mathbb{C}[[z, z^{-1}]]$ as follows ($\gamma := (qt^{-1})^{-1/2}$):

$$\eta(z) := :\exp\left(-\sum_{n\neq 0}(1-t^n)a_n\frac{z^{-n}}{n}\right):, \qquad \xi(z) := :\exp\left(\sum_{n\neq 0}(1-t^n)\gamma^{|n|}a_n\frac{z^{-n}}{n}\right):.$$

We can check that $\eta(z)$, $\xi(z)$ satisfy the relations

$$\eta(z)\eta(w) = \frac{(1-w/z)(1-qt^{-1}w/z)}{(1-qw/z)(1-t^{-1}w/z)}:\eta(z)\eta(w):, \tag{2.4}$$

$$\xi(z)\xi(w) = \frac{(1-w/z)(1-q^{-1}tw/z)}{(1-q^{-1}w/z)(1-tw/z)}:\xi(z)\xi(w):. \tag{2.5}$$

Define $\phi(z) : \mathcal{F} \to \mathcal{F} \otimes \mathbb{C}[[z, z^{-1}]]$ and $\phi^*(z) : \mathcal{F}^* \to \mathcal{F}^* \otimes \mathbb{C}[[z, z^{-1}]]$ as follows:

$$\phi(z) := \exp\biggl(\sum_{n>0} \frac{1-t^n}{1-q^n} a_{-n} \frac{z^n}{n}\biggr), \qquad \phi^*(z) := \exp\biggl(\sum_{n>0} \frac{1-t^n}{1-q^n} a_n \frac{z^n}{n}\biggr).$$

Then we have the relations

$$\eta(z)\phi(w) = \frac{1-w/z}{1-tw/z} {:}\eta(z)\phi(w){:}, \qquad {:}\eta(tz)\phi(z){:}|0\rangle = \phi(qz)|0\rangle,$$
$$\xi(z)\phi(w) = \frac{1-t\gamma w/z}{1-\gamma w/z} {:}\xi(z)\phi(w){:}, \quad {:}\xi(\gamma z)\phi(z){:}|0\rangle = \phi(q^{-1}z)|0\rangle.$$

They are shown in the following way. By Wick's theorem we have

$$\begin{aligned}
\eta(z)\phi(w) &= \exp\biggl(-\sum_{m>0}(1-t^m)\frac{1-t^m}{1-q^m} \cdot m \frac{1-q^m}{1-t^m} \frac{(w/z)^m}{m \cdot m}\biggr) {:}\eta(z)\phi(w){:} \\
&= \exp\biggl(-\sum_{m>0}(1-t^m)\frac{(w/z)^m}{m}\biggr) {:}\eta(z)\phi(w){:} \\
&= \frac{1-w/z}{1-tw/z} {:}\eta(z)\phi(w){:}, \\
\xi(z)\phi(w) &= \exp\biggl(\sum_{m>0}(1-t^m)\gamma^m \frac{(w/z)^m}{m}\biggr) {:}\xi(z)\phi(w){:} \\
&= \frac{1-t\gamma w/z}{1-\gamma w/z} {:}\xi(z)\phi(w){:},
\end{aligned}$$

where we use $\log(1-x) = -\sum_{n>0} x^n/n$ ($|x| < 1$). The other equations follow from simple calculations.

Here and in what follows, we denote the plus and minus parts of operators by $(\cdot)_+$, $(\cdot)_-$, respectively. For example,

$$(\eta(z))_+ = \exp\biggl(-\sum_{n>0}(1-t^n)a_n \frac{z^{-n}}{n}\biggr), \qquad (\eta(z))_- = \exp\biggl(-\sum_{n<0}(1-t^n)a_n \frac{z^{-n}}{n}\biggr).$$

In the following, $[f(z)]_1 = \oint \frac{dz}{2\pi i z} f(z)$ denotes the constant term of $f(z)$ in z.

Set $\phi_N(x) := \prod_{j=1}^N \phi(x_j)$ ($N \in \mathbb{Z}_{>0}$). Then we have the following.

Proposition 2.6. *The constant terms of $\eta(z)$, $\xi(z)$ act on $\phi_N(x)|0\rangle$ as follows:*

(2.6) $$[\eta(z)]_1 \phi_N(x)|0\rangle = t^{-N}\{(t-1)H_N(q,t) + 1\}\phi_N(x)|0\rangle,$$

(2.7) $$[\xi(z)]_1 \phi_N(x)|0\rangle = t^{N}\{(t^{-1}-1)H_N(q^{-1},t^{-1}) + 1\}\phi_N(x)|0\rangle.$$

Proof. We show (2.6). From the relation of $\eta(z)$ and $\phi(z)$, we have

$$\eta(z)\phi_N(x) = \prod_{i=1}^N \frac{1-x_i/z}{1-tx_i/z} {:}\eta(z)\phi_N(x){:}.$$

By partial fraction expansion,

$$\prod_{i=1}^{N} \frac{1-x_i/z}{1-tx_i/z} = \frac{1-t}{1-t^N} \sum_{i=1}^{N} \frac{1-t^{-N}tx_i/z}{1-tx_i/z} \prod_{j\neq i} \frac{tx_i-x_j}{x_i-x_j}.$$

Furthermore, we use the formal expression of the delta function $\delta(x)$:

$$\delta(x) = \sum_{n\in\mathbb{Z}} x^n = \frac{1}{1-x} + \frac{x^{-1}}{1-x^{-1}}.$$

Then we obtain

$$\prod_{i=1}^{N} \frac{1-x_i/z}{1-tx_i/z} = t^{-N}(t-1) \sum_{i=1}^{N} \prod_{j\neq i} \frac{tx_i-x_j}{x_i-x_j} \delta\left(t\frac{x_i}{z}\right) + t^{-N} \prod_{i=1}^{N} \frac{1-z/x_i}{1-t^{-1}z/x_i}.$$

Hence

$$\begin{aligned}\eta(z)\phi_N(x)|0\rangle = {} & t^{-N}(t-1) \sum_{i=1}^{N} \prod_{j\neq i} \frac{tx_i-x_j}{x_i-x_j} \delta\left(t\frac{x_i}{z}\right) T_{q,x_i}\phi_N(x)|0\rangle \\ & + t^{-N} \prod_{i=1}^{N} \frac{1-z/x_i}{1-t^{-1}z/x_i} (\eta(z))_- \phi_N(x)|0\rangle,\end{aligned}$$

where we use the relation $(\eta(tz))_-\phi(z) = \phi(qz) = T_{q,z}\phi(z)$. From this equation,

$$\begin{aligned}&[\eta(z)]_1\phi_N(x)|0\rangle \\ &= \left\{ t^{-N}(t-1) \sum_{i=1}^{N} \prod_{j\neq i} \frac{tx_i-x_j}{x_i-x_j} T_{q,x_i} + t^{-N} \left[\prod_{i=1}^{N} \frac{1-z/x_i}{1-t^{-1}z/x_i} (\eta(z))_- \right]_1 \right\} \phi_N(x)|0\rangle \\ &= t^{-N}\{(t-1)H_N(q,t)+1\}\phi_N(x)|0\rangle,\end{aligned}$$

where we use the equation

$$\left[\prod_{i=1}^{N} \frac{1-z/x_i}{1-t^{-1}z/x_i} (\eta(z))_- \right]_1 = 1.$$

The proof of (2.7) is similar, so we omit it. □

Remark 2.7. (1) Set $\phi_N^*(x) := \prod_{j=1}^{N} \phi^*(x_j)$ $(N \in \mathbb{Z}_{>0})$. Then the kernel function $\Pi(q,t)(x,y)$ can be recovered from the operators $\phi_N^*(x)$, $\phi_N(y)$:

$$\langle 0|\phi_N^*(x)\phi_N(y)|0\rangle = \Pi(q,t)(x,y) = \prod_{1\leq i,j\leq N} \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty}.$$

(2) Let us recall that the kernel function $\Pi(q,t)(x,y)$ determines relation (2.3). Therefore the free field realization of the Macdonald operator is based on the kernel function $\Pi(q,t)(x,y)$.

§2.3. Ding–Iohara algebra $\mathcal{U}(q,t)$

As is seen in the previous subsection, we can represent the Macdonald operator by using $\eta(z)$, $\xi(z)$. Applying Wick's theorem, we can show the following.

Proposition 2.8 (Relations of $\eta(z)$, $\xi(z)$ and $\varphi^{\pm}(z)$). *Set* $\gamma = (qt^{-1})^{-1/2}$ *and define operators* $\varphi^{\pm}(z) : \mathcal{F} \to \mathcal{F} \otimes \mathbb{C}[[z,z^{-1}]]$ *as*

$$\varphi^{+}(z) := {:}\eta(\gamma^{1/2}z)\xi(\gamma^{-1/2}z){:}, \qquad \varphi^{-}(z) := {:}\eta(\gamma^{-1/2}z)\xi(\gamma^{1/2}z){:}.$$

Define the structure function $g(x) \in \mathbb{C}[[x]]$ *as*

$$g(x) := \frac{(1-qx)(1-t^{-1}x)(1-q^{-1}tx)}{(1-q^{-1}x)(1-tx)(1-qt^{-1}x)}. \tag{2.8}$$

Then the operators $\eta(z)$, $\xi(z)$ *and* $\varphi^{\pm}(z)$ *satisfy the following relations:*

(2.9)
$$[\varphi^{\pm}(z), \varphi^{\pm}(w)] = 0, \qquad \varphi^{+}(z)\varphi^{-}(w) = \frac{g(\gamma z/w)}{g(\gamma^{-1}z/w)}\varphi^{-}(w)\varphi^{+}(z),$$
$$\varphi^{\pm}(z)\eta(w) = g\left(\gamma^{\pm\frac{1}{2}}\frac{z}{w}\right)\eta(w)\varphi^{\pm}(z), \qquad \varphi^{\pm}(z)\xi(w) = g\left(\gamma^{\mp\frac{1}{2}}\frac{z}{w}\right)^{-1}\xi(w)\varphi^{\pm}(z),$$
$$\eta(z)\eta(w) = g\left(\frac{z}{w}\right)\eta(w)\eta(z), \qquad \xi(z)\xi(w) = g\left(\frac{z}{w}\right)^{-1}\xi(w)\xi(z),$$
$$[\eta(z), \xi(w)] = \frac{(1-q)(1-t^{-1})}{1-qt^{-1}}\left\{\delta\left(\gamma\frac{w}{z}\right)\varphi^{+}(\gamma^{1/2}w) - \delta\left(\gamma^{-1}\frac{w}{z}\right)\varphi^{-}(\gamma^{-1/2}w)\right\}.$$

Remark 2.9. (1) In Section 3, we will prove an elliptic version of Proposition 2.8. Therefore we omit the proof of the proposition.

(2) As $[\varphi^{\pm}(z)]_1 = 1$, the above leads to $[[\eta(z)]_1, [\xi(w)]_1] = 0$. This corresponds to the commutativity of the Macdonald operators, $[H_N(q,t), H_N(q^{-1},t^{-1})] = 0$.

It is important that relations (2.9) are similar to the relations in the Drinfeld realization of $U_q(\widehat{sl_2})$ [22], [7]. From this fact, we can view (2.9) as a kind of quantum group structure. In this way, we can define the Ding–Iohara algebra $\mathcal{U}(q,t)$ as follows [11].

Definition 2.10 (Ding–Iohara algebra $\mathcal{U}(q,t)$). Let $g(x)$ be as defined in (2.8):

$$g(x) = \frac{(1-qx)(1-t^{-1}x)(1-q^{-1}tx)}{(1-q^{-1}x)(1-tx)(1-qt^{-1}x)}.$$

Let C be a central, invertible element and let $x^{\pm}(z) := \sum_{n\in\mathbb{Z}} x_n^{\pm} z^{-n}$ and $\psi^{\pm}(z) := \sum_{n\in\mathbb{Z}} \psi_n^{\pm} z^{-n}$ satisfy the relations

(2.10)

$$[\psi^{\pm}(z), \psi^{\pm}(w)] = 0, \qquad \psi^{+}(z)\psi^{-}(w) = \frac{g(Cz/w)}{g(C^{-1}z/w)}\psi^{-}(w)\psi^{+}(z),$$

$$\psi^{\pm}(z)x^{+}(w) = g\left(C^{\pm\frac{1}{2}}\frac{z}{w}\right)x^{+}(w)\psi^{\pm}(z), \ \ \psi^{\pm}(z)x^{-}(w) = g\left(C^{\mp\frac{1}{2}}\frac{z}{w}\right)^{-1}x^{-}(w)\psi^{\pm}(z),$$

$$x^{\pm}(z)x^{\pm}(w) = g\left(\frac{z}{w}\right)^{\pm 1}x^{\pm}(w)x^{\pm}(z),$$

$$[x^{+}(z), x^{-}(w)] = \frac{(1-q)(1-t^{-1})}{1-qt^{-1}}\left\{\delta\left(C\frac{w}{z}\right)\psi^{+}\left(C^{1/2}w\right) - \delta\left(C^{-1}\frac{w}{z}\right)\psi^{-}\left(C^{-1/2}w\right)\right\}.$$

Then we define the *Ding–Iohara algebra* $\mathcal{U}(q,t)$ to be the associative $\mathbb{C}$-algebra generated by $\{x_n^{\pm}\}_{n\in\mathbb{Z}}$, $\{\psi_n^{\pm}\}_{n\in\mathbb{Z}}$ and C with the above relations.

Due to Proposition 2.8, the map

$$C \mapsto \gamma, \qquad x^{+}(z) \mapsto \eta(z), \qquad x^{-}(z) \mapsto \xi(z), \qquad \psi^{\pm}(z) \mapsto \varphi^{\pm}(z)$$

gives a representation of the Ding–Iohara algebra (the *free field realization*).

Remark 2.11. It is known that $\mathcal{U}(q,t)$ has the coproduct

$$\Delta : \mathcal{U}(q,t) \to \mathcal{U}(q,t) \otimes \mathcal{U}(q,t)$$

defined as follows [11]:

$$\begin{aligned} &\Delta(C^{\pm 1}) = C^{\pm 1} \otimes C^{\pm 1}, \qquad \Delta(\psi^{\pm}(z)) = \psi^{\pm}(C_{(2)}^{\pm 1/2}z) \otimes \psi^{\pm}(C_{(1)}^{\mp 1/2}z), \\ &\Delta(x^{+}(z)) = x^{+}(z) \otimes 1 + \psi^{-}(C_{(1)}^{1/2}z) \otimes x^{+}(C_{(1)}z), \\ &\Delta(x^{-}(z)) = x^{-}(C_{(2)}z) \otimes \psi^{+}(C_{(2)}^{1/2}z) + 1 \otimes x^{-}(z). \end{aligned} \tag{2.11}$$

Here we define $C_{(1)} := C \otimes 1$, $C_{(2)} := 1 \otimes C$.

