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Bar Construction and Tannakization

by

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Abstract

We continue our development of tannakizations of symmetric monoidal ∞-categories, begun in [19]. In this note we calculate the tannakizations of some examples of symmetric monoidal stable ∞-categories with fiber functors. We consider the case of symmetric monoidal ∞-categories of perfect complexes on perfect derived stacks. The first main result in particular says that our tannakization includes the bar construction for an augmented commutative ring spectrum and its equivariant version as a special case. We apply it to the study of the tannakization of the stable ∞-category of mixed Tate motives over a perfect field. We prove that its tannakization can be obtained from the $\mathbb{G}_{\mathrm{m}}$-equivariant bar construction of a commutative differential graded algebra equipped with the $\mathbb{G}_{\mathrm{m}}$-action. Moreover, under the Beilinson–Soulé vanishing conjecture, we prove that the underlying group scheme of the tannakization is the motivic Galois group for mixed Tate motives, constructed in [4], [25], [26]. The case of Artin motives is also included.

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§1. Introduction

In [19] we have constructed tannakizations of symmetric monoidal ∞-categories. Let R be a commutative ring spectrum. Let $\mathcal{C}^{\otimes}$ be a small symmetric monoidal ∞-category, equipped with a symmetric monoidal functor $F : \mathcal{C}^{\otimes} \to \mathrm{PMod}_R^{\otimes}$ where $\mathrm{PMod}_R^{\otimes}$ denotes the symmetric monoidal ∞-category of compact R-spectra. (Although we use the machinery of quasi-categories in the text, here by an ∞-category we informally mean an $(\infty, 1)$-category in the sense of [3].) In [19], given $F : \mathcal{C}^{\otimes} \to \mathrm{PMod}_R^{\otimes}$ we construct a derived affine group scheme G over R which represents the automorphism group of F and has a certain universality (see Theorem 3.1). A derived affine group scheme is an analogue of an affine group scheme in derived algebraic geometry [37], [29]. For simplicity, we shall call it the tannakization of $F : \mathcal{C}^{\otimes} \to \mathrm{PMod}_R^{\otimes}$. This construction was applied to the stable ∞-category

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of mixed motives to construct derived motivic Galois groups and underived motivic Galois groups. For the reader who is not familiar with higher category theory it is worth emphasizing that the ∞-categorical framework is crucial for the nice representability of automorphism groups, whereas coarse machinery, such as triangulated categories, does not enable getting it.

The purpose of this note is to calculate tannakizations of some examples of $F : \mathcal{C}^{\otimes} \to \mathrm{PMod}_R^{\otimes}$; our principal interest here is the case when $\mathcal{C}^{\otimes}$ is the symmetric monoidal ∞-category $\mathrm{PMod}_Y^{\otimes}$ of perfect complexes on a derived stack Y and F is induced by $\mathrm{Spec}\, R \to Y$. We will study tannakization under the assumption of perfectness on derived stacks, introduced in [2], which in particular includes two cases:

(i) Y is an affine derived scheme over R, that is, $Y = \mathrm{Spec}\, A$ over $\mathrm{Spec}\, R$ with A a commutative ring spectrum,

(ii) Y is the quotient stack $[X/G]$ where X is an affine derived scheme $X = \mathrm{Spec}\, A$ and G is an algebraic group in characteristic zero.

We note that for our purpose the assumption of affineness of Y in (i) and X in (ii) is not essential since $\mathrm{PMod}_Y^{\otimes} \to \mathrm{PMod}_R^{\otimes}$ only depends on a Zariski neighborhood of the image of $\mathrm{Spec}\, R \to Y$. Also, we remark that A in (i) and (ii) can be nonconnective. Our result may be expressed as follows (see Theorem 4.8, Corollary 4.9):

Theorem 1. *Let Y be a derived stack over R and* $\mathrm{Spec}\, R \to Y$ *a section of the structure map $Y \to$* $\mathrm{Spec}\, R$*. Let* $\mathrm{PMod}_Y^{\otimes} \to \mathrm{PMod}_R^{\otimes}$ *be the associated pullback symmetric monoidal functor. Suppose that Y is perfect* (*cases* (i) *and* (ii) *have this property*)*. Let G be the derived affine group scheme arising from the Čech nerve associated to* $\mathrm{Spec}\, R \to Y$*. Then the tannakization of* $\mathrm{PMod}_Y^{\otimes} \to \mathrm{PMod}_R^{\otimes}$ *is equivalent to G.*

Bar construction and equivariant bar construction. One of our motivations arises from comparison between derived group schemes obtained by tannakization and by bar construction and its variants. Bar construction has been an important device in various contexts of homotopy theory, mixed Tate motives and nonabelian Hodge theory, etc. In case (i), the Čech nerve in Aff_R associated to $\mathrm{Spec}\, R \to Y = \mathrm{Spec}\, A$, which we can regard as a derived affine group scheme over R, is known as the bar construction of an augmented commutative ring spectrum (or commutative differential graded algebra), whose explicit construction can be given by bar resolutions. In case (ii), we can think of the Čech nerve as the G-equivariant version of the bar construction. As a matter of fact, our actual aim is to study the relationship between our tannakization and bar constructions and their equivariant versions; Theorem 1 in particular means that our method of tannakization includes bar constructions and their equivariant versions as a special case. This allows one to link bar constructions and their variants to a more general method of tannakization. It will be of use in subsequent work as a key ingredient (see the end of this introduction).

Toward applications to Galois groups of mixed motives. It is worth mentioning that the equivariant versions are also important in applications to motivic contexts: for instance, in order to take weight structures into account, one often uses the $\mathbb{G}_{\mathrm{m}}$-equivariant version of bar construction. Our results fit very naturally in with the structure of mixed Tate motives and (hopefully) arbitrary mixed motives (in the general case, one should replace $\mathbb{G}_{\mathrm{m}}$ by the Tannaka dual of the (conjecturally tannakian) category of numerical motives).

To describe our aim, consider the symmetric monoidal stable ∞-category $\mathsf{DM}_\vee^\otimes := \mathsf{DM}_\vee^\otimes(k)$ of (dualizable) mixed motives over a base scheme Spec k, where k is a perfect field (see Section 6.1 for our conventions). For a mixed Weil cohomology theory (such as singular, étale, de Rham cohomology) with coefficient field $\mathbf{K}$ of characteristic zero, there exists a homological realization functor $\mathsf{R} : \mathsf{DM}_\vee^\otimes \to \mathrm{PMod}_{H\mathbf{K}}^\otimes$, which is a symmetric monoidal exact functor. In our previous paper [19], we introduced a new approach of tannakization of ∞-categories to motivic Galois groups: we focus on the automorphism group of the realization functor R and define the derived motivic Galois group to be the tannakization of $\mathsf{DM}_\vee^\otimes$ endowed with the realization functor R (and we also construct the underived motivic Galois group as its coarse moduli space) in an unconditional way.

As a first step to our detailed study, we apply Theorem 1 to compare the conventional motivic Galois group of mixed Tate motives with ours. Let $\mathsf{DTM}_\vee^\otimes \subset \mathsf{DM}_\vee^\otimes$ be the small symmetric monoidal stable ∞-category of mixed Tate motives (see Section 6.2). By applying the above theorem, we deduce Theorem 6.12 which informally says:

Theorem 2. *Let* $\mathsf{MTG} = \operatorname{Spec} B$ *be the derived affine group scheme obtained as the tannakization of* $\mathsf{R}_T : \mathsf{DTM}_\vee^\otimes \to \mathrm{PMod}_{H\mathbf{K}}^\otimes$. (*Here* B *is a commutative differential graded* $\mathbf{K}$*-algebra.*) *Then* MTG *is obtained from the* $\mathbb{G}_{\mathrm{m}}$*-equivariant bar construction of a commutative differential graded* $\mathbf{K}$*-algebra* $\overline{Q}$ *equipped with the* $\mathbb{G}_{\mathrm{m}}$*-action. Namely, it is the Čech nerve of a morphism of derived stacks* $\operatorname{Spec} H\mathbf{K} \to [\operatorname{Spec} \overline{Q}/\mathbb{G}_{\mathrm{m}}]$.

We remark that the underlying complex $\overline{Q}$ can be described in terms of Bloch's cycle complexes. The proof of Theorem 2 requires two ingredients: one is Theorem 1, and the other is to identify $\mathsf{R}_T : \mathsf{DTM}_\vee^\otimes \to \mathrm{PMod}_{H\mathbf{K}}^\otimes$ with a certain pullback functor between ∞-categories of perfect complexes on derived stacks, which makes use of the module-theoretic (i.e. Morita-theoretic) presentation theorem for the stable ∞-category $\mathsf{DTM}_\vee^\otimes$ (see [34]).

If the Beilinson–Soulé vanishing conjecture holds for the base field k (e.g. if k is a number field), there is a traditional method of passing to a group scheme. Under the vanishing conjecture, one can define the motivic t-structure on $\mathsf{DTM}_\vee$. The heart of this t-structure is a neutral Tannakian category (cf. [33], [11]), and we can extract an affine group scheme MTG over $\mathbf{K}$ from it. The so-called motivic Galois group for mixed Tate motives MTG was constructed by Bloch–Kriz, Kriz–May, and Levine [4], [25], [26]. The vanishing conjecture does not imply that the stable ∞-category of complexes of the heart recovers the original ∞-category $\mathsf{DTM}_\vee$. However, the following result gives a quite nice relation between MTG and MTG and also provides a conceptual understanding of MTG as a coarse moduli space of the automorphism group of R (it is important to notice that this characterization does not refer to a motivic t-structure).

Theorem 3. *Suppose that the Beilinson–Soulé vanishing conjecture holds for k. Then the group scheme MTG is an excellent coarse moduli space* (*cf. Definition* 7.15) *of* MTG.

This result is proved in Section 7 (Theorem 7.16). In view of Theorems 2 and 3, we can say that the derived motivic Galois group constructed from $\mathsf{DM}_\vee^\otimes$ in [19] is a natural generalization of MTG to all the mixed motives. Also, it gives a clear relation between the conventional theory (in the case of mixed Tate motives) and our approach.

Application to future work. The result (Theorem 1) has already found nice applications in the study of the motivic Galois group of mixed motives generated by an abelian variety (not to be confused with 1-motives)—see [20]. In [20], it connects certain based loop stacks with the representability of automorphisms, which allows one to use various techniques such as Galois representations, rational homotopy theory, etc. Combining this with the general method of perfect adjoint pairs discussed in [20, Section 3] one may expect to acchieve more in this and other directions.

This article is organized as follows: In Section 2, we will review some notions and notation which we need in this note. In Section 3, we recall the definitions of representation of derived affine group schemes, automorphism group of symmetric functors, etc. Section 4 contains the proof of Theorem 1. In Section 5, we give a brief exposition of bar constructions from our viewpoint. In Sections 6 and 7, we give applications to examples. Sections 6 and 7 are devoted to the study of the tannakization of the stable ∞-category of mixed Tate motives; we prove Theorems 2 and 3. In Section 8, for the sake of completeness we will also treat the stable subcategory of Artin motives in DM, which is generated by motives of smooth 0-dimensional varieties. We show that the tannakization of the stable ∞-category of Artin motives is the absolute Galois group $\mathrm{Gal}(\bar{k}/k)$ (see Proposition 8.3).

§2. Notation and conventions

We fix our notation and conventions.

∞-categories. Throughout this note we use the theory of *quasi-categories*. A quasi-category is a simplicial set which satisfies the weak Kan condition of Boardman–Vogt. The theory of quasi-categories from the viewpoint of higher category theory was extensively developed by Joyal and Lurie [21], [27], [28]. Following [27] we shall refer to quasi-categories as *∞-categories*. Our main references are [27] and [28]. We often refer to a map $S \to T$ of ∞-categories as a *functor*. We call a vertex (resp. an edge) in an ∞-category S an object (resp. a morphism). For a rapid introduction to ∞-categories, we refer to [27, Chapter 1], [14], [13, Section 2]. For a quick survey of various approaches to $(\infty,1)$-categories and their relations, we refer to [3].

- Δ: the category of linearly ordered finite sets (consisting of $[0],[1],\dots,[n] = \{0,\dots,n\},\dots$).
- Δ^n: the standard n-simplex.
- N: the simplicial nerve functor (cf. [27, 1.1.5]).
- $\mathcal{C}^{\mathrm{op}}$: the opposite ∞-category of an ∞-category $\mathcal{C}$.
- Let $\mathcal{C}$ be an ∞-category and suppose that we are given an object c. Then $\mathcal{C}_{c/}$ and $\mathcal{C}_{/c}$ denote the undercategory and overcategory respectively (cf. [27, 1.2.9]).
- Cat_∞: the ∞-category of small ∞-categories in a fixed Grothendieck universe $\mathbb{U}$; we employ the axioms of ZFC together with the axiom of Grothendieck universes. We fix a sequence of universes $(\mathbb{N} \in)\mathbb{U} \in \mathbb{V} \in \mathbb{W}$ and refer to sets belonging to $\mathbb{U}$ (resp. $\mathbb{V}$, $\mathbb{W}$) *as small sets* (resp. *large sets*, *super-large sets*).
- $\widehat{\mathrm{Cat}}_\infty$: the ∞-category of large ∞-categories.
- $\mathcal{S}$: the ∞-category of small spaces. We denote by $\widehat{\mathcal{S}}$ the ∞-category of large ∞-categories (cf. [27, 1.2.16]).
- $\mathrm{h}(\mathcal{C})$: the homotopy category of an ∞-category (cf. [27, 1.2.3.1]).
- $\mathrm{Fun}(A,B)$: the function complex for simplicial sets A and B.
- $\mathrm{Fun}_C(A,B)$: the simplicial subset of $\mathrm{Fun}(A,B)$ classifying maps which are compatible with given projections $A \to C$ and $B \to C$.

- $\mathrm{Map}(A, B)$: the largest Kan complex of $\mathrm{Fun}(A, B)$ when A and B are ∞-categories.
- $\mathrm{Map}_{\mathcal{C}}(C, C')$: the mapping space from an object $C \in \mathcal{C}$ to $C' \in \mathcal{C}$ where $\mathcal{C}$ is an ∞-category. We usually view it as an object in $\mathcal{S}$ (cf. [27, 1.2.2]).

Stable ∞-categories, symmetric monoidal ∞-categories and spectra. For the definitions of (symmetric) monoidal ∞-categories and ∞-operads, and their algebra objects, we refer to [28]. The theory of stable ∞-categories is developed in [28, Chapter 1]. We list some notation:

- $\mathbb{S}$: the sphere spectrum.
- Sp: the ∞-category of spectra; we denote the smash product by $\otimes$.
- PSp: the full subcategory of Sp spanned by compact spectra.
- Mod_A: the ∞-category of A-module spectra for a commutative ring spectrum A.
- PMod_A: the full subcategory of Mod_A spanned by compact objects (in Mod_A, an object is compact if and only if it is dualizable, see [2]). We refer to objects in PMod_A as *perfect A-module* (*spectra*).
- Fin_*: the category of pointed finite sets $\langle 0\rangle_* = \{*\}, \langle 1\rangle_* = \{1, *\}, \ldots, \langle n\rangle_* = \{1 \ldots, n, *\}, \ldots$. A morphism is a map $f : \langle n\rangle_* \to \langle m\rangle_*$ such that $f(*) = *$. Note that f is not assumed to be order-preserving.
- Let $\mathcal{M}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads. We denote by $\mathrm{Alg}_{/\mathcal{O}^{\otimes}}(\mathcal{M}^{\otimes})$ the ∞-category of algebra objects (cf. [28, 2.1.3.1]). We often write $\mathrm{Alg}(\mathcal{M}^{\otimes})$ or $\mathrm{Alg}(\mathcal{M})$ for $\mathrm{Alg}_{/\mathcal{O}^{\otimes}}(\mathcal{M}^{\otimes})$. Suppose that $\mathcal{P}^{\otimes} \to \mathcal{O}^{\otimes}$ is a map of ∞-operads. Then $\mathrm{Alg}_{\mathcal{P}^{\otimes}/\mathcal{O}^{\otimes}}(\mathcal{M}^{\otimes})$ is the ∞-category of $\mathcal{P}$-algebra objects.
- $\mathrm{CAlg}(\mathcal{M}^{\otimes})$: the ∞-category of commutative algebra objects in a symmetric monoidal ∞-category $\mathcal{M}^{\otimes} \to \mathrm{N}(\mathrm{Fin}_*)$.
- CAlg_R: the ∞-category of commutative algebra objects in the symmetric monoidal ∞-category $\mathrm{Mod}_R^{\otimes}$ where R is a commutative ring spectrum. When $R = \mathbb{S}$, we set $\mathrm{CAlg} = \mathrm{CAlg}_{\mathbb{S}}$.
- $\mathrm{Mod}_A^{\otimes}(\mathcal{M}^{\otimes}) \to \mathrm{N}(\mathrm{Fin}_*)$: the symmetric monoidal ∞-category of A-module objects, where $\mathcal{M}^{\otimes}$ is a symmetric monoidal ∞-category such that (1) the underlying ∞-category admits a colimit for any simplicial diagram, and (2) its tensor product functor $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$ preserves colimits of simplicial diagrams separately in each variable. Here A belongs to $\mathrm{CAlg}(\mathcal{M}^{\otimes})$ (cf. [28, 3.3.3, 4.4.2]).

Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category. We usually denote by $\mathcal{C}$ its underlying ∞-category. We say that an object X in $\mathcal{C}$ is dualizable if there exist an object $X^{\vee}$ and two morphisms $e : X \otimes X^{\vee} \to \mathbf{1}$ and $c : \mathbf{1} \to X \otimes X^{\vee}$ with $\mathbf{1}$ a unit such that the composition

$$X \xrightarrow{\mathrm{Id}_X \otimes c} X \otimes X^\vee \otimes X \xrightarrow{e \otimes \mathrm{Id}_X} X$$

is equivalent to the identity, and

$$X^\vee \xrightarrow{c \otimes \mathrm{Id}_{X^\vee}} X^\vee \otimes X \otimes X^\vee \xrightarrow{\mathrm{Id}_{X^\vee} \otimes e} X^\vee$$

is also equivalent to the identity. The symmetric monoidal structure of $\mathcal{C}$ induces that of the homotopy category $\mathrm{h}(\mathcal{C})$. If we consider X to be an object also in $\mathrm{h}(\mathcal{C})$, then X is dualizable in $\mathcal{C}$ if and only if X is dualizable in $\mathrm{h}(\mathcal{C})$. For example, for $R \in \mathrm{CAlg}$, compact and dualizable objects coincide in the symmetric monoidal ∞-category $\mathrm{Mod}_R^\otimes$ (cf. [2]).

Let us recall the symmetric monoidal ∞-categories $\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}}$ and $\mathrm{Cat}_\infty^{\mathrm{st}}$ (see [2, Section 4], [28, 6.3]). Let $\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}}$ be the subcategory of $\widehat{\mathrm{Cat}}_\infty$ spanned by stable presentable ∞-categories, in which morphisms are functors which preserve small colimits. For $\mathcal{C}, \mathcal{D} \in \widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}}$, $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$ is defined to be the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by functors which preserve small colimits. Then $\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}}$ has a symmetric monoidal structure $\otimes : \widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}} \times \widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}} \to \widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}}$ such that for $\mathcal{C}, \mathcal{D} \in \widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}}$, $\mathcal{C} \otimes \mathcal{D}$ has the following universality: there exists a functor $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ which induces an equivalence $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \simeq \mathrm{Fun}'(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ for every $\mathcal{E} \in \widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}}$, where the right-hand side indicates the full subcategory of $\mathrm{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ spanned by functors which preserve small colimits separately in each variable. A unit is equivalent to Sp. Let $\mathrm{Cat}_\infty^{\mathrm{st}}$ denote the subcategory of Cat_∞ which consists of small stable idempotent complete ∞-categories. Morphisms in $\mathrm{Cat}_\infty^{\mathrm{st}}$ are functors that preserve finite colimits, that is, exact functors. There is a symmetric monoidal structure on $\mathrm{Cat}_\infty^{\mathrm{st}}$. For $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}_\infty^{\mathrm{st}}$ the tensor product $\mathcal{C} \otimes \mathcal{D}$ has the following universality: there is a functor $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ which preserves finite colimits separately in each variable, such that if $\mathcal{E} \in \mathrm{Cat}_\infty^{\mathrm{st}}$ and $\mathrm{Fun}_{\mathrm{fc}}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ spanned by functors which preserve finite colimits separately in each variable, then the composition induces a categorical equivalence

$$\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \to \mathrm{Fun}_{\mathrm{fc}}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$$

where $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E})$ is the full subcategory of $\mathrm{Fun}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E})$ spanned by exact functors. A unit is equivalent to PSp. An object (resp. a morphism) in $\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})$ can be regarded as a symmetric monoidal stable presentable ∞-category whose tensor operation preserves small colimits separately in each variable (resp. a symmetric monoidal functor which preserves small colimits). Similarly, an object (resp. a morphism) in $\mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{st}})$ can be regarded as a symmetric monoidal small stable idempotent complete ∞-category whose tensor operation preserves finite colimits separately in each variable (resp. a symmetric monoidal functor which preserves finite colimits). If R is a commutative ring spectrum, we refer to an object in $\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})_{\mathrm{Mod}_R^\otimes/}$ (resp. $\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{st}})_{\mathrm{PMod}_R^\otimes/}$) simply as an *$R$-linear symmetric monoidal stable presentable ∞-category* (resp. an *R-linear symmetric monoidal small stable idempotent complete ∞-category*). We refer to morphisms in $\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})_{\mathrm{Mod}_R^\otimes/}$ (or $\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{st}})_{\mathrm{PMod}_R^\otimes/}$) as *$R$-linear symmetric monoidal functors*.

§3. Derived group schemes and ∞-categories of representations

In this section we recall the definitions of ∞-categories of representations of derived affine group schemes and the tannakization of symmetric monoidal ∞-categories.

§3.1. Derived affine group schemes G and the ∞-categories Rep_G and PRep_G

We recall some basic definitions of derived group schemes. We refer to [19, Appendix] for the generalities concerning derived group schemes. Roughly speaking, the notion of derived group scheme is the direct generalization of group scheme to derived algebraic geometry [29], [37]. Informally, a derived scheme is a geometric object, realized as a locally ringed ∞-topos, which locally looks like an affine object $\mathrm{Spec}\,A$ where A is a commutative ring spectrum instead of the Zariski spectrum of a usual commutative ring (depending on the situation, one may choose connective ring spectra, simplicial rings and other variants as ring objects). In this note, we only treat derived *affine* schemes and their quotients by algebraic groups, and thus we only recall the definition of derived affine schemes. Let R be a commutative ring spectrum. The ∞-category Aff_R of *derived affine schemes* over R is the opposite category of CAlg_R. We shall denote by $\mathrm{Spec}\,A$ an object corresponding to $A \in \mathrm{CAlg}_R$. From Grothendieck's viewpoint of "functor of points", through the Yoneda embedding $\mathrm{Aff}_R \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$, we may also consider $\mathrm{Spec}\,A$ to be a functor $\mathrm{CAlg}_R \to \widehat{\mathcal{S}}$ corepresented by A. We equip Aff_R with the étale topology (see [19, Appendix]). Let $\mathrm{Sh}(\mathrm{Aff}_R)$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$ spanned by sheaves with respect to the étale topology. It is an ∞-topos. We have inclusions $\mathrm{Aff}_R \subset \mathrm{Sh}(\mathrm{Aff}_R) \subset \mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$ of full subcategories. Derived stacks are objects in $\mathrm{Sh}(\mathrm{Aff}_R)$ which satisfy a certain geometric condition (see Section 4).

Remembering that an affine group scheme is a group object in the category of affine schemes, we will define derived affine group schemes in a similar way. To begin, let us recall the notion of group object in an arbitrary ∞-category $\mathcal{T}$ having finite products and a final object. A *monoid object* in $\mathcal{T}$ is a simplicial diagram

$M : \mathrm{N}(\Delta)^{\mathrm{op}} \to \mathcal{T}$ such that for each $n \geq 0$ the inclusions $\{i, i+1\} \hookrightarrow [n]$ induce an equivalence

$$M([n]) \to M(\{0,1\}) \times \cdots \times M(\{n-1, n\})$$

and $M([0])$ is a final object in $\mathcal{T}$. The underlying object of M is $M([1])$. A *group object* in $\mathcal{T}$ is a monoid object $G : \mathrm{N}(\Delta)^{\mathrm{op}} \to \mathcal{T}$ such that the inclusions $\{0,2\} \hookrightarrow [2]$ and $\{0,1\} \hookrightarrow [2]$ induce an equivalence

$$G([2]) \to G(\{0,2\}) \times G(\{0,1\}).$$

The ∞-category of group objects in $\mathcal{T}$ is the full subcategory of $\mathrm{Fun}(\mathrm{N}(\Delta)^{\mathrm{op}}, \mathcal{T})$ spanned by group objects. We denote it by $\mathrm{Grp}(\mathcal{T})$. A *derived affine group scheme* G is defined to be a group object in Aff_R, that is, a functor $G : \mathrm{N}(\Delta)^{\mathrm{op}} \to \mathrm{Aff}_R$ satisfying the above condition. The object $G([1])$ in Aff_R is the underlying derived affine group scheme $\operatorname{Spec} B$ of G. For ease of notation, we often indicate with $G = \operatorname{Spec} B$ (or simply $\operatorname{Spec} B$) the derived affine group scheme with the underlying derived affine scheme $\operatorname{Spec} B$ (not only the underlying derived affine scheme). The ∞-category of derived affine group schemes over R is the full subcategory of $\mathrm{Fun}(\mathrm{N}(\Delta)^{\mathrm{op}}, \mathrm{Aff}_R)$ spanned by derived affine group schemes, that is, $\mathrm{Grp}(\mathrm{Aff}_R)$. The Yoneda embedding $\mathrm{Aff}_R \subset \mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$ preserves small limits, and thus a derived affine group scheme $G : \mathrm{N}(\Delta)^{\mathrm{op}} \to \mathrm{Aff}_R$ induces a group object $\mathrm{N}(\Delta)^{\mathrm{op}} \xrightarrow{G} \mathrm{Aff}_R \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$. Using adjunctions we easily see that there is a natural equivalence

$$\mathrm{Grp}(\mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})) \simeq \mathrm{Fun}(\mathrm{CAlg}_R, \mathrm{Grp}(\widehat{\mathcal{S}})).$$

Consequently, one may say that a derived affine group scheme (over R) is a functor $\mathrm{CAlg}_R \to \mathrm{Grp}(\widehat{\mathcal{S}})$ such that the composition $\mathrm{CAlg}_R \to \widehat{\mathcal{S}}$ with the forgetful functor $\mathrm{Grp}(\widehat{\mathcal{S}}) \to \widehat{\mathcal{S}}$ is represented by some derived affine scheme $\operatorname{Spec} B$ (here $\operatorname{Spec} B$ is the underlying derived affine scheme). This description is an analogue of the definition of group schemes as group-valued functors. If the base ring spectrum R is the Eilenberg–MacLane spectrum Hk for a commutative ring k, and $\operatorname{Spec} C$ is the usual *flat* affine group scheme over k, then the derived affine scheme $\operatorname{Spec} HC$ over Hk is endowed with the structure of a group object in the obvious way; the category of usual affine group schemes flat over k can be naturally regarded as the full subcategory of the ∞-category of derived affine group schemes over Hk. We will often regard usual flat affine group schemes over k as derived affine group schemes over Hk.

