

Convergences and Projection Markov Property of Markov Processes on Ultrametric Spaces

by

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Abstract

Let (S, ρ) be an ultrametric space satisfying certain conditions and S^k be the quotient space of S with respect to the partition by balls with a fixed radius $\phi(k)$. We prove that, for a Hunt process X on S associated with a Dirichlet form $(\mathcal{E}, \mathcal{F})$, a Hunt process X^k on S^k associated with the averaged Dirichlet form $(\mathcal{E}^k, \mathcal{F}^k)$ is Mosco convergent to X , and under certain additional conditions, X^k converges weakly to X . Moreover, we give a sufficient condition for the Markov property of X to be preserved under the canonical projection π^k to S^k . In this case, we see that the projected process $\pi^k \circ X$ is identical in law to X^k and converges almost surely to X .

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§1. Introduction

A metric space (S, ρ) is said to be *ultrametric* if the metric ρ satisfies the inequality

$$(1.1) \quad \rho(x, z) \leq \max\{\rho(x, y), \rho(y, z)\} \quad (\forall x, y, z \in S),$$

which is obviously stronger than the usual triangle inequality. In this paper, we always assume the following conditions:

(U.1) (S, ρ) is a locally compact complete ultrametric space.

(U.2) Any closed ball in S is compact.

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(U.3) There exist an integer valued function $r : S \times S \rightarrow \mathbb{Z} \cup \{\infty\}$ and a strictly decreasing function $\phi : \mathbb{Z} \cup \{\infty\} \rightarrow \mathbb{R}$ with $\phi(\infty) = 0$ such that

$$\rho(x, y) = \phi(r(x, y)) \quad (\forall x, y \in S).$$

(U.4) There exist a Radon measure μ on S assigning strictly positive finite values to all closed balls of positive radius.

Note that separability of S follows from the above conditions. We denote $B_x^k := \{y \in S : \rho(x, y) \leq \phi(k)\}$.

Ultrametric spaces have many important examples in various fields of mathematics. One of the best known examples is the field \mathbb{Q}_p of p -adic numbers equipped with the metric $\rho(x, y) = \|x - y\|_p$ where $\|\cdot\|_p$ is the p -adic norm. The field \mathbb{Q}_p originated in number theory and is now investigated in various fields. A lot of interesting studies of Markov processes on \mathbb{Q}_p (or more generally, on local fields) are due to Albeverio, Kaneko, Karwowski, Kochubei, Yasuda, Zhao and others (see [1, 3, 4, 5, 16, 20, 25, 26] and references therein). Recently Karwowski–Yasuda [17] studied an application of Markov processes on \mathbb{Q}_p to *spin glasses*, which is an important subject in physics. The study of Markov processes on \mathbb{Q}_p is important both from the mathematical and the physical viewpoint. In the above studies, not only the ultrametric structure of \mathbb{Q}_p was essentially used, but also algebraic structures, such as rings or topological groups. However, recently several authors studied Markov processes on more general ultrametric spaces without any algebraic structures, such as the endpoints of locally-finite trees (called *leaves of multibranching trees* in Albeverio–Karwowski [2], and *non-compact Cantor sets* in Kigami [19]). For more details, see Albeverio–Karwowski [2], Kigami [18, 19], Bendikov–Grigor’yan–Pittet [6], Woess [24] and Bendikov–Grigor’yan–Pittet–Woess [7] and references therein. In this paper, we do not assume any algebraic structures.

Ultrametric spaces have a remarkable geometrical property, which is quite different from the usual Euclidean spaces: any two balls with the same radius are either disjoint or identical. From this fact and the conditions (U.1)–(U.3), it follows that the family of balls with radius $\phi(k)$ forms a countable partition of the whole space S . Let S^k denote the quotient space with respect to this partition. For later arguments, we embed S^k in S via a fixed map I^k satisfying $I^k([x]) \in B_x^k$ where $[x]$ denotes the equivalence class containing x . Note that later arguments do not depend on a particular choice of I^k . Then the following question arises:

(Q1) Can we approximate a Markov process on S by Markov chains on S^k ?

Since ultrametric spaces are totally disconnected, we can only consider pure jump processes on S . Let us consider the following bilinear form $(\mathcal{E}, \mathcal{F})$ on $L^2(S; \mu)$:

$$\mathcal{E}(u, v) = \frac{1}{2} \int_{S \times S \setminus d} (u(x) - u(y))(v(x) - v(y))J(x, y) \mu(dx) \mu(dy),$$

with domain $\mathcal{F} = \overline{D_0}^{\mathcal{E}_1}$ and a non-negative Borel measurable function $J(x, y)$ on $S \times S \setminus d$. Here D_0 stands for the set of finite linear combinations of indicator functions of closed balls and d denotes the diagonal of $S \times S$. We assume that J satisfies the following conditions:

- (A.1) For all $k \in \mathbb{Z}$ and $i \in S^k$, $\int_{B_i^k \times (B_i^k)^c} J(x, y) \mu(dx) \mu(dy) < \infty$.
- (A.2) For all $(x, y) \in S \times S \setminus d$, $J(x, y) = J(y, x)$.

In the above setting, $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form (see Section 2). There exists a Hunt process $(\mathcal{M}_t, X_t, \mathbb{P}_x)$ on S associated with $(\mathcal{E}, \mathcal{F})$ (see, e.g., [12, Theorem 7.21]). We write $(\mathcal{E}^k, \mathcal{F}^k)$ for the following bilinear form:

$$\mathcal{E}^k(u, v) = \frac{1}{2} \sum_{i, j \in S^k} (u(i) - u(j))(v(i) - v(j))J^k(i, j)\mu^k(i)\mu^k(j),$$

with domain $\mathcal{F}^k = \overline{C_0^k}^{\mathcal{E}_1^k}$, where C_0^k denotes the set of functions on S^k with finite support, $\mu^k(i) := \mu(B_i^k)$ for $i \in S^k$, and

$$J^k(i, j) := \begin{cases} \frac{1}{\mu^k(i)\mu^k(j)} \int_{B_i^k \times B_j^k} J(x, y) \mu(dx) \mu(dy) & (i \neq j), \\ 0 & (i = j). \end{cases}$$

We call $(\mathcal{E}^k, \mathcal{F}^k)$ the *averaged Dirichlet form (of level k)*. The averaged Dirichlet form $(\mathcal{E}^k, \mathcal{F}^k)$ is also a regular Dirichlet form; let (X_t^k, \mathbb{P}_i^k) be a Hunt process on S^k associated with it. We obtain the following result:

Theorem 1.1. *Suppose (A.1) and (A.2) hold. Then the averaged Dirichlet form $(\mathcal{E}^k, \mathcal{F}^k)$ is Mosco convergent in the generalized sense to $(\mathcal{E}, \mathcal{F})$ as $k \rightarrow \infty$.*

The definition of Mosco convergence in the generalized sense will be given in Section 3 following Chen–Kim–Kumagai [9, Definition 8.1]. Theorem 1.1 is quite a general result applicable to very wide classes of symmetric Markov processes on S . For example, (A.1) and (A.2) are satisfied by the class constructed by Albeverio–Karwowski [2], the more general class constructed in Kigami [19] and a new class constructed in this paper (called the *mixed class*). See Section 6.2 for details.