§3. Elliptic Ding–Iohara algebra

In this section, we are going to show that: 1) from the elliptic kernel function we can construct elliptic currents, 2) from relations among the elliptic currents, an elliptic analog of the Ding–Iohara algebra arises.

In the following, we use parameters $q, t, p \in \mathbb{C}$ which satisfy $|q|, |p| < 1$.

§3.1. Kernel function introduced by Komori, Noumi and Shiraishi

First fix $N \in \mathbb{Z}_{>0}$. The *elliptic kernel function* introduced by Komori, Noumi and Shiraishi [15] is defined as

$$\Pi(q,t,p)(x,y) := \prod_{1\le i,j\le N} \frac{\Gamma_{q,p}(x_i y_j)}{\Gamma_{q,p}(t x_i y_j)}.$$

Since $\Gamma_{q,p}(x) \to (x;q)_\infty^{-1}$ as $p \to 0$, the elliptic kernel function degenerates to $\Pi(q,t)(x,y)$ in the limit $p \to 0$:

$$\Pi(q,t,p)(x,y) \xrightarrow[p\to 0]{} \Pi(q,t)(x,y) = \prod_{1\le i,j\le N} \frac{(t x_i y_j;q)_\infty}{(x_i y_j;q)_\infty}.$$

Remark 3.1. In [15], it is shown that $\Pi(q,t,p)(x,y)$ and the elliptic Macdonald operator $H_N(q,t,p)$ of (1.1) satisfy the relation

$$H_N(q,t,p)_x \Pi(q,t,p)(x,y) = H_N(q,t,p)_y \Pi(q,t,p)(x,y).$$

We can check the following expression of $\Gamma_{q,p}(x)$:

$$\Gamma_{q,p}(x) = \exp\left(-\sum_{n>0} \frac{(qp)^n}{(1-q^n)(1-p^n)} \frac{x^{-n}}{n}\right) \exp\left(\sum_{n>0} \frac{1}{(1-q^n)(1-p^n)} \frac{x^n}{n}\right).$$

Then we can rewrite $\Pi(q,t,p)(x,y)$ by using power sums:

$$\begin{aligned}\Pi(q,t,p)(x,y) = &\exp\left(\sum_{n>0} \frac{(1-t^n)(qt^{-1}p)^n}{(1-q^n)(1-p^n)} \frac{p_n(\overline{x})p_n(\overline{y})}{n}\right)\\ &\cdot \exp\left(\sum_{n>0} \frac{1-t^n}{(1-q^n)(1-p^n)} \frac{p_n(x)p_n(y)}{n}\right).\end{aligned} \tag{3.1}$$

Here $p_n(\overline{x}) := \sum_{i=1}^N x_i^{-n}$ ($n \in \mathbb{Z}_{>0}$) denotes the negative power sum, and for a partition λ, we set $p_\lambda(\overline{x}) := p_{\lambda_1}(\overline{x}) \cdots p_{\lambda_{\ell(\lambda)}}(\overline{x})$. We also define

$$z_\lambda(q,t,p) := z_\lambda \prod_{i=1}^{\ell(\lambda)} (1-p^{\lambda_i}) \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}, \qquad \overline{z}_\lambda(q,t,p) := z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-p^{\lambda_i}}{(qt^{-1}p)^{\lambda_i}} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}.$$

Then we can expand $\Pi(q,t,p)(x,y)$ as follows:

$$\Pi(q,t,p)(x,y) = \sum_{\lambda\in\mathcal{P}} \frac{1}{\overline{z}_\lambda(q,t,p)} p_\lambda(\overline{x}) p_\lambda(\overline{y}) \sum_{\mu\in\mathcal{P}} \frac{1}{z_\mu(q,t,p)} p_\mu(x) p_\mu(y). \tag{3.2}$$

§3.2. Operator $\phi(p;z)$ and elliptic currents $\eta(p;z)$, $\xi(p;z)$ and $\varphi^{\pm}(p;z)$

In this subsection we define elliptic currents and study their properties. Keeping the expression (3.2) of $\Pi(q,t,p)(x,y)$ in mind, we introduce an algebra $\mathcal{B}_{a,b}$ of bosons generated by $\{a_n\}_{n\in\mathbb{Z}\setminus\{0\}}$, $\{b_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ with the relations

$$[a_m,a_n]=m(1-p^{|m|})\frac{1-q^{|m|}}{1-t^{|m|}}\delta_{m+n,0},\qquad [b_m,b_n]=m\frac{1-p^{|m|}}{(qt^{-1}p)^{|m|}}\frac{1-q^{|m|}}{1-t^{|m|}}\delta_{m+n,0},$$
$$[a_m,b_n]=0.$$

As in the trigonometric case, let $|0\rangle$ be the vacuum vector which satisfies the conditions $a_n|0\rangle=b_n|0\rangle=0$ $(n>0)$ and define the boson Fock space $\mathcal{F}$ as a left $\mathcal{B}_{a,b}$-module:

$$\mathcal{F}:=\mathrm{span}\{a_{-\lambda}b_{-\mu}|0\rangle:\lambda,\mu\in\mathcal{P}\}. \tag{3.3}$$

The dual vacuum vector $\langle 0|$ is defined by the conditions $\langle 0|a_n=\langle 0|b_n=0$ $(n<0)$ and we define the dual boson Fock space $\mathcal{F}^*$ as a right $\mathcal{B}_{a,b}$-module:

$$\mathcal{F}^*:=\mathrm{span}\{\langle 0|a_\lambda b_\mu:\lambda,\mu\in\mathcal{P}\}. \tag{3.4}$$

We define a bilinear form $\langle\cdot|\cdot\rangle:\mathcal{F}^*\times\mathcal{F}\to\mathbb{C}$ by the conditions

$$\langle 0|0\rangle=1,\qquad \langle 0|a_{\lambda_1}b_{\lambda_2}a_{-\mu_1}b_{-\mu_2}|0\rangle=\delta_{\lambda_1\mu_1}\delta_{\lambda_2\mu_2}z_{\lambda_1}(q,t,p)\overline{z}_{\lambda_2}(q,t,p).$$

We also define the normal ordering $:\cdot:$ as usual:

$$:a_m a_n:=\begin{cases}a_m a_n & (m<n),\\ a_n a_m & (m\ge n),\end{cases}\qquad :b_m b_n:=\begin{cases}b_m b_n & (m<n),\\ b_n b_m & (m\ge n).\end{cases}$$

Remark 3.2. The above defined algebra of bosons leads to consider the space $\Lambda_N(q,t,p):=\mathbb{C}[[x_i,x_i^{-1}:1\le i\le N]]^{\mathfrak{S}_N}$ of symmetric functions. But it is not clear whether elliptic analogs of the Macdonald symmetric functions live in $\Lambda_N(q,t,p)$.

Define $\phi(p;z):\mathcal{F}\to\mathcal{F}\otimes\mathbb{C}[[z,z^{-1}]]$ and $\phi^*(p;z):\mathcal{F}^*\to\mathcal{F}^*\otimes\mathbb{C}[[z,z^{-1}]]$ by

$$\phi(p;z):=\exp\Bigg(\sum_{n>0}\frac{(1-t^n)(qt^{-1}p)^n}{(1-q^n)(1-p^n)}b_{-n}\frac{z^{-n}}{n}\Bigg)\exp\Bigg(\sum_{n>0}\frac{1-t^n}{(1-q^n)(1-p^n)}a_{-n}\frac{z^n}{n}\Bigg),$$
$$\phi^*(p;z):=\exp\Bigg(\sum_{n>0}\frac{(1-t^n)(qt^{-1}p)^n}{(1-q^n)(1-p^n)}b_n\frac{z^{-n}}{n}\Bigg)\exp\Bigg(\sum_{n>0}\frac{1-t^n}{(1-q^n)(1-p^n)}a_n\frac{z^n}{n}\Bigg).$$

Set $\phi_N(p;x):=\prod_{j=1}^N\phi(p;x_j)$, $\phi_N^*(p;x):=\prod_{j=1}^N\phi^*(p;x_j)$ $(N\in\mathbb{Z}_{>0})$. Then

$$\langle 0|\phi_N^*(p;x)\phi_N(p;y)|0\rangle=\Pi(q,t,p)(x_1,\dots,x_N,y_1,\dots,y_N). \tag{3.5}$$

To check this, we first observe that

$$\begin{aligned}
&\phi^*(p;z)\phi(p;w)\\
&\quad= \exp\left(\sum_{m>0}\frac{(1-t^m)(qt^{-1}p)^m}{(1-q^m)(1-p^m)}\frac{(1-t^m)(qt^{-1}p)^m}{(1-q^m)(1-p^m)}\cdot m\frac{1-p^m}{(qt^{-1}p)^m}\frac{1-q^m}{1-t^m}\frac{(zw)^{-m}}{m}\right)\\
&\qquad\times\exp\left(\sum_{m>0}\frac{1-t^m}{(1-q^m)(1-p^m)}\frac{1-t^m}{(1-q^m)(1-p^m)}\cdot m(1-p^m)\frac{1-q^m}{1-t^m}\frac{(zw)^m}{m}\right)\\
&\qquad\qquad\times :\phi^*(p;z)\phi(p;w):\\
&\quad=\exp\left(\sum_{m>0}\frac{(1-t^m)(qt^{-1}p)^m}{(1-q^m)(1-p^m)}\frac{(zw)^{-m}}{m}\right)\exp\left(\sum_{m>0}\frac{1-t^m}{(1-q^m)(1-p^m)}\frac{(zw)^m}{m}\right)\\
&\qquad\qquad\times :\phi^*(p;z)\phi(p;w):.
\end{aligned}$$

From this equation and the expression (3.1) of the kernel function, we obtain (3.5).

Next, let us construct an operator $\eta(p;z)$ which satisfies the conditions

$$\begin{aligned}
&(1)\ :\eta(p;tz)\phi(p;z):|0\rangle=\phi(p;qz)|0\rangle,\\
&(2)\ \langle 0|:\phi^*(p;z)\eta(p;z^{-1}):=\langle 0|\phi^*(p;qz).
\end{aligned}\tag{3.6}$$

These conditions are satisfied by the following operator, which we would like to call the *elliptic current*.

Proposition 3.3 (Elliptic current $\eta(p;z)$). *Let $\eta(p;z):\mathcal{F}\to\mathcal{F}\otimes\mathbb{C}[[z,z^{-1}]]$ be defined as follows:*

$$\eta(p;z):=:\exp\left(-\sum_{n\neq 0}\frac{1-t^{-n}}{1-p^{|n|}}p^{|n|}b_n\frac{z^n}{n}\right)\exp\left(-\sum_{n\neq 0}\frac{1-t^n}{1-p^{|n|}}a_n\frac{z^{-n}}{n}\right):.$$

Then:

(i) *$\eta(p;z)$ satisfies (3.6).*

(ii) *We have*

$$\eta(p;z)\eta(p;w)=\frac{\Theta_p(w/z)\Theta_p(qt^{-1}w/z)}{\Theta_p(qw/z)\Theta_p(t^{-1}w/z)}:\eta(p;z)\eta(p;w):\tag{3.7}$$
$$(|p|<|qw/z|<1,\ |p|<|t^{-1}w/z|<1).$$

Proof. (i) We first show that (1) of (3.6) is satisfied. We have

$$:\eta(p;tz)\phi(p;z):|0\rangle=(\eta(p;tz))_-\phi(p;z)|0\rangle.$$

Hence we have to show that $(\eta(p;tz))_-\phi(p;z)=\phi(p;qz)$. The proof is straightforward: since

$$(\eta(p;z))_-=\exp\left(\sum_{n>0}\frac{1-t^n}{1-p^n}p^nb_{-n}\frac{z^{-n}}{n}\right)\exp\left(\sum_{n>0}\frac{1-t^{-n}}{1-p^n}a_{-n}\frac{z^n}{n}\right),$$

we have

$$\begin{aligned}&(\eta(p;tz))_-\phi(p;z)\\&\quad=\exp\left(\sum_{n>0}\frac{(qt^{-1}p)^n(1-t^n)}{(1-q^n)(1-p^n)}\{t^{-n}(1-q^n)(qt^{-1})^{-n}+1\}b_{-n}\frac{z^{-n}}{n}\right)\\&\qquad\times\exp\left(\sum_{n>0}\frac{1-t^n}{(1-q^n)(1-p^n)}\{-t^{-n}(1-q^n)t^n+1\}a_{-n}\frac{z^n}{n}\right)\\&\quad=\exp\left(\sum_{n>0}\frac{(qt^{-1}p)^n(1-t^n)}{(1-q^n)(1-p^n)}q^{-n}b_{-n}\frac{z^{-n}}{n}\right)\exp\left(\sum_{n>0}\frac{1-t^n}{(1-q^n)(1-p^n)}q^na_{-n}\frac{z^n}{n}\right)\\&\quad=\phi(p;qz).\end{aligned}$$

Next we show that $\eta(p;z)$ satisfies (2) of (3.6). Due to the relation

$$\langle 0|{:}\phi^*(p;z)\eta(p;z^{-1}){:}=\langle 0|\phi^*(p;z)(\eta(p;z^{-1}))_+,$$

we have to show that $\phi^*(p;z)(\eta(p;z^{-1}))_+=\phi^*(p;qz)$. Now,

$$(\eta(p;z^{-1}))_+=\exp\left(-\sum_{n>0}\frac{1-t^{-n}}{1-p^n}p^nb_n\frac{z^{-n}}{n}\right)\exp\left(-\sum_{n>0}\frac{1-t^n}{1-p^n}a_n\frac{z^n}{n}\right),$$

hence

$$\begin{aligned}\phi^*(p;z)(\eta(p;z^{-1}))_+&=\exp\left(\sum_{n>0}\frac{(1-t^n)(qt^{-1}p)^n}{(1-q^n)(1-p^n)}\{1+q^{-n}(1-q^n)\}b_n\frac{z^{-n}}{n}\right)\\&\qquad\times\exp\left(\sum_{n>0}\frac{1-t^n}{(1-q^n)(1-p^n)}\{1-(1-q^n)\}a_n\frac{z^n}{n}\right)\\&=\phi^*(p;qz).\end{aligned}$$

