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Variations of Mixed Hodge Structure and Semipositivity Theorems

by

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Abstract

We discuss variations of mixed Hodge structure for cohomology with compact support of quasi-projective simple normal crossing pairs. We show that they are graded polarizable admissible variations of mixed Hodge structure. Then we prove a generalization of the Fujita–Kawamata semipositivity theorem.

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Contents

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§1. Introduction

Let X be a simple normal crossing divisor on a smooth projective variety M and let B be a simple normal crossing divisor on M such that $X + B$ is simple normal crossing on M and that X and B have no common irreducible components. Then the pair (X, D), where $D = B|_X$, is a typical example of simple normal crossing pairs. In this situation, a stratum of (X, D) is an irreducible component of $T_{i_1} \cap \cdots \cap T_{i_k} \subset X$ for some $\{i_1, \ldots, i_k\} \subset I$, where $X + B = \sum_{i \in I} T_i$ is the irreducible decomposition of $X + B$. For the precise definition of simple normal crossing pairs, see Definition 2.1 below. We note that simple normal crossing pairs frequently appear in the study of the log minimal model program for higher dimensional algebraic varieties with bad singularities. The first author has already investigated the mixed Hodge structures for $H^\bullet_c(X \setminus D, \mathbb{Q})$ in [F7, Chapter 2] to obtain various vanishing theorems (see also [F17]). In this paper, we show that their variations are graded polarizable admissible variations of mixed Hodge structure. Then we prove a generalization of the Fujita–Kawamata semipositivity theorem. Our formulation of the theorem is different from Kawamata's original one. However, it is more suited for our studies of simple normal crossing pairs.

The following theorem is a corollary of Theorems 7.1 and 7.3, which are our main results in this paper (cf. [Kw1, Theorem 5], [Ko2, Theorem 2.6], [N1, Theorem 1], [F4, Theorems 3.4 and 3.9], [Kw3, Theorem 1.1], and so on).

Theorem 1.1 (Semipositivity theorem; cf. Theorems 7.1 and 7.3). *Let (X, D) be a simple normal crossing pair such that D is reduced and let $f : X \to Y$ be a projective surjective morphism onto a smooth complete algebraic variety Y. Assume that every stratum of (X, D) is dominant onto Y. Let Σ be a simple normal crossing divisor on Y such that every stratum of (X, D) is smooth over $Y^* = Y \setminus \Sigma$. Then $R^p f_* \omega_{X/Y}(D)$ is locally free for every p. Set $X^* = f^{-1}(Y^*)$, $D^* = D|_{X^*}$, and $d = \dim X - \dim Y$. Further assume that all the local monodromies on $R^{d-i}(f|_{X^* \setminus D^*})_! \mathbb{Q}_{X^* \setminus D^*}$ around Σ are unipotent. Then $R^i f_* \omega_{X/Y}(D)$ is a semipositive locally free sheaf on Y.*

We note the following definition.

Definition 1.2 (Semipositivity in the sense of Fujita–Kawamata). A locally free sheaf $\mathcal{E}$ of finite rank on a complete algebraic variety X is said to be *semipositive* (in the sense of Fujita–Kawamata) if $\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(1)$ is nef on $\mathbb{P}_X(\mathcal{E})$.

In [Kw3], Kawamata obtained a weaker result similar to Theorem 1.1 (see [Kw3, Theorem 1.1]). It is not surprising because both [Kw3] and this paper grew out from the same question raised by Valery Alexeev and Christopher Hacon. For details on Kawamata's approach, we recommend [FFS], where we use the theory of mixed Hodge modules to give an alternative proof of Theorem 1.1. Note that [FFS] was written after this paper was circulated.

The semipositivity of $R^i f_* \omega_{X/Y}(D)$ in Theorem 1.1 follows from a purely Hodge-theoretic semipositivity theorem (Theorem 5.21). In the proof of Theorem 1.1, we use the semipositivity of $(\mathrm{Gr}_F^a \mathcal{V})^*$ in Theorem 1.3, where $(\mathrm{Gr}_F^a \mathcal{V})^* = \mathcal{H}om_{\mathcal{O}_Y}(\mathrm{Gr}_F^a \mathcal{V}, \mathcal{O}_Y)$. We do not need the semipositivity of $F^b\mathcal{V}$ in Theorem 1.3 for Theorem 1.1. For more details, see the discussion in 1.6 below.

Theorem 1.3 (Hodge-theoretic semipositivity theorem; cf. Theorem 5.21). *Let X be a smooth complete complex algebraic variety, D a simple normal crossing divisor on X, and $\mathcal{V}$ a locally free $\mathcal{O}_X$-module of finite rank equipped with a finite increasing filtration W and a finite decreasing filtration F. Assume the following:*

(1) *$F^a\mathcal{V} = \mathcal{V}$ and $F^{b+1}\mathcal{V} = 0$ for some $a < b$.*

(2) *$\mathrm{Gr}_F^p \mathrm{Gr}_m^W \mathcal{V}$ is a locally free $\mathcal{O}_X$-module of finite rank for all m, p.*

(3) *For all m, $\mathrm{Gr}_m^W \mathcal{V}$ admits an integrable logarithmic connection ∇_m with nilpotent residue morphisms which satisfies the conditions*

$$\nabla_m(F^p \mathrm{Gr}_m^W \mathcal{V}) \subset \Omega_X^1(\log D) \otimes F^{p-1}\mathrm{Gr}_m^W \mathcal{V} \quad \text{for all } p.$$

(4) *The triple $(\mathrm{Gr}_m^W \mathcal{V}, F\mathrm{Gr}_m^W \mathcal{V}, \nabla_m)|_{X \setminus D}$ underlies a polarizable variation of $\mathbb{R}$-Hodge structure of weight m for every integer m.*

Then $(\mathrm{Gr}_F^a \mathcal{V})^*$ and $F^b\mathcal{V}$ are semipositive.

In this paper, we concentrate on the Hodge-theoretic aspect of the Fujita–Kawamata semipositivity theorem (cf. [Z], [Kw1], [Ko2], [N1], [F4], and [FFS]). On the other hand, there are many results related to that theorem from the analytic viewpoint (cf. [Ft], [Be], [BeP], [MT], and so on). Note that Griffiths's pioneering work on the variation of Hodge structure (cf. [G]) is a starting point of the Fujita–Kawamata semipositivity theorem. For a related topic, see [Moc]. Mochizuki's approach is completely different from ours and has a more arithmetic-geometrical flavor.

As a special case of Theorem 1.1, we obtain the following (Theorem 7.7).

Theorem 1.4 (cf. [Kw1, Theorem 5], [Ko2, Theorem 2.6], and [N1, Theorem 1]). *Let $f : X \to Y$ be a projective morphism between smooth complete algebraic varieties which satisfies the following conditions:*

(i) *There is a Zariski open subset Y^* of Y such that $\Sigma = Y \setminus Y^*$ is a simple normal crossing divisor on Y.*

(ii) *Set* $X^* = f^{-1}(Y^*)$. *Then* $f|_{X^*}$ *is smooth.*

(iii) *All local monodromies of* $R^{d+i}(f|_{X^*})_*\mathbb{C}_{X^*}$ *around* Σ *are unipotent, where* $d = \dim X - \dim Y$.

Then $R^i f_*\omega_{X/Y}$ *is a semipositive locally free sheaf on* Y.

We note that Theorem 1.4 was first proved by Kawamata (cf. [Kw1, Theorem 5]) under the extra assumptions that $i = 0$ and that f has connected fibers. The above statement follows from [Ko2, Theorem 2.6] or [N1, Theorem 1] (see also [F5, Theorem 5.4]). We also note that, by Poincaré–Verdier duality, $R^{d+i}(f|_{X^*})_*\mathbb{C}_{X^*}$ is the dual local system of $R^{d-i}(f|_{X^*})_*\mathbb{C}_{X^*}$ in Theorem 1.4. In [Ko2] and [N1], variations of Hodge structure on $R^{d+i}(f|_{X^*})_*\mathbb{C}_{X^*}$ are investigated for the proof of Theorem 1.4. On the other hand, in this paper, we concentrate on variations of Hodge structure on $R^{d-i}(f|_{X^*})_*\mathbb{C}_{X^*}$ for Theorem 1.4.

The following example shows that the assumption (2) in Theorem 1.3 is indispensable. For related examples, see [SZ, (3.15) and (3.16)]. In the proof of Theorem 1.1, the admissibility of the graded polarizable variation of $\mathbb{Q}$-mixed Hodge structure on $R^{d-i}(f|_{X^*\setminus D^*})_!\mathbb{Q}_{X^*\setminus D^*}$, which is proved in Theorem 4.15, ensures the existence of the extension of the Hodge filtration satisfying (2). Note that the notion of *admissibility* is due to Steenbrink–Zucker [SZ] and Kashiwara [Ks].

Example 1.5. Let $\mathbb{V}$ be a 2-dimensional $\mathbb{Q}$-vector space with basis $\{e_1, e_2\}$. We define an increasing filtration W on $\mathbb{V}$ by $W_{-1}\mathbb{V} = 0$, $W_0\mathbb{V} = W_1\mathbb{V} = \mathbb{Q}e_1$, and $W_2\mathbb{V} = \mathbb{V}$. The constant sheaf on $\mathbb{P}^1$ whose fibers are $\mathbb{V}$ is again denoted by $\mathbb{V}$. An increasing filtration W on $\mathbb{V}$ is given as above. We consider $\mathcal{V} = \mathcal{O}_{\mathbb{P}^1} \otimes \mathbb{V} = \mathcal{O}e_1 \oplus \mathcal{O}e_2$ on $\mathbb{P}^1$. We define a decreasing filtration F on $\mathcal{V}|_{\mathbb{C}^*}$ by

$$F^0(\mathcal{V}|_{\mathbb{C}^*}) = \mathcal{V}|_{\mathbb{C}^*},\ F^1(\mathcal{V}|_{\mathbb{C}^*}) = \mathcal{O}_{\mathbb{C}^*}(t^{-1}e_1 + e_2),\ F^2(\mathcal{V}|_{\mathbb{C}^*}) = 0,$$

where t is the coordinate function of $\mathbb{C} \subset \mathbb{P}^1$. We can easily check that

$$((\mathbb{V}, W)|_{\mathbb{C}^*}, (\mathcal{V}|_{\mathbb{C}^*}, F))$$

is a graded polarizable variation of $\mathbb{Q}$-mixed Hodge structure on $\mathbb{C}^*$. In this case, we cannot extend the Hodge filtration F on $\mathcal{V}|_{\mathbb{C}^*}$ to the filtration F on $\mathcal{V}$ satisfying assumption (2) of Theorem 1.3. In particular, the above variation of $\mathbb{Q}$-mixed Hodge structure is not admissible.

We note that we can extend the Hodge filtration F on $\mathcal{V}|_{\mathbb{C}^*}$ to a filtration F on $\mathcal{V}$ such that $F^2\mathcal{V} = 0$, $F^1\mathcal{V} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$, and $F^0\mathcal{V} = \mathcal{V}$ with $\mathrm{Gr}_F^0\mathcal{V} \simeq \mathcal{O}_{\mathbb{P}^1}(1)$. In this case, $F^1\mathcal{V}$ and $(\mathrm{Gr}_F^0\mathcal{V})^*$ are not semipositive. This means that a naive generalization of the Fujita–Kawamata semipositivity theorem to graded polarizable variations of $\mathbb{Q}$-mixed Hodge structure is false.

As an application of Theorem 1.1, the first author proved a semipositivity theorem for families of singular varieties in [F16]. It is a generalization of [Ko4, 4.12. Theorem] and implies that the moduli functor of stable varieties is semipositive in the sense of Kollár (see [Ko4, 2.4. Definition]). Therefore, it will play a crucial role for the projectivity of the moduli spaces of higher-dimensional stable varieties. For details, see [Ko4], [F15], and [F16].

We give a sketch of the proof of Theorem 1.1 for the reader's convenience.

1.6 (Sketch of the proof of Theorem 1.1). In Theorem 1.1, we see that the local system $R^{d-i}(f|_{X^*\setminus D^*})_!\mathbb{Q}_{X^*\setminus D^*}$ underlies an admissible variation of $\mathbb{Q}$-mixed Hodge structure by Theorem 4.15. Let $\mathcal{V}$ be the canonical extension of the locally free sheaf $(R^{d-i}(f|_{X^*\setminus D^*})_!\mathbb{Q}_{X^*\setminus D^*})\otimes \mathcal{O}_{Y^*}$. Then we can prove $R^{d-i}f_*\mathcal{O}_X(-D)\simeq \mathrm{Gr}^0_F\mathcal{V}$ where F is the canonical extension of the Hodge filtration. Note that the *admissibility* ensures the existence of the canonical extensions of the Hodge bundles (cf. Proposition 3.12 and Remark 7.4). We also note that we use an explicit description of the canonical extension of the Hodge filtration in order to prove $R^{d-i}f_*\mathcal{O}_X(-D)\simeq \mathrm{Gr}^0_F\mathcal{V}$ when Y is a curve. By Grothendieck duality, we obtain $R^if_*\omega_{X/Y}(D)\simeq (\mathrm{Gr}^0_F\mathcal{V})^*$. Therefore, $R^if_*\omega_{X/Y}(D)$ is semipositive by Theorem 1.3. It is important to note that the local system $R^{d-i}(f|_{X^*\setminus D^*})_!\mathbb{Q}_{X^*\setminus D^*}$ is not necessarily the dual local system of $R^{d+i}(f|_{X^*\setminus D^*})_*\mathbb{Q}_{X^*\setminus D^*}$ because X is not a smooth variety but a simple normal crossing variety. In the proof of Theorem 1.1, we use the recent developments of the theory of partial resolution of singularities for *reducible* varieties (see [BiM] and [BiP]) to reduce the problem to simpler cases.

We quickly explain the reason why we use mixed Hodge structures for cohomology with compact support.

1.7 (Mixed Hodge structure for cohomology with compact support). Let X be a smooth projective variety and let D be a simple normal crossing divisor on X. After Iitaka introduced the notion of logarithmic Kodaira dimension, $\mathcal{O}_X(K_X+D)$ plays an important role in birational geometry, where K_X is the canonical divisor of X. In the traditional birational geometry, $\mathcal{O}_X(K_X+D)$ is recognized to be $\Omega_X^{\dim X}(\log D)$. Therefore, the Hodge to de Rham spectral sequence

$$E_1^{p,q}=H^q(X,\Omega^p_X(\log D))\Rightarrow H^{p+q}(X\setminus D,\mathbb{C})$$

arising from the mixed Hodge structures on $H^\bullet(X\setminus D,\mathbb{C})$ is useful. The first author sees $\mathcal{O}_X(K_X+D)$ as

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(-D),\mathcal{O}_X(K_X))$$

or

$$R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(-D),\omega^\bullet_X)[-\dim X]$$

where $\omega^\bullet_X = \mathcal{O}_X(K_X)[\dim X]$ is the dualizing complex of X. Furthermore, $\mathcal{O}_X(-D)$ can be interpreted as the 0-th term of the complex

$$\Omega^\bullet_X(\log D) \otimes \mathcal{O}_X(-D).$$

By this observation, we can use the Hodge to de Rham spectral sequence

$$E_1^{p,q} = H^q(X, \Omega^p_X(\log D) \otimes \mathcal{O}_X(-D)) \Rightarrow H^{p+q}_c(X \setminus D, \mathbb{C})$$

arising from the mixed Hodge structures on the cohomology groups $H^\bullet_c(X \setminus D, \mathbb{C})$ of $X \setminus D$ with compact support and obtain various powerful vanishing theorems. For details and many applications, see [F7, Chapter 2], [F8], [F10], [F11, Section 5], [F12], [F15], and [F17]. Therefore, it is natural to consider variations of such mixed Hodge structures.

We summarize the contents of this paper. Section 2 is a preliminary section. Section 3 collects some generalities on variations of mixed Hodge structure. In Section 4, we discuss variations of mixed Hodge structure for simple normal crossing pairs. We show that they are graded polarizable and admissible. Theorem 4.15 is the main result of Section 4. In Section 5, we discuss a purely Hodge-theoretic aspect of the Fujita–Kawamata semipositivity theorem. Our formulation is different from Kawamata's but is well suited for our results in Section 7. In Section 6, we discuss some generalizations of vanishing and torsion-free theorems for quasi-projective simple normal crossing pairs. They are necessary for the arguments in Section 7. Section 7 is the main part of this paper. Here, we characterize higher direct images of log canonical divisors by using canonical extensions of Hodge bundles (Theorems 7.1 and 7.3). This is a generalization of the results by Yujiro Kawamata, Noboru Nakayama, János Kollár, Morihiko Saito, and Osamu Fujino. In Section 8, we treat some examples which help us understand the Fujita–Kawamata semipositivity theorem, Viehweg's weak positivity theorem, etc.

Let us recall basic definitions and notation.

Notation. For a proper morphism $f : X \to Y$, the *exceptional locus*, denoted by $\mathrm{Exc}(f)$, is the locus where f is not an isomorphism.

1.8 (Divisors, $\mathbb{Q}$-divisors, and $\mathbb{R}$-divisors). For an $\mathbb{R}$-Weil divisor $D = \sum_{j=1}^r d_j D_j$ such that D_i is a prime divisor for every i and $D_i \neq D_j$ for $i \neq j$, we define the *round-up* $\lceil D \rceil = \sum_{j=1}^r \lceil d_j \rceil D_j$ (resp. the *round-down* $\lfloor D \rfloor = \sum_{j=1}^r \lfloor d_j \rfloor D_j$), where for any real number x, $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) is the integer defined by $x \leq \lceil x \rceil < x + 1$ (resp. $x - 1 < \lfloor x \rfloor \leq x$). The *fractional part* $\{D\}$ of D is $D - \lfloor D \rfloor$. We call D a *boundary* (resp. *subboundary*) $\mathbb{R}$-divisor if $0 \leq d_j \leq 1$ (resp. $d_j \leq 1$) for every j. $\mathbb{Q}$-*linear equivalence* (resp. $\mathbb{R}$-*linear equivalence*) of two $\mathbb{Q}$-divisors (resp. $\mathbb{R}$-*divisors*) B_1 and B_2 is denoted by $B_1 \sim_{\mathbb{Q}} B_2$ (resp. $B_1 \sim_{\mathbb{R}} B_2$).

1.9 (Singularities of pairs). Let X be a normal variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $f : X \to Y$ be a resolution such that $\mathrm{Exc}(f) \cup f_*^{-1}\Delta$ has a simple normal crossing support, where $f_*^{-1}\Delta$ is the strict transform of Δ on Y. We can write

$$K_Y = f^*(K_X + \Delta) + \sum_i a_i E_i.$$

We say that (X, Δ) is *log canonical* (*lc* for short) if $a_i \geq -1$ for every i. We usually write $a_i = a(E_i, X, \Delta)$ and call it the *discrepancy coefficient* of E with respect to (X, Δ).

If (X, Δ) is log canonical and there exist a resolution $f : Y \to X$ and a divisor E on Y such that $a(E, X, \Delta) = -1$, then $f(E)$ is called a *log canonical center* (*lc center* for short) with respect to (X, Δ).

It is important to understand the following example.

1.10 (A basic example). Let X be a smooth variety and let Δ be a reduced simple normal crossing divisor on X. Then the pair (X, Δ) is log canonical. Let $\Delta = \sum_{i \in I} \Delta_i$ be the irreducible decomposition of Δ. Then a subvariety W of X is a log canonical center with respect to (X, Δ) if and only if W is an irreducible component of $\Delta_{i_1} \cap \cdots \cap \Delta_{i_k}$ for some $\{i_1, \ldots, i_k\} \subset I$.

Throughout we will work over the field $\mathbb{C}$ of complex numbers.

§2. Preliminaries

Let us recall the definition of *simple normal crossing pairs.*

Definition 2.1 (Simple normal crossing pairs). We say that the pair (X, D) is *simple normal crossing* at a point $a \in X$ if a has a Zariski open neighborhood U in X that can be embedded in a smooth variety Y with a regular system of parameters $(x_1, \ldots, x_p, y_1, \ldots, y_r)$ at $a = 0$ in which U is defined by a monomial equation

$$x_1 \cdots x_p = 0$$

and

$$D = \sum_{i=1}^{r} \alpha_i (y_i = 0)|_U, \quad \alpha_i \in \mathbb{R}.$$

We say that (X, D) is a *simple normal crossing pair* if it is simple normal crossing at every point of X. When D is the zero divisor for a simple normal crossing pair (X, D), X is called a *simple normal crossing variety.* If (X, D) is a simple normal crossing pair, then X has only Gorenstein singularities. Thus, it has an invertible dualizing sheaf ω_X. Therefore, we can define the *canonical divisor* K_X such that $\omega_X \simeq \mathcal{O}_X(K_X)$. It is a Cartier divisor on X and is well-defined up to linear equivalence.

We say that a simple normal crossing pair (X, D) is *embedded* if there exists a closed embedding $\iota : X \hookrightarrow M$, where M is a smooth variety of dimension $\dim X + 1$.

Let X be a simple normal crossing variety and let D be a Cartier divisor on X. If (X, D) is a simple normal crossing pair and D is reduced, then D is called a *simple normal crossing divisor* on X.

We note that a simple normal crossing pair is called a *semi-snc pair* in [Ko6, Definition 1.10].

Definition 2.2 (Strata and permissibility). Let X be a simple normal crossing variety and let $X = \bigcup_{i \in I} X_i$ be the irreducible decomposition of X. A *stratum* of X is an irreducible component of $X_{i_1} \cap \cdots \cap X_{i_k}$ for some $\{i_1, \ldots, i_k\} \subset I$. A Cartier divisor B on X is *permissible* if B contains no strata of X in its support. A finite $\mathbb{Q}$-linear (resp. $\mathbb{R}$-linear) combination of permissible Cartier divisors is called a *permissible* $\mathbb{Q}$-*divisor* (resp. $\mathbb{R}$-*divisor*) on X.

Let (X, D) be a simple normal crossing pair such that D is a boundary $\mathbb{R}$-divisor on X. Let $\nu : X^\nu \to X$ be the normalization. We define Θ by the formula

$$K_{X^\nu} + \Theta = \nu^*(K_X + D).$$

Then a *stratum* of (X, D) is an irreducible component of X or the ν-image of a log canonical center of (X^ν, Θ). We note that (X^ν, Θ) is log canonical (cf. 1.10). When $D = 0$, this definition is compatible with the aforementioned case. A Cartier divisor B on X is *permissible with respect to* (X, D) if B contains no strata of (X, D) in its support. A finite $\mathbb{Q}$-linear (resp. $\mathbb{R}$-linear) combination of permissible Cartier divisors with respect to (X, D) is called a *permissible* $\mathbb{Q}$-*divisor* (resp. $\mathbb{R}$-*divisor*) *with respect to* (X, D).

The notion of *globally embedded simple normal crossing pairs* is very useful for the proof of vanishing and torsion-free theorems (cf. [F7, Chapter 2]).

Definition 2.3 (Globally embedded simple normal crossing pairs). Let X be a simple normal crossing divisor on a smooth variety M and let B be an $\mathbb{R}$-divisor on M such that $\mathrm{Supp}(B + X)$ is a simple normal crossing divisor, and B and X have no common irreducible components. We set $D = B|_X$ and call (X, D) a *globally embedded simple normal crossing pair*. In this case, it is obvious that (X, D) is an embedded simple normal crossing pair.

In Section 6, we will discuss some vanishing and torsion-free theorems for *quasi-projective* simple normal crossing pairs, which will play a crucial role in Section 7. See also [F14], [F15], and [F17].

Finally, let us recall the definition of *semidivisorial log terminal pairs* in the sense of Kollár (see [Ko6, Definition 5.19] and [F14, Definition 4.1]).

Definition 2.4 (Semidivisorial log terminal pairs). Let X be an equidimensional variety which satisfies Serre's S_2 condition and is normal crossing in codimension one. Let Δ be a boundary $\mathbb{R}$-divisor on X whose support does not contain any irreducible components of the conductor of X. Assume that $K_X + \Delta$ is $\mathbb{R}$-Cartier. The pair (X, Δ) is *semidivisorial log terminal* if $a(E, X, \Delta) > -1$ for every exceptional divisor E over X such that (X, Δ) is not a simple normal crossing pair at the generic point of $c_X(E)$, where $c_X(E)$ is the center of E on X.

Remark 2.5. The definition of semidivisorial log terminal pairs in [F1, Definition 1.1] is different from Definition 2.4.

For the details of semidivisorial log terminal pairs, see [Ko6, Section 5.4] and [F14, Section 4].

§3. Generalities on variation of mixed Hodge structure

3.1. Let X be a complex analytic variety. For a point $x \in X$, $\mathbb{C}(x)$ $(\simeq \mathbb{C})$ denotes the residue field at x. For a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules and $x \in X$ the morphism

$$\varphi \otimes \mathrm{id} : \mathcal{F} \otimes \mathbb{C}(x) \to \mathcal{G} \otimes \mathbb{C}(x)$$

is denoted by $\varphi(x)$.

Remark 3.2. For a complex K equipped with a finite decreasing filtration F and for an integer q, the following four conditions are equivalent:

(3.2.1) $d : K^q \to K^{q+1}$ is strictly compatible with the filtration F,
(3.2.2) the canonical morphism $H^{q+1}(F^p K) \to H^{q+1}(K)$ is injective for all p,
(3.2.3) the canonical morphism $H^{q+1}(F^{p+1} K) \to H^{q+1}(F^p K)$ is injective for all p,
(3.2.4) the canonical morphism $H^q(F^p K) \to H^q(\mathrm{Gr}_F^p K)$ is surjective for all p.

Therefore the strict compatibility in (3.2.1) makes sense in the filtered derived category.

On a complex variety X, a complex of $\mathcal{O}_X$-modules K is called *perfect* if, locally on X, it is isomorphic in the derived category to a bounded complex consisting of free $\mathcal{O}_X$-modules of finite rank (see e.g. [FGAE, 8.3.6.3]). The following definition is an analogue of the notion of perfect complex.

Definition 3.3. Let X be a complex variety. A complex of $\mathcal{O}_X$-modules K equipped with a finite decreasing filtration F is called *filtered perfect* if $\mathrm{Gr}_F^p K$ is a perfect complex for all p.

Lemma 3.4. *Let X be a complex manifold.*

(i) *For a perfect complex K on X, the function $X \ni x \mapsto \dim H^q(K \otimes \mathbb{C}(x))$ is upper semicontinuous for all q.*

(ii) *Let K be a perfect complex on X. If there exists an integer q_0 such that $H^q(K)$ is locally free of finite rank for all $q \geq q_0$, then the canonical morphism*

$$H^q(K) \otimes \mathcal{F} \to H^q(K \otimes^L \mathcal{F})$$

is an isomorphism for any $\mathcal{O}_X$-module $\mathcal{F}$ and for all $q \geq q_0$.

(iii) *Fix an integer q. For a perfect complex K on X, the following two conditions are equivalent:*

(3.4.1) *The function $X \ni x \mapsto \dim H^q(K \otimes^L \mathbb{C}(x))$ is locally constant.*

(3.4.2) *The sheaf $H^q(K)$ is locally free of finite rank and the canonical morphism $H^q(K) \otimes \mathcal{F} \to H^q(K \otimes^L \mathcal{F})$ is an isomorphism for any $\mathcal{O}_X$-module $\mathcal{F}$.*

Moreover, if these equivalent conditions are satisfied, then the canonical morphism

$$H^{q-1}(K) \otimes \mathcal{F} \to H^{q-1}(K \otimes^L \mathcal{F})$$

is an isomorphism for any $\mathcal{O}_X$-module $\mathcal{F}$.

(iv) *Let (K, F) be a filtered perfect complex on X. Assume that the function $X \ni x \mapsto \dim H^q(K \otimes^L \mathbb{C}(x))$ is locally constant. If the morphisms*

$$d(x) : (K \otimes^L \mathbb{C}(x))^{q-1} \to (K \otimes^L \mathbb{C}(x))^q,$$
$$d(x) : (K \otimes^L \mathbb{C}(x))^q \to (K \otimes^L \mathbb{C}(x))^{q+1}$$

are strictly compatible with the filtration $F(K \otimes^L \mathbb{C}(x))$ for every $x \in X$, then $H^q(\mathrm{Gr}_F^p K)$ is locally free of finite rank, the canonical morphism

$$H^q(\mathrm{Gr}_F^p K) \otimes \mathbb{C}(x) \simeq H^q(\mathrm{Gr}_F^p(K \otimes^L \mathbb{C}(x))) \tag{3.4.3}$$

is an isomorphism for all p and $x \in X$, and $d : K^q \to K^{q+1}$ is strictly compatible with the filtration F.