We want to know when X^k converges weakly to X . We consider the following condition for tightness of $\{X^k\}_{k \in \mathbb{Z}}$:

(A.3) For any $k_1 \in \mathbb{Z}$,

$$\sup_{k \geq k_1} \sup_{x \in S} \frac{1}{\mu(B_x^k)} \int_{B_x^k \times (B_x^{k_1})^c} J(y, z) \mu(dy) \mu(dz) < \infty.$$

To ensure conservativeness of $(\mathcal{E}, \mathcal{F})$, we introduce the following condition:

(A.4)
$$\sup_{x \in S} \int_{\{y: \rho(x, y) \geq 1\}} J(x, y) \mu(dy) < \infty.$$

Note that since the ultrametric inequality does not allow processes to exit from a unit ball only by jumps whose sizes are smaller than one, we do not need other assumptions for small jumps and volume growth conditions for conservativeness. See Theorem 4.1.

Let $0 < T < \infty$ and let $\mathbb{D}_S[0, T]$ denote the set of right-continuous paths $[0, T] \rightarrow S$ having left limits, which is equipped with the Skorokhod topology (see e.g. [11]). Let $C_0^+(S)$ denote the family of non-negative real-valued continuous functions on S with compact support. For each $\psi \in C_0^+(S)$, define the function ψ^k on S^k as $\psi^k(i) := \frac{1}{\mu^k(i)} \int_{B_i^k} \psi(x) \mu(dx)$. Define

$$\mathbb{P}_\psi(\cdot) = \frac{1}{\mu(\psi)} \int_S \mathbb{P}_x(\cdot) \psi(x) \mu(dx) \quad \text{and} \quad \mathbb{P}_{\psi^k}^k(\cdot) = \frac{1}{\mu^k(\psi^k)} \sum_{i \in S^k} \mathbb{P}_i^k(\cdot) \psi^k(i) \mu^k(i),$$

where $\mu(\psi) = \int_S \psi(x) \mu(dx)$ and $\mu^k(\psi^k) = \sum_{i \in S^k} \psi^k(i) \mu^k(i)$. Then (X, \mathbb{P}_ψ) and $(X^k, \mathbb{P}_{\psi^k}^k)$ are called *Hunt processes with initial distributions $\psi\mu$ and $\psi^k\mu^k$* , respectively. Now we obtain the following main theorem:

Theorem 1.2. *Suppose (A.1)–(A.4). Let $\psi \in C_0^+(S)$ and $0 < T < \infty$. Then, as $k \rightarrow \infty$, the Markov chain $(X^k, \mathbb{P}_{\psi^k}^k)$ converges in law on $\mathbb{D}_S[0, T]$ to the Hunt process (X, \mathbb{P}_ψ) .*

The initial distributions are restricted to be absolutely continuous with respect to the reference measure μ because we use the Lyons–Zheng decomposition in the proof of tightness, following Chen–Kim–Kumagai [9].

Remark 1.1. There are some related works:

- (i) The result of Theorem 1.2 and its proof are very similar to [9, Theorem 6.1]. The formulations, however, are so different that we cannot reduce Theorem 1.2 to [9, Theorem 6.1]. In [9], the authors studied a metric measure space (E, ρ, m) under several conditions where *approximation graphs* $\{(V_k, \Theta_k)\}_{k \in \mathbb{Z}}$ can be constructed. They proved that Markov chains X^k on (V_k, Θ_k) converge weakly to a Markov process X on (E, ρ, m) , provided that X^k 's satisfy several conditions. In particular they assumed that the jump density of X^k

is controlled by the graph metric of (V_k, Θ_k) . In our setting, however, the approximation set S^k does not need any graph structures and actually we essentially use only topological properties in the proof of Theorem 1.2.

- (ii) Martínez–Remenik–Martín [21] investigated convergence of Markov processes on a compact ultrametric space. Let T be a locally-finite rooted tree and cut this tree at each finite level k (we write T^k for the cut tree). They considered some random walk W on T and W^k on T^k . They showed that W and W^k induce Markov processes X and X^k on their Martin boundaries S and S^k , which are compact ultrametric spaces. Then, they showed that X^k converges weakly to X as $k \rightarrow \infty$. In our setting, the state space S is not necessarily compact, and thus tightness of $\{X^k\}_k$ is not obvious. Moreover, our Markov processes need not be induced by random walks on trees.
- (iii) In Yasuda [26], a different type of convergence theorem of Lévy processes on local fields K was proved. Let X_t be a semi-stable process on K with an epoch $a < 1$ and a span b . The author showed that if ξ_i is identically distributed as X_1 , $a_n = a^{-n}$, and $b_n = b^n$, then $(1/b_n) \sum_{i=1}^{[a_n t]} \xi_i$ as $n \rightarrow \infty$ converges weakly to X , where $[a_n t]$ stands for the integral part of $a_n t$. This result uses the algebraic structure of local fields. We do not know the relationship of our result and theirs.

We introduce an interesting property of Markov processes on S . Let π^k be the canonical map from S to S^k . We call the following property the *projection Markov property of level k* (we write $(\text{pMp})_k$ for short):

$(\text{pMp})_k$ There exists a transition probability $\tilde{p}_t^k(x, y)$ such that

$$\mathbb{P}_x(\pi^k \circ X_{t+s} = y \mid \mathcal{M}_s) = \tilde{p}_t^k(\pi^k \circ X_s, y)$$

for quasi-every $x \in S$ and all $y \in S^k$; consequently, the projected process $\pi^k \circ X_t$ under $(\mathcal{M}_t, \mathbb{P}_x)$ is also a Markov process for quasi-every $x \in S$.

This property cannot be expected in the usual Euclidean spaces. For example, let X be the 2-dimensional Brownian motion and let us take a countable partition of \mathbb{R}^2 by $\{[n, n + 1) \times [m, m + 1)\}_{n, m \in \mathbb{Z}}$. Let \mathbb{R}^2/\sim denote the quotient space with respect to this partition and let π^1 be the canonical map from \mathbb{R}^2 to \mathbb{R}^2/\sim . Then we can see that $\pi^1 \circ X$ is not Markov.

We consider the following question:

- (Q2) When do Markov processes have the projection Markov property of level k ?

We introduce several additional conditions: For a fixed k ,

(BC) $_k$ (Ball-wise constance of level k) For each $i, j \in S^k$ with $i \neq j$,

$$J|_{B_i^k \times B_j^k} \equiv C_{ij}^k$$

for some constant C_{ij}^k depending only on i, j, k .

(BC) $_\infty$ The condition (BC) $_k$ is satisfied for all $k \in \mathbb{Z}$.

Note that these conditions are natural for ultrametric spaces because there are many locally constant functions on such spaces.

Let $\{P_t^k\}_{t \geq 0}$ be the transition semigroup on $L^2(S^k; \mu^k)$ associated with the averaged Dirichlet form $(\mathcal{E}^k, \mathcal{F}^k)$, and E^k be the extension operator defined in (3.4).

Theorem 1.3. (1) *Suppose (A.1) and (A.2) hold. The following two assertions are equivalent:*

- (i) *The Hunt process (X_t, \mathbb{P}_x) has the projection Markov property of level k .*
- (ii) *$P_t E^k f = E^k P_t^k f$ for all $f \in L^2(S^k; \mu^k)$ and $t \geq 0$.*

If one (and hence each) of assertions (i) and (ii) holds, the projected process $(\pi^k \circ X, \mathbb{P}_x)$ is equal in law to (X^k, \mathbb{P}_i^k) .