(ii) By Wick's theorem,

$$\begin{aligned}&\eta(p;z)\eta(p;w)\\&\quad=\exp\left(\sum_{m>0}\frac{1-t^{-m}}{1-p^m}p^m\frac{1-t^m}{1-p^m}p^m\cdot m\frac{1-p^m}{(qt^{-1}p)^m}\,\frac{1-q^m}{1-t^m}\,\frac{(z/w)^m}{m(-m)}\right)\\&\qquad\times\exp\left(\sum_{m>0}\frac{1-t^m}{1-p^m}\,\frac{1-t^{-m}}{1-p^m}\cdot m(1-p^m)\frac{1-q^m}{1-t^m}\,\frac{(w/z)^m}{m(-m)}\right){:}\eta(p;z)\eta(p;w){:}\end{aligned}$$

$$= \exp\left(-\sum_{m>0}\frac{(1-q^m)(1-t^{-m})(qt^{-1})^{-m}}{1-p^m}p^m\frac{(z/w)^m}{m}\right)$$
$$\times\exp\left(-\sum_{m>0}\frac{(1-q^m)(1-t^{-m})}{1-p^m}\frac{(w/z)^m}{m}\right):\eta(p;z)\eta(p;w):$$
$$=\frac{(q^{-1}tpz/w;p)_\infty(pz/w;p)_\infty}{(tpz/w;p)_\infty(q^{-1}pz/w;p)_\infty}\frac{(w/z;p)_\infty(qt^{-1}w/z;p)_\infty}{(qw/z;p)_\infty(t^{-1}w/z;p)_\infty}:\eta(p;z)\eta(p;w):$$
$$=\frac{\Theta_p(w/z)\Theta_p(qt^{-1}w/z)}{\Theta_p(qw/z)\Theta_p(t^{-1}w/z)}:\eta(p;z)\eta(p;w):. \qquad \square$$

Since (3.7) is an elliptic analog of the trigonometric case (2.4), we can view $\eta(p;z)$ as an elliptic analog of $\eta(z)$. In similar way, we can define an elliptic analog $\xi(p;z):\mathcal{F}\to\mathcal{F}\otimes\mathbb{C}[[z,z^{-1}]]$ of $\xi(z)$.

Proposition 3.4 (Elliptic current $\xi(p;z)$). *Let $\xi(p;z):\mathcal{F}\to\mathcal{F}\otimes\mathbb{C}[[z,z^{-1}]]$ be defined as follows:*

$$\xi(p;z):=\,:\exp\left(\sum_{n\neq 0}\frac{1-t^{-n}}{1-p^{|n|}}\gamma^{-|n|}p^{|n|}b_n\frac{z^n}{n}\right)\exp\left(\sum_{n\neq 0}\frac{1-t^n}{1-p^{|n|}}\gamma^{|n|}a_n\frac{z^{-n}}{n}\right):.$$

Then

$$(\xi(p;\gamma z))_-\phi(p;z)=\phi(p;q^{-1}z),$$
$$\phi^*(p;z)(\xi(p;t\gamma^{-1}z^{-1}))_+=\phi^*(p;q^{-1}z), \tag{3.8}$$

$$\xi(p;z)\xi(p;w)=\frac{\Theta_p(w/z)\Theta_p(q^{-1}tw/z)}{\Theta_p(q^{-1}w/z)\Theta_p(tw/z)}:\xi(p;z)\xi(p;w): \tag{3.9}$$
$$(|p|<|q^{-1}w/z|<1,\ |p|<|tw/z|<1).$$

Proof. Since the proof of (3.8) is quite similar to the proof of Proposition 3.3(i), we omit it. (3.9) is shown as follows:

$\xi(p;z)\xi(p;w)$

$$=\exp\left(\sum_{m>0}\frac{1-t^{-m}}{1-p^m}\gamma^{-m}p^m\,\frac{1-t^m}{1-p^m}\gamma^{-m}p^m\cdot m\frac{1-p^m}{(qt^{-1}p)^m}\,\frac{1-q^m}{1-t^m}\,\frac{(z/w)^m}{m(-m)}\right)$$
$$\times\exp\left(\sum_{m>0}\frac{1-t^m}{1-p^m}\gamma^m\frac{1-t^{-m}}{1-p^m}\gamma^m\cdot m(1-p^m)\frac{1-q^m}{1-t^m}\,\frac{(w/z)^m}{m(-m)}\right):\xi(p;z)\xi(p;w):$$
$$=\exp\left(-\sum_{m>0}\frac{(1-q^m)(1-t^{-m})}{1-p^m}p^m\frac{(z/w)^m}{m}\right)$$
$$\times\exp\left(-\sum_{m>0}\frac{(1-q^m)(1-t^{-m})(qt^{-1})^{-m}}{1-p^m}\frac{(w/z)^m}{m}\right):\xi(p;z)\xi(p;w):$$
$$=\frac{\Theta_p(w/z)\Theta_p(q^{-1}tw/z)}{\Theta_p(q^{-1}w/z)\Theta_p(tw/z)}:\xi(p;z)\xi(p;w):. \qquad \square$$

As in the trigonometric case, it is natural to find a commutation relation between $\eta(p;z)$ and $\xi(p;z)$. For this, we need a lemma which gives a relation between the theta function and the delta function.

Lemma 3.5. *The theta function $\Theta_p(x)$ and the delta function $\delta(x)$ satisfy*

$$\frac{1}{\Theta_p(x)}+\frac{x^{-1}}{\Theta_p(x^{-1})}=\frac{1}{(p;p)_\infty^3}\delta(x), \tag{3.10}$$

$$\frac{1}{\Theta_p(x)}+\frac{x^{-1}}{\Theta_p(px)}=\frac{1}{(p;p)_\infty^3}\delta(x). \tag{3.11}$$

This leads to

$$\frac{1}{\Theta_p(px)}=\frac{1}{\Theta_p(x^{-1})}. \tag{3.12}$$

Proof. To prove (3.10), let us recall the formal expression

$$\delta(x)=\sum_{n\in\mathbb{Z}}x^n=\frac{1}{1-x}+\frac{x^{-1}}{1-x^{-1}}.$$

From this expression, we have

$$\begin{aligned}\frac{1}{\Theta_p(x)}&=\frac{1}{(p;p)_\infty(x;p)_\infty(px^{-1};p)_\infty}=\frac{1}{(p;p)_\infty}\frac{1}{(1-x)(px;p)_\infty(px^{-1};p)_\infty}\\&=\frac{1}{(p;p)_\infty}\left(\delta(x)-\frac{x^{-1}}{1-x^{-1}}\right)\frac{1}{(px;p)_\infty(px^{-1};p)_\infty}\\&=\frac{1}{(p;p)_\infty^3}\delta(x)-\frac{x^{-1}}{\Theta_p(x^{-1})}.\end{aligned}$$

Relation (3.11) is shown in a similar way:

$$\begin{aligned}\frac{1}{\Theta_p(px)}&=\frac{1}{(p;p)_\infty(px;p)_\infty(x^{-1};p)_\infty}=\frac{1}{(p;p)_\infty(px;p)_\infty}\frac{1}{(1-x^{-1})(px^{-1};p)_\infty}\\&=\frac{1}{(p;p)_\infty(px;p)_\infty}\left(\delta(x)-\frac{x}{1-x}\right)\frac{1}{(px^{-1};p)_\infty}=\frac{1}{(p;p)_\infty^3}\delta(x)-\frac{x}{\Theta_p(x)}.\end{aligned}$$

By the subtraction (3.10) − (3.11), we have $1/\Theta_p(px)=1/\Theta_p(x^{-1})$. □

Remark 3.6. Relations (3.10), (3.11) should be understood in the context of the Sato hyperfunction [19].

From this lemma, we can calculate $[\eta(p;z),\xi(p;w)]$

Proposition 3.7 (Commutator $[\eta(p;z),\xi(p;w)]$). *Define $\varphi^{\pm}(p;z):\mathcal{F}\to\mathcal{F}\otimes\mathbb{C}[[z,z^{-1}]]$ as follows:*

$$\varphi^+(p;z):=\boldsymbol{:}\eta(p;\gamma^{1/2}z)\xi(p;\gamma^{-1/2}z)\boldsymbol{:},\qquad \varphi^-(p;z):=\boldsymbol{:}\eta(p;\gamma^{-1/2}z)\xi(p;\gamma^{1/2}z)\boldsymbol{:}.$$

Then

(3.13)
$$[\eta(p;z),\xi(p;w)] = \frac{\Theta_p(q)\Theta_p(t^{-1})}{(p;p)_\infty^3\Theta_p(qt^{-1})}\left\{\delta\left(\gamma\frac{w}{z}\right)\varphi^+(p;\gamma^{1/2}w)-\delta\left(\gamma^{-1}\frac{w}{z}\right)\varphi^-(p;\gamma^{-1/2}w)\right\}.$$

Proof. By Wick's theorem,

$$\begin{aligned}\eta(p;z)\xi(p;w) &= \exp\left(\sum_{m>0}\frac{(1-q^m)(1-t^{-m})}{1-p^m}\gamma^m p^m\frac{(z/w)^m}{m}\right)\\ &\quad\times\exp\left(\sum_{m>0}\frac{(1-q^m)(1-t^{-m})}{1-p^m}\gamma^m\frac{(w/z)^m}{m}\right):\eta(p;z)\xi(p;w):\\ &= \frac{\Theta_p(q\gamma w/z)\Theta_p(q^{-1}\gamma^{-1}w/z)}{\Theta_p(\gamma w/z)\Theta_p(\gamma^{-1}w/z)}:\eta(p;z)\xi(p;w):\end{aligned}$$

and

$$\begin{aligned}\xi(p;w)\eta(p;z) &= \exp\left(\sum_{m>0}\frac{(1-q^m)(1-t^{-m})}{1-p^m}\gamma^m p^m\frac{(w/z)^m}{m}\right)\\ &\quad\times\exp\left(\sum_{m>0}\frac{(1-q^m)(1-t^{-m})}{1-p^m}\gamma^m\frac{(z/w)^m}{m}\right):\eta(p;z)\xi(p;w):\\ &= \frac{\Theta_p(q\gamma z/w)\Theta_p(q^{-1}\gamma^{-1}z/w)}{\Theta_p(\gamma z/w)\Theta_p(\gamma^{-1}z/w)}:\eta(p;z)\xi(p;w):.\end{aligned}$$

Hence

$$\begin{aligned}&[\eta(p;z),\xi(p;w)]\\ &= \left\{\frac{\Theta_p(q\gamma w/z)\Theta_p(q^{-1}\gamma^{-1}w/z)}{\Theta_p(\gamma w/z)\Theta_p(\gamma^{-1}w/z)}-\frac{\Theta_p(q\gamma z/w)\Theta_p(q^{-1}\gamma^{-1}z/w)}{\Theta_p(\gamma z/w)\Theta_p(\gamma^{-1}z/w)}\right\}:\eta(p;z)\xi(p;w):\\ &= \Theta_p(q\gamma w/z)\Theta_p(q^{-1}\gamma^{-1}w/z)\left\{\frac{1}{\Theta_p(\gamma w/z)\Theta_p(\gamma^{-1}w/z)}-\frac{(z/w)^2}{\Theta_p(\gamma z/w)\Theta_p(\gamma^{-1}z/w)}\right\}\\ &\qquad\times:\eta(p;z)\xi(p;w):.\end{aligned}$$

From Lemma 3.5,

$$\begin{aligned}&\frac{1}{\Theta_p(\gamma x)\Theta_p(\gamma^{-1}x)}-\frac{x^{-2}}{\Theta_p(\gamma x^{-1})\Theta_p(\gamma^{-1}x^{-1})}\\ &= \left\{\frac{1}{\Theta_p(\gamma x)}+\frac{\gamma^{-1}x^{-1}}{\Theta_p(\gamma^{-1}x^{-1})}\right\}\frac{1}{\Theta_p(\gamma^{-1}x)}-\frac{\gamma^{-1}x^{-1}}{\Theta_p(\gamma^{-1}x^{-1})}\left\{\frac{1}{\Theta_p(\gamma^{-1}x)}+\frac{\gamma x^{-1}}{\Theta_p(\gamma x^{-1})}\right\}\\ &= \frac{1}{(p;p)_\infty^3}\delta(\gamma x)\frac{1}{\Theta_p(\gamma^{-1}x)}-\frac{\gamma^{-1}x^{-1}}{\Theta_p(\gamma^{-1}x^{-1})}\frac{1}{(p;p)_\infty^3}\delta(\gamma^{-1}x)\\ &= \frac{1}{(p;p)_\infty^3\Theta_p(qt^{-1})}\{\delta(\gamma x)-\gamma^{-2}\delta(\gamma^{-1}x)\}.\end{aligned}$$

This leads to

$$\frac{\Theta_p(q\gamma x)\Theta_p(q^{-1}\gamma^{-1}x)}{\Theta_p(\gamma x)\Theta_p(\gamma^{-1}x)} - \frac{\Theta_p(q\gamma x^{-1})\Theta_p(q^{-1}\gamma^{-1}x^{-1})}{\Theta_p(\gamma x^{-1})\Theta_p(\gamma^{-1}x^{-1})} = \frac{\Theta_p(q)\Theta_p(t^{-1})}{(p;p)^3_\infty \Theta_p(qt^{-1})}\{\delta(\gamma x) - \delta(\gamma^{-1}x)\}.$$

From this relation and the definition of $\varphi^{\pm}(p;z)$, we have (3.13). □

The commutation relation (3.13) is also an elliptic analog of the trigonometric case in (2.9). Furthermore, we can show the following theorem by applying Wick's theorem.

