

Non-symmetrizable Quantum Groups: Defining Ideals and Specialization

by

Xin FANG

Abstract

Two generating sets of the defining ideal of a Nichols algebra of diagonal type were proposed, which are then applied to study the bar involution and the specialization problem for quantum groups associated to non-symmetrizable generalized Cartan matrices.

2010 Mathematics Subject Classification: Primary 16T20; Secondary 81R50.

Keywords: Nichols algebra, quantum group, defining ideal, specialization.

§1. Introduction

§1.1. Motivations

Quantized enveloping algebras (quantum groups) $U_q(\mathfrak{g})$ were constructed by V. Drinfel'd and M. Jimbo in the eighties of the last century by deforming the usual enveloping algebras associated to symmetrizable Kac–Moody Lie algebras \mathfrak{g} with the aim of finding solutions of the Yang–Baxter equation. They motivate much research in the last three decades in areas such as pointed Hopf algebras, canonical (crystal) bases, quantum knot invariants, quiver representations and Hall algebras, (quantum) cluster algebras, Hecke algebras, quantum affine and toroidal algebras, and so on.

In the original definition of a quantum group in terms of generators and relations, the symmetrizable condition on the Cartan matrix is essential in writing down explicitly the quantized Serre relations. With this explicit expression, it is not difficult to construct a specialization map [21] sending the quantum parameter q to 1 to recover the enveloping algebra $U(\mathfrak{g})$; the map is shown to be an isomorphism of Hopf algebras. It should be remarked that the well-definedness of

Communicated by H. Nakajima. Received September 30, 2013. Revised March, 14, 2014.

X. Fang: Département de Mathématiques, Bâtiment 425, Faculté des Sciences d'Orsay, Université Paris-Sud, F-91405 Orsay Cedex, France;
e-mail: xinfang.math@gmail.com

the specialization map depends on the knowledge of the quantized Serre relations and the Gabber–Kac theorem [10] in Kac–Moody Lie algebras.

In a survey article [17], M. Kashiwara asked: does a crystal graph for non-symmetrizable \mathfrak{g} have a meaning? He also remarked that the definition of the quantum group $U_q(\mathfrak{g})$ associated to an arbitrary Kac–Moody Lie algebra \mathfrak{g} was not known at that time.

This problem was recently solved at the combinatorial level by Joseph and Lamrou [14]: they constructed the abstract crystal $\mathcal{B}(\infty)$ associated to a generalized Cartan–Borcherds matrix (not necessarily symmetrizable) without passing to the quantized enveloping algebra but adopting the path model construction after Littelmann [20] by using the action of root operators on a sort of good paths. This construction is combinatorial and it is natural to ask for a true algebra bearing it and the globalization of these local crystals. This is the main motivation of our study of non-symmetrizable quantum groups. This project is divided into three steps:

1. Define the quantum group associated to a non-symmetrizable generalized Cartan matrix, study their structures, specializations and the existence of the bar involutions.
2. Define the q -Boson algebra, its action on the negative part of this quantum group and its semisimplicity; then define the Kashiwara operators associated to simple roots.
3. Establish the local crystal structure and its globalization, and compare the former with the construction of Joseph–Lamrou.

In this paper we will tackle the first step. The second step is almost achieved, the main tool being the construction in [7]; details will be given in a forthcoming paper.

The first functorial (coordinate-free) construction of (the positive or negative part of) the quantum group appears in the work of M. Rosso [23], [24] with the name “quantum shuffle algebras” and then interpreted in a dual language by Andruskiewitsch and Schneider [4] and named “Nichols algebras”. These constructions broadly generalize the definition of the usual quantum group and can be applied in particular to the non-symmetrizable case to obtain half of the quantum group. The quantum double construction can then be applied to combine the positive and negative parts to yield the whole quantum group.

In summary, we can associate a Hopf algebra (the quantum double of the bosonized Nichols algebra) to a generalized Cartan matrix C which is not necessarily symmetrizable. It is natural to ask whether there exists a specialization map from this Hopf algebra to the enveloping algebra of the Kac–Moody Lie algebra $\mathfrak{g}(C)$ associated to C ; this is not easy since in general, neither the Nichols

algebra nor the Kac–Moody Lie algebra has an explicit presentation by generators and relations.