Next we will recall the definition of the symmetric monoidal ∞-category $\mathrm{Rep}_G^{\otimes}$. Let $G = \operatorname{Spec} B$ be a derived affine group scheme over R such that B is a cosimplicial object $\phi := G^{\mathrm{op}} : \mathrm{N}(\Delta) \to \mathrm{CAlg}_R$. We here abuse notation by indicating with B the underlying object $\phi([1])$ in CAlg_R. Let

$$\Theta : \mathrm{CAlg} \to \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})$$

be a functor which carries $A \in \mathrm{CAlg}$ to the symmetric monoidal ∞-category Mod_A and sends a map $A \to A'$ in CAlg to a colimit-preserving symmetric monoidal base change functor $\mathrm{Mod}_A \to \mathrm{Mod}'_A : M \mapsto M \otimes_A A'$ (see [19, Appendix A.6]). This functor induces

$$\Theta_R : \mathrm{CAlg}_R \simeq \mathrm{CAlg}_{R/} \to \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})_{\mathrm{Mod}_R^\otimes /}.$$

Consider the composition $\mathrm{N}(\Delta) \xrightarrow{\phi} \mathrm{CAlg}_R \xrightarrow{\Theta_R} \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})_{\mathrm{Mod}_R^\otimes /}$. We define $\mathrm{Rep}_G^\otimes$ to be the limit of this composition. We call it the symmetric monoidal ∞-category of *representations* of G. The underlying ∞-category is stable and presentable. Since the forgetful functor $\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})_{\mathrm{Mod}_R^\otimes /} \to \widehat{\mathrm{Cat}}_\infty$ is limit-preserving, we see that the underlying ∞-category of $\mathrm{Rep}_G^\otimes$, which we denote by Rep_G, is a limit of the composition $\mathrm{N}(\Delta) \xrightarrow{\Theta_R \circ \phi} \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})_{\mathrm{Mod}_R^\otimes /} \to \widehat{\mathrm{Cat}}_\infty$. There is the natural symmetric monoidal functor $\mathrm{Rep}_G^\otimes \to \mathrm{Mod}_R^\otimes$ and we let $\mathrm{PRep}_G^\otimes$ be the inverse image of the full subcategory $\mathrm{PMod}_R^\otimes$. Alternatively, there is a natural categorical equivalence $\mathrm{PRep}_G \simeq \lim_{[n]\in\Delta} \mathrm{PMod}_{\phi([n])}$ and $\mathrm{PRep}_G^\otimes$ is a symmetric monoidal full subcategory of $\mathrm{Rep}_G^\otimes$ spanned by dualizable objects. We call it the symmetric monoidal ∞-category of *perfect representations* of G.

§3.2. ∞-categories of modules over presheaves

Let $(\mathrm{CAlg}_R)^{\mathrm{op}} \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$ be the Yoneda embedding, where $\widehat{\mathcal{S}}$ denotes the ∞-category of (not necessarily small) spaces, i.e. Kan complexes. We shall refer to objects in $\mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$ as presheaves on CAlg_R or simply functors. By left Kan extension of Θ_R, we have a colimit-preserving functor

$$\overline{\Theta}_R : \mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}}) \to (\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty)_{\mathrm{Mod}_R^\otimes /})^{\mathrm{op}}.$$

Let $\mathrm{N}(\Delta)^{\mathrm{op}} \xrightarrow{G} (\mathrm{CAlg}_R)^{\mathrm{op}} \hookrightarrow \mathrm{Sh}(\mathrm{Aff}_R)$ be the composition and let $\mathsf{B}G$ denote the colimit. Recall $\overline{\Theta}_R(\mathsf{B}G) = \mathrm{Mod}_{\mathsf{B}G}^\otimes \simeq \mathrm{Rep}_G^\otimes$. The second equivalence follows from flat descent theory (see [37, II, 1.3.7.2], [29, VII, 6.13, VIII, 2.7.14]).

Let $X \in \mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$. Let $\mathrm{PMod}_X^\otimes$ denote the symmetric monoidal full subcategory of the underlying symmetric monoidal ∞-category $\overline{\Theta}_R$ spanned by dualizable objects. Suppose that $\mathrm{PMod}_X^\otimes$ is a small stable idempotent complete symmetric monoidal ∞-category whose tensor operation $\otimes : \mathrm{PMod}_X \times \mathrm{PMod}_X \to \mathrm{PMod}_X$ preserves finite colimits separately in each variable. We refer to $\mathrm{PMod}_X^\otimes$ as the symmetric monoidal ∞-category of *perfect complexes* on X. We here call presheaves enjoying this condition *admissible* presheaves (functors). For example, affine derived schemes and $\mathsf{B}G$ with G a derived affine group scheme are admissible. Indeed, $\mathsf{B}G$ is described as the colimit of a simplicial affine derived scheme $a : \mathrm{N}(\Delta)^{\mathrm{op}} \to \mathrm{Aff}_R$ and $\mathrm{Cat}_\infty^{\mathrm{st}} \hookrightarrow \mathrm{Cat}_\infty$ preserves small limits. It follows that $\mathrm{PMod}_{\mathsf{B}G} = \mathrm{PRep}_G \simeq \lim_{[n]} \mathrm{PMod}_{a([n])}$ is stable and idempotent complete, where $\lim_{[n]\in\Delta} \mathrm{PMod}_{a([n])}$ is the limit of the cosimplicial diagram of ∞-categories. Let $\mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})^{\mathrm{adm}}$ be the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$ spanned by admissible presheaves. Applying $\overline{\Theta}_R$ and taking full subcategories of $\overline{\Theta}_R(X)$ spanned by dualizable objects we have the functor

$$\overline{\theta}_R : \mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})^{\mathrm{adm}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{st}})^{\mathrm{op}}$$

which carries X to $\mathrm{PMod}_X^\otimes$. We remark that by [27, 3.3.3.2, 5.1.2.2], P in $\lim_{\mathrm{Spec}\, A\to X} \mathrm{PMod}_A$ ($\mathrm{Spec}\, A \to X$ run over $(\mathrm{Aff}_R)_{/X}$) is a finite colimit of a (finite) diagram $I \to \mathrm{PMod}_X$ if and only if for each $\mathrm{Spec}\, A \to X$ the image of P in PMod_A is a finite colimit of the induced diagram.

§3.3. Automorphisms

Let us review the automorphism group of a symmetric monoidal functor. Let $\mathcal{C}^\otimes$ be a symmetric monoidal small ∞-category. We write $\mathcal{C}$ for its underlying ∞-category. Let $\theta_{\mathcal{C}^\otimes} : \mathrm{CAlg}(\mathrm{Cat}_\infty) \to \mathcal{S}$ be the functor corresponding to $\mathcal{C}^\otimes$ via the Yoneda embedding $\mathrm{CAlg}(\mathrm{Cat}_\infty)^{\mathrm{op}} \subset \mathrm{Fun}(\mathrm{CAlg}(\mathrm{Cat}_\infty), \mathcal{S})$. We denote the restriction of $\bar{\theta}_R$ to Aff_R by θ_R. Then the composite

$$\xi : \mathrm{CAlg}_R \xrightarrow{\theta_R^{\mathrm{op}}} \mathrm{CAlg}(\mathrm{Cat}_\infty) \xrightarrow{\theta_{\mathcal{C}^\otimes}} \mathcal{S}$$

carries A to the space equivalent to $\mathrm{Map}^\otimes(\mathcal{C}^\otimes, \mathrm{PMod}_A^\otimes)$. Here if $\mathcal{A}^\otimes$ and $\mathcal{B}^\otimes$ are symmetric monoidal small ∞-categories, we write $\mathrm{Map}^\otimes(\mathcal{A}^\otimes, \mathcal{B}^\otimes)$ for $\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Cat}_\infty)}(\mathcal{A}^\otimes, \mathcal{B}^\otimes)$.

Let $\omega : \mathcal{C}^\otimes \to \mathrm{PMod}_R^\otimes$ be a symmetric monoidal functor. For any $A \in \mathrm{CAlg}_R$, one has the composite

$$\omega_A : \mathcal{C}^\otimes \xrightarrow{\omega} \mathrm{PMod}_R^\otimes \to \mathrm{PMod}_A^\otimes,$$

where the second functor is the base change by $R \to A$. Consequently, we can associate to each $A \in \mathrm{CAlg}_R$ the map $b_A : \Delta^0 \to \mathrm{Map}^\otimes(\mathcal{C}^\otimes, \mathrm{PMod}_A^\otimes)$ corresponding to the composite ω_A. If $\mathcal{S}_*$ denotes the ∞-category of pointed spaces, that is, $\mathcal{S}_{\Delta^0/}$, we can extend ξ to $\xi_* : \mathrm{CAlg}_R \to \mathcal{S}_*$ that carries A to $\Delta^0 \to \mathrm{Map}^\otimes(\mathcal{C}^\otimes, \mathrm{PMod}_A^\otimes)$. The ξ_* is constructed as follows: Let $\mathcal{M} \to \mathrm{CAlg}_R$ be a left fibration corresponding to ξ. Extending ξ to ξ_* amounts to giving a section $\mathrm{CAlg}_R \to \mathcal{M}$ of the left fibration $\mathcal{M} \to \mathrm{CAlg}_R$, that is, a map from the "trivial" left fibration $\mathrm{CAlg}_R \to \mathrm{CAlg}_R$ to $\mathcal{M} \to \mathrm{CAlg}_R$ (over CAlg_R). According to [27, 3.3.3.4] there is a natural categorical equivalence

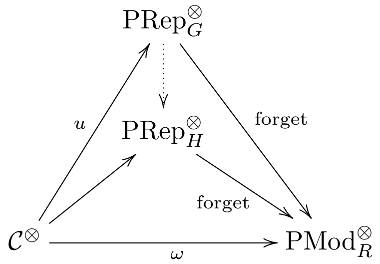
$$\mathcal{L} := \lim_{A \in \mathrm{CAlg}_R} \mathrm{Map}^{\otimes}(\mathcal{C}^{\otimes}, \mathrm{PMod}_A^{\otimes}) \simeq \mathrm{Map}_{(\widehat{\mathrm{Cat}}_{\infty})_{/\mathrm{CAlg}_R}}(\mathrm{CAlg}_R, \mathcal{M}).$$

Thus a section $\mathrm{CAlg}_R \to \mathcal{M}$ corresponds to an object in $\mathcal{L}$. If $\lim \mathrm{PMod}_A^{\otimes}$ denotes the limit of $\bar{\theta}_R^{\mathrm{op}} : \mathrm{CAlg}_R \to \mathrm{CAlg}(\mathrm{Cat}_{\infty})$, then $\mathcal{L}$ and $\mathrm{Map}^{\otimes}(\mathcal{C}^{\otimes}, \lim \mathrm{PMod}_A^{\otimes})$ are equivalent as ∞-categories (or equivalently spaces). The natural functor $\mathrm{PMod}_R^{\otimes} \xrightarrow{\sim} \lim \mathrm{PMod}_A^{\otimes}$ induces $p : \mathrm{Map}^{\otimes}(\mathcal{C}^{\otimes}, \mathrm{PMod}_R^{\otimes}) \to \mathrm{Map}^{\otimes}(\mathcal{C}^{\otimes}, \lim \mathrm{PMod}_A^{\otimes}) \simeq \mathcal{L}$. The image $p(\omega)$ in $\mathcal{L}$ gives rise to a section $\mathrm{CAlg}_R \to \mathcal{M}$. Consequently, we have $\xi_* : \mathrm{CAlg}_R \to \mathcal{S}_*$ which extends ξ. We define $\mathrm{Aut}(\omega)$ to be the composite $\mathrm{CAlg}_R \xrightarrow{\xi_*} \mathcal{S}_* \xrightarrow{\Omega_*} \mathrm{Grp}(\mathcal{S})$, where the second functor is the based loop functor. The functor $\mathrm{Aut}(\omega)$ sends A to the base loop space of $\mathrm{Map}^{\otimes}(\mathcal{C}^{\otimes}, \mathrm{PMod}_A^{\otimes})$ equipped with the base point b_A. The base loop space can be thought of as the "automorphism group space" of ω_A. We refer to $\mathrm{Aut}(\omega)$ as the *automorphism group functor* of $\omega : \mathcal{C}^{\otimes} \to \mathrm{PMod}_R^{\otimes}$.

§3.4. Tannakization

We recall one of the main results of [19, Section 4].

Theorem 3.1. *Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal small ∞-category. Let $\omega : \mathcal{C}^{\otimes} \to \mathrm{PMod}_R^{\otimes}$ be a symmetric monoidal functor. There exists a derived affine group scheme G over R which represents the automorphism group functor* $\mathrm{Aut}(\omega)$*. Moreover, there is a symmetric monoidal functor $u : \mathcal{C}^{\otimes} \to \mathrm{PRep}_G^{\otimes}$ which makes the outer triangle in*



commute in the ∞-category of symmetric monoidal ∞-categories such that these have the following universality: for any inner triangle consisting of solid arrows in the above diagram where H is a derived affine group scheme, there exists a unique (*in an appropriate sense*) *morphism $f : H \to G$ of derived affine group schemes which induces $\mathrm{PRep}_G^{\otimes} \to \mathrm{PRep}_H^{\otimes}$* (*indicated by the dotted arrow*) *filling the above diagram.*

We usually refer to G as the *tannakization* of $\omega : \mathcal{C}^{\otimes} \to \mathrm{PMod}_R^{\otimes}$. (In this note, we do not use this theorem in an essential way.)

Remark 3.2. We here make a remark on why the usual affine group schemes are not sufficient for our purpose. Many readers might find the appearance of derived affine groups unpleasant, since they are familiar with algebraic groups, but not with derived groups. Our point of view is different: rather, derived affine group schemes are natural objects. To explain this, recall that in the light of classical Tannaka duality, an affine group scheme G over a field corresponds to a neutral Tannakian category $\mathcal{A}$, i.e., a symmetric monoidal abelian category satisfying certain conditions. Namely, the symmetric monoidal stable ∞-category arising from G is the derived ∞-category of $\mathcal{A}$, that is, a natural ∞-categorical enhancement of the derived category of $\mathcal{A}$. On the other hand, there are a lot of important symmetric monoidal stable ∞-categories which are not derived ∞-categories of abelian categories. The ∞-category of spectra is such an example. Put another way, if we denote by Hk the Eilenberg–MacLane spectrum of a field k, then there are many examples of symmetric monoidal functors $\mathcal{C}^\otimes \to \mathrm{PMod}_{Hk}^\otimes$ whose automorphism groups are not representable by affine group schemes over k. For example, let A be a free commutative differential graded algebra generated by the shifted nonzero k-vector space $V[-2]$ over a field k of characteristic zero, and put $\mathcal{C}^\otimes := \mathrm{PMod}_A^\otimes$. The natural augmentation $A \to k$ induces a base change symmetric monoidal functor $f : \mathrm{PMod}_A^\otimes \to \mathrm{PMod}_{Hk}^\otimes$, but one can prove that f has nontrivial higher automorphisms (moreover, f is conservative, that is, $f(M) \simeq 0$ implies $M \simeq 0$). Hence $\mathrm{Aut}(f)$ is not representable by an affine group scheme. Intuitively, derived affine group schemes also take account of higher automorphisms, i.e., automorphisms between automorphisms, and automorphisms of them and so on (furthermore, derived affine group schemes consist of data of derived deformations of automorphisms of fiber functors, but we will not explain what derived deformations are).

Even in the case when $\mathcal{C}^\otimes$ is conjecturally expected to be the derived ∞-category of a symmetric monoidal abelian category, the notion of derived affine group scheme will play a useful role in the procedure of tannakization.

§4. Automorphisms of fiber functors

Let Y be a derived stack over R (we fix our conventions below) and $\mathrm{PMod}_Y^\otimes$ the ∞-category of perfect complexes on Y, which we regard as an object in $\mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{st}})_{\mathrm{PMod}_R^\otimes /}$. Let $\mathrm{Spec}\, R \to Y$ be a section of the structure morphism $Y \to \mathrm{Spec}\, R$. There is the pullback functor $\mathrm{PMod}_Y^\otimes \to \mathrm{PMod}_R^\otimes$ in $\mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{st}})$. In this section, we study the automorphisms of this functor. Our goal is Theorem 4.8 and Corollary 4.9.

We start with our setup of derived stacks. A sheaf $Y : \mathrm{CAlg}_R \to \widehat{\mathcal{S}}$ is said to be a *derived stack* over R if there exists a groupoid object $Y_\bullet : \mathrm{N}(\Delta)^{\mathrm{op}} \to \mathrm{Aff}_R$

(cf. [19, Definition A.2]) such that Y is equivalent to the colimit of the composite $\mathrm{N}(\Delta)^{\mathrm{op}} \to \mathrm{Aff}_R \to \mathrm{Sh}(\mathrm{Aff}_R)$.

When Y is a colimit of a simplicial diagram $Y_\bullet$, we refer to $Y_\bullet$ as a *simplicial covering* for Y.

Our definition of derived stacks is different from standard ones (compare [37], [29]; for example we do not assume that the natural morphism $Y_\bullet([0]) \to Y$ is smooth or étale). We note that our derived stacks are admissible functors.

Example 4.1. We present quotient stacks arising from the action of a derived affine group scheme on an affine scheme as examples of derived stacks. Let $F : \mathrm{N}(\Delta)^{\mathrm{op}} \to \mathrm{Aff}_R$ be a groupoid object, which we regard as a derived stack. Let $G : \mathrm{N}(\Delta)^{\mathrm{op}} \to \mathrm{Aff}_R$ be a group object, that is, a derived affine group scheme. Let $F \to G$ be a morphism (i.e., natural transformation) which induces a cartesian diagram

$$\begin{array}{ccc} F([n]) & \longrightarrow & F([m]) \\ \downarrow & & \downarrow \\ G([n]) & \longrightarrow & G([m]) \end{array}$$

in Aff_R for each $[m] \to [n]$. If we write X for $F([0])$, then we can think that the morphism $F \to G$ with the above property means an action of G on X. In this situation, we say that G *acts on* X and denote by $[X/G]$ the colimit of $\mathrm{N}(\Delta)^{\mathrm{op}} \xrightarrow{F} \mathrm{Aff}_R \hookrightarrow \mathrm{Sh}(\mathrm{Aff}_R)$. We refer to $[X/G]$ as the *quotient stack*. We can think of $\mathsf{B}G$ as the quotient stack $[\mathrm{Spec}\, R/G]$ where G acts trivially on $\mathrm{Spec}\, R$.

Let $\pi : \mathrm{Spec}\, R \to Y$ denote a fixed morphism. Let $\pi^* : \mathrm{Mod}_Y^\otimes \to \mathrm{Mod}_R^\otimes$ be the associated symmetric monoidal functor which preserves small colimits. Since Mod_Y and Mod_R are presentable, by the adjoint functor theorem (see [27, 5.5.2.9]) there is a right adjoint functor $\pi_* : \mathrm{Mod}_R \to \mathrm{Mod}_Y$. Moreover, according to [28, 8.3.2.6] the right adjoint functor extends to a right adjoint functor relative to $\mathrm{N}(\mathrm{Fin}_*)$ (see [28, 8.3.2.2])

$$\begin{array}{ccc} \mathrm{Mod}_R^\otimes & \longrightarrow & \mathrm{Mod}_Y^\otimes \\ & \searrow \quad \swarrow & \\ & \mathrm{N}(\mathrm{Fin}_*) & \end{array}$$

It yields a right adjoint functor

$$\mathrm{CAlg}(\mathrm{Mod}_R^\otimes) \to \mathrm{CAlg}(\mathrm{Mod}_Y^\otimes)$$

of the functor $\mathrm{CAlg}(\mathrm{Mod}_Y^\otimes) \to \mathrm{CAlg}(\mathrm{Mod}_R^\otimes)$ determined by π^*.

Let $\phi : \mathrm{N}(\Delta) \to \mathrm{CAlg}_R$ be a cosimplicial diagram such that $Y_\bullet : \mathrm{N}(\Delta)^{\mathrm{op}} \xrightarrow{\phi^{\mathrm{op}}} \mathrm{Aff}_R \hookrightarrow \mathrm{Sh}(\mathrm{Aff}_R)$ is a simplicial covering for Y. In the argument below, we use the following properties:

- $Y_n \times_Y \operatorname{Spec} R$ belongs to Aff_R where $Y_n = Y_\bullet([n])$ for $n \geq 0$,
- $\operatorname{Spec} R \times_Y \operatorname{Spec} R$ belongs to Aff_R.

These are special cases of the more general fact that for any two morphisms $\operatorname{Spec} A \to Y$ and $\operatorname{Spec} B \to Y$, the fiber product $\operatorname{Spec} A \times_Y \operatorname{Spec} B$ is affine. To see this, we note that $Y_1 \simeq Y_0 \times_Y Y_0$ is affine. Since $Y_0 \to Y$ is an effective epimorphism, taking suitable étale coverings $\operatorname{Spec} A' \to \operatorname{Spec} A$ and $\operatorname{Spec} B' \to \operatorname{Spec} B$ we have a derived affine scheme $\operatorname{Spec} A' \times_Y \operatorname{Spec} B'$. By the étale local character of affine representability [37, II, 1.3.2.8, 2.4.1.8], we see that $\operatorname{Spec} A \times_Y \operatorname{Spec} B$ belongs to Aff_R.

Next recall from Section 3.1 the functor $\Theta_R : \mathrm{CAlg}_R \to \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})_{\mathrm{Mod}_R^\otimes/}$. Note that by definition $\mathrm{Mod}_Y^\otimes$ is a limit of the composition $\phi'' : \mathrm{N}(\Delta) \xrightarrow{\phi} \mathrm{CAlg}_R \xrightarrow{\Theta_R} \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})_{\mathrm{Mod}_R^\otimes/} \to \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})$ where the last functor is the forgetful functor. Let $p : \mathcal{M}_{\phi'} \to \mathrm{N}(\Delta)$ be the coCartesian fibration corresponding to the composition $\phi' : \mathrm{N}(\Delta) \xrightarrow{\phi''} \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}}) \to \widehat{\mathrm{Cat}}_\infty$ where the last functor is the forgetful functor. We denote by $\mathrm{Fun}'_{\mathrm{N}(\Delta)}(\mathrm{N}(\Delta), \mathcal{M}_{\phi'})$ the full subcategory of $\mathrm{Fun}_{\mathrm{N}(\Delta)}(\mathrm{N}(\Delta), \mathcal{M}_{\phi'})$ spanned by those sections $\mathrm{N}(\Delta) \to \mathcal{M}_{\phi'}$ which carry all edges of $\mathrm{N}(\Delta)$ to p-coCartesian edges. Then by [27, 3.3.3.2], Mod_Y is equivalent to $\mathrm{Fun}'_{\mathrm{N}(\Delta)}(\mathrm{N}(\Delta), \mathcal{M}_{\phi'})$ as ∞-categories. Consider the base change of $\mathrm{N}(\Delta)^{\mathrm{op}} \xrightarrow{\phi^{\mathrm{op}}} \mathrm{Aff}_R \hookrightarrow \mathrm{Sh}(\mathrm{Aff}_R)$, where the second functor is the Yoneda embedding, by $\pi : \operatorname{Spec} R \to Y$. Let $Y_n = \phi^{\mathrm{op}}([n]) \in \mathrm{Aff}_R$ for each $[n] \in \Delta$. The n-th term of this base change $\tau : \mathrm{N}(\Delta)^{\mathrm{op}} \to \mathrm{Sh}(\mathrm{Aff}_R)$ is equivalent to $Y_n \times_Y \operatorname{Spec} R$, and in particular it factors through $\mathrm{Aff}_R \subset \mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$. Taking the opposite categories we have $\psi : \mathrm{N}(\Delta) \to \mathrm{CAlg}_R$. Note that $\operatorname{Spec} R$ is a colimit of τ since in the ∞-topos $\mathrm{Sh}(\mathrm{Aff}_R)$ colimits are universal (see [27, Chapter 6]). Thus the natural transformation $\psi^{\mathrm{op}} \to \phi^{\mathrm{op}}$ induces $\pi : \operatorname{Spec} R \to Y$, and we can informally depict our situation as follows:

$$\begin{array}{ccccccc}
\cdots \; \substack{\to\\[-2pt]\to\\[-2pt]\to} & Y_1 \times_Y \operatorname{Spec} R & \rightleftarrows & Y_0 \times_Y \operatorname{Spec} R & \longrightarrow & \operatorname{Spec} R \\
 & \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi} \\
\cdots \; \substack{\to\\[-2pt]\to\\[-2pt]\to} & Y_1 & \rightleftarrows & Y_0 & \longrightarrow & Y
\end{array}$$

(here $\psi^{\mathrm{op}}, \phi^{\mathrm{op}} : \mathrm{N}(\Delta)^{\mathrm{op}} \to \mathrm{Aff}_R$). We define $\psi' : \mathrm{N}(\Delta) \to \widehat{\mathrm{Cat}}_\infty$ in the same way that we define ϕ', and we let $q : \mathcal{M}_{\psi'} \to \mathrm{N}(\Delta)$ be the coCartesian fibration corresponding to ψ'. The natural transformation $\phi \to \psi$ corresponds to a map between coCartesian fibrations $\mathcal{M}_{\phi'} \to \mathcal{M}_{\psi'}$ over $\mathrm{N}(\Delta)$, which carries coCartesian edges to coCartesian edges. Again by [28, 8.3.2.6] there is a right adjoint functor $\mathcal{M}_{\psi'} \to \mathcal{M}_{\phi'}$ of $\mathcal{M}_{\phi'} \to \mathcal{M}_{\psi'}$ relative to $\mathrm{N}(\Delta)$. Let us observe the following:

Lemma 4.2. *The map $\mathcal{M}_{\psi'} \to \mathcal{M}_{\phi'}$ of coCartesian fibrations over $\mathrm{N}(\Delta)$ carries q-coCartesian edges to p-coCartesian edges.*

Proof. It suffices to show that if for any map $r : [m] \to [n]$ in Δ we write the diagram induced by $\psi^{\mathrm{op}} \to \phi^{\mathrm{op}}$ as

$$\begin{array}{ccc} Y_n \times_Y \operatorname{Spec} R & \xrightarrow{a} & Y_m \times_Y \operatorname{Spec} R \\ \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\ Y_n & \xrightarrow{d} & Y_m \end{array}$$

then the natural base change morphism $d^* \circ c_* \to b_* \circ a^*$ is an equivalence. This follows from [2, Lemma 3.14]. □

Let
$$\alpha : \mathrm{Fun}'_{\mathrm{N}(\Delta)}(\mathrm{N}(\Delta), \mathcal{M}_{\phi'}) \rightleftarrows \mathrm{Fun}'_{\mathrm{N}(\Delta)}(\mathrm{N}(\Delta), \mathcal{M}_{\psi'}) : \beta$$
be functors induced by the adjunction $\mathcal{M}_{\phi'} \rightleftarrows \mathcal{M}_{\psi'}$, where $\mathrm{Fun}'_{\mathrm{N}(\Delta)}(\mathrm{N}(\Delta), \mathcal{M}_{\phi'})$ is the full subcategory of $\mathrm{Fun}_{\mathrm{N}(\Delta)}(\mathrm{N}(\Delta), \mathcal{M}_{\phi'})$ spanned by those sections which carry all edges to coCartesian edges, and we define $\mathrm{Fun}'_{\mathrm{N}(\Delta)}(\mathrm{N}(\Delta), \mathcal{M}_{\phi'})$ in a similar way. Note that by [27, 3.3.3.2],

$$\mathrm{Fun}'_{\mathrm{N}(\Delta)}(\mathrm{N}(\Delta), \mathcal{M}_{\phi'}) \simeq \mathrm{Mod}_Y \quad \text{and} \quad \mathrm{Fun}'_{\mathrm{N}(\Delta)}(\mathrm{N}(\Delta), \mathcal{M}_{\psi'}) \simeq \mathrm{Mod}_R,$$

and $\mathrm{Fun}'_{\mathrm{N}(\Delta)}(\mathrm{N}(\Delta), \mathcal{M}_{\phi'}) \to \mathrm{Fun}'_{\mathrm{N}(\Delta)}(\mathrm{N}(\Delta), \mathcal{M}_{\psi'})$ is equivalent to $\pi^* : \mathrm{Mod}_Y \to \mathrm{Mod}_R$ as functors. Then observe that the pair (α, β) forms an adjunction. Namely,

$$\begin{aligned} \mathrm{Map}_{\mathrm{Fun}'_{\mathrm{N}(\Delta)}(\mathrm{N}(\Delta), \mathcal{M}_{\psi'})}(\alpha(a), b) &\simeq \lim_{[n]\in\Delta} \mathrm{Map}_{\psi'([n])}(\alpha(a_n), b_n) \\ &\to \lim_{[n]\in\Delta} \mathrm{Map}_{\phi'([n])}(\beta(\alpha(a_n)), \beta(b_n)) \\ &\xrightarrow{x} \lim_{[n]\in\Delta} \mathrm{Map}_{\phi'([n])}(a_n, \beta(b_n)) \\ &\simeq \mathrm{Map}_{\mathrm{Fun}'_{\mathrm{N}(\Delta)}(\mathrm{N}(\Delta), \mathcal{M}_{\phi'})}(a, \beta(b)) \end{aligned}$$

is an equivalence in $\mathcal{S}$, where a_n (resp. b_n) is the projection of a (resp. b) to $\phi'([n])$ (resp. $\psi'([n])$) and x is induced by the unit map of the adjunction $\mathcal{M}_{\phi'} \rightleftarrows \mathcal{M}_{\psi'}$. (The fiber of the adjunction $\mathcal{M}_{\phi'} \rightleftarrows \mathcal{M}_{\psi'}$ over each object of $\mathrm{N}(\Delta)$ forms an adjunction.) Notice that $\mathrm{Fun}_{\mathrm{N}(\Delta)}(\mathrm{N}(\Delta), \mathcal{M}_{\psi'}) \to \mathrm{Fun}_{\mathrm{N}(\Delta)}(\mathrm{N}(\Delta), \mathcal{M}_{\phi'})$ and $\pi_* : \mathrm{Mod}_R \to \mathrm{Mod}_Y$ are equivalent as functors. Consequently, we have

Lemma 4.3. *Let*

$$\begin{array}{ccc} Y_n \times_Y \operatorname{Spec} R & \xrightarrow{s_n} & \operatorname{Spec} R \\ \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi} \\ Y_n & \xrightarrow{t_n} & Y \end{array}$$

be the pullback diagram induced by $\psi^{\mathrm{op}}([n]) \to \phi^{\mathrm{op}}([n])$*. Then the natural base change morphism* $(t_n)^* \circ \pi_* \to (\pi_n)_* \circ (s_n)^*$ *is an equivalence of functors from* Mod_R *to* Mod_{Y_n}*.*

Corollary 4.4. *Abusing notation, write* $(t_n)^* \circ \pi_* \to (\pi_n)_* \circ (s_n)^*$ *for the natural base change morphism from* $\mathrm{CAlg}(\mathrm{Mod}_R^{\otimes})$ *to* $\mathrm{CAlg}(\mathrm{Mod}_{Y_n}^{\otimes})$ *which is determined by adjunctions* (π^*, π_*) *and* $((\pi_n)^*, (\pi_n)_*)$ *relative to* $\mathrm{N}(\mathrm{Fin}_*)$*. Then* $(t_n)^* \circ \pi_* \to (\pi_n)_* \circ (s_n)^*$ *is an equivalence of functors.*

Let $\mathbf{1}_R$ be a unit of Mod_R which we here regard as an object in $\mathrm{CAlg}_R = \mathrm{CAlg}(\mathrm{Mod}_R)$. Then there is a lax symmetric monoidal functor $\mathrm{Mod}_R^{\otimes} \to \mathrm{Mod}_{\pi_*\mathbf{1}_R}^{\otimes}(\mathrm{Mod}_Y^{\otimes})$ of symmetric monoidal ∞-categories induced by π_*, by the construction of the ∞-operad of module objects [28, 3.3.3.8]. For the notation $\mathrm{Mod}_{\pi_*\mathbf{1}_R}^{\otimes}(\mathrm{Mod}_Y^{\otimes})$, see Section 2.