(2) *Suppose (A.1), (A.2) and (BC) $_k$ hold. Then assertions (i) and (ii) hold.*

In Dynkin [10], Rogers–Pitman [22] and Glover [13], functions preserving the Markov property were called *Markov functions* and those authors studied several sufficient conditions of different types for a function to be a Markov function. Theorem 1.3 asserts that the canonical projection π^k is a Markov function under (A.1), (A.2) and (BC) $_k$. The key to the proof is to verify Dynkin’s sufficient condition [10].

As a corollary of Theorem 1.3, under condition (BC) $_\infty$, we can show that X^k converges to X almost surely for $k \rightarrow \infty$, which is a stronger result than Theorem 1.2:

Corollary 1.1. *Suppose (A.1), (A.2) and (BC) $_\infty$ hold. Then (X^k, \mathbb{P}_i^k) is equal in law to $(\pi^k \circ X, \mathbb{P}_x)$ for all k , and $\pi^k \circ X_t$ converges to X_t as $k \rightarrow \infty$ uniformly on compact intervals in t , \mathbb{P}_x -almost everywhere for quasi-every $x \in S$.*

This paper is organized as follows. In Section 2, we give preliminary facts on ultrametric spaces and Dirichlet forms on ultrametric spaces. In Section 3, we prove Theorem 1.1. First, we recall the definition of Mosco convergence in the generalized sense given in Chen–Kim–Kumagai [9], and we introduce extension and restriction operators in our setting. Second, we relate $(\mathcal{E}, \mathcal{F})$ and the averaged

Dirichlet form $(\mathcal{E}^k, \mathcal{F}^k)$ by using the extension operator. Finally, we complete the proof of Theorem 1.1. In Section 4, we prove Theorem 1.2. First, we give a sufficient condition for conservativeness and obtain convergence of finite-dimensional distributions. Second, we prove tightness of X^k , and we complete the proof of Theorem 1.2. In Section 5, we prove Theorem 1.3. In Section 6, we introduce a mixed class, a kind of generalization of the class constructed in [19].

Throughout this paper, we denote by \mathbb{Z}, \mathbb{N} and \mathbb{N}_0 the set of integers, positive integers and non-negative integers, respectively. Sometimes we write $\sum_i A_i$ for $\bigcup_i A_i$ whenever $\{A_i\}_{i \in \mathbb{N}}$ is disjoint. We write $C_0(S)$, $C_b(S)$ and $C_\infty(S)$ for the class of real-valued continuous functions on S with compact support, bounded and vanishing at infinity, respectively. Write C_0^k for the class of continuous functions on S^k with finite support.

§2. Preliminary facts

We recall the following facts on ultrametric spaces:

- Fact 2.1.** (i) *If $B_a^k \cap B_b^k \neq \emptyset$, then $B_a^k = B_b^k$.*
 (ii) *The metric ρ is constant on $B_a^k \times B_b^k$ whenever $B_a^k \neq B_b^k$.*
 (iii) *Any closed ball is open.*
 (iv) *The indicator function of every closed (or open) ball is continuous.*

We shall utilize the following basic properties of ultrametric spaces under conditions (U.1)–(U.3).

Proposition 2.1. *Suppose (U.1)–(U.3) hold. Then:*

- (i) *For any $k \in \mathbb{Z}$ and $a \in S$, there exists a finite subset $\{a_i\}_i \subset B_a^k$ such that $\{B_{a_i}^k\}_i$ is disjoint and $B_a^k = \sum_i B_{a_i}^{k+1}$. If, moreover, there exists another such finite subset $\{b_i\}_i$, then $B_{a_i}^k = B_{b_i}^{k+1}$ for all i after suitable rearrangement.*
 (ii) *For any $k \in \mathbb{Z}$, there exists a countable subset $\{a_i\}_i \subset E$ such that $\{B_{a_i}^k\}_i$ is disjoint and $S = \sum_i B_{a_i}^k$. If, moreover, there exists another such countable subset $\{b_i\}_i$, then $B_{a_i}^k = B_{b_i}^k$ for all $i \in \mathbb{N}$ after suitable rearrangement.*
 (iii) *(S, ρ) is separable.*

Proof. (i) Let $a_0 = a \in S$. We take the open covering $B_a^k = \bigcup_{x \in B_a^k} B_x^{k+1}$. By (U.2) and Fact 2.1(i), we can extract a finite subcover $B_a^k = \sum_i B_{a_i}^{k+1}$. The last assertion is obvious by Fact 2.1(i).

(ii) Let $x \in S$ be fixed. By (i), there exists a finite subset $\{a_i^1\}_i \subset B_x^{k-1}$ such that $B_x^{k-1} = \sum_i B_{a_i^1}^k$. By using (i) again, there exists a finite subset $\{a_i^2\}_i \subset B_x^{k-2}$ such that $B_x^{k-2} \setminus B_x^{k-1} = \sum_i B_{a_i^2}^k$. Using this argument inductively, we see that

$S = \bigcup_{l=1}^{\infty} B_x^{k-l} = \sum_{i \in \mathbb{N}} B_{a_i}^k$. The last part of the statement can be shown by the same argument as (i).

(iii) Let $\{a_i^k\}_{i \in \mathbb{N}}$ be a countable subset of S such that $S = \sum_{i \in \mathbb{N}} B_{a_i^k}^k$. Define $S^k = \{a_i^k\}_i$ and $S = \bigcup_{k \in \mathbb{Z}} S^k$. Clearly, S is countable. For any $x \in S$, there exists a unique sequence $\{i(k)\}_{k \in \mathbb{Z}}$ such that $\cdots \supset B_{a_{i(k)}}^k \supset B_{a_{i(k+1)}}^{k+1} \supset \cdots \supset \{x\}$. This means that $a_{i(k)} \rightarrow x$ as $k \rightarrow \infty$, which completes the proof. \square

We prepare several classes of functions on S and S^k . Let $\mathbf{1}_i^k$ denote the indicator function of B_i^k . Define

$$D^k = \left\{ \sum_{i \in S^k} c_i \mathbf{1}_i^k : c_i \neq 0 \text{ for finitely many } i\text{'s only} \right\}$$

for each $k \in \mathbb{Z}$. It is obvious by Proposition 2.1(i) that $D^k \subset D^{k+1}$. We denote

$$(2.1) \quad D_0 = \bigcup_{k \in \mathbb{Z}} D^k.$$

In other words, D_0 is the set of finite linear combinations of indicator functions of balls. Since $D^k \subset D^{k+1}$, for each $u \in D_0$, we write

$$(2.2) \quad m(u) = \inf\{k : u \in D^k\}.$$

The class D_0 is a dense subset both of $C_0(S)$ with respect to the uniform norm and of $L^2(S; \mu)$ with respect to the L^2 -norm.

Assume (A.1) and (A.2) hold. Then $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{E}^k, \mathcal{F}^k)$ are symmetric regular Dirichlet forms. We refer the readers to [12, Example 1.2.4], for example.