Proof. We can easily obtain (i)–(iii) by the arguments in [Mu, Chapter 5].

The strict compatibility conditions in (iv) imply the exactness of the sequence

$$0 \to H^q(F^{p+1}(K \otimes^L \mathbb{C}(x))) \to H^q(F^p(K \otimes^L \mathbb{C}(x))) \to H^q(\mathrm{Gr}_F^p K \otimes^L \mathbb{C}(x)) \to 0$$

for all p and $x \in X$. Thus

$$\sum_p \dim H^q(\mathrm{Gr}_F^p K \otimes^L \mathbb{C}(x)) = \dim H^q(K \otimes^L \mathbb{C}(x))$$

for every x, which implies that $\dim H^q(\mathrm{Gr}_F^p K \otimes^L \mathbb{C}(x))$ is locally constant with respect to $x \in X$. Applying (iii), $H^q(\mathrm{Gr}_F^p K)$ is locally free and (3.4.3) is an isomorphism for all p and $x \in X$. By using the isomorphisms (3.4.3) for all p, we can easily deduce the surjectivity of the canonical morphism

$$H^q(F^p K) \otimes \mathbb{C}(x) \to H^q(\mathrm{Gr}_F^p K) \otimes \mathbb{C}(x)$$

for any $x \in X$, and then the canonical morphism

$$H^q(F^p K) \to H^q(\mathrm{Gr}_F^p K)$$

is surjective for every p. Thus the morphism $d : K^q \to K^{q+1}$ is strictly compatible with the filtration F by Remark 3.2. □

Definition 3.5. Let X be a complex manifold. A *pre-variation of* $\mathbb{Q}$*-Hodge structure* of weight m on X is a triple $V = (\mathbb{V}, (\mathcal{V}, F), \alpha)$ such that

- $\mathbb{V}$ is a local system of finite-dimensional $\mathbb{Q}$-vector spaces on X,
- $\mathcal{V}$ is an $\mathcal{O}_X$-module and F is a finite decreasing filtration on $\mathcal{V}$,
- $\alpha : \mathbb{V} \to \mathcal{V}$ is a morphism of $\mathbb{Q}$-sheaves,

satisfying the conditions:

(3.5.1) α induces an isomorphism $\mathcal{O}_X \otimes \mathbb{V} \simeq \mathcal{V}$ of $\mathcal{O}_X$-modules,
(3.5.2) $\mathrm{Gr}_F^p \mathcal{V}$ is a locally free $\mathcal{O}_X$-module of finite rank for every p,
(3.5.3) $(\mathbb{V}_x, F(\mathcal{V}(x)))$ is a Hodge structure of weight m for every $x \in X$, where we identify $\mathbb{V}_x \otimes \mathbb{C}$ with $\mathcal{V}(x)$ by the isomorphism $\alpha(x)$.

We denote $(\mathbb{V}_x, F(\mathcal{V}(x)))$ by $V(x)$ for $x \in X$.

We identify $\mathcal{O}_X \otimes \mathbb{V}$ with $\mathcal{V}$ by the isomorphism in (3.5.1) if there is no danger of confusion. Under this identification, we write $V = (\mathbb{V}, F)$ for a pre-variation of $\mathbb{Q}$-Hodge structure.

We define the notion of a morphism of pre-variations in the trivial way.

Remark 3.6. A variation of $\mathbb{Q}$-Hodge structure of weight m on X is nothing but a pre-variation $V = (\mathbb{V}, F)$ of $\mathbb{Q}$-Hodge structure of weight m such that the canonical integrable connection ∇ on $\mathcal{V} = \mathcal{O}_X \otimes \mathbb{V}$ satisfies *Griffiths transversality*

$$\nabla(F^p) \subset \Omega_X^1 \otimes F^{p-1} \tag{3.6.1}$$

for every p. A morphism of variations of $\mathbb{Q}$-Hodge structure is a morphism of underlying pre-variations of $\mathbb{Q}$-Hodge structure.

Remark 3.7. (i) Let $V_1 = (\mathbb{V}_1, F)$ and $V_2 = (\mathbb{V}_2, F)$ be pre-variations of $\mathbb{Q}$-Hodge structure of weight m_1 and m_2 respectively. Then the local systems $\mathbb{V}_1 \otimes \mathbb{V}_2$ and $\mathcal{H}om(\mathbb{V}_1, \mathbb{V}_2)$ underlie pre-variations of $\mathbb{Q}$-Hodge structure of weight $m_1 + m_2$ and $m_2 - m_1$ respectively, denoted by $V_1 \otimes V_2$ and $\mathcal{H}om(V_1, V_2)$.

(ii) For an integer n, $\mathbb{Q}_X(n)$ denotes as usual the pre-variation of $\mathbb{Q}$-Hodge structure of Tate. This is, in fact, a variation of $\mathbb{Q}$-Hodge structure of weight $-2n$ on X. For a pre-variation V of $\mathbb{Q}$-Hodge structure of weight m, $V(n) = V \otimes \mathbb{Q}_X(n)$ is a pre-variation of $\mathbb{Q}$-Hodge structure of weight $m - 2n$, which is called the *Tate twist* of V as usual.

Definition 3.8. Let X be a complex manifold and $V = (\mathbb{V}, F)$ a pre-variation of $\mathbb{Q}$-Hodge structure of weight m on X. A *polarization* on V is a morphism of pre-variations of $\mathbb{Q}$-Hodge structure

$$V \otimes V \to \mathbb{Q}_X(-m)$$

which induces a polarization on $V(x)$ for every point $x \in X$. A pre-variation of $\mathbb{Q}$-Hodge structure of weight m is said to be *polarizable* if there exists a polarization on it. A morphism of polarizable pre-variations of $\mathbb{Q}$-Hodge structure is a morphism of the underlying pre-variations of $\mathbb{Q}$-Hodge structure.

Definition 3.9. Let X be a complex manifold.

(i) A p*re-variation of* $\mathbb{Q}$-*mixed Hodge structure* on X is a triple

$$V = ((\mathbb{V}, W), (\mathcal{V}, W, F), \alpha)$$

consisting of

- a local system $\mathbb{V}$ of finite-dimensional $\mathbb{Q}$-vector spaces, equipped with a finite increasing filtration W by local subsystems,
- an $\mathcal{O}_X$-module $\mathcal{V}$ equipped with a finite increasing filtration W and a finite decreasing filtration F,
- a morphism $\alpha : \mathbb{V} \to \mathcal{V}$ of $\mathbb{Q}$-sheaves preserving the filtration W

such that the triple $\mathrm{Gr}_m^W V = (\mathrm{Gr}_m^W \mathbb{V}, (\mathrm{Gr}_m^W \mathcal{V}, F), \mathrm{Gr}_m^W \alpha)$ is a pre-variation of $\mathbb{Q}$-Hodge structure of weight m for every m.

We identify $(\mathcal{O}_X \otimes \mathbb{V}, W)$ and $(\mathcal{V}, W)$ by the isomorphism induced by α as before, if there is no danger of confusion. Under this identification, we use the notation $V = (\mathbb{V}, W, F)$ for a pre-variation of $\mathbb{Q}$-mixed Hodge structure.

(ii) A pre-variation $V = (\mathbb{V}, W, F)$ of $\mathbb{Q}$-mixed Hodge structure on X is called *graded polarizable* if $\mathrm{Gr}_m^W V$ is a polarizable pre-variation of $\mathbb{Q}$-Hodge structure for every m.

(iii) We define a morphism of pre-variations of $\mathbb{Q}$-mixed Hodge structure in the trivial way. A morphism of polarizable pre-variations of $\mathbb{Q}$-mixed Hodge structure is a morphism of the underlying pre-variations.

Now, let us recall the definition of *graded polarizable variation of* $\mathbb{Q}$*-mixed Hodge structure* (GPVMHS for short). See, for example, [SZ, §3], [SSU, Part I, Section 1], [BZ, Section 7], [PS, Definitions 14.44 and 14.45], etc.

Definition 3.10 (GPVMHS). Let X be a complex manifold.

(i) A pre-variation $V = (\mathbb{V}, W, F)$ of $\mathbb{Q}$-mixed Hodge structure on X is said to be a *variation of* $\mathbb{Q}$*-mixed Hodge structure* if the canonical integrable connection ∇ on $\mathcal{V} \simeq \mathcal{O}_X \otimes \mathbb{V}$ satisfies Griffiths transversality (3.6.1).

(ii) A variation of $\mathbb{Q}$-mixed Hodge structure is called *graded polarizable* if the underlying pre-variation is graded polarizable.

(iii) A *morphism* of (graded polarizable) *variations of* $\mathbb{Q}$*-mixed Hodge structure* is a morphism of the underlying pre-variations.

The following definition of *admissibility* was given by Steenbrink–Zucker [SZ, (3.13) Properties] in the one-dimensional case and by Kashiwara [Ks, 1.8, 1.9] in the general case. See also [PS, Definition 14.49].

Definition 3.11 (Admissibility, cf. [Ks, 1.8, 1.9]). (i) A variation of $\mathbb{Q}$-mixed Hodge structure $V = (\mathbb{V}, W, F)$ over $\Delta^* = \Delta \setminus \{0\}$, where $\Delta = \{t \in \mathbb{C} \mid |t| < 1\}$, is said to be *pre-admissible* if it satisfies:

(3.11.1) The monodromy around the origin is quasi-unipotent.

(3.11.2) Let $\widetilde{\mathcal{V}}$ and $W_k\widetilde{\mathcal{V}}$ be the upper canonical extensions of $\mathcal{V} = \mathcal{O}_{\Delta^*} \otimes \mathbb{V}$ and of $\mathcal{O}_{\Delta^*} \otimes W_k\mathbb{V}$ in the sense of Deligne [D1, Remarques 5.5(i)] (see also [Ko2, Section 2] and Remark 7.4). Then the filtration F on $\mathcal{V}$ extends to the filtration F on $\widetilde{\mathcal{V}}$ such that $\mathrm{Gr}_F^p \mathrm{Gr}_k^W \widetilde{\mathcal{V}}$ is a locally free $\mathcal{O}_\Delta$-module of finite rank for all k, p.

(3.11.3) The logarithm of the unipotent part of the monodromy admits a weight filtration relative to W.

(ii) Let X be a complex variety and U a nonsingular Zariski open subset of X. A variation of $\mathbb{Q}$-mixed Hodge structure V on U is said to be *admissible* (with respect to X) if for every morphism $i : \Delta \to X$ with $i(\Delta^*) \subset U$, the variation i^*V on Δ^* is pre-admissible.

We can analogously define an $\mathbb{R}$-mixed Hodge structure, a variation of $\mathbb{R}$-mixed Hodge structure, etc.

In Section 5 we frequently use the following lemma which is a special case of [Ks, Proposition 1.11.3] (see also Remark 7.4 below).

Proposition 3.12 (cf. [Ks]). *Let X be a complex manifold, U the complement of a normal crossing divisor on X, and $V = (\mathbb{V}, W, F)$ a variation of $\mathbb{R}$-mixed Hodge structure on U. The upper canonical extensions of $\mathcal{V} = \mathcal{O}_U \otimes \mathbb{V}$ and of $W_k\mathcal{V} = \mathcal{O}_U \otimes W_k\mathbb{V}$ are denoted by $\widetilde{\mathcal{V}}$ and $W_k\widetilde{\mathcal{V}}$ respectively. If V is admissible on U with respect to X, then the filtration F on $\mathcal{V}$ extends to a finite filtration F on $\widetilde{\mathcal{V}}$ by subbundles such that $\mathrm{Gr}_F^p \mathrm{Gr}_k^W \widetilde{\mathcal{V}}$ is a locally free $\mathcal{O}_X$-module of finite rank for all k, p.*

We give an elementary but useful remark on the quasi-unipotency of monodromy.

Remark 3.13 (Quasi-unipotency). If the local system $\mathbb{V}$ has a $\mathbb{Z}$-structure, that is, there is a local system $\mathbb{V}_\mathbb{Z}$ on X of $\mathbb{Z}$-modules of finite rank such that $\mathbb{V} = \mathbb{V}_\mathbb{Z} \otimes \mathbb{Q}$, in Definition 3.11, then quasi-unipotency automatically follows from Borel's theorem (cf. [Sc, (4.5) Lemma (Borel)]).

The following lemma states the fundamental results on pre-variations of $\mathbb{Q}$-Hodge structure.

Lemma 3.14. *Let X be a complex manifold.*

(i) *The category of the pre-variations of $\mathbb{Q}$-Hodge structure of weight m on X is an abelian category for every m.*

(ii) *Let V_1 and V_2 be pre-variations of $\mathbb{Q}$-Hodge structure of weight m_1 and m_2 respectively, and $\varphi : V_1 \to V_2$ a morphism of pre-variations. If $m_1 > m_2$, then $\varphi = 0$.*

(iii) *Let $\varphi : V_1 \to V_2$ be a morphism of pre-variations $V_1 = (\mathbb{V}_1, F)$ and $V_2 = (\mathbb{V}_2, F)$ of $\mathbb{Q}$-Hodge structure of weight m on X. Then the induced morphism $\varphi \otimes \mathrm{id} : \mathbb{V}_1 \otimes \mathcal{O}_X \to \mathbb{V}_2 \otimes \mathcal{O}_X$ is strictly compatible with the filtration F.*

(iv) *The functor from the category of pre-variations of $\mathbb{Q}$-Hodge structure of weight m to the category of $\mathbb{Q}$-Hodge structures of weight m which sends V to $V(x)$ is an exact functor for every $x \in X$.*

(v) *The category of polarizable variations of $\mathbb{Q}$-Hodge structure of weight m on X is an abelian category for every m.*

Proof. Statements (i), (iii) and (iv) are easy consequences of Lemma 3.4(iv), and (ii) is easily deduced from the corresponding result for $\mathbb{Q}$-Hodge structures. So we prove (v) now.

Let $V_1 = (\mathbb{V}_1, F)$ and $V_2 = (\mathbb{V}_2, F)$ be polarizable pre-variations of $\mathbb{Q}$-Hodge structure of weight m on X, and $\varphi : V_1 \to V_2$ a morphism. We fix polarizations on V_1 and V_2 respectively. Taking (i) into the account, it is sufficient to prove that $\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$ are polarizable. The case of $\mathrm{Ker}(\varphi)$ is trivial, so we discuss the case of $\mathrm{Coker}(\varphi)$.

The morphism φ induces a morphism

$$\varphi^* : \mathcal{H}om(V_2, \mathbb{Q}_X(-m)) \to \mathcal{H}om(V_1, \mathbb{Q}_X(-m)),$$

which is clearly a morphism of pre-variations of $\mathbb{Q}$-Hodge structure of weight m. On the other hand, the polarizations on V_1 and V_2 induce identifications

$$V_1 \simeq \mathcal{H}om(V_1, \mathbb{Q}_X(-m)), \quad V_2 \simeq \mathcal{H}om(V_2, \mathbb{Q}_X(-m)),$$

which are isomorphisms of pre-variations of $\mathbb{Q}$-Hodge structure. By these identifications the morphism φ^* above can be considered as a morphism of pre-variations $V_2 \to V_1$, denoted again by φ^* by abuse of notation. Then the inclusion $\mathrm{Ker}(\varphi^*) \hookrightarrow V_2$ induces an isomorphism $\mathrm{Ker}(\varphi^*) \simeq \mathrm{Coker}(\varphi)$ of pre-variations. Thus we obtain the expected polarization. □

Here we make a brief remark on the dual of a variation of $\mathbb{Q}$-mixed Hodge structure.

Remark 3.15. Let $V = ((\mathbb{V}, W), (\mathcal{V}, W, F), \alpha)$ be a pre-variation of $\mathbb{Q}$-mixed Hodge structure on a complex manifold X. On the dual local system $\mathbb{V}^* = \mathcal{H}om_{\mathbb{Q}}(\mathbb{V}, \mathbb{Q})$,

$$W_m \mathbb{V}^* = (\mathbb{V}/W_{-m-1})^* \subset \mathbb{V}^*$$

defines an increasing filtration W. Similarly, on $\mathcal{V}^* = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{V}, \mathcal{O}_X)$,

$$W_m \mathcal{V}^* = (\mathcal{V}/W_{-m-1})^*, \quad F^p \mathcal{V}^* = (\mathcal{V}/F^{1-p})^*$$

define increasing and decreasing filtrations. We have

$$\mathrm{Gr}_F^p \mathrm{Gr}_m^W \mathcal{V}^* \simeq (\mathrm{Gr}_F^{-p} \mathrm{Gr}_{-m}^W \mathcal{V})^*$$

for all m, p by definition. In view of the isomorphism $\mathcal{O}_X \otimes \mathbb{V}^* \simeq \mathcal{V}^*$, it turns out that $((\mathbb{V}^*, W), (\mathcal{V}^*, W, F))$ is a pre-variation of $\mathbb{Q}$-mixed Hodge structure on X. It is denoted by V^* for short and called the *dual* of V. It is easy to see that V^* is graded polarizable or a variation of $\mathbb{Q}$-mixed Hodge structure if V is so. If X is a Zariski open subset of another variety, then V^* is admissible if V is so.

We close this section with a lemma concerning the relative monodromy weight filtration for a filtered $\mathbb{Q}$-mixed Hodge complex. For the definition, see, for example, [E2, 6.1.4 Définition].

Remark 3.16. We set

$$W[m]_k = W_{k-m}$$

as in [D2], [E2]. Our notation is different from the one in [CKS].

Lemma 3.17. *Let* $((A_{\mathbb{Q}}, W^f, W), (A_{\mathbb{C}}, W^f, W, F), \alpha)$ *be a filtered* $\mathbb{Q}$*-mixed Hodge complex such that the spectral sequence* $E_r^{p,q}(A_{\mathbb{C}}, W^f)$ *degenerates at* E_2*-terms, and* $\nu : A_{\mathbb{C}} \to A_{\mathbb{C}}$ *a morphism of complexes preserving the filtration* W^f *and satisfying the condition* $\nu(W_m A_{\mathbb{C}}) \subset W_{m-2} A_{\mathbb{C}}$ *for every* m*. If the filtration* $W[-m]$ *on* $H^n(\mathrm{Gr}_m^{W^f} A_{\mathbb{C}})$ *is the monodromy weight filtration of the endomorphism* $H^n(\mathrm{Gr}_m^{W^f} \nu)$ *for all* m, n*, then the filtration* W *on* $H^n(A_{\mathbb{C}})$ *is the relative weight monodromy filtration of the endomorphism* $H^n(\nu)$ *with respect to the filtration* W^f *for all* n*.*

Proof. The assumption implies that the morphism $H^{p+q}(\mathrm{Gr}_{-p}^{W^f} \nu)^k$ induces an isomorphism

$$\mathrm{Gr}_{q+k}^{W[p+q]} E_1^{p,q}(A_{\mathbb{C}}, W^f) \to \mathrm{Gr}_{q-k}^{W[p+q]} E_1^{p,q}(A_{\mathbb{C}}, W^f)$$

for all p, q and $k \geq 0$, by the isomorphism $E_1^{p,q}(A_{\mathbb{C}}, W^f) \simeq H^{p+q}(\mathrm{Gr}_{-p}^{W^f} A_{\mathbb{C}})$. On the other hand, the E_2-degeneracy for the filtration W^f gives the isomorphism

$$E_2^{p,q}(A_{\mathbb{C}}, W^f) \simeq \mathrm{Gr}_{-p}^{W^f} H^{p+q}(A_{\mathbb{C}})$$

for all p, q, under which the filtration W_{rec} on the left hand side coincides with the filtration W on the right hand side by [E2, 6.1.8 Théorème]. Since the morphism d_1 of E_1-terms induces a morphism of mixed Hodge structures

$$d_1 : (E_1^{p,q}(A_{\mathbb{C}}, W^f), W[p+q], F) \to (E_1^{p+1,q}(A_{\mathbb{C}}, W^f), W[p+q+1], F)$$

for all p, q by [E2, 6.1.8 Théorème] again, the morphism $(H^{p+q}(\nu))^k$ induces an isomorphism

$$\mathrm{Gr}_{q+k}^{W[p+q]} \mathrm{Gr}_{-p}^{W^f} H^{p+q}(A_{\mathbb{C}}) \to \mathrm{Gr}_{q-k}^{W[p+q]} \mathrm{Gr}_{-p}^{W^f} H^{p+q}(A_{\mathbb{C}})$$

for all p, q and $k \geq 0$. Now we can easily check the conclusion. □

§4. Variations of mixed Hodge structure of geometric origin

In this section, we discuss variations of mixed Hodge structure arising from mixed Hodge structures on cohomology with compact support for simple normal crossing pairs. We will check that those variations are *graded polarizable* and *admissible* (see Theorem 4.15). These properties will play a crucial role in the subsequent sections.

4.1. For a morphism $f : X \to Y$ of topological spaces, we always use the Godement resolution to compute the higher direct image Rf_* of abelian sheaves on X. This means that $Rf_*\mathcal{F}$ is the genuine *complex* $f_*\mathcal{C}^\bullet_{\mathrm{Gdm}}\mathcal{F}$ for an abelian sheaf $\mathcal{F}$ on X, where $\mathcal{C}^\bullet_{\mathrm{Gdm}}$ stands for the Godement resolution as in Peters–Steenbrink [PS, B.2.1]. If $\mathcal{F}$ carries a filtration F, $Rf_*\mathcal{F}$ is the genuine *filtered complex*. For a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of sheaves on X, the morphism

$$Rf_*(\varphi) : Rf_*\mathcal{F} \to Rf_*\mathcal{G}$$

is the genuine *morphism of complexes* defined by using the Godement resolution. We use the same notation for complexes of abelian sheaves, $\mathbb{Q}$-sheaves, $\mathbb{C}$-sheaves, $\mathcal{O}_X$-modules etc.

4.2. Let $f : X_\bullet \to Y$ be an augmented semisimplicial topological space. The morphism $X_p \to Y$ induced by f is denoted by f_p for every p. For an abelian sheaf $\mathcal{F}^\bullet$ on $X_\bullet$.

$$Rf_{\bullet *}\mathcal{F}^\bullet = \{Rf_{p*}\mathcal{F}^p\}_{p\geq 0}$$

defines a co-semisimplicial *complex* on Y by what we mentioned in 4.1. Then we define

$$Rf_*\mathcal{F}^\bullet = sRf_{\bullet *}\mathcal{F}^\bullet$$

as in [D3, (5.2.6.1)]. More precisely, $Rf_*\mathcal{F}^\bullet$ is the single complex associated to the double complex

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & (Rf_{p*}\mathcal{F}^p)^q & \xrightarrow{\delta} & (Rf_{p+1*}\mathcal{F}^{p+1})^q & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle (-1)^p d} & & \downarrow{\scriptstyle (-1)^{p+1} d} & & \\
\cdots & \longrightarrow & (Rf_{p*}\mathcal{F}^p)^{q+1} & \xrightarrow{\delta} & (Rf_{p+1*}\mathcal{F}^{p+1})^{q+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & &
\end{array}$$

where δ in the horizontal lines denotes the Čech type morphism and d in the vertical lines denotes the differential of the *complexes* $Rf_{p*}\mathcal{F}^p$ for every p. The increasing filtration L on $Rf_*\mathcal{F}^\bullet$ is defined by

$$L_m(Rf_*\mathcal{F}^\bullet)^n = \bigoplus_{p\geq -m}(Rf_{p*}\mathcal{F}^p)^{n-p} \tag{4.2.1}$$

for all m, n (cf. Deligne [D3, (5.1.9.3)]). Thus we have

$$\mathrm{Gr}^L_m Rf_*\mathcal{F}^\bullet = Rf_{-m*}\mathcal{F}^{-m}[m] \tag{4.2.2}$$

for every m. Whenever $\mathcal{F}^\bullet$ admits an increasing filtration W, we set

$$W_m(Rf_*\mathcal{F}^\bullet)^n = \bigoplus_p W_m(Rf_{p*}\mathcal{F}^p)^{n-p}, \tag{4.2.3}$$

$$\delta(W,L)_m(Rf_*\mathcal{F}^\bullet)^n = \bigoplus_p W_{m+p}(Rf_{p*}\mathcal{F}^p)^{n-p} \tag{4.2.4}$$

for all n, p. Then we have

$$\mathrm{Gr}_m^{\delta(W,L)} Rf_*\mathcal{F} = \bigoplus_p \mathrm{Gr}_{m+p}^W Rf_{p*}\mathcal{F}^p[-p]$$

for every m. The case of a decreasing filtration F on $\mathcal{F}$ is transformed to the case of an increasing filtration by setting $W_m\mathcal{F} = F^{-m}\mathcal{F}$. We use the same convention for complexes of abelian sheaves, $\mathbb{Q}$-sheaves etc.

4.3. Let X and Y be complex manifolds and $f : X \to Y$ a smooth projective morphism. The de Rham complexes of X and Y are denoted by Ω_X and Ω_Y respectively, and the relative de Rham complex for f is denoted by $\Omega_{X/Y}$. Moreover, F denotes the stupid filtration on Ω_X and $\Omega_{X/Y}$. The inclusion $\mathbb{Q}_X \to \mathcal{O}_X$ induces a morphism of complexes $\mathbb{Q}_X \to \Omega_{X/Y}$. Then we obtain a morphism

$$R^i f_*\mathbb{Q}_X \to R^i f_*\Omega_{X/Y}$$

for every i, denoted by $\alpha_{X/Y}$. Then

$$(R^i f_*\mathbb{Q}_X, (R^i f_*\Omega_{X/Y}, F), \alpha_{X/Y})$$

is a polarizable variation of $\mathbb{Q}$-Hodge structure of weight i on Y. Here we recall the proof of Griffiths transversality following [KO].

A finite decreasing filtration G on Ω_X is defined by

$$G^p\Omega_X = \mathrm{Im}(f^{-1}\Omega_Y^p \otimes_{f^{-1}\mathcal{O}_Y} \Omega_X[-p] \to \Omega_X) \tag{4.3.1}$$

for all p. Then we have isomorphisms

$$\mathrm{Gr}_G^p\Omega_X \simeq f^{-1}\Omega_Y^p \otimes_{f^{-1}\mathcal{O}_Y} \Omega_{X/Y}[-p]$$

for all p, which induce isomorphisms

$$E_1^{p,q}(Rf_*\Omega_X, G) \simeq \Omega_Y^p \otimes R^q f_*\Omega_{X/Y}$$

for all p, q. Thus the morphisms of the E_1-terms give us

$$\nabla : \Omega_Y^p \otimes R^q f_*\Omega_{X/Y} \to \Omega_Y^{p+1} \otimes R^q f_*\Omega_{X/Y}$$

for all p, q. It is easy to see that

$$\nabla : R^q f_* \Omega_{X/Y} \to \Omega^1_Y \otimes R^q f_* \Omega_{X/Y}$$

satisfies Griffiths transversality.