Lemma 4.5. *The functor* $\mathrm{Mod}_R^{\otimes} \to \mathrm{Mod}_{\pi_*\mathbf{1}_R}^{\otimes}(\mathrm{Mod}_Y^{\otimes})$ *is a symmetric monoidal equivalence.*

Proof. We first observe that $\mathrm{Mod}_R^{\otimes} \to \mathrm{Mod}_{\pi_*\mathbf{1}_R}^{\otimes}(\mathrm{Mod}_Y^{\otimes})$ is symmetric monoidal. Since it is lax symmetric monoidal, by Lemma 4.3 we are reduced to showing the following obvious claim: for a morphism $x : \operatorname{Spec} A \to \operatorname{Spec} B$ of affine derived schemes and $M, N \in \mathrm{Mod}_A$, the natural map $x_*(M) \otimes_A x_*(N) \to x_*(M \otimes_A N)$ is an equivalence, where $x_* : \mathrm{Mod}_A \to \mathrm{Mod}_A(\mathrm{Mod}_B^{\otimes})$ is the natural pushforward functor.

We now adopt notation similar to Lemma 4.3. By the natural equivalence $(t_n)^* \circ \pi_*\mathbf{1}_R \simeq (\pi_n)_* \circ (s_n)^*\mathbf{1}_R$ in view of the above result, we have

$$\begin{aligned} (\pi_n)_* : \mathrm{Mod}_{\psi([n])} = \mathrm{Mod}_{Y_n \times_Y \operatorname{Spec} R} &\simeq \mathrm{Mod}_{(\pi_n)_* \circ (s_n)^* \mathbf{1}_R}(\mathrm{Mod}_{\phi([n])}^{\otimes}) \\ &\simeq \mathrm{Mod}_{(t_n)^* \circ \pi_* \mathbf{1}_R}(\mathrm{Mod}_{\phi([n])}^{\otimes}) \end{aligned}$$

for each n. Then we identify $\mathrm{Mod}_R \to \mathrm{Mod}_{\pi_*\mathbf{1}_R}(\mathrm{Mod}_Y^{\otimes})$ with the limit

$$\lim_{[n]\in\Delta} \mathrm{Mod}_{\psi([n])} \simeq \lim_{[n]\in\Delta} \mathrm{Mod}_{Y_n \times_Y \operatorname{Spec} R} \simeq \lim_{[n]\in\Delta} \mathrm{Mod}_{(t_n)^* \circ \pi_* \mathbf{1}_R}(\mathrm{Mod}_{\phi([n])}^{\otimes}),$$

which is an equivalence in $\widehat{\mathrm{Cat}}_\infty$. It follows that $\mathrm{Mod}_R^{\otimes} \to \mathrm{Mod}_{\pi_*\mathbf{1}_R}^{\otimes}(\mathrm{Mod}_Y^{\otimes})$ is a symmetric monoidal equivalence. □

Let $\mathrm{Aut}(\pi^*) : \mathrm{CAlg}_R \to \mathrm{Grp}(\widehat{\mathcal{S}})$ be the automorphism group functor of π^* (defined as in the previous section), which carries $A \in \mathrm{CAlg}_R$ to the automorphisms of the composition $\mathrm{Mod}_Y^{\otimes} \to \mathrm{Mod}_R^{\otimes} \to \mathrm{Mod}_A^{\otimes}$ in $\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})$ where the second functor is the base change by $R \to A$.

Let Δ_+ be the category of finite (possibly empty) linearly ordered sets and we write $[-1]$ for the empty set. Let $\iota : \Delta^1 \to \mathrm{N}(\Delta_+)$ be a map which carries $\{0\}$ and $\{1\}$ to $[-1]$ and $[0]$ respectively. It is a fully faithful functor. Let $(\Delta^1)^{\mathrm{op}} \to \mathrm{Sh}(\mathrm{Aff}_R)$ be the map corresponding to $\pi : \mathrm{Spec}\, R \to Y$. Let $\rho : \mathrm{N}(\Delta_+)^{\mathrm{op}} \to \mathrm{Sh}(\mathrm{Aff}_R)$ be a right Kan extension along $\iota^{\mathrm{op}} : (\Delta^1)^{\mathrm{op}} \to \mathrm{N}(\Delta_+)^{\mathrm{op}}$; ρ is called the *Čech nerve* (cf. [27, 6.1.2.11]). By our assumption, for each $n \geq 0$, $\rho([n])$ belongs to Aff_R and the restriction of ρ to $\mathrm{N}(\Delta)^{\mathrm{op}}$ is a derived affine group scheme which we denote by G_π. By the definition of G_π and $\mathrm{Mod}_{G_\pi}^{\otimes}$, we see that $\pi^* : \mathrm{Mod}_Y^{\otimes} \to \mathrm{Mod}_R^{\otimes}$ factors through the forgetful functor $\mathrm{Rep}_{G_\pi}^{\otimes} \to \mathrm{Mod}_R^{\otimes}$. Note that the derived group scheme $G_\pi : (\mathrm{Aff}_R)^{\mathrm{op}} \to \mathrm{Grp}(\mathcal{S})$ represents the automorphism group $\mathrm{Aut}(\pi) : \mathrm{CAlg}_R \to \mathrm{Grp}(\mathcal{S})$ of $\pi : \mathrm{Spec}\, R \to Y$. Here for any $A \in \mathrm{CAlg}_R$, $\mathrm{Aut}(\pi)(A)$ is the mapping space in $\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})}(\mathrm{Spec}\, A, Y)$ from $\mathrm{Spec}\, A \to \mathrm{Spec}\, R \xrightarrow{\pi} Y$ to itself, endowed with the group structure (the construction is similar to that of $\mathrm{Aut}(\omega)$ in the previous section). We have the natural morphism $G_\pi \simeq \mathrm{Aut}(\pi) \to \mathrm{Aut}(\pi^*)$.

Proposition 4.6. *The natural morphism $G_\pi \to \mathrm{Aut}(\pi^*)$ is an equivalence, that is,* $\mathrm{Aut}(\pi^*)$ *is represented by G_π.*

Proof. For simplicity, let $G := G_\pi$. Let $G_1 : \mathrm{CAlg}_R \to \widehat{\mathcal{S}}$ and (resp. $\mathrm{Aut}(\pi^*)_1$) be the composite of $G : \mathrm{CAlg}_R \to \mathrm{Grp}(\widehat{\mathcal{S}})$ (resp. $\mathrm{Aut}(\pi^*)$) and the forgetful functor $\mathrm{Grp}(\widehat{\mathcal{S}}) \to \widehat{\mathcal{S}}$. For each $A \in \mathrm{CAlg}_R$, it will suffice to show that the induced map $G_1(A) \to \mathrm{Aut}(\pi^*)_1(A)$ is an equivalence in $\widehat{\mathcal{S}}$.

For $A \in \mathrm{CAlg}_R$, let $\pi_A : \mathrm{Spec}\, A \to \mathrm{Spec}\, R \to Y$ denote the composition. Let $\mathbf{1}_A$ be the unit of Mod_A which we here think of as an object of $\mathrm{CAlg}(\mathrm{Mod}_A^{\otimes})$. Applying [28, 6.3.5.18] together with Lemma 4.5 and adjunction we deduce

$$\begin{aligned}\mathrm{Map}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})_{\mathrm{Mod}_Y^{\otimes}/}}(\mathrm{Mod}_A^{\otimes}, \mathrm{Mod}_A^{\otimes}) &\simeq \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Mod}_Y^{\otimes})}((\pi_A)_*\mathbf{1}_A, (\pi_A)_*\mathbf{1}_A) \\ &\simeq \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Mod}_A)}((\pi_A)^*(\pi_A)_*\mathbf{1}_A, \mathbf{1}_A).\end{aligned}$$

Unwinding the definitions we have

$$\begin{aligned}\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Mod}_A)}((\pi_A)^*(\pi_A)_*\mathbf{1}_A, \mathbf{1}_A) &\simeq \mathrm{Map}_{(\mathrm{Aff})_{/\mathrm{Spec}\, A}}(\mathrm{Spec}\, A, \mathrm{Spec}\, A \times_Y \mathrm{Spec}\, A) \\ &\simeq \mathrm{Map}_{(\mathrm{Aff})_{/\mathrm{Spec}\, A}}(\mathrm{Spec}\, A, G_1 \times_R A \times_R A) \\ &\simeq \mathrm{Map}_{\mathrm{Aff}_{/Y}}(\mathrm{Spec}\, A, \mathrm{Spec}\, A)\end{aligned}$$

where G_1 is $\operatorname{Spec} R \times_Y \operatorname{Spec} R \simeq \rho([1])$, and $G_1 \times_R A \times_R A \to \operatorname{Spec} A \in (\mathrm{Aff})_{/\operatorname{Spec} A}$ is the second projection. Note that through natural equivalences a morphism $\operatorname{Spec} A \to \operatorname{Spec} A$ over Y, which we regard as an object of $\mathrm{Map}_{\mathrm{Aff}_{/Y}}(\operatorname{Spec} A, \operatorname{Spec} A)$, induces a symmetric monoidal functor $\mathrm{Mod}_A^{\otimes} \to \mathrm{Mod}_A^{\otimes}$ under $\mathrm{Mod}_Y^{\otimes}$ which we think of as an object of $\mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\infty}^{\mathrm{L,st}})_{\mathrm{Mod}_Y^{\otimes}/}$.

Next using the natural equivalence

$$\mathrm{Map}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\infty}^{\mathrm{L,st}})_{\mathrm{Mod}_Y^{\otimes}/}}(\mathrm{Mod}_A^{\otimes}, \mathrm{Mod}_A^{\otimes}) \simeq \mathrm{Map}_{\mathrm{Aff}_{/Y}}(\operatorname{Spec} A, \operatorname{Spec} A)$$

we consider the automorphisms of π^*. To this end let T_A be the fiber product

$$\mathrm{Map}_{\mathrm{Aff}_{/Y}}(\operatorname{Spec} A, \operatorname{Spec} A) \times_{\mathrm{Map}_{\mathrm{Aff}}(\operatorname{Spec} A, \operatorname{Spec} A)} \{\mathrm{Id}_{\operatorname{Spec} A}\}$$

in $\widehat{\mathcal{S}}$ where the diagram is induced by the forgetful functor

$$\mathrm{Map}_{\mathrm{Aff}_{/Y}}(\operatorname{Spec} A, \operatorname{Spec} A) \to \mathrm{Map}_{\mathrm{Aff}}(\operatorname{Spec} A, \operatorname{Spec} A).$$

Similarly, we define S_A to be the fiber product

$$\mathrm{Map}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\infty}^{\mathrm{L,st}})_{\mathrm{Mod}_Y^{\otimes}/}}(\mathrm{Mod}_A^{\otimes}, \mathrm{Mod}_A^{\otimes}) \times_{\mathrm{Map}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\infty}^{\mathrm{L,st}})}(\mathrm{Mod}_A^{\otimes}, \mathrm{Mod}_A^{\otimes})} \{\mathrm{Id}\}$$

in $\widehat{\mathcal{S}}$, which is equivalent to T_A. There are natural equivalences

$$\begin{aligned} T_A &\simeq \mathrm{Map}'_{(\mathrm{Aff})_{/\operatorname{Spec} A}}(\operatorname{Spec} A, G_1 \times_R A \times_R A) \\ &\simeq \mathrm{Map}_{(\mathrm{Aff})_{/\operatorname{Spec} A}}(\operatorname{Spec} A, G_1 \times_R A) \simeq \mathrm{Map}_{\mathrm{Aff}}(\operatorname{Spec} A, G_1) \end{aligned}$$

in $\widehat{\mathcal{S}}$ where $\mathrm{Map}'_{(\mathrm{Aff})_{/\operatorname{Spec} A}}(\operatorname{Spec} A, G_1 \times_R A \times_R A)$ is the fiber product

$$\mathrm{Map}_{(\mathrm{Aff})_{/\operatorname{Spec} A}}(\operatorname{Spec} A, G_1 \times_R A \times_R A) \times_{\mathrm{Map}_{\mathrm{Aff}}(\operatorname{Spec} A, \operatorname{Spec} A)} \{\mathrm{Id}_{\operatorname{Spec} A}\}$$

in $\widehat{\mathcal{S}}$ where the diagram is induced by the projection $\mathrm{pr}_3 : G_1 \times_R A \times_R A \to \operatorname{Spec} A$. Thus we have an equivalence $\mathrm{Map}_{\mathrm{Aff}}(\operatorname{Spec} A, G_1) \simeq S_A$. Hence we have the required equivalence $G_1(A) \simeq \mathrm{Aut}(\pi^*)_1(A)$. □

Let $\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes} \in \mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\infty}^{\mathrm{L,st}})$. Suppose that $\mathcal{C}$ is compactly generated, that is, the natural colimit-preserving functor $\mathrm{Ind}(\mathcal{C}_\circ) \to \mathcal{C}$ is a categorical equivalence, and $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ induces $\mathcal{C}_\circ \times \mathcal{C}_\circ \to \mathcal{C}_\circ$, which makes $\mathcal{C}_\circ$ a symmetric monoidal ∞-category, where $\mathcal{C}_\circ$ is the full subcategory of compact objects in $\mathcal{C}$ and $\mathrm{Ind}(-)$ indicates the Ind-category (see [27, 5.3.5]). Note that under this assumption, a unit object is compact. Recall the following result which follows from [27, 5.3.6.8] and [28, 6.3.1.10].

Proposition 4.7. *Let* $\mathrm{Map}^{\otimes,\mathrm{L}}(\mathcal{C}^{\otimes},\mathcal{D}^{\otimes})$ *be* $\mathrm{Map}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}^{\mathrm{L,st}}_{\infty})}(\mathcal{C}^{\otimes},\mathcal{D}^{\otimes})$*, and let* $\mathrm{Map}^{\otimes,\mathrm{ex}}(\mathcal{C}^{\otimes}_{\circ},\mathcal{D}^{\otimes})$ *be the full subcategory of* $\mathrm{Map}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\infty})}(\mathcal{C}^{\otimes}_{\circ},\mathcal{D}^{\otimes})$ *spanned by symmetric monoidal functors which preserve finite colimits. The natural inclusion* $\mathcal{C}^{\otimes}_{\circ}\to\mathcal{C}^{\otimes}$ *induces an equivalence*

$$\mathrm{Map}^{\otimes,\mathrm{L}}(\mathcal{C}^{\otimes},\mathcal{D}^{\otimes})\to\mathrm{Map}^{\otimes,\mathrm{ex}}(\mathcal{C}^{\otimes}_{\circ},\mathcal{D}^{\otimes})$$

in $\widehat{\mathcal{S}}$*.*

Let us recall the definition of perfectness of stacks introduced by Ben-Zvi, Francis, and Nadler in their work on derived Morita theory [2] (this notion is also important to our previous paper [13]). We say that a derived stack Y is *perfect* if the natural functor $\mathrm{Ind}(\mathrm{PMod}_Y)\to\mathrm{Mod}_Y$ is a categorical equivalence. As a corollary of the results of this section, we have:

Theorem 4.8. *Let* Y *be a perfect derived stack over* R*, and* $\pi:\mathrm{Spec}\,R\to Y$ *a section of the structure morphism* $Y\to\mathrm{Spec}\,R$*. Let* $\pi^*:\mathrm{Mod}^{\otimes}_Y\to\mathrm{Mod}^{\otimes}_R$ *be the morphism in* $\mathrm{CAlg}(\widehat{\mathrm{Cat}}^{\mathrm{L,st}}_{\infty})$ *induced by* $\pi:\mathrm{Spec}\,R\to Y$*, and* $\pi^*_{\circ}:\mathrm{PMod}^{\otimes}_Y\to\mathrm{PMod}^{\otimes}_R$ *its restriction which belongs to* $\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{st}}_{\infty})$*. Let* $\mathrm{Aut}(\pi^*_{\circ}):\mathrm{CAlg}_R\to\mathrm{Grp}(\mathcal{S})$ *be the automorphism functor of* $\pi^*_{\circ}$*. Then the restriction induces an equivalence of functors* $\mathrm{Aut}(\pi^*)\to\mathrm{Aut}(\pi^*_{\circ})$*. In particular, the tannakization of* $\pi^*_{\circ}:\mathrm{PMod}^{\otimes}_Y\to\mathrm{PMod}^{\otimes}_R$ *is equivalent to* G_{π} *(see the setup before Proposition 4.6 for the notation* G_{π}*).*

Proof. Combine Propositions 4.6 and 4.7. □

Corollary 4.9. *Let* Y *be a derived stack over* R *equipped with* $\pi:\mathrm{Spec}\,R\to Y$ *as in Theorem 4.8. Suppose that either*

(i) Y *belongs to* Aff_R*, or*

(ii) $Y=[X/G]$ *is the quotient stack (see Example 4.1) where* G *is an affine group scheme of finite type over a field* k *of characteristic zero, which we regard as a derived affine group scheme over* $R=Hk$*, and we suppose that* G *acts on* $X\in\mathrm{Aff}_R$*.*

Then the tannakization of $\pi^*_{\circ}:\mathrm{PMod}^{\otimes}_Y\to\mathrm{PMod}^{\otimes}_R$ *is equivalent to* G_{π}*.*

Proof. According to Proposition 4.6 and Theorem 4.8, it will suffice to show that Y is perfect, that is, the natural functor $\mathrm{Ind}(\mathrm{PMod}_Y)\to\mathrm{Mod}_Y$ is a categorical equivalence. This follows from [2, 3.19, 3.22]. □

§5. Bar constructions

This section contains no new result. We review the relation between bar constructions and case (i) of Corollary 4.9. Let $A \in \mathrm{CAlg}_R$ and let $s : R \to A$ be the natural morphism in CAlg_R (note that R is an initial object in CAlg_R). Suppose that $t : A \to R$ is a *cosection* of s, that is, $t \circ s$ is equivalent to the identity of R. Recall that Δ_+ is the category of finite (possibly empty) linearly ordered sets and we write $[-1]$ for the empty set. Let $\iota : \Delta^1 \to \mathrm{N}(\Delta_+)$ be the map which carries $\{0\}$ and $\{1\}$ to $[-1]$ and $[0]$ respectively. It is a fully faithful functor. Let $f : \Delta^1 \to \mathrm{CAlg}_R$ be the map corresponding to $A \to R$. Since CAlg_R admits small colimits, there is a left Kan extension

$$g : \mathrm{N}(\Delta_+) \to \mathrm{CAlg}_R$$

of f along ι. We refer to $g^{\mathrm{op}} : \mathrm{N}(\Delta_+)^{\mathrm{op}} \to \mathrm{Aff}_R$ as the Čech nerve of $f^{\mathrm{op}} : (\Delta^1)^{\mathrm{op}} \to \mathrm{Aff}_R$. This construction is called the *bar construction* for $t : A \to R$. The underlying simplicial object $\mathrm{N}(\Delta)^{\mathrm{op}} \to \mathrm{N}(\Delta_+)^{\mathrm{op}} \to \mathrm{Aff}_R$ is a group object. Let G be a derived affine group scheme corresponding to the simplicial object.

Let $t_\circ^* : \mathrm{PMod}_A^\otimes \to \mathrm{PMod}_R^\otimes$ be the morphism in $\mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{st}})_{\mathrm{PMod}_R^\otimes /}$. Case (i) of Corollary 4.9 says:

Theorem 5.1. $\mathrm{Aut}(t_\circ^*)$ *is represented by* G.

Remark 5.2. For the readers who are familiar with commutative differential graded algebras (dg-algebras for short), we relate the bar construction of commutative dg-algebras to G. Let k be a field of characteristic zero. Let dga_k be the category of commutative dg-algebras over k (cf. [16]). A morphism $P^\bullet \to Q^\bullet$ in dga_k is a weak equivalence (resp. fibration) if it induces a bijection $H^n(P^\bullet) \to H^n(Q^\bullet)$ for each $n \in \mathbb{Z}$ (resp. $P^n \to Q^n$ is a surjective morphism of k-vector spaces for each $n \in \mathbb{Z}$). There is a model category structure on dga_k whose weak equivalences and fibrations are defined in this way (see [16, 2.2.1]). Let $\mathrm{N}(\mathsf{dga}_k^c)_\infty$ be the ∞-category obtained from the full subcategory dga_k^c spanned by cofibrant objects by inverting weak equivalences (see [28, 1.3.4.15]). According to [28, 8.1.4.11], there is a categorical equivalence $\mathrm{N}(\mathsf{dga}_k^c)_\infty \simeq \mathrm{CAlg}_{Hk}$. Let $R = Hk$ and let $t : A \to k$ be an augmentation in dga_k. Abusing notation, we denote by $t : A \to R$ the induced morphism in CAlg_R. The underlying derived scheme of G is the fiber product $\mathrm{Spec}\, R \times_{\mathrm{Spec}\, A} \mathrm{Spec}\, R$ in Aff_R. By this equivalence, the pushout $R \otimes_A R$ in CAlg_R corresponds to a homotopy pushout $k \otimes_A^{\mathbb{L}} k$ in the model category dga_k, which is weak equivalent to a homotopy pushout $A \otimes_{A \otimes_k A}^{\mathbb{L}} k$ of

$$\begin{array}{ccc} A \otimes_k A & \xrightarrow{t\otimes t} & k \\ \downarrow{\scriptstyle m} & & \\ A & & \end{array}$$

where m is multiplication. We will review the construction of the concrete model of a homotopy pushout $A \otimes^{\mathbb{L}}_{A\otimes_k A} k$ in dga_k, which is known as the *bar construction of a commutative dg-algebra* (see for example [31], [36]). Consider the adjoint pair

$$T : \mathsf{dga}_{k,A/} \rightleftarrows \mathsf{dga}_{k,A\otimes_k A/} : U$$

where U is the forgetful functor induced by $A \to A \otimes_k A$, $x \mapsto x \otimes 1$, and T is given by $M \mapsto M \otimes_A (A \otimes_k A)$. Let $\alpha : \mathrm{Id} \to UT$ and $\beta : TU \to \mathrm{Id}$ be the unit map and counit map respectively. To an object $C \in \mathsf{dga}_{k,A\otimes_k A/}$ one associates a simplicial diagram $(T,U)_\bullet(C)$ in $\mathsf{dga}_{k,A/}$ as follows: Define

$$(T,U)_n(C) = (TU)^{\circ(n+1)}(C) = (TU) \circ \cdots \circ (TU)(C)$$

(an $(n+1)$-fold composition). For $0 \le i \le n+1$,

$$\begin{aligned} d_i : (T,U)_{n+1}(C) = (TU)^{\circ i} \circ (TU) \circ (TU)^{\circ(n+1-i)}(C) \\ \to (TU)^{\circ i} \circ \mathrm{Id} \circ (TU)^{\circ(n+1-i)}(C) = (T,U)_n(C) \end{aligned}$$

is induced by β in the middle term. For $0 \le i \le n$,

$$\begin{aligned} s_i : (T,U)_n(C) = (TU)^{\circ i} \circ T \circ \mathrm{Id} \circ U \circ (TU)^{\circ(n-i)}(C) \\ \to (TU)^{\circ i} \circ T \circ (UT) \circ U \circ (TU)^{\circ(n-i)}(C) = (T,U)_{n+1}(C) \end{aligned}$$

is induced by $\alpha : \mathrm{Id} \to (UT)$ in the middle term. Let us consider A to be an object in $\mathsf{dga}_{k,A\otimes_k A/}$ via $m : A \otimes_k A \to A$. Then by the above construction we obtain the simplicial object $(T,U)_\bullet(A) \otimes_{A\otimes_k A} k$ in dga_k. Its totalization $\mathrm{tot}((T,U)_\bullet(A) \otimes_{A\otimes_k A} k) \in \mathsf{dga}_k$, which we call the *bar complex*, represents the homotopy pushout $A \otimes^{\mathbb{L}}_{A\otimes_k A} k$.

Let us explain how usual bar constructions (which are computable by means of bar spectral sequences) appear in applications we have in mind. A typical application of Corollary 4.9 goes as follows: Given a symmetric monoidal stable (presentable) ∞-category $\mathcal{C}^\otimes$ we first show that $\mathcal{C}^\otimes$ is equivalent to $\mathrm{Mod}_X^\otimes$ where X is a quotient stack of the form $[\operatorname{Spec} A/G]$. It is natural to have such a presentation when we deal with ∞-categories of tannakian nature (e.g., see the next section, or [20]). Suppose that they are all defined over Hk. If $\omega : \mathcal{C}^\otimes \to \mathrm{Mod}_{Hk}^\otimes$ is a symmetric monoidal (colimit-preserving) functor, we can construct a point

$\pi : \operatorname{Spec} Hk \to [\operatorname{Spec} A/G]$ such that ω can be identified with the pullback functor $\pi^* : \mathrm{Mod}_X^{\otimes} \to \mathrm{Mod}_{Hk}^{\otimes}$. Then we apply Corollary 4.9 to deduce that $\mathrm{Aut}(\omega)$ is representable by G_π. In many interested cases, we can further take a point $\tilde{\pi}$: $\operatorname{Spec} Hk \to \operatorname{Spec} A$ such that π decomposes into $\operatorname{Spec} Hk \xrightarrow{\tilde{\pi}} \operatorname{Spec} A \to [\operatorname{Spec} A/G]$ where the second morphism is the natural projection. This gives rise to a fiber sequence (pullback square) of derived affine group schemes

$$\begin{array}{ccc} G_{\tilde{\pi}} & \longrightarrow & G_\pi \\ \downarrow & & \downarrow \\ \operatorname{Spec} Hk & \longrightarrow & G \end{array}$$

When k is the field of characteristic zero and we regard A as a commutative differential graded algebra, $G_{\tilde{\pi}} = \operatorname{Spec} Hk \otimes_A Hk$ can be described by the bar complex $\mathrm{tot}((T,U)_\bullet(A) \otimes_{A \otimes_k A} k)$.