Theorem 3.8 (Relations of $\eta(p;z)$, $\xi(p;z)$ and $\varphi^{\pm}(p;z)$). *Define the structure function $g_p(x) \in \mathbb{C}[[x,x^{-1}]]$ as*

$$g_p(x) := \frac{\Theta_p(qx)\Theta_p(t^{-1}x)\Theta_p(q^{-1}tx)}{\Theta_p(q^{-1}x)\Theta_p(tx)\Theta_p(qt^{-1}x)}. \tag{3.14}$$

Then $\eta(p;z)$, $\xi(p;z)$ and $\varphi^{\pm}(p;z)$ satisfy the relations

$$\begin{aligned} &[\varphi^{\pm}(p;z), \varphi^{\pm}(p;w)] = 0, \\ &\varphi^{+}(p;z)\varphi^{-}(p;w) = \frac{g_p(\gamma z/w)}{g_p(\gamma^{-1}z/w)}\varphi^{-}(p;w)\varphi^{+}(p;z), \end{aligned} \tag{3.15}$$

$$\varphi^{\pm}(p;z)\eta(p;w) = g_p\left(\gamma^{\pm\frac{1}{2}}\frac{z}{w}\right)\eta(p;w)\varphi^{\pm}(p;z), \tag{3.16}$$

$$\varphi^{\pm}(p;z)\xi(p;w) = g_p\left(\gamma^{\mp\frac{1}{2}}\frac{z}{w}\right)^{-1}\xi(p;w)\varphi^{\pm}(p;z), \tag{3.17}$$

$$\eta(p;z)\eta(p;w) = g_p\left(\frac{z}{w}\right)\eta(p;w)\eta(p;z), \tag{3.18}$$

$$\xi(p;z)\xi(p;w) = g_p\left(\frac{z}{w}\right)^{-1}\xi(p;w)\xi(p;z), \tag{3.19}$$

$$\begin{aligned} &[\eta(p;z), \xi(p;w)] \\ &= \frac{\Theta_p(q)\Theta_p(t^{-1})}{(p;p)^3_\infty \Theta_p(qt^{-1})}\left\{\delta\left(\gamma\frac{w}{z}\right)\varphi^{+}(p;\gamma^{1/2}w) - \delta\left(\gamma^{-1}\frac{w}{z}\right)\varphi^{-}(p;\gamma^{-1/2}w)\right\}. \end{aligned} \tag{3.20}$$

Proof. Relations (3.18) and (3.19) follow from (3.7), (3.9). By the definition of $\varphi^{\pm}(p;z)$,

$$\varphi^+(p;z) = \exp\left(-\sum_{n>0}\frac{1-t^{-n}}{1-p^n}p^n(1-\gamma^{-2n})\gamma^{n/2}b_n\frac{z^n}{n}\right)$$
$$\times\exp\left(-\sum_{n>0}\frac{1-t^n}{1-p^n}(1-\gamma^{2n})\gamma^{-n/2}a_n\frac{z^{-n}}{n}\right),$$
$$\varphi^-(p;z) = \exp\left(-\sum_{n<0}\frac{1-t^{-n}}{1-p^{-n}}p^{-n}(1-\gamma^{2n})\gamma^{-n/2}b_n\frac{z^n}{n}\right)$$
$$\times\exp\left(-\sum_{n<0}\frac{1-t^n}{1-p^{-n}}(1-\gamma^{-2n})\gamma^{n/2}a_n\frac{z^{-n}}{n}\right).$$

From these expressions, the relation $[\varphi^\pm(p;z),\varphi^\pm(p;w)]=0$ is trivial. Next we show the second relation in (3.15). Here we can check that

$$g_p(x) = \exp\left(-\sum_{n>0}\frac{(1-q^n)(1-t^{-n})(1-\gamma^{2n})}{1-p^n}p^n\frac{x^{-n}}{n}\right)$$
$$\times\exp\left(\sum_{n>0}\frac{(1-q^n)(1-t^{-n})(1-\gamma^{2n})}{1-p^n}\frac{x^n}{n}\right).$$

We can also check that $g_p(x^{-1})=g_p(x)^{-1}$. From these facts, we have

$$\begin{aligned}
&\varphi^+(p;z)\varphi^-(p;w)\\
&\quad=\exp\left(-\sum_{m>0}\frac{(1-q^m)(1-t^{-m})(1-\gamma^{2m})}{1-p^m}p^m(\gamma^{-m}-\gamma^m)\frac{(z/w)^m}{m}\right)\\
&\qquad\times\exp\left(\sum_{m>0}\frac{(1-q^m)(1-t^{-m})(1-\gamma^{2m})}{1-p^m}(\gamma^m-\gamma^{-m})\frac{(w/z)^m}{m}\right)\varphi^-(p;w)\varphi^+(p;z)\\
&\quad=\frac{g_p(\gamma w/z)}{g_p(\gamma^{-1}w/z)}\varphi^-(p;w)\varphi^+(p;z)\\
&\quad=\frac{g_p(\gamma z/w)}{g_p(\gamma^{-1}z/w)}\varphi^-(p;w)\varphi^+(p;z)\qquad(\because\ g_p(x^{-1})=g_p(x)^{-1}).
\end{aligned}$$

Next we show (3.16). By Wick's theorem,

$$\begin{aligned}
&\varphi^+(p;z)\eta(p;z)\\
&\quad=\exp\left(\sum_{m>0}\frac{(1-q^m)(1-t^{-m})(1-\gamma^{2m})}{1-p^m}p^m\gamma^{m/2}\frac{(z/w)^m}{m}\right)\\
&\qquad\times\exp\left(-\sum_{m>0}\frac{(1-q^m)(1-t^{-m})(1-\gamma^{2m})}{1-p^m}\gamma^{-m/2}\frac{(w/z)^m}{m}\right)\eta(p;w)\varphi^+(p;z)\\
&\quad=g_p\left(\gamma^{-1/2}\frac{w}{z}\right)^{-1}\eta(p;w)\varphi^+(p;z)=g_p\left(\gamma^{1/2}\frac{z}{w}\right)\eta(p;w)\varphi^+(p;z).
\end{aligned}$$

Similarly,

$$
\begin{aligned}
&\eta(p;w)\varphi^-(p;z)\\
&\quad = \exp\left(\sum_{m>0}\frac{(1-q^m)(1-t^{-m})(1-\gamma^{2m})}{1-p^m}p^m\gamma^{m/2}\frac{(w/z)^m}{m}\right)\\
&\qquad \times \exp\left(-\sum_{m>0}\frac{(1-q^m)(1-t^{-m})(1-\gamma^{2m})}{1-p^m}\gamma^{-m/2}\frac{(z/w)^m}{m}\right)\varphi^-(p;z)\eta(p;w)\\
&\quad = g_p\left(\gamma^{-1/2}\frac{z}{w}\right)^{-1}\varphi^-(p;z)\eta(p;w).
\end{aligned}
$$

Consequently, $\varphi^\pm(p;z)\eta(p;w) = g_p\big(\gamma^{\pm\frac{1}{2}}\frac{z}{w}\big)\eta(p;w)\varphi^\pm(p;z)$.

Finally, we show (3.17). Similarly to the above calculations, we have

$$
\begin{aligned}
&\varphi^+(p;z)\xi(p;z)\\
&\quad = \exp\left(-\sum_{m>0}\frac{(1-q^m)(1-t^{-m})(1-\gamma^{2m})}{1-p^m}p^m\gamma^{-m/2}\frac{(z/w)^m}{m}\right)\\
&\qquad \times \exp\left(\sum_{m>0}\frac{(1-q^m)(1-t^{-m})(1-\gamma^{2m})}{1-p^m}\gamma^{m/2}\frac{(w/z)^m}{m}\right)\xi(p;w)\varphi^+(p;z)\\
&\quad = g_p\left(\gamma^{1/2}\frac{w}{z}\right)\xi(p;w)\varphi^+(p;z) = g_p\left(\gamma^{-1/2}\frac{z}{w}\right)^{-1}\xi(p;w)\varphi^+(p;z)
\end{aligned}
$$

and

$$
\begin{aligned}
&\xi(p;w)\varphi^-(p;z)\\
&\quad = \exp\left(-\sum_{m>0}\frac{(1-q^m)(1-t^{-m})(1-\gamma^{2m})}{1-p^m}p^m\gamma^{-m/2}\frac{(w/z)^m}{m}\right)\\
&\qquad \times \exp\left(\sum_{m>0}\frac{(1-q^m)(1-t^{-m})(1-\gamma^{2m})}{1-p^m}\gamma^{m/2}\frac{(z/w)^m}{m}\right)\varphi^-(p;z)\xi(p;w)\\
&\quad = g_p\left(\gamma^{1/2}\frac{z}{w}\right)\varphi^-(p;z)\xi(p;w).
\end{aligned}
$$

Therefore $\varphi^\pm(p;z)\xi(p;w) = g_p\big(\gamma^{\mp\frac{1}{2}}\frac{z}{w}\big)^{-1}\xi(p;w)\varphi^\pm(p;z)$, completing the proof of the theorem. □

§3.3. Elliptic Ding–Iohara algebra $\mathcal{U}(q,t,p)$

Having Theorem 3.8, we can define the elliptic Ding–Iohara algebra.

Definition 3.9 (Elliptic Ding–Iohara algebra $\mathcal{U}(q,t,p)$). Let $g_p(x)$ be the structure function defined by (3.14):

$$g_p(x) = \frac{\Theta_p(qx)\Theta_p(t^{-1}x)\Theta_p(q^{-1}tx)}{\Theta_p(q^{-1}x)\Theta_p(tx)\Theta_p(qt^{-1}x)}.$$

Let C be a central, invertible element and let $x^\pm(p;z) := \sum_{n\in\mathbb{Z}} x_n^\pm(p) z^{-n}$ and $\psi^\pm(p;z) := \sum_{n\in\mathbb{Z}} \psi_n^\pm(p) z^{-n}$ be operators subject to the relations

(3.21)

$$[\psi^\pm(p;z), \psi^\pm(p;w)] = 0, \qquad \psi^+(p;z)\psi^-(p;w) = \frac{g_p(Cz/w)}{g_p(C^{-1}z/w)}\psi^-(p;w)\psi^+(p;z),$$

$$\psi^\pm(p;z)x^+(p;w) = g_p\left(C^{\pm\frac{1}{2}}\frac{z}{w}\right)x^+(p;w)\psi^\pm(p;z),$$

$$\psi^\pm(p;z)x^-(p;w) = g_p\left(C^{\mp\frac{1}{2}}\frac{z}{w}\right)^{-1}x^-(p;w)\psi^\pm(p;z),$$

$$x^\pm(p;z)x^\pm(p;w) = g_p\left(\frac{z}{w}\right)^{\pm1}x^\pm(p;w)x^\pm(p;z),$$

$$[x^+(p;z), x^-(p;w)] = \frac{\Theta_p(q)\Theta_p(t^{-1})}{(p;p)_\infty^3\Theta_p(qt^{-1})}\left\{\delta\left(C\frac{w}{z}\right)\psi^+(p;C^{1/2}w) - \delta\left(C^{-1}\frac{w}{z}\right)\psi^-(p;C^{-1/2}w)\right\}.$$

Then we define the *elliptic Ding–Iohara algebra* $\mathcal{U}(q,t,p)$ to be the associative $\mathbb{C}$-algebra generated by $\{x_n^\pm(p)\}_{n\in\mathbb{Z}}$, $\{\psi_n^\pm(p)\}_{n\in\mathbb{Z}}$ and C with these relations.

Similarly to the trigonometric case, the map defined by

$$C \mapsto \gamma, \qquad x^+(p;z) \mapsto \eta(p;z), \qquad x^-(p;z) \mapsto \xi(p;z), \qquad \psi^\pm(p;z) \mapsto \varphi^\pm(p;z)$$

gives a representation, or the *free field realization*, of the elliptic Ding–Iohara algebra $\mathcal{U}(q,t,p)$ (Theorem 1.2).

Remark 3.10. (1) By the definition, in the trigonometric limit $p \to 0$, the elliptic Ding–Iohara algebra $\mathcal{U}(q,t,p)$ degenerates to the Ding–Iohara algebra $\mathcal{U}(q,t)$.

(2) Since relations (3.21) take the same forms as in the trigonometric case (2.10), we can define the coproduct $\Delta : \mathcal{U}(q,t,p) \to \mathcal{U}(q,t,p) \otimes \mathcal{U}(q,t,p)$ similarly to the trigonometric case (see (2.11)):

$$\Delta(C^{\pm1}) = C^{\pm1} \otimes C^{\pm1}, \qquad \Delta(\psi^\pm(p;z)) = \psi^\pm(p;C_{(2)}^{\pm1/2}z) \otimes \psi^\pm(p;C_{(1)}^{\mp1/2}z),$$

$$\Delta(x^+(p;z)) = x^+(p;z) \otimes 1 + \psi^-(p;C_{(1)}^{1/2}z) \otimes x^+(p;C_{(1)}z),$$

$$\Delta(x^-(p;z)) = x^-(p;C_{(2)}z) \otimes \psi^+(p;C_{(2)}^{1/2}z) + 1 \otimes x^-(p;z).$$

(3) In [11], another elliptic Ding–Iohara algebra is defined based on the idea of quasi-Hopf deformation. Then the same structure function (3.14) arises.

§4. Free field realization of the elliptic Macdonald operator

In this section, we study the relations between the elliptic currents $\eta(p;z)$, $\xi(p;z)$ and the elliptic Macdonald operators $H_N(q,t,p)$, $H_N(q^{-1},t^{-1},p)$.

§4.1. Preparations

The elliptic Macdonald operator $H_N(q,t,p)$ ($N\in\mathbb{Z}_{>0}$) is defined by

$$H_N(q,t,p):=\sum_{i=1}^{N}\prod_{j\neq i}\frac{\Theta_p(tx_i/x_j)}{\Theta_p(x_i/x_j)}T_{q,x_i}.$$

First, we need a lemma to calculate the constant term of a product of theta functions.

Lemma 4.1. (1) *We have the following partial fraction expansion:*

$$\prod_{i=1}^{N}\frac{\Theta_p(t^{-1}x_iz)}{\Theta_p(x_iz)}=\frac{\Theta_p(t)}{\Theta_p(t^N)}\sum_{i=1}^{N}\frac{\Theta_p(t^{-N}x_iz)}{\Theta_p(x_iz)}\prod_{j\neq i}\frac{\Theta_p(tx_i/x_j)}{\Theta_p(x_i/x_j)}. \tag{4.1}$$

(2) *From Ramanujan's summation formula*

$$\sum_{n\in\mathbb{Z}}\frac{(a;p)_n}{(b;p)_n}z^n=\frac{(az;p)_\infty(p/az;p)_\infty(b/a;p)_\infty(p;p)_\infty}{(z;p)_\infty(b/az;p)_\infty(p/a;p)_\infty(b;p)_\infty}\qquad(|a^{-1}b|<|z|<1), \tag{4.2}$$

we have the expansion

$$\frac{\Theta_p(az)}{\Theta_p(z)}=\frac{\Theta_p(a)}{(p;p)_\infty^3}\sum_{n\in\mathbb{Z}}\frac{z^n}{1-ap^n}\qquad(|p|<|z|<1). \tag{4.3}$$

Proof. (1) From the partial fraction expansion (6.10) in Appendix B,

$$\prod_{i=1}^{N}\frac{\Theta_p(t_ix_i^{-1}z)}{\Theta_p(x_i^{-1}z)}=\sum_{i=1}^{N}\frac{\Theta_p(t_i)}{\Theta_p(t_{(N)})}\frac{\Theta_p(t_{(N)}x_i^{-1}z)}{\Theta_p(x_i^{-1}z)}\prod_{j\neq i}\frac{\Theta_p(t_jx_i/x_j)}{\Theta_p(x_i/x_j)},$$

by setting $t_j=t^{-1}$ and substituting $x_j\to x_j^{-1}$, we obtain (4.1).