On the other hand, we consider the complexes Ω_Y and $f^{-1}\Omega_Y$. The stupid filtration on Ω_Y is denoted by G for a while. We have

$$\begin{aligned}\mathrm{Gr}^p_G f^{-1}\Omega_Y &\simeq f^{-1}\Omega^p_Y[-p] = f^{-1}\Omega^p_Y \otimes_{\mathbb{Q}} \mathbb{Q}_X[-p] \\ &= f^{-1}\Omega^p_Y \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{O}_Y[-p]\end{aligned}$$

for every p. Therefore the E_1-terms of the associated spectral sequence is identified with

$$E_1^{p,q}(Rf_* f^{-1}\Omega_Y, G) = \Omega^p_Y \otimes_{\mathbb{Q}} R^q f_* \mathbb{Q}_X$$

under which the morphisms of E_1-terms are identified with

$$d \otimes \mathrm{id} : \Omega^p_Y \otimes_{\mathbb{Q}} R^q f_* \mathbb{Q}_X \to \Omega^{p+1}_Y \otimes_{\mathbb{Q}} R^q f_* \mathbb{Q}_X$$

for all p, q. On the other hand, the canonical morphism

$$f^{-1}\Omega_Y \to \Omega_X$$

is a filtered quasi-isomorphism by the relative Poincaré lemma. Thus we obtain a commutative diagram

$$\begin{array}{ccc}\mathcal{O}_Y \otimes_{\mathbb{Q}} R^q f_* \mathbb{Q}_X & \xrightarrow{\simeq} & R^q f_* \Omega_{X/Y} \\ {\scriptstyle d\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \nabla} \\ \Omega^1_Y \otimes_{\mathbb{Q}} R^q f_* \mathbb{Q}_X & \xrightarrow{\simeq} & \Omega^1_Y \otimes R^q f_* \Omega_{X/Y}\end{array}$$

which shows that $d \otimes \mathrm{id}$ on $\mathcal{O}_Y \otimes_{\mathbb{Q}} R^i f_* \mathbb{Q}_X$ satisfies Griffiths transversality.

Notation 4.4. A semisimplicial variety $X_\bullet$ is said to be *strict* if there exists a non-negative integer p_0 such that $X_p = \emptyset$ for all $p \geq p_0$.

For an augmented semisimplicial variety $f : X_\bullet \to Y$, we say f is smooth, projective etc. if $f_p : X_p \to Y$ is smooth, projective etc. for all p.

Lemma 4.5. *Let $f : X_\bullet \to Y$ be a smooth projective augmented strict semisimplicial variety. Moreover, assume that Y is smooth. Then $R^i f_* \mathbb{Q}_{X_\bullet}$ underlies a graded polarizable variation of $\mathbb{Q}$-mixed Hodge structure on Y for all i.*

Proof. The morphism

$$R^i f_*(\alpha_{X_\bullet/Y}) : R^i f_* \mathbb{Q}_{X_\bullet} \to R^i f_* \Omega_{X_\bullet/Y}$$

is induced by the canonical morphism $\alpha_{X_\bullet/Y} : \mathbb{Q}_{X_\bullet} \to \Omega_{X_\bullet/Y}$. We can easily see that $R^i f_*(\alpha_{X_\bullet/Y})$ induces an isomorphism

$$R^i f_* \mathbb{Q}_{X_\bullet/Y} \otimes \mathcal{O}_Y \to R^i f_* \Omega_{X_\bullet/Y} \tag{4.5.1}$$

by the relative Poincaré lemma.

The filtration L on $R^i f_* \mathbb{Q}_{X_\bullet}$ and $R^i f_* \Omega_{X_\bullet/Y}$ is defined by (4.2.1). Moreover, the stupid filtration F on $\Omega_{X_\bullet/Y}$ induces the filtration F on $R^i f_* \Omega_{X_\bullet/Y}$ in the same way as in (4.2.3). It is sufficient to prove that

$$((R^i f_* \mathbb{Q}_{X_\bullet}, L[i]), (R^i f_* \Omega_{X_\bullet/Y}, L[i], F), R^i f_*(\alpha_{X_\bullet/Y}))$$

is a graded polarizable variation of $\mathbb{Q}$-mixed Hodge structure on Y.

To this end, we consider the data

$$K = ((Rf_* \mathbb{Q}_{X_\bullet}, L), (Rf_* \Omega_{X_\bullet/Y}, L, F), Rf_*(\alpha_{X_\bullet/Y}))$$

and the spectral sequence associated to the filtration L. By (4.2.2), we have

$$E_1^{p,q}(K, L) = (R^q f_{p*} \mathbb{Q}_{X_p}, (R^q f_{p*} \Omega_{X_p/Y}, F), R^q f_{p*}(\alpha_{X_p/Y})),$$

which is a polarizable variation of $\mathbb{Q}$-Hodge structure of weight q for every p, q. Moreover, the morphism

$$d_1 : E_1^{p,q}(K, L) \to E_1^{p+1,q}(K, L)$$

is a morphism of variations of $\mathbb{Q}$-Hodge structure. Therefore $E_2^{p,q}(K, L)$ is a polarizable variation of $\mathbb{Q}$-Hodge structure of weight q for all p, q, and $F_{\mathrm{rec}} = F_d = F_{d^*}$ on $E_2^{p,q}(K, L)$ by the lemma on two filtrations [D3]. Then the morphism

$$d_2 : E_2^{p,q}(K, L) \to E_2^{p+2,q-1}(K, L)$$

between E_2-terms is a morphism of variations of $\mathbb{Q}$-Hodge structure of weight q and of weight $q-1$ respectively, which implies that $d_2 = 0$ (see Lemma 3.14(ii)). Therefore the spectral sequence $E_r^{p,q}(K, L)$ degenerates at E_2-terms and $F = F_{\mathrm{rec}} = F_d = F_{d^*}$ on $E_\infty^{p,q}(K, L) = \mathrm{Gr}_{-p}^L H^{p+q}(K)$ by the lemma on two filtrations again. Thus it turns out that

$$\mathrm{Gr}_m^{L[i]} H^i(K) = (\mathrm{Gr}_m^{L[i]} R^i f_* \mathbb{Q}_{X_\bullet}, (\mathrm{Gr}_m^{L[i]} R^i f_* \Omega_{X_\bullet/Y}, F), \mathrm{Gr}_m^{L[i]} R^i f_*(\alpha_{X_\bullet/Y}))$$

is a polarizable pre-variation of $\mathbb{Q}$-Hodge structure of weight m on Y for every i, m.

It remains to prove that the morphism

$$d \otimes \mathrm{id} : \mathcal{O}_Y \otimes R^i f_* \mathbb{Q}_{X_\bullet} \to \Omega_Y^1 \otimes R^i f_* \mathbb{Q}_{X_\bullet} \tag{4.5.2}$$

satisfies Griffiths transversality under the identification (4.5.1).

Now we consider $Rf_*f_\bullet^{-1}\Omega_Y$ and $Rf_*\Omega_{X_\bullet}$, where $f_\bullet^{-1}\Omega_Y$ denotes the complex $\{f_p^{-1}\Omega_Y\}_{p\geq 0}$ on $X_\bullet$. The filtration G on Ω_Y and $\Omega_{X_\bullet}$ induces a filtration G on $Rf_*f_\bullet^{-1}\Omega_Y$ and $Rf_*\Omega_{X_\bullet}$. Moreover, the filtration F on $\Omega_{X_\bullet}$ induces a filtration F on $Rf_*\Omega_{X_\bullet}$. The canonical morphism

$$\gamma : f_\bullet^{-1}\Omega_Y \to \Omega_{X_\bullet},$$

which is a filtered quasi-isomorphism with respect to the filtration G by the relative Poincaré lemma, induces the filtered quasi-isomorphism

$$Rf_*(\gamma) : Rf_*f_\bullet^{-1}\Omega_Y \to Rf_*\Omega_{X_\bullet}$$

with respect to G. Now we consider the spectral sequences associated to G, and the morphism of spectral sequences induced by $Rf_*(\gamma)$.

We have

$$\operatorname{Gr}^p_G Rf_*f_\bullet^{-1}\Omega_Y \simeq Rf_*f_\bullet^{-1}\Omega^p_Y[-p] \simeq \Omega^p_Y \otimes Rf_*\mathbb{Q}_{X_\bullet}[-p] \tag{4.5.3}$$

and

$$\begin{aligned} \operatorname{Gr}^p_G Rf_*\Omega_{X_\bullet} &\simeq Rf_*(f_\bullet^{-1}\Omega^p_Y \otimes \Omega_{X_\bullet/Y}[-p]) \\ &\simeq \Omega^p_Y \otimes Rf_*\Omega_{X_\bullet/Y}[-p] \end{aligned} \tag{4.5.4}$$

for every p, so that the diagram

$$\begin{array}{ccc} \operatorname{Gr}^p_G Rf_*f_\bullet^{-1}\Omega_Y & \xrightarrow{\simeq} & \Omega^p_Y \otimes Rf_*\mathbb{Q}_{X_\bullet}[-p] \\ {\scriptstyle \operatorname{Gr}^p_G Rf_*(\gamma)}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes Rf_*(\alpha_{X_\bullet/Y})[-p]} \\ \operatorname{Gr}^p_G Rf_*\Omega_{X_\bullet} & \xrightarrow{\simeq} & \Omega^p_Y \otimes Rf_*\Omega_{X_\bullet/Y}[-p] \end{array} \tag{4.5.5}$$

is commutative.

The morphism

$$\nabla : R^i f_*\Omega_{X_\bullet/Y} \to \Omega^1_Y \otimes R^i f_*\Omega_{X_\bullet/Y} \tag{4.5.6}$$

is induced by the morphism of E_1-terms

$$d_1 : E_1^{0,i}(Rf_*\Omega_{X_\bullet}, G) \to E_1^{1,i}(Rf_*\Omega_{X_\bullet}, G)$$

via the identification (4.5.4). On the other hand, the morphism of E_1-terms

$$d_1 : E_1^{0,i}(Rf_*f_\bullet^{-1}\Omega_Y, G) \to E_1^{1,i}(Rf_*f_\bullet^{-1}\Omega_Y, G)$$

is identified with

$$d\otimes \mathrm{id} : \mathcal{O}_Y \otimes R^i f_*\mathbb{Q}_{X_\bullet} \to \Omega^1_Y \otimes R^i f_*\mathbb{Q}_{X_\bullet}$$

by the isomorphism (4.5.3). By the commutativity of the diagram (4.5.5), the morphisms ∇ and $d \otimes \mathrm{id}$ are identified under the isomorphism (4.5.1). Because ∇ satisfies Griffiths transversality, so does $d \otimes \mathrm{id}$. □

Remark 4.6. The spectral sequence $E_r^{p,q}(Rf_*\Omega_{X_\bullet/Y}, F)$ degenerates at E_1-terms by the lemma on two filtrations [D3, Proposition (7.2.8)].

Remark 4.7. The construction above is functorial and compatible with the pull-back by the morphism $Y' \to Y$.

Lemma 4.8. *Let $f : X_\bullet \to Y$ and $g : Z_\bullet \to Y$ be smooth projective augmented strict semisimplicial varieties and $\varphi : Z_\bullet \to X_\bullet$ a morphism of semisimplicial varieties compatible with the augmentations $X_\bullet \to Y$ and $Z_\bullet \to Y$. The morphism φ induces a morphism of complexes*

$$\varphi^{-1} : Rf_*\mathbb{Q}_{X_\bullet} \to Rg_*\mathbb{Q}_{Z_\bullet}$$

by using the Godement resolutions, as mentioned in 4.1. The cone of the morphism φ^{-1} is denoted by $C(\varphi^{-1})$. Then $H^i(C(\varphi^{-1}))$ underlies a graded polarizable variation of $\mathbb{Q}$-mixed Hodge structure for every i.

Proof. A filtration L on $C(\varphi^{-1})$ is defined by

$$L_mC(\varphi^{-1})^n = L_{m-1}(Rf_*\mathbb{Q}_{X_\bullet})^{n+1} \oplus L_m(Rg_*\mathbb{Q}_{Z_\bullet})^n$$

where L on the right hand side denotes the filtrations defined in the proof of Lemma 4.5.

The morphism $\varphi : Z_\bullet \to X_\bullet$ induces another morphism of *complexes*

$$\varphi^* : Rf_*\Omega_{X_\bullet/Y} \to Rg_*\Omega_{Z_\bullet/Y}$$

which makes the diagram of *complexes*

$$\begin{array}{ccc} Rf_*\mathbb{Q}_{X_\bullet} & \xrightarrow{\varphi^{-1}} & Rg_*\mathbb{Q}_{Z_\bullet} \\ {\scriptstyle Rf_*(\alpha_{X_\bullet/Y})}\downarrow & & \downarrow{\scriptstyle Rg_*(\alpha_{Z_\bullet/Y})} \\ Rf_*\Omega_{X_\bullet/Y} & \xrightarrow{\varphi^*} & Rg_*\Omega_{Z_\bullet/Y} \end{array} \tag{4.8.1}$$

commute, where $\alpha_{X_\bullet/Y}$ and $\alpha_{Z_\bullet/Y}$ are the canonical morphisms as in the proof of Lemma 4.5. Now we consider the mixed cone of the morphism φ^* (see e.g. [PS, 3.4]), that is, the cone $C(\varphi^*)$ equipped with the filtrations

$$\begin{aligned} L_mC(\varphi^*)^n &= L_{m-1}(Rf_*\Omega_{X_\bullet/Y})^{n+1} \oplus L_m(Rg_*\Omega_{Z_\bullet/Y})^n, \\ F^pC(\varphi^*)^n &= F^p(Rf_*\Omega_{X_\bullet/Y})^{n+1} \oplus F^p(Rg_*\Omega_{Z_\bullet/Y})^n, \end{aligned} \tag{4.8.2}$$

where L and F on the right hand sides denote the filtrations defined in the proof of Lemma 4.5. Then the commutative diagram (4.8.1) induces a morphism of filtered complexes $\alpha : (C(\varphi^{-1}), L) \to (C(\varphi^*), L)$ which induces a filtered quasi-isomorphism $(C(\varphi^{-1}), L) \otimes \mathcal{O}_Y \to (C(\varphi^*), L)$. Then we have

$$\begin{aligned}\mathrm{Gr}_m^L C(\varphi^{-1}) &= \mathrm{Gr}_{m-1}^L Rf_*\mathbb{Q}_{X_\bullet}[1] \oplus \mathrm{Gr}_m^L Rg_*\mathbb{Q}_{Z_\bullet} \\ &= R(f_{-m+1})_*\mathbb{Q}_{X_{-m}}[m] \oplus R(g_{-m})_*\mathbb{Q}_{Z_{-m}}[m]\end{aligned}$$

and

$$\begin{aligned}(\mathrm{Gr}_m^L C(\varphi^*), F) &= (\mathrm{Gr}_{m-1}^L Rf_*\Omega_{X_\bullet/Y}[1], F) \oplus (\mathrm{Gr}_m^L Rg_*\Omega_{Z_\bullet/Y}, F) \\ &= (R(f_{-m+1})_*\Omega_{X_{-m+1}/Y}[m], F) \oplus (R(g_{-m})_*\Omega_{Z_{-m}/Y}[m], F)\end{aligned}$$

for every m. Therefore the data

$$(E_1^{p,q}(C(\varphi^{-1}), L), (E_1^{p,q}(C(\varphi^*), L), F_{\mathrm{rec}}), E_1^{p,q}(\alpha))$$

is a polarizable variation of $\mathbb{Q}$-Hodge structure of weight q. Now the same argument in the proof of Lemma 4.5 implies that the spectral sequences $E_r^{p,q}(C(\varphi^{-1}), L)$ and $E_r^{p,q}(C(\varphi^*), L)$ degenerate at E_2-terms, the spectral sequence $E_r^{p,q}(C(\varphi^*), F)$ degenerates at E_1-terms, and the data

$$((H^i(C(\varphi^{-1})), L[i]), (H^i(C(\varphi^*)), L[i], F), H^i(\alpha))$$

is a graded polarizable pre-variation of $\mathbb{Q}$-mixed Hodge structure on Y for every i.

It remains to prove Griffiths transversality. We consider the complexes $\Omega_{X_\bullet}$, $\Omega_{Z_\bullet}$, $g_\bullet^{-1}\Omega_Y$ and $f_\bullet^{-1}\Omega_Y$ with the decreasing filtration G as in the proof of Lemma 4.5. We have the commutative diagram

$$\begin{array}{ccc} Rf_*f_\bullet^{-1}\Omega_Y & \longrightarrow & Rg_*g_\bullet^{-1}\Omega_Y \\ \downarrow & & \downarrow \\ Rf_*\Omega_{X_\bullet} & \longrightarrow & Rg_*\Omega_{Z_\bullet} \end{array}$$

where the vertical arrows are filtered quasi-isomorphisms with respect to the filtration G. The top horizontal arrow is denoted by ψ^{-1} and the bottom by ψ^* for a while. Considering the cones $C(\psi^{-1})$ and $C(\psi^*)$ with the filtration G defined just as F in (4.8.2), we obtain a commutative diagram of quasi-isomorphisms

$$\begin{array}{ccc} C(\varphi^{-1})[-p] \otimes \Omega_Y^p & \longrightarrow & \mathrm{Gr}_G^p C(\psi^{-1}) \\ {\scriptstyle \alpha\otimes\mathrm{id}}\downarrow & & \downarrow \\ C(\varphi^*)[-p] \otimes \Omega_Y^p & \longrightarrow & \mathrm{Gr}_G^p C(\psi^*) \end{array}$$

for every p. Then we can check Griffiths transversality as in the proof of Lemma 4.5.

□

4.9. Now we review Steenbrink's results [St1], [St2] and fix some notation.

Let X be a smooth complex variety and $f : X \to \Delta$ a projective surjective morphism. We set $X^* = f^{-1}(\Delta^*)$ and $E = f^{-1}(0)$. The coordinate function on Δ is denoted by t. We assume that E_{red} is a simple normal crossing divisor on X and $f : X^* \to \Delta^*$ is a smooth morphism. Moreover, we assume that $R^i f_* \mathbb{Q}_{X^*}$ are of unipotent monodromy for all i for simplicity.

A finite decreasing filtration G on $\Omega_X(\log E)$ is defined by

$$\begin{aligned} &G^0\Omega_X(\log E) = \Omega_X(\log E),\\ &G^1\Omega_X(\log E) = \mathrm{Im}(f^{-1}\Omega^1_\Delta(\log 0) \otimes \Omega_X(\log E)[-1] \to \Omega_X(\log E)),\\ &G^2\Omega_X(\log E) = 0, \end{aligned}$$

as in (4.3.1). Then the morphism

$$\nabla : R^i f_* \Omega_{X/\Delta}(\log E) \to \Omega^1_\Delta(\log 0) \otimes R^i f_* \Omega_{X/\Delta}(\log E)$$

is obtained as the morphism of E_1-terms of the spectral sequence

$$E_r^{p,q}(Rf_*\Omega_{X/\Delta}(\log E), G).$$

The restriction $\nabla|_{\Delta^*}$ is identified with $d \otimes \mathrm{id}$ on $\mathcal{O}_{\Delta^*} \otimes R^i f_* \mathbb{Q}_{X^*}$ via the isomorphisms

$$R^i f_* \Omega_{X/\Delta}(\log E)|_{\Delta^*} \simeq R^i f_* \Omega_{X^*/\Delta^*} \simeq \mathcal{O}_{\Delta^*} \otimes R^i f_* \mathbb{Q}_{X^*}$$

by definition.

Steenbrink proved that $R^i f_* \Omega_{X/\Delta}(\log E)$ is a locally free coherent $\mathcal{O}_\Delta$-module and $\mathrm{Res}_0(\nabla)$ is nilpotent. Therefore $R^i f_* \Omega_{X/\Delta}(\log E)$ is the canonical extension of $\mathcal{O}_{\Delta^*} \otimes R^i f_* \mathbb{Q}_{X^*}$ for every i. Once we know the local freeness of $R^i f_* \Omega_{X/\Delta}(\log E)$, the canonical morphism

$$R^i f_* \Omega_{X/\Delta}(\log E) \otimes \mathbb{C}(0) \xrightarrow{\simeq} H^i(E, \Omega_{X/\Delta}(\log E) \otimes \mathcal{O}_E) \tag{4.9.1}$$

is an isomorphism for every i.

The filtration G on $\Omega_X(\log E)$ induces a filtration on $\Omega_X(\log E) \otimes \mathcal{O}_E$, denoted by G again. Then

$$\mathrm{Gr}^1_G \Omega_X(\log E) \otimes \mathcal{O}_E \simeq (\Omega_{X/\Delta}(\log E) \otimes \mathcal{O}_E)[-1] \tag{4.9.2}$$

because we have the identification

$$G^1\Omega_X(\log E) = d\log t \wedge \Omega_X(\log E)[-1] \simeq \Omega_{X/\Delta}(\log E)[-1]$$

where $d \log t = dt/t$. Therefore

$$\begin{aligned} &E_1^{0,i}(R\Gamma(E, \Omega_X(\log E) \otimes \mathcal{O}_E), G) \simeq H^i(E, \Omega_{X/\Delta}(\log E) \otimes \mathcal{O}_E),\\ &E_1^{1,i}(R\Gamma(E, \Omega_X(\log E) \otimes \mathcal{O}_E), G) \simeq H^i(E, \Omega_{X/\Delta}(\log E) \otimes \mathcal{O}_E), \end{aligned}$$

for every i. Then the morphism of E_1-terms

$$H^i(E, \Omega_{X/\Delta}(\log E) \otimes \mathcal{O}_E) \to H^i(E, \Omega_{X/\Delta}(\log E) \otimes \mathcal{O}_E)$$

coincides with $\mathrm{Res}_0(\nabla)$ under the identification (4.9.1).

In [St1], Steenbrink constructed a cohomological $\mathbb{Q}$-mixed Hodge complex, denoted by

$$A_{X/\Delta} = ((A^{\mathbb{Q}}_{X/\Delta}, W), (A^{\mathbb{C}}_{X/\Delta}, W, F), \alpha_{X/\Delta}),$$

which admits a filtered quasi-isomorphism

$$\theta_{X/\Delta} : (\Omega_{X/\Delta}(\log E) \otimes \mathcal{O}_{E_{\mathrm{red}}}, F) \to (A^{\mathbb{C}}_{X/\Delta}, F) \tag{4.9.3}$$

where the F on the left hand side denotes the filtration induced by the stupid filtration on $\Omega_{X/\Delta}(\log E)$. Because the canonical morphism

$$H^i(E, \Omega_{X/\Delta}(\log E) \otimes \mathcal{O}_E) \to H^i(E_{\mathrm{red}}, \Omega_{X/\Delta}(\log E) \otimes \mathcal{O}_{E_{\mathrm{red}}})$$

is an isomorphism for every i by the unipotent monodromy condition, we have the isomorphisms

$$\begin{aligned} R^i f_* \Omega_{X/\Delta}(\log E) \otimes \mathbb{C}(0) &\xrightarrow{\simeq} H^i(E, \Omega_{X/\Delta}(\log E) \otimes \mathcal{O}_E) \\ &\xrightarrow{\simeq} H^i(E_{\mathrm{red}}, \Omega_{X/\Delta}(\log E) \otimes \mathcal{O}_{E_{\mathrm{red}}}) \\ &\xrightarrow{\simeq} H^i(E_{\mathrm{red}}, A^{\mathbb{C}}_{X/\Delta}) \end{aligned} \tag{4.9.4}$$

for every i.

Here we just recall the definition of $A^{\mathbb{C}}_{X/\Delta}$ from [St1]. $W_X(E)$ denotes the increasing filtration on $\Omega_X(\log E)$ defined by the order of poles along E as usual. The complex $A^{\mathbb{C}}_{X/\Delta}$ is the single complex associated to the double complex $((A^{\mathbb{C}}_{X/\Delta})^{p,q}, d', d'')$ given by

$$(A^{\mathbb{C}}_{X/\Delta})^{p,q} = \Omega^{p+q+1}_X(\log E)/W_X(E)_q$$

with the differentials

$$\begin{aligned} d' &= -d : (A^{\mathbb{C}}_{X/\Delta})^{p,q} \to (A^{\mathbb{C}}_{X/\Delta})^{p+1,q}, \\ d'' &= -d\log t \wedge : (A^{\mathbb{C}}_{X/\Delta})^{p,q} \to (A^{\mathbb{C}}_{X/\Delta})^{p,q+1}, \end{aligned}$$

where d is the morphism induced from the differential of $\Omega_X(\log E)$, and $d\log t\wedge$ denotes the morphism given by the wedge product with $d\log t = dt/t$. For the definitions of W and F on $A^{\mathbb{C}}_{X/\Delta}$, see [St1, 4.17]. The morphism given by

$$\Omega^p_X(\log E) \ni \omega \mapsto d\log t \wedge \omega \in (A^{\mathbb{C}}_{X/\Delta})^{p,0}$$

induces the morphism (4.9.3)

$$\theta_{X/\Delta} : \Omega_{X/\Delta}(\log E) \otimes \mathcal{O}_{E_{\mathrm{red}}} \to A^{\mathbb{C}}_{X/\Delta}$$

which turns out to be a filtered quasi-isomorphism with respect to the filtrations F on both sides (see [St1, Lemma (4.15)]). The composite

$$\Omega_X(\log E) \otimes \mathcal{O}_E \to \Omega_X(\log E) \otimes \mathcal{O}_{E_{\mathrm{red}}} \xrightarrow{\theta_{X/\Delta}} A^{\mathbb{C}}_{X/\Delta}$$

is also denoted by $\theta_{X/\Delta}$ by abuse of notation.

On the other hand, the projection

$$\Omega^p_X(\log E) \to \Omega^p_X(\log E)/W_X(E)_0 = (A^{\mathbb{C}}_{X/\Delta})^{p-1,0} \subset (A^{\mathbb{C}}_{X/\Delta}[-1])^p$$

induces a morphism

$$\pi_{X/\Delta} : \Omega^p_X(\log E) \otimes \mathcal{O}_E \to (A^{\mathbb{C}}_{X/\Delta}[-1])^p$$

for every p. Moreover, the projection

$$\begin{aligned}(A^{\mathbb{C}}_{X/\Delta})^{p,q} = \Omega^{p+q+1}_X(\log D)/W_X(E)_q \\ \to \Omega^{p+q+1}_X(\log E)/W_X(E)_{q+1} = (A^{\mathbb{C}}_{X/\Delta})^{p-1,q+1}\end{aligned}$$

induces a morphism of bifiltered complexes

$$(A^{\mathbb{C}}_{X/\Delta}, W, F) \to (A^{\mathbb{C}}_{X/\Delta}, W[-2], F[-1])$$

denoted by $\nu_{X/\Delta}$ (see [St1, (4.22), Proposition (4.23)]). Then we have

$$d\pi_{X/\Delta} = \pi_{X/\Delta} d + \nu_{X/\Delta}\theta_{X/\Delta} : \Omega^p_X(\log E) \otimes \mathcal{O}_E \to (A^{\mathbb{C}}_{X/\Delta})^p \tag{4.9.5}$$

for every p, where d on the left hand side is the differential of $A^{\mathbb{C}}_{X/\Delta}[-1]$.

We set

$$N_{X/\Delta} = H^i(\nu_{X/\Delta}) : H^i(E_{\mathrm{red}}, A^{\mathbb{C}}_{X/\Delta}) \to H^i(E_{\mathrm{red}}, A^{\mathbb{C}}_{X/\Delta})$$

for every i. It is known that the morphism

$$N^k_{X/\Delta} : \mathrm{Gr}^W_k H^i(E_{\mathrm{red}}, A^{\mathbb{C}}_{X/\Delta}) \to \mathrm{Gr}^W_{-k} H^i(E_{\mathrm{red}}, A^{\mathbb{C}}_{X/\Delta}) \tag{4.9.6}$$

is an isomorphism for every $k \geq 0$ (see Steenbrink [St1], El Zein [E1], Saito [Sa], Guillen–Navarro Aznar [GN], Usui [U]).

The complex $B_{X/\Delta}$ is defined by

$$B^p_{X/\Delta} = (A^{\mathbb{C}}_{X/\Delta})^{p-1} \oplus (A^{\mathbb{C}}_{X/\Delta})^p$$

with the differential

$$d(x, y) = (-dx - \nu_{X/\Delta}(y), dy)$$

for $x \in (A^{\mathbb{C}}_{X/\Delta})^{p-1}$ and $y \in (A^{\mathbb{C}}_{X/\Delta})^p$, where d denotes the differential of the complex $A^{\mathbb{C}}_{X/\Delta}$. We define a filtration G on $B_{X/\Delta}$ by

$$G^0 B_{X/\Delta} = B_{X/\Delta}, \quad G^1 B_{X/\Delta} = A^{\mathbb{C}}_{X/\Delta}[-1], \quad G^2 B_{X/\Delta} = 0,$$

where $A^{\mathbb{C}}_{X/\Delta}[-1]$ is regarded as a subcomplex of $B_{X/\Delta}$ by the inclusion $(A^{\mathbb{C}}_{X/\Delta})^{p-1} \to B^p_{X/\Delta}$ for every p.