§6. Mixed Tate motives

In this section, as an application of the results we have proved, in particular Theorem 4.8 and Corollary 4.9, we will describe the tannakization of the stable ∞-category of mixed Tate motives equipped with the realization functor as the $\mathbb{G}_{\mathrm{m}}$-equivariant bar construction of a commutative dg-algebra. The main goal of this section is Theorem 6.12. We emphasize that we do not assume the Beilinson–Soulé vanishing conjecture. In what follows, we often use model categories. Our references for them are [18] and [27, Appendix].

§6.1. Review of the ∞-category of mixed motives

Let $\mathbf{K}$ be a field of characteristic zero. Let $\mathcal{A}$ be the abelian category of $\mathbf{K}$-vector spaces. We equip the category of complexes of $\mathbf{K}$-vector spaces, denoted by $\mathrm{Comp}(\mathcal{A})$, with the projective model structure, in which weak equivalences are quasi-isomorphisms, and fibrations are degreewise surjective maps (cf. e.g. [18, Section 2.3], [27, Appendix], [6]).

Let k be a perfect field. Let $\mathrm{DM}^{\mathrm{eff}}(k)$ be the category of complexes of $\mathcal{A}$-valued Nisnevich sheaves with transfers (see [30] and [9]). For a smooth scheme X separated of finite type over k, we denote by $L(X)$ the $\mathcal{A}$-valued Nisnevich sheaf with transfer which is represented by X (cf. [30, p. 15]). We equip $\mathrm{DM}^{\mathrm{eff}}(k)$ with the symmetric monoidal model structure in [6, Example 4.12]. The triangulated subcategory of the homotopy category of this model category $\mathrm{DM}^{\mathrm{eff}}(k)$, spanned by right bounded complexes, is equivalent to the triangulated category $\mathbf{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k,\mathbf{K})$ constructed in [30, Lecture 14].

The pointed algebraic torus $\mathrm{Spec}(k) \to \mathbb{G}_{\mathrm{m}}$ over k induces a split monomorphism $L(\mathrm{Spec}(k)) \to L(\mathbb{G}_{\mathrm{m}})$ in $\mathrm{DM}^{\mathrm{eff}}(k)$. Then we define $\mathbf{K}(1)$ to be

$$\mathrm{Coker}\big(L(\mathrm{Spec}(k)) \to L(\mathbb{G}_{\mathrm{m}})\big)[-1].$$

Let $\mathrm{DM}(k)$ be the category of symmetric $\mathbf{K}(1)$-spectra in $(\mathrm{DM}^{\mathrm{eff}}(k))^{\mathfrak{S}}$ (cf. [6, Section 7]) which is endowed with a symmetric monoidal model structure in [6, Example 7.15]. Then we have a sequence of left Quillen symmetric monoidal functors

$$\mathrm{Comp}(\mathcal{A}) \to \mathrm{DM}^{\mathrm{eff}}(k) \xrightarrow{\Sigma^\infty} \mathrm{DM}(k),$$

where the first functor sends the unit to $L(\mathrm{Spec}(k))$, and the second is the infinite suspension functor.

Recall the localization method in [28, 1.3.4.1, 1.3.1.15, 4.1.3.4] (see also [12], [19, Section 5]); it associates to any (symmetric monoidal) model category $\mathbb{M}$ a (symmetric monoidal) ∞-category $\mathrm{N}(\mathbb{M}^c)_\infty$. Here $\mathbb{M}^c$ is the full subcategory spanned by cofibrant objects (this restriction is due to a technical reason in the construction of symmetric monoidal ∞-categories). We shall refer to the associated (symmetric monoidal) ∞-category as the (symmetric monoidal) ∞-category obtained from the model category $\mathbb{M}$ *by inverting weak equivalences*. Applying this localization, we obtain symmetric monoidal functors of symmetric monoidal ∞-categories

$$\mathrm{Mod}^{\otimes}_{H\mathbf{K}} \simeq \mathrm{N}(\mathrm{Comp}(\mathcal{A})^c)_\infty \to \mathrm{N}(\mathrm{DM}^{\mathrm{eff}}(k)^c)_\infty \to \mathrm{N}(\mathrm{DM}(k)^c)_\infty$$

where the first equivalence follows from [28, 8.1.2.13]. Here $H\mathbf{K}$ denotes the Eilenberg–MacLane spectrum. We shall write DM and $\mathsf{DM}^{\mathrm{eff}}$ for $\mathrm{N}(\mathrm{DM}(k)^c)_\infty$ and $\mathrm{N}(\mathrm{DM}^{\mathrm{eff}}(k)^c)_\infty$ respectively. When we wish to indicate that DM is a symmetric monoidal ∞-category, we denote it by $\mathsf{DM}^{\otimes}$. The functor $\mathrm{Mod}^{\otimes}_{H\mathbf{K}} \to \mathsf{DM}^{\otimes}$ is considered to be an $H\mathbf{K}$-linear structure. For our proof of Theorem 6.12, the $H\mathbf{K}$-structure is not needed. But $H\mathbf{K}$-linear structures are useful in other situations, thus we will take into accout such structures in some lemmata and propositions. In [19, Section 5] we have constructed another symmetric monoidal stable presentable ∞-category $\mathsf{Sp}^{\otimes}_{\mathrm{Tate}}(\mathbf{HK})$ by using the recipe in [7] and [32]. We do not recall the construction; we just mention that there is an equivalence $\mathsf{DM}^{\otimes} \simeq \mathsf{Sp}^{\otimes}_{\mathrm{Tate}}(\mathbf{HK})$ (cf. [28, Remark 6.6]).

It should be emphasized that there are several (quite different but equivalent) constructions of the category of mixed motives as differential graded categories and model categories. One can obtain ∞-categories from differential graded categories and model categories. In our work, it is important to treat "the category of mixed motives" as a *symmetric monoidal* ∞-category, and therefore we choose the symmetric monoidal model category $\mathrm{DM}(k)$ constructed by Cisinski–Déglise.

§6.2. ∞-category of mixed Tate motives

Let us recall the stable ∞-category of mixed Tate motives. We also denote by $\mathbf{K}(1)$ the image of $\mathbf{K}(1) \in \mathrm{DM}^{\mathrm{eff}}(k)$ in $\mathrm{DM}(k)$. It is a cofibrant object and $\mathbf{K}(1)$ can be regarded as an object in the ∞-category DM. There exists the dual object of $\mathbf{K}(1)$ in DM, which we will denote by $\mathbf{K}(-1)$. Let DTM be the presentable stable subcategory generated by $\mathbf{K}(1)^{\otimes n} = \mathbf{K}(n)$ for $n \in \mathbb{Z}$, where $\mathbf{K}(1)^{\otimes n}$ is the n-fold tensor product in $\mathsf{DM}^{\otimes}$. Namely, DTM is the smallest stable subcategory in DM, which admits coproducts (thus all small colimits) and contains $\mathbf{K}(n)$ for all $n \in \mathbb{Z}$. The tensor product functor $\otimes : \mathsf{DM} \times \mathsf{DM} \to \mathsf{DM}$ preserves small colimits and translations (suspensions and loops) separately in each variable, and thus the symmetric monoidal structure of DM induces a symmetric monoidal structure on DTM. We denote by $\mathsf{DTM}^{\otimes}$ the resulting symmetric monoidal stable presentable ∞-category. Note that the inclusion $\mathsf{DTM} \hookrightarrow \mathsf{DM}$ preserves small colimits. Let $\mathsf{DTM}_{\mathrm{gm}}$ be the smallest stable subcategory containing $\mathbf{K}(n)$ for $n \in \mathbb{Z}$. Since $\mathbf{K}(n)$ is compact in DM for every $n \in \mathbb{Z}$, every object in $\mathsf{DTM}_{\mathrm{gm}}$ is compact in DM. Let $\mathrm{Ind}(\mathsf{DTM}_{\mathrm{gm}}) \to \mathsf{DTM}$ be a (colimit-preserving) left Kan extension of $\mathsf{DTM}_{\mathrm{gm}} \to \mathsf{DTM}$, which is fully faithful by [27, 5.3.5.11]. Hence it identifies $\mathrm{Ind}(\mathsf{DTM}_{\mathrm{gm}})$ with DTM. The symmetric monoidal functor $\mathrm{Mod}^{\otimes}_{H\mathbf{K}} \to \mathsf{DM}^{\otimes}$ factors through $\mathsf{DTM}^{\otimes} \subset \mathsf{DM}^{\otimes}$ since $\mathsf{DTM}^{\otimes} \hookrightarrow \mathsf{DM}^{\otimes}$ preserves small colimits, and DTM contains the unit of DM. The factorization $\mathrm{Mod}^{\otimes}_{H\mathbf{K}} \to \mathsf{DTM}^{\otimes} \hookrightarrow \mathsf{DM}^{\otimes}$ is regarded as a map in $\mathrm{CAlg}(\widehat{\mathrm{Cat}}^{\mathrm{L,st}}_{\infty})_{\mathrm{Mod}^{\otimes}_{H\mathrm{K}}/}$, which we also denote by $\mathsf{DTM}^{\otimes} \hookrightarrow \mathsf{DM}^{\otimes}$.

Lemma 6.1. *Let $\mathsf{DTM}_{\vee}$ be the full subcategory of $\mathsf{DTM}^{\otimes}$ spanned by dualizable objects. Let $\mathsf{DTM}_{\circ}$ be the full subcategory of DTM spanned by compact objects. Then $\mathsf{DTM}_{\circ} = \mathsf{DTM}_{\vee}$.*

Proof. Observe that every object in $\mathsf{DTM}_{\vee}$ is compact in DTM. To see this, it is enough to show that the unit object of $\mathsf{DTM}^{\otimes}$ is compact (cf. [7, 2.5.1]). This follows from [7, Theorem 2.7.10]. For any $n \in \mathbb{Z}$, $\mathbf{K}(n)$ belongs to $\mathsf{DTM}_{\vee}$. Therefore $\mathsf{DTM}_{\mathrm{gm}} \subset \mathsf{DTM}_{\vee} \subset \mathsf{DTM}_{\circ}$. Notice that $\mathsf{DTM}_{\mathrm{gm}} \subset \mathsf{DTM}_{\circ}$ can be viewed as an idempotent completion (see e.g. [5, Lemma 2.14]). Moreover DTM is idempotent complete by [27, 4.4.5.16]. It will suffice to prove that the inclusion $\mathsf{DTM}_{\vee} \subset \mathsf{DTM}$ is closed under retracts. This easily follows from the definition of dualizable objects. □

Let $\prod_S \mathrm{DM}$ be a product of the category DM, indexed by a small set S. There is a combinatorial model structure on $\prod_S \mathrm{DM}$, called the projective model structure (cf. [27, A. 2.8.2]), in which weak equivalences (resp. fibrations) are termwise weak equivalences (resp. termwise fibrations) in DM. Notice that cofibrations in

$\prod_S \mathrm{DM}$ are termwise cofibrations. When $S = \mathbb{N}$, $\prod_{\mathbb{N}} \mathrm{DM}$ has a symmetric monoidal structure defined as follows: Let $(M_i)_{i\in\mathbb{N}}$ and $(N_j)_{j\in\mathbb{N}}$ be two objects in $\prod_{\mathbb{N}} \mathrm{DM}$. Then $(M_i)_{i\in\mathbb{N}} \otimes (N_j)_{j\in\mathbb{N}}$ is defined to be $(\bigoplus_{i+j=k} M_i \otimes N_j)_{k\in\mathbb{N}}$.

Lemma 6.2. *With the above symmetric monoidal structure, $\prod_{\mathbb{N}} \mathrm{DM}$ is a symmetric monoidal model category in the sense of* [27, A 3.1.2].

Proof. We must prove that cofibrations $\alpha : (M_i) = (M_i)_{i\in\mathbb{N}} \to (M_i) = (M'_i)_{i\in\mathbb{N}}$ and $\beta : (N_i) = (N_i)_{i\in\mathbb{N}} \to (N_i) = (N'_i)_{i\in\mathbb{N}}$ induce a cofibration

$$\alpha \wedge \beta : (M_i) \otimes (N'_i) \amalg_{(M_i)\otimes(N_i)} (M'_i) \otimes (N_i) \to (M'_i) \otimes (N'_i),$$

and moreover if either α or β is a trivial cofibration, then so is $\alpha \wedge \beta$. Unwinding the definition, we are reduced to showing that

$$\bigoplus_{i+j=k} (M_i \otimes N'_j \amalg_{M_i\otimes N_j} M'_i \otimes N_j) \to \bigoplus_{i+j=k} M'_i \otimes N'_j$$

is a cofibration in DM, and moreover it is a trivial cofibration if either α or β. This follows from the left lifting property of (trivial) cofibrations and the fact that DM is a symmetric monoidal model category. □

Consider the symmetric monoidal functor $\xi : \prod_{\mathbb{N}} \mathrm{DM} \to \mathrm{DM}$, which carries (M_i) to $\bigoplus_i M_i \otimes \mathbf{K}(-i)$. Here $\mathbf{K}(-1)$ is a cofibrant "model" of the dual of $\mathbf{K}(1)$, and $\mathbf{K}(-i)$ is the i-fold tensor product of $\mathbf{K}(-1)$ in the symmetric monoidal category DM. Since $\mathbf{K}(-i)$ is cofibrant, we see that ξ is a left Quillen adjoint functor. By localization, we obtain a symmetric monoidal left adjoint functor

$$f := \mathrm{N}(\xi) : \mathsf{DM}_{\mathbb{N}}^{\otimes} := \mathrm{N}\Big(\Big(\prod_{\mathbb{N}} \mathrm{DM}\Big)^c\Big)_\infty \to \mathrm{N}(\mathrm{DM}^c)_\infty = \mathsf{DM}^{\otimes}.$$

By the relative version of the adjoint functor theorem [28, 8.3.2.6] (see also [29, VIII 3.2.1]), f has a lax symmetric monoidal right adjoint functor which we denote by $g : \mathsf{DM}^{\otimes} \to \mathsf{DM}_{\mathbb{N}}^{\otimes}$. It yields $g : \mathrm{CAlg}(\mathsf{DM}^{\otimes}) \to \mathrm{CAlg}(\mathsf{DM}_{\mathbb{N}}^{\otimes})$. We set $A := g(1_{\mathsf{DM}})$ in $\mathrm{CAlg}(\mathsf{DM}_{\mathbb{N}}^{\otimes})$, where 1_{DM} is a unit in $\mathsf{DM}^{\otimes}$. The adjoint pair

$$f : \mathsf{DM}_{\mathbb{N}} \rightleftarrows \mathsf{DM} : g$$

induces the adjoint pair

$$f : \mathrm{h}(\mathsf{DM}_{\mathbb{N}}) \rightleftarrows \mathrm{h}(\mathsf{DM}) : g$$

of homotopy categories. Let $\mathsf{Hom}(N, -)$ denote the internal Hom object given by the right adjoint of $(-) \otimes N : \mathsf{DM} \to \mathsf{DM}$. Then g is given by $M \mapsto (\mathsf{Hom}(\mathbf{K}(-i), M))_{i\in\mathbb{N}}$. Thus the underlying object A in $\mathrm{h}(\mathsf{DM})$ is $(\mathbf{K}(i))_{i\in\mathbb{N}}$, that is, the i-th term is $\mathbf{K}(i)$. Moreover, by straightforward calculation of adjunction maps, we see that the commutative algebra structure of A in the symmetric monoidal homotopy category $\mathrm{h}(\mathsf{DM})$ is given by

$$(\mathbf{K}(i))_{i\in\mathbb{N}}\otimes(\mathbf{K}(j))_{j\in\mathbb{N}}=\Big(\bigoplus_{i+j=k}\mathbf{K}(i)\otimes\mathbf{K}(j)\Big)_{k\in\mathbb{N}}\to(\mathbf{K}(k))_{k\in\mathbb{N}}$$

where the right-hand map is induced by the identity maps $\mathbf{K}(i)\otimes\mathbf{K}(j)\simeq\mathbf{K}(k)\to\mathbf{K}(k)$.

Now recall from [34] the notion of "periodic" commutative ring object (in [34] the notion of "periodizable" is introduced, and we use this notion in a slightly modified form). Let $\prod_{\mathbb{Z}}\mathrm{DM}$ be the product of DM indexed by $\mathbb{Z}$, which is a combinatorial model category defined as above. With the tensor product $(M_i)_{i\in\mathbb{Z}}\otimes(N_j)_{j\in\mathbb{Z}}=(\bigoplus_{i+j=k}M_i\otimes N_j)_{k\in\mathbb{Z}}$, $\prod_{\mathbb{Z}}\mathrm{DM}$ is a symmetric monoidal model category in the same way as $\prod_{\mathbb{N}}\mathrm{DM}$ is. Let $\mathsf{DM}^{\otimes}_{\mathbb{Z}}$ be the symmetric monoidal ∞-category obtained from $(\prod_{\mathbb{Z}}\mathrm{DM})^c$ by inverting weak equivalences. A commutative algebra object X in $\mathsf{DM}^{\otimes}_{\mathbb{Z}}$ is said to be *periodic* if the underlying object is of the form

$$(\ldots,\mathbf{K}(-1),\mathbf{K}(0),\mathbf{K}(1),\ldots),$$

that is, $\mathbf{K}(i)$ sits in the i-th degree, and the commutative algebra structure of X in $\mathrm{h}(\mathsf{DM}^{\otimes}_{\mathbb{Z}})$ induced by that in $\mathsf{DM}^{\otimes}_{\mathbb{Z}}$ is determined by the identity maps $\mathbf{K}(i)\otimes\mathbf{K}(j)\to\mathbf{K}(i+j)$.

A periodic commutative algebra object actually exists. To construct it, we let $i:\mathsf{DM}^{\otimes}_{\mathbb{N}}\to\mathsf{DM}^{\otimes}_{\mathbb{Z}}$ be the symmetric monoidal functor informally given by $(M_i)_{i\in\mathbb{N}}\mapsto(\ldots,0,0,M_0,M_1,\ldots)$, that is, determined by inserting 0 in each negative degree. Then $P_+:=i(A)$ belongs to $\mathrm{CAlg}(\mathsf{DM}^{\otimes}_{\mathbb{Z}})$. According to [34, Proposition 4.2] and its proof, we have:

Proposition 6.3 ([34]). *There exists a morphism $P_+\to P$ in $\mathrm{CAlg}(\mathsf{DM}^{\otimes}_{\mathbb{Z}})$ such that P is periodic.*

Remark 6.4. Let $\mathbf{K}(1)_1$ be the object of the form $(\ldots,0,\mathbf{K}(1),0,\ldots)$ where $\mathbf{K}(1)$ sits in the first degree. Let $\mathrm{Sym}^*_{P_+}:\mathrm{Mod}_{P_+}(\mathsf{DM}^{\otimes}_{\mathbb{Z}})\to\mathrm{CAlg}(\mathrm{Mod}^{\otimes}_{P_+}(\mathsf{DM}^{\otimes}_{\mathbb{Z}}))$ be the left adjoint of the forgetful functor. Let

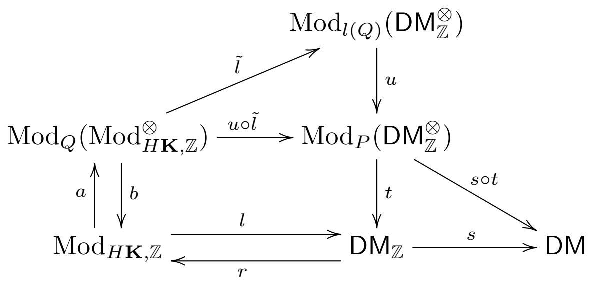
$$\mathrm{CAlg}(\mathrm{Mod}^{\otimes}_{P_+}(\mathsf{DM}^{\otimes}_{\mathbb{Z}}))\rightleftarrows\mathrm{CAlg}(\mathrm{Mod}^{\otimes}_{P_+}(\mathsf{DM}^{\otimes}_{\mathbb{Z}}))[\mathrm{Sym}^*_{P_+}(\kappa)^{-1}]$$

be the localization adjoint pair (cf. [27, 5.2.7.2, 5.5.4]) which inverts $\mathrm{Sym}^*_{P_+}(\kappa)$, where $\kappa:\mathbf{K}(1)_1\otimes P_+\to P_+$ in $\mathrm{Mod}_{P_+}(\mathsf{DM}^{\otimes}_{\mathbb{Z}})$ is induced by the natural embedding $\mathbf{K}(1)_1\to P_+$ in the first degree. The morphism $P_+\to P$ is obtained as the unit map of this adjoint pair.

Let $\prod_{\mathbb{Z}}\mathrm{Comp}(\mathcal{A})$ be the product of the category $\mathrm{Comp}(\mathcal{A})$, which is endowed with the projective model structure. As in Lemma 6.2, we see that $\prod_{\mathbb{Z}}\mathrm{Comp}(\mathcal{A})$ is a symmetric monoidal model category, whose tensor product is given by $(A_i)_{i\in\mathbb{Z}}\otimes(B_j)_{j\in\mathbb{Z}}=(\bigoplus_{i+j=k}A_i\otimes B_j)_{k\in\mathbb{Z}}$. Then the natural left Quillen adjoint symmetric monoidal functor $\mathrm{Comp}(\mathcal{A})\to\mathrm{DM}$ naturally extends to a left Quillen adjoint symmetric monoidal functor $l:\prod_{\mathbb{Z}}\mathrm{Comp}(\mathcal{A})\to\prod_{\mathbb{Z}}\mathrm{DM}$. It gives rise to the symmetric monoidal left adjoint functor of ∞-categories

$$l:\mathrm{Mod}^{\otimes}_{\boldsymbol{H}\mathbf{K},\mathbb{Z}}:=\mathrm{N}\Big(\prod_{\mathbb{Z}}\mathrm{Comp}(\mathcal{A})^c\Big)^{\otimes}_{\infty}\to\mathsf{DM}^{\otimes}_{\mathbb{Z}}.$$

According to the relative version of the adjoint functor theorem [28, 8.3.2.6] (see also [29, VIII 3.2.1]), l has a lax symmetric monoidal right adjoint functor r. Let $Q:=r(P)\in\mathrm{CAlg}(\mathrm{Mod}^{\otimes}_{\boldsymbol{H}\mathbf{K},\mathbb{Z}})$. Let $\mathrm{DM}\to\prod_{\mathbb{Z}}\mathrm{DM}_{\mathbb{Z}}$ be the left Quillen symmetric monoidal functor which carries M to (M_i) where $M_0=M$ and $M_i=0$ if $i\neq 0$. Thus we have a symmetric monoidal functor $\mathsf{DM}\to\mathsf{DM}_{\mathbb{Z}}$, and again by the relative version of the adjoint functor theorem we obtain a lax symmetric monoidal functor $s:\mathsf{DM}_{\mathbb{Z}}\to\mathsf{DM}$ as the right adjoint. Therefore there exists a diagram of symmetric monoidal ∞-categories



such that

- $\tilde{l}$ is a symmetric monoidal functor induced by l,
- u is the symmetric monoidal base change functor induced by the counit map $l(Q)=l(r(P))\to P$,
- t is the forgetful monoidal functor which is a lax symmetric monoidal functor,
- a is the base change functor, and b is the forgetful functor.

Let $z:=s\circ t\circ u\circ\tilde{l}$. We recall the theorem by Spitzweck [34, Theorem 4.3] (see also its proof):

Theorem 6.5 ([34]). *The composite* $z:\mathrm{Mod}_Q(\mathrm{Mod}^{\otimes}_{\boldsymbol{H}\mathbf{K},\mathbb{Z}})\to\mathsf{DM}$ *gives an equivalence*

$$\mathrm{Mod}_Q(\mathrm{Mod}^{\otimes}_{\boldsymbol{H}\mathbf{K},\mathbb{Z}})\simeq\mathsf{DTM}$$

of symmetric monoidal ∞-categories.

Remark 6.6. This result is extended in [20] to a more general situation by a different method.

Furthermore, we can see that z gives an equivalence of the above categories as $H\mathbf{K}$-*linear* symmetric monoidal ∞-categories. To see this, it is enough to show that z can be promoted to an $H\mathbf{K}$-linear symmetric monoidal functor. To treat problems of this type, the following lemma is useful.

Lemma 6.7. *Let $\mathcal{C}^{\otimes}$ be in $\mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\infty}^{\mathrm{L,st}})$. Denote by $\mathcal{C}$ the underlying ∞-category. Suppose that a unit $\mathbf{1}$ of $\mathcal{C}^{\otimes}$ is compact in $\mathcal{C}$. Let $\mathcal{C}_{\mathbf{1}} \subset \mathcal{C}$ be the smallest stable subcategory which admits small colimits and contains $\mathbf{1}$. The ∞-category $\mathcal{C}_{\mathbf{1}}$ admits a symmetric monoidal structure induced by that of $\mathcal{C}^{\otimes}$. Then there exist A in* CAlg *and an equivalence $\mathrm{Mod}_A^{\otimes} \simeq \mathcal{C}^{\otimes}$ of symmetric monoidal ∞-categories. Moreover, if R is a commutative ring spectrum and $p : \mathrm{Mod}_R^{\otimes} \to \mathcal{C}^{\otimes}$ is a symmetric monoidal colimit-preserving functor, then p factors through $\mathcal{C}_{\mathbf{1}}^{\otimes} \subset \mathcal{C}^{\otimes}$ and there exists a morphism $R \to A$ in* CAlg*, up to the contractible space of choices, which induces* $\mathrm{Mod}_R^{\otimes} \to \mathcal{C}_{\mathbf{1}}^{\otimes} \simeq \mathrm{Mod}_A^{\otimes}$ *(as the base change).*

Proof. The first assertion follows from [28, 8.1.2.7], the characterization of symmetric monoidal stable ∞-categories of module spectra. Since p preserves small colimits, p factors through $\mathcal{C}_{\mathbf{1}}^{\otimes} \subset \mathcal{C}^{\otimes}$. The last assertion can be deduced from [28, 6.3.5.18]. □

Remark 6.8. Under the assumption of Lemma 6.7, A is considered to be the "endomorphism algebra" of the unit, and we can say that giving an R-linear structure, that is, a symmetric monoidal colimit-preserving functor $\mathrm{Mod}_R^{\otimes} \to \mathcal{C}^{\otimes}$, is equivalent to giving a morphism $R \to A$ in CAlg.

We return to the case of $H\mathbf{K}$-linear symmetric monoidal ∞-category $\mathsf{DTM}^{\otimes}$. The endomorphism algebra of the unit of $\mathsf{DTM}^{\otimes}$ is $H\mathbf{K}$ (i.e. $\mathbf{K}$), and its $H\mathbf{K}$-linear structure is determined by the identity $H\mathbf{K} \to H\mathbf{K}$. Thus, to promote z to an $H\mathbf{K}$-linear symmetric monoidal functor, it is enough to show that $f \circ a \circ q : \mathrm{Mod}_{H\mathbf{K}}^{\otimes} \to \mathsf{DTM}^{\otimes}$ induces the identity morphism $H\mathbf{K} \to H\mathbf{K}$ of the endomorphism algebras of units, where q is the inclusion $\mathrm{Mod}_{H\mathbf{K}}^{\otimes} \to \mathrm{Mod}_{H\mathbf{K},\mathbb{Z}}^{\otimes}$ into the degree zero part. This claim is clear from our construction.