§3. Proof of Theorem 1.1

We adopt the generalized Mosco convergence following Chen–Kim–Kumagai [9, Appendix]. For $k \in \mathbb{Z}$, let $(\mathcal{H}^k, \langle \cdot, \cdot \rangle_k)$ and $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be Hilbert spaces whose norms are denoted by $\|\cdot\|_k$ and $\|\cdot\|$. Suppose that $(a^k, \mathcal{D}[a^k])$ and $(a, \mathcal{D}[a])$ are positive densely defined closed symmetric contraction bilinear forms on \mathcal{H}^k and \mathcal{H} , respectively. We extend the definition of $a^k(u, u)$ to all $u \in \mathcal{H}^k$ by setting $a^k(u, u) = \infty$ for $u \in \mathcal{H}^k \setminus \mathcal{D}[a^k]$. Similar extension is done for a as well.

Let $E^k : \mathcal{H}^k \rightarrow \mathcal{H}$ and $\Pi^k : \mathcal{H} \rightarrow \mathcal{H}^k$ be bounded linear operators. If the following conditions are satisfied, we call E^k the *extension operator* and Π^k the *restriction operator*, respectively:

$$(ER.1) \quad \langle \Pi^k u, v \rangle_k = \langle u, E^k v \rangle \text{ for } u \in \mathcal{H} \text{ and } v \in \mathcal{H}^k.$$

$$(ER.2) \quad \Pi^k E^k u = u \text{ for } u \in \mathcal{H}^k.$$

(ER.3) $\sup_{k \in \mathbb{Z}} \|\Pi^k\|_{\text{op}} < \infty$, where $\|\cdot\|_{\text{op}}$ denotes the operator norm.

(ER.4) For every $u \in \mathcal{H}$, $\lim_{k \rightarrow \infty} \|\Pi^k u\|_k = \|u\|$.

It follows immediately that E^k is an isometry, i.e., $\langle E^k u, E^k v \rangle = \langle u, v \rangle_k$ for every $k \in \mathbb{Z}$ and $u, v \in \mathcal{H}^k$.

Now the generalized Mosco convergence is defined as follows:

Definition 3.1. In the above setting, we say that the closed bilinear form a^k is *Mosco-convergent to a in the generalized sense* if the following conditions are satisfied :

(i) If $u_k \in \mathcal{H}^k, u \in \mathcal{H}$ and $E^k u_k \rightarrow u$ weakly in \mathcal{H} , then

$$(3.1) \quad \liminf_{k \rightarrow \infty} a^k(u_k, u_k) \geq a(u, u).$$

(ii) For every $u \in \mathcal{H}$, there exists a sequence $\{u_k\}$ such that $u_k \in \mathcal{H}^k$ for all k , $E^k u_k \rightarrow u$ strongly in \mathcal{H} and

$$(3.2) \quad \limsup_{k \rightarrow \infty} a^k(u_k, u_k) \leq a(u, u).$$

Now we introduce extension and restriction operators in our setting. Set $\mathcal{H}^k = L^2(S^k; \mu^k)$ and $\mathcal{H} = L^2(S; \mu)$ equipped with L^2 -inner products

$$\langle u, v \rangle_k := \sum_{i \in S^k} u(i)v(i)\mu^k(i) \quad \text{and} \quad \langle u, v \rangle := \int_S u(x)v(x)\mu(dx).$$

Define bounded linear operators $E^k : L^2(S^k; \mu^k) \rightarrow L^2(S; \mu)$ and $\Pi^k : L^2(S; \mu) \rightarrow L^2(S^k; \mu^k)$ as follows:

$$(3.3) \quad \Pi^k u(i) = \frac{1}{\mu^k(i)} \int_{B_i^k} u(x)\mu(dx),$$

$$(3.4) \quad E^k v(x) = v([x]_k) = \sum_{i \in S^k} v(i)\mathbf{1}_{B_i^k}(x),$$

for $u \in L^2(S; \mu)$ and $v \in L^2(S^k; \mu^k)$. It is easy to check that E^k and Π^k satisfy (ER.1)–(ER.4) (see [9, Lemma 4.1]).

The following proposition relates the averaged Dirichlet form $(\mathcal{E}^k, \mathcal{F}^k)$ to $(\mathcal{E}, \mathcal{F})$.

Proposition 3.1. *Suppose (A.1) and (A.2) hold. Then:*

- (i) For all $u \in C_0^k, E^k u \in D_0$,
- (ii) For all $u \in \mathcal{F}^k, E^k u \in \mathcal{F}$,
- (iii) For all $u \in \mathcal{F}^k, \mathcal{E}^k(u, u) = \mathcal{E}(E^k u, E^k u)$.

Proof. (i) is clear by definition.

(ii) and (iii): Let $u \in C_0^k$. Let d^k denote the diagonal of $S^k \times S^k$. Then

$$\begin{aligned}
 (3.5) \quad \mathcal{E}^k(u, u) &= \frac{1}{2} \sum_{(i,j) \in S^k \times S^k \setminus d^k} |u(i) - u(j)|^2 J^k(i, j) \mu^k(i) \mu^k(j) \\
 &= \frac{1}{2} \sum_{(i,j) \in S^k \times S^k \setminus d^k} |u(i) - u(j)|^2 \int_{B_i^k \times B_j^k} J(x, y) \mu(dx) \mu(dy) \\
 &= \frac{1}{2} \sum_{(i,j) \in S^k \times S^k \setminus d^k} \int_{B_i^k \times B_j^k} |E^k u(x) - E^k u(y)|^2 J(x, y) \mu(dx) \mu(dy) \\
 &= \frac{1}{2} \int_{S \times S \setminus d} |E^k u(x) - E^k u(y)|^2 J(x, y) \mu(dx) \mu(dy) = \mathcal{E}(E^k u, E^k u).
 \end{aligned}$$

Let now $u \in \mathcal{F}^k$. Then we can take an \mathcal{E}_1^k -Cauchy sequence $\{u_n\}_{n \in \mathbb{N}} \subset C_0^k$ such that $u_n \rightarrow u$ in $L^2(S^k; \mu^k)$ and

$$(3.6) \quad \lim_{n \rightarrow \infty} \mathcal{E}^k(u_n, u_n) = \mathcal{E}^k(u, u).$$

By (3.5), we see that $\{u_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence with respect to $\mathcal{E}(E^k \cdot, E^k \cdot)$. From this fact and $E^k u_n \rightarrow E^k u$ in $L^2(S; \mu)$, we have

$$(3.7) \quad \lim_{n \rightarrow \infty} \mathcal{E}(E^k u_n, E^k u_n) = \mathcal{E}(E^k u, E^k u).$$

Hence, $E^k u \in \mathcal{F}$. Furthermore, by (3.6)–(3.7), we have

$$\mathcal{E}^k(u, u) = \lim_{n \rightarrow \infty} \mathcal{E}^k(u_n, u_n) = \lim_{n \rightarrow \infty} \mathcal{E}(E^k u_n, E^k u_n) = \mathcal{E}(E^k u, E^k u),$$

completing the proof. □

To prove the Mosco convergence of $(\mathcal{E}^k, \mathcal{F}^k)$ to $(\mathcal{E}, \mathcal{F})$, we need Fatou’s lemma for $(\mathcal{E}, \mathcal{F})$.

Lemma 3.1. *If $u_n \in \mathcal{F}$ converges μ -almost everywhere to $u \in L^2(S; \mu)$, then $\mathcal{E}(u, u) \leq \liminf_n \mathcal{E}(u_n, u_n)$.*

This is a direct application of Proposition 1 of Schmuland [23], so we omit the proof.