A morphism

$$\Omega^p_X(\log E) \otimes \mathcal{O}_E \ni \omega \mapsto (\pi_{X/\Delta}(\omega), \theta_{X/\Delta}(\omega)) \in (A^{\mathbb{C}}_{X/\Delta})^{p-1} \oplus (A^{\mathbb{C}}_{X/\Delta})^p$$

defines a morphism of complexes

$$\eta_{X/\Delta} : \Omega_X(\log E) \otimes \mathcal{O}_E \to B_{X/\Delta} \tag{4.9.7}$$

by (4.9.5). It is easy to check that the morphism $\eta_{X/\Delta}$ preserves the filtration G on both sides. Note that the diagrams

$$\begin{array}{ccc} \mathrm{Gr}^0_G \Omega_X(\log E) \otimes \mathcal{O}_E & \xrightarrow{\mathrm{Gr}^0_G \eta_{X/\Delta}} & \mathrm{Gr}^0_G B_{X/\Delta} \\ \| & & \| \\ \Omega_{X/\Delta}(\log E) \otimes \mathcal{O}_E & \xrightarrow{\theta_{X/\Delta}} & A^{\mathbb{C}}_{X/\Delta} \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Gr}^1_G \Omega_X(\log E) \otimes \mathcal{O}_E & \xrightarrow{\mathrm{Gr}^1_G \eta_{X/\Delta}} & \mathrm{Gr}^1_G B_{X/\Delta} \\ \| & & \| \\ \Omega_{X/\Delta}(\log E) \otimes \mathcal{O}_E[-1] & \xrightarrow{\theta_{X/\Delta}[-1]} & A^{\mathbb{C}}_{X/\Delta}[-1] \end{array}$$

are commutative. Considering the morphisms between E_1-terms induced by $\eta_{X/\Delta}$, we have $\mathrm{Res}_0(\nabla) = -N_{X/\Delta}$ on $H^i(E_{\mathrm{red}}, A^{\mathbb{C}}_{X/\Delta})$ under the isomorphism (4.9.4).

We remark that the construction above is functorial. For the rational structure $A^{\mathbb{Q}}_{X/\Delta}$, we can use the construction by Steenbrink–Zucker [SZ].

In [St2], the local freeness of $R^i f_* \Omega^p_{X/\Delta}(\log E)$ is proved.

Lemma 4.10. *Let $f : X_\bullet \to Y$ be a projective surjective augmented strict semi-simplicial variety to a smooth algebraic variety Y. Moreover, assume that X_p is smooth for every p. Then there exists a Zariski open dense subset Y^* of Y such that $(R^i f_* \mathbb{Q}_{X_\bullet})|_{Y^*}$ underlies an admissible graded polarizable variation of $\mathbb{Q}$-mixed Hodge structure for every i.*

Proof. There exists a non-empty Zariski open subset Y^* of Y such that the morphism $f : X_\bullet \to Y$ is smooth over Y^*. We set $X^*_\bullet = f^{-1}(Y^*)$ and denote the induced morphisms $X^*_\bullet \to Y^*$ again by f. Then $(R^i f_* \mathbb{Q}_{X_\bullet})|_{Y^*}$ underlies a graded polarizable variation of $\mathbb{Q}$-mixed Hodge structure by Lemma 4.5.

It is sufficient to prove that the graded polarizable variation of $\mathbb{Q}$-mixed Hodge structure above is admissible. Because the graded polarizable variation of $\mathbb{Q}$-mixed Hodge structure constructed in Lemma 4.5 commutes with the base change as in Remark 4.7, we may assume $Y = \Delta$ and $Y^* = \Delta^*$.

Our variation of graded polarizable $\mathbb{Q}$-mixed Hodge structure has a $\mathbb{Z}$-structure. Therefore, the quasi-unipotency of the monodromy around the origin is obvious by Remark 3.13. Thus we have the property (3.11.1). Once we know the quasi-unipotency of the monodromy, Lemma 1.9.1 in [Ks] allows us to assume that $f : X_\bullet \to \Delta$ is of unipotent monodromy. Moreover we may assume that $f^{-1}(0)_{\mathrm{red}}$ is a simple normal crossing divisor on the smooth semisimplicial complex variety $X_\bullet$. We set $E_\bullet = f_\bullet^{-1}(0)$. Note that $E_\bullet$ and $E_{\bullet\mathrm{red}} = \{(E_p)_{\mathrm{red}}\}_{p\geq 0}$ are strict semisimplicial subvarieties of $X_\bullet$.

Consider the bifiltered complex

$$(Rf_*\Omega_{X_\bullet/\Delta}(\log E_\bullet), L, F)$$

where L is defined in (4.2.1) and F as in (4.2.3). We trivially have

$$(Rf_*\Omega_{X_\bullet/\Delta}(\log E_\bullet), L, F)|_{\Delta^*} = (Rf_*\Omega_{X^*_\bullet/\Delta^*}, L, F)$$

by definition.

Step 1. In this first step, we prove the local freeness of several coherent $\mathcal{O}_\Delta$-modules.

Consider the spectral sequence

$$(E_r^{p,q}(Rf_*\Omega_{X_\bullet/\Delta}(\log E_\bullet), L), F_{\mathrm{rec}})$$

associated to the filtration L on the complex $Rf_*\Omega_{X_\bullet/\Delta}(\log E_\bullet)$.

By (4.2.2), we have

$$E_1^{p,q}(Rf_*\Omega_{X_\bullet/\Delta}(\log E_\bullet), L) \simeq R^q f_{p*}\Omega_{X_p/\Delta}(\log E_p), \tag{4.10.1}$$

which is the canonical extension of

$$\begin{aligned} E_1^{p,q}(Rf_*\Omega_{X^*_\bullet/\Delta^*}, L) &\simeq R^q f_{p*}\Omega_{X^*_p/\Delta^*} \\ &\simeq \mathcal{O}_{\Delta^*} \otimes R^q f_{p*}\mathbb{Q}_{X^*_p} \simeq \mathcal{O}_{\Delta^*} \otimes E_1^{p,q}(Rf_*\mathbb{Q}_{X^*_\bullet}, L) \end{aligned}$$

by [St1]. Because taking the canonical extension is an exact functor by [D1, Proposition 5.2(d)], $E_2^{p,q}(Rf_*\Omega_{X_\bullet/\Delta}(\log E_\bullet), L)$ is the canonical extension of

$$E_2^{p,q}(Rf_*\Omega_{X^*_\bullet/\Delta^*}, L) \simeq \mathcal{O}_{\Delta^*} \otimes_{\mathbb{Q}} E_2^{p,q}(Rf_*\mathbb{Q}_{X^*_\bullet}, L).$$

Therefore $E_2^{p,q}(Rf_*\Omega_{X_\bullet/\Delta}(\log E_\bullet), L)$ is a locally free coherent $\mathcal{O}_\Delta$-module. Once we know the local freeness of E_2-terms, the spectral sequence

$$E_r^{p,q}(Rf_*\Omega_{X_\bullet/\Delta}(\log E_\bullet), L)$$

degenerates at E_2-terms because its restriction on Δ^* degenerates at E_2-terms. Thus we see that

$$\begin{aligned} E_2^{p,q}(Rf_*\Omega_{X_\bullet/\Delta}(\log E_\bullet), L) &\simeq E_\infty^{p,q}(Rf_*\Omega_{X_\bullet/\Delta}(\log E_\bullet), L) \\ &\simeq \mathrm{Gr}^L_{-p} R^{p+q} f_*\Omega_{X_\bullet/\Delta}(\log E_\bullet) \end{aligned}$$

is locally free for all p, q. In particular, $R^i f_*\Omega_{X_\bullet/\Delta}(\log E_\bullet)$ is a locally free coherent $\mathcal{O}_\Delta$-module with

$$R^i f_*\Omega_{X_\bullet/\Delta}(\log E_\bullet)|_{\Delta^*} \simeq R^i f_*\Omega_{X^*_\bullet/\Delta} \simeq \mathcal{O}_{\Delta^*} \otimes R^i f_*\mathbb{Q}_{X^*_\bullet}$$

for every i.

The morphism of E_1-terms

$$d_1 : R^q f_{p*}\Omega_{X_p/\Delta}(\log E_p) \to R^q f_{p+1*}\Omega_{X_{p+1}/\Delta}(\log E_{p+1}) \tag{4.10.2}$$

preserves the filtration F as $F_{\mathrm{rec}} = F_d = F_{d^*}$ on the E_1-terms in general and F_{rec} on $E_1^{p,q}(Rf_*\Omega_{X_\bullet/\Delta}(\log E_\bullet), L)$ coincides with F on $R^q f_{p*}\Omega_{X_p/\Delta}(\log E_p)$ under the isomorphism (4.10.1). On the other hand, the filtration F on $R^q f_{p*}\Omega_{X_p/\Delta}(\log E_p)$ coincides with the filtration obtained by the nilpotent orbit theorem of [Sc] because

$$\mathrm{Gr}^r_F R^q f_{p*}\Omega_{X_p/\Delta}(\log E_p) \simeq R^{q-r} f_{p*}\Omega^r_{X_p/\Delta}(\log E_p)$$

is locally free for every r (see Corollary 5.2 below). By the SL_2-orbit theorem [Sc],

$$(R^q f_{p*}\Omega_{X_p/\Delta}(\log E_p) \otimes \mathbb{C}(0), W, F)$$

underlies a $\mathbb{Q}$-mixed Hodge structure for all p, q, where W denotes the monodromy weight filtration. On the other hand, the morphism

$$d_1(0) : R^q f_{p*}\Omega_{X_p/\Delta}(\log E_p) \otimes \mathbb{C}(0) \to R^q f_{p+1*}\Omega_{X_{p+1}/\Delta}(\log E_{p+1}) \otimes \mathbb{C}(0)$$

induced by the morphism d_1 in (4.10.2) underlies a morphism of $\mathbb{Q}$-mixed Hodge structures because the restriction of d_1 on Δ^* preserves the $\mathbb{Q}$-structures $R^q f_{p*}\mathbb{Q}_{X^*_p}$ and $R^q f_{p+1*}\mathbb{Q}_{X^*_{p+1}}$. Therefore $d_1(0)$ is strictly compatible with the filtrations F

on $R^q f_{p*}\Omega_{X_p/\Delta}(\log E_p) \otimes \mathbb{C}(0)$ and $R^q f_{p+1*}\Omega_{X_{p+1}/\Delta}(\log E_{p+1}) \otimes \mathbb{C}(0)$. In other words, the morphism

$$d_1(0) : E_1^{p,q}(Rf_*\Omega_{X_\bullet/\Delta}(\log E_\bullet), L) \otimes \mathbb{C}(0) \\ \to E_1^{p+1,q}(Rf_*\Omega_{X_\bullet/\Delta}(\log E_\bullet), L) \otimes \mathbb{C}(0)$$

is strictly compatible with the filtrations F_{rec} on both sides.

Applying Lemma 3.4(iv) to the complex

$$(E_1^{\bullet,q}(Rf_*\Omega_{X_\bullet/\Delta}(\log E_\bullet), L), F_{\mathrm{rec}})$$

we conclude that

$$\mathrm{Gr}^r_{F_{\mathrm{rec}}} E_2^{p,q}(Rf_*\Omega_{X_\bullet/\Delta}(\log E_\bullet), L)$$

is a locally free coherent $\mathcal{O}_\Delta$-module for every p, q, r, and $F_{\mathrm{rec}} = F_d = F_{d^*}$ on $E_2^{p,q}(Rf_*\Omega_{X_\bullet/\Delta}(\log E_\bullet), L)$. Therefore

$$\mathrm{Gr}^r_{F_{\mathrm{rec}}} E_2^{p,q}(Rf_*\Omega_{X_\bullet/\Delta}(\log E_\bullet), L) \simeq \mathrm{Gr}^r_F \mathrm{Gr}^L_{-p} R^{p+q} f_*\Omega_{X_\bullet/\Delta}(\log E_\bullet)$$

is locally free for all p, q, r. Moreover, the spectral sequence

$$E_r^{p,q}(Rf_*\Omega_{X_\bullet/\Delta}(\log E_\bullet), F)$$

degenerates at E_1-terms by the lemma on two filtrations as before.

Step 2. The canonical morphism

$$\Omega_{X_\bullet/\Delta}(\log E_\bullet) \to \Omega_{X_\bullet/\Delta}(\log E_\bullet) \otimes \mathcal{O}_{E_\bullet}$$

induces a morphism of complexes

$$Rf_*\Omega_{X_\bullet/\Delta}(\log E_\bullet) \otimes \mathbb{C}(0) \to R\Gamma(E_\bullet, \Omega_{X_\bullet/\Delta}(\log E_\bullet) \otimes \mathcal{O}_{E_\bullet}) \tag{4.10.3}$$

preserving the filtration L on both sides. Then the morphism of spectral sequences induces a morphism

$$E_r^{p,q}(Rf_*\Omega_{X_\bullet/\Delta}(\log E_\bullet), L) \otimes \mathbb{C}(0) \\ \to E_r^{p,q}(R\Gamma(E_\bullet, \Omega_{X_\bullet/\Delta}(\log E_\bullet) \otimes \mathcal{O}_{E_\bullet}), L) \tag{4.10.4}$$

for all p, q, r. For $r = 1$, the morphism above coincides with the canonical morphism

$$R^q f_{p*}\Omega_{X_p/\Delta}(\log E_p) \otimes \mathbb{C}(0) \to H^q(E_p, \Omega_{X_p/\Delta}(\log E_p) \otimes \mathcal{O}_{E_p}),$$

which is an isomorphism for all p, q as mentioned in 4.9. Therefore the morphism

$$H^p(E_1^{\bullet,q}(Rf_*\Omega_{X_\bullet/\Delta}(\log E_\bullet), L) \otimes \mathbb{C}(0)) \\ \to E_2^{p,q}(R\Gamma(E_\bullet, \Omega_{X_\bullet/\Delta}(\log E_\bullet) \otimes \mathcal{O}_{E_\bullet}), L)$$

is an isomorphism for all p, q. Moreover, the canonical morphism

$$E_2^{p,q}(Rf_*\Omega_{X_\bullet/\Delta}(\log E_\bullet), L) \otimes \mathbb{C}(0) \\ \to H^p(E_1^{\bullet,q}(Rf_*\Omega_{X_\bullet/\Delta}(\log E_\bullet), L) \otimes \mathbb{C}(0))$$

is an isomorphism for all p, q, because $E_2^{p,q}(Rf_*\Omega_{X_\bullet/\Delta}(\log E_\bullet), L)$ is locally free for all p, q as proved in Step 1. Thus we know that the morphism (4.10.4) is an isomorphism for all p, q and $r = 2$. Therefore the E_2-degeneracy of $E_r^{p,q}(Rf_*\Omega_{X_\bullet/\Delta}(\log E_\bullet), L)$ implies the E_2-degeneracy of

$$E_r^{p,q}(R\Gamma(E_\bullet, \Omega_{X_\bullet/\Delta}(\log E_\bullet) \otimes \mathcal{O}_{E_\bullet}), L).$$

Moreover, we have the canonical isomorphism

$$\mathrm{Gr}_m^L R^i f_*\Omega_{X_\bullet/\Delta}(\log E_\bullet) \otimes \mathbb{C}(0) \to \mathrm{Gr}_m^L H^i(E_\bullet, \Omega_{X_\bullet/\Delta}(\log E_\bullet) \otimes \mathcal{O}_{E_\bullet})$$

for all i, m. In particular, the canonical morphism

$$R^i f_*\Omega_{X_\bullet/\Delta}(\log E_\bullet) \otimes \mathbb{C}(0) \to H^i(E_\bullet, \Omega_{X_\bullet/\Delta}(\log E_\bullet) \otimes \mathcal{O}_{E_\bullet}) \tag{4.10.5}$$

is an isomorphism for every i.

Step 3. Considering the filtered complex $(Rf_*\Omega_{X_\bullet}(\log E_\bullet), G)$ we obtain the integrable logarithmic connection

$$\nabla : R^i f_*\Omega_{X_\bullet/\Delta}(\log E_\bullet) \to \Omega^1_\Delta(\log 0) \otimes R^i f_*\Omega_{X_\bullet/\Delta}(\log E_\bullet)$$

as the morphism of E_1-terms of the spectral sequence. It is clear that $\nabla|_{\Delta^*}$ coincides with the connection (4.5.6).

On the other hand, we consider

$$(R\Gamma(E_\bullet, \Omega_{X_\bullet}(\log E_\bullet) \otimes \mathcal{O}_{E_\bullet}), G)$$

with the identification

$$G^1\Omega_{X_\bullet}(\log E_\bullet) \otimes \mathcal{O}_{E_\bullet} \simeq (\Omega_{X_\bullet/\Delta}(\log E_\bullet) \otimes \mathcal{O}_{E_\bullet})[-1]$$

as in (4.9.2). The same procedure as in 4.9 shows the fact that the morphism of E_1-terms

$$H^i(E_\bullet, \Omega_{X_\bullet/\Delta}(\log E_\bullet) \otimes \mathcal{O}_{E_\bullet}) \to H^i(E_\bullet, \Omega_{X_\bullet/\Delta}(\log E_\bullet) \otimes \mathcal{O}_{E_\bullet}) \tag{4.10.6}$$

coincides with $\mathrm{Res}_0(\nabla)$ via the isomorphism (4.10.5).

Step 4. The data $A_{X_\bullet/\Delta}$ gives us an object on the semisimplicial variety $E_{\bullet\mathrm{red}}$ because Steenbrink's construction in 4.9 is functorial, as mentioned there. Then the data

$$(R\Gamma(E_{\bullet\mathrm{red}}, A_{X_\bullet/\Delta}), L, \delta(W, L), F) \tag{4.10.7}$$

is obtained. We set

$$A_{\mathbb{C}} = R\Gamma(E_{\bullet\mathrm{red}}, A^{\mathbb{C}}_{X_\bullet/\Delta})$$

for simplicity. The morphism

$$\theta_{X_\bullet/\Delta} : \Omega_{X_\bullet/\Delta}(\log E_\bullet) \otimes \mathcal{O}_{E_\bullet} \to A^{\mathbb{C}}_{X_\bullet/\Delta}$$

induces a morphism

$$\theta : R\Gamma(E_\bullet, \Omega_{X_\bullet/\Delta}(\log E_\bullet) \otimes \mathcal{O}_{E_\bullet}) \to A_{\mathbb{C}} \tag{4.10.8}$$

which preserves the filtrations L and F. Because

$$H^i(\mathrm{Gr}^L_m \theta) : H^i(\mathrm{Gr}^L_m R\Gamma(E_\bullet, \Omega_{X_\bullet/\Delta}(\log E_\bullet) \otimes \mathcal{O}_{E_\bullet})) \to H^i(\mathrm{Gr}^L_m A_{\mathbb{C}})$$

coincides with the isomorphism

$$H^i(E_{-m}, \Omega_{X_{-m}/\Delta}(\log E_{-m}) \otimes \mathcal{O}_{E_{-m}})) \to H^i((E_{-m})_{\mathrm{red}}, A^{\mathbb{C}}_{X_{-m}/\Delta})$$

in (4.9.4) for all i, m, the morphism θ is a filtered quasi-isomorphism with respect to L. In particular, θ is a quasi-isomorphism, that is,

$$H^i(\theta) : H^i(E_\bullet, \Omega_{X_\bullet/\Delta}(\log E_\bullet) \otimes \mathcal{O}_{E_\bullet}) \to H^i(A_{\mathbb{C}}) \tag{4.10.9}$$

is an isomorphism for every i.

Moreover, the morphism

$$\nu_{X_\bullet/\Delta} : (A^{\mathbb{C}}_{X_\bullet/\Delta}, W, F) \to (A^{\mathbb{C}}_{X_\bullet/\Delta}, W[-2], F[-1])$$

induces a morphism

$$(A_{\mathbb{C}}, L, \delta(W, L), F) \to (A_{\mathbb{C}}, L, \delta(W, L)[-2], F[-1]) \tag{4.10.10}$$

which we simply denote by ν. Since

$$\nu(\delta(W, L)_m A_{\mathbb{C}}) \subset \delta(W, L)_{m-2} A_{\mathbb{C}},$$

ν is a nilpotent endomorphism. We set

$$N = H^i(\nu) : H^i(A_{\mathbb{C}}) \to H^i(A_{\mathbb{C}})$$

for every i.

On the other hand, we obtain $(R\Gamma(E_{\bullet\mathrm{red}}, B_{X_\bullet/\Delta}), G)$ from $B_{X_\bullet/\Delta}$ with the filtration G. By definition, we have

$$R\Gamma(E_{\bullet\mathrm{red}}, B_{X_\bullet/\Delta})^n = A_{\mathbb{C}}^{n-1} \oplus A_{\mathbb{C}}^n$$

and

$$d(x, y) = (-dx - \nu(y), dy)$$

for $x \in A_{\mathbb{C}}^{n-1}$ and $y \in A_{\mathbb{C}}^n$, where d is the differential of the complex $A_{\mathbb{C}}$. Moreover, the filtration G on $R\Gamma(E_{\bullet\mathrm{red}}, B_{X_\bullet/\Delta})$ satisfies

$$G^1 R\Gamma(E_{\bullet\mathrm{red}}, B_{X_\bullet/\Delta}) = A_{\mathbb{C}}[-1]$$

as in 4.9. The morphism

$$\eta_{X_\bullet/\Delta} : \Omega_{X_\bullet}(\log E_\bullet) \otimes \mathcal{O}_{E_\bullet} \to B_{X_\bullet/\Delta} \tag{4.10.11}$$

induced by (4.9.7) gives a morphism

$$\eta : R\Gamma(E_\bullet, \Omega_{X_\bullet}(\log E_\bullet) \otimes \mathcal{O}_{E_\bullet}) \to R\Gamma(E_{\bullet\mathrm{red}}, B_{X_\bullet/\Delta})$$

preserving the filtration G. Note that the diagrams

$$\begin{array}{ccc} \mathrm{Gr}_G^0 R\Gamma(E_\bullet, \Omega_{X_\bullet}(\log E_\bullet) \otimes \mathcal{O}_{E_\bullet}) & \xrightarrow{\mathrm{Gr}_G^0 \eta} & \mathrm{Gr}_G^0 B_{X_\bullet/\Delta} \\ \| & & \| \\ R\Gamma(E_\bullet, \Omega_{X_\bullet/\Delta}(\log E_\bullet) \otimes \mathcal{O}_{E_\bullet}) & \xrightarrow{\theta} & A_{\mathbb{C}} \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Gr}_G^1 R\Gamma(E_\bullet, \Omega_{X_\bullet}(\log E_\bullet) \otimes \mathcal{O}_{E_\bullet}) & \xrightarrow{\mathrm{Gr}_G^1 \eta} & \mathrm{Gr}_G^1 B_{X_\bullet/\Delta} \\ \| & & \| \\ R\Gamma(E_\bullet, \Omega_{X_\bullet/\Delta}(\log E_\bullet) \otimes \mathcal{O}_{E_\bullet})[-1] & \xrightarrow{\theta[-1]} & A_{\mathbb{C}}[-1] \end{array}$$

are commutative by definition. Thus the morphism of E_1-terms (4.10.6) coincides with $-N$ under the isomorphism (4.10.9). Therefore $\mathrm{Res}_0(\nabla)$ is identified with $-N$ via the isomorphisms (4.10.5) and (4.10.9). Because ν is nilpotent, the morphism N is nilpotent, and hence so is $\mathrm{Res}_0(\nabla)$. Thus we conclude that $R^i f_* \Omega_{X_\bullet/\Delta}(\log E_\bullet)$ is the canonical extension of $R^i f_* \Omega_{X_\bullet^*/\Delta^*} \simeq \mathcal{O}_{\Delta^*} \otimes R^i f_* \mathbb{Q}_{X_\bullet^*}$.

Step 5. We can easily see that the data (4.10.7) is a $\mathbb{Q}$-mixed Hodge complex filtered by L (for the definition of filtered $\mathbb{Q}$-mixed Hodge complex, see e.g. [E2, 6.1.4 Définition]). Moreover, the spectral sequence associated to the filtration L degenerates at E_2-terms because θ in (4.10.8) is a filtered quasi-isomorphism and because the spectral sequence for $(R\Gamma(E_\bullet, \Omega_{X_\bullet/\Delta}(\log E_\bullet) \otimes \mathcal{O}_{E_\bullet}), L)$ degenerates at E_2-terms. The filtration induced by $\delta(W, L)$ on

$$\mathrm{Gr}_m^L A_{\mathbb{C}} \simeq R\Gamma((E_{-m})_{\mathrm{red}}, A^{\mathbb{C}}_{X_{-m}/\Delta})$$

coincides with $W[m]$ and the morphism $\mathrm{Gr}_m^L \nu$ coincides with $\nu_{X_{-m}/\Delta}$ for every m. Hence the filtration $\delta(W, L)[-m]$ on $H^i(\mathrm{Gr}_m^L A_{\mathbb{C}})$ is the monodromy weight filtration by the isomorphism (4.9.6). Therefore $\delta(W, L)$ on $H^i(A_{\mathbb{C}})$ is the monodromy weight filtration of $N = H^i(\nu)$ relative to the filtration L by Lemma 3.17.

Thus the condition (3.11.2) is obtained by the local freeness in Step 1 and by the fact that $R^i f_{\bullet *}\Omega_{X_\bullet/\Delta}(\log E_\bullet)$ is the canonical extension of $R^i f_{\bullet *}\Omega_{X^*_\bullet/\Delta^*} \simeq \mathcal{O}_{\Delta^*} \otimes R^i f_{\bullet *}\mathbb{Q}_{X^*_\bullet}$ for every i in Step 3. Moreover, the condition (3.11.3) is deduced from the existence of the monodromy weight filtration of N relative to L in Step 5 and from the fact that N coincides with $-\mathrm{Res}_0(\nabla)$ in Step 3. □

Remark 4.11. Let (V, W) be a finite-dimensional $\mathbb{Q}$-vector space equipped with a finite increasing filtration W, and N a nilpotent endomorphism of V preserving the filtration W. On the $\mathbb{C}$-vector space $V_{\mathbb{C}} = \mathbb{C} \otimes V$, the filtration W and the nilpotent endomorphism $N_{\mathbb{C}} = \mathrm{id} \otimes N$ are induced in the trivial way. Then the existence of the monodromy weight filtration of N relative to W on V is equivalent to the existence of the monodromy weight filtration of $N_{\mathbb{C}}$ relative to W on $V_{\mathbb{C}}$. We can check this equivalence by using Theorem (2.20) of [SZ].

Lemma 4.12 (GPVMHS for relative cohomology). *Let $f : X_\bullet \to Y$ and $g : Z_\bullet \to Y$ be projective augmented strict semisimplicial varieties and $\varphi : Z_\bullet \to X_\bullet$ a morphism of semisimplicial varieties compatible with the augmentations $X_\bullet \to Y$ and $Z_\bullet \to Y$. The cone of the canonical morphism $\varphi^{-1} : Rf_*\mathbb{Q}_{X_\bullet} \to Rg_*\mathbb{Q}_{Z_\bullet}$ is denoted by $C(\varphi^{-1})$ as in Lemma 4.8. Take an open subset Y^* such that $f : X_\bullet \to Y$ and $g : Z_\bullet \to Y$ are smooth over Y^*. Then $H^i(C(\varphi^{-1}))|_{Y^*}$ underlies an admissible graded polarizable variation of $\mathbb{Q}$-mixed Hodge structure for every i.*

Proof. By Lemma 4.8, $H^i(C(\varphi^{-1}))|_{Y^*}$ is a graded polarizable variation of $\mathbb{Q}$-mixed Hodge structure. We will prove its admissibility.

As in Lemma 4.10, we may assume the following:

- $Y = \Delta$, $Y^* = \Delta^*$.
- $f_p : X_p \to \Delta$ and $g_q : Z_q \to \Delta$ are of unipotent monodromy for all p, q.
- $f^{-1}(0)_{\mathrm{red}}$ and $g^{-1}(0)_{\mathrm{red}}$ are simple normal crossing divisors on $X_\bullet$ and $Z_\bullet$ respectively.