§6.3. Realization functor and augmentation

Let E be a mixed Weil theory with $\mathbf{K}$-coefficients (cf. [7, Definition 2.1]). A mixed Weil theory is a presheaf of commutative dg $\mathbf{K}$-algebras on the category of smooth affine schemes over k, which satisfies the Nisnevich descent property, $\mathbb{A}^1$-homotopy, Künneth formula and axioms of dimensions, etc. (for a precise definition see [7, 2.1.4]). For example, algebraic de Rham cohomology determines a mixed Weil theory with $\mathbf{K} = k$: to any smooth affine scheme X we associate a commutative dg k-algebra $\Gamma(X, \Omega^*_{X/k})$ where $\Omega^*_{X/k}$ is the algebraic de Rham complex arising from the exterior $\mathcal{O}_X$-algebra generated by $\Omega^1_{X/k}$. Another example is l-adic étale cohomology with $\mathbf{K} = \mathbb{Q}_l$ (see [7, Section 3]). To a mixed Weil theory E we can associate a morphism

$$\mathsf{R}_E : \mathsf{DM}^{\otimes} \to \mathrm{Mod}^{\otimes}_{H\mathbf{K}}$$

in $\mathrm{CAlg}(\widehat{\mathrm{Cat}}^{\mathrm{L,st}}_{\infty})_{\mathrm{Mod}^{\otimes}_{H\mathbf{K}}/}$, which we call the *homological realization functor with respect to* E (see [19, Section 5.1, 5.2], [7, 2.6]). From now on we usually omit the subscript E. According to [7, 2.7.14], when E is the mixed Weil theory associated to algebraic de Rham cohomology, for any smooth affine scheme X the image $\mathsf{R}(h(X))$ in $\mathrm{Mod}_{H\mathbf{K}}$ is equivalent to the dual complex of derived global sections $\mathbf{R}\Gamma(X, \Omega^*_{X/k})$ where by [19, 5.10] we identify $\mathrm{Mod}_{H\mathbf{K}}$ with the ∞-category of unbounded complexes of $\mathbf{K}$-vector spaces. We denote by R_T the composition

$$\mathsf{DTM}^{\otimes} \hookrightarrow \mathsf{DM}^{\otimes} \to \mathrm{Mod}^{\otimes}_{H\mathbf{K}},$$

which we call the *homological realization of Tate motives* (with respect to E). By restrictions, it gives rise to a morphism $\mathsf{DTM}^{\otimes}_{\vee} \to \mathrm{PMod}^{\otimes}_{H\mathbf{K}}$ in $\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{st}}_{\infty})_{\mathrm{PMod}^{\otimes}_{H\mathbf{K}}/}$, which we also denote by R_T.

Combining this with Theorem 6.5 we have a sequence of symmetric monoidal colimit-preserving functors

$$\mathrm{Mod}^{\otimes}_{H\mathbf{K},\mathbb{Z}} \xrightarrow{a} \mathrm{Mod}^{\otimes}_{Q}(\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}}) \simeq \mathsf{DTM}^{\otimes} \xrightarrow{\mathsf{R}_T} \mathrm{Mod}^{\otimes}_{H\mathbf{K}}\,.$$

By the relative version of the adjoint functor theorem, the composition admits a lax symmetric monoidal right adjoint functor ξ. In particular, if we set $R = \xi(1_{H\mathbf{K}})$ with $1_{H\mathbf{K}}$ the unit of $\mathrm{Mod}^{\otimes}_{H\mathbf{K}}$, then R belongs to $\mathrm{CAlg}(\mathrm{Mod}^{\otimes}_{H\mathbf{K},\mathbb{Z}})$. By functoriality and the construction of Q, we have a natural morphism $Q \to R$ in $\mathrm{CAlg}(\mathrm{Mod}^{\otimes}_{H\mathbf{K},\mathbb{Z}})$. There is a commutative diagram (up to homotopy) of symmetric monoidal ∞-categories

$$\begin{array}{ccccc}
\mathrm{Mod}^{\otimes}_{Q}(\mathrm{Mod}^{\otimes}_{H\mathbf{K},\mathbb{Z}}) & \xrightarrow[z]{\sim} & \mathsf{DTM}^{\otimes} & \xrightarrow[\mathsf{R}_T]{} & \mathrm{Mod}^{\otimes}_{H\mathbf{K}} \\
\downarrow & & \downarrow & & \uparrow \\
\mathrm{Mod}^{\otimes}_{R}(\mathrm{Mod}^{\otimes}_{H\mathbf{K},\mathbb{Z}}) & \xrightarrow[\tilde{z}]{} & \mathrm{Mod}^{\otimes}_{f(R)}(\mathsf{DTM}^{\otimes}) & \xrightarrow[\tilde{\mathsf{R}}_T]{} & \mathrm{Mod}^{\otimes}_{\mathsf{R}_T(z(R))}(\mathrm{Mod}^{\otimes}_{H\mathbf{K}})
\end{array}$$

where $\tilde{z}$ and $\tilde{\mathsf{R}}_T$ are induced by z and R_T respectively, the left and central vertical arrows are base change functors, and the right vertical arrow is the counit map

$\mathsf{R}_T(z(R)) \to H\mathbf{K}$ in $\mathrm{CAlg}(\mathrm{Mod}^{\otimes}_{H\mathbf{K}})$. Note that all functors in the diagram are $H\mathbf{K}$-linear symmetric monoidal functors. The commutativity of the right square follows from the observation that the counit map $\mathsf{R}_T(z(R)) \to H\mathbf{K}$ is an augmentation of the structure map $H\mathbf{K} \to \mathsf{R}_T(z(R))$.

Lemma 6.9. *The composite $h : \mathcal{C}^{\otimes} := \mathrm{Mod}^{\otimes}_R(\mathrm{Mod}^{\otimes}_{H\mathbf{K},\mathbb{Z}}) \to \mathcal{D}^{\otimes} := \mathrm{Mod}^{\otimes}_{H\mathbf{K}}$ in the above diagram gives an equivalence of $H\mathbf{K}$-linear symmetric monoidal ∞-categories.*

Proof. It will suffice to show that the underlying functor is a categorical equivalence.

The symmetric monoidal functor h is $H\mathbf{K}$-linear. Thus h is essentially surjective.

Next we will show that h is fully faithful. Let $\mathbf{K}_n := (\dots, 0, \mathbf{K}, 0, \dots)$ be the object in $\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}}$ such that $\mathbf{K}$ sits in the n-th degree. Let $R(n)$ be the image of $\mathbf{K}_n$ by the base change functor $\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}} \to \mathrm{Mod}_R(\mathrm{Mod}^{\otimes}_{H\mathbf{K},\mathbb{Z}})$. (For any $n \in \mathbb{Z}$, $h(R(n)) \simeq H\mathbf{K}$.) It is enough to prove that

$$\mathrm{Map}_{\mathcal{C}}(R(i), R(j)) \to \mathrm{Map}_{\mathcal{D}}(h(R(i)), h(R(j)))$$

is an equivalence in $\mathcal{S}$. Indeed, $\mathcal{C}$ is generated by the sets $\{R(i)\}_{i\in\mathbb{Z}}$ under finite (co)limits, translations, and filtered colimits. Since $R(i)$ and $h(R(i))$ are compact for each $i \in \mathbb{Z}$ and h is colimit-preserving, we are reduced to showing that the above map is an equivalence in $\mathcal{S}$. (Assuming this to hold, note first that $\mathrm{Map}_{\mathcal{C}}(R(i), N) \to \mathrm{Map}_{\mathcal{D}}(h(R(i)), h(N))$ is an equivalence in $\mathcal{S}$ for N being in the smallest stable subcategory $\mathcal{C}'$ generated by $\{R(i)\}_{i\in\mathbb{Z}}$. Then since $R(i)$ and $h(R(i))$ are compact, $\mathrm{Ind}(\mathcal{C}') \simeq \mathcal{C}$, and h preserves small colimits, thus for any $N \in \mathcal{C}$, $\mathrm{Map}_{\mathcal{C}}(R(i), N) \to \mathrm{Map}_{\mathcal{D}}(h(R(i)), h(N))$ is an equivalence. Since $\mathcal{C}$ is generated by $\{R(i)\}_{i\in\mathbb{Z}}$ under finite colimits, translations and filtered colimits, we conclude that for any $M, N \in \mathcal{C}$, $\mathrm{Map}_{\mathcal{C}}(M, N) \to \mathrm{Map}_{\mathcal{D}}(h(M), h(N))$ is an equivalence.) Note that $\mathrm{Map}_{\mathcal{C}}(R(i), R(j)) \simeq \mathrm{Map}_{\mathcal{C}}(R(i-j), R)$, and therefore we may and will assume that $j = 0$. Then by using adjunctions we can identify $\mathrm{Map}_{\mathcal{C}}(R(i), R) \to \mathrm{Map}_{\mathcal{D}}(h(R(i)), h(R))$ with the composition

$$\begin{aligned}\mathrm{Map}_{\mathcal{C}}(R(i), R) &\xrightarrow{\sim} \mathrm{Map}_{\mathrm{Mod}_Q(\mathrm{Mod}^{\otimes}_{H\mathbf{K},\mathbb{Z}})}(Q(i), R)\\ &\xrightarrow{\sim} \mathrm{Map}_{\mathrm{Mod}_{H\mathbf{K}}}(\mathsf{R}_T(z(Q(i))), H\mathbf{K})\\ &\xrightarrow{\sim} \mathrm{Map}_{\mathrm{Mod}_{H\mathbf{K}}}(H\mathbf{K}, H\mathbf{K}).\end{aligned}$$

This proves our lemma. □

Proposition 6.10. *There exists an $H\mathbf{K}$-linear symmetric monoidal equivalence*

$$\mathrm{Mod}^{\otimes}_{H\mathbf{K},\mathbb{Z}} \to \mathrm{Mod}^{\otimes}_{\mathrm{B}\mathbb{G}_{\mathrm{m}}}.$$

Proof. We will construct a symmetric monoidal functor $\mathrm{Mod}^{\otimes}_{H\mathbf{K},\mathbb{Z}} \to \mathrm{Mod}^{\otimes}_{\mathrm{B}\mathbb{G}_{\mathrm{m}}}$ which preserves colimits.

For this purpose, we will construct $\mathrm{Mod}^{\otimes}_{\mathrm{B}\mathbb{G}_{\mathrm{m}}}$ in an explicit way. Regard the group scheme $\mathbb{G}_{\mathrm{m}}$ over $\mathbf{K}$ as a simplicial scheme, denoted by $G_{\bullet}$, such that G_i is the i-fold product $\mathbb{G}_{\mathrm{m}}^{\times i}$. This corresponds to the cosimplicial $\mathbf{K}$-algebra $\Gamma(G)^{\bullet}$ such that $\Gamma(G)^i \simeq \mathbf{K}[t_1^{\pm}, \ldots, t_i^{\pm}]$. The cosimplicial $\mathbf{K}$-algebra $\Gamma(G)^{\bullet}$ naturally induces a cosimplicial diagram $\rho : \mathrm{N}(\Delta) \to \widehat{\mathrm{Cat}}_{\infty}$ such that $\rho([i]) = \mathrm{N}(\mathrm{Comp}(\Gamma(G)^i)^c)$. Here $\mathrm{Comp}(\Gamma(G)^i)$ denotes the category of chain complexes of $\Gamma(G)^i$-modules which is endowed with the projective model structure, and $\mathrm{Comp}(\Gamma(G)^i)^c$ is its full subcategory of cofibrant objects. Each category $\mathrm{Comp}(\Gamma(G)^i)^c$ has a (natural) symmetric monoidal structure, and thus ρ is promoted to $\rho : \mathrm{N}(\Delta) \to \mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\infty})$, where $\mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\infty})$ is the ∞-category of symmetric monoidal ∞-categories (i.e., commutative algebra objects in the Cartesian symmetric monoidal ∞-category $\widehat{\mathrm{Cat}}_{\infty}$). The symmetric monoidal category $\mathrm{Comp}(\Gamma(G)^i)^c$ admits the subset of edges of weak equivalences. Inverting weak equivalences in $\mathrm{Comp}(\Gamma(G)^i)^c$, we have $\rho' : \mathrm{N}(\Delta) \to \mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\infty})$ and the natural transformation $\rho \to \rho'$ such that $\rho'([i])$ is the symmetric monoidal ∞-category obtained from $\mathrm{Comp}(\Gamma(G)^i)^c$ by inverting weak equivalences.

By the explicit unstraightening functor [27, 3.2.5.2], the maps $\rho, \rho' : \mathrm{N}(\Delta) \rightrightarrows \mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\infty})$ give rise to coCartesian fibrations $\mathcal{C}^{\otimes}_{\mathrm{pre}} \to \mathrm{N}(\mathrm{Fin}_*) \times \mathrm{N}(\Delta)$ and $\mathcal{C}^{\otimes} \to \mathrm{N}(\mathrm{Fin}_*) \times \mathrm{N}(\Delta)$. The natural transformation $\rho \to \rho'$ induces a map of coCartesian fibrations

$$\begin{array}{ccc} \mathcal{C}^{\otimes}_{\mathrm{pre}} & \xrightarrow{\quad\sigma\quad} & \mathcal{C}^{\otimes} \\ & \searrow \quad \swarrow & \\ & \mathrm{N}(\mathrm{Fin}_*) \times \mathrm{N}(\Delta) & \end{array}$$

which preserves coCartesian edges. Note that for each $[i] \in \Delta$, the fiber $\rho^{-1}([i]) \to \mathrm{N}(\mathrm{Fin}_*) \times \{[i]\} \cong \mathrm{N}(\mathrm{Fin}_*)$ is the symmetric monoidal ∞-category associated to the diagram of $\mathrm{Comp}(\Gamma(G)^i)^c$'s. The fiber $(\rho')^{-1}([i]) \to \mathrm{N}(\mathrm{Fin}_*)$ is the symmetric monoidal ∞-category obtained from $\mathrm{Comp}(\Gamma(G)^i)^c$ by inverting weak equivalences.

Next we define a map of simplicial sets $\overline{\mathrm{Sec}}(\mathcal{C}^{\otimes}_{\mathrm{pre}}) \to \mathrm{N}(\mathrm{Fin}_*)$ as follows. For any $a : T \to \mathrm{N}(\mathrm{Fin}_*)$, giving a map $T \to \overline{\mathrm{Sec}}(\mathcal{C}^{\otimes}_{\mathrm{pre}})$ over $\mathrm{N}(\mathrm{Fin}_*)$ amounts to giving $\phi : T \times \mathrm{N}(\Delta) \to \mathcal{C}^{\otimes}_{\mathrm{pre}}$ which commutes with $a \times \mathrm{Id} : T \times \mathrm{N}(\Delta) \to \mathrm{N}(\mathrm{Fin}_*) \times \mathrm{N}(\Delta)$ and $\mathcal{C}^{\otimes}_{\mathrm{pre}} \to \mathrm{N}(\mathrm{Fin}_*) \times \mathrm{N}(\Delta)$. Let $\mathrm{Sec}(\mathcal{C}^{\otimes}_{\mathrm{pre}})$ be the largest subcomplex of

$\overline{\mathrm{Sec}}(\mathcal{C}^{\otimes}_{\mathrm{pre}})$ which consists of the following vertexes: a vertex $v \in \overline{\mathrm{Sec}}(\mathcal{C}^{\otimes}_{\mathrm{pre}})$ lying over $\langle i\rangle$ belongs to $\mathrm{Sec}(\mathcal{C}^{\otimes}_{\mathrm{pre}})$ exactly when $v : \{\langle i\rangle\} \times \mathrm{N}(\Delta) \to \mathcal{C}^{\otimes}_{\mathrm{pre}}$ carries all edges in $\{\langle i\rangle\} \times \mathrm{N}(\Delta)$ to coCartesian edges in $\mathcal{C}^{\otimes}_{\mathrm{pre}}$. We define $\overline{\mathrm{Sec}}(\mathcal{C}^{\otimes}) \to \mathrm{N}(\mathrm{Fin}_*)$ and $\mathrm{Sec}(\mathcal{C}^{\otimes}) \to \mathrm{N}(\mathrm{Fin}_*)$ in a similar way. According to [27, 3.1.2.1(1)], we see that $\overline{\mathrm{Sec}}(\mathcal{C}^{\otimes}_{\mathrm{pre}}) \to \mathrm{N}(\mathrm{Fin}_*)$ and $\overline{\mathrm{Sec}}(\mathcal{C}^{\otimes}) \to \mathrm{N}(\mathrm{Fin}_*)$ are coCartesian fibrations (notice that $\overline{\mathrm{Sec}}(\mathcal{C}^{\otimes}_{\mathrm{pre}}) = \mathrm{N}(\mathrm{Fin}_*) \times_{\mathrm{Fun}(\mathrm{N}(\Delta),\mathrm{N}(\mathrm{Fin}_*)\times\mathrm{N}(\Delta))} \mathrm{Fun}(\mathrm{N}(\Delta), \mathcal{C}^{\otimes}_{\mathrm{pre}}))$ where $\mathrm{N}(\mathrm{Fin}_*) \to \mathrm{Fun}(\mathrm{N}(\Delta), \mathrm{N}(\mathrm{Fin}_*) \times \mathrm{N}(\Delta))$ is induced by the identity $\mathrm{N}(\mathrm{Fin}_*) \times \mathrm{N}(\Delta) \to \mathrm{N}(\mathrm{Fin}_*) \times \mathrm{N}(\Delta))$. Moreover, by [27, 3.1.2.1(2)], $\mathrm{Sec}(\mathcal{C}^{\otimes}_{\mathrm{pre}}) \to \mathrm{N}(\mathrm{Fin}_*)$ and $\mathrm{Sec}(\mathcal{C}^{\otimes}) \to \mathrm{N}(\mathrm{Fin}_*)$ are coCartesian fibrations. By construction, $\mathrm{Sec}(\mathcal{C}^{\otimes}_{\mathrm{pre}}) \to \mathrm{N}(\mathrm{Fin}_*)$ is furthermore a symmetric monoidal ∞-category. Since the procedure of inverting weak equivalences commutes with finite products [28, 4.1.3.2], we see that $\mathrm{Sec}(\mathcal{C}^{\otimes}) \to \mathrm{N}(\mathrm{Fin}_*)$ is also a symmetric monoidal ∞-category. We will abuse notation and denote by $\mathrm{Sec}(\mathcal{C}^{\otimes}_{\mathrm{pre}})$ and $\mathrm{Sec}(\mathcal{C}^{\otimes})$ the underlying ∞-categories. Note that σ (which preserves coCartesian edges) induces a symmetric monoidal functor $\mathrm{Sec}(\mathcal{C}^{\otimes}_{\mathrm{pre}}) \to \mathrm{Sec}(\mathcal{C}^{\otimes})$.

Observe that the symmetric monoidal ∞-category $\mathrm{Sec}(\mathcal{C}^{\otimes}) \to \mathrm{N}(\mathrm{Fin}_*)$ is equivalent to the symmetric monoidal ∞-category $\mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}}$. By [27, 3.3.3.2] and [28, 3.2.2.4], the symmetric monoidal ∞-category $\mathrm{Sec}(\mathcal{C}^{\otimes})$ is a limit of the diagram $\rho' : \mathrm{N}(\Delta) \to \mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\infty})$. Note that by [28, 8.1.2.13], $\rho'([i])$ is equivalent to $\mathrm{Mod}^{\otimes}_{\Gamma(G)^i}$. Moreover, the functor $\Theta : \mathrm{CAlg} \to \mathrm{CAlg}(\widehat{\mathrm{Cat}}^{\mathrm{L,st}}_{\infty})$ which carries A to $\mathrm{Mod}^{\otimes}_A$ (see Section 3.1) is fully faithful [28, 6.3.5.18]. For a symmetric monoidal functor $\phi : \mathrm{Mod}^{\otimes}_A \to \mathrm{Mod}^{\otimes}_B$ in $\mathrm{CAlg}(\widehat{\mathrm{Cat}}^{\mathrm{L,st}}_{\infty})$, one can recover $f : A \to B$ with $\Theta(f) \simeq \phi$ as the induced morphism from the endomorphism spectrum of a unit of $\mathrm{Mod}^{\otimes}_A$ to that of the unit in $\mathrm{Mod}^{\otimes}_B$. Therefore from the construction of ρ' (and ρ) and the definition of $\mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}}$, we conclude that $\mathrm{Sec}(\mathcal{C}^{\otimes}) \to \mathrm{N}(\mathrm{Fin}_*)$ is equivalent to $\mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}}$.

Therefore, to construct $\mathrm{Mod}^{\otimes}_{H\mathbf{K},\mathbb{Z}} \to \mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}}$ it suffices to construct a symmetric monoidal functor from $\prod_{\mathbb{Z}} \mathrm{Comp}(\mathcal{A})^c$ to $\mathrm{Sec}(\mathcal{C}^{\otimes}_{\mathrm{pre}})$ which carries weak equivalences in $\prod_{\mathbb{Z}} \mathrm{Comp}(\mathcal{A})^c$ to edges in $\mathrm{Sec}(\mathcal{C}^{\otimes}_{\mathrm{pre}})$ whose images in $\mathrm{Sec}(\mathcal{C}^{\otimes})$ are equivalences (note the universality of $\mathrm{Mod}^{\otimes}_{H\mathbf{K},\mathbb{Z}}$ [28, 4.1.3.4]). Let $\mathbf{K}_n$ in $\prod_{\mathbb{Z}} \mathrm{Comp}(\mathcal{A})^c$ be the $\mathbf{K}$ which sits in the n-th degree with respect to $\prod_{\mathbb{Z}}$. To $\mathbf{K}_n$ we attach the weight n representation of $\mathbb{G}_{\mathrm{m}}$ on $\mathbf{K}$. The weight n representation gives rise, in the obvious way, to an object of $\mathrm{Sec}(\mathcal{C}^{\otimes}_{\mathrm{pre}})$ which we denote by $\mathbf{K}'_n$. To $(M_i)_{i\in\mathbb{Z}} \in \prod_{\mathbb{Z}} \mathrm{Comp}(\mathcal{A})^c$, we attach $\bigoplus_{i\in\mathbb{Z}} M_i \otimes \mathbf{K}'_i$. Here we consider M_i to be an object in $\mathrm{Sec}(\mathcal{C}^{\otimes}_{\mathrm{pre}})$, that is, a complex endowed with the trivial action of $\mathbb{G}_{\mathrm{m}}$. This naturally induces a symmetric monoidal functor having the desired property. To prove that the induced functor $\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}} \to \mathrm{Mod}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}}$ preserves small colimits, it is enough to show that the composite $\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}} \to \mathrm{Mod}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}} \to \mathrm{Mod}_{H\mathbf{K}}$, where the second functor is forgetful, preserves small colimits since the forgetful functor is conservative and preserves small colimits (an exact functor $p : \mathcal{K} \to \mathcal{L}$ between stable ∞-categories is said to be *conservative* if for any $K \in \mathcal{K}$, $p(K) \simeq 0$ implies that $K \simeq 0$). The composite carries $(M_i)_{i\in\mathbb{Z}}$ to $\bigoplus_{i\in\mathbb{Z}} M_i$ and thus we conclude that the composite preserves small colimits. To prove that $\mathrm{Mod}^{\otimes}_{H\mathbf{K},\mathbb{Z}} \to \mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}}$ can be promoted to an $H\mathbf{K}$-linear symmetric monoidal functor, according to Lemma 6.7 (see also the discussion at the end of 6.3), it suffices to observe that $\mathrm{Mod}^{\otimes}_{H\mathbf{K},\mathbb{Z}} \to \mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}}$ induces the identity morphism $H\mathbf{K} \to H\mathbf{K}$ of the endomorphism algebras of units. To see this, we are reduced to showing that the composite $\mathrm{Mod}^{\otimes}_{H\mathbf{K},\mathbb{Z}} \to \mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}} \to \mathrm{Mod}^{\otimes}_{H\mathbf{K}}$, where the second functor is the forgetful functor, induces the identity morphism $H\mathbf{K} \to H\mathbf{K}$ of the endomorphism algebras of units. This is clear.

We have constructed a symmetric monoidal colimit-preserving functor $\mathrm{Mod}^{\otimes}_{H\mathbf{K},\mathbb{Z}} \to \mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}}$ with the (lax symmetric monoidal) right adjoint functor (the existence is ensured by the relative version of the adjoint functor theorem). To see that $\mathrm{Mod}^{\otimes}_{H\mathbf{K},\mathbb{Z}} \to \mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}}$ is an equivalence of symmetric monoidal ∞-categories, it is enough to show that it induces a categorical equivalence $\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}} \to \mathrm{Mod}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}}$ of the underlying ∞-categories. Moreover, by [19, 5.8], it suffices to check that it induces an equivalence $\mathrm{h}(\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}}) \to \mathrm{h}(\mathrm{Mod}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}})$ of their homotopy categories. The desired equivalence now follows from [35, Section 8, Theorem 8.5] (see also the strictification theorem [17, 18.7]). □

Let A be an object in $\mathrm{CAlg}(\mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}})$. Let $\overline{A}$ denote the image of A in $\mathrm{CAlg}(\mathrm{Mod}^{\otimes}_{H\mathbf{K}})$ (via the pullback of $\mathrm{Spec}\, H\mathbf{K} \to \mathsf{B}\mathbb{G}_{\mathrm{m}}$). With the notation in the proof of Proposition 6.10, there is a natural augmented simplicial diagram $G_{\bullet} \to \mathsf{B}\mathbb{G}_{\mathrm{m}}$. This induces a natural functor $\mathrm{CAlg}(\mathrm{Mod}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}}) \to \lim_{[i]\in\Delta} \mathrm{CAlg}(\mathrm{Mod}_{H\Gamma(G)^i})$. We write $A^{\bullet}$ for the image of A in $\lim_{[i]\in\Delta} \mathrm{CAlg}(\mathrm{Mod}_{H\Gamma(G)^i})$. It gives rise to the quotient stack $[\mathrm{Spec}\, \overline{A}/\mathbb{G}_{\mathrm{m}}]$ (see Example 4.1). The construction of the quotient derived stack is as follows: The cosimplicial diagram $\{\Gamma(G)^i\}_{[i]\in\Delta}$ of ordinary commutative $\mathbf{K}$-algebras has a natural map from the constant simplicial diagram $\{\mathbf{K}\}$. Both cosimplicial diagrams naturally induce $c_{\mathbf{K}}, c_G : \mathrm{N}(\Delta) \to \widehat{\mathrm{Cat}}_{\infty}$ such that $c_{\mathbf{K}}$ is the constant diagram of $\mathrm{CAlg}_{H\mathbf{K}}$, and $c_G([i]) = \mathrm{CAlg}_{H\Gamma(G)^i}$ and $[i] \to [j]$ maps to $\mathrm{CAlg}_{H\Gamma(G)^i} \to \mathrm{CAlg}_{H\Gamma(G)^j}; R \mapsto H\Gamma(G)^j \otimes_{H\Gamma(G)^i} R$. By [27, 3.2.0.1, 4.2.4.4] the cosimplicial diagrams $c_{\mathbf{K}}$ and c_G give rise to coCartesian fibrations $\mathrm{pr}_2 : \mathrm{CAlg}_{H\mathbf{K}} \times \mathrm{N}(\Delta) \to \mathrm{N}(\Delta)$ and $\overline{\mathrm{CAlg}}_G \to \mathrm{N}(\Delta)$ respectively. The morphism $c_{\mathbf{K}} \to c_G$ induced by $\{\mathbf{K}\} \to \{\Gamma(G)^i\}_{[i]\in\Delta}$ gives rise to a morphism of coCartesian fibrations $\alpha : \mathrm{CAlg}_{H\mathbf{K}} \times \mathrm{N}(\Delta) \to \overline{\mathrm{CAlg}}_G$ over $\mathrm{N}(\Delta)$ that preserves coCartesian edges. By [28, 8.3.2.7], there is a right adjoint β of α relative to $\mathrm{N}(\Delta)$. Let $s : \mathrm{N}(\Delta) \to \overline{\mathrm{CAlg}}_G$ be a section corresponding to $A^{\bullet}$ (cf. [27, 3.3.3.2]). Then the composite

$$\xi : \mathrm{N}(\Delta) \xrightarrow{s} \overline{\mathrm{CAlg}}_G \xrightarrow{\beta} \mathrm{CAlg}_{H\mathbf{K}} \times \mathrm{N}(\Delta) \xrightarrow{\mathrm{pr}_1} \mathrm{CAlg}_{H\mathbf{K}}$$

gives rise to a simplicial diagram $\xi^{\mathrm{op}} : \mathrm{N}(\Delta)^{\mathrm{op}} \to \mathrm{Aff}_{H\mathbf{K}}$. We define $[\mathrm{Spec}\,\overline{A}/\mathbb{G}_{\mathrm{m}}]$ to be a colimit (geometric realization) of the simplicial diagram ξ^{op} in $\mathrm{Sh}(\mathrm{Aff}_{H\mathbf{K}})$. If $s_1 : \mathrm{N}(\Delta) \to \overline{\mathrm{CAlg}}_G$ is the section corresponds to the initial object of $\lim_{[i]\in\Delta} \mathrm{CAlg}(\mathrm{Mod}_{H\Gamma(G)^i})$, then the composite $\mathrm{pr}_1 \circ \beta \circ s_1 : \mathrm{N}(\Delta) \to \mathrm{CAlg}_{H\mathbf{K}}$ is equivalent to the cosimplicial diagram $\{H\Gamma(G)^i\}$. Thus we have a natural morphism $\pi : [\mathrm{Spec}\,\overline{A}/\mathbb{G}_{\mathrm{m}}] \to B\mathbb{G}_{\mathrm{m}}$. It is easy to see that this morphism makes $[\mathrm{Spec}\,\overline{A}/\mathbb{G}_{\mathrm{m}}]$ a quotient stack.