Suppose that (i) of Definition 3.1 holds. Then the following lemma yields a sufficient condition for (ii) of Definition 3.1 to hold (see [9, Lemma 8.2]):

Lemma 3.2. *Suppose that (A.1) and (A.2) hold. Then:*

- (i) D_0 is dense in \mathcal{F} with respect to the \mathcal{E}_1 -norm.
- (ii) For every $u \in D_0$, $\Pi^k u \in C_0^k$.
- (iii) For every $u \in D_0$, $\limsup_{k \rightarrow \infty} \mathcal{E}^k(\Pi^k u, \Pi^k u) = \mathcal{E}(u, u)$.

Proof. (i) is clear by the definition of $(\mathcal{E}, \mathcal{F})$, and (ii) is clear by the definition of $(\mathcal{E}^k, \mathcal{F}^k)$.

(iii) For $k \geq m(u)$, we have $E^k \Pi^k u = u$. Hence, from Proposition 3.1,

$$\limsup_{k \rightarrow \infty} \mathcal{E}^k(\Pi^k u, \Pi^k u) = \limsup_{k \rightarrow \infty} \mathcal{E}(E^k \Pi^k u, E^k \Pi^k u) = \mathcal{E}(u, u). \quad \square$$

Now we show that $(\mathcal{E}^k, \mathcal{F}^k)$ is Mosco convergent to $(\mathcal{E}, \mathcal{F})$.

Proof of Theorem 1.1. By Lemma 3.2, we only have to show condition (i) of Definition 3.1. It suffices to prove inequality (3.1) for sequences $u_k \in L^2(S^k; \mu^k)$ such that $E^k u_k$ converges weakly to $u \in L^2(S; \mu)$ and $\liminf_{k \rightarrow \infty} \mathcal{E}^k(u_k, u_k) =: L < \infty$. Taking a subsequence if necessary, we may assume that $\lim_{k \rightarrow \infty} \mathcal{E}^k(u_k, u_k) = L$. Since $E^k u_k$ converges to u weakly, $\{E^k u_k\}_{k \in \mathbb{Z}}$ is a bounded sequence in $L^2(S; \mu)$.

By the Banach–Saks Theorem (see e.g. [8, Theorem A.4.1]), by taking a subsequence if necessary, we may assume that $v_k := (1/k) \sum_{i=1}^k E^i u_i$ converges to some $v_\infty \in L^2(S; \mu)$. Since $E^k u_k$ converges weakly to u , we see that $v_\infty = u$ μ -a.e. on S . Then, by the triangle inequality with respect to $\mathcal{E}(\cdot, \cdot)^{1/2}$ and by Proposition 3.1, we have

$$\mathcal{E}(v_k, v_k)^{1/2} \leq \frac{1}{k} \sum_{i=1}^k \mathcal{E}(E^i u_i, E^i u_i)^{1/2} = \frac{1}{k} \sum_{i=1}^k \mathcal{E}^i(u_i, u_i)^{1/2} \rightarrow L^{1/2} \quad (k \rightarrow \infty).$$

In addition, from Lemma 3.1, we have $\mathcal{E}(u, u) \leq \liminf_{k \rightarrow \infty} \mathcal{E}(v_k, v_k) \leq L$. This completes the proof. \square

§4. Proof of Theorem 1.2

Now we give a sufficient condition for conservativeness:

Theorem 4.1. *Suppose that (A.1), (A.2) and (A.4) hold. Then $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{E}^k, \mathcal{F}^k)$ are conservative.*

Proof. Let $J'(x, y) = J(x, y) \mathbf{1}_{\rho(x, y) \leq 1}$ and $J''(x, y) = J(x, y) \mathbf{1}_{\rho(x, y) \geq 1}$. Let $(\mathcal{E}', \mathcal{F}')$ and $(\mathcal{E}'', \mathcal{F}'')$ be the Dirichlet forms corresponding to J' and J'' , respectively. Using (A.1), (A.2), (A.4) and the fact that D_0 is a core for $(\mathcal{E}, \mathcal{F})$, and by the same argument as in [14, Theorem 2.2], we see that the conservativeness of $(\mathcal{E}, \mathcal{F})$ is equivalent to that of the truncated Dirichlet form $(\mathcal{E}', \mathcal{F}')$.

Now it suffices to show that $(\mathcal{E}', \mathcal{F}')$ is conservative. However, by the ultrametric inequality, the corresponding process cannot exit from a unit ball only by small jumps (i.e., jumps whose sizes are smaller than 1). Since the jump density J' has only small jumps, $(\mathcal{E}', \mathcal{F}')$ is conservative. Using Proposition 3.1, the conservativeness of $(\mathcal{E}^k, \mathcal{F}^k)$ can be shown by the same argument. \square

Now we obtain convergence of finite-dimensional distributions by Theorems 1.1 and 4.1.

Corollary 4.1. *Suppose that (A.1), (A.2) and (A.4) hold. Then, for any $\psi \in C_0^+(S)$, finite-dimensional distributions of the Hunt process $(X_t^k, \mathbb{P}_\psi^k)$ converge to those of (X_t, \mathbb{P}_ψ) as $k \rightarrow \infty$.*

Proof. By Theorem 4.1, $(\mathcal{E}, \mathcal{F})$ is conservative. Thus, by Theorem 1.1, we can use the same argument as Chen–Kim–Kumagai [9, Theorem 5.1]. \square

To prove tightness, we adopt the same strategy as in [9]. The following lemma plays a key role in proving tightness; it corresponds to [9, Lemma 3.3] and the following proof is a modification of the one in [9].

Lemma 4.1. *Suppose that (A.1)–(A.3) hold. Then, for any $g \in D_0$, there exist $C > 0$ and $k_0 \in \mathbb{Z}$ such that, for any $k \geq k_0$ and any $0 \leq t \leq s < \infty$,*

$$\int_s^t \sum_{j \in S^k} (g(X_u^k) - g(j))^2 J^k(X_u^k, j) \mu^k(j) du \leq C(t - s).$$

Proof. Let $C_g := \max_{x, y \in \text{supp}(g)} |g(x) - g(y)|$. Take $k_0 = \max\{m(g), k_1\}$, where k_1 has been defined in (A.3), and $m(g)$ has been defined in (2.2). Note that we have $|g(x) - g(y)| = 0$ if $r(x, y) \geq k_0$, i.e., $\rho(x, y) \leq \phi(k_0)$. For any $k \geq k_0$,

$$\begin{aligned} & \sup_{i \in S^k} \sum_{j \in S^k} (g(i) - g(j))^2 J^k(i, j) \mu^k(j) \\ &= \sup_{i \in S^k} \left(\sum_{j: g(j)=0} g(i)^2 J^k(i, j) \mu^k(j) + \sum_{j: g(j) \neq 0} (g(i) - g(j))^2 J^k(i, j) \mu^k(j) \right) \\ &\leq \|g\|_\infty^2 \sup_{i: g(i) \neq 0} \sum_{j: g(j)=0} J^k(i, j) \mu^k(j) + C_g^2 \sup_{i: g(i) \neq 0} \sum_{\substack{j: g(j) \neq 0 \\ r(i, j) \leq k_0}} J^k(i, j) \mu^k(j) \\ &\quad + \|g\|_\infty^2 \sup_{i: g(i)=0} \sum_{j: g(j) \neq 0} J^k(i, j) \mu^k(j) \\ &\leq (2\|g\|_\infty^2 \vee C_g^2) \sup_{i: g(i) \neq 0} \sum_{\substack{j \in S^k \\ r(i, j) \leq k_0}} J^k(i, j) \mu^k(j) \\ &\quad + \|g\|_\infty^2 \sup_{i: g(i)=0} \sum_{j: g(j) \neq 0} J^k(i, j) \mu^k(j) \\ &=: \text{(I)}_k + \text{(II)}_k. \end{aligned}$$