We set $E_\bullet = f^{-1}(0)$ and $F_\bullet = g^{-1}(0)$. The morphism $\varphi : Z_\bullet \to X_\bullet$ induces a morphism of *complexes*

$$\varphi^* : Rf_*\Omega_{X_\bullet/\Delta}(\log E_\bullet) \to Rg_*\Omega_{Z_\bullet/\Delta}(\log F_\bullet)$$

as in the proof of Lemma 4.8. Then we consider $C(\varphi^*)$ equipped with filtrations L and F defined as in (4.8.2) from the filtrations L and F on the complexes $Rf_*\Omega_{X_\bullet/\Delta}(\log E_\bullet)$ and $Rg_*\Omega_{Z_\bullet/\Delta}(\log F_\bullet)$. Then $C(\varphi^*)|_{\Delta^*}$ induces a mixed Hodge structure on $H^i(C(\varphi^{-1}))$ as in the proof of Lemma 4.8.

Because

$$\begin{aligned}(E_1^{p,q}&(C(\varphi^*), L), F)\\ &= (R^q(f_{p+1})_*\Omega_{X_{p+1}/\Delta}(\log E_p), F) \oplus (R^q(g_p)_*\Omega_{Z_p/\Delta}(\log F_p), F)\end{aligned}$$

as in the proof of Lemma 4.8, the same argument as in Step 1 of the proof of Lemma 4.10 shows that the spectral sequence $E_r^{p,q}(C(\varphi^*), L)$ degenerates at E_2-terms, and

$$\mathrm{Gr}^r_{F_{\mathrm{rec}}} E_2^{p,q}(C(\varphi^*), L) \simeq \mathrm{Gr}^r_F \mathrm{Gr}^L_{-p} H^{p+q}(C(\varphi^*))$$

are locally free coherent $\mathcal{O}_\Delta$-modules for all p, q, r. Moreover, the spectral sequence $E_r^{p,q}(C(\varphi^*), F)$ degenerates at E_1-terms by the lemma on two filtrations as usual.

Let $A^{\mathbb{C}}_{X_\bullet/\Delta}$ and $A^{\mathbb{C}}_{Z_\bullet/\Delta}$ be the complexes defined in Step 4 of the proof of Lemma 4.10. The morphism φ induces a morphism of trifiltered *complexes*

$$(R\Gamma(E_{\bullet\mathrm{red}}, A^{\mathbb{C}}_{X_\bullet/\Delta}), L, \delta(W, L), F) \to (R\Gamma(F_{\bullet\mathrm{red}}, A^{\mathbb{C}}_{Z_\bullet/\Delta}), L, \delta(W, L), F)$$

by using the Godement resolution as in 4.1. This morphism of *complexes* is denoted by ψ for a while. On the complex $C(\psi)$, the filtrations L and F are defined in the same way as in (4.8.2), and the filtration $\delta(W, L)$ in the same way as L. We can easily check that $(C(\psi), L, \delta(W, L), F)$ underlies a filtered $\mathbb{Q}$-mixed Hodge complex. The composites of the morphisms (4.10.3) and (4.10.8) for $X_\bullet$ and $Z_\bullet$ fit in the commutative diagram

$$\begin{array}{ccc} Rf_*\Omega_{X_\bullet/\Delta}(\log E_\bullet) \otimes \mathbb{C}(0) & \longrightarrow & R\Gamma(E_{\bullet\mathrm{red}}, A^{\mathbb{C}}_{X_\bullet/\Delta}) \\ {\scriptstyle \varphi^*\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \psi} \\ Rg_*\Omega_{Z_\bullet/\Delta}(\log F_\bullet) \otimes \mathbb{C}(0) & \longrightarrow & R\Gamma(F_{\bullet\mathrm{red}}, A^{\mathbb{C}}_{Z_\bullet/\Delta}) \end{array}$$

from which a morphism of complexes $C(\varphi^*) \otimes \mathbb{C}(0) \to C(\psi)$ preserving the filtrations L and F is obtained. This morphism induces an isomorphism

$$E_1^{p,q}(C(\varphi^*), L) \otimes \mathbb{C}(0) \to E_1^{p,q}(C(\psi), L)$$

because the morphism between E_1-terms coincides with the direct sum of the isomorphisms (4.10.5) for $X_\bullet$ and $Z_\bullet$. Then the canonical morphism

$$E_2^{p,q}(C(\varphi^*), L) \otimes \mathbb{C}(0) \to E_2^{p,q}(C(\psi), L)$$

is an isomorphism because the local freeness of $E_2^{p,q}(C(\varphi^*), L)$ implies that the canonical morphism

$$E_2^{p,q}(C(\varphi^*), L) \otimes \mathbb{C}(0) \to H^p(E_1^{\bullet,q}(C(\varphi^*), L) \otimes \mathbb{C}(0))$$

is an isomorphism. Therefore the spectral sequence $E_r^{p,q}(C(\psi), L)$ degenerates at E_2-terms and the canonical morphism

$$H^i(C(\varphi^*)) \otimes \mathbb{C}(0) \to H^i(C(\psi))$$

is an isomorphism for every i, under which the filtration L on both sides coincides.

The morphisms (4.10.10) for $X_\bullet$ and $Z_\bullet$ induce a morphism of complexes

$$(C(\psi), L, \delta(W, L), F) \to (C(\psi), L, \delta(W, L)[-2], F[-1]),$$

denoted by ν again.

By using the mapping cone of the canonical morphism

$$(Rf_*\Omega_{X_\bullet}(\log E_\bullet), G) \to (Rg_*\Omega_{Z_\bullet}(\log F_\bullet), G)$$

with the decreasing filtrations G defined in the same way as F in (4.8.2), we obtain the integrable logarithmic connection ∇ on $H^i(C(\varphi^*))$ for every i in the same way as in the proof of Lemma 4.8. By definition, the restriction of this ∇ onto Δ^* coincides with the original ∇ on $H^i(C(\varphi^*))|_{\Delta^*}$. Similarly, the morphism $\varphi : Z_\bullet \to X_\bullet$ induces a morphism of filtered complexes $(B_{X_\bullet/\Delta}, G) \to (B_{Z_\bullet/\Delta}, G)$ such that the diagram

$$\begin{array}{ccc} \Omega_{X_\bullet}(\log E_\bullet) & \longrightarrow & \Omega_{Z_\bullet}(\log F_\bullet) \\ \downarrow & & \downarrow \\ B_{X_\bullet/\Delta} & \longrightarrow & B_{Z_\bullet/\Delta} \end{array}$$

is commutative. By considering the cone of the morphism

$$R\Gamma(E_{\bullet\,\mathrm{red}}, B_{X_\bullet/\Delta}) \to R\Gamma(F_{\bullet\,\mathrm{red}}, B_{Z_\bullet/\Delta})$$

with the decreasing filtration G, the residue $\mathrm{Res}_0(\nabla)$ on $H^i(C(\varphi^*)) \otimes \mathbb{C}(0)$ is identified with $-H^i(\nu)$ for every i. Because the morphism ν is trivially nilpotent, we conclude that $\mathrm{Res}_0(\nabla)$ is nilpotent. Therefore $H^i(C(\varphi^*))$ is the canonical extension of $H^i(C(\varphi^*))|_{\Delta^*}$. Thus the condition (3.11.2) is satisfied by $H^i(C(\varphi^*))$ for every i. Moreover, we can easily see that the filtration $\delta(W, L)[-m]$ on $H^i(\mathrm{Gr}_m^L C(\psi))$ is the monodromy weight filtration of $\mathrm{Gr}_m^L H^i(\nu)$ for every i, m. Therefore Lemma 3.17 implies that the filtration $\delta(W, L)$ is the monodromy weight filtration of $H^i(\nu)$ relative to L on $H^i(C(\psi)) \simeq H^i(C(\varphi^*)) \otimes \mathbb{C}(0)$. □

Theorem 4.13 (GPVMHS for cohomology with compact support). *Let $f: X \to Y$ be a projective surjective morphism from a complex variety X onto a smooth complex variety Y, and Z a closed subset of X. Then there exists a Zariski open dense subset Y^* of Y such that $(R^i(f|_{X \setminus Z})_! \mathbb{Q}_{X \setminus Z})|_{Y^*}$ underlies an admissible graded polarizable variation of $\mathbb{Q}$-mixed Hodge structure for every i.*

Proof. The open immersion $X \setminus Z \to X$ and the closed immersion $Z \to X$ are denoted by ι and j respectively. We set $g = fj : Z \to Y$. Take cubical hyperresolutions $\varepsilon_Z : Z_\bullet \to Z$ and $\varepsilon_X : X_\bullet \to X$ which fit in the commutative diagram

$$\begin{array}{ccc} Z_\bullet & \xrightarrow{\varphi} & X_\bullet \\ \varepsilon_Z \downarrow & & \downarrow \varepsilon_X \\ Z & \xrightarrow{j} & X \end{array}$$

for some morphism $\varphi : Z_\bullet \to X_\bullet$ of cubical varieties. The cone of the canonical morphism $\varphi^{-1} : R(f\varepsilon_X)_* \mathbb{Q}_{X_\bullet} \to R(g\varepsilon_Z)_* \mathbb{Q}_{Z_\bullet}$ is denoted by $C(\varphi^{-1})$ as in Lemma 4.12. Then the composite of the canonical morphisms

$$R(f|_{X \setminus Z})_! \mathbb{Q}_{X \setminus Z} \simeq Rf_* \iota_! \mathbb{Q}_{X \setminus Z} \to Rf_* \mathbb{Q}_X \to R(f\varepsilon_X)_* \mathbb{Q}_{X_\bullet}$$

induces a quasi-isomorphism $R(f|_{X \setminus Z})_! \mathbb{Q}_{X \setminus Z} \to C(\varphi^{-1})[-1]$ from which we obtain the conclusion by considering the filtration $L[-1]$ on $C(\varphi^{-1})[-1]$. □

4.14. Let (X, D) be a simple normal crossing pair with D reduced. The irreducible decompositions of X and D are given by

$$X = \bigcup_{i \in I} X_i, \quad D = \bigcup_{\lambda \in \Lambda} D_\lambda$$

respectively. Fixing orders $<$ on Λ and I, we set

$$D_k \cap X_l = \coprod_{\substack{\lambda_0 < \lambda_1 < \cdots < \lambda_k \\ i_0 < i_1 < \cdots < i_l}} D_{\lambda_0} \cap D_{\lambda_1} \cap \cdots \cap D_{\lambda_k} \cap X_{i_0} \cap X_{i_1} \cap \cdots \cap X_{i_l}$$

for $k, l \geq 0$. Here we use the convention

$$D_k = D_k \cap X_{-1} = \coprod_{\lambda_0 < \lambda_1 < \cdots < \lambda_k} D_{\lambda_0} \cap D_{\lambda_1} \cap \cdots \cap D_{\lambda_k},$$

$$X_l = D_{-1} \cap X_l = \coprod_{i_0 < i_1 < \cdots < i_l} X_{i_0} \cap X_{i_1} \cap \cdots \cap X_{i_l},$$

for $k, l \geq 0$. Moreover, we set

$$(D \cap X)_n = \coprod_{k+l+1=n} D_k \cap X_l$$

for $n \geq 0$. Thus we obtain projective augmented strict semisimplicial varieties $\varepsilon_X : (D \cap X)_\bullet \to X$, $\varepsilon_D : D_\bullet \to D$ and a morphism of semisimplicial varieties $\varphi : D_\bullet \to (D \cap X)_\bullet$ compatible with the augmentations ε_X and ε_D. We remark that $D_k \cap X_l$ are smooth for all k, l by the definition of simple normal crossing pair. Therefore $\varepsilon_D : D_\bullet \to D$ is a hyperresolution of D. Now we will see that $\varepsilon_X : (D \cap X)_\bullet \to X$ is also a hyperresolution of X. It is sufficient to prove that ε_X is of cohomological descent. The cone of the canonical morphism

$$\mathbb{Q}_X \to (\varepsilon_X)_* \mathbb{Q}_{(D \cap X)_\bullet} = R(\varepsilon_X)_* \mathbb{Q}_{(D \cap X)_\bullet}$$

is the single complex associated to the double complex

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Q}_X & \longrightarrow & \mathbb{Q}_{X_0} & \longrightarrow & \mathbb{Q}_{X_1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Q}_{D_0} & \longrightarrow & \mathbb{Q}_{D_0 \cap X_0} & \longrightarrow & \mathbb{Q}_{D_0 \cap X_1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Q}_{D_1} & \longrightarrow & \mathbb{Q}_{D_1 \cap X_0} & \longrightarrow & \mathbb{Q}_{D_1 \cap X_1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

shifted by 1. All the lines of the diagram above are exact because they are the Mayer–Vietoris exact sequences for X and for D_k. Then the single complex associated to the double complex above is acyclic. Thus we conclude that the canonical morphism $\mathbb{Q}_X \to (\varepsilon_X)_* \mathbb{Q}_{(D \cap X)_\bullet}$ is a quasi-isomorphism.

Theorem 4.15 (GPVMHS for a snc pair). *Let (X, D) be a simple normal crossing pair with D reduced and $f : X \to Y$ a projective surjective morphism to a smooth algebraic variety Y. Let Y^* be a non-empty Zariski open subset of Y such that all the strata of (X, D) are smooth over Y^*. Then $(R^i(f|_{X \setminus D})_! \mathbb{Q}_{X \setminus D})|_{Y^*}$ underlies an admissible graded polarizable variation of $\mathbb{Q}$-mixed Hodge structure for every i.*

Proof. As mentioned in 4.14, we have the commutative diagram

$$\begin{array}{ccc} D_{\bullet} & \xrightarrow{\varphi} & (D\cap X)_{\bullet} \\ {\scriptstyle \varepsilon_D}\downarrow & & \downarrow{\scriptstyle \varepsilon_X} \\ D & \longrightarrow & X \end{array}$$

such that ε_D and ε_X are hyperresolutions. Then we obtain the conclusion in the same way as in the proof of Theorem 4.13 from Lemma 4.12. □

Remark 4.16. In the situation above, the inverse images of the open subset Y^* are indicated by the superscript *, such as $X^* = f^{-1}(Y^*)$. From the proof of Lemma 4.8, we can check that $\mathrm{Gr}_F^p(\mathcal{O}_{Y^*} \otimes (R^i(f|_{X\setminus D})_!\mathbb{Q}_{X\setminus D})|_{Y^*})$ coincides with the $(i-p)$-th direct image of the single complex associated to the double complex

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^p_{X_0^*/Y^*} & \longrightarrow & \Omega^p_{X_1^*/Y^*} & \longrightarrow & \Omega^p_{X_2^*/Y^*} & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^p_{D_0^*\cap X_0^*/Y^*} & \longrightarrow & \Omega^p_{D_0^*\cap X_1^*/Y^*} & \longrightarrow & \Omega^p_{D_0^*\cap X_2^*/Y^*} & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^p_{D_1^*\cap X_0^*/Y^*} & \longrightarrow & \Omega^p_{D_1^*\cap X_1^*/Y^*} & \longrightarrow & \Omega^p_{D_1^*\cap X_2^*/Y^*} & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

by f. Therefore we have the canonical isomorphism

$$R^i f_*\mathcal{O}_X(-D) \simeq \mathrm{Gr}_F^0(\mathcal{O}_{Y^*} \otimes (R^i(f|_{X\setminus D})_!\mathbb{Q}_{X\setminus D})|_{Y^*}) \quad \text{for every } i.$$

§5. Semipositivity theorem

In this section, we discuss a purely Hodge-theoretic aspect of the Fujita–Kawamata semipositivity theorem (cf. [Z] and [Kw1, §4 Semi-positivity]). Our formulation is different from Kawamata's but is indispensable for our main theorem, Theorem 7.1(4). For related topics, see [Mor, Section 5], [F5, Section 5], [F4, 3.2. Semipositivity theorem], and [Ko5, 8.10]. We use the theory of integrable logarithmic connections. For the basic properties and results on integrable logarithmic connections, see [D1], [Kt], and [Bo, IV. Regular connections, after Deligne] by Bernard Malgrange. For a different approach, see [FFS, Section 4].

We start with easy observations.

Lemma 5.1. *Let X be a complex manifold, U a dense open subset of X, and $\mathcal{V}$ a locally free $\mathcal{O}_X$-module of finite rank. Assume that two $\mathcal{O}_X$-submodules $\mathcal{F}$ and $\mathcal{G}$ satisfy the following conditions:*

(5.1.1) *$\mathcal{G}$ and $\mathcal{V}/\mathcal{G}$ are locally free $\mathcal{O}_X$-modules of finite rank.*
(5.1.2) *$\mathcal{F}|_U = \mathcal{G}|_U$.*

Then $\mathcal{F} \subset \mathcal{G}$ on X.

Corollary 5.2. *Let X, U and $\mathcal{V}$ be as above. Suppose that two finite decreasing filtrations F and G on $\mathcal{V}$ satisfy the following conditions:*

- *$\mathrm{Gr}_G^p \mathcal{V}$ is a locally free $\mathcal{O}_X$-module of finite rank for every p.*
- *$F^p\mathcal{V}|_U = G^p\mathcal{V}|_U$ for every p.*

Then $F^p\mathcal{V} \subset G^p\mathcal{V}$ on X for every p. In particular, $F^p\mathcal{V} = G^p\mathcal{V}$ for every p if, in addition, $\mathrm{Gr}_F^p \mathcal{V}$ is locally free of finite rank for every p.

5.3. Let X be a complex manifold and $D = \sum_{i \in I} D_i$ a simple normal crossing divisor on X, where D_i is a smooth irreducible divisor on X for every $i \in I$. We set

$$D(J) = \bigcap_{i \in J} D_i, \quad D_J = \sum_{i \in J} D_i$$

for any subset J. Note that $D(\emptyset) = X$ and $D_\emptyset = 0$ by definition. Moreover we set $D(J)^* = D(J) \setminus D(J) \cap D_{I \setminus J}$ for $J \subset I$. For $J = \emptyset$, we set $X^* = D(\emptyset)^* = X \setminus D$.

Let $\mathcal{V}$ be a locally free $\mathcal{O}_X$-module of finite rank and

$$\nabla : \mathcal{V} \to \Omega_X^1(\log D) \otimes \mathcal{V}$$

an integrable logarithmic connection on $\mathcal{V}$. The residue of ∇ along D_i is denoted by

$$\mathrm{Res}_{D_i}(\nabla) : \mathcal{O}_{D_i} \otimes \mathcal{V} \to \mathcal{O}_{D_i} \otimes \mathcal{V}.$$

We assume the following condition throughout this section:

(5.3.1) $\mathrm{Res}_{D_i}(\nabla) : \mathcal{O}_{D_i} \otimes \mathcal{V} \to \mathcal{O}_{D_i} \otimes \mathcal{V}$ is nilpotent for every $i \in I$.

This is equivalent to the local system $\mathrm{Ker}(\nabla)|_{X^*}$ being of unipotent local monodromy.

5.4. In the situation above, the morphism

$$\mathrm{id} \otimes \mathrm{Res}_{D_i}(\nabla) : \mathcal{O}_{D(J)} \otimes \mathcal{V} \to \mathcal{O}_{D(J)} \otimes \mathcal{V}$$

for a subset J of I and for $i \in J$ is denoted by $N_{i,D(J)}$. We simply write N_i if there is no danger of confusion. We have

$$N_{i,D(J)}N_{j,D(J)} = N_{j,D(J)}N_{i,D(J)}$$

for all $i, j \in J$. For two subsets J, K of I with $K \subset J$, we set $N_{K,D(J)} = \sum_{i \in K} N_{i,D(J)}$, which is nilpotent by the assumption above. Once a subset J is fixed, we use the symbols N_K for short. We have the monodromy weight filtration $W(K)$ on $\mathcal{O}_{D(J)} \otimes \mathcal{V}$ which is characterized by the condition that N_K^q induces an isomorphism

$$\mathrm{Gr}_q^{W(K)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}) \xrightarrow{\simeq} \mathrm{Gr}_{-q}^{W(K)}(\mathcal{O}_{D(J)} \otimes \mathcal{V})$$

for all $q \geq 0$. For $K = \emptyset$, $W(\emptyset)$ is trivial, that is, $W(\emptyset)_{-1}\mathcal{O}_{D(J)} \otimes \mathcal{V} = 0$ and $W(\emptyset)_0\mathcal{O}_{D(J)} \otimes \mathcal{V} = \mathcal{O}_{D(J)} \otimes \mathcal{V}$.

For $J = K$, we set

$$\mathcal{P}_k(J) = \mathrm{Ker}\big(N_J^{k+1} : \mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}) \to \mathrm{Gr}_{-k-2}^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V})\big)$$

for every non-negative integer k, which is called the *primitive part* of $\mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V})$ with respect to N_J. Then we have the primitive decomposition

$$\mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}) = \bigoplus_{l \geq \max(0,-k)} N_J^l(\mathcal{P}_{k+2l}(J))$$

for every k, and N_J^l induces an isomorphism

$$\mathcal{P}_{k+2l}(J) \to N_J^l(\mathcal{P}_{k+2l}(J))$$

for all k, l with $l \geq \max(0, -k)$.

Lemma 5.5. *In the situation above,* $\mathrm{Gr}_k^{W(K)}(\mathcal{O}_{D(J)} \otimes \mathcal{V})$ *is a locally free* $\mathcal{O}_{D(J)}$*-module of finite rank for every* k *and for any subsets* J, K *of* I *with* $K \subset J$*.*

Proof. Easy by the local description of an integrable logarithmic connection (see e.g. Deligne [D1], Katz [Kt]). □

Corollary 5.6. *In the situation above, fix a subset* J *of* I*. For any subset* K *of* J *we have*

$$W(K) = W(N_K(x))$$

on $\mathcal{V}(x) = \mathcal{V} \otimes \mathbb{C}(x)$ *for every* $x \in D(J)$*, where the left hand side denotes the filtration on* $\mathcal{V}(x)$ *induced by* $W(K)$*.*

Remark 5.7. Let $(\mathcal{V}_1, \nabla_1)$ and $(\mathcal{V}_2, \nabla_2)$ be pairs of locally free sheaves of $\mathcal{O}_X$-modules of finite rank and integrable logarithmic connections on them. We assume that they satisfy the condition in 5.3. If a morphism $\varphi : \mathcal{V}_1 \to \mathcal{V}_2$ of $\mathcal{O}_X$-modules is compatible with the connections ∇_1 and ∇_2, then the diagram

$$\begin{array}{ccc} \mathcal{O}_{D(J)} \otimes \mathcal{V}_1 & \xrightarrow{\mathrm{id}\otimes\varphi} & \mathcal{O}_{D(J)} \otimes \mathcal{V}_2 \\ {\scriptstyle N_{i,D(J)}}\downarrow & & \downarrow{\scriptstyle N_{i,D(J)}} \\ \mathcal{O}_{D(J)} \otimes \mathcal{V}_1 & \xrightarrow{\mathrm{id}\otimes\varphi} & \mathcal{O}_{D(J)} \otimes \mathcal{V}_2 \end{array}$$

is commutative for every subset J of I and for every $i \in J$. Therefore $\mathrm{id} \otimes \varphi$ preserves the filtration $W(K)$ for every $K \subset J$.

5.8. Let m be an integer. For a finite decreasing filtration F on $\mathcal{V}$, we consider the following condition:

(mMH) The triple

$$(\mathcal{V}(x), W(J)[m], F)$$

underlies an $\mathbb{R}$-mixed Hodge structure for any subset J of I and for any point $x \in D(J)^*$.

Here we remark that we do not assume the local freeness of $\mathrm{Gr}_F^p \mathcal{V}$ at the beginning.

The following lemma is a counterpart of Schmid's results [Sc].

Lemma 5.9. *Let U be an open subset of $X \setminus D$ such that $X \setminus U$ is a nowhere dense analytic subspace of X. Moreover, we are given a finite decreasing filtration F on $\mathcal{V}|_U$. If $(\mathcal{V}, F, \nabla)|_U$ underlies a polarizable variation of $\mathbb{R}$-Hodge structure of weight m on U, then there exists a finite decreasing filtration $\widetilde{F}$ on $\mathcal{V}$ satisfying:*

(i) *$\widetilde{F}^p \mathcal{V}|_U = F^p \mathcal{V}|_U$ for every p.*
(ii) *$\mathrm{Gr}_{\widetilde{F}}^p \mathcal{V}$ is a locally free $\mathcal{O}_X$-module of finite rank for every p.*
(iii) *$\widetilde{F}$ satisfies the condition (mMH) of 5.8.*

Proof. See [Sc]. □

Lemma 5.10. *Let U be as above, and F a finite decreasing filtration on $\mathcal{V}$ in the situation 5.3. Assume that $(\mathcal{V}, F, \nabla)|_U$ underlies a polarizable variation of $\mathbb{R}$-Hodge structure of weight m on U. Then $\mathrm{Gr}_F^p \mathcal{V}$ is locally free of finite rank for every p if and only if F satisfies the condition (mMH) of 5.8.*

Proof. By the lemma above, there exists a finite decreasing filtration $\widetilde{F}$ on $\mathcal{V}$ satisfying (i)–(iii). By Corollary 5.2, the local freeness of $\mathrm{Gr}_F^p \mathcal{V}$ for every p is equivalent to the equality $F^p \mathcal{V} = \widetilde{F}^p \mathcal{V}$ for every p. If $F = \widetilde{F}$ on $\mathcal{V}$, then F satisfies the condition (mMH) by the lemma above. Thus it suffices to prove the equality

$F = \widetilde{F}$ on $\mathcal{V}$ under the assumption that F satisfies (mMH). By Corollary 5.2 again, we have $F^p\mathcal{V} \subset \widetilde{F}^p\mathcal{V}$ for every p. On the other hand, $(\mathcal{V}(x), W(J)[m], F)$ and $(\mathcal{V}(x), W(J)[m], \widetilde{F})$ are $\mathbb{R}$-mixed Hodge structures for every $x \in D(J)^*$, if F satisfies (mMH). Therefore $F(\mathcal{V}(x)) = \widetilde{F}(\mathcal{V}(x))$ for every $x \in X$, which implies $F = \widetilde{F}$ on $\mathcal{V}$. □

5.11. In addition to the situation 5.3, we assume that we are given a finite decreasing filtration F on $\mathcal{V}$ satisfying the following three conditions:

- Griffiths transversality holds, that is, $\nabla(F^p) \subset \Omega^1_X(\log E) \otimes F^{p-1}$ for every p.
- $(\mathcal{V}, F, \nabla)|_{X^*}$ underlies a polarizable variation of $\mathbb{R}$-Hodge structure of weight m.
- $\mathrm{Gr}^p_F\mathcal{V}$ is locally free of finite rank for every p, or equivalently F satisfies the condition (mMH).

For a subset J of I, Griffiths transversality implies that

$$N_i(F^p(\mathcal{O}_{D(J)} \otimes \mathcal{V})) \subset F^{p-1}(\mathcal{O}_{D(J)} \otimes \mathcal{V})$$

for all p and $i \in J$.

Lemma 5.12. *In the situation above, we have*

(1) $N_i(W(K)_k) \subset W(K)_{k-1}$ *for all* $i \in K$ *and* k,
(2) $W(J)$ *is the monodromy weight filtration of* N_K *relative to the filtration* $W(J \setminus K)$

on $\mathcal{O}_{D(J)} \otimes \mathcal{V}$ *for any two subsets* J, K *of* I *with* $K \subset J$.

Proof. See Cattani–Kaplan [CK, (3.3) Theorem, (3,4)] and Steenbrink–Zucker [SZ, (3.12) Theorem]. □

Corollary 5.13. *In the situation* 5.3 *and* 5.11*, the induced filtration* F *on*

$$\mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V})$$

satisfies $((m+k)\mathrm{MH})$ *for any subset* J *of* I.