Proposition 6.11. *There exists a natural equivalence*

$$\mathrm{Mod}_A^{\otimes}(\mathrm{Mod}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}}^{\otimes}) \simeq \mathrm{Mod}^{\otimes}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_{\mathrm{m}}]}\,.$$

Proof. We first construct a symmetric monoidal colimit-preserving functor

$$\mathrm{Mod}_A^{\otimes}(\mathrm{Mod}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}}^{\otimes}) \to \mathrm{Mod}^{\otimes}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_{\mathrm{m}}]}\,.$$

Let $\pi^* : \mathrm{Mod}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}}^{\otimes} \to \mathrm{Mod}^{\otimes}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_{\mathrm{m}}]}$ be the symmetric monoidal functor induced by the natural morphism $\pi : [\mathrm{Spec}\,\overline{A}/\mathbb{G}_{\mathrm{m}}] \to \mathsf{B}\mathbb{G}_{\mathrm{m}}$. By the relative version of the adjoint functor theorem, there is a lax symmetric monoidal right adjoint fucntor $\pi_* : \mathrm{Mod}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_{\mathrm{m}}]} \to \mathrm{Mod}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}}$. If $\mathsf{1}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_{\mathrm{m}}]}$ is a unit of $\mathrm{Mod}^{\otimes}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_{\mathrm{m}}]}$, by the definition of $[\mathrm{Spec}\,\overline{A}/\mathbb{G}_{\mathrm{m}}]$ and the base-change formula, $\pi_*(\mathsf{1}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_{\mathrm{m}}]})$ is equivalent to A in $\mathrm{CAlg}(\mathrm{Mod}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}}^{\otimes})$. Thus we have the composition of symmetric monoidal colimit-preserving functors

$$h : \mathrm{Mod}_A^{\otimes}(\mathrm{Mod}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}}^{\otimes}) \to \mathrm{Mod}_{\pi^*(A)}(\mathrm{Mod}^{\otimes}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_{\mathrm{m}}]}) \to \mathrm{Mod}^{\otimes}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_{\mathrm{m}}]}$$

where the second functor is induced by the counit map $\pi^*(A) \simeq \pi^*(\pi_*(\mathsf{1}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_{\mathrm{m}}]})) \to \mathsf{1}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_{\mathrm{m}}]}$. Note that the composite is naturally an $H\mathbf{K}$-linear symmetric monoidal functor.

Next we will show that h gives an equivalence of symmetric monoidal ∞-categories. It will suffice to prove that the underlying functor of ∞-categories is a categorical equivalence. We first show that h is fully faithful. Let $\mathsf{1}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}}(i) \in \mathrm{Mod}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}}^{\otimes}$ be the object corresponding to $\mathbf{K}_n$ in the proof of Lemma 6.9. Let $A(i)$ be the image of $\mathsf{1}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}}(i)$ under the natural functor $\mathrm{Mod}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}} \to \mathrm{Mod}_A(\mathrm{Mod}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}}^{\otimes})$. Unwinding the definition of h and using adjunctions, we see that

$$\mathrm{Map}_{\mathrm{Mod}_A^{\otimes}(\mathrm{Mod}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}}^{\otimes})}(A(i), A(j)) \to \mathrm{Map}_{\mathrm{Mod}^{\otimes}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_{\mathrm{m}}]}}\big(h(A(i)), h(A(j))\big)$$

can be identified with

$$\begin{aligned}\mathrm{Map}_{\mathrm{Mod}_A(\mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}})}(A(i),A(j)) &\simeq \mathrm{Map}_{\mathrm{Mod}_A(\mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}})}(A(i-j),A)\\ &\simeq \mathrm{Map}_{\mathrm{Mod}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}}}(1_{\mathsf{B}\mathbb{G}_{\mathrm{m}}}(i-j),A)\\ &\simeq \mathrm{Map}_{\mathrm{Mod}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_{\mathrm{m}}]}}\big(\pi^*(1_{\mathsf{B}\mathbb{G}_{\mathrm{m}}}(i-j)),1_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_{\mathrm{m}}]}\big)\\ &\simeq \mathrm{Map}_{\mathrm{Mod}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_{\mathrm{m}}]}}\big(1_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_{\mathrm{m}}]}(i),1_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_{\mathrm{m}}]}(j)\big).\end{aligned}$$

Note that $A(i)$ and $h(A(i))$ are compact for each i, and h preserves small colimits. The stable presentable ∞-category $\mathrm{Mod}_A(\mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}})$ is generated by $\{A(i)\}_{i\in\mathbb{Z}}$, that is, $\mathrm{Mod}^{\otimes}_A(\mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}})$ is the smallest stable subcategory which contains the set $\{A(i)\}_{i\in\mathbb{Z}}$ of objects and admits filtered colimits. Therefore for any $N\in\mathrm{Mod}_A(\mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}})$,

$$\mathrm{Map}_{\mathrm{Mod}_A(\mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}})}(A(i),N)\to \mathrm{Map}_{\mathrm{Mod}^{\otimes}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_{\mathrm{m}}]}}\big(h(A(i)),h(N)\big)$$

is an equivalence in $\mathcal{S}$. Furthermore, it follows from the fact that h is colimit-preserving that for any $M,N\in\mathrm{Mod}_A(\mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}})$,

$$\mathrm{Map}_{\mathrm{Mod}_A(\mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}})}(M,N)\to \mathrm{Map}_{\mathrm{Mod}^{\otimes}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_{\mathrm{m}}]}}(h(M),h(N))$$

is an equivalence in $\mathcal{S}$. It remains to show that h is essentially surjective. To this end, note that $\mathrm{Mod}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_{\mathrm{m}}]}\simeq\mathrm{Ind}(\mathcal{E})$ where $\mathcal{E}$ is the smallest stable subcategory which contains $\{1_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_{\mathrm{m}}]}(i)\}_{i\in\mathbb{Z}}$. To see this, since $1_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_{\mathrm{m}}]}(i)$ are compact, by [2, Definition 3.7] it is enough to observe that the right orthogonal of $\{1_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_{\mathrm{m}}]}(i)\}_{i\in\mathbb{Z}}$ is zero, where $1_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_{\mathrm{m}}]}(i)=\pi^*(1_{\mathsf{B}\mathbb{G}_{\mathrm{m}}}(i))$. The condition that

$$\mathrm{Map}_{\mathrm{Mod}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_{\mathrm{m}}]}}(1_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_{\mathrm{m}}]}(i),N)\simeq \mathrm{Map}_{\mathrm{Mod}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}}}(1_{\mathsf{B}\mathbb{G}_{\mathrm{m}}}(i),\pi_*(N))=0$$

for any $i\in\mathbb{Z}$ implies that $\pi_*(N)=0$. Since π_* is conservative we deduce that $N=0$, as desired. Since the set $\{1_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_{\mathrm{m}}]}(i)\}_{i\in\mathbb{Z}}$ of compact objects generates $\mathrm{Mod}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_{\mathrm{m}}]}$ (in the above sense), we have $\mathrm{Ind}(h(\mathcal{D}))\simeq\mathrm{Mod}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_{\mathrm{m}}]}$ (see [27, 5.3.5.11]) where $\mathcal{D}$ is the smallest stable subcategory in $\mathrm{Mod}_A(\mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}})$ which contains $\{A(i)\}_{i\in\mathbb{Z}}$. It follows that h is essentially surjective, by noting that h is colimit-preserving and fully faithful. □

§6.4. Tannakization and derived stack of mixed Tate motives

Propositions 6.10, 6.11 and Lemma 6.9 allow us to identify the realization functor $\mathsf{R}_T:\mathsf{DTM}^{\otimes}\to\mathrm{Mod}^{\otimes}_{H\mathbf{K}}$ with

$$\rho^*:\mathrm{Mod}^{\otimes}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_{\mathrm{m}}]}\to\mathrm{Mod}^{\otimes}_{[\mathrm{Spec}\,\overline{R}/\mathbb{G}_{\mathrm{m}}]}$$

induced by the morphism of derived stacks $\rho : [\operatorname{Spec}\overline{R}/\mathbb{G}_{\mathrm{m}}] \to [\operatorname{Spec}\overline{Q}/\mathbb{G}_{\mathrm{m}}]$. Here $\overline{R}$ is the image of R in $\mathrm{CAlg}(\mathrm{Mod}_{H\mathbf{K}}^{\otimes})$.

Observe that $[\operatorname{Spec}\overline{R}/\mathbb{G}_{\mathrm{m}}] \simeq \operatorname{Spec} H\mathbf{K}$. To see this, note that by the property of the realization functor the composite of symmetric monoidal left adjoint functors

$$\mathrm{Mod}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}} \to \mathrm{Mod}_{H\mathbf{K},\mathbb{Z}} \to \mathrm{Mod}_Q(\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}}^{\otimes}) \simeq \mathsf{DTM} \to \mathrm{Mod}_{H\mathbf{K}}$$

is equivalent to the forgetful functor (since the heart of the standard t-structure on $\mathrm{Mod}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}}$ maps to the heart of the standard t-structure on $\mathrm{Mod}_{H\mathbf{K}}$ as the forgetful functor). Thus if $\pi : \operatorname{Spec} H\mathbf{K} \to \mathsf{B}\mathbb{G}_{\mathrm{m}}$ denotes the natural projection, its right adjoint functor sends $\mathbf{1}_{H\mathbf{K}}$ to the object $R = \pi_*(\mathbf{1}_{H\mathbf{K}})$, i.e., the coordinate ring of $\mathbb{G}_{\mathrm{m}}$ (equipped with the natural action of $\mathbb{G}_{\mathrm{m}}$). Hence $[\operatorname{Spec}\overline{R}/\mathbb{G}_{\mathrm{m}}] \simeq \operatorname{Spec} H\mathbf{K}$.

We refer to $[\operatorname{Spec}\overline{Q}/\mathbb{G}_{\mathrm{m}}]$ and $\rho : \operatorname{Spec} H\mathbf{K} \to [\operatorname{Spec}\overline{Q}/\mathbb{G}_{\mathrm{m}}]$ as the *derived stack of mixed Tate motives* and the *point determined by the mixed Weil cohomology E* respectively.

Theorem 6.12. *Let* MTG *be the derived affine group scheme over* $H\mathbf{K}$ *which represents the automorphism group functor of* $\mathsf{R}_T : \mathsf{DTM}_{\vee}^{\otimes} \to \mathrm{PMod}_{H\mathbf{K}}^{\otimes}$*, that is, the tannakization. Then* MTG *is equivalent to the derived affine group scheme arising from the Čech nerve of* $\rho : \operatorname{Spec} H\mathbf{K} \to [\operatorname{Spec}\overline{Q}/\mathbb{G}_{\mathrm{m}}]$.

Proof. Apply Corollary 4.9 to ρ. □

§6.5. Cycle complex and Q

We describe the ($\mathbb{Z}$-graded) complex Q in terms of Bloch's cycle complexes. We here regard Q as the object in the ∞-category $\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}}$. (The results of this subsection will not be used in other sections and the reader may skip them.)

For this purpose, we need an explicit right adjoint functor $r : \mathsf{DM}_{\mathbb{Z}} \to \mathrm{Mod}_{H\mathbf{K},\mathbb{Z}}$ of $l : \mathrm{Mod}_{H\mathbf{K},\mathbb{Z}} \to \mathsf{DM}_{\mathbb{Z}}$. To this end, recall the Quillen adjoint pair

$$1 \otimes (-) : \mathrm{Comp}(\mathcal{A}) \rightleftarrows \mathrm{DM}^{\mathrm{eff}}(k) : \Gamma$$

where the right-hand side is the model category in [6, Example 4.12] (cf. Section 6.1) and the left adjoint functor carries a complex M to the tensor product $1 \otimes M$ with the (cofibrant) unit 1 of $\mathrm{DM}^{\mathrm{eff}}(k)$. Here the tensor product $1 \otimes M$ is considered to be a complex of sheaves with transfers $U \mapsto L(\operatorname{Spec} k)(U) \otimes_{\mathbf{K}} M$. The right adjoint functor sends a complex of Nisnevich sheaves with transfers P to the complex $\Gamma(P)$ of sections at $\operatorname{Spec} k$. Let F be a Nisnevich sheaf with transfers. Let $\triangle^{\bullet}$ be the cosimplicial scheme where $\triangle^n = \operatorname{Spec} k[x_0, \ldots, x_n]/(\sum_{i=0}^{n} x_i = 0)$ and the j-th face $\triangle^n \hookrightarrow \triangle^{n+1}$ is determined by $x_j = 0$ (see e.g. [30]). We then have the Suslin complex $C_*(F)$ in $\mathrm{DM}^{\mathrm{eff}}(k)$, that is, the complex of sheaves with transfers, defined by $X \mapsto F(\triangle^{\bullet} \times_k X)$ (take the Moore complex).

Lemma 6.13. *Let F be a Nisnevich sheaf with transfers. Let $F \to F'$ be a fibrant replacement in* $\mathrm{DM}^{\mathrm{eff}}(k)$. *Then the global section* $F'(\operatorname{Spec} k)$ *is quasi-isomorphic to* $C_*(F)(\operatorname{Spec} k)$.

Proof. It is well-known that the natural morphism $F \to C_*(F)$ is a weak equivalence in $\mathrm{DM}^{\mathrm{eff}}(k)$ (cf. [30, 14.4]). Let $C_*(F) \to C_*(F)'$ be a fibrant replacement. Then $F' \to C_*(F)'$ (induced by the functorial fibrant replacement) is a weak equivalence. According to [30, 2.19, 13.8], cohomology sheaves of $C_*(F)$ are homotopy invariant. By [30, 13.8, 14.8] and the definition of $\mathbb{A}^1$-local objects [6, 4.12], $C_*(F)$, $C_*(F)'$ and F' are $\mathbb{A}^1$-local. Thus both $C_*(F) \to C_*(F)'$ and $F' \to C_*(F)'$ induce isomorphisms of cohomology sheaves. Therefore, taking the Nisnevich topology of $\operatorname{Spec} k$ into account, we deduce that $C_*(F)(\operatorname{Spec} k)$ is quasi-isomorphic to $F'(\operatorname{Spec} k)$. □

For an equidimensional scheme X over k, we denote by $z^n(X,*)$ the Bloch cycle complex of X (cf. e.g. [30, Lecture 17]).

Corollary 6.14. *Let $n \geq 0$. The total right Quillen derived functor $\mathbb{R}\Gamma$ sends $\mathbf{K}(n)$ to a complex which is quasi-isomorphic to* $z^n(\operatorname{Spec} k,*)[-2n]$.

Proof. The comparison theorems [30, 16.7, 19.8] together with Lemma 6.13 implies that $\mathbb{R}\Gamma(\mathbf{K}(n))$ is quasi-isomorphic to $z^n(\mathbb{A}^n,*)[-2n]$, where $\mathbb{A}^n$ is the n-dimensional affine space. The homotopy invariance of higher Chow groups (cf. [30, 17.4 (4)]) shows that $z^n(\mathbb{A}^n,*)[-2n]$ is quasi-isomorphic to $z^n(\operatorname{Spec} k,*)[-2n]$. □

Remark 6.15. Let n be a negative integer. Then every morphism from $\mathbf{K}$ to $\mathbf{K}(n)[i]$ in DM is null-homotopic for any $i \in \mathbb{Z}$. Thus by adjunction, the right adjoint functor of the canonical functor $\mathrm{Mod}_{H\mathbf{K}} \to \mathsf{DM}$ carries $\mathbf{K}(n)$ to zero in $\mathrm{Mod}_{H\mathbf{K}}$.

Proposition 6.16. *Let $Q_n \in \mathrm{Mod}_{H\mathbf{K}}$ denote the complex of the n-th degree of $Q \in \mathrm{Mod}_{H\mathbf{K},\mathbb{Z}}$ (this is not the homological degree). Then Q_n is equivalent to $z^n(\operatorname{Spec} k,*)[-2n]$ for any $n \geq 0$, and $Q_n \simeq 0$ for $n < 0$.*

Proof. Recall that Q is the image of

$$\mathbf{K}(*) := (\ldots, \mathbf{K}(-1), \mathbf{K}(0), \mathbf{K}(1), \ldots)$$

under $r : \mathsf{DM}_{\mathbb{Z}} \to \mathrm{Mod}_{H\mathbf{K},\mathbb{Z}}$ (we adopt the notation of Section 6.2). The natural functor $\Sigma^\infty : \mathsf{DM}^{\mathrm{eff}} \to \mathsf{DM}$ is fully faithful by Voevodsky's cancellation theorem, and thus the right adjoint $\Omega^\infty : \mathsf{DM} \to \mathsf{DM}^{\mathrm{eff}}$ sends $\mathbf{K}(i)$ to $\mathbf{K}(i)$ for $i \geq 0$. Now our claim follows from Corollary 6.14 and Remark 6.15. □

§7. Mixed Tate motives assuming the Beilinson–Soulé vanishing conjecture

In this section, we adopt the notation of Section 6. In contrast to the previous section, in this section we will assume the Beilinson–Soulé vanishing conjecture for the base field k: the motivic cohomology

$$H^{n,i}(\operatorname{Spec} k, \mathbf{K})$$

is zero for $n \leq 0$ and $i > 0$. Here $H^{n,i}(\operatorname{Spec} k, \mathbf{K})$ denotes the motivic cohomology (following the notation of [30, Definition 3.4]). What we need is that this condition implies that Q is cohomologically connective, that is, $\pi_n(\overline{Q}) = 0$ for $n > 0$, and $\pi_0(\overline{Q}) = \mathbf{K}$. For example, the Beilinson–Soulé vanishing conjecture holds when k is a number field. The goal of this section is to prove Theorem 7.16, which relates our tannakization MTG of $\mathsf{DTM}_{\vee}^{\otimes}$ to the Galois group of mixed Tate motives constructed by Bloch–Kriz [4], Kriz–May [25], Levine [26] (all group schemes are known to be equivalent to one another) under this vanishing conjecture.

§7.1. Motivic t-structure on DTM

Under the Beilinson–Soulé vanishing conjecture, one can define a motivic t-structure on DTM, as proved by Levine [26] and Kriz–May [25]. We will construct a t-structure in our setting (we do not claim any originality).

We fix our convention on t-structures. Let $\mathcal{C}$ be a stable ∞-category. A *t-structure on* $\mathcal{C}$ is a t-structure on the triangulated category $\mathrm{h}(\mathcal{C})$ (the homotopy category is naturally endowed with the structure of triangulated category, see [28, Chapter 1]). That is, there is a pair of full subcategories $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ of $\mathcal{C}$ such that

- $\mathcal{C}_{\geq 0}[1] \subset \mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\leq 0}[-1] \subset \mathcal{C}_{\leq 0}$,
- for $X \in \mathcal{C}_{\geq 0}$ and $Y \in \mathcal{C}_{\leq 0}$, the group $\operatorname{Hom}_{\mathrm{h}(\mathcal{C})}(X, Y[-1])$ is zero,
- for $X \in \mathcal{C}$, there exists a distinguished triangle

$$X' \to X \to X''$$

in $\mathrm{h}(\mathcal{C})$ such that $X' \in \mathcal{C}_{\geq 0}$ and $X'' \in \mathcal{C}_{\leq 0}[-1]$.

We here assume that full subcategories are stable under equivalences. We use homological indexing. Our references on t-structures are [28] and [23]. We shall write $\mathcal{C}_{\geq n}$ and $\mathcal{C}_{\leq n}$ for $\mathcal{C}_{\geq 0}[n]$ and $\mathcal{C}_{\leq 0}[n]$ respectively. We denote by $\tau_{\geq n}$ the right adjoint to $\mathcal{C}_{\geq n} \subset \mathcal{C}$. Similarly, we denote by $\tau_{\leq n}$ the left adjoint to $\mathcal{C}_{\leq n} \subset \mathcal{C}$.

Let $\mathsf{R}_T : \mathsf{DTM} \to \mathrm{Mod}_{H\mathbf{K}}$ be the realization functor of a fixed mixed Weil theory E. Let $(\mathrm{Mod}_{H\mathbf{K},\geq 0}, \mathrm{Mod}_{H\mathbf{K},\leq 0})$ be the standard t-structure of $\mathrm{Mod}_{H\mathbf{K}}$ such that X belongs to $\mathrm{Mod}_{H\mathbf{K},\geq 0}$ (resp. $\mathrm{Mod}_{H\mathbf{K},\leq 0}$) exactly when the homotopy group $\pi_n(X)$ of the underlying spectra is zero for $n < 0$ (resp. $n > 0$).

Proposition 7.1. *Let*

$$\mathsf{DTM}_{\vee,\geq 0} := \mathsf{R}_T^{-1}(\mathrm{Mod}_{H\mathbf{K},\geq 0}) \cap \mathsf{DTM}_\vee,$$
$$\mathsf{DTM}_{\vee,\leq 0} := \mathsf{R}_T^{-1}(\mathrm{Mod}_{H\mathbf{K},\leq 0}) \cap \mathsf{DTM}_\vee.$$

Then the pair $(\mathsf{DTM}_{\vee,\geq 0}, \mathsf{DTM}_{\vee,\leq 0})$ *is a bounded t-structure on* $\mathsf{DTM}_\vee$*. (Of course, the realization functor is t-exact.)*

Proof. Since R_T is exact, $\mathsf{DTM}_{\vee,\geq 0}[1] \subset \mathsf{DTM}_{\vee,\geq 0}$ and $\mathsf{DTM}_{\vee,\leq 0}[-1] \subset \mathsf{DTM}_{\vee,\leq 0}$.

We next claim that the realization functor induces a conservative functor $\mathsf{DTM}_\vee \to \mathrm{Mod}_{H\mathbf{K}}$. (Recall again that an exact functor $p : \mathcal{K} \to \mathcal{L}$ between stable ∞-categories is said to be conservative if for any $K \in \mathcal{K}$, $p(K) \simeq 0$ implies that $K \simeq 0$.) Note that the realization functor $\mathsf{DTM} \simeq \mathrm{Mod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_{\mathrm{m}}]} \xrightarrow{\rho^*} \mathrm{Mod}_{H\mathbf{K}}$ is induced by $\rho : \mathrm{Spec}\, H\mathbf{K} \to [\mathrm{Spec}\,\overline{Q}/\mathbb{G}_{\mathrm{m}}]$ (see Section 6.4). The morphism ρ extends to $\overline{\rho} : \mathrm{Spec}\, H\mathbf{K} \to \mathrm{Spec}\,\overline{Q}$. Thus the realization functor decomposes into

$$\mathsf{DTM} \simeq \mathrm{Mod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_{\mathrm{m}}]} \to \mathrm{Mod}_{\mathrm{Spec}\,\overline{Q}} \xrightarrow{\overline{\rho}^*} \mathrm{Mod}_{H\mathbf{K}}.$$

By the definition, the pullback of the projection $\mathrm{Mod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_{\mathrm{m}}]} \to \mathrm{Mod}_{\mathrm{Spec}\,\overline{Q}}$ is conservative. Furthermore, the stable ∞-category $\mathrm{Mod}_{\overline{Q}}$ admits a t-structure $(\mathrm{Mod}_{\overline{Q},\geq 0}, \mathrm{Mod}_{\overline{Q},\leq 0})$ such that X in $\mathrm{Mod}_{\overline{Q}}$ belongs to $\mathrm{Mod}_{\overline{Q},\leq 0}$ if and only if $\pi_n(X) = 0$ for $n > 0$ (see, [29, VIII, 4.5.4]). According to [29, VIII, 4.1.11], the composite $\bigcup_{n\in\mathbb{Z}} \mathrm{Mod}_{\overline{Q},\leq n} \to \mathrm{Mod}_{H\mathbf{K}}$ is conservative. Observe that every object $X \in \mathrm{PMod}_{\overline{Q}}$ lies in $\bigcup_{n\in\mathbb{Z}} \mathrm{Mod}_{\overline{Q},\leq n}$. To see this, note that $\mathrm{PMod}_{\overline{Q}}$ is the smallest stable subcategory which contains $\overline{Q}$ and is closed under retracts. Since $\overline{Q}$ belongs to $\bigcup_{n\in\mathbb{Z}} \mathrm{Mod}_{\overline{Q},\leq n}$, and $\bigcup_{n\in\mathbb{Z}} \mathrm{Mod}_{\overline{Q},\leq n}$ is closed under retracts, we see that $\mathrm{PMod}_{\overline{Q}} \subset \bigcup_{n\in\mathbb{Z}} \mathrm{Mod}_{\overline{Q},\leq n}$. Therefore the composite $\mathsf{DTM}_\vee \simeq \mathrm{PMod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_{\mathrm{m}}]} \to \mathrm{Mod}_{H\mathbf{K}}$ is conservative. By using this fact, we verify the second condition of the definition of t-structure.

It remains to show the third condition of t-structure. For this purpose, note first that if $Z \subset \mathrm{Mod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_{\mathrm{m}}]}$ denotes the inverse image of $\bigcup_{n\in\mathbb{Z}} \mathrm{Mod}_{\overline{Q},\leq n}$ and $f : Z \to \mathrm{Mod}_{H\mathbf{K}}$ denotes the restriction of the realization functor, we have $f^{-1}(\mathrm{PMod}_{H\mathbf{K}}) = \mathrm{PMod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_{\mathrm{m}}]}$. Clearly, $f^{-1}(\mathrm{PMod}_{H\mathbf{K}}) \supset \mathrm{PMod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_{\mathrm{m}}]}$ since the realization functor is symmetric monoidal. An object in $\mathrm{Mod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_{\mathrm{m}}]}$ is dualizable if and only if its image in $\mathrm{Mod}_{\overline{Q}}$ is dualizable. Thus it is enough to show that $g^{-1}(\mathrm{PMod}_{H\mathbf{K}}) = \mathrm{PMod}_{\overline{Q}}$ where $g : \bigcup_{n\in\mathbb{Z}} \mathrm{Mod}_{\overline{Q},\geq n} \to \mathrm{Mod}_{H\mathbf{K}}$. According to [28, VIII 4.5.2(7)], we have a natural symmetric monoidal fully faithful functor $\bigcup_{n\in\mathbb{Z}} \mathrm{Mod}_{\overline{Q},\leq n} \to \lim_{\overline{Q}\to B} \mathrm{Mod}_B$ where B runs over connective commutative ring spectra under $\overline{Q}$. An object $M \in \lim_{\overline{Q}\to B} \mathrm{Mod}_B$ belongs to its essential image if and only if the image $M(H\mathbf{K})$ of M in $\mathrm{Mod}_{H\mathbf{K}}$ under the natural projection has trivial homotopy groups, $\pi_m(M(H\mathbf{K})) = 0$, for sufficiently large $m \gg 0$. Note that every morphism $\overline{Q} \to B$ factors through $\overline{Q} \to H\mathbf{K}$ since $\overline{Q}$ is cohomologically connected. Consequently, we deduce that $g^{-1}(\mathrm{PMod}_{H\mathbf{K}}) \simeq \lim_{\overline{Q}\to B} \mathrm{PMod}_B$. Thus all objects in $g^{-1}(\mathrm{PMod}_{H\mathbf{K}})$ are dualizable. It follows that $g^{-1}(\mathrm{PMod}_{H\mathbf{K}}) = \mathrm{PMod}_{\overline{Q}}$. Next consider

$$\mathrm{Mod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_{\mathrm{m}}],\geq 0} := \mathrm{Mod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_{\mathrm{m}}]} \times_{\mathrm{Mod}_{\overline{Q}}} \mathrm{Mod}_{\overline{Q},\geq 0}.$$

This category is presentable, by [27, 5.5.3.13]. Define $\mathrm{Mod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_{\mathrm{m}}],\leq 0}$ by replacing ≥ 0 on the right-hand side by ≤ 0. Then the comonad of $\mathrm{Mod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_{\mathrm{m}}]} \rightleftarrows \mathrm{Mod}_{\overline{Q}}$ is given by $M \mapsto M \otimes_{H\mathbf{K}} H\mathbf{K}[t^{\pm}]$ (this is checked by using right adjointability; Lemma 4.3). Therefore we can apply [29, VII 6.20] to deduce that

$$(\mathrm{Mod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_{\mathrm{m}}],\geq 0}, \mathrm{Mod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_{\mathrm{m}}],\leq 0})$$

is a t-structure. Note that since $\mathrm{Mod}_{\overline{Q}} \to \mathrm{Mod}_{H\mathbf{K}}$ is t-exact (by [29, VIII, 4.1.10, 4.5.4 (2)]), $\mathrm{Mod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_{\mathrm{m}}]} \to \mathrm{Mod}_{H\mathbf{K}}$ is also t-exact. We now claim that $\mathrm{PMod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_{\mathrm{m}}]}$ is stable under the truncations $\tau_{\geq 0}$ and $\tau_{\leq 0}$. Let $M \in \mathrm{PMod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_{\mathrm{m}}]}$. Then $\tau_{\geq 0}M$ and $\tau_{\leq 0}M$ are contained in Z. Thus, to prove that $\tau_{\geq 0}M$ and $\tau_{\leq 0}M$ belong to $\mathrm{PMod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_{\mathrm{m}}]}$, it will suffice to prove that $g(\tau_{\geq 0}M)$ and $g(\tau_{\leq 0}M)$ belong to $\mathrm{PMod}_{H\mathbf{K}}$. Let $H_i = \tau_{\geq i} \circ \tau_{\leq i} = \tau_{\leq i} \circ \tau_{\geq i}$ (this notation slightly differs from the standard one). Using t-exactness, we have

$$H_i(g(\tau_{\geq 0}M)) = g(H_i \circ \tau_{\geq 0}M) = g(\tau_{\leq i} \circ \tau_{\geq i} \circ \tau_{\geq 0}M) = g(H_i(M)) = H_i(g(M))$$

for $i \geq 0$. It follows that $H_i(g(\tau_{\geq 0}M))[-i]$ is equivalent to a finite-dimensional $\mathbf{K}$-vector space, and the set

$$\{i \in \mathbb{Z} \mid H_i(g(\tau_{\geq 0}M))[-i] \neq 0\}$$

is finite. This implies that $g(\tau_{\geq 0}M)$ lies in $\mathrm{PMod}_{H\mathbf{K}}$. Similarly, $g(\tau_{\leq 0}M)$ lies in $\mathrm{PMod}_{H\mathbf{K}}$. Therefore for any $M \in \mathrm{PMod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_{\mathrm{m}}]}$ we have the distinguished triangle (at the level of homotopy category)

$$\tau_{\geq 0}M \to M \to \tau_{\leq -1}M$$

such that $\mathsf{R}_T(\tau_{\geq 0}M) \in \mathrm{Mod}_{H\mathbf{K},\geq 0}$ and $\mathsf{R}_T(\tau_{\leq -1}M) \in \mathrm{Mod}_{H\mathbf{K},\leq 0}[-1]$, as desired.