(2) From Ramanujan's summation formula (4.2) (proved in Appendix B), by setting $b/a=p$ we have

$$\text{LHS (4.2)}=\sum_{n\in\mathbb{Z}}\frac{(a;p)_n}{(ap;p)_n}z^n=(1-a)\sum_{n\in\mathbb{Z}}\frac{z^n}{1-ap^n},$$

$$\text{RHS (4.2)} = \frac{(az;p)_\infty (p(az)^{-1};p)_\infty (p;p)_\infty^2}{(z;p)_\infty (pz;p)_\infty (pa^{-1};p)_\infty (pa;p)_\infty} = \frac{(1-a)(p;p)_\infty^3}{\Theta_p(a)} \frac{\Theta_p(az)}{\Theta_p(z)}. \qquad \square$$

Remark 4.2. Using (6.10) and the relation between the theta function and the delta function, we have

$$\prod_{i=1}^N \frac{\Theta_p(t_i x_i^{-1} z)}{\Theta_p(x_i^{-1} z)} = \sum_{i=1}^N \frac{\Theta_p(t_i)}{(p;p)_\infty^3} \prod_{j\neq i} \frac{\Theta_p(t_j x_i/x_j)}{\Theta_p(x_i/x_j)} \delta(x_i^{-1} z) + t_{(N)} \prod_{i=1}^N \frac{\Theta_p(t_i^{-1} x_i/z)}{\Theta_p(x_i/z)}$$
$$= \sum_{i=1}^N \frac{\Theta_p(t_i)}{(p;p)_\infty^3} \prod_{j\neq i} \frac{\Theta_p(t_j x_i/x_j)}{\Theta_p(x_i/x_j)} \delta(x_i^{-1} z) + t_{(N)} \prod_{i=1}^N \frac{\Theta_p(p t_i x_i^{-1} z)}{\Theta_p(p x_i^{-1} z)},$$

where we have used the relation $1/\Theta_p(px) = 1/\Theta_p(x^{-1})$. Taking the constant term of the above relation in z, we obtain

$$\sum_{i=1}^N \Theta_p(t_i) \prod_{j\neq i} \frac{\Theta_p(t_j x_i/x_j)}{\Theta_p(x_i/x_j)} = (1-t_{(N)})(p;p)_\infty^3 \left[\prod_{i=1}^N \frac{\Theta_p(t_i x_i^{-1} z)}{\Theta_p(x_i^{-1} z)} \right]_1.$$

§4.2. The case of using $\eta(p;z)$

Theorem 4.3 (Free field realization of the elliptic Macdonald operator (1)). *Set $\phi_N(p;x) := \prod_{j=1}^N \phi(p;x_j)$ ($N \in \mathbb{Z}_{>0}$). Then the elliptic current $\eta(p;z)$ reproduces the elliptic Macdonald operator $H_N(q,t,p)$ in the sense that*

$$[\eta(p;z) - t^{-N} (\eta(p;z))_- (\eta(p;p^{-1}z))_+]_1 \phi_N(p;x)|0\rangle$$
$$= \frac{t^{-N+1}\Theta_p(t^{-1})}{(p;p)_\infty^3} H_N(q,t,p) \phi_N(p;x)|0\rangle.$$

Proof. First, we prove that

$$\eta(p;z)\phi(p;w) = \frac{\Theta_p(w/z)}{\Theta_p(tw/z)} {:}\eta(p;z)\phi(p;w){:}.$$

This follows from by Wick's theorem:

$$\eta(p;z)\phi(p;w)$$
$$= \exp\left(-\sum_{m>0} \frac{1-t^{-m}}{1-p^m} p^m \frac{(1-t^m)(qt^{-1}p)^m}{(1-q^m)(1-p^m)} \cdot m \frac{1-p^m}{(qt^{-1}p)^m} \frac{1-q^m}{1-t^m} \frac{(z/w)^m}{m\cdot m} \right)$$
$$\times \exp\left(-\sum_{m>0} \frac{1-t^m}{1-p^m} \frac{1-t^m}{(1-q^m)(1-p^m)} \cdot m(1-p^m) \frac{1-q^m}{1-t^m} \frac{(w/z)^m}{m\cdot m} \right) {:}\eta(p;z)\phi(p;w){:}$$

$$= \exp\left(-\sum_{m>0}\frac{1-t^{-m}}{1-p^m}p^m\frac{(z/w)^m}{m}\right)\exp\left(-\sum_{m>0}\frac{1-t^{m}}{1-p^m}\frac{(w/z)^m}{m}\right):\eta(p;z)\phi(p;w):$$
$$= \frac{(pz/w;p)_\infty}{(t^{-1}pz/w;p)_\infty}\frac{(w/z;p)_\infty}{(tw/z;p)_\infty}:\eta(p;z)\phi(p;w): = \frac{\Theta_p(w/z)}{\Theta_p(tw/z)}:\eta(p;z)\phi(p;w):.$$

From this relation, we deduce that

$$\eta(p;z)\phi_N(p;x) = \prod_{i=1}^N \frac{\Theta_p(x_i/z)}{\Theta_p(tx_i/z)}:\eta(p;z)\phi_N(p;x):.$$

Using (3.10) and (4.1), we can check that

$$\prod_{i=1}^N \frac{\Theta_p(x_i/z)}{\Theta_p(tx_i/z)} = \frac{t^{-N+1}\Theta_p(t^{-1})}{(p;p)_\infty^3}\sum_{i=1}^N\prod_{j\neq i}\frac{\Theta_p(tx_i/x_j)}{\Theta_p(x_i/x_j)}\delta\left(t\frac{x_i}{z}\right)+t^{-N}\prod_{i=1}^N\frac{\Theta_p(z/x_i)}{\Theta_p(t^{-1}z/x_i)}.$$

From these relations we have

$$\begin{aligned}\eta(p;z)\phi_N(p;x)|0\rangle &= \prod_{i=1}^N \frac{\Theta_p(x_i/z)}{\Theta_p(tx_i/z)}(\eta(p;z))_-\phi_N(p;x)|0\rangle\\
&= \frac{t^{-N+1}\Theta_p(t^{-1})}{(p;p)_\infty^3}\sum_{i=1}^N\prod_{j\neq i}\frac{\Theta_p(tx_i/x_j)}{\Theta_p(x_i/x_j)}\delta\left(t\frac{x_i}{z}\right)(\eta(p;tx_i))_-\phi_N(p;x)|0\rangle\\
&\quad + t^{-N}\prod_{i=1}^N\frac{\Theta_p(z/x_i)}{\Theta_p(t^{-1}z/x_i)}(\eta(p;z))_-\phi_N(p;x)|0\rangle\\
&= \frac{t^{-N+1}\Theta_p(t^{-1})}{(p;p)_\infty^3}\sum_{i=1}^N\prod_{j\neq i}\frac{\Theta_p(tx_i/x_j)}{\Theta_p(x_i/x_j)}\delta\left(t\frac{x_i}{z}\right)T_{q,x_i}\phi_N(p;x)|0\rangle\\
&\quad + t^{-N}\prod_{i=1}^N\frac{\Theta_p(z/x_i)}{\Theta_p(t^{-1}z/x_i)}(\eta(p;z))_-\phi_N(p;x)|0\rangle,\end{aligned}$$

where we have used the relation $(\eta(p;tz))_-\phi(p;z) = \phi(p;qz) = T_{q,z}\phi(p;z)$. Let us recall the relation $1/\Theta_p(px) = 1/\Theta_p(x^{-1})$. This leads to

$$\prod_{i=1}^N\frac{\Theta_p(z/x_i)}{\Theta_p(t^{-1}z/x_i)} = \prod_{i=1}^N\frac{\Theta_p(px_i/z)}{\Theta_p(ptx_i/z)}.$$

Hence

$$\prod_{i=1}^N\frac{\Theta_p(z/x_i)}{\Theta_p(t^{-1}z/x_i)}(\eta(p;z))_-\phi_N(p;x)|0\rangle = (\eta(p;z))_-(\eta(p;p^{-1}z))_+\phi_N(p;x)|0\rangle.$$

Finally,

$$\begin{aligned}&[\eta(p;z) - t^{-N}(\eta(p;z))_-(\eta(p;p^{-1}z))_+]_1\phi_N(p;x)|0\rangle\\
&\qquad = \frac{t^{-N+1}\Theta_p(t^{-1})}{(p;p)_\infty^3}H_N(q,t,p)\phi_N(p;x)|0\rangle. \qquad \square\end{aligned}$$

Remark 4.4. Let us define

$$C_N(p;x,y) := \langle 0|\phi_N^*(p;x)[(\eta(p;z))_-(\eta(p;p^{-1}z))_+]_1\phi_N(p;x)|0\rangle/\Pi(q,t,p)(x,y).$$

By Wick's theorem, we have

$$C_N(p;x,y) = \left[\prod_{i=1}^{N}\frac{\Theta_p(t^{-1}x_iz)\Theta_p(z/y_i)}{\Theta_p(x_iz)\Theta_p(t^{-1}z/y_i)}\right]_1.$$

By (4.3), the explicit form of $C_N(p;x,y)$ is

$$C_N(p;x,y) = \left(\frac{t^{-N+1}\Theta_p(t^{-1})}{(p;p)_\infty^3}\right)^2 \sum_{\substack{1\le i\le N\\ 1\le k\le N}}\prod_{j\ne i}\frac{\Theta_p(tx_i/x_j)}{\Theta_p(x_i/x_j)}\prod_{\ell\ne k}\frac{\Theta_p(ty_k/y_\ell)}{\Theta_p(y_k/y_\ell)}\sum_{m\in\mathbb{Z}}\frac{(tpx_iy_k)^m}{(1-t^{-N}p^m)^2}.$$

In the trigonometric limit, $C_N(p;x,y)$ degenerates to 1: $C_N(p;x,y)\to 1$ as $p\to 0$.

§4.3. The case of using $\xi(p;z)$

We can carry out similar calculations by using $\xi(p;z)$ instead of $\eta(p;z)$. Then we obtain the following theorem.

Theorem 4.5 (Free field realization of the elliptic Macdonald operator (2)). *The elliptic current $\xi(p;z)$ reproduces the elliptic Macdonald operator $H_N(q^{-1},t^{-1},p)$ in the sense that*

$$[\xi(p;z) - t^N(\xi(p;z))_-(\xi(p;p^{-1}z))_+]_1\phi_N(p;x)|0\rangle = \frac{t^{N-1}\Theta_p(t)}{(p;p)_\infty^3}H_N(q^{-1},t^{-1},p)\phi_N(p;x)|0\rangle.$$

The proof is similar to that of Theorem 4.3.

§4.4. Other forms of Theorems 4.3 and 4.5

Let us introduce zero mode generators a_0, Q satisfying

$$[a_0,Q]=1,\qquad [a_n,a_0]=[b_n,a_0]=0,\qquad [a_n,Q]=[b_n,Q]=0\qquad (n\in\mathbb{Z}\backslash\{0\}).$$

For a complex number α, we define $|\alpha\rangle := e^{\alpha Q}|0\rangle$. Then $a_0|\alpha\rangle = \alpha|\alpha\rangle$.

By using the zero modes, we can reformulate the free field realization of the elliptic Macdonald operator as follows.

Theorem 4.6. *Set*

$$\widetilde{\eta}(p;z) := (\eta(p;z))_-(\eta(p;p^{-1}z))_+, \qquad \widetilde{\xi}(p;z) := (\xi(p;z))_-(\xi(p;p^{-1}z))_+.$$

Define

$$E(p;z) := \eta(p;z) - \widetilde{\eta}(p;z)t^{-a_0}, \qquad F(p;z) := \xi(p;z) - \widetilde{\xi}(p;z)t^{a_0}. \tag{4.4}$$

Then the elliptic Macdonald operators $H_N(q,t,p)$, $H_N(q^{-1},t^{-1},p)$ can be recovered from the operators $E(p;z)$, $F(p;z)$ as follows:

$$[E(p;z)]_1\phi_N(p;x)|N\rangle = \frac{t^{-N+1}\Theta_p(t^{-1})}{(p;p)_\infty^3}H_N(q,t,p)\phi_N(p;x)|N\rangle,$$
$$[F(p;z)]_1\phi_N(p;x)|N\rangle = \frac{t^{N-1}\Theta_p(t)}{(p;p)_\infty^3}H_N(q^{-1},t^{-1},p)\phi_N(p;x)|N\rangle.$$

§5. Some observations and remarks

To end this paper, we indicate what remains unclear or should be clarified, and give some comments.

§5.1. The method of elliptic deformation

Looking at the construction of elliptic currents such as $\eta(p;z)$, $\xi(p;z)$ again, we can define a procedure of elliptic deformation as follows.