Proof. Take a subset K of $I \setminus J$. For any point $x \in D(J \cup K)^*$, the triple

$$(\mathcal{V}(x), W(J \cup K)[m], F)$$

underlies an $\mathbb{R}$-mixed Hodge structure because F satisfies (mMH) by assumption. Moreover, $(2\pi\sqrt{-1})^{-1}N_J(x)$ is a morphism of $\mathbb{R}$-mixed Hodge structures of type $(-1,-1)$ by condition (2) in the lemma above and by Griffiths transversality. Therefore

$$(\mathrm{Gr}_k^{W(J)}\mathcal{V}(x), W(J \cup K)[m], F)$$

is an $\mathbb{R}$-mixed Hodge structure. On the other hand, we have

$$W(J\cup K)(\mathrm{Gr}_k^{W(J)}\mathcal{V}(x)) = W(K)(\mathrm{Gr}_k^{W(J)}\mathcal{V}(x))[k],$$

by (2) in the lemma above. Thus

$$(\mathrm{Gr}_k^{W(J)}\mathcal{V}(x), W(K)[m+k], F)$$

underlies an $\mathbb{R}$-mixed Hodge structure. □

5.14. In the situation 5.3 and 5.11 we fix a subset J of I. We have an exact sequence

$$0 \to \Omega^1_{D(J)}(\log D(J)\cap D_{I\setminus J}) \to \Omega^1_X(\log D)\otimes\mathcal{O}_{D(J)} \to \mathcal{O}_{D(J)}^{\oplus|J|} \to 0,$$

where $|J|$ denotes the cardinality of J. On the other hand, the integrable logarithmic connection ∇ induces a commutative diagram

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\nabla} & \Omega^1_X(\log D)\otimes\mathcal{V} \\ \downarrow & & \downarrow \\ \mathcal{O}_{D(J)}\otimes\mathcal{V} & \longrightarrow & \mathcal{O}_{D(J)}^{\oplus|J|}\otimes\mathcal{V} \end{array}$$

where the bottom horizontal arrow coincides with $\bigoplus_{i\in J} N_{i,D(J)}$ under the identification $\mathcal{O}_{D(J)}^{\oplus|J|}\otimes\mathcal{V}\simeq(\mathcal{O}_{D(J)}\otimes\mathcal{V})^{|J|}$. Because ∇ preserves the filtration $W(J)$ on $\mathcal{O}_{D(J)}\otimes\mathcal{V}$ by the local description in [D1], [Kt] and because $N_{i,D(J)}(W(J)_k)\subset W(J)_{k-1}$ for every k by Lemma 5.12(1), we obtain a morphism

$$\mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)}\otimes\mathcal{V}) \to \Omega^1_{D(J)}(\log D(J)\cap D_{I\setminus J})\otimes\mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)}\otimes\mathcal{V})$$

for every k. It is denoted by $\nabla_k(J)$, or simply $\nabla(J)$. It is easy to see that $\nabla(J)$ is an integrable logarithmic connection on $\mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)}\otimes\mathcal{V})$ satisfying $\nabla(J)(F^p)\subset F^{p-1}$ for every p for the induced filtration F on $\mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)}\otimes\mathcal{V})$. We can easily see that the residue of $\nabla(J)$ along $D(J)\cap D_i$ coincides with $N_{i,D(J\cup\{i\})}$ for $i\in I\setminus J$. Thus $\nabla(J)$ satisfies the condition in 5.3.

5.15. Let $(\mathcal{V},F,\nabla)$ be as in 5.3 and 5.11. Then $(\mathcal{V},F,\nabla)|_{X^*}$ is a polarizable variation of $\mathbb{R}$-Hodge structure of weight m. An integrable logarithmic connection on $\mathcal{V}\otimes\mathcal{V}$ is defined by $\nabla\otimes\mathrm{id}+\mathrm{id}\otimes\nabla$ as usual. Assume that we are given a morphism

$$S:\mathcal{V}\otimes\mathcal{V}\to\mathcal{O}_X$$

satisfying the following:

- S is $(-1)^m$-symmetric.
- S is compatible with the connections, where $\mathcal{O}_X$ is equipped with the trivial connection d.

- $S(F^p\mathcal{V}\otimes F^q\mathcal{V})=0$ if $p+q>m$.
- $S|_{X^*}$ underlies a polarization of the variation of $\mathbb{R}$-Hodge structure $(\mathcal{V},F,\nabla)|_{X^*}$.

Now we fix a subset J of I. Then S induces a morphism

$$\mathcal{O}_{D(J)}\otimes\mathcal{V}\otimes\mathcal{V}\simeq(\mathcal{O}_{D(J)}\otimes\mathcal{V})\otimes(\mathcal{O}_{D(J)}\otimes\mathcal{V})\to\mathcal{O}_{D(J)},$$

denoted by S_J.

Lemma 5.16. *In the situation above, we have*

$$S_J(W(K)_a\otimes W(K)_b)=0$$

for every $K\subset J$ and all a,b with $a+b<0$.

Proof. We fix a subset K of J. It is sufficient to prove that

$$S_J(W(K)_a\otimes W(K)_{-a-1})=0$$

for every non-negative integer a.

Since S is compatible with the connections, we have

$$S_J\cdot(N_i\otimes\mathrm{id}+\mathrm{id}\otimes N_i)=0$$

for every $i\in J$, which yields

$$S_J\cdot(N_K\otimes\mathrm{id}+\mathrm{id}\otimes N_K)=0.$$

Then we have

$$\begin{aligned}S_J(W(K)_a\otimes W(K)_{-a-1})&=(S_J\cdot\mathrm{id}\otimes N_K^{a+1})(W(K)_a\otimes W(K)_{a+1})\\&=(-1)^{a+1}(S_J\cdot N_K^{a+1}\otimes\mathrm{id})(W(K)_a\otimes W(K)_{a+1})\\&=(-1)^{a+1}S_J(W(K)_{-a-2}\otimes W(K)_{a+1})\\&=(-1)^{a+1+m}S_J(W(K)_{a+1}\otimes W(K)_{-a-2})\end{aligned}$$

by using the equality $W(K)_{-k}=N^k(W(K)_k)$ for $k\geq 0$ (see e.g. [SZ, (2.2) Corollary]). Thus we obtain the conclusion by descending induction on a. □

Corollary 5.17. *In the situation above, S_J induces a morphism*

$$\mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)}\otimes\mathcal{V})\otimes\mathrm{Gr}_{-k}^{W(J)}(\mathcal{O}_{D(J)}\otimes\mathcal{V})\to\mathcal{O}_{D(J)}$$

for every non-negative integer k.

5.18. In the situation above, we define a morphism

$$\overline{S}_k(J):\mathcal{P}_k(J)\otimes\mathcal{P}_k(J)\to\mathcal{O}_{D(J)}$$

by $\overline{S}_k(J)=S_J\cdot(\mathrm{id}\otimes N_J^k)$ for every $J\subset I$ and every non-negative integer k.

By using the direct sum decomposition

$$\mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}) = \bigoplus_{l \geq 0} N^l(\mathcal{P}_{k+2l}(J))$$

for every non-negative integer k, we obtain a morphism

$$S_k(J) : \mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}) \otimes \mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}) \to \mathcal{O}_{D(J)}$$

which is characterized by the following properties:

- For non-negative integers $a \neq b$ we have

$$S_k(J)(N^a(\mathcal{P}_{k+2a}(J)) \otimes N^b(\mathcal{P}_{k+2b}(J))) = 0.$$

- The diagram

$$\begin{array}{ccc} \mathcal{P}_{k+2l}(J) \otimes \mathcal{P}_{k+2l}(J) & \xrightarrow{\overline{S}_{k+2l}(J)} & \mathcal{O}_{D(J)} \\ {\scriptstyle N^l \otimes N^l} \downarrow & & \| \\ N^l(\mathcal{P}_{k+2l}(J)) \otimes N^l(\mathcal{P}_{k+2l}(J)) & \xrightarrow{S_k(J)} & \mathcal{O}_{D(J)} \end{array}$$

  is commutative for every non-negative integer l.

For a positive integer k, the morphism

$$S_{-k}(J) : \mathrm{Gr}_{-k}^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}) \otimes \mathrm{Gr}_{-k}^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}) \to \mathcal{O}_{D(J)}$$

is defined by identifying $\mathrm{Gr}_{-k}^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V})$ with $\mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V})$ via the morphism $N(J)^k$. More precisely, $S_{-k}(J)$ is the unique morphism such that the diagram

$$\begin{array}{ccc} \mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}) \otimes \mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}) & \xrightarrow{S_k(J)} & \mathcal{O}_{D(J)} \\ {\scriptstyle N(J)^k \otimes N(J)^k} \downarrow & & \| \\ \mathrm{Gr}_{-k}^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}) \otimes \mathrm{Gr}_{-k}^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}) & \xrightarrow{S_{-k}(J)} & \mathcal{O}_{D(J)} \end{array}$$

is commutative.

The following proposition plays an essential role in the inductive argument for the proof of the semipositivity theorem.

Proposition 5.19. *In the situation 5.3, 5.11 and 5.15, the data*

$$(\mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}), F, \nabla(J), S_k(J))$$

satisfies the conditions in 5.3, 5.11 and 5.15 again.

Proof. By Lemma 5.5 and 5.14, the pair $(\mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}), \nabla(J))$ satisfies the condition of 5.3. As remarked in 5.14, we have $\nabla(J)(F^p) \subset F^{p-1}$ for every p. Moreover, the filtration F on $\mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V})$ satisfies the condition $((m+k)\mathrm{MH})$ by Corollary 5.13.

By definition, the morphism

$$S_J : \mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}) \otimes \mathrm{Gr}_{-k}^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}) \to \mathcal{O}_{D(J)}$$

is compatible with the connections on both sides. Therefore $\overline{S}_k(J)$ is compatible with the connections because

$$N_J : \mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}) \to \mathrm{Gr}_{k-2}^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V})$$

is compatible with the connection $\nabla(J)$ on both sides. Thus $S_k(J)$ is compatible with the connection. Moreover we can check the equality

$$S_k(J)(F^p \otimes F^q) = 0$$

for $p + q > m + k$ by using $N_J^k(F^q) \subset F^{q-k}$.

There exists an open subset U of $D(J)^*$ such that $\mathrm{Gr}_F^p \mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V})$ is a locally free $\mathcal{O}_{D(J)}$-module of finite rank for every p, and $D(J) \setminus U$ is a nowhere dense closed analytic subspace of $D(J)$.

By the local description as in Deligne [D1], Katz [Kt] and by the property (1) in Lemma 5.12, we can easily check that $\mathrm{Ker}(\nabla_k(J))|_{D(J)^*}$ admits an $\mathbb{R}$-structure, that is, there exists a local system $\mathbb{V}_k(J)$ of finite-dimensional $\mathbb{R}$-vector spaces with $\mathbb{C} \otimes \mathbb{V}_k(J) \simeq \mathrm{Ker}(\nabla_k(J))|_{D(J)^*}$. Then the data

$$(\mathbb{V}_k(J), (\mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}), F), \nabla(J), S_k(J))|_U$$

is a polarized variation of $\mathbb{R}$-Hodge structure of weight $m+k$, by Schmid [Sc]. By Lemma 5.10, $\mathrm{Gr}_F^p \mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V})$ turns out to be locally free for all k, p and then

$$(\mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}), F, \nabla(J), S_k(J))|_{D(J)^*}$$

underlies a polarized variation of $\mathbb{R}$-Hodge structure of weight $m+k$ as desired. By continuity, $S_k(J)$ is $(-1)^{m+k}$-symmetric. □

Let us recall the definition of semipositive vector bundles in the sense of Fujita–Kawamata. Example 8.2 below helps us understand the Fujita–Kawamata semipositivity.

Definition 5.20 (Semipositivity). A locally free sheaf (or a vector bundle) $\mathcal{E}$ of finite rank on a complete algebraic variety X is said to be *semipositive* if for every smooth curve C, for every morphism $\varphi : C \to X$, and for every quotient invertible sheaf (or line bundle) $\mathcal{Q}$ of $\varphi^*\mathcal{E}$, we have $\deg_C \mathcal{Q} \geq 0$.

It is easy to see that $\mathcal{E}$ is semipositive if and only if $\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(1)$ is nef where $\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(1)$ is the tautological line bundle on $\mathbb{P}_X(\mathcal{E})$.

The following theorem is the main result of this section (cf. [Kw1, Theorem 5]). It is a completely Hodge-theoretic result.

Theorem 5.21 (Semipositivity theorem). *Let X be a smooth complete complex variety, D a simple normal crossing divisor on X, and $\mathcal{V}$ a locally free $\mathcal{O}_X$-module of finite rank equipped with a finite increasing filtration W and a finite decreasing filtration F. Assume that:*

(1) *$F^a\mathcal{V} = \mathcal{V}$ and $F^{b+1}\mathcal{V} = 0$ for some $a < b$.*
(2) *$\mathrm{Gr}_F^p\mathrm{Gr}_m^W\mathcal{V}$ is a locally free $\mathcal{O}_X$-module of finite rank for all m, p.*
(3) *For all m, $\mathrm{Gr}_m^W\mathcal{V}$ admits an integrable logarithmic connection ∇_m with nilpotent residue morphisms which satisfies*

$$\nabla_m(F^p\mathrm{Gr}_m^W\mathcal{V}) \subset \Omega_X^1(\log D) \otimes F^{p-1}\mathrm{Gr}_m^W\mathcal{V} \quad \textit{for all } p.$$

(4) *The triple $(\mathrm{Gr}_m^W\mathcal{V}, F\mathrm{Gr}_m^W\mathcal{V}, \nabla_m)|_{X\setminus D}$ underlies a polarizable variation of $\mathbb{R}$-Hodge structure of weight m for every integer m.*

Then $(\mathrm{Gr}_F^a\mathcal{V})^*$ and $F^b\mathcal{V}$ are semipositive.

Proof. Since a vector bundle which is an extension of two semipositive vector bundles is also semipositive, we may assume without loss of generality that $\mathcal{V}$ is pure of weight m, that is, $W_m\mathcal{V} = \mathcal{V}, W_{m-1}\mathcal{V} = 0$ for some integer m. Then $\mathcal{V}$ carries an integrable logarithmic connection ∇ whose residue morphisms are nilpotent. Thus the data $(\mathcal{V}, F, \nabla)$ satisfies the conditions in 5.3 and 5.11. Note that $\mathcal{V}$ is the canonical extension of $\mathcal{V}|_{X\setminus D}$ because the residue morphisms of ∇ are nilpotent.

By the assumption (4) above, $\mathcal{V}|_{X\setminus D}$ carries a polarization which extends to a morphism

$$S : \mathcal{V} \otimes \mathcal{V} \to \mathcal{O}_X$$

by functoriality of the canonical extensions. We can easily see that the data $(\mathcal{V}, F, \nabla)$ and S satisfies the conditions in 5.3, 5.11 and 5.15.

For $\dim X = 1$, we obtain the conclusion by Zucker [Z] (see also Kawamata [Kw1] and the proof of [Ko3, Theorem 5.20]).

Next, we study the case of $\dim X > 1$. Let $\varphi : C \to X$ be a morphism from a smooth projective curve. The irreducible decomposition of D is denoted by $D = \sum_{i\in I} D_i$ as in 5.3. We set $J = \{i \in I : \varphi(C) \subset D_i\} \subset I$. Then $\varphi(C) \subset D(J)$,

$\varphi(C)\cap D(J)^*\neq\emptyset$ and $\varphi^*D_{I\setminus J}$ is an effective divisor on C. By Proposition 5.19, the locally free sheaf $\mathcal{O}_{D(J)}\otimes\mathcal{V}$ with the finite increasing filtration $W(J)$ and the finite decreasing filtration F satisfies assumptions (1)–(4) for $D(J)$ with the simple normal crossing divisor $D(J)\cap D_{I\setminus J}$. Therefore $\varphi^*\mathcal{V}=\varphi^*(\mathcal{O}_{D(J)}\otimes\mathcal{V})$ with the induced filtrations W and F satisfies (1)–(4) for C with the effective divisor $\varphi^*D_{I\setminus J}$. Now we deduce the desired semipositivity from the case of $\dim X=1$. □

Remark 5.22. In Theorem 5.21, if X is not complete, then the following holds. Let V be a complete subvariety of X. Then $(\mathrm{Gr}^a_F\mathcal{V})^*|_V$ and $(F^b\mathcal{V})|_V$ are semipositive locally free sheaves on V. This is obvious by the proof of Theorem 5.21.

Corollary 5.23. *Let X and D be as in Theorem 5.21. Assume that we are given an admissible graded polarizable variation of $\mathbb{R}$-mixed Hodge structure $V=((\mathbb{V},W),F)$ on $X\setminus D$ of unipotent monodromy. Assume that $F^a\mathcal{V}=\mathcal{V}$ and $F^{b+1}\mathcal{V}=0$. Denote by $\widetilde{\mathcal{V}}$ and $W_k\widetilde{\mathcal{V}}$ the canonical extensions of $\mathcal{V}=\mathcal{O}_{X\setminus D}\otimes\mathbb{V}$ and of $W_k\mathcal{V}=\mathcal{O}_{X\setminus D}\otimes W_k$ for all k. As stated in Proposition 3.12, the Hodge filtration F extends to $\widetilde{\mathcal{V}}$ such that $\mathrm{Gr}^p_F\mathrm{Gr}^W_k\widetilde{\mathcal{V}}$ is locally free of finite rank for all k,p. Then $(\mathrm{Gr}^a_F\widetilde{\mathcal{V}})^*$ and $F^b\widetilde{\mathcal{V}}$ are semipositive.*

We learned the following from Hacon.

Remark 5.24. The proof of the semipositivity theorem in [Ko5, Theorem 8.10.12] contains some ambiguities. In the notation there, if D is a simple normal crossing divisor but is not a *smooth* divisor, then it is not clear how to express $R^mf_*\omega_{X/Y}(D)$ as an extension of $R^mf_*\omega_{D_J/Y}$'s. The case when $D=F$ is a *smooth* divisor is treated in the proof of [Ko5, Theorem 8.10.12]. The same argument does not seem to be sufficient for the general case.

Fortunately, [F4, Theorem 3.9] is sufficient for all applications in [Ko5] (see also [FG2]). For some related topics, see [FFS].

§6. Vanishing and torsion-free theorems

In this section, we discuss some generalizations of torsion-free and vanishing theorems for *quasi-projective* simple normal crossing pairs.

First, let us recall the following very useful lemma. For a proof, see, for example, [F14, Lemma 3.3].

Lemma 6.1 (Relative vanishing lemma). *Let $f:Y\to X$ be a proper morphism from a simple normal crossing pair (Y,Δ) to an algebraic variety X such that Δ is a boundary $\mathbb{R}$-divisor on Y. Assume that f is an isomorphism at the generic point of any stratum of the pair (Y,Δ). Let L be a Cartier divisor on Y such that $L\sim_{\mathbb{R}}K_Y+\Delta$. Then $R^qf_*\mathcal{O}_Y(L)=0$ for every $q>0$.*

As an application of Lemma 6.1, we obtain Lemma 6.2 below. We will use it several times in Section 7.

Lemma 6.2 (cf. [F13, Lemma 2.7]). *Let (V_1, D_1) and (V_2, D_2) be simple normal crossing pairs such that D_1 and D_2 are reduced. Let $f : V_1 \to V_2$ be a proper morphism. Assume that there is a Zariski open subset U_1 (resp. U_2) of V_1 (resp. V_2) such that U_1 (resp. U_2) contains the generic point of any stratum of (V_1, D_1) (resp. (V_2, D_2)) and that f induces an isomorphism between U_1 and U_2. Then $R^i f_*\omega_{V_1}(D_1) = 0$ for every $i > 0$ and $f_*\omega_{V_1}(D_1) \simeq \omega_{V_2}(D_2)$. By Grothendieck duality, we obtain $R^i f_*\mathcal{O}_{V_1}(-D_1) = 0$ for every $i > 0$ and $f_*\mathcal{O}_{V_1}(-D_1) \simeq \mathcal{O}_{V_2}(-D_2)$.*

Proof. We can write

$$K_{V_1} + D_1 = f^*(K_{V_2} + D_2) + E$$

so that E is f-exceptional. We consider the commutative diagram

$$\begin{array}{ccc} V_1^\nu & \xrightarrow{f^\nu} & V_2^\nu \\ {\scriptstyle \nu_1}\downarrow & & \downarrow{\scriptstyle \nu_2} \\ V_1 & \xrightarrow{f} & V_2 \end{array}$$

where $\nu_1 : V_1^\nu \to V_1$ and $\nu_2 : V_2^\nu \to V_2$ are the normalizations. We can write $K_{V_1^\nu} + \Theta_1 = \nu_1^*(K_{V_1} + D_1)$ and $K_{V_2^\nu} + \Theta_2 = \nu_2^*(K_{V_2} + D_2)$. By pulling back $K_{V_1} + D_1 = f^*(K_{V_2} + D_2) + E$ to V_1^ν by ν_1, we have

$$K_{V_1^\nu} + \Theta_1 = (f^\nu)^*(K_{V_2^\nu} + \Theta_2) + \nu_1^* E.$$

Note that V_2^ν is smooth and Θ_2 is a reduced simple normal crossing divisor on V_2^ν. By assumption, f^ν is an isomorphism over the generic point of any lc center of the pair (V_2^ν, Θ_2) (cf. 1.10). Therefore, $\nu_1^* E$ is effective since $K_{V_2^\nu} + \Theta_2$ is Cartier. Thus, E is effective. Since V_2 satisfies Serre's S_2 condition, we can check that $\mathcal{O}_{V_2} \simeq f_*\mathcal{O}_{V_1}$ and $f_*\mathcal{O}_{V_1}(K_{V_1} + D_1) \simeq \mathcal{O}_{V_2}(K_{V_2} + D_2)$. On the other hand, we obtain $R^i f_*\mathcal{O}_{V_1}(K_{V_1} + D_1) = 0$ for every $i > 0$ by Lemma 6.1. Therefore, $Rf_*\mathcal{O}_{V_1}(K_{V_1} + D_1) \simeq \mathcal{O}_{V_2}(K_{V_2} + D_2)$ in the derived category of coherent sheaves on V_2. Since V_1 and V_2 are Gorenstein, we have

$$\begin{aligned} Rf_*\mathcal{O}_{V_1}(-D_1) &\simeq R\mathcal{H}om(Rf_*\omega^\bullet_{V_1}(D_1), \omega^\bullet_{V_2}) \simeq R\mathcal{H}om(Rf_*\omega_{V_1}(D_1), \omega_{V_2}) \\ &\simeq R\mathcal{H}om(\omega_{V_2}(D_2), \omega_{V_2}) \simeq \mathcal{O}_{V_2}(-D_2) \end{aligned}$$

in the derived category of coherent sheaves on V_2 by Grothendieck duality. Therefore, $R^i f_*\mathcal{O}_{V_1}(-D_1) = 0$ for every $i > 0$ and $f_*\mathcal{O}_{V_1}(-D_1) \simeq \mathcal{O}_{V_2}(-D_2)$. □

Next, we prove the following theorem, which was proved for *embedded simple normal crossing pairs* in [F7, Theorem 2.39] and [F7, Theorem 2.47]. We note that here we do not assume the existence of ambient spaces. However, we need the assumption that X is *quasi-projective*.

Theorem 6.3 (cf. [F7, Theorems 2.39 and 2.47]). *Let* (X, B) *be a* quasi-projective *simple normal crossing pair such that* B *is a boundary* $\mathbb{R}$*-divisor on* X*. Let* $f : X \to Y$ *be a proper morphism between algebraic varieties and let* L *be a Cartier divisor on* X*. Let* q *be an arbitrary integer.*

(i) *Assume that* $L-(K_X+B)$ *is* f*-semiample, that is,* $L-(K_X+B)=\sum_i a_iD_i$ *where* D_i *is an* f*-semiample Cartier divisor on* X *and* a_i *is a positive real number for every* i*. Then every associated prime of* $R^qf_*\mathcal{O}_X(L)$ *is the generic point of the* f*-image of some stratum of* (X, B)*.*

(ii) *Let* $\pi : Y \to Z$ *be a proper morphism. Assume that* $L-(K_X+B)\sim_{\mathbb{R}} f^*A$ *for some* $\mathbb{R}$*-Cartier* $\mathbb{R}$*-divisor* A *on* Y *such that* A *is nef and log big over* Z *with respect to* $f : (X, B) \to Y$ *(see* [F7, *Definition* 2.46]*). Then* $R^qf_*\mathcal{O}_X(L)$ *is* π_**-acyclic, that is,* $R^p\pi_*R^qf_*\mathcal{O}_X(L)=0$ *for every* $p>0$*.*

Proof. Since X is quasi-projective, we can embed X into a smooth projective variety V. By Lemma 6.5 below, we can replace (X, B) and L with (X_k, B_k) and σ^*L and assume that there exists an $\mathbb{R}$-divisor D on V such that $B=D|_X$. Then, by using Bertini's theorem, we can take a general complete intersection $W\subset V$ such that $\dim W=\dim X+1$, $X\subset W$, and W is smooth at the generic point of every stratum of (X, B) (cf. the proof of [Ko6, Proposition 10.59]). We take a suitable resolution $\psi : M \to W$ with the following properties:

(A) The strict transform X' of X is a simple normal crossing divisor on M.

(B) We can write
$$K_{X'}+B'=\varphi^*(K_X+B)+E$$
so that $\varphi=\psi|_{X'}$, $(X', B'-E)$ is a globally embedded simple normal crossing pair (cf. Definition 2.3), B' is a boundary $\mathbb{R}$-divisor on X', $\lceil E\rceil$ is effective and φ-exceptional, and the φ-image of every stratum of (X', B') is a stratum of (X, B).

(C) φ is an isomorphism over the generic point of every stratum of (X, B).

(D) φ is an isomorphism at the generic point of every stratum of (X', B').

Then
$$K_{X'}+B'+\{-E\}=\varphi^*(K_X+B)+\lceil E\rceil,$$
$$\varphi_*\mathcal{O}_{X'}(\varphi^*L+\lceil E\rceil)\simeq\mathcal{O}_X(L),$$

and

$$R^q\varphi_*\mathcal{O}_{X'}(\varphi^*L+\lceil E\rceil)=0$$

for every $q>0$ by Lemma 6.1. We note that

$$\varphi^*L+\lceil E\rceil-(K_{X'}+B'+\{-E\})=\varphi^*(L-(K_X+B))$$

and φ is an isomorphism at the generic point of every stratum of $(X', B'+\{-E\})$.

Therefore, by replacing (X,B) and L with $(X', B'+\{-E\})$ and $\varphi^*L+\lceil E\rceil$, we may assume that (X,B) is a quasi-projective globally embedded simple normal crossing pair (cf. Definition 2.3). In this case, the claims have already been established in [F7, Theorems 2.39 and 2.47]. □

For some generalizations of Theorem 6.3 to *semi log canonical pairs*, see [F15].

Remark 6.4. Theorem 6.3(i) is contained in [F14, Theorem 1.1(i)]. In [F14, Theorem 1.1], X is not assumed to be quasi-projective. On the other hand, we do not know how to remove the quasi-projectivity assumption from Theorem 6.3(ii).

By direct calculations, we can obtain the following elementary lemma. It was used in the proof of Theorem 6.3.