By (A.3), (I)_k is bounded in k because

$$\begin{aligned} \sup_{i: g(i) \neq 0} \sum_{\substack{j \in S^k \\ r(i,j) \leq k_0}} J^k(i,j) \mu^k(j) &= \sup_{i: g(i) \neq 0} \sum_{\substack{j \in S^k \\ r(i,j) \leq k_0}} \frac{1}{\mu^k(i)} \int_{B_i^k \times B_j^k} J(x,y) \mu(dx) \mu(dy) \\ &< \sup_{k \geq k_0} \sup_{i \in S^k} \frac{1}{\mu^k(i)} \int_{B_i^k \times (B_i^{k_0})^c} J(x,y) \mu(dx) \mu(dy) < \infty. \end{aligned}$$

By (A.3), (II)_k is bounded in k because

$$\begin{aligned} \sup_{i: g(i) = 0} \sum_{j: g(j) \neq 0} J^k(i,j) \mu^k(j) \\ < \sup_{k \geq k_0} \sup_{i \in S^k} \sum_{j: g(j) \neq 0} \frac{1}{\mu^k(i)} \int_{B_i^k \times (B_i^{k_0})^c} J(x,y) \mu(dx) \mu(dy) < \infty. \end{aligned}$$

This completes the proof. □

Lemma 4.1 enables us to follow the same argument as in [9, Proposition 3.4] to obtain the following proposition:

Proposition 4.1. *Suppose that (A.1)–(A.4) hold. Let $\psi \in C_0^+(S)$ and $0 < T < \infty$. Then, for any finite subset $\{g_1, \dots, g_m\} \subset D_0^+$, the family of the laws of the processes $\{(g_1, \dots, g_m) \circ X^k, \mathbb{P}_{\psi^k}^k\}_{k \in \mathbb{Z}}$ is tight in $\mathbb{D}_{\mathbb{R}^m}[0, T]$.*

Now we prove Theorem 1.2.

Proof of Theorem 1.2. By Corollary 4.1 and Proposition 4.1, for any finite subset $\{g_1, \dots, g_m\} \subset D_0^+$, we infer that $((g_1, \dots, g_m) \circ X^k, \mathbb{P}_{\psi^k}^k)$ converges as $k \rightarrow \infty$ to $((g_1, \dots, g_m) \circ X, \mathbb{P}_\psi)$ in law on $\mathbb{D}_{\mathbb{R}^m}[0, 1]$. Since D_0^+ strongly separates points in S , we complete the proof by using [11, Corollary 3.9.2]. □

§5. Proof of Theorem 1.3

Let $\{P_t\}_{t \geq 0}$ (resp. $\{P_t^k\}_{t \geq 0}$) and $\{G_\alpha\}_{\alpha \geq 0}$ (resp. $\{G_\alpha^k\}_{\alpha \geq 0}$) denote the semigroups and resolvents corresponding to the Dirichlet form $(\mathcal{E}, \mathcal{F})$ (resp. the averaged Dirichlet form $(\mathcal{E}^k, \mathcal{F}^k)$). Define $\mathcal{E}_\lambda(u, v) = \mathcal{E}(u, v) + \lambda \langle u, v \rangle$ for $\lambda \geq 0, u, v \in \mathcal{F}$, and $\mathcal{E}_\lambda^k(u, v) = \mathcal{E}^k(u, v) + \lambda \langle u, v \rangle_k$ for $\lambda \geq 0, u, v \in \mathcal{F}^k$. Recall that, in Section 3, we have defined $\langle u, v \rangle = \int_S u(x)v(x) \mu(dx)$ and $\langle u, v \rangle_k = \sum_{i \in S^k} u(i)v(i) \mu^k(i)$.

We now prove Theorem 1.3.

Proof of Theorem 1.3. (1) (i)⇒(ii): Let \tilde{P}_t denote the transition semigroup of $(\pi^k \circ X, \mathbb{P}_x)$. By (pMp)_k, we have

$$(5.1) \quad P_t E^k f(x) = \mathbb{E}_x f(\pi^k \circ X_t) = \tilde{P}_t^k f(\pi^k x) = E^k \tilde{P}_t^k f(x).$$

Let $\tilde{\mathcal{E}}^k(u) = \lim_{t \downarrow 0} (1/t)(u - \tilde{P}_t^k u, u)$ for $u \in D^k$. Then, by the isometry of E^k , (5.1) and Proposition 3.1, we have

$$\begin{aligned} \tilde{\mathcal{E}}^k(u) &= \lim_{t \downarrow 0} \frac{(E^k u - E^k \tilde{P}_t^k u, E^k u)}{t} = \lim_{t \downarrow 0} \frac{(E^k u - P_t E^k u, E^k u)}{t} = \mathcal{E}(E^k u) \\ &= \mathcal{E}^k(u), \end{aligned}$$

where $\mathcal{E}(E^k u)$ and $\mathcal{E}^k(u)$ denote $\mathcal{E}(E^k u, E^k u)$ and $\mathcal{E}^k(u, u)$, respectively. This implies that the generator matrices of $\pi^k \circ X$ and X^k are the same. Since both $\pi^k \circ X$ and X^k are minimal processes (i.e., do not come back from the cemetery ∂ to S^k), by the general theory of Markov chains, we have $\tilde{P}_t^k = P_t^k$. Thus, by (5.1), we obtain $E^k P_t^k f(x) = P_t E^k f(x)$ for any $f \in L^2(S^k; \mu^k)$. Thus, we have completed the proof of (i) \Rightarrow (ii) and of the last assertion of (1).

(ii) \Rightarrow (i): Let S_∂^k and S_∂ be the one-point compactification of S^k and S , respectively. For any real-valued bounded $\mathcal{B}(S_\partial^k)/\mathcal{B}(\mathbb{R})$ -measurable function f on S_∂^k with $f(\partial) = 0$, we have

$$\mathbb{E}_x(f(\pi^k \circ X_{t+s}) | \mathcal{M}_t) = P_s E^k f(X_t) = E^k P_s^k f(X_t) = P_s^k f(\pi^k \circ X_t).$$

Taking $\tilde{p}_t^k(x, y) = P_t^k \mathbf{1}_y(x)$, we have shown (pMp) $_k$, proving (ii) \Rightarrow (i).

(2) We will show that (ii) holds under the assumptions of (2). By the standard argument of Laplace transform, it suffices to show that $E^k G_\lambda^k = G_\lambda E^k$ on $L^2(S^k; \mu^k)$ for any $\lambda \geq 0$.