The goal of this paper is twofold: on the one hand, tackling the specialization problem in the non-symmetrizable case by studying the defining ideal of the corresponding Nichols algebra; on the other hand, defining the bar involution in the non-symmetrizable case. As a byproduct, we get an estimate on the size of the defining ideal.

§1.2. Defining ideals in Nichols algebras

Let (V, σ) be a braided vector space. The tensor algebra $T(V)$ admits a braided Hopf algebra structure with coproduct making elements in V primitive; it can then be extended to the entire $T(V)$ by replacing the usual flip with the braiding.

If the braiding σ arises from an H -Yetter–Drinfel’d module structure on V over a Hopf algebra H , the Nichols algebra can be defined as the quotient of $T(V)$ by some maximal ideal and coideal $\mathfrak{J}(V)$ contained in the set of elements of degree no less than 2. We call $\mathfrak{J}(V)$ the *defining ideal* of the Nichols algebra $\mathfrak{B}(V)$.

As an example, for a symmetrizable Kac–Moody Lie algebra \mathfrak{g} , the negative part $U_q^-(\mathfrak{g})$ of the corresponding quantum group is a Nichols algebra, in which case the defining ideal $\mathfrak{J}(V)$ is generated as a Hopf ideal by quantized Serre relations. In general, it is very difficult to find a minimal generating set of $\mathfrak{J}(V)$ as a Hopf ideal in $T(V)$.

In [1], Andruskiewitsch asked some questions which guide research in this domain, including the following ones concerning defining ideals:

1. For those $\mathfrak{B}(V)$ having finite Gel’fand–Kirillov dimension, find a minimal generating set of $\mathfrak{J}(V)$.
2. When is the ideal $\mathfrak{J}(V)$ finitely generated?

The first general result on the defining ideal is due to M. Rosso [24] and P. Schauenburg [25]: they characterize it as the kernel of the total symmetrization operator. Recently, for Nichols algebras of diagonal type with finite root system, a minimal generating set of the defining ideal has been found by I. Angiono [5]. In this case, the corresponding Lyndon words and their symmetries (Lusztig’s isomorphisms) [11] play an essential role.

In [8], we proposed the notion of “level n ” elements with the help of a decomposition of the total symmetrization operators in the braid groups and proved their primitivity. These elements can be easily computed and the degrees where they appear are strongly restricted. This construction demands no concrete restriction on the braiding, hence is quite general, but we pay the price that these elements may not generate the defining ideal.

Once restricted to the diagonal case where the braiding is a twist by scalars of the usual flip, with some modifications on the conditions posed on “level n ” elements, we obtain a generating set formed by some “pre-relations”.

§1.3. Main ideas and results

The main part of this paper is devoted to some methods of studying a slightly modified version of the above problems. First, we will restrict ourselves to infinite-dimensional Nichols algebras of diagonal type having not necessarily finite Gel’fand–Kirillov dimension. Second, our principle has a pragmatic feature: we do not always desire a minimal generating set of the defining ideal but are satisfied with finding generating subsets suitable for solving concrete problems.

We propose four subsets of the defining ideal $\mathfrak{J}(V)$: left and right constants, left and right pre-relations. The first two sets are defined as the intersection of the kernels of left and right differential operators, and the last two are their subsets obtained by selecting elements which are contained in the images of the Dynkin operators. The two main results (Theorems 4.4 and 5.8) state that both are generators of the defining ideal.

These results are then applied to the study of the specialization problem. In general, if the generalized Cartan matrix C is not symmetrizable, we show by an example that the natural specialization map may not be well-defined. Therefore in our approach, the first step is to pass to a symmetric matrix \bar{C} by taking the average of the Cartan matrix. A result due to Andruskiewitsch and Schneider ensures that this procedure does not lose too much information.

Having passed to the averaged matrix, we prove in Theorem 7.4 that the specialization map $U_q(\bar{C}) \rightarrow U(\mathfrak{g}(\bar{C}))$ is well-defined and surjective.