Lemma 6.5 (cf. [F7, Lemma 3.60]). *Let (X,B) be a simple normal crossing pair such that B is a boundary $\mathbb{R}$-divisor. Let V be a smooth variety such that $X\subset V$. Then we can construct a sequence of blow-ups*

$$V_k\to V_{k-1}\to\cdots\to V_0=V$$

such that:

(1) *$\sigma_{i+1}:V_{i+1}\to V_i$ is the blow-up along a smooth irreducible component of* $\operatorname{Supp}B_i$ *for every $i\geq 0$.*
(2) *Set $X_0=X$, $B_0=B$, and X_{i+1} is the strict transform of X_i for every $i\geq 0$.*
(3) *Set $K_{X_{i+1}}+B_{i+1}=\sigma_{i+1}^*(K_{X_i}+B_i)$ for every $i\geq 0$.*
(4) *There exists an $\mathbb{R}$-divisor D on V_k such that $D|_{X_k}=B_k$ and B_k is a boundary $\mathbb{R}$-divisor on X_k.*
(5) *$\sigma_*\mathcal{O}_{X_k}\simeq\mathcal{O}_X$ and $R^q\sigma_*\mathcal{O}_{X_k}=0$ for every $q>0$, where $\sigma:V_k\to V_{k-1}\to\cdots\to V_0=V$.*

Proof. All we have to do is to check property (5). We note that

$$\sigma_{i+1*}\mathcal{O}_{V_{i+1}}(K_{V_{i+1}})\simeq\mathcal{O}_{V_{i+1}}(K_{V_{i+1}})$$

and $R^q\sigma_{i+1*}\mathcal{O}_{V_{i+1}}(K_{V_{i+1}})=0$ for every q and for each step $\sigma_{i+1}:V_{i+1}\to V_i$ by Lemma 6.2. Therefore we obtain $R^q\sigma_*\mathcal{O}_{X_k}(K_{X_k})=0$ for every $q>0$ and $\sigma_*\mathcal{O}_{X_k}(K_{X_k})\simeq\mathcal{O}_X(K_X)$. Thus by Grothendieck duality, $R^q\sigma_*\mathcal{O}_{X_k}=0$ for every $q>0$ and $\sigma_*\mathcal{O}_{X_k}\simeq\mathcal{O}_X$ as in the proof of Lemma 6.2. □

As a special case of Theorem 6.3(i), we have:

Corollary 6.6 (Torsion-freeness). *Let (X, D) be a quasi-projective simple normal crossing pair such that D is reduced and let $f : X \to Y$ be a projective surjective morphism onto a smooth algebraic variety Y. Assume that every stratum of (X, D) is dominant onto Y. Then $R^i f_*\omega_{X/Y}(D)$ is torsion-free for every i.*

Proof. It is sufficient to prove that $R^i f_*\mathcal{O}_X(K_X + D)$ is torsion-free for every i since $\mathcal{O}_Y(K_Y)$ is locally free. By Theorem 6.3(i), every associated prime of $R^i f_*\mathcal{O}_X(K_X+D)$ is the generic point of Y for every i. This means $R^i f_*\mathcal{O}_X(K_X+D)$ is torsion-free for every i. □

We will use this corollary in Section 7.

§7. Higher direct images of log canonical divisors

This section is the main part of this paper. The following theorem is our main result (cf. [Kw1, Theorem 5], [Ko2, Theorem 2.6], [N1, Theorem 1], [F4, Theorems 3.4 and 3.9], and [Kw3, Theorem 1.1]), which is a *natural* generalization of the Fujita–Kawamata semipositivity theorem to simple normal crossing pairs.

Theorem 7.1. *Let (X, D) be a simple normal crossing pair such that D is reduced and let $f : X \to Y$ be a projective surjective morphism onto a smooth algebraic variety Y. Assume that every stratum of (X, D) is dominant onto Y. Let Σ be a simple normal crossing divisor on Y such that every stratum of (X, D) is smooth over $Y^* = Y \setminus \Sigma$. Set $X^* = f^{-1}(Y^*)$, $D^* = D|_{X^*}$, and $d = \dim X - \dim Y$. Let $\iota : X^* \setminus D^* \to X^*$ be the natural open immersion. Then:*

(1) *$R^k(f|_{X^*})_*\iota_!\mathbb{Q}_{X^*\setminus D^*} \simeq R^k(f|_{X^*\setminus D^*})_!\mathbb{Q}_{X^*\setminus D^*}$ underlies a graded polarizable variation of $\mathbb{Q}$-mixed Hodge structure on Y^* for every k. Moreover, it is* admissible.

Set $\mathcal{V}^k_{Y^*} = R^k(f|_{X^*})_*\iota_!\mathbb{Q}_{X^*\setminus D^*} \otimes \mathcal{O}_{Y^*}$ for every k. Let

$$\cdots \subset F^{p+1}(\mathcal{V}^k_{Y^*}) \subset F^p(\mathcal{V}^k_{Y^*}) \subset F^{p-1}(\mathcal{V}^k_{Y^*}) \subset \cdots$$

be the Hodge filtration. Assume that all the local monodromies on the local system $R^{d-i}(f|_{X^*})_*\iota_!\mathbb{Q}_{X^*\setminus D^*}$ around Σ are unipotent. Then

(2) *$R^{d-i}f_*\mathcal{O}_X(-D)$ is isomorphic to the canonical extension of*

$$\mathrm{Gr}^0_F(\mathcal{V}^{d-i}_{Y^*}) = F^0(\mathcal{V}^{d-i}_{Y^*})/F^1(\mathcal{V}^{d-i}_{Y^*}),$$

denoted by $\mathrm{Gr}^0_F(\mathcal{V}^{d-i}_Y)$. In particular, $R^{d-i}f_*\mathcal{O}_X(-D)$ is locally free.

By Grothendieck duality, we obtain:

(3) *$R^i f_* \omega_{X/Y}(D)$ is isomorphic to the canonical extension of*

$$(\mathrm{Gr}_F^0(\mathcal{V}_{Y^*}^{d-i}))^* = \mathcal{H}om_{\mathcal{O}_{Y^*}}(\mathrm{Gr}_F^0(\mathcal{V}_{Y^*}^{d-i}), \mathcal{O}_{Y^*}).$$

Thus, $R^i f_* \omega_{X/Y}(D) \simeq (\mathrm{Gr}_F^0(\mathcal{V}_Y^{d-i}))^*$. In particular, $R^i f_* \omega_{X/Y}(D)$ is locally free.

(4) *Assume further that Y is complete. Then $R^i f_* \omega_{X/Y}(D)$ is semipositive.*

Even the following very special case of Theorem 7.1 has never been checked before. It does not follow from [Kw3, Theorem 1.1].

Corollary 7.2. *Let $f : X \to Y$ be a projective morphism from a simple normal crossing variety X to a smooth complete algebraic variety Y. Assume that every stratum of X is smooth over Y. Then $R^i f_* \omega_{X/Y}$ is a semipositive locally free sheaf for every i.*

It is natural to prove Theorem 7.3 below, which is a slight generalization of (2) and (3) in Theorem 7.1, simultaneously with Theorem 7.1.

Theorem 7.3 (cf. [Ko2, Theorem 2.6]). *Under the notation and assumptions of Theorem 7.1, if we do not assume that the local monodromies on the local system $R^{d-i}(f|_{X^*})_* \iota_! \mathbb{Q}_{X^* \setminus D^*}$ around Σ are unipotent, then:*

(a) *$R^{d-i} f_* \mathcal{O}_X(-D)$ is isomorphic to the lower canonical extension of*

$$\mathrm{Gr}_F^0(\mathcal{V}_{Y^*}^{d-i}) = F^0(\mathcal{V}_{Y^*}^{d-i})/F^1(\mathcal{V}_{Y^*}^{d-i}).$$

In particular, $R^{d-i} f_* \mathcal{O}_X(-D)$ is locally free.

By Grothendieck duality, we obtain

(b) *$R^i f_* \omega_{X/Y}(D)$ is isomorphic to the upper canonical extension of*

$$(\mathrm{Gr}_F^0(\mathcal{V}_{Y^*}^{d-i}))^* = \mathcal{H}om_{\mathcal{O}_{Y^*}}(\mathrm{Gr}_F^0(\mathcal{V}_{Y^*}^{d-i}), \mathcal{O}_{Y^*}).$$

In particular, $R^i f_* \omega_{X/Y}(D)$ is locally free.

Before we start the proof of Theorems 7.1 and 7.3, we make a remark on canonical extensions.

Remark 7.4 (Upper and lower canonical extensions of Hodge bundles). Let ${}^l\mathcal{V}_{Y^*}^k$ (resp. ${}^u\mathcal{V}_{Y^*}^k$) be the *Deligne extension* of $\mathcal{V}_{Y^*}^k$ on Y such that the eigenvalues of the residue of the connection are contained in $[0,1)$ (resp. $(-1,0]$). We call it the *lower* *canonical extension* (resp. *upper canonical extension*) of $\mathcal{V}^k_{Y^*}$ following [Ko2, Definition 2.3]. If the local monodromies on $R^k(f|_{X^*})_*\iota_!\mathbb{Q}_{X^*\setminus D^*} \simeq R^k(f|_{X^*\setminus D^*})_!\mathbb{Q}_{X^*\setminus D^*}$ around Σ are unipotent, then

$$ {}^l\mathcal{V}^k_{Y^*} = {}^u\mathcal{V}^k_{Y^*}. $$

In this case, we set

$$ \mathcal{V}^k_Y = {}^l\mathcal{V}^k_{Y^*} = {}^u\mathcal{V}^k_{Y^*} $$

and call it the *canonical extension* of $\mathcal{V}^k_{Y^*}$. Let $j : Y^* \to Y$ be the natural open immersion. We set

$$ {}^lF^p(\mathcal{V}^k_{Y^*}) = j_*F^p(\mathcal{V}^k_{Y^*}) \cap {}^l\mathcal{V}^k_{Y^*} $$

and call it the *lower canonical extension* of $F^p(\mathcal{V}^k_{Y^*})$ on Y. We can define the *upper canonical extension* ${}^uF^p(\mathcal{V}^k_{Y^*})$ of $F^p(\mathcal{V}^k_{Y^*})$ on Y similarly. As above, when the local monodromies on $R^k(f|_{X^*})_*\iota_!\mathbb{Q}_{X^*\setminus D^*}$ around Σ are unipotent, we write $F^p(\mathcal{V}^k_Y)$ for ${}^lF^p(\mathcal{V}^k_{Y^*}) = {}^uF^p(\mathcal{V}^k_{Y^*})$ and call it the *canonical extension* of $F^p(\mathcal{V}^k_{Y^*})$. Theorem 7.3(a) means that the short exact sequence

$$ 0 \to F^1(\mathcal{V}^{d-i}_{Y^*}) \to F^0(\mathcal{V}^{d-i}_{Y^*}) \to \mathrm{Gr}^0_F(\mathcal{V}^{d-i}_{Y^*}) \to 0 \tag{7.4.1} $$

extends to a short exact sequence

$$ 0 \to {}^lF^1(\mathcal{V}^{d-i}_{Y^*}) \to {}^lF^0(\mathcal{V}^{d-i}_{Y^*}) \to R^{d-i}f_*\mathcal{O}_X(-D) \to 0. \tag{7.4.2} $$

Let us consider the dual variation of mixed Hodge structure (cf. Remark 3.15). The dual local system of $R^k(f|_{X^*})_*\iota_!\mathbb{Q}_{X^*\setminus D^*}$ underlies $(\mathcal{V}^k_{Y^*})^*$. The locally free sheaf $(\mathcal{V}^k_{Y^*})^*$ carries the Hodge filtration F defined in Remark 3.15. Theorem 7.3(b) means that the short exact sequence

$$ 0 \to F^1((\mathcal{V}^{d-i}_{Y^*})^*) \to F^0((\mathcal{V}^{d-i}_{Y^*})^*) \to \mathrm{Gr}^0_F((\mathcal{V}^{d-i}_{Y^*})^*) \to 0 \tag{7.4.3} $$

extends to a short exact sequence

$$ 0 \to {}^uF^1((\mathcal{V}^{d-i}_{Y^*})^*) \to {}^uF^0((\mathcal{V}^{d-i}_{Y^*})^*) \to R^if_*\omega_{X/Y}(D) \to 0. \tag{7.4.4} $$

We note that

$$ \mathrm{Gr}^{-p}_F((\mathcal{V}^{d-i}_{Y^*})^*) \simeq (\mathrm{Gr}^p_F(\mathcal{V}^{d-i}_{Y^*}))^* $$

for every p as in Remark 3.15. We also note that all the terms in (7.4.2) and (7.4.4) are locally free sheaves by [Ks, Proposition 1.11.3] since $R^k(f|_{X^*})_*\iota_!\mathbb{Q}_{X^*\setminus D^*}$ underlies an admissible graded polarized variation of $\mathbb{Q}$-mixed Hodge structure on Y^* for every k by Theorem 7.1(1). See also Proposition 3.12. Let us see the relationship between (7.4.2) and (7.4.4) in detail for the reader's convenience. By definition, it is easy to see that

$$ ({}^l\mathcal{V}^k_{Y^*})^* = {}^u((\mathcal{V}^k_{Y^*})^*) $$

for every k. We can check that

$$0 \to {}^{u}F^{p}((\mathcal{V}^{k}_{Y^*})^*) \to ({}^{l}\mathcal{V}^{k}_{Y^*})^* \to ({}^{l}F^{1-p}(\mathcal{V}^{k}_{Y^*}))^* \to 0$$

is exact for all p and k (cf. Lemma 5.1). Then we have the following big commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & R^i f_*\omega_{X/Y}(D) & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & {}^{u}F^{1}((\mathcal{V}^{d-i}_{Y^*})^*) & \longrightarrow & ({}^{l}\mathcal{V}^{d-i}_{Y^*})^* & \longrightarrow & ({}^{l}F^{0}(\mathcal{V}^{d-i}_{Y^*}))^* & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & {}^{u}F^{0}((\mathcal{V}^{d-i}_{Y^*})^*) & \longrightarrow & ({}^{l}\mathcal{V}^{d-i}_{Y^*})^* & \longrightarrow & ({}^{l}F^{1}(\mathcal{V}^{d-i}_{Y^*}))^* & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & R^i f_*\omega_{X/Y}(D) & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

The first vertical line is nothing but (7.4.4) and the third vertical line is the dual of (7.4.2).

Theorem 7.1(2) (resp. (3)) is a special case of Theorem 7.3(a) (resp. (b)).

Let us start the proof of Theorems 7.1 and 7.3.

Proof of Theorems 7.1 and 7.3. Statement (1) in Theorem 7.1 follows from Theorem 4.15. We also note that (4) in Theorem 7.1 follows from Theorem 5.21 and Corollary 5.23 by (3) in Theorem 7.1.

Without loss of generality, by [BiP, Theorem 1.4] and Lemma 6.2, we may assume that $\mathrm{Supp}(f^*\Sigma \cup D)$ is a simple normal crossing divisor on X.

In Steps 1 and 2, we prove (2) and (3) of Theorem 7.1 for every i under the assumption that all the local monodromies on $R^k f_*\mathbb{C}_{S^*}$, where S is a stratum of (X, D), $S^* = S|_{X^*}$, and k is any integer, around Σ are unipotent. In Steps 3 and 4, we prove Theorem 7.3, which contains (2) and (3) of Theorem 7.1.

From now on, we assume that all the local monodromies on $R^k f_*\mathbb{C}_{S^*}$, where S is a stratum of (X, D), $S^* = S|_{X^*}$, and k is any integer, around Σ are unipotent.

Step 1 ($\dim Y = 1$). By shrinking Y, we may assume that Y is the unit disc Δ in $\mathbb{C}$ and $\Sigma = \{0\}$ in Δ. We set $E = f^{-1}(0)$. By considering $\Omega_{(D\cap X)_\bullet/Y}(\log E_\bullet)$ as in the proof of Lemma 4.10, we find that $R^{d-i}f_*\mathcal{O}_X(-D)$ is isomorphic to the canonical extension of $\mathrm{Gr}_F^0(\mathcal{V}_{Y^*}^{d-i})$ for every i (see also Remark 4.16). Therefore, $R^if_*\omega_{X/Y}(D) \simeq (\mathrm{Gr}_F^0(\mathcal{V}_Y^{d-i}))^*$ for every i by Grothendieck duality.

Step 2 ($l := \dim Y \geq 2$). We shall prove statement (3) by induction on l for every i.

By Step 1, there is an open subset Y_1 of Y such that $\mathrm{codim}(Y \setminus Y_1) \geq 2$ and

$$R^if_*\omega_{X/Y}(D)|_{Y_1} \simeq (\mathrm{Gr}_F^0(\mathcal{V}_Y^{d-i}))^*|_{Y_1}.$$

Since $(\mathrm{Gr}_F^0(\mathcal{V}_Y^{d-i}))^*$ is locally free (see Remark 7.4), we obtain a homomorphism

$$\varphi_Y^i : R^if_*\omega_{X/Y}(D) \to (\mathrm{Gr}_F^0(\mathcal{V}_Y^{d-i}))^*.$$

We will prove that φ_Y^i is an isomorphism. Without loss of generality, we may assume that X and Y are quasi-projective by shrinking Y. By Corollary 6.6, $R^if_*\omega_{X/Y}(D)$ is torsion-free. Therefore, $\mathrm{Ker}\varphi_Y^i = 0$. We put $G_Y^i := \mathrm{Coker}\varphi_Y^i$. Taking a general hyperplane cut, we see that $\mathrm{Supp}\, G_Y^i$ is a finite set by the induction hypothesis. Assume that $G_Y^i \neq 0$. We may also assume that $\mathrm{Supp}\, G_Y^i = \{P\}$ by shrinking Y. Let $\mu : W \to Y$ be the blowing up at P and set $E = \mu^{-1}(P)$. Then $E \simeq \mathbb{P}^{l-1}$. By [BiM, Theorem 1.5] and [BiP, Theorem 1.4], we can take a projective birational morphism $\pi : X' \to X$ from a simple normal crossing variety X' with the following properties:

(i) The composition $X' \to X \to Y \dashrightarrow W$ is a morphism.

(ii) π is an isomorphism over X^*.

(iii) $\mathrm{Exc}(\pi) \cup D'$ is a simple normal crossing divisor on X', where D' is the strict transform of D.

We obtain $R^qf_*\omega_{X/Y}(D) \simeq R^q(f \circ \pi)_*\omega_{X'/Y}(D')$ for every q as $R\pi_*\omega_{X'}(D') \simeq \omega_X(D)$ in the derived category of coherent sheaves on X by Lemma 6.2. We note that every stratum of (X', D') is dominant onto Y. We also note the following commutative diagram:

$$\begin{array}{ccc} X' & \xrightarrow{\pi} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ W & \xrightarrow{\mu} & Y \end{array}$$

By replacing (X, D) with (X', D'), we may assume that there is a morphism $g : X \to W$ such that $f = \mu \circ g$. Since $g : X \to W$ is in the same situation as f, we obtain the exact sequence

$$0 \to R^i g_*\omega_{X/W}(D) \to (\mathrm{Gr}_F^0(\mathcal{V}_W^{d-i}))^* \to G_W^i \to 0.$$

Tensoring $\mathcal{O}_W(\nu E)$ for $0 \leq \nu \leq l-1$ and applying $R^j\mu_*$ for $j \geq 0$ to each ν, we have an exact sequence

$$\begin{aligned} 0 &\to \mu_*(R^i g_*\omega_{X/W}(D) \otimes \mathcal{O}_W(\nu E)) \to \mu_*((\mathrm{Gr}_F^0(\mathcal{V}_W^{d-i}))^* \otimes \mathcal{O}_W(\nu E)) \\ &\to \mu_*(G_W^i \otimes \mathcal{O}_W(\nu E)) \to R^1\mu_*(R^i g_*\omega_{X/W}(D) \otimes \mathcal{O}_W(\nu E)) \\ &\to R^1\mu_*((\mathrm{Gr}_F^0(\mathcal{V}_W^{d-i}))^* \otimes \mathcal{O}_W(\nu E)) \to 0 \end{aligned}$$

and $R^q\mu_*(R^i g_*\omega_{X/W}(D) \otimes \mathcal{O}_W(\nu E)) \simeq R^q\mu_*((\mathrm{Gr}_F^0(\mathcal{V}_W^{d-i}))^* \otimes \mathcal{O}_W(\nu E))$ for $q \geq 2$.

By [Kw2, Proposition 1] and Remark 7.4, we obtain

$$(\mathrm{Gr}_F^0(\mathcal{V}_W^{d-i}))^* \simeq \mu^*(\mathrm{Gr}_F^0(\mathcal{V}_Y^{d-i}))^*.$$

We have

$$\mu_*((\mathrm{Gr}_F^0(\mathcal{V}_W^{d-i}))^* \otimes \mathcal{O}_W(\nu E)) \simeq (\mathrm{Gr}_F^0(\mathcal{V}_Y^{d-i}))^*$$

and

$$R^q\mu_*((\mathrm{Gr}_F^0(\mathcal{V}_W^{d-i}))^* \otimes \mathcal{O}_W(\nu E)) = 0$$

for $q \geq 1$. Therefore, $R^q\mu_*(R^i g_*\omega_{X/W}(D) \otimes \mathcal{O}_W(\nu E)) = 0$ for $q \geq 2$ and

$$\begin{aligned} 0 &\to \mu_*(R^i g_*\omega_{X/W}(D) \otimes \mathcal{O}_W(\nu E)) \to \mu_*((\mathrm{Gr}_F^0(\mathcal{V}_W^{d-i}))^* \otimes \mathcal{O}_W(\nu E)) \\ &\to \mu_*(G_W^i \otimes \mathcal{O}_W(\nu E)) \to R^1\mu_*(R^i g_*\omega_{X/W}(D) \otimes \mathcal{O}_W(\nu E)) \to 0 \end{aligned}$$

is exact. Since $\omega_W = \mu^*\omega_Y \otimes \mathcal{O}_W((l-1)E)$, we have a spectral sequence

$$E_2^{p,q} = R^p\mu_*(R^q g_*\omega_{X/W}(D) \otimes \mathcal{O}_W((l-1)E)) \Rightarrow R^{p+q} f_*\omega_{X/Y}(D).$$

However, $E_2^{p,q} = 0$ for $p \geq 2$ by the above argument. Thus

$$\begin{aligned} 0 \to R^1\mu_* R^{i-1} g_*\omega_{X/Y}(D) &\to R^i f_*\omega_{X/Y}(D) \\ &\to \mu_*(R^i g_*\omega_{X/W}(D) \otimes \mathcal{O}_W((l-1)E)) \to 0. \end{aligned}$$

By Corollary 6.6, $R^i f_*\omega_{X/Y}(D)$ is torsion-free. So, we obtain

$$R^1\mu_* R^{i-1} g_*\omega_{X/Y}(D) = 0.$$

Therefore, for $q \geq 1$ and for every i, we obtain

(A) $R^i f_*\omega_{X/Y}(D) \simeq \mu_*(R^i g_*\omega_{X/W}(D) \otimes \mathcal{O}_W((l-1)E))$ and
(B) $R^q\mu_*(R^i g_*\omega_{X/W}(D) \otimes \mathcal{O}_W((l-1)E)) = 0$.

Next, we shall consider the commutative diagram

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ R^i g_*\omega_{X/W}(D)\otimes\mathcal{O}_W((l-2)E) & \to & R^i g_*\omega_{X/W}(D)\otimes\mathcal{O}_W((l-1)E) \\ \downarrow & & \downarrow \\ (\mathrm{Gr}^0_F(\mathcal{V}^{d-i}_W))^*\otimes\mathcal{O}_W((l-2)E) & \to & (\mathrm{Gr}^0_F(\mathcal{V}^{d-i}_W))^*\otimes\mathcal{O}_W((l-1)E) \\ \downarrow & & \downarrow \\ G^i_W\otimes\mathcal{O}_W((l-2)E) & \to & G^i_W\otimes\mathcal{O}_W((l-1)E) \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

By applying μ_*, we obtain the commutative diagram

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \mu_*(R^i g_*\omega_{X/W}(D)\otimes\mathcal{O}_W((l-2)E)) & \to & \mu_*(R^i g_*\omega_{X/W}(D)\otimes\mathcal{O}_W((l-1)E)) \\ \downarrow & & \downarrow \\ (\mathrm{Gr}^0_F(\mathcal{V}^{d-i}_Y))^* & \simeq & (\mathrm{Gr}^0_F(\mathcal{V}^{d-i}_Y))^* \\ \downarrow & & \downarrow \\ \mu_*(G^i_W\otimes\mathcal{O}_W((l-2)E)) & \to & \mu_*(G^i_W\otimes\mathcal{O}_W((l-1)E)) \\ & & \downarrow \\ & & 0 \end{array}$$

By (A) and (B), $G^i_Y \simeq \mu_*(G^i_W\otimes\mathcal{O}_W((l-1)E))$ and

$$\mu_*(G^i_W\otimes\mathcal{O}_W((l-2)E)) \to \mu_*(G^i_W\otimes\mathcal{O}_W((l-1)E))$$

is surjective. Since $\dim\operatorname{Supp}G^i_W = 0$ and $E\cap\operatorname{Supp}G^i_W \neq \emptyset$, it follows that $G^i_W = 0$ by Nakayama's lemma. Therefore, $G^i_Y = 0$. This implies $R^i f_*\omega_{X/Y}(D) \simeq (\mathrm{Gr}^0_F(\mathcal{V}^{d-i}_Y))^*$. By Grothendieck duality, $R^{d-i}f_*\mathcal{O}_X(-D)\simeq \mathrm{Gr}^0_F(\mathcal{V}^{d-i}_Y)$.

From now on, we treat the general case, that is, we do not assume that local monodromies are unipotent.

Step 3. In this step, we prove the local freeness of $R^i f_*\omega_{X/Y}(D)$ for every i. We use the unipotent reduction with respect to all the local systems after shrinking Y suitably. This means that, shrinking Y, we have the commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{\alpha} & X' & \xleftarrow{\beta} & \widetilde{X} \\ {\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle \widetilde{f}} \\ Y & \xleftarrow{\tau} & Y' & = & Y' \end{array}$$

with the following properties:

(i) $\tau : Y' \to Y$ is a finite Kummer covering from a non-singular variety Y' and τ ramifies only along Σ.

(ii) $f' : X' \to Y'$ is the base change of $f : X \to Y$ by τ over $Y \setminus \Sigma$.

(iii) $(X', \alpha^* D)$ is a semidivisorial log terminal pair in the sense of Kollár (see Definition 2.4). Let X_j be any irreducible component of X. Then $X'_j = \alpha^{-1}(X_j)$ is the normalization of the base change of $X_j \to Y$ by $\tau : Y' \to Y$ and $X' = \bigcup_j X'_j$. We note that X'_j is a V-manifold for every j. More precisely, X'_j is toroidal for every j.

(iv) β is a projective birational morphism from a simple normal crossing variety $\widetilde{X}$ and $\widetilde{D} \cup \mathrm{Exc}(\beta)$ is a simple normal crossing divisor on $\widetilde{X}$, where $\widetilde{D}$ is the strict transform of $\alpha^* D$ (cf. [BiM, Theorem 1.5] and [BiP, Theorem 1.4]). We may further assume that β is an isomorphism over the largest Zariski open set U of X' such that $(X', \alpha^* D)|_U$ is a simple normal crossing pair.

(v) $\widetilde{f} : \widetilde{X} \to Y'$, $\widetilde{D}$, and $\tau^{-1}\Sigma$ satisfy the conditions and assumptions in Theorem 7.1 and all the local monodromies on all the local systems around $\tau^{-1}\Sigma$ are unipotent.

Therefore, $R^i \widetilde{f}_* \omega_{\widetilde{X}}(\widetilde{D})$ is locally free by Steps 1 and 2. On the other hand, we can prove

$$R^p \widetilde{f}_* \omega_{\widetilde{X}}(\widetilde{D}) \simeq R^p f'_* \omega_{X'}(\alpha^* D)$$

for every $p \geq 0$. We note that

$$K_{\widetilde{X}} + \widetilde{D} = \beta^*(K_{X'} + \alpha^* D) + F$$

where F is β-exceptional, F is permissible on $\widetilde{X}$, $\mathrm{Supp}\, F$ is a simple normal crossing divisor on $\widetilde{X}$, and $\lceil F \rceil$ is effective. Hence $\beta_* \omega_{\widetilde{X}}(\widetilde{D}) \simeq \omega_{X'}(\alpha^* D)$ and $R^q \beta_* \omega_{\widetilde{X}}(\widetilde{D}) = 0$ for every $q > 0$ by Lemma 6.1. Thus, $R^i f'_* \omega_{X'}(\alpha^* D)$ is locally free for every i. Since $R^i f_* \omega_X(D)$ is a direct summand of

$$\tau_* R^i f'_* \omega_{X'}(\alpha^* D) \simeq R^i f_*(\alpha_* \omega_{X'}(\alpha^* D)),$$

we deduce that $R^i f_* \omega_X(D)$ is locally free, equivalently, $R^i f_* \omega_{X/Y}(D)$ is locally free for every i. We note that, by Grothendieck duality, $R^{d-i} f_* \mathcal{O}_X(-D)$ is also locally free for every i.