Definition 5.1 (The method of elliptic deformation). Suppose $X(z)$ is an operator of the form

$$X(z) = \exp\Bigl(\sum_{n<0} X_n^- a_n z^{-n}\Bigr)\exp\Bigl(\sum_{n>0} X_n^+ a_n z^{-n}\Bigr) \qquad (X_n^\pm \in \mathbb{C}),$$

where $\{a_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ are boson generators which satisfy the relations

$$[a_m,a_n] = m\frac{1-q^{|m|}}{1-t^{|m|}}\delta_{m+n,0}.$$

Then the *method of elliptic deformation* is the following procedure:

Step 1. Change the boson generators to ones satisfying

$$[a_m,a_n] = m(1-p^{|m|})\frac{1-q^{|m|}}{1-t^{|m|}}\delta_{m+n,0}, \qquad [b_m,b_n] = m\frac{1-p^{|m|}}{(qt^{-1}p)^{|m|}}\,\frac{1-q^{|m|}}{1-t^{|m|}}\delta_{m+n,0},$$

$$[a_m,b_n] = 0.$$

Step 2. Set $X(p;z) := X_b(p;z)X_a(p;z)$, where

$$X_b(p;z) := \exp\left(-\sum_{n<0} \frac{p^{|n|}}{1-p^{|n|}} X^-_{-n} b_n z^n\right) \exp\left(-\sum_{n>0} \frac{p^{|n|}}{1-p^{|n|}} X^+_{-n} b_n z^n\right),$$
$$X_a(p;z) := \exp\left(\sum_{n<0} \frac{1}{1-p^{|n|}} X^-_n a_n z^{-n}\right) \exp\left(\sum_{n>0} \frac{1}{1-p^{|n|}} X^+_n a_n z^{-n}\right).$$

§5.2. Commutator of $E(p;z)$, $F(p;z)$

In Proposition 3.7, we showed that

$$[\eta(p;z),\xi(p;w)] = \frac{\Theta_p(q)\Theta_p(t^{-1})}{(p;p)^3_\infty \Theta_p(qt^{-1})} \left\{ \delta\left(\gamma \frac{w}{z}\right)\varphi^+(p;\gamma^{1/2}w) - \delta\left(\gamma^{-1}\frac{w}{z}\right)\varphi^-(p;\gamma^{-1/2}w)\right\}.$$

Since $[\varphi^+(p;z)]_1 \neq [\varphi^-(p;z)]_1$, we have $[[\eta(p;z)]_1, [\xi(p;w)]_1] \neq 0$. This can be compared with the following statement for the operators $E(p;z)$, $F(p;z)$ defined in (4.4).

Proposition 5.2. (1) *We have*

$$E(p;z)E(p;w) = g_p\left(\frac{z}{w}\right) E(p;w)E(p;z), \tag{5.1}$$

$$F(p;z)F(p;w) = g_p\left(\frac{z}{w}\right)^{-1} F(p;w)F(p;z). \tag{5.2}$$

(2) *The commutator of* $E(p;z)$, $F(p;z)$ *is*

$$[E(p;z),F(p;w)] = \frac{\Theta_p(q)\Theta_p(t^{-1})}{(p;p)^3_\infty \Theta_p(qt^{-1})} \delta\left(\gamma\frac{w}{z}\right)\{\varphi^+(p;\gamma^{1/2}w) - \varphi^+(p;\gamma^{1/2}p^{-1}w)\}. \tag{5.3}$$

Proof. (1) We will only show (5.1). First we observe that

$$\eta(p;z)\widetilde{\eta}(p;w) = \frac{\Theta_p(w/z)\Theta_p(qt^{-1}w/z)}{\Theta_p(qw/z)\Theta_p(t^{-1}w/z)} :\eta(p;z)\widetilde{\eta}(p;w):,$$
$$\widetilde{\eta}(p;z)\eta(p;w) = \frac{\Theta_p(w/p^{-1}z)\Theta_p(qt^{-1}w/p^{-1}z)}{\Theta_p(qw/p^{-1}z)\Theta_p(t^{-1}w/p^{-1}z)} :\widetilde{\eta}(p;z)\widetilde{\eta}(p;w):$$
$$= \frac{\Theta_p(z/w)\Theta_p(q^{-1}tz/w)}{\Theta_p(q^{-1}z/w)\Theta_p(tz/w)} :\widetilde{\eta}(p;z)\eta(p;w):,$$
$$\widetilde{\eta}(p;z)\widetilde{\eta}(p;w) = \frac{\Theta_p(z/w)\Theta_p(q^{-1}tz/w)}{\Theta_p(q^{-1}z/w)\Theta_p(tz/w)} :\widetilde{\eta}(p;z)\widetilde{\eta}(p;w):.$$

From these we have

$$\eta(p;z)\widetilde{\eta}(p;w) = g_p\left(\frac{z}{w}\right)\widetilde{\eta}(p;w)\eta(p;z),$$
$$\widetilde{\eta}(p;z)\eta(p;w) = g_p\left(\frac{z}{w}\right)\eta(p;w)\widetilde{\eta}(p;z),$$
$$\widetilde{\eta}(p;z)\widetilde{\eta}(p;w) = g_p\left(\frac{z}{w}\right)\widetilde{\eta}(p;w)\widetilde{\eta}(p;z).$$

Hence

$$\begin{aligned}E(p;z)E(p;w) &= (\eta(p;z) - \widetilde{\eta}(p;z)t^{-a_0})(\eta(p;w) - \widetilde{\eta}(p;w)t^{-a_0})\\ &= \eta(p;z)\eta(p;w) - \eta(p;z)\widetilde{\eta}(p;w)t^{-a_0} - \widetilde{\eta}(p;z)\eta(p;w)t^{-a_0} + \widetilde{\eta}(p;z)\widetilde{\eta}(p;w)t^{-2a_0}\\ &= g_p\left(\frac{z}{w}\right)(\eta(p;w)\eta(p;z) - \eta(p;w)\widetilde{\eta}(p;z)t^{-a_0}\\ &\quad - \widetilde{\eta}(p;w)\eta(p;z)t^{-a_0} + \widetilde{\eta}(p;w)\widetilde{\eta}(p;z)t^{-2a_0})\\ &= g_p\left(\frac{z}{w}\right)E(p;w)E(p;z).\end{aligned}$$

(2) Let us recall the relations shown in Proposition 3.7:

$$\eta(p;z)\xi(p;w) = \frac{\Theta_p(q\gamma w/z)\Theta_p(q^{-1}\gamma^{-1}w/z)}{\Theta_p(\gamma w/z)\Theta_p(\gamma^{-1}w/z)} :\eta(p;z)\xi(p;w):,$$
$$\xi(p;w)\eta(p;z) = \frac{\Theta_p(q\gamma z/w)\Theta_p(q^{-1}\gamma^{-1}z/w)}{\Theta_p(\gamma z/w)\Theta_p(\gamma^{-1}z/w)} :\eta(p;z)\xi(p;w):.$$

We define

$$A(x) := \frac{\Theta_p(q\gamma x)\Theta_p(q^{-1}\gamma^{-1}x)}{\Theta_p(\gamma x)\Theta_p(\gamma^{-1}x)}.$$

Then we have

$$\begin{aligned}E(p;z)F(p;w) &= (\eta(p;z) - \widetilde{\eta}(p;z)t^{-a_0})(\xi(p;w) - \widetilde{\xi}(p;w)t^{a_0})\\ &= A\left(\frac{w}{z}\right):\eta(p;z)\xi(p;w): - A\left(\frac{w}{z}\right):\eta(p;z)\widetilde{\xi}(p;w):t^{a_0}\\ &\quad - A\left(\frac{w}{p^{-1}z}\right):\widetilde{\eta}(p;z)\xi(p;w):t^{-a_0} + A\left(\frac{w}{p^{-1}z}\right):\widetilde{\eta}(p;z)\widetilde{\xi}(p;w):\\ &= A\left(\frac{w}{z}\right):\eta(p;z)\xi(p;w): - A\left(\frac{w}{z}\right):\eta(p;z)\widetilde{\xi}(p;w):t^{a_0}\\ &\quad - A\left(\frac{z}{w}\right):\widetilde{\eta}(p;z)\xi(p;w):t^{-a_0} + A\left(\frac{z}{w}\right):\widetilde{\eta}(p;z)\widetilde{\xi}(p;w):\end{aligned}$$

and

$$\begin{aligned}F(p;w)E(p;z) &= (\xi(p;w)-\widetilde{\xi}(p;w)t^{a_0})(\eta(p;z)-\widetilde{\eta}(p;z)t^{-a_0})\\ &= A\left(\frac{z}{w}\right){:}\eta(p;z)\xi(p;w){:} - A\left(\frac{z}{w}\right){:}\widetilde{\eta}(p;z)\xi(p;w){:}t^{-a_0}\\ &\quad - A\left(\frac{z}{p^{-1}w}\right){:}\eta(p;z)\widetilde{\xi}(p;w){:}t^{a_0} + A\left(\frac{z}{p^{-1}w}\right){:}\widetilde{\eta}(p;z)\widetilde{\xi}(p;w){:}\\ &= A\left(\frac{z}{w}\right){:}\eta(p;z)\xi(p;w){:} - A\left(\frac{z}{w}\right){:}\widetilde{\eta}(p;z)\xi(p;w){:}t^{-a_0}\\ &\quad - A\left(\frac{w}{z}\right){:}\eta(p;z)\widetilde{\xi}(p;w){:}t^{a_0} + A\left(\frac{w}{z}\right){:}\widetilde{\eta}(p;z)\widetilde{\xi}(p;w){:}.\end{aligned}$$

Here we have used the relation $A(px)=A(x^{-1})$. Hence

$$[E(p;z),F(p;w)] = \left\{A\left(\frac{w}{z}\right) - A\left(\frac{z}{w}\right)\right\}({:}\eta(p;z)\xi(p;w){:} - {:}\widetilde{\eta}(p;z)\widetilde{\xi}(p;w){:}).$$

Let us recall that

$$A(x)-A(x^{-1}) = \frac{\Theta_p(q)\Theta_p(t^{-1})}{(p;p)_\infty^3\Theta_p(qt^{-1})}\{\delta(\gamma x)-\delta(\gamma^{-1}x)\}.$$

Using this relation we obtain

$$\begin{aligned}&[E(p;z),F(p;w)]\\ &\quad= \frac{\Theta_p(q)\Theta_p(t^{-1})}{(p;p)_\infty^3\Theta_p(qt^{-1})}\left\{\delta\left(\gamma\frac{w}{z}\right) - \delta\left(\gamma^{-1}\frac{w}{z}\right)\right\}({:}\eta(p;z)\xi(p;w){:} - {:}\widetilde{\eta}(p;z)\widetilde{\xi}(p;w){:})\\ &\quad= \frac{\Theta_p(q)\Theta_p(t^{-1})}{(p;p)_\infty^3\Theta_p(qt^{-1})}\delta\left(\gamma\frac{w}{z}\right)({:}\eta(p;\gamma w)\xi(p;w){:} - {:}\widetilde{\eta}(p;\gamma w)\widetilde{\xi}(p;w){:})\\ &\qquad - \frac{\Theta_p(q)\Theta_p(t^{-1})}{(p;p)_\infty^3\Theta_p(qt^{-1})}\delta\left(\gamma^{-1}\frac{w}{z}\right)({:}\eta(p;\gamma^{-1}w)\xi(p;w){:} - {:}\widetilde{\eta}(p;\gamma^{-1}w)\widetilde{\xi}(p;w){:}).\end{aligned}$$

Then ${:}\eta(p;\gamma w)\xi(p;w){:} = \varphi^+(p;\gamma^{1/2}w)$, ${:}\eta(p;\gamma^{-1}w)\xi(p;w){:} = \varphi^-(p;\gamma^{-1/2}w)$ and also

$$\begin{aligned}{:}\widetilde{\eta}(p;\gamma w)\widetilde{\xi}(p;w){:} &= (\eta(p;\gamma w))_-(\xi(p;w))_-(\eta(p;\gamma p^{-1}w))_+(\xi(p;p^{-1}w))_+\\ &= \varphi^+(p;\gamma^{1/2}p^{-1}w),\\ {:}\widetilde{\eta}(p;\gamma^{-1}w)\widetilde{\xi}(p;w){:} &= (\eta(p;\gamma^{-1}w))_-(\xi(p;w))_-(\eta(p;\gamma^{-1}p^{-1}w))_+(\xi(p;p^{-1}w))_+\\ &= \varphi^-(p;\gamma^{-1/2}w).\end{aligned}$$

Therefore we have (5.3). □

Remark 5.3. From formula (5.3), we have the commutativity of constant terms: $[[E(p;z)]_1, [F(p;w)]_1] = 0$. This corresponds to the commutativity of the elliptic Macdonald operators:

$$[H_N(q,t,p), H_N(q^{-1},t^{-1},p)] = 0.$$

It seems that for the free field realization of the elliptic Macdonald operator, we should use the operators $E(p;z)$ and $F(p;z)$.

§5.3. Perspectives

In this paper, we have considered an elliptic analog of the Ding–Iohara algebra and a possibility of the free field realization of the elliptic Macdonald operator. In the following, we mention some ideas which can be cultivated in the future.

5.3.1. Elliptic q-Virasoro algebra, elliptic q-W_N algebra. As we have shown, starting from the elliptic kernel function $\Pi(q,t,p)(x,y)$ we can construct elliptic currents $\eta(p;z)$, $\xi(p;z)$ and $\varphi^{\pm}(p;z)$ which satisfy the relations of the elliptic Ding–Iohara algebra. Furthermore we obtain a procedure of producing elliptic currents, namely the method of elliptic deformation. Actually, we can apply the method of elliptic deformation to the free field realization of the q-Virasoro algebra, which yields an elliptic analog of the q-Virasoro algebra. Similarly, we can also construct a free field realization of an elliptic analog of the q-W_N algebra. In the near future, we intend to report on this in a follow-up paper [21].

5.3.2. Elliptic Macdonald symmetric functions. To construct an elliptic analog of Macdonald symmetric functions (in the following, we call them elliptic Macdonald symmetric functions for short) is required for a good understanding of some objects, for example the elliptic Ruijsenaars model [20], the superconformal index [2], [25], [26], etc. To construct the elliptic Macdonald symmetric functions, a possibility would be to find an elliptic analog of the integral representation of Macdonald symmetric functions. That representation shows that Macdonald symmetric functions can be recovered from the kernel function $\Pi(q,t)(x,y)$ and the weight function $\Delta(q,t)(x)$ defined by

$$\Delta(q,t)(x) := \prod_{i \neq j} \frac{(x_i/x_j;q)_\infty}{(tx_i/x_j;q)_\infty}$$

and the "seed" of Macdonald symmetric functions [6], which is formed by monomials. As is seen in the previous sections, we already have the elliptic kernel function $\Pi(q,t,p)(x,y)$, and the elliptic weight function $\Delta(q,t,p)(x)$ is also known [14]:

$$\Delta(q,t,p)(x) := \prod_{i \neq j} \frac{\Gamma_{q,p}(tx_i/x_j)}{\Gamma_{q,p}(x_i/x_j)}.$$

But we do not know what is the seed of elliptic Macdonald symmetric functions, i.e. the simplest and nontrivial eigenfunctions of the elliptic Macdonald operator are not known. Therefore the construction of an elliptic analog of the integral representation of Macdonald symmetric functions is not accomplished.