Finally, this t-structure is clearly bounded. □

Remark 7.2. The definition of t-structure in Proposition 7.1 is compatible with the definition of motivic t-structure on the triangulated category of (all) mixed motives developed by Hanamura [15] (up to anti-equivalence). In [15], the expected motivic t-structure is constructed using Grothendieck's standard conjectures, the Bloch–Beilinson–Murre conjecture and the Beilinson–Soulé vanishing conjecture for smooth projective varieties.

In Proposition 7.1, by the extension of coefficients $\mathbb{Q} \to \mathbf{K}$ we can replace $\mathbf{K}$ by $\mathbb{Q}$.

We refer to $(\mathsf{DTM}_{\vee,\geq 0}, \mathsf{DTM}_{\vee,\leq 0})$ as the *motivic t-structure* on $\mathsf{DTM}_\vee$. We let $\mathsf{DTM}_\vee^\heartsuit := \mathsf{DTM}_{\vee,\geq 0} \cap \mathsf{DTM}_{\vee,\leq 0}$ be the heart. At first sight, it depends on the choice of our realization functor. But the mapping space $\mathrm{Map}(\mathrm{Spec}\, H\mathbf{K}, \mathrm{Spec}\, \overline{Q})$ is connected since $\overline{Q}$ is cohomologically connected (cf. [29, VIII, 4.1.7]). Therefore $\rho^* : \mathrm{Mod}^\otimes_{[\mathrm{Spec}\, \overline{Q}/\mathbb{G}_{\mathrm{m}}]} \to \mathrm{Mod}^\otimes_{H\mathbf{K}}$ is unique up to equivalence.

As a by-product of the proof, we have

Corollary 7.3. *In the notation of the proof of Proposition 7.1, the realization functor induces a conservative functor* $f : \bigcup_{n\in\mathbb{Z}} \mathrm{Mod}_{[\mathrm{Spec}\, \overline{Q}/\mathbb{G}_{\mathrm{m}}],\leq n} \to \mathrm{Mod}_{H\mathbf{K}}$. *In particular,* $\mathsf{DTM}_\vee \to \mathrm{PMod}_{H\mathbf{K}}$ *is conservative. Moreover,* $f^{-1}(\mathrm{PMod}_{H\mathbf{K}})$ *coincides with* $\mathsf{DTM}_\vee$.

Recall DTM is compactly generated. Namely, we have a natural equivalence $\mathrm{Ind}(\mathsf{DTM}_\circ) \simeq \mathrm{Ind}(\mathsf{DTM}_\vee) \simeq \mathsf{DTM}$.

Corollary 7.4. *Let* $\mathsf{DTM}_{\geq 0} := \mathrm{Ind}(\mathsf{DTM}_{\vee,\geq 0})$ *and* $\mathsf{DTM}_{\leq 0} := \mathrm{Ind}(\mathsf{DTM}_{\vee,\leq 0})$. *Then* $(\mathsf{DTM}_{\geq 0}, \mathsf{DTM}_{\leq 0})$ *is an accessible right complete t-structure on* DTM.

Proof. This follows from Proposition 7.1, [29, VIII, 5.4.1] and [28, 1.4.4.13]. □

Let $(\mathrm{Mod}^\heartsuit_{H\mathbf{K}})^\otimes$ be the symmetric monoidal abelian category such that the underlying category is $\mathrm{Mod}_{H\mathbf{K},\geq 0} \cap \mathrm{Mod}_{H\mathbf{K},\leq 0}$ and its symmetric monoidal structure is induced by that of $\mathrm{Mod}^\otimes_{H\mathbf{K}}$. It is (the nerve of) the symmetric monoidal category of $\mathbf{K}$-vector spaces. For an affine group scheme G over $\mathbf{K}$ (which can be viewed as a derived affine group scheme over $H\mathbf{K}$), we let $\mathrm{Rep}(G)^\otimes$ be the symmetric monoidal full subcategory $z^{-1}((\mathrm{Mod}^\heartsuit_{H\mathbf{K}})^\otimes)$ of $\mathrm{Mod}^\otimes_{\mathsf{B}G}$ where $z : \mathrm{Mod}^\otimes_{\mathsf{B}G} \to \mathrm{Mod}^\otimes_{H\mathbf{K}}$ is the natural projection determined by $\mathrm{Spec}\, H\mathbf{K} \to \mathsf{B}G$. We denote by $\mathrm{Rep}(G)^\otimes_\vee$ the symmetric monoidal full subcategory of $\mathrm{Rep}(G)^\otimes$ which consists of dualizable objects. Applying the classical Tannaka duality by Saavedra, Deligne-Milne, Deligne [33], [11], [10] to the faithful symmetric monoidal exact functor of abelain categories $(\mathsf{DTM}_\vee^\heartsuit)^\otimes \to (\mathrm{Mod}^\heartsuit_{H\mathbf{K}})^\otimes$ induced by the realization functor, we have

Corollary 7.5. *There are an affine group scheme MTG over $\mathbf{K}$ and an equivalence* $(\mathsf{DTM}_\vee^\heartsuit)^\otimes \xrightarrow{\sim} \mathrm{Rep}(MTG)^\otimes_\vee$ *of symmetric monoidal ∞-categories.*

We here give a symmetric monoidal equivalence between the abelian category $\mathsf{DTM}_\vee^\heartsuit$ and the abelian category $\mathbf{TM}_k$ which is constructed via the axiomatic formulation in [26]. Let i be an integer. Let $W_{\ge i}\mathsf{DTM}_{\mathrm{gm}} \subset \mathsf{DTM}_{\mathrm{gm}}$ (resp. $W_{\le i}\mathsf{DTM}_{\mathrm{gm}} \subset \mathsf{DTM}_{\mathrm{gm}}$) be the smallest stable subcategory generated by $\mathbf{K}(n)$ for $-2n \ge i$ (resp. $\mathbf{K}(n)$ for $-2n \le i$). Then according to [26, Lemma 1.2], the pair $(W_{\ge i}\mathsf{DTM}_{\mathrm{gm}}, W_{\le i}\mathsf{DTM}_{\mathrm{gm}})$ is a t-structure. Let $\mathrm{Gr}_i^W : \mathsf{DTM}_{\mathrm{gm}} \to W_i\mathsf{DTM}_{\mathrm{gm}} := W_{\ge i}\mathsf{DTM}_{\mathrm{gm}} \cap W_{\le i}\mathsf{DTM}_{\mathrm{gm}}$ be the functor H_0 with respect to this t-structure. When i is even, the ∞-category $W_i\mathsf{DTM}_{\mathrm{gm}}$ is equivalent to the full subcategory $\mathrm{h}(\mathrm{PMod}_{H\mathbf{K}})$ of $\mathrm{h}(\mathrm{Mod}_{H\mathbf{K}})$ spanned by bounded complexes of $\mathbf{K}$-vector spaces whose (co)homology is finite-dimensional. This equivalence is given by the exact functor $\mathrm{h}(\mathrm{PMod}_{H\mathbf{K}}) \to W_i\mathsf{DTM}_{\mathrm{gm}}$ which carries $\mathbf{K}[r]$ to $\mathbf{K}(-i/2)[r]$. If i is odd, $W_i\mathsf{DTM}_{\mathrm{gm}}$ is zero. This gives rise to a natural symmetric monoidal exact functor $\mathrm{Gr} : \mathrm{h}(\mathsf{DTM}_{\mathrm{gm}}) \to \mathrm{h}(\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}})$, which sends X to $\{\mathrm{Gr}_i^W(X)\}_{i\in\mathbb{Z}}$, of homotopy categories (which are furthermore triangulated categories). The triangulated category $\mathrm{h}(\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}}) \simeq \prod_{\mathbb{Z}} \mathrm{h}(\mathrm{Mod}_{H\mathbf{K}})$ has the standard t-structure determined by the product of the pair $(\mathrm{Mod}_{H\mathbf{K},\ge 0}, \mathrm{Mod}_{H\mathbf{K},\le 0})$. We denote it by $(\mathrm{h}(\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}})_{\ge 0}, \mathrm{h}(\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}})_{\le 0})$. Let $\mathsf{DTM}_{\mathrm{gm},\ge 0} := \mathrm{Gr}^{-1}(\mathrm{h}(\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}})_{\ge 0})$ and $\mathsf{DTM}_{\mathrm{gm},\le 0} := \mathrm{Gr}^{-1}(\mathrm{h}(\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}})_{\le 0})$. Then by [26, Theorem 1.4], we have

Lemma 7.6 ([26]). *The pair* $(\mathsf{DTM}_{\mathrm{gm},\ge 0}, \mathsf{DTM}_{\mathrm{gm},\le 0})$ *is a bounded t-structure, and* Gr *is t-exact and conservative.*

Let $\mathbf{TM}_k$ be its heart.

Lemma 7.7. *The realization functor* $\mathsf{R}_{\mathrm{gm}} : \mathsf{DTM}_{\mathrm{gm}} \to \mathrm{Mod}_{H\mathbf{K}}$ (*induced by* $\mathsf{R}_T : \mathsf{DTM} \to \mathrm{Mod}_{H\mathbf{K}}$) *is t-exact.*

Proof. We will show that the essential image of $\mathsf{DTM}_{\mathrm{gm},\le 0}$ is contained in $\mathrm{Mod}_{H\mathbf{K},\le 0}$. The dual case is similar. Let $X \in \mathsf{DTM}_{\mathrm{gm},\le 0}$. Let m be the cardinality of the set of integers i such that $H_i(X)[-i]$ is not zero (recall our (nonstandard) notation $H_i = \tau_{\le i} \circ \tau_{\ge i}$). We proceed by induction on m. If $m = 0$, we conclude that $X \simeq 0$ (since the t-structure on $\mathsf{DTM}_{\mathrm{gm}}$ is bounded). Hence this case is clear. By [26, Theorem 1.4(iii)] we see that the essential image of $\mathbf{TM}_k$ are contained in $\mathrm{Mod}_{H\mathbf{K}}^\heartsuit$. Hence the case $m = 1$ follows. Suppose that our claim holds for $m \le n$. To prove the case when $m = n + 1$, consider the distinguished triangle

$$H_i(X) \to X \to \tau_{\le i-1}X$$

where i is the largest number such that $H_i(X)[-i] \ne 0$. Note that the functor $\mathsf{DTM}_{\mathrm{gm}} \to \mathrm{Mod}_{H\mathbf{K}}$ is exact, and the images of $H_i(X)$ and $\tau_{\le i-1}X$ are contained in $\mathrm{Mod}_{H\mathbf{K},\le 0}$. Thus the image of X is also contained in $\mathrm{Mod}_{H\mathbf{K},\le 0}$. □

Lemma 7.8. $\mathsf{DTM}_{\mathrm{gm},\geq 0}$ *(resp.* $\mathsf{DTM}_{\mathrm{gm},\leq 0}$*) is the inverse image of* $\mathrm{Mod}_{H\mathbf{K},\geq 0}$ *(resp.* $\mathrm{Mod}_{H\mathbf{K},\leq 0}$*) under* $\mathsf{R}_{\mathrm{gm}} : \mathsf{DTM}_{\mathrm{gm}} \to \mathrm{Mod}_{H\mathbf{K}}$*.*

Proof. We will deal with the case $\mathsf{DTM}_{\mathrm{gm},\leq 0}$. The other case is similar. We have already proved that R_{gm} is t-exact in the previous lemma. It will suffice to show that if X does not belong to $\mathsf{DTM}_{\mathrm{gm},\leq 0}$, then $\mathsf{R}_{\mathrm{gm}}(X)$ does not lie in $\mathrm{Mod}_{H\mathbf{K},\leq 0}$. For such X, there exists $i \geq 1$ such that $H_i(X) \neq 0$. According to Corollary 7.3, R_{gm} is conservative. By t-exactness, we deduce that $H_i(\mathsf{R}_{\mathrm{gm}}(X))[-i] \neq 0$. This implies that $\mathsf{R}_{\mathrm{gm}}(X)$ is not in $\mathrm{Mod}_{H\mathbf{K},\leq 0}$, as required. □

By Lemma 7.8, we have a t-exact fully faithful functor $\mathsf{DTM}_{\mathrm{gm}} \to \mathsf{DTM}_{\vee}$, and it induces a natural fully faithful functor $\mathbf{TM}_k \to \mathsf{DTM}_{\vee}^{\heartsuit}$ between (the nerves of) symmetric monoidal abelian categories.

Proposition 7.9. *The natural inclusion* $\mathbf{TM}_k \to \mathsf{DTM}_{\vee}^{\heartsuit}$ *is an equivalence.*

Proof. Since $\mathbf{TM}_k$ is (the nerve of) an abelian category, and in particular it is idempotent complete, it is enough to prove that $\mathbf{TM}_k \to \mathsf{DTM}_{\vee}^{\heartsuit}$ is an idempotent completion. Recall that $\mathsf{DTM}_{\mathrm{gm}} \to \mathsf{DTM}_{\vee}$ is an idempotent completion. Let $X \in \mathbf{TM}_k$. The direct summand of X (which automatically belongs to $\mathsf{DTM}_{\vee}$) lies in $\mathsf{DTM}_{\vee}^{\heartsuit}$ by the definition of t-structure of $\mathsf{DTM}_{\vee}$. Conversely, if $Y \in \mathsf{DTM}_{\vee}^{\heartsuit}$, then there exists $X \in \mathsf{DTM}_{\mathrm{gm}}$ such that Y is equivalent to a direct summand of X. Then Y is a direct summand of $H_0(X) \in \mathbf{TM}_k$ (note that we here use the t-exactness of $\mathsf{DTM}_{\mathrm{gm}} \to \mathsf{DTM}_{\vee}$). Consequently, $\mathbf{TM}_k \to \mathsf{DTM}_{\vee}^{\heartsuit}$ is an idempotent completion. □

Corollary 7.10. *The Tannaka dual of* $\mathbf{TM}_k$ *(endowed with the realization functor) is equivalent to* MTG*.*

Warning 7.10.1. In [26], one works over rational coefficients. In this note, we work over $\mathbf{K}$. Therefore MTG is the base change of the Tannaka dual of the abelian category of mixed Tate motives in [26] over $\mathbb{Q}$ to $\mathbf{K}$.

§7.2. Completion and locally dimensional ∞-category

Let $\mathsf{DTM}^{\otimes} \to \overline{\mathsf{DTM}}^{\otimes}$ be the left completion of $\mathsf{DTM}^{\otimes}$ with respect to the t-structure $(\mathsf{DTM}_{\geq 0}, \mathsf{DTM}_{\leq 0})$ (we refer the reader to [28, 1.2.1.17] and [29, VIII, 4.6.17] for the notions of left completeness and left completion). It is symmetric monoidal, t-exact and colimit-preserving. Here, the ∞-category $\overline{\mathsf{DTM}}$ is the limit of the diagram indexed by $\mathbb{Z}$

$$\cdots \to \mathsf{DTM}_{\leq n+1} \xrightarrow{\tau_{\leq n}} \mathsf{DTM}_{\leq n} \xrightarrow{\tau_{\leq n-1}} \mathsf{DTM}_{\leq n-1} \xrightarrow{\tau_{\leq n-2}} \cdots$$

of ∞-categories. Note that according to [27, 3.3.3] the ∞-category $\overline{\mathsf{DTM}}$ can be identified with the full subcategory of $\mathrm{Fun}(\mathrm{N}(\mathbb{Z}), \mathsf{DTM})$ spanned by functors $\phi : \mathrm{N}(\mathbb{Z}) \to \mathsf{DTM}$ such that

- for any $n \in \mathbb{Z}$, $\phi([n])$ belongs to $\mathsf{DTM}_{\le -n}$,
- for any $m \le n \in \mathbb{Z}$, the associated map $\phi([m]) \to \phi([n])$ gives an equivalence $\tau_{\le -n}\phi([m]) \to \phi([n])$.

Let $\overline{\mathsf{DTM}}_{\ge 0}$ (resp. $\overline{\mathsf{DTM}}_{\le 0}$) be the full subcategory of $\overline{\mathsf{DTM}}$ spanned by $\phi : \mathrm{N}(\mathbb{Z}) \to \mathsf{DTM}$ such that $\phi([n])$ belongs to $\mathsf{DTM}_{\ge 0}$ (resp. $\mathsf{DTM}_{\le 0}$) for each $n \in \mathbb{Z}$. The functor $\mathsf{DTM} \to \overline{\mathsf{DTM}}$ induces an equivalence $\mathsf{DTM}_{\le 0} \to \overline{\mathsf{DTM}}_{\le 0}$. The pair $(\overline{\mathsf{DTM}}_{\ge 0}, \overline{\mathsf{DTM}}_{\le 0})$ is an accessible, left complete and right complete t-structure of $\overline{\mathsf{DTM}}$.

Proposition 7.11. (i) *$\overline{\mathsf{DTM}}_{\le 0}$ is closed under filtered colimits.*

(ii) *The unit* 1 *belongs to the heart* $\overline{\mathsf{DTM}}^{\heartsuit} := \overline{\mathsf{DTM}}_{\ge 0} \cap \overline{\mathsf{DTM}}_{\le 0}$.

(iii) *$\overline{\mathsf{DTM}}_{\ge 0}$ and $\overline{\mathsf{DTM}}_{\le 0}$ are closed under the tensor product $\overline{\mathsf{DTM}} \times \overline{\mathsf{DTM}} \to \overline{\mathsf{DTM}}$.*

(iv) *The unit* 1 *is compact in* $\mathsf{DTM}_{\le n}$ *for each* $n \ge 0$.

(v) *There exists a full subcategory $\overline{\mathsf{DTM}}^{\heartsuit}_{\mathrm{fd}}$ of $\overline{\mathsf{DTM}}^{\heartsuit}$ such that every object in $\overline{\mathsf{DTM}}^{\heartsuit}_{\mathrm{fd}}$ has the dual in $\overline{\mathsf{DTM}}^{\heartsuit}_{\mathrm{fd}}$, and $\overline{\mathsf{DTM}}^{\heartsuit}_{\mathrm{fd}}$ generates $\overline{\mathsf{DTM}}^{\heartsuit}$ under filtered colimits.*

(vi) $\pi_0(\mathrm{Map}_{\overline{\mathsf{DTM}}}(1, 1)) = \mathbf{K}$.

(vii) *For any $X \in \overline{\mathsf{DTM}}^{\heartsuit}_{\mathrm{fd}}$, the composite*

$$1 \to X \otimes X^{\vee} \to 1$$

of the coevaluation map and the evaluation map corresponds to a nonnegative integer $\dim(X) \in \mathbb{Z} \subset \mathbf{K}$.

Proof. From our construction and $\overline{\mathsf{DTM}}_{\le 0} = \mathsf{DTM}_{\le 0}$, (i) is clear. Since the unit of DTM lies in $\mathsf{DTM}^{\heartsuit} := \mathsf{DTM}_{\ge 0} \cap \mathsf{DTM}_{\le 0}$, (ii) follows.

Next we will prove (iii). Taking into account the definition of $\mathsf{DTM}_{\vee, \le 0}$ and $\mathsf{DTM}_{\vee, \ge 0}$ and the conservativity of $\mathsf{R}_T : \mathsf{DTM}_{\vee} \to \mathrm{Mod}_{H\mathbf{K}}$, we see that $\mathsf{DTM}_{\vee, \le 0}$ and $\mathsf{DTM}_{\vee, \ge 0}$ are stable under the tensor operation. Since $\mathrm{Ind}(\mathsf{DTM}_{\vee, \le 0}) = \overline{\mathsf{DTM}}_{\le 0}$ and the tensor operation preserves colimits in each variable, we deduce that $\overline{\mathsf{DTM}}_{\le 0}$ is stable under the tensor operation. Since $\mathsf{DTM}_{\ge 0}$ is stable under the tensor operation of DTM, by definition we also see that $\overline{\mathsf{DTM}}_{\ge 0}$ is stable under the tensor operation.

The unit 1 is compact in DTM, and so is in $\mathsf{DTM}_{\le n}$ for any $n \in \mathbb{Z}$. Noting that $\overline{\mathsf{DTM}}_{\le n} = \mathsf{DTM}_{\le n}$, we have (iv).

To prove (v), note first that $\mathsf{DTM} \to \overline{\mathsf{DTM}}$ induces equivalences $\bigcup_{n\in\mathbb{Z}} \mathsf{DTM}_{\le n} \to \bigcup_{n\in\mathbb{Z}} \overline{\mathsf{DTM}}_{\le n}$ and $\mathsf{DTM}^{\heartsuit} \simeq \overline{\mathsf{DTM}}^{\heartsuit}$. In particular, $\mathsf{DTM}_{\vee} \to \overline{\mathsf{DTM}}$ is fully faithful. Let $X \in \overline{\mathsf{DTM}}^{\heartsuit} = \mathsf{DTM}^{\heartsuit}$. Then X is the filtered colimit of a diagram $I \to \mathsf{DTM}_{\vee,\ge 0}$ in DTM (or in $\overline{\mathsf{DTM}}$): $\mathrm{colim}_{\lambda\in I} X_\lambda \simeq X$. (Recall $\mathsf{DTM}^{\heartsuit} \subset \mathrm{Ind}(\mathsf{DTM}_{\vee,\ge 0})$.) Note that $X_\lambda \in \mathsf{DTM}_{\vee}$ and by definition $\mathsf{DTM}_{\vee,\ge 0}$, $\mathsf{DTM}_{\vee,\le 0}$ and their hearts are stable under the tensor operation. The heart is stable under taking dual objects. It follows that $\tau_{\le 0}(X_\lambda) = H_0(X_\lambda)$ is dualizable, that is, it belongs to $\mathsf{DTM}^{\heartsuit}_{\mathrm{fd}} := \mathsf{DTM}_{\vee} \cap \mathsf{DTM}^{\heartsuit}$ and the dual of $H_0(X_\lambda)$ lies in $\mathsf{DTM}^{\heartsuit}_{\mathrm{fd}}$. Since $\tau_{\le 0}$ is a left adjoint, the natural morphism $\mathrm{colim}_\lambda \tau_{\le 0}(X_\lambda) \to \tau_{\le 0}(\mathrm{colim}_\lambda X_\lambda)$ is an equivalence. This shows that $\mathsf{DTM}^{\heartsuit}_{\mathrm{fd}}$ generates $\mathsf{DTM}^{\heartsuit} = \overline{\mathsf{DTM}}^{\heartsuit}$ under filtered colimits.

We remark that $H^{0,0}(\mathrm{Spec}\, k, \mathbf{K}) = \mathbf{K}$. Hence (vi) holds. Finally, we prove (vii). For any $X \in \overline{\mathsf{DTM}}^{\heartsuit}_{\mathrm{fd}}$, the element in $\mathbf{K}$ corresponding to the composite $1 \to X \otimes X^{\vee} \to 1$ is equal to the element in $\mathbf{K}$ corresponding to $\mathsf{R}_T(1) \to \mathsf{R}_T(X) \otimes \mathsf{R}_T(X)^{\vee} \to \mathsf{R}_T(1)$. The latter element is nothing but the dimension of $\mathsf{R}_T(X)$, which lies in $\mathbb{Z}$. □

Remark 7.12. Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal stable subcategory of $\mathsf{DM}^{\otimes}_{\vee}$ which is closed under taking retracts and dual objects. Suppose that $\mathcal{C}^{\otimes}$ admits a nondegenerate t-structure $(\mathcal{C}_{\ge 0}, \mathcal{C}_{\le 0})$ such that

- the realization functor $\mathcal{C}^{\otimes} \subset \mathsf{DM}^{\otimes}_{\vee} \to \mathrm{Mod}^{\otimes}_{H\mathbf{K}}$ is t-exact,
- both $\mathcal{C}_{\ge 0}$ and $\mathcal{C}_{\le 0}$ are stable under the tensor operation $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

As observed in [1, 1.3], the heart $\mathcal{C}_{\ge 0} \cap \mathcal{C}_{\le 0}$ is a tannakian category equipped with the realization functor as a fiber functor, and the realization functor $\mathcal{C} \to \mathrm{Mod}_{H\mathbf{K}}$ is conservative. Let $\widehat{\mathcal{C}}_{\ge 0}$ (resp. $\widehat{\mathcal{C}}_{\le 0}$) be the left completion of $\mathrm{Ind}(\mathcal{C}_{\ge 0})$ (resp. $\mathrm{Ind}(\mathcal{C}_{\le 0})$). Then as above the pair $(\widehat{\mathcal{C}}_{\ge 0}, \widehat{\mathcal{C}}_{\le 0})$ is an accessible, both right complete and left complete t-structure on the left completion $\widehat{\mathcal{C}}$ of $\mathrm{Ind}\,\mathcal{C}$ (with respect to $(\mathrm{Ind}(\mathcal{C}_{\ge 0}), \mathrm{Ind}(\mathcal{C}_{\le 0}))$). The argument of the above proof shows that the analogous assertions to Proposition 7.11 also hold for $(\widehat{\mathcal{C}}_{\ge 0}, \widehat{\mathcal{C}}_{\le 0})$. (Consequently, analogues of Corollary 7.13 and Proposition 7.14 also hold.)

Corollary 7.13. *The symmetric monoidal ∞-category $\overline{\mathsf{DTM}}^{\otimes}$ endowed with the t-structure $(\overline{\mathsf{DTM}}_{\ge 0}, \overline{\mathsf{DTM}}_{\le 0})$ is a locally dimensional ∞-category in the sense of* [29, VIII, 5.6].