Step 4. In this last step, we prove that $R^{d-i} f_* \mathcal{O}_X(-D)$ is the lower canonical extension for every i. By Grothendieck duality and Step 3, $R^{d-i} \widetilde{f}_* \mathcal{O}_{\widetilde{X}}(-\widetilde{D})$ is locally free. By Step 3, we obtain $R\beta_* \omega_{\widetilde{X}}(\widetilde{D}) \simeq \omega_{X'}(\alpha^* D)$ in the derived category of coherent sheaves on X'. Therefore, we obtain

$$\begin{aligned} R\beta_*\mathcal{O}_{\widetilde{X}}(-\widetilde{D}) &\simeq R\mathcal{H}om(R\beta_*\omega^\bullet_{\widetilde{X}}(\widetilde{D}), \omega^\bullet_{X'}) \\ &\simeq R\mathcal{H}om(\omega^\bullet_{X'}(\alpha^*D), \omega^\bullet_{X'}) \simeq \mathcal{O}_{X'}(-\alpha^*D) \end{aligned}$$

in the derived category of coherent sheaves on X'. Note that X' is Cohen–Macaulay (cf. [F14, Theorem 4.2]) and that $\omega^\bullet_{X'} \simeq \omega_{X'}[\dim X']$. Thus, we have

$$R^p\widetilde{f}_*\mathcal{O}_{\widetilde{X}}(-\widetilde{D}) \simeq R^pf'_*\mathcal{O}_{X'}(-\alpha^*D)$$

for every p. Let G be the Galois group of $\tau : Y' \to Y$. Then we have

$$(\tau_*R^pf'_*\mathcal{O}_{X'}(-\alpha^*D))^G \simeq R^pf_*(\alpha_*\mathcal{O}_{X'}(-\alpha^*D))^G \simeq R^pf_*\mathcal{O}_X(-D).$$

Thus, $R^{d-i}f_*\mathcal{O}_X(-D)$ is the lower canonical extension for every i (cf. [Ko2, Notation 2.5(iii)]). By Grothendieck duality, $R^if_*\omega_{X/Y}(D)$ is the upper canonical extension for every i.

We have finished the proof of Theorems 7.1 and 7.3. □

The following theorem is a generalization of [Ko1, Proposition 7.6].

Theorem 7.5. *Let $f : X \to Y$ be a projective surjective morphism from a simple normal crossing variety to a smooth algebraic variety Y with connected fibers. Assume that every stratum of X is dominant onto Y. Then $R^df_*\omega_X \simeq \omega_Y$ where $d = \dim X - \dim Y$.*

Proof. By [BiM, Theorem 1.5] and [BiP, Theorem 1.4], we can construct a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\pi} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ W & \xrightarrow{p} & Y \end{array}$$

with the following properties:

(i) $p : W \to Y$ is a projective birational morphism from a smooth quasi-projective variety W.

(ii) V is a simple normal crossing variety.

(iii) π is projective birational and π induces an isomorphism $\pi^0 = \pi|_{V^0} : V^0 \to X^0$ where X^0 (resp. V^0) is a Zariski open set of X (resp. V) which contains the generic point of any stratum of X (resp. V).

(iv) g is projective.

(v) There is a simple normal crossing divisor Σ on W such that every stratum of V is smooth over $W \setminus \Sigma$.

We note that $R^j g_* \omega_V$ is locally free for every j by Theorem 7.3. By Grothendieck duality,

$$Rg_* \mathcal{O}_V \simeq R\mathcal{H}om_{\mathcal{O}_W}(Rg_* \omega_V^\bullet, \omega_W^\bullet).$$

Therefore,

$$\mathcal{O}_W \simeq \mathcal{H}om_{\mathcal{O}_W}(R^d g_* \omega_V, \omega_W).$$

Note that, by Zariski's main theorem, $f_* \mathcal{O}_X \simeq \mathcal{O}_Y$ since every stratum of X is dominant onto Y. Therefore, $g_* \mathcal{O}_V \simeq \mathcal{O}_W$. Thus, $R^d g_* \omega_V \simeq \omega_W$. By applying p_*, we have $p_* R^d g_* \omega_V \simeq p_* \omega_W \simeq \omega_Y$. We note that $p_* R^d g_* \omega_V \simeq R^d (p \circ g)_* \omega_V$ since $R^i p_* R^d g_* \omega_V = 0$ for every $i > 0$ (cf. Theorem 6.3(ii)). On the other hand,

$$R^d (p \circ g)_* \omega_V \simeq R^d (f \circ \pi)_* \omega_V \simeq R^d f_* \omega_X$$

since $R^i \pi_* \omega_V = 0$ for every $i > 0$ by Lemma 6.1 and $\pi_* \omega_V \simeq \omega_X$ (cf. Lemma 6.2). Therefore, $R^d f_* \omega_X \simeq \omega_Y$. □

In geometric applications, we sometimes have a projective surjective morphism $f : X \to Y$ from a simple normal crossing variety to a smooth variety Y with *connected fibers* such that every stratum of X is mapped onto Y. The example below shows that in general there is no stratum S of X such that general fibers of $S \to Y$ are connected. Therefore, Kawamata's result [Kw3, Theorem 1.1] is very restrictive. He assumes that $S \to Y$ has connected fibers for every stratum S of X.

Example 7.6. We consider $W = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Let $p_i : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the i-th projection for $i = 1, 2, 3$. We take general members $X_1 \in |p_1^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes p_2^* \mathcal{O}_{\mathbb{P}^1}(2)|$ and $X_2 \in |p_1^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes p_3^* \mathcal{O}_{\mathbb{P}^1}(2)|$. We define $X = X_1 \cup X_2$, $Y = \mathbb{P}^1$, and $f = p_1|_X : X \to Y$. Then f is a projective morphism from a simple normal crossing variety X to a smooth projective curve Y. We can directly check that

$$H^1(W, \mathcal{O}_W(-X_1)) = H^1(W, \mathcal{O}_W(-X_2)) = 0,$$
$$H^1(W, \mathcal{O}_W(-X_1 - X_2)) = H^2(W, \mathcal{O}_W(-X_1 - X_2)) = 0.$$

Therefore, by using

$$0 \to \mathcal{O}_W(-X_1 - X_2) \to \mathcal{O}_W(-X_2) \to \mathcal{O}_{X_1}(-X_2) \to 0,$$

we obtain $H^1(X_1, \mathcal{O}_{X_1}(-X_2)) = 0$. By using

$$0 \to \mathcal{O}_{X_1}(-X_2) \to \mathcal{O}_{X_1} \to \mathcal{O}_{X_1 \cap X_2} \to 0,$$

we obtain $H^0(X_1 \cap X_2, \mathcal{O}_{X_1 \cap X_2}) = \mathbb{C}$ since $H^0(X_1, \mathcal{O}_{X_1}) = \mathbb{C}$. This means that $C = X_1 \cap X_2$ is a smooth connected curve. Therefore, every stratum of X is mapped onto Y by f. We note that general fibers of $f : X_1 \to Y$, $f : X_2 \to Y$, and $f : C \to Y$ are disconnected.

As a special case of Theorem 7.1, we obtain the following theorem.

Theorem 7.7 (cf. [Kw1, Theorem 5], [Ko2, Theorem 2.6], and [N1, Theorem 1]). *Let $f : X \to Y$ be a projective morphism between smooth complete algebraic varieties which satisfies the following conditions:*

(i) *There is a Zariski open subset Y^* of Y such that $\Sigma = Y \setminus Y^*$ is a simple normal crossing divisor on Y.*
(ii) *Set $X^* = f^{-1}(Y^*)$. Then $f|_{X^*}$ is smooth.*
(iii) *The local monodromies of $R^{d+i}(f|_{X^*})_*\mathbb{C}_{X^*}$ around Σ are unipotent, where $d = \dim X - \dim Y$.*

Then $R^if_*\omega_{X/Y}$ is a semipositive locally free sheaf on Y.

Proof. By Poincaré–Verdier duality (see, for example, [PS, Theorem 13.9]), the local system $R^{d-i}(f|_{X^*})_*\mathbb{C}_{X^*}$ is the dual local system of $R^{d+i}(f|_{X^*})_*\mathbb{C}_{X^*}$. Therefore, the local monodromies of $R^{d-i}(f|_{X^*})_*\mathbb{C}_{X^*}$ around Σ are unipotent. Thus, by Theorem 7.1, $R^if_*\omega_{X/Y} \simeq (R^{d-i}f_*\mathcal{O}_X)^*$ is a semipositive locally free sheaf on Y. □

Similarly, the semipositivity theorem of [F4, Theorem 3.9] can be recovered from Theorem 7.1. We note that [Kw3, Theorem 1.1] does not cover [F4, Theorem 3.9]. This is because Kawamata's theorem requires $S \to Y$ to have connected fibers for every stratum S of (X, D) (cf. Example 7.6).

Remark 7.8. Let $f : X \to Y$ be a projective morphism between smooth projective varieties. Assume that there exists a simple normal crossing divisor Σ on Y such that f is smooth over $Y \setminus \Sigma$. Then $R^if_*\omega_{X/Y}$ is locally free for every i (cf. Theorem 7.3 and [Ko2, Theorem 2.6]). We note that $R^if_*\omega_{X/Y}$ is not always semipositive if we assume nothing on monodromies around Σ.

We close this section with an easy example.

Example 7.9 (Double cover). We consider $\pi : Y = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \to \mathbb{P}^1$. Let E and G be the sections of π such that $E^2 = -2$ and $G^2 = 2$. We note that $E + 2F \sim G$ where F is a fiber of π. We put $\mathcal{L} = \mathcal{O}_Y(E + F)$. Then $E + G \in |\mathcal{L}^{\otimes 2}|$. Let $f : X \to Y$ be the double cover constructed by $E + G \in |\mathcal{L}^{\otimes 2}|$. Then $f : X \to Y$

is étale outside $\Sigma = E + G$ and

$$f_*\omega_{X/Y} \simeq \mathcal{O}_Y \oplus \mathcal{L}.$$

In this case, $f_*\omega_{X/Y}$ is not semipositive since $\mathcal{L} \cdot E = -1$. We note that the local monodromies on $(f|_{X^*})_*\mathbb{C}_{X^*}$ around Σ are not unipotent, where $Y^* = Y \setminus \Sigma$ and $X^* = f^{-1}(Y^*)$.

In Example 7.9, $f : X \to Y$ is finite and the general fibers of f are disconnected. In Section 8, we discuss an example $f : X \to Y$ whose general fibers are elliptic curves such that $f_*\omega_{X/Y}$ is not semipositive (cf. Corollary 8.10 and Example 8.16).

§8. Examples

In this final section, we give supplementary examples for the Fujita–Kawamata semipositivity theorem (cf. [Kw1, Theorem 5]), Viehweg's weak positivity theorem, and the Fujino–Mori canonical bundle formula (cf. [FM]). For details of the original Fujita–Kawamata semipositivity theorem, see, for example, [Mor, §5] and [F5, Section 5].

8.1 (Semipositivity in the sense of Fujita–Kawamata). The following example is due to Takeshi Abe. It is a small remark on Definition 5.20.

Example 8.2. Let C be an elliptic curve and let E be a stable vector bundle on C such that the degree of E is -1 and the rank of E is two. Let $f_m : C \to C$ be multiplication by m where m is a positive integer. In this case, every quotient line bundle L of E has non-negative degree. However, $\mathcal{O}_{\mathbb{P}(E)}(1)$ is not nef, because we can find a quotient line bundle M of f_m^*E whose degree is negative for some positive integer m.

8.3 (Canonical bundle formula). We give sample computations of our canonical bundle formula obtained in [FM]. We will freely use the notation in [FM]. For details of our canonical bundle formula, see [FM], [F2, §3], and [F3, §3, §4, §5, and §6].

8.4 (Kummer manifolds). Let E be an elliptic curve and let E^n be the n-fold direct product of E. Let G be the cyclic group of order two of analytic automorphisms of E^n generated by the automorphism

$$\tau : E^n \to E^n : (z_1, \ldots, z_n) \mapsto (-z_1, \ldots, -z_n).$$

The automorphism τ has 2^{2n} fixed points. Each singular point is terminal for $n \geq 3$ and is canonical for $n \geq 2$.

8.5 (Kummer surfaces). First, we consider $q : E^2/G \to E/G \simeq \mathbb{P}^1$, which is induced by the first projection, and $g = q \circ \mu : Y \to \mathbb{P}^1$, where $\mu : Y \to E^2/G$ is the minimal resolution of sixteen A_1-singularities. It is easy to see that Y is a $K3$ surface. In this case, it is obvious that

$$g_*\mathcal{O}_Y(mK_{Y/\mathbb{P}^1}) \simeq \mathcal{O}_{\mathbb{P}^1}(2m)$$

for every $m \geq 1$. Thus, we can put $L_{Y/\mathbb{P}^1} = D$ for any degree two Weil divisor D on $\mathbb{P}^1$. For the definition of $L_{Y/\mathbb{P}^1}$, see [FM, Definition 2.3]. We obtain $K_Y = g^*(K_{\mathbb{P}^1} + L_{Y/\mathbb{P}^1})$. Let Q_i be the branch point of $E \to E/G \simeq \mathbb{P}^1$ for $1 \leq i \leq 4$. Then

$$L^{\mathrm{ss}}_{Y/\mathbb{P}^1} = D - \sum_{i=1}^{4}\left(1 - \frac{1}{2}\right)Q_i = D - \sum_{i=1}^{4}\frac{1}{2}Q_i$$

by the definition of the semistable part $L^{\mathrm{ss}}_{Y/\mathbb{P}^1}$ (see [FM, Proposition 2.8, Definition 4.3, and Proposition 4.7]). Therefore,

$$K_Y = g^*\left(K_{\mathbb{P}^1} + L^{\mathrm{ss}}_{Y/\mathbb{P}^1} + \sum_{i=1}^{4}\frac{1}{2}Q_i\right).$$

Thus,

$$L^{\mathrm{ss}}_{Y/\mathbb{P}^1} = D - \sum_{i=1}^{4}\frac{1}{2}Q_i \nsim 0$$

but

$$2L^{\mathrm{ss}}_{Y/\mathbb{P}^1} = 2D - \sum_{i=1}^{4}Q_i \sim 0.$$

Note that $L^{\mathrm{ss}}_{Y/\mathbb{P}^1}$ is not a Weil divisor but a $\mathbb{Q}$-Weil divisor on $\mathbb{P}^1$.

8.6 (Elliptic fibrations). Next, we consider E^3/G and E^2/G. We consider the morphism $p : E^3/G \to E^2/G$ induced by the projection $E^3 \to E^2 : (z_1, z_2, z_3) \mapsto (z_1, z_2)$. Let $\nu : X' \to E^3/G$ be the weighted blow-up of E^3/G at sixty-four $\frac{1}{2}(1,1,1)$-singularities. Thus

$$K_{X'} = \nu^*K_{E^3/G} + \sum_{j=1}^{64}\frac{1}{2}E_j,$$

where $E_j \simeq \mathbb{P}^2$ is the exceptional divisor for every j. Let P_i be an A_1-singularity of E^2/G for $1 \leq i \leq 16$. Let $\psi : X \to X'$ be the blow-up of X' along the strict transform of $p^{-1}(P_i)$, which is isomorphic to $\mathbb{P}^1$, for every i. Then we obtain the commutative diagram

$$\begin{array}{ccc} E^3/G & \xleftarrow{\phi := \nu\circ\psi} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle f} \\ E^2/G & \xleftarrow{\mu} & Y \end{array}$$

Note that

$$K_X = \phi^* K_{E^3/G} + \sum_{j=1}^{64} \frac{1}{2} E_j + \sum_{k=1}^{16} F_k,$$

where E_j is the strict transform of E_j on X and F_k is the ψ-exceptional prime divisor for every k. We can check that X is a smooth projective threefold. We put $C_i = \mu^{-1}(P_i)$ for every i. It can be checked that C_i is a (-2)-curve for every i. It is easily checked that f is smooth outside $\sum_{i=1}^{16} C_i$ and that the degeneration of f is of type I_0^* along C_i for every i. We renumber $\{E_j\}_{j=1}^{64}$ as $\{E_i^j\}$, where $f(E_i^j) = C_i$ for every $1 \le i \le 16$ and $1 \le j \le 4$. We note that f is flat since f is equidimensional.

Let us recall the following theorem (cf. [Kw2, Theorem 20] and [N2, Corollary 3.2.1 and Theorem 3.2.3]).

Theorem 8.7 (. . . , Kawamata, Nakayama, . . .). *We have*

$$(f_*\omega_{X/Y})^{\otimes 12} \simeq \mathcal{O}_Y\Big(\sum_{i=1}^{16} 6C_i\Big),$$

where $\omega_{X/Y} \simeq \mathcal{O}_X(K_{X/Y}) = \mathcal{O}_X(K_X - f^*K_Y)$.

The proof of Theorem 8.7 depends on the investigation of the upper canonical extension of the Hodge filtration and the period map. It is obvious that

$$2K_X = f^*\Big(2K_Y + \sum_{i=1}^{16} C_i\Big)$$

and

$$2mK_X = f^*\Big(2mK_Y + m\sum_{i=1}^{16} C_i\Big)$$

for all $m \ge 1$ since $f^*C_i = 2F_i + \sum_{j=1}^{4} E_i^j$ for every i. Therefore, $2L_{X/Y} \sim \sum_{i=1}^{16} C_i$. On the other hand, $f_*\omega_{X/Y} \simeq \mathcal{O}_Y(\lfloor L_{X/Y} \rfloor)$. Note that Y is a smooth surface and f is flat. Since

$$\mathcal{O}_Y(12\lfloor L_{X/Y} \rfloor) \simeq (f_*\omega_{X/Y})^{\otimes 12} \simeq \mathcal{O}_Y\Big(\sum_{i=1}^{16} 6C_i\Big),$$

we have

$$12L_{X/Y} \sim 6\sum_{i=1}^{16} C_i \sim 12\lfloor L_{X/Y} \rfloor.$$

Thus, $L_{X/Y}$ is a Weil divisor on Y, because the fractional part $\{L_{X/Y}\}$ is effective and linearly equivalent to zero. So, $L_{X/Y}$ is numerically equivalent to $\frac{1}{2}\sum_{i=1}^{16} C_i$.

We have $g^*Q_i = 2G_i + \sum_{j=1}^4 C_i^j$ for every i. Here, we renumbered $\{C_j\}_{j=1}^{16}$ as $\{C_i^j\}_{i,j=1}^4$ so that $g(C_i^j) = Q_i$ for every i and j. More precisely, we set $2G_i = g^*Q_i - \sum_{j=1}^4 C_i^j$ for every i. We note that we used notations in 8.5. We consider $A := g^*D - \sum_{i=1}^4 G_i$. Then A is a Weil divisor and $2A \sim \sum_{i=1}^{16} C_i$. Thus, A is numerically equivalent to $\frac{1}{2}\sum_{i=1}^{16} C_i$. Since $H^1(Y, \mathcal{O}_Y) = 0$, we can set $L_{X/Y} = A$. So, we have

$$L_{X/Y}^{\mathrm{ss}} = g^*D - \sum_{i=1}^{4} G_i - \sum_{j=1}^{16} \frac{1}{2} C_j.$$

We obtain the following canonical bundle formula.

Theorem 8.8. *We have*

$$K_X = f^*\Big(K_Y + L_{X/Y}^{\mathrm{ss}} + \sum_{j=1}^{16} \frac{1}{2} C_j\Big),$$

where $L_{X/Y}^{\mathrm{ss}} = g^*D - \sum_{i=1}^4 G_i - \sum_{j=1}^{16} \frac{1}{2} C_j$.

We note that $2L_{X/Y}^{\mathrm{ss}} \sim 0$ but $L_{X/Y}^{\mathrm{ss}} \not\sim 0$. The semistable part $L_{X/Y}^{\mathrm{ss}}$ is not a Weil divisor but a $\mathbb{Q}$-divisor on Y.

The next lemma is obvious since the index of $K_{E^3/G}$ is two. We give a direct proof here.

Lemma 8.9. $H^0(Y, L_{X/Y}) = 0$.

Proof. Suppose that there exists an effective Weil divisor B on Y such that $L_{X/Y} \sim B$. Since $B \cdot C_i = -1$, we have $B \geq \frac{1}{2} C_i$ for all i. Thus $B \geq \sum_{i=1}^{16} \frac{1}{2} C_i$. This implies that $B - \sum_{i=1}^{16} \frac{1}{2} C_i$ is an effective $\mathbb{Q}$-divisor and is numerically equivalent to zero. Thus $B = \sum_{i=1}^{16} \frac{1}{2} C_i$, a contradiction. □

We can easily check the following corollary.

Corollary 8.10. *We have*

$$f_*\omega_{X/Y}^{\otimes m} \simeq \begin{cases} \mathcal{O}_Y(\sum_{i=1}^{16} nC_i) & \text{if } m = 2n, \\ \mathcal{O}_Y(L_{X/Y} + \sum_{i=1}^{16} nC_i) & \text{if } m = 2n+1. \end{cases}$$

In particular, $f_*\omega_{X/Y}^{\otimes m}$ *is not nef for any* $m \geq 1$. *We can also check that*

$$H^0(Y, f_*\omega_{X/Y}^{\otimes m}) \simeq \begin{cases} \mathbb{C} & \text{if } m \text{ is even}, \\ 0 & \text{if } m \text{ is odd}. \end{cases}$$

Corollary 8.10 shows that [T, Theorem 1.9(1)] is sharp.

8.11 (Weak positivity). Let us recall the definition of Viehweg's weak positivity (cf. [V1, Definition 1.2] and [V3, Definition 2.11]). The reader can find some interesting applications of a generalization of Viehweg's weak positivity theorem in [FG1].

Definition 8.12 (Weak positivity). Let W be a smooth quasi-projective variety and let $\mathcal{F}$ be a locally free sheaf on W. Let U be an open subvariety of W. Then $\mathcal{F}$ is *weakly positive over* U if for every ample invertible sheaf $\mathcal{H}$ and every positive integer α there exists some positive integer β such that $S^{\alpha\cdot\beta}(\mathcal{F})\otimes\mathcal{H}^{\beta}$ is generated by global sections over U where S^k denotes the k-th symmetric product for every positive integer k. This means that the natural map

$$H^0(W, S^{\alpha\cdot\beta}(\mathcal{F})\otimes\mathcal{H}^{\beta})\otimes\mathcal{O}_W\to S^{\alpha\cdot\beta}(\mathcal{F})\otimes\mathcal{H}^{\beta}$$

is surjective over U.

Remark 8.13 (cf. [V1, (1.3) Remark. iii)]). In Definition 8.12, it is enough to check the condition for one invertible sheaf $\mathcal{H}$, not necessarily ample, and all $\alpha>0$. For details, see [V3, Lemma 2.14 a)].

Remark 8.14. In [V2, Definition 3.1], $S^{\alpha\cdot\beta}(\mathcal{F})\otimes\mathcal{H}^{\otimes\beta}$ is only required to be generically generated. See also [Mor, (5.1) Definition].

We explicitly check the weak positivity for the elliptic fibration constructed in 8.6 (cf. [V1, Theorem 4.1 and Theorem III] and [V3, Theorem 2.41 and Corollary 2.45]).

Proposition 8.15. *Let m be a positive integer. Let $f: X\to Y$ be the elliptic fibration constructed in 8.6. Then $f_*\omega^{\otimes m}_{X/Y}$ is weakly positive over $Y_0=Y\setminus\sum_{i=1}^{16}C_i$. Let U be a Zariski open set such that $U\not\subset Y_0$. Then $f_*\omega^{\otimes m}_{X/Y}$ is not weakly positive over U.*

Proof. Let H be a very ample Cartier divisor on Y such that $L_{X/Y}+H$ is very ample. Set $\mathcal{H}=\mathcal{O}_Y(H)$. Let α be an arbitrary positive integer. Then

$$S^{\alpha}(f_*\omega^{\otimes m}_{X/Y})\otimes\mathcal{H}\simeq\mathcal{O}_Y\left(\alpha\sum_{i=1}^{16}nC_i+H\right)$$

if $m=2n$. When $m=2n+1$, we have

$$S^{\alpha}(f_*\omega^{\otimes m}_{X/Y})\otimes\mathcal{H}\simeq\begin{cases}\mathcal{O}_Y\left(\alpha\sum_{i=1}^{16}nC_i+H+L_{X/Y}+\left\lfloor\frac{\alpha}{2}\right\rfloor\sum_{i=1}^{16}C_i\right) & \text{if } \alpha \text{ is odd,}\\ \mathcal{O}_Y\left(\alpha\sum_{i=1}^{16}nC_i+H+\frac{\alpha}{2}\sum_{i=1}^{16}C_i\right) & \text{if } \alpha \text{ is even.}\end{cases}$$

Thus, $S^{\alpha}(f_*\omega_{X/Y}^{\otimes m})\otimes \mathcal{H}$ is generated by global sections over Y_0 for every $\alpha>0$. Therefore, $f_*\omega_{X/Y}^{\otimes m}$ is weakly positive over Y_0.

Let $\mathcal{A}$ be an ample invertible sheaf on Y. We put $k=\max_j(C_j\cdot\mathcal{A})$. Let α be a positive integer with $\alpha>k/2$. We note that

$$S^{2\alpha\cdot\beta}(f_*\omega_{X/Y}^{\otimes m})\otimes \mathcal{A}^{\otimes\beta}\simeq\Big(\mathcal{O}_Y\Big(\alpha\sum_{i=1}^{16}mC_i\Big)\otimes\mathcal{A}\Big)^{\otimes\beta}.$$

If $H^0(Y,S^{2\alpha\cdot\beta}(f_*\omega_{X/Y}^{\otimes m})\otimes\mathcal{A}^{\otimes\beta})\neq 0$, then we can take

$$G\in\Big|\Big(\mathcal{O}_Y\Big(\alpha\sum_{i=1}^{16}mC_i\Big)\otimes\mathcal{A}\Big)^{\otimes\beta}\Big|.$$

In this case, $G\cdot C_i<0$ for every i because $\alpha>k/2$. Therefore, $G\geq\sum_{i=1}^{16}C_i$. Thus, $S^{2\alpha\cdot\beta}(f_*\omega_{X/Y}^{\otimes m})\otimes\mathcal{A}^{\otimes\beta}$ is not generated by global sections over U for any $\beta\geq 1$. This means that $f_*\omega_{X/Y}^{\otimes m}$ is not weakly positive over U. □

Proposition 8.15 implies that [V3, Corollary 2.45] is the best possible result.

Example 8.16. Let $f:X\to Y$ be the elliptic fibration constructed in 8.6. Let $Z:=C\times X$, where C is a smooth projective curve with genus $g(C)=r\geq 2$. Let $\pi_1:Z\to C$ (resp. $\pi_2:Z\to X$) be the first (resp. second) projection. Set $h:=f\circ\pi_2:Z\to Y$. In this case, $K_Z=\pi_1^*K_C\otimes\pi_2^*K_X$. Therefore,

$$h_*\omega_{Z/Y}^{\otimes m}=f_*\pi_{2*}(\pi_1^*\omega_C^{\otimes m}\otimes\pi_2^*\omega_X^{\otimes m})\otimes\omega_Y^{\otimes -m}=(f_*\omega_{X/Y}^{\otimes m})^{\oplus l},$$

where $l=\dim H^0(C,\mathcal{O}_C(mK_C))$. Thus, $l=(2m-1)r-2m+1$ if $m\geq 2$ and $l=r$ if $m=1$. So, $h_*\omega_{Z/Y}$ is a rank $r\geq 2$ vector bundle on Y such that $h_*\omega_{Z/Y}$ is not semipositive. We note that h is smooth over $Y_0=Y\setminus\sum_{i=1}^{16}C_i$. Moreover, $h_*\omega_{Z/Y}^{\otimes m}$ is weakly positive over Y_0 for every $m\geq 1$ by [V3, Theorem 2.41 and Corollary 2.45].

Example 8.16 shows that the assumption on the local monodromies around $\sum_{i=1}^{16}C_i$ is indispensable for the semipositivity theorem.

We close this section with a comment on [FM].

8.17 (Comment). We give a remark on [FM, Section 4]. In [FM, 4.4], $g:Y\to X$ is a log resolution of (X,Δ). However, it is better to assume that g is a log resolution of $(X,\Delta-(1/b)B^{\Delta})$ for the proof of [FM, Theorem 4.8].

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