On the other hand, it is known that singular vectors of the q-Virasoro algebra and of the q-W_N algebra correspond to Macdonald symmetric functions [24], [3], [4]. Perhaps there would be a way to construct elliptic Macdonald symmetric functions from the elliptic analog of the q-Virasoro algebra. In [21], we construct an elliptic analog of the screening currents of the q-Virasoro algebra, and the correlation function of the product of the elliptic screening currents reproduces the elliptic kernel function $\Pi(q,t,p)(x,y)$ as well as the elliptic weight function $\Delta(q,t,p)(x)$. But as mentioned above, an elliptic analog of the integral representation of Macdonald symmetric functions has not been obtained yet.

§6. Appendix

§6.1. Appendix A: Boson calculus

In this subsection we review some basic facts of boson calculus.

Proposition 6.1. *Let $\mathcal{A}$ be an associative $\mathbb{C}$-algebra. For $A \in \mathcal{A}$, define the exponential of A by*

$$e^A := \exp(A) := \sum_{n\geq 0} \frac{1}{n!} A^n.$$

Then for $A, B \in \mathcal{A}$,

$$e^A B e^{-A} = e^{\mathrm{ad}(A)} B,$$

where $\mathrm{ad}(A)B := AB - BA$.

Proof. Let us define $F(t) := e^{tA} B e^{-tA}$ $(t \in \mathbb{C})$. Then we can check that

$$\left.\frac{d^n}{dt^n} F(t)\right|_{t=0} = \mathrm{ad}(A)^n B \qquad (n \geq 0).$$

By the Taylor expansion of $F(t)$ around $t = 0$, we have

$$F(t) = \sum_{n\geq 0} \frac{t^n}{n!} \left.\frac{d^n}{dt^n} F(t)\right|_{t=0} = \sum_{n\geq 0} \frac{t^n}{n!} \mathrm{ad}(A)^n B = e^{t\mathrm{ad}(A)} B.$$

From this expression, we obtain $F(1) = e^A B e^{-A} = e^{\mathrm{ad}(A)} B$. □

By this proposition, we have $e^A e^B e^{-A} = \exp(e^{\mathrm{ad}(A)} B)$, which yields

Corollary 6.2. *For $A, B \in \mathcal{A}$, if $[A,B] \in \mathbb{C}$ then*

$$e^A e^B = e^{[A,B]} e^B e^A.$$

This corollary is essentially Wick's theorem which we use frequently in this paper.

Next we are going to prove Wick's theorem. First we define $\mathcal{B}$ to be an associative $\mathbb{C}$-algebra generated by $\{a_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ with the relations

$$[a_m, a_n] = A(m)\delta_{m+n,0} \qquad (A(m) \in \mathbb{C}).$$

We call such algebras *bosons.* For example, if we choose $A(m) = m\frac{1-q^{|m|}}{1-t^{|m|}}$ then

$$[a_m, a_n] = m\frac{1-q^{|m|}}{1-t^{|m|}}\delta_{m+n,0}, \tag{6.1}$$

which is one of boson algebras used in this paper. We define the normal ordering $:\cdot:$ by

$$:a_m a_n: = \begin{cases} a_m a_n & (m < n), \\ a_n a_m & (m \geq n). \end{cases}$$

For $\{X_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ $(X_n \in \mathbb{C})$, we define $X(z) \in \mathcal{B}\otimes\mathbb{C}[[z,z^{-1}]]$ to be the formal power series

$$X(z) := \sum_{n\neq 0} X_n a_n z^{-n}.$$

We define its *plus part* $(X(z))_+$ and *minus part* $(X(z))_-$ by

$$(X(z))_+ := \sum_{n>0} X_n a_n z^{-n}, \qquad (X(z))_- := \sum_{n<0} X_n a_n z^{-n}.$$

In this notation,

$$:\exp(X(z)): = \exp((X(z))_-)\exp((X(z))_+).$$

Proposition 6.3 (Wick's theorem). *For boson operators $X(z) \in \mathcal{B}\otimes\mathbb{C}[[z,z^{-1}]]$ and $Y(w) \in \mathcal{B}\otimes\mathbb{C}[[w,w^{-1}]]$, if $[(X(z))_+, (Y(w))_-] \in \mathbb{C}[[w/z]]$ exists, define*

$$\langle X(z), Y(w)\rangle := [(X(z))_+, (Y(w))_-].$$

Then

$$:\exp(X(z))::\exp(Y(w)): = \exp(\langle X(z), Y(w)\rangle):\exp(X(z))\exp(Y(w)):.$$

As an example of the use of Wick's theorem, we consider the boson algebra with (6.1) and define

$$\eta(z) := :\exp\left(-\sum_{n\neq 0}(1-t^n)a_n\frac{z^{-n}}{n}\right):.$$

Let us show that

$$\eta(z)\eta(w)=\frac{(1-w/z)(1-qt^{-1}w/z)}{(1-qw/z)(1-t^{-1}w/z)}{:}\eta(z)\eta(w){:}.$$

By Wick's theorem, we have

$$\begin{aligned}\eta(z)\eta(w)&=\exp\biggl(\sum_{m>0}\sum_{n<0}(1-t^m)(1-t^n)[a_m,a_n]\frac{z^{-m}w^{-n}}{mn}\biggr){:}\eta(z)\eta(w){:}\\&=\exp\biggl(\sum_{m>0}\sum_{n<0}(1-t^m)(1-t^n)m\frac{1-q^{|m|}}{1-t^{|m|}}\delta_{m+n,0}\frac{z^{-m}w^{-n}}{mn}\biggr){:}\eta(z)\eta(w){:}\\&=\exp\biggl(\sum_{m>0}(1-t^m)(1-t^{-m})m\frac{1-q^m}{1-t^m}\,\frac{z^{-m}w^m}{m(-m)}\biggr){:}\eta(z)\eta(w){:}\\&=\exp\biggl(-\sum_{m>0}(1-q^m)(1-t^{-m})\frac{(w/z)^m}{m}\biggr){:}\eta(z)\eta(w){:}\\&=\frac{(1-w/z)(1-qt^{-1}w/z)}{(1-qw/z)(1-t^{-1}w/z)}{:}\eta(z)\eta(w){:},\end{aligned}$$

where we use $\log(1-x)=-\sum_{n>0}x^n/n$ $(|x|<1)$.

§6.2. Appendix B: Some formulas

In this subsection, we show some formulas which are used in this paper.

6.2.1. Ramanujan's summation formula. We prove Ramanujan's summation formula which is used in Section 4. As a preparation, we show the q-binomial theorem. In the following, we assume that the base $q\in\mathbb{C}$ satisfies $|q|<1$. Set

$$(x;q)_\infty:=\prod_{n\geq 0}(1-xq^n),\qquad (x;q)_n:=\frac{(x;q)_\infty}{(q^nx;q)_\infty}\qquad (n\in\mathbb{Z}).$$

Proposition 6.4 (q-Binomial theorem). *For $a\in\mathbb{C}$,*

$$\frac{(az;q)_\infty}{(z;q)_\infty}=\sum_{n\geq 0}\frac{(a;q)_n}{(q;q)_n}z^n\qquad (|z|<1).$$

Proof. We expand $(az;q)_\infty/(z;q)_\infty$ as

$$\frac{(az;q)_\infty}{(z;q)_\infty}=\sum_{n\geq 0}c_nz^n.$$

We have to show that $c_n=(a;q)_n/(q;q)_n$. Since

$$\frac{(aqz;q)_\infty}{(qz;q)_\infty}=\frac{1-z}{1-az}\,\frac{(az;q)_\infty}{(z;q)_\infty},$$

we have

$$q^n c_n - aq^{n-1}c_{n-1} = c_n - c_{n-1}, \quad \text{that is,} \quad c_n = \frac{1-aq^{n-1}}{1-q^n}c_{n-1}.$$

By using this relation repeatedly, we obtain

$$c_n = \frac{1-aq^{n-1}}{1-q^n}\frac{1-aq^{n-2}}{1-q^{n-1}}\cdots\frac{1-a}{1-q}c_0 = \frac{(a;q)_n}{(q;q)_n}c_0.$$

Then $c_0 = 1$, hence $c_n = (a;q)_n/(q;q)_n$. □

Setting $a = 0$ in the q-binomial theorem gives

$$\frac{1}{(z;q)_\infty} = \sum_{n\geq 0}\frac{1}{(q;q)_n}z^n.$$

Similarly to the proof of the q-binomial theorem, one can show Euler's formula

$$(z;q)_\infty = \sum_{n\geq 0}\frac{(-1)^n q^{n(n-1)/2}}{(q;q)_n}z^n. \tag{6.2}$$

Before giving the proof of Ramanujan's summation formula, we prove Jacobi's triple product formula by using Euler's formula and the q-binomial theorem.

Proposition 6.5 (Jacobi's triple product formula).

$$(q;q)_\infty(z;q)_\infty(qz^{-1};q)_\infty = \sum_{n\in\mathbb{Z}}(-1)^n z^n q^{n(n-1)/2}. \tag{6.3}$$

Proof. First, we rewrite $(z;q)_\infty$ as

$$\begin{aligned}(z;q)_\infty &= \sum_{n\geq 0}\frac{(-1)^n q^{n(n-1)/2}}{(q;q)_n}z^n = \sum_{n\geq 0}(-1)^n q^{n(n-1)/2}\frac{(q^{n+1};q)_\infty}{(q;q)_\infty}z^n \\ &= \frac{1}{(q;q)_\infty}\sum_{n\in\mathbb{Z}}(-1)^n q^{n(n-1)/2}(q^{n+1};q)_\infty z^n \quad (\because (q^{n+1};q)_\infty = 0 \text{ for } n<0).\end{aligned}$$

Furthermore, by applying Euler's formula (6.2) to $(q^{n+1};q)_\infty$ we have

$$\begin{aligned}(z;q)_\infty &= \frac{1}{(q;q)_\infty}\sum_{n\in\mathbb{Z}}(-1)^n q^{n(n-1)/2}\sum_{r\geq 0}\frac{(-1)^r q^{r(r-1)/2}}{(q;q)_r}(q^{n+1})^r z^n \\ &= \frac{1}{(q;q)_\infty}\sum_{\substack{n\in\mathbb{Z}\\ r\geq 0}}(-1)^{n+r}z^{n+r}q^{(n+r)(n+r-1)/2}\frac{1}{(q;q)_r}(qz^{-1})^r \\ &= \frac{1}{(q;q)_\infty}\sum_{r\geq 0}\frac{1}{(q;q)_r}(qz^{-1})^r\sum_{n\in\mathbb{Z}}(-1)^{n+r}z^{n+r}q^{(n+r)(n+r-1)/2}.\end{aligned}$$

Then by the substitution $n \to n - r$, we have

$$(z;q)_\infty = \frac{1}{(q;q)_\infty} \sum_{r \geq 0} \frac{1}{(q;q)_r} (qz^{-1})^r \sum_{n \in \mathbb{Z}} (-1)^n z^n q^{n(n-1)/2}$$
$$= \frac{1}{(q;q)_\infty (qz^{-1};q)_\infty} \sum_{n \in \mathbb{Z}} (-1)^n z^n q^{n(n-1)/2}.$$

Finally,

$$(q;q)_\infty (z;q)_\infty (qz^{-1};q)_\infty = \sum_{n \in \mathbb{Z}} (-1)^n z^n q^{n(n-1)/2}. \qquad \square$$

Jacobi's triple product formula (6.3) means that

$$\Theta_p(z) = (p;p)_\infty (z;p)_\infty (pz^{-1};p)_\infty = \sum_{n \in \mathbb{Z}} (-1)^n z^n p^{n(n-1)/2}.$$

Next let us prove Ramanujan's summation formula. We define the bilateral series ${}_1\psi_1\left({a \atop b}; z\right)$ by

$${}_1\psi_1\left(\begin{matrix} a \\ b \end{matrix}; z\right) := \sum_{n \in \mathbb{Z}} \frac{(a;q)_n}{(b;q)_n} z^n.$$

Then we can check that

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}),$$
$$(a;q)_{-n} = \frac{1}{(1-aq^{-n})(1-aq^{-n+1})\cdots(1-aq^{-1})} = \frac{q^{n(n+1)/2}}{(-1)^n a^n (q/a;q)_n} \qquad (n > 0).$$

Using these relations, the series ${}_1\psi_1\left({a \atop b}; z\right)$ can be rewritten as follows:

$$\sum_{n \in \mathbb{Z}} \frac{(a;q)_n}{(b;q)_n} z^n = \sum_{n \geq 0} \frac{(a;q)_n}{(b;q)_n} z^n + \sum_{n \geq 1} \frac{(q/b;q)_n}{(q/a;q)_n} \left(\frac{b}{az}\right)^n,$$

which converges in $|a^{-1}b| < |z| < 1$.