To state the next result which follows from the theory of locally dimensional ∞-categories, we prepare some notation. We say that a commutative ring spectrum S is *discrete* if $\pi_i(S) = 0$ for $i \neq 0$. This property is equivalent to the property that there exists a (usual) commutative ring R such that $HR \simeq S$ in CAlg. Let $\mathrm{CAlg}^{\mathrm{dis}}$

be the ∞-category of discrete commutative ring spectra. The ∞-category $\mathrm{CAlg}^{\mathrm{dis}}$ is equivalent to the nerve of the category of (usual) commutative rings (via Eilenberg–MacLane spectra). Let $\mathfrak{S} : \mathrm{CAlg}^{\mathrm{dis}} \to \widehat{\mathcal{S}}$ be the functor which carries $A \in \mathrm{CAlg}^{\mathrm{dis}}$ to the space $\mathrm{Map}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})}(\overline{\mathsf{DTM}}^{\otimes}, \mathrm{Mod}_A^{\otimes})$ (which can be constructed from Θ of Section 3.1 and Yoneda embedding). Let $\xi : \mathrm{CAlg}^{\mathrm{dis}} \to \widehat{\mathcal{S}}$ be the functor which carries $A \in \mathrm{CAlg}^{\mathrm{dis}}$ to the space $\mathrm{Map}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})}(\mathrm{Mod}_{H\mathbf{K}}^{\otimes}, \mathrm{Mod}_A^{\otimes})$. Since there exists a natural equivalence

$$\mathrm{Map}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})}(\mathrm{Mod}_{H\mathbf{K}}^{\otimes}, \mathrm{Mod}_A^{\otimes}) \simeq \mathrm{Map}_{\mathrm{CAlg}}(H\mathbf{K}, A)$$

(cf. [13, Section 5], [28, 6.3.5.18]), ξ is corepresented by $H\mathbf{K}$. We here write $\mathrm{Spec}\, H\mathbf{K}$ for ξ. There is a sequence of functors $\mathrm{Mod}_{H\mathbf{K}}^{\otimes} \to \overline{\mathsf{DTM}}^{\otimes} \to \mathrm{Mod}_{H\mathbf{K}}^{\otimes}$ whose composite is equivalent to the identity. Therefore we have $\mathrm{Spec}\, H\mathbf{K} \xrightarrow{\eta} \mathfrak{S} \to \mathrm{Spec}\, H\mathbf{K}$ whose composite is the identity. Let $V : \mathrm{CAlg}^{\mathrm{dis}} \to \widehat{\mathcal{S}}$ be a functor equipped with $V \to \mathrm{Spec}\, H\mathbf{K}$. To $f : H\mathbf{K} \to A$ in $\mathrm{CAlg}_{H\mathbf{K}}^{\mathrm{dis}} := (\mathrm{CAlg}^{\mathrm{dis}})_{H\mathbf{K}/}$ we associate $\{f\} \times_{\mathrm{Spec}\, H\mathbf{K}(A)} V(A)$. This yields a functor $V_0 : \mathrm{CAlg}_{H\mathbf{K}}^{\mathrm{dis}} \to \widehat{\mathcal{S}}$. The morphism $\eta : \mathrm{Spec}\, H\mathbf{K} \to \mathfrak{S}$ induces $\eta_0 : (\mathrm{Spec}\, H\mathbf{K})_0 \to \mathfrak{S}_0$. Note that $(\mathrm{Spec}\, H\mathbf{K})_0$ is equivalent to the constant functor taking the value Δ^0, that is, the final object.

The following result was essentially proved by Lurie in the context of locally dimensional ∞-categories (see [29, VIII, 5.2.12, 5.6.1, 5.6.19 and their proofs]). We here state only the version related to Corollary 7.13, which fits in with our need.

Proposition 7.14 ([29]). *Let* $\mathrm{Grp}^{\mathrm{dis}}$ *be the nerve of the category of (usual) groups. Consider the functor* $\pi_1(\mathfrak{S}_0, \eta_0) : \mathrm{CAlg}_{H\mathbf{K}}^{\mathrm{dis}} \to \mathrm{Grp}^{\mathrm{dis}}$ *which is given by* $A \mapsto \pi_1(\mathfrak{S}_0(A), \eta_0)$*. Then* $\pi_1(\mathfrak{S}_0, \eta_0)$ *is represented by MTG, that is, the Tannaka dual of* $(\mathsf{DTM}_\vee^\heartsuit)^{\otimes}$.

§7.3. Comparison theorem

Definition 7.15. Let $G : \mathrm{CAlg}_{H\mathbf{K}} \to \mathrm{Grp}(\mathcal{S})$ be a derived affine group scheme over $H\mathbf{K}$. Let $\pi_0 : \mathrm{Grp}(\mathcal{S}) \to \mathrm{Grp}^{\mathrm{dis}}$ be the truncation functor given by $G \mapsto \pi_0(G)$. Here $\mathrm{Grp}^{\mathrm{dis}}$ denotes the nerve of the category of groups. If the composition

$$G' : \mathrm{CAlg}_{H\mathbf{K}}^{\mathrm{dis}} \hookrightarrow \mathrm{CAlg}_{H\mathbf{K}} \xrightarrow{G} \mathrm{Grp}(\mathcal{S}) \xrightarrow{\pi_0} \mathrm{Grp}^{\mathrm{dis}}$$

is represented by an affine group scheme G_0 over $\mathbf{K}$, we say that G_0 is an *excellent coarse moduli space* of G. If there is an affine group scheme G_0 (considered as $\mathrm{CAlg}_{H\mathbf{K}}^{\mathrm{dis}} \to \mathrm{Grp}(\mathcal{S})$) and a morphism $G|_{\mathrm{CAlg}_{H\mathbf{K}}^{\mathrm{dis}}} \to G_0$ that is universal among morphisms into affine group schemes over $\mathbf{K}$, we say that G_0 is a *coarse moduli space* of G. We remark that an excellent coarse moduli space is a coarse moduli space.

Theorem 7.16. *Let* MTG *denote the tannakization of* $\mathsf{R}_T : \mathsf{DTM}^{\otimes}_{\vee} \to \mathrm{PMod}^{\otimes}_{H\mathbf{K}}$ (*cf. Theorem* 6.12). *Then MTG is an excellent coarse moduli space of* MTG.

Proof. For $A \in \mathrm{CAlg}^{\mathrm{dis}}$, we set $\mathrm{Mod}_{A,\geq 0} = \{X \in \mathrm{Mod}_A \mid \pi_i(X) = 0 \text{ for } i < 0\}$ and $\mathrm{Mod}_{A,\leq 0} = \{X \in \mathrm{Mod}_A \mid \pi_i(X) = 0 \text{ for } i > 0\}$. Then the pair $(\mathrm{Mod}_{A,\geq 0}, \mathrm{Mod}_{A,\leq 0})$ is an accessible, left and right complete t-structure. Thus we have

$$\begin{aligned}\mathrm{Map}^{\mathrm{rex}}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}^{\mathrm{L,st}}_{\infty})}(\overline{\mathsf{DTM}}^{\otimes}, \mathrm{Mod}^{\otimes}_A) &\simeq \mathrm{Map}^{\mathrm{rex}}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}^{\mathrm{L,st}}_{\infty})}(\mathsf{DTM}^{\otimes}, \mathrm{Mod}^{\otimes}_A)\\ &\hookrightarrow \mathrm{Map}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\infty})}(\mathsf{DTM}^{\otimes}_{\vee}, \mathrm{Mod}^{\otimes}_A)\end{aligned}$$

where $\mathrm{Map}^{\mathrm{rex}}$ indicates the full subcategory spanned by right t-exact functors, and the second arrow is fully faithful by Proposition 4.7. (The essential image consists of symmetric monoidal exact functors which are right t-exact.) Note that $\mathsf{R}_T : \mathsf{DTM}^{\otimes} \to \mathrm{Mod}^{\otimes}_{H\mathbf{K}}$ is t-exact, and it belongs to $\mathrm{Map}^{\mathrm{rex}}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}^{\mathrm{L,st}}_{\infty})}(\mathsf{DTM}^{\otimes}, \mathrm{Mod}^{\otimes}_A)$.

Consider the automorphism group functor $\mathrm{Aut}(\mathsf{R}_T) : \mathrm{CAlg}_{H\mathbf{K}} \to \mathrm{Grp}(\mathcal{S})$ of $\mathsf{R}_T : \mathsf{DTM}^{\otimes}_{\vee} \to \mathrm{PMod}^{\otimes}_{H\mathbf{K}}$ in $\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{st}}_{\infty})$ (we abuse notation for R_T). According to Theorem 6.12, $\mathrm{Aut}(\mathsf{R}_T)$ is represented by MTG. On the other hand, using the above equivalence and unfolding the definitions of $\pi_1(\mathfrak{S}_0, \eta_0)$ and $\mathrm{Aut}(\mathsf{R}_T)$, we see that the composite

$$\mathrm{CAlg}^{\mathrm{dis}}_{H\mathbf{K}} \hookrightarrow \mathrm{CAlg}_{H\mathbf{K}} \xrightarrow{\mathrm{Aut}(\mathsf{R}_T)} \mathrm{Grp}(\mathcal{S}) \xrightarrow{\pi_0} \mathrm{Grp}^{\mathrm{dis}}$$

is equivalent to $\pi_1(\mathfrak{S}_0, \eta_0)$. Combining this with Proposition 7.14 we complete the proof. □

Remark 7.17. The truncation procedure given in [19, Section 5] also allows us to construct the (usual) affine group scheme $MTG' = \operatorname{Spec} H^0(\tau B)$ over $\mathbf{K}$ from $\mathsf{MTG} = \operatorname{Spec} B$ where B is a commutative differential graded algebra. Here we adopt the notation of [19, Section 5]. This MTG' coincides with the above MTG. To observe this, we invoke τ of [19] to have $\operatorname{Spec} \tau B$, and we regard $\operatorname{Spec} \tau B$ as a functor $\mathrm{CAlg}^{\mathrm{dis}}_{H\mathbf{K}} \to \mathrm{Grp}(\mathcal{S})$. As functors $\mathrm{CAlg}^{\mathrm{dis}}_{H\mathbf{K}} \to \mathrm{Grp}(\mathcal{S})$, $\operatorname{Spec} \tau B$ and $\operatorname{Spec} B$ are equivalent. Also, let us regard MTG as a functor $\mathrm{CAlg}^{\mathrm{dis}}_{H\mathbf{K}} \to \mathrm{Grp}(\mathcal{S})$. Then we have a natural morphism $\operatorname{Spec} \tau B \to MTG$. Also, the composition with $\operatorname{Spec} \tau B \to MTG$ induces an equivalence of spaces

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{dis}}_{H\mathbf{K}}, \mathrm{Grp}(\mathcal{S}))}(MTG, F) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{dis}}_{H\mathbf{K}}, \mathrm{Grp}(\mathcal{S}))}(\operatorname{Spec} \tau B, F)$$

for any $F : \mathrm{CAlg}^{\mathrm{dis}}_{H\mathbf{K}} \to \mathrm{Grp}^{\mathrm{dis}} \subset \mathrm{Grp}(\mathcal{S})$. On the other hand, by the construction in [19], $\operatorname{Spec} \tau B \to MTG'$ is universal among morphisms to usual affine group schemes over $\mathbf{K}$. Hence $MTG \simeq MTG'$.

§8. Artin motives and absolute Galois group

Let G_k denote the absolute Galois group $\mathrm{Gal}(\bar{k}/k)$ with $\bar{k}$ an algebraic closure of a perfect field k. For the sake of completeness, we will construct a natural homomorphism

$$\mathsf{MG}_E \to G_k$$

where MG_E is the derived motivic Galois group. This represents the automorphism functor of $\mathsf{R}_E : \mathsf{DM}^{\otimes}_{\vee} \to \mathrm{PMod}^{\otimes}_{H\mathbf{K}}$. Here $\mathsf{DM}^{\otimes}_{\vee}$ denotes the symmetric monoidal full subcategory of $\mathsf{DM}^{\otimes}$ spanned by dualizable objects. To this end, we consider the full subcategory of DM which consists of Artin motives, and we will finish by proving that its tannakization is the absolute Galois group (Proposition 8.3).

Let $\mathrm{Cor}_{\mathbf{K},0}$ be the full subcategory of $\mathrm{Cor}_{\mathbf{K}}$ spanned by smooth schemes X which are étale over $\mathrm{Spec}\, k$. We write simply Cor_0 and Cor for $\mathrm{Cor}_{\mathbf{K},0}$ and $\mathrm{Cor}_{\mathbf{K}}$ respectively. The classical Galois theory says that the category of schemes which are étale over k is equivalent to the category of finite G_k-sets. Consequently, we easily see that there is a fully faithful functor $\mathrm{Cor}_0 \to \mathbf{K}[G_k]$-Mod which carries X to the $\mathbf{K}$-vector space generated by the set $X(\bar{k})$ endowed with an action of G_k. Here $\mathbf{K}[G_k]$-Mod denotes the category of $\mathbf{K}[G_k]$-modules, i.e. abelian groups equipped with (left) actions of $\mathbf{K}[G_k]$. The essential image consists of permutational representations (see [38, p. 216]).

Let $\iota : \mathrm{Cor} \to \mathrm{Cor}_0$ be the left adjoint of the inclusion $\mathrm{Cor}_0 \hookrightarrow \mathrm{Cor}$. The functor ι carries X to the Zariski spectrum of the integral closure of k in $\Gamma(X)$. Let $\mathrm{PSh}(\mathrm{Cor}_0)$ be the category of presheaves (of $\mathbf{K}$-vector spaces) with transfers, that is, the category of $\mathbf{K}$-linear functors $(\mathrm{Cor}_0)^{\mathrm{op}} \to \mathbf{K}$-Vect where $\mathbf{K}$-Vect is the category of $\mathbf{K}$-vector spaces. Note that $\mathrm{PSh}(\mathrm{Cor}_0)$ contains Cor_0 as a full subcategory by the enriched Yoneda lemma [24]. There is a symmetric monoidal structure on $\mathrm{PSh}(\mathrm{Cor}_0)$ which makes $\mathrm{Cor}_0 \hookrightarrow \mathrm{PSh}(\mathrm{Cor}_0)$ symmetric monoidal such that the tensor product $\mathrm{PSh}(\mathrm{Cor}_0) \times \mathrm{PSh}(\mathrm{Cor}_0) \to \mathrm{PSh}(\mathrm{Cor}_0)$ preserves small colimits separately in each variable. Such a symmetric monoidal structure is usually called *Day convolution* [8]. This exhibits $\mathrm{PSh}(\mathrm{Cor}_0)$ as a symmetric monoidal abelian category. We define $\mathrm{Sh}(\mathrm{Cor})$ to be the symmetric monoidal category of Nisnevich sheaves with transfer (see [6]). Composition with ι and sheafification induces a symmetric monoidal functor $\mathrm{PSh}(\mathrm{Cor}_0) \to \mathrm{Sh}(\mathrm{Cor})$. This gives rise to a functor

$$\mathrm{Comp}(\mathrm{PSh}(\mathrm{Cor}_0)) \to \mathrm{Comp}(\mathrm{Sh}(\mathrm{Cor})).$$

Let us equip the category $\mathrm{Comp}(\mathrm{Sh}(\mathrm{Cor}))$ with the model structure given in [6, 2.4], in which weak equivalences are quasi-isomorphisms. We equip $\mathrm{Comp}(\mathrm{PSh}(\mathrm{Cor}_0))$ with the model structure of [6, 2.5] by choosing the descent structure $(\mathcal{G}, \mathcal{H})$ in [6, 2.2] as $\mathcal{G} :=$ sheaves represented by objects in Cor_0, and $\mathcal{H} = \{0\}$. Then by [6, 2.14] the above functor is a left Quillen adjoint symmetric monoidal functor. Hence we take its localization and obtain the symmetric monoidal colimit-preserving functor

$$\mathrm{N}(\mathrm{Comp}(\mathrm{PSh}(\mathrm{Cor}_0))^c)^{\otimes}_{\infty} \to \mathrm{N}(\mathrm{Comp}(\mathrm{PSh}(\mathrm{Cor}))^c)^{\otimes}_{\infty}$$

of symmetric monoidal ∞-categories. By the construction of DM (cf. [6, 7.15]) there is a natural symmetric monoidal colimit-preserving functor

$$\mathrm{N}(\mathrm{Comp}(\mathrm{Sh}(\mathrm{Cor}))^c)^{\otimes}_{\infty} \to \mathsf{DM}^{\otimes}$$

which is induced by localization with respect to $\mathbb{A}^1$-homotopy equivalence and by stabilization with respect to the Tate sphere. Thus by composition we obtain the symmetric monoidal functor

$$\mathfrak{A} : \mathsf{Q}^{\otimes} := \mathrm{N}(\mathrm{Comp}(\mathrm{PSh}(\mathrm{Cor}_0))^c)^{\otimes}_{\infty} \to \mathsf{DM}^{\otimes} \simeq \mathsf{Sp}_{\mathrm{Tate}}(\mathbf{HK})^{\otimes}.$$

The image of the inclusion $\mathrm{Cor}_0 \hookrightarrow \mathrm{Comp}(\mathrm{PSh}(\mathrm{Cor}_0))$ is contained in $\mathrm{Comp}(\mathrm{PSh}(\mathrm{Cor}_0)^c)$. Let $\mathsf{Art}(k)$ be the smallest stable idempotent complete subcategory which contains its essential image. Alternatively, if we let A be the triangulated thick subcategory of $\mathrm{h}(\mathsf{Q})$ generated by the essential image of $\mathrm{Cor}_0 \to \mathrm{h}(\mathsf{Q})$, then $\mathsf{Art}(k) \simeq \mathsf{Q} \times_{\mathrm{h}(\mathsf{Q})} A$. Observe that by elementary representation theory and the fully faithful embedding $\mathrm{Cor}_0 \subset G_k$-Mod, the idempotent completion $\mathrm{Cor}_0^{\sim}$ of Cor_0 (in $\mathrm{PSh}(\mathrm{Cor}_0)$) can be identified with the abelian category of discrete representations of G_k (that is, actions $\rho : G_k \to \mathrm{Aut}(V)$ of G_k on finite-dimensional $\mathbf{K}$-vector spaces V that factor through some finite quotient $G_k \to H$). The abelian category $\mathrm{Cor}_0^{\sim}$ is semisimple. Hence the stable subcategory $\mathsf{Art}(k)$ of Q is spanned by bounded complexes C such that C^n belongs to $\mathrm{Cor}_0^{\sim}$ for each $n \in \mathbb{Z}$ (indeed such complexes are cofibrant). Note that the symmetric monoidal structure of $\mathsf{Q}^{\otimes}$ induces the symmetric monoidal structure of $\mathsf{Art}(k)$. According to [38, 3.4.1] and Voevodsky's cancellation theorem together with [19, Lemma 5.8], we deduce:

Lemma 8.1. *The natural functor* $\mathsf{Art}(k) \to \mathsf{DM}$ *is fully faithful.*

We identify $\mathsf{Art}(k)^{\otimes}$ with a symmetric monoidal full subcategory of $\mathsf{DM}^{\otimes}$ and refer to it as the ∞-*category of Artin motives*. We remark that $\mathsf{Art}(k)$ is contained in the full subcategory of DM spanned by compact objects.

We regard G_k as a limit $\lim(\mathrm{Gal}(L/k))$ where L runs through all finite Galois extensions L of k. Let $\mathrm{Gal}(L/k)$-Perm be the $\mathbf{K}$-linear category of permutational representations. We define $\mathrm{PSh}(\mathrm{Gal}(L/k)\text{-Perm})$ to be the symmetric monoidal category of presheaves (of $\mathbf{K}$-vector spaces) on $\mathrm{Gal}(L/k)$-Perm in the same way as $\mathrm{PSh}(\mathrm{Cor}_0)$. Let us equip the category $\mathrm{Comp}(\mathrm{PSh}(\mathrm{Gal}(L/k)\text{-Perm}))$ with the symmetric monoidal model structure given in [6, 2.5, 3.2] by choosing the descent structure $(\mathcal{G}, \mathcal{H})$ of [6, 2.2] as $\mathcal{G}$:= sheaves represented by objects of $\mathrm{Gal}(L/k)$-Perm and $\mathcal{H} = \{0\}$. Let $\mathrm{Gal}(L/k)\text{-Perm}^{\sim}$ be the idempotent completion of $\mathrm{Gal}(L/k)$-Perm which can be identified with the abelian category of finite-dimensional representations of $\mathrm{Gal}(L/k)$. Let A_L be the stable subcategory of $\mathsf{B}_L := \mathrm{N}(\mathrm{Comp}(\mathrm{PSh}(\mathrm{Gal}(L/k)\text{-Perm}))^c)_\infty$ spanned by bounded complexes C such that C^n lies in $\mathrm{Gal}(L/k)\text{-Perm}^{\sim}$ for each $n \in \mathbb{Z}$. The full subcategory A_L is a symmetric monoidal full subcategory of $\mathsf{B}_L^{\otimes}$ (spanned by dualizable objects). The quotient homomorphism $\pi_L : G_k \to \mathrm{Gal}(L/k)$ naturally induces a symmetric monoidal functor $\mathrm{PSh}(\mathrm{Gal}(L/k)\text{-Perm}) \to \mathrm{PSh}(\mathrm{Cor}_0)$ which is a left Kan extension of the natural functor $\xi : \mathrm{Gal}(L/k)\text{-Perm} \hookrightarrow \mathrm{Cor}_0 \hookrightarrow \mathrm{PSh}(\mathrm{Cor}_0)$. This left adjoint is given by the formula $M \mapsto \mathrm{colim}_{H_\sigma \to M}\, \xi(H_\sigma)$ where H_σ is a presheaf represented by $\sigma \in \mathrm{Gal}(L/k)$-Perm and $H_\sigma \to M$ runs through the overcategory $\mathrm{Gal}(L/k)\text{-Perm}_{/M}$. This gives rise to a symmetric monoidal colimit-preserving functor

$$\mathrm{Comp}(\mathrm{PSh}(\mathrm{Gal}(L/k)\text{-Perm})) \to \mathrm{Comp}(\mathrm{PSh}(\mathrm{Cor}_0)).$$

By the definition of the model structures this is a left Quillen adjoint, and we obtain a symmetric monoidal colimit-preserving functor

$$\mathrm{N}(\mathrm{Comp}(\mathrm{PSh}(\mathrm{Gal}(L/k)\text{-Perm}))^c)_\infty^{\otimes} \to \mathsf{Q}^{\otimes} = \mathrm{N}(\mathrm{Comp}(\mathrm{PSh}(\mathrm{Cor}_0))^c)_\infty^{\otimes}.$$

Taking account of all finite Galois extensions L we have

$$f : \mathrm{colim}_L(\mathsf{A}_L^{\otimes}) \to \mathsf{Art}(k)^{\otimes}.$$

Lemma 8.2. *The functor f is an equivalence of symmetric monoidal ∞-categories.*

Proof. By [28, 3.2.3.1, 4.2.3.5] we can regard the underlying ∞-category of $\mathrm{colim}_L(\mathsf{A}_L^{\otimes})$ as a colimit of the diagram of the underlying ∞-categories A_L in Cat_∞. Moreover, the filtered colimit of the stable ∞-categories A_L is also stable [28, 1.1.4.6]. Thus by [19, Lemma 5.8] it is enough to observe that $f : \mathrm{colim}_L(\mathsf{A}_L) \to \mathsf{Art}(k)$ induces an equivalence of their homotopy categories. Clearly, f is essentially surjective. By computing the hom sets in the homotopy category we see that f induces a fully faithful functor $\mathrm{h}(\mathsf{A}_L) \to \mathrm{h}(\mathsf{Art}(k))$ of homotopy categories for each L. □

Let $\mathsf{R}' : \mathsf{Art}(k)^{\otimes} \to \mathrm{PMod}_{H\mathbf{K}}^{\otimes}$ be the composition of $\mathsf{Art}(k)^{\otimes} \to \mathsf{DM}_{\vee}^{\otimes}$ and the realization functor $\mathsf{R} : \mathsf{DM}_{\vee}^{\otimes} \to \mathrm{PMod}_{H\mathbf{K}}^{\otimes}$ associated to a mixed Weil theory E. We study the automorphism group of R'. We will show that it is represented by G_k. Here for a finite Galois extension L, we regard $\mathrm{Gal}(L/k)$ as the constant derived affine group scheme over $H\mathbf{K}$ and we think of G_k as the limit of derived affine group schemes $\mathrm{Gal}(L/k)$.

Let $\acute{\mathrm{E}}\mathrm{t}/k$ be the category of étale schemes over k. There is a natural functor $\acute{\mathrm{E}}\mathrm{t}/k \to \mathrm{Cor}_0$ determined by graphs. Then we have the composition

$$\acute{\mathrm{E}}\mathrm{t}/k \to \mathrm{Cor}_0 \to \mathsf{Art}(k) \to \mathrm{PMod}_{H\mathbf{K}}$$

where the second functor is the natural functor induced by $\mathrm{Cor}_0 \to \mathrm{Comp}(\mathrm{PSh}(\mathrm{Cor}_0))$. (We often do not indicate taking the simplicial nerves of the ordinary categories.) Note that the second functor is fully faithful. The essential image is contained in the heart of $\mathrm{PMod}_{H\mathbf{K}}$ with respect to the standard t-structure, that is, the category of $\mathbf{K}$-vector spaces (the standard t-structure is determined by a pair of full subcategories: the first consists of spectra which are concentrated in nonnegative degrees, and the second consists of spectra which are concentrated in nonpositive degrees). Then this gives rise to $\acute{\mathrm{E}}\mathrm{t}/k \to \mathbf{K}\text{-Vect}$.

Now suppose that the mixed Weil theory E is either l-adic étale cohomology theory or Betti cohomology, see [7] ($\mathbf{K}$ depends on the choice of a mixed Weil cohomology theory). Then $\acute{\mathrm{E}}\mathrm{t}/k \to \mathbf{K}\text{-Vect}$ carries X to the $\mathbf{K}$-vector space generated by the set of $X(\bar{k})$ ($\bar{k}$ is the algebraic closure). Applying [30, 6.5] (after taking the dual vector spaces) we see that there exists a unique extension $\mathrm{Cor}_0 \to \mathbf{K}\text{-Vect}$ of $\acute{\mathrm{E}}\mathrm{t}/k \to \mathbf{K}\text{-Vect}$. Such a functor $\mathrm{Cor}_0 \to \mathbf{K}\text{-Vect}$ is given by $\mathrm{Cor}_0 \simeq \mathbf{K}[G_k]\text{-Perm} \to \mathbf{K}\text{-Vect}$ where $\mathbf{K}[G_k]\text{-Perm}$ denotes the category of permutational representations and the second functor is the forgetful functor (it is also symmetric monoidal). Consequently, the restriction $\mathrm{Cor}_0^{\sim} \to \mathbf{K}\text{-Vect}$ of $\mathsf{Art}(k) \to \mathrm{PMod}_{H\mathbf{K}}$ to $\mathrm{Cor}_0^{\sim}$ (contained in $\mathsf{Art}(k)$ as a symmetric monoidal full subcategory) is equivalent to the forgetful functor $\mathbf{K}[G_k]\text{-rep} \to \mathbf{K}\text{-Vect}$ as a symmetric monoidal functor. Here $\mathbf{K}[G_k]\text{-rep}$ is the category of finite-dimensional discrete representations of G_k. The stable ∞-category A_L has the standard t-structure, whose heart is $\mathrm{Gal}(L/k)\text{-Perm}^{\sim}$. Recall that this idempotent completion is equivalent to the category of finite-dimensional representations of $\mathrm{Gal}(L/k)$. The composition $\mathsf{A}_L^{\otimes} \to \mathsf{Art}(k)^{\otimes} \to \mathrm{PMod}_{H\mathbf{K}}^{\otimes}$ induces a ($\mathbf{K}$-linear) symmetric monoidal functor $(\mathrm{Gal}(L/k)\text{-Perm}^{\sim})^{\otimes} \to \mathbf{K}\text{-Vect}^{\otimes}$ which we can identify with the forgetful functor. According to the main theorem in [13, Section 5] together with the classical Tannaka duality (cf. [11]), $\mathsf{A}_L^{\otimes} \to \mathrm{PMod}_{H\mathbf{K}}^{\otimes}$ is equivalent to the forgetful functor $\mathrm{PRep}_{\mathrm{Gal}(L/k)}^{\otimes} \to \mathrm{PMod}_{H\mathbf{K}}^{\otimes}$.

Proposition 8.3. *The absolute Galois group G_k is the tannakization of $\mathsf{Art}^{\otimes}(k) \to \mathrm{PMod}_{H\mathbf{K}}^{\otimes}$.*

Proof. By Lemma 8.2, we are reduced to showing that the tannakization of the forgetful functor $\mathsf{A}_L^{\otimes} \to \mathrm{PMod}_{H\mathbf{K}}^{\otimes}$ is the constant finite group scheme $\mathrm{Gal}(L/k)$ over $H\mathbf{K}$. Our claim follows from Corollary 4.9. □

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