Note that, for $g \in L^2(S; \mu)$, the image $G_\lambda g$ is a unique function $h \in \mathcal{F}$ such that $\mathcal{E}_\lambda(h, v) = \langle g, v \rangle$ for all $v \in \mathcal{F}$. Since D_0 is a core of \mathcal{F} , to prove (i) it suffices to show that, for $f \in L^2(S^k; \mu^k)$, $\mathcal{E}_\lambda(E^k G_\lambda^k f, v) = \langle E^k f, v \rangle$ for all $v \in D_0$. To do this, we first prove

$$(5.2) \quad \mathcal{E}_\lambda(E^k G_\lambda^k f, v) = \mathcal{E}_\lambda(E^k G_\lambda^k f, E^k \Pi^k v).$$

Since Π^k is the adjoint of E^k , and E^k is isometric, we have

$$(5.3) \quad \langle E^k G_\lambda^k f, v \rangle = \langle G_\lambda^k f, \Pi^k v \rangle_k = \langle E^k G_\lambda^k f, E^k \Pi^k v \rangle.$$

By (BC) $_k$,

$$\begin{aligned} \mathcal{E}(E^k G_\lambda^k f, v) &= \int_{S \times S \setminus d} (E^k G_\lambda^k f(x) - E^k G_\lambda^k f(y))(v(x) - v(y))J(x, y) \mu(dx) \mu(dy) \\ &= \sum_{\substack{i, j \in S^k \\ i \neq j}} (E^k G_\lambda^k f(i) - E^k G_\lambda^k f(j))J(i, j) \int_{B_i^k \times B_j^k} (v(x) - v(y)) \mu(dx) \mu(dy) \\ &= \sum_{i \neq j} (E^k G_\lambda^k f(i) - E^k G_\lambda^k f(j))J(i, j)(\Pi^k v(i) - \Pi^k v(j))\mu^k(i)\mu^k(j) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i \neq j} (E^k G_\lambda^k f(i) - E^k G_\lambda^k f(j)) J(i, j) \int_{B_i^k \times B_j^k} (E^k \Pi^k v(x) - E^k \Pi^k v(y)) \mu(dx) \mu(dy) \\
 &= \int_{S \times S \setminus d} (E^k G_\lambda^k f(x) - E^k G_\lambda^k f(y)) (E^k \Pi^k v(x) - E^k \Pi^k v(y)) J(x, y) \mu(dx) \mu(dy) \\
 &= \mathcal{E}(E^k G_\lambda^k f, E^k \Pi^k v).
 \end{aligned}$$

Thus, we obtain (5.2). Hence,

$$\mathcal{E}_\lambda(E^k G_\lambda^k f, v) = \mathcal{E}_\lambda^k(G_\lambda^k f, \Pi^k v) = \langle f, \Pi^k v \rangle_k = \langle E^k f, v \rangle.$$

Here we have used Proposition 3.1(iii). This completes the proof of (2). □

§6. Examples

§6.1. Examples of ultrametric spaces

We give several examples of ultrametric spaces included in our setting.

Example 6.1 (*p*-adic ring). Fix an integer $p \geq 2$. Note that p need not be prime. Define the *p*-adic ring \mathbb{Q}_p by

$$\mathbb{Q}_p = \{(x_i)_{i \in \mathbb{Z}} \in \{0, 1, \dots, p - 1\}^{\mathbb{Z}} : x_i = 0 \text{ for all } i \leq M \text{ for some } M\}.$$

From the algebraic viewpoint, \mathbb{Q}_p has a natural ring structure obtained by identifying $(x_i) \in \mathbb{Q}_p$ with the formal power series $\sum_{i=-\infty}^{\infty} x_i p^i$. However, we do not use any algebraic structures of \mathbb{Q}_p in this paper. Let us equip \mathbb{Q}_p with the ultrametric ρ_p defined by

$$\rho_p(x, y) = p^{-r(x, y)},$$

where $r(x, y) := \min\{i \in \mathbb{Z} : x_i \neq y_i\}$, $\min \emptyset := \infty$ and $p^{-\infty} := 0$. Let μ_p be the Haar measure on \mathbb{Q}_p normalized as $\mu(B_0^0) = 1$. The set $(\mathbb{Q}_p, \rho_p, \mu_p)$ satisfies (U.1)–(U.4).

Example 6.2 (leaves of a multibranching tree). We introduce *leaves of a multibranching tree*, which is a generalization of *p*-adic rings. Define

$$\mathbb{S}^\infty = \{x = (x_i)_{i \in \mathbb{Z}} \in \mathbb{N}_0^{\mathbb{Z}} : x_i = 0 \text{ for all } i \leq M \text{ for some } M\}.$$

For a fixed $k \in \mathbb{Z}$,

$$\mathbb{S}^k = \{x = (x_i)_{i \leq k} : x_i = 0 \text{ for all } i \leq M \text{ for some } M\}.$$

Define a map $\{\cdot\}_k : \mathbb{S}^\infty \rightarrow \mathbb{S}^k$ by

$$\{x\}_k := (\dots, x_{k-1}, x_k) \quad \text{for } x = (x_i)_{i \in \mathbb{Z}} \in \mathbb{S}.$$

Let $\mathbb{S} := \prod_{k \in \mathbb{Z}} \mathbb{S}^k$. Let V be an arbitrary function from \mathbb{S} to \mathbb{N} . We define

$$\mathbb{S}_V = \{(x_i)_{i \in \mathbb{Z}} \in \mathbb{S}^\infty : 0 \leq x_i \leq V(\{x\}_{i-1}) \text{ for all } i\}.$$

The set \mathbb{S}_V is called the *leaves of a multibranching tree*. Let $q > 1$ be given. We metrize \mathbb{S}_V as follows: for all $x, y \in \mathbb{S}_V$, we define

$$\rho_q(x, y) = q^{-r(x, y)}$$

where $r(x, y) := \min\{i \in \mathbb{Z} : x_i \neq y_i\}$. If no confusion may occur, we drop the subscript q on ρ . We define a Radon measure μ_V on \mathbb{S}_V such that

$$(6.1) \quad \mu_V(B(0, 1)) = 1, \quad \mu_V(B_x^k) = V(\{x\}_k) \mu_V(B_x^{k+1}) \quad (\forall x \in \mathbb{S}_V, \forall k \in \mathbb{Z}).$$

In Proposition 6.1 below, we show that $(\mathbb{S}_V, \rho_q, \mu_V)$ satisfies (U.1)–(U.4). This is a generalization of $(\mathbb{Q}_p, \rho_p, \mu_p)$. Indeed, we can obtain $(\mathbb{Q}_p, \rho_p, \mu_p)$ from $(\mathbb{S}_V, \rho_q, \mu_V)$ by setting $q = p$ and $V \equiv p - 1$.

Now we show that $(\mathbb{S}_V, \rho_q, \mu_V)$ satisfies (U.1)–(U.4).

Proposition 6.1. $(\mathbb{S}_V, \rho_q, \mu_V)$ satisfies (U.1)–(U.4).