Proposition 6.6 (Ramanujan's summation formula). *For $a, b \in \mathbb{C}$,*

$$\sum_{n \in \mathbb{Z}} \frac{(a;q)_n}{(b;q)_n} z^n = \frac{(az;q)_\infty (q/az;q)_\infty (b/a;q)_\infty (q;q)_\infty}{(z;q)_\infty (b/az;q)_\infty (q/a;q)_\infty (b;q)_\infty} \qquad (|a^{-1}b| < |z| < 1). \tag{6.4}$$

Proof. We follow the proof due to Gasper and Rahman [13]. We set

$$f(b) := {}_1\psi_1\left(\begin{matrix} a \\ b \end{matrix}; z\right) = \sum_{n \in \mathbb{Z}} \frac{(a;q)_n}{(b;q)_n} z^n.$$

We are going to show (6.4) by using a difference equation for $f(b)$. First, we have

$$\begin{aligned}
&{}_1\psi_1\left(\begin{matrix}a\\b\end{matrix};z\right) - a\,{}_1\psi_1\left(\begin{matrix}a\\b\end{matrix};qz\right)\\
&\quad= \sum_{n\in\mathbb{Z}}\left\{\frac{(a;q)_n}{(b;q)_n} - a\frac{(a;q)_n}{(b;q)_n}q^n\right\}z^n = \sum_{n\in\mathbb{Z}}\frac{(a;q)_{n+1}}{(b;q)_n}z^n\\
&\quad= (1-b/q)\sum_{n\in\mathbb{Z}}\frac{(a;q)_{n+1}}{(b/q;q)_{n+1}}z^n = (1-b/q)z^{-1}\sum_{n\in\mathbb{Z}}\frac{(a;q)_{n+1}}{(b/q;q)_{n+1}}z^{n+1}\\
&\quad= (1-b/q)z^{-1}{}_1\psi_1\left(\begin{matrix}a\\b/q\end{matrix};z\right),
\end{aligned}$$

therefore

$$f(b) - (1-b/q)z^{-1}f(q^{-1}b) = a\,{}_1\psi_1\left(\begin{matrix}a\\b\end{matrix};qz\right).$$

Making the substitution $b \to qb$, we obtain

$$f(qb) - (1-b)z^{-1}f(b) = a\,{}_1\psi_1\left(\begin{matrix}a\\qb\end{matrix};qz\right).$$

Second, we have

$$\begin{aligned}
a\,{}_1\psi_1\left(\begin{matrix}a\\qb\end{matrix};qz\right) &= a\sum_{n\in\mathbb{Z}}\frac{(a;q)_n}{(qb;q)_n}q^nz^n = -ab^{-1}\sum_{n\in\mathbb{Z}}\frac{(a;q)_n(1-bq^n-1)}{(qb;q)_n}z^n\\
&= -ab^{-1}\sum_{n\in\mathbb{Z}}\frac{(a;q)_n(1-bq^n)}{(qb;q)_n}z^n + ab^{-1}\sum_{n\in\mathbb{Z}}\frac{(a;q)_n}{(qb;q)_n}z^n\\
&= -ab^{-1}(1-b)\sum_{n\in\mathbb{Z}}\frac{(a;q)_n}{(b;q)_n}z^n + ab^{-1}f(qb)\\
&= -ab^{-1}(1-b)f(b) + ab^{-1}f(qb).
\end{aligned}$$

Therefore

$$f(qb) - (1-b)z^{-1}f(b) = -ab^{-1}(1-b)f(b) + ab^{-1}f(qb),$$

so

$$f(b) = \frac{1-b/a}{(1-b)(1-b/az)}f(qb).$$

By using this relation repeatedly, we obtain

$$f(b) = \frac{(b/a;q)_\infty}{(b;q)_\infty(b/az;q)_\infty}f(0).$$

Instead of $f(0)$, we determine $f(q)$:

$$
\begin{aligned}
f(q) &= \sum_{n\in\mathbb{Z}} \frac{(a;q)_n}{(q;q)_n} z^n \\
&= \sum_{n\geq 0} \frac{(a;q)_n}{(q;q)_n} z^n \overset{\substack{q\text{-binomial}\\ \text{theorem}}}{=} \frac{(az;q)_\infty}{(z;q)_\infty} \qquad \left(\because \frac{1}{(q;q)_n} = 0 \text{ for } n<0\right) \\
&= \frac{(q/a;q)_\infty}{(q;q)_\infty (q/az;q)_\infty} f(0).
\end{aligned}
$$

Thus

$$
f(0) = \frac{(az;q)_\infty (q/az;q)_\infty (q;q)_\infty}{(z;q)_\infty (q/a;q)_\infty}.
$$

Consequently,

$$
f(b) = \sum_{n\in\mathbb{Z}} \frac{(a;q)_n}{(b;q)_n} z^n = \frac{(az;q)_\infty (q/az;q)_\infty (b/a;q)_\infty (q;q)_\infty}{(z;q)_\infty (b/az;q)_\infty (b;q)_\infty (q/a;q)_\infty}. \qquad \square
$$

6.2.2. Partial fraction expansion formula. We give a partial fraction expansion formula for the theta function.

Proposition 6.7 (Partial fraction expansion [11]). *Let* $[u]$ $(u \in \mathbb{C})$ *be an entire function which satisfies the following relations:*

(1) *Odd function:* $[-u] = -[u]$,

(2) *The Riemann relation:*

$$
[x+z][x-z][y+w][y-w]-[x+w][x-w][y+z][y-z] = [x+y][x-y][z+w][z-w]. \tag{6.5}
$$

For $N \in \mathbb{Z}_{>0}$ *and parameters* q_i, c_i $(1 \leq i \leq N)$, *set* $c_{(N)} := \sum_{i=1}^N c_i$. *Then*

$$
\prod_{i=1}^N \frac{[u-q_i+c_i]}{[u-q_i]} = \sum_{i=1}^N \frac{[c_i]}{[c_{(N)}]} \frac{[u-q_i+c_{(N)}]}{[u-q_i]} \prod_{j\neq i} \frac{[q_i-q_j+c_j]}{[q_i-q_j]}. \tag{6.6}
$$

Proof. The proposition follows from the Riemann relation by induction on N. In the case $N=2$, we are going to show

$$
\begin{aligned}
&\frac{[u-q_1+c_1]}{[u-q_1]} \frac{[u-q_2+c_2]}{[u-q_2]} \\
&\quad = \frac{[c_1]}{[c_{(2)}]} \frac{[u-q_1+c_{(2)}]}{[u-q_1]} \frac{[q_1-q_2+c_2]}{[q_1-q_2]} + \frac{[c_2]}{[c_{(2)}]} \frac{[u-q_2+c_{(2)}]}{[u-q_2]} \frac{[q_2-q_1+c_1]}{[q_2-q_1]}.
\end{aligned}
$$

By multiplying both sides by $[c_{(2)}][u-q_1][u-q_2][q_1-q_2]$, this takes the form

$$
\begin{aligned}
&[c_{(2)}][q_1-q_2][u-q_1+c_1][u-q_2+c_2] \\
&\quad = [c_1][u-q_1+c_{(2)}][u-q_2][q_1-q_2+c_2]-[c_2][u-q_2+c_{(2)}][u-q_1][q_2-q_1+c_1].
\end{aligned} \tag{6.7}
$$

Now we show RHS (6.7)) = LHS (6.7) by the Riemann relation. Define x, y, z, w by

$$x + y = c_{(2)}, \quad x - y = q_1 - q_2, \quad z + w = u - q_1 + c_1, \quad z - w = u - q_2 + c_2.$$

It is clear that LHS (6.7) $= [x+y][x-y][z+w][z-w]$. On the other hand,

$$x = \frac{q_1 - q_2 + c_{(2)}}{2}, \qquad y = \frac{q_2 - q_1 + c_{(2)}}{2},$$
$$z = \frac{2u - q_1 - q_2 + c_{(2)}}{2}, \quad w = \frac{-q_1 + q_2 + c_1 - c_2}{2},$$

hence

$$\begin{aligned}\text{RHS}\,(6.7) &= [x+w][y+z][z-y][x-w] - [y-w][x+z][z-x][y+w] \\ &= [x+z][x-z][y+w][y-w] - [x+w][x-w][y+z][y-z] \\ &\overset{\substack{\text{Riemann}\\ \text{relation}}}{=} [x+y][x-y][z+w][z-w],\end{aligned}$$

so (6.7) is satisfied.

Next we suppose that (6.6) holds for some $N \geq 2$. Then

$$\begin{aligned}\prod_{i=1}^{N+1} \frac{[u-q_i+c_i]}{[u-q_i]} &= \frac{[u-q_{N+1}+c_{N+1}]}{[u-q_{N+1}]} \prod_{i=1}^{N} \frac{[u-q_i+c_i]}{[u-q_i]} \\ &= \frac{[u-q_{N+1}+c_{N+1}]}{[u-q_{N+1}]} \sum_{i=1}^{N} \frac{[c_i]}{[c_{(N)}]} \frac{[u-q_i+c_{(N)}]}{[u-q_i]} \prod_{1\leq j\leq N,\, j\neq i} \frac{[q_i-q_j+c_j]}{[q_i-q_j]} \\ &= \sum_{i=1}^{N} \frac{[c_i]}{[c_{(N)}]} \left\{ \frac{[c_{N+1}]}{[c_{(N+1)}]} \frac{[u-q_{N+1}+c_{(N+1)}]}{[u-q_{N+1}]} \frac{[q_{N+1}-q_i+c_{(N)}]}{[q_{N+1}-q_i]} \right. \\ &\quad \left. + \frac{[c_{(N)}]}{[c_{(N+1)}]} \frac{[u-q_i+c_{(N+1)}]}{[u-q_i]} \frac{[q_i-q_{N+1}+c_{N+1}]}{[q_i-q_{N+1}]} \right\} \prod_{1\leq j\leq N,\, j\neq i} \frac{[q_i-q_j+c_j]}{[q_i-q_j]} \\ &= \frac{[c_{N+1}]}{[c_{(N+1)}]} \frac{[u-q_{N+1}+c_{(N+1)}]}{[u-q_{N+1}]} \sum_{i=1}^{N} \frac{[c_i]}{[c_{(N)}]} \frac{[q_{N+1}-q_i+c_{(N)}]}{[q_{N+1}-q_i]} \prod_{1\leq j\leq N,\, j\neq i} \frac{[q_i-q_j+c_j]}{[q_i-q_j]} \\ &\quad + \sum_{i=1}^{N} \frac{[c_i]}{[c_{(N+1)}]} \frac{[u-q_i+c_{(N+1)}]}{[u-q_i]} \prod_{1\leq j\leq N+1,\, j\neq i} \frac{[q_i-q_j+c_j]}{[q_i-q_j]}.\end{aligned}$$

From the induction hypothesis, we have

$$\sum_{i=1}^{N} \frac{[c_i]}{[c_{(N)}]} \frac{[q_{N+1}-q_i+c_{(N)}]}{[q_{N+1}-q_i]} \prod_{1\leq j\leq N,\, j\neq i} \frac{[q_i-q_j+c_j]}{[q_i-q_j]} = \prod_{j=1}^{N} \frac{[q_{N+1}-q_j+c_j]}{[q_{N+1}-q_j]}.$$

Therefore

$$\prod_{i=1}^{N+1}\frac{[u-q_i+c_i]}{[u-q_i]}=\frac{[c_{N+1}]}{[c_{(N+1)}]}\frac{[u-q_{N+1}+c_{(N+1)}]}{[u-q_{N+1}]}\prod_{j=1}^{N}\frac{[q_{N+1}-q_j+c_j]}{[q_{N+1}-q_j]}$$
$$+\sum_{i=1}^{N}\frac{[c_i]}{[c_{(N+1)}]}\frac{[u-q_i+c_{(N+1)}]}{[u-q_i]}\prod_{1\le j\le N+1,\, j\ne i}\frac{[q_i-q_j+c_j]}{[q_i-q_j]}$$
$$=\sum_{i=1}^{N+1}\frac{[c_i]}{[c_{(N)}]}\frac{[u-q_i+c_{(N)}]}{[u-q_i]}\prod_{1\le j\le N+1,\, j\ne i}\frac{[q_i-q_j+c_j]}{[q_i-q_j]},$$

which proves (6.6) in the case of $N+1$. □

Proposition 6.7 is written in additive variables. Let us rewrite it in multiplicative variables. The theta function is defined by

$$\Theta_p(x)=(p;p)_\infty(x;p)_\infty(px^{-1};p)_\infty.$$

Proposition 6.8 (The Riemann relation for the theta function $\Theta_p(x)$). *For the theta function $\Theta_p(x)$, the Riemann relation is as follows:*

$$\Theta_p(xz)\Theta_p(x/z)\Theta_p(yw)\Theta_p(y/w)-\Theta_p(xw)\Theta_p(x/w)\Theta_p(yz)\Theta_p(y/z)$$
$$=\frac{y}{z}\Theta_p(xy)\Theta_p(x/y)\Theta_p(zw)\Theta_p(z/w). \tag{6.8}$$

Sketch of proof. For $x\in\mathbb{C}$, we let $f(x)$ be the ratio of the right hand side and the left hand side of (6.8):

(6.9)
$$f(x):=\frac{(y/z)\Theta_p(xy)\Theta_p(x/y)\Theta_p(zw)\Theta_p(z/w)}{\Theta_p(xz)\Theta_p(x/z)\Theta_p(yw)\Theta_p(y/w)-\Theta_p(xw)\Theta_p(x/w)\Theta_p(yz)\Theta_p(y/z)}.$$

Then we can check that $f(px)=f(x)$ using $\Theta_p(px)=-x^{-1}\Theta_p(x)$. Moreover $f(x)$ has no poles in the region $|p|\le|x|\le 1$. This shows that $f(x)$ is bounded on $\mathbb{C}^\times$. By the Liouville theorem, $f(x)$ is constant, i.e. the ratio (6.9) does not depend on x. Hence $f(x)=f(w)=1$, so (6.8) holds. □

For a variable $z\in\mathbb{C}$, we define the additive variable $u\in\mathbb{C}$ as $z=e^{2\pi iu}$, and we set

$$[u]:=-z^{-1/2}\Theta_p(z).$$

Using this notation, the Riemann relation (6.8) takes the form (6.5). Consequently, we have

Proposition 6.9. *For $1\le j\le N$ set*

$$z:=e^{2\pi iu},\qquad x_j:=e^{2\pi iq_j},\qquad t_j:=e^{2\pi ic_j},\qquad t_{(N)}:=t_1t_2\cdots t_N,$$

where u, q_j, c_j $(1 \le j \le N)$ are variable and parameters as in Proposition 6.7. *From* (6.6), *we have the partial fraction decomposition*

$$\prod_{i=1}^{N} \frac{\Theta_p(t_i x_i^{-1} z)}{\Theta_p(x_i^{-1} z)} = \sum_{i=1}^{N} \frac{\Theta_p(t_i)}{\Theta_p(t_{(N)})} \frac{\Theta_p(t_{(N)} x_i^{-1} z)}{\Theta_p(x_i^{-1} z)} \prod_{j \neq i} \frac{\Theta_p(t_j x_i / x_j)}{\Theta_p(x_i / x_j)}. \tag{6.10}$$

The theta function $\Theta_p(x)$ satisfies $\Theta_p(x) \to 1 - x$ as $p \to 0$. From the trigonometric limit of (6.10), we have

$$\prod_{i=1}^{N} \frac{1 - t_i x_i^{-1} z}{1 - x_i^{-1} z} = \sum_{i=1}^{N} \frac{1 - t_i}{1 - t_{(N)}} \frac{1 - t_{(N)} x_i^{-1} z}{1 - x_i^{-1} z} \prod_{j \neq i} \frac{1 - t_j x_i / x_j}{1 - x_i / x_j}.$$

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