Proof. Conditions (U.3) and (U.4) clearly hold. To show (U.2), we note that, for all $x \in \mathbb{S}_V$ and $k \in \mathbb{Z}$, there exists a finite sequence $\{a_i\}_{0 \leq i \leq p-1} \subset B_x^k$ such that $\{B_{a_i}^{k+1}\}_{0 \leq i \leq p-1}$ is disjoint and

$$(6.2) \quad B_x^k = \sum_{i: \text{finite}} B_{a_i}^{k+1}$$

(see Alberverio–Karwowski [2, Section 2]). We also note that (\mathbb{S}_V, ρ_q) is complete (see [2, Proposition 2.8]). By completeness, it suffices to see that any closed ball is totally bounded. For any $\epsilon > 0$, let $l \in \mathbb{Z}$ be such that $0 < q^{-l} < \epsilon$. By using equality (6.2) inductively, we find that there exists a finite sequence $\{b_i\}_i \subset B_x^k$ such that $\{B_{b_i}^l\}_i$ is disjoint and $B_x^k = \sum_{i: \text{finite}} B_{b_i}^l$. Since $B_{b_i}^l$ is open by Fact 2.1(iii), we have checked (U.2).

Now we show (U.1). We have already seen above that (\mathbb{S}_V, ρ_q) is a complete ultrametric space. Local compactness is clear by (U.2). Thus, (U.1) is satisfied. \square

§6.2. Mixed class

We introduce a new class of Hunt processes on \mathbb{S}_V . This class is a kind of generalization of the class constructed by Kigami [19]. We recall the class of Hunt processes on \mathbb{S}_V constructed in [19]. We only consider the conservative case. Let $(S, \rho) = (\mathbb{S}_V, \rho)$ be as in Example 6.2. Let $\mathbf{B} = \{\{x\}_k \in \mathbb{S}^k : x \in \mathbb{S}_V, k \in \mathbb{Z}\}$.

Let $\lambda : \mathbb{B} \rightarrow [0, \infty]$ be a function such that

$$(6.3) \quad \sum_{i=-\infty}^0 |\lambda(\{0\}_{m+1}) - \lambda(\{0\}_m)| < \infty,$$

and, for all $x, y \in \mathbb{S}_V$ with $x \neq y$, define

$$(6.4) \quad J(x, y) = J_{\lambda, \mu}(x, y) := \sum_{m=-\infty}^{r(x,y)} \frac{\lambda(\{x\}_m) - \lambda(\{x\}_{m-1})}{\mu(B_x^m)} \geq 0.$$

For a fixed $x \in \mathbb{S}_V$, we see that $J(x, \cdot)$ depends only on $r(x, \cdot)$. We sometimes write $J_x^k = J(x, y)$ for y such that $r(x, y) = k$.

Let us define $(\mathcal{E}, \mathcal{F})$ as the following symmetric bilinear form:

$$\begin{aligned} \mathcal{E}(u, v) &= \frac{1}{2} \int_{\mathbb{S}_V \times \mathbb{S}_V \setminus d} (u(x) - u(y))(v(x) - v(y)) J(x, y) \mu(dx) \mu(dy), \\ \mathcal{F} &= \{f \in L^2(\mathbb{S}_V; \mu) : \mathcal{E}(f, f) < \infty\}. \end{aligned}$$

By [19, Section 3], $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form with core D_0 . Hence the domain \mathcal{F} is equal to the closure of D_0 with respect to the \mathcal{E}_1 -norm. The above Dirichlet form is determined by the following pair (λ, μ) :

$$\Theta_{K,c} = \{(\lambda, \mu) \in l^+(\mathbb{B}) \times \mathcal{M}(\mathbb{S}_V) : (6.3) \text{ and } (6.4) \text{ hold}\},$$

where $\mathcal{M}(\mathbb{S}_V)$ stands for the set of Radon measures on (\mathbb{S}_V, ρ) and $l^+(\mathbb{B})$ stands for the set of functions $\lambda : \mathbb{B} \rightarrow [0, \infty]$. Note that Dirichlet forms corresponding to elements of $\Theta_{K,c}$ are conservative and the subscript c indicates this.

Now we introduce the new class of Hunt processes. Take pairs

$$(6.5) \quad (\lambda_1, \mu), \dots, (\lambda_l, \mu) \in \Theta_{K,c}.$$

Define the jump densities as follows (see (6.4)):

$$J_1 = J_{\lambda_1, \mu}, \quad J_2 = J_{\lambda_2, \mu}, \quad \dots, \quad J_l = J_{\lambda_l, \mu}.$$

Let $N_l(\mathbb{S}_V)$ denote the set of functions $f : \mathbb{S}_V \times \mathbb{S}_V \setminus d \rightarrow \{1, \dots, l\}$. Let $\Gamma_l \in N_l(\mathbb{S}_V)$ be such that, for each $k \in \mathbb{Z}$ and $i, j \in \mathbb{S}_V^k$ with $i \neq j$,

$$(6.6) \quad \Gamma_l|_{B_i^k \times B_j^k} = N_{ij}^k,$$

where $N_{ij}^k \in \{1, \dots, l\}$ denotes a constant which depends only on k, i and j . Define

the mixed jump density function as follows:

$$J_{\Gamma_l}(x, y) = J_{\Gamma_l(x,y)}(x, y).$$

Let $(\mathcal{E}_{\Gamma_l}, D_0)$ denote the following symmetric bilinear form:

$$\mathcal{E}_{\Gamma_l}(u, v) = \frac{1}{2} \int_{\mathbb{S}_V \times \mathbb{S}_V \setminus d} (u(x) - u(y))(v(x) - v(y)) J_{\Gamma_l}(x, y) \mu(dx) \mu(dy) \quad (u, v \in D_0).$$

Proposition 6.2. $(\mathcal{E}_{\Gamma_l}, D_0)$ satisfies (A.1), (A.2) and $(BC)_\infty$.

Proof. Conditions (A.2) and $(BC)_\infty$ are obvious. We check (A.1). By definition,

$$J_{\Gamma_l}(x, y) \leq \sum_{i=1}^l J_i(x, y) \quad \text{for all } (x, y) \in \mathbb{S}_V \times \mathbb{S}_V \setminus d.$$

Note that the jump densities corresponding to $\Theta_{K,c}$ satisfy (A.1) (see [19, Theorem 3.7]). Thus, for all $k \in \mathbb{Z}$ and $i \in \mathbb{S}_V^k$, we have

$$\int_{B_i^k \times (B_i^k)^c} J_{\Gamma_l}(x, y) \mu(dx) \mu(dy) \leq \sum_{i=1}^l \int_{B_i^k \times (B_i^k)^c} J_i(x, y) \mu(dx) \mu(dy) < \infty,$$

proving (A.1). □

It is easy to see that $(\mathcal{E}_{\Gamma_l}, \overline{D_0}^{\mathcal{E}_{\Gamma_l}})$ is a regular Dirichlet form. Note that it is determined by $\mu, \lambda_1, \dots, \lambda_l$ and Γ_l . Now we define a new class, a generalization of $\Theta_{K,c}$:

Definition 6.1. The following class is called the *mixed class*:

$$\Theta_{\text{Mix}} = \{(\mu, \lambda_1, \dots, \lambda_l, \Gamma_l) \in \mathcal{M}(\mathbb{S}_V) \times l^+(\mathbf{B}) \times \dots \times l^+(\mathbf{B}) \times N_l(\mathbb{S}_V) : \text{(6.5) and (6.6) hold}\}.$$

By Proposition 6.2 and Theorem 1.3, the class of Hunt processes associated with the mixed class has the projection Markov property at any level. By Corollary 1.1, Hunt processes associated with the mixed class can be approximated almost surely by the projected processes, which are equal in law to Markov chains associated with the averaged forms.

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