As another application, we relate the degrees where pre-relations may appear to integral points of some quadratic forms arising from the action of the centre of the braid group. This allows us

1. to reprove some well-known results in a completely different way which we hope could shed light on the finite generation problem for $\mathfrak{J}(V)$;
2. to explain that the sets of left and of right pre-relations are not too large.

§1.4. Organization of this paper

After some recollections on Nichols algebras and braid groups in Sections 2 and 3, we define the constants and pre-relations in Sections 4 and 5 and show that they are indeed generating sets. These results are then applied to study the specialization problem in Sections 6 and 7. Another application to the finite generation property is given in Section 8.

§2. Recollections on Nichols algebras

Let \mathbb{K} be a field of characteristic 0 and $\mathbb{K}^\times = \mathbb{K} \setminus \{0\}$. All algebras and vector spaces, if not specified otherwise, are over the field \mathbb{K} .

§2.1. Nichols algebras

Let H be a Hopf algebra and ${}^H_H\mathcal{YD}$ be the category of H -Yetter–Drinfel’d modules. The category ${}^H_H\mathcal{YD}$ is a braided category; for any $V, W \in {}^H_H\mathcal{YD}$, we let $\sigma_{V,W} : V \otimes W \rightarrow W \otimes V$ denote the braiding. With this notation, $(V, \sigma_{V,V})$ is a braided vector space. Readers unfamiliar with these constructions are directed to [4] for a survey.

Definition 2.1 ([4]). A graded braided Hopf algebra $R = \bigoplus_{n=0}^{\infty} R(n)$ is called the *Nichols algebra* of $V \in {}^H_H\mathcal{YD}$ if:

1. $R(0) \cong \mathbb{K}$, $R(1) \cong V$.
2. R is generated as an algebra by $R(1)$.
3. $R(1)$ is the set of all primitive elements of R .

We let $\mathfrak{B}(V)$ denote this braided Hopf algebra.

The Nichols algebra $\mathfrak{B}(V)$ can be realized concretely as a quotient of the braided tensor Hopf algebra $T(V)$.

Remark 2.2 ([4]). Let $V \in {}^H_H\mathcal{YD}$ be an H -Yetter–Drinfel’d module. There exists a braided tensor Hopf algebra structure on the tensor space

$$T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}.$$

1. Multiplication on $T(V)$ is given by concatenation.
2. The coalgebra structure is defined on V by $\Delta(v) = v \otimes 1 + 1 \otimes v$ and $\varepsilon(v) = 0$ for any $v \in V$. Then it can be extended to the whole $T(V)$ by the universal property of $T(V)$ as an algebra.

For $k \geq 2$, let $T^{\geq k}(V) = \bigoplus_{n \geq k} V^{\otimes n}$ and let $\mathfrak{J}(V)$ be the maximal coideal of $T(V)$ contained in $T^{\geq 2}(V)$; it is also a two-sided ideal ([4]). The Nichols algebra $\mathfrak{B}(V)$ associated to V is isomorphic to $T(V)/\mathfrak{J}(V)$ as a braided Hopf algebra. We let S denote the convolution inverse of the identity map on $\mathfrak{B}(V)$.

Remark 2.3. The construction of a Nichols algebra $\mathfrak{B}(V)$ is still valid when (V, σ) is a braided vector space.

§2.2. Nichols algebras of diagonal type

Definition 2.4 ([4]). The Nichols algebra $\mathfrak{B}(V)$ associated to a braided vector space (V, σ) is called *of diagonal type* if there exists a basis $\{v_1, \dots, v_N\}$ of V and a matrix $(q_{ij})_{1 \leq i, j \leq N} \in M_N(\mathbb{K}^\times)$ of non-zero scalars such that for any $1 \leq i, j \leq N$, $\sigma(v_i \otimes v_j) = q_{ij} v_j \otimes v_i$. The scalar matrix is called the *braiding matrix* of (V, σ) .

In the situation of Remark 2.2, we will abuse language by saying that $T(V)$ is of diagonal type if $\mathfrak{B}(V)$ is so.

The following example of a Nichols algebra of diagonal type is the main object we will study in this paper. Let $G = \mathbb{Z}^N$ be the additive group, $H = \mathbb{K}[G]$ be its group algebra and \widehat{G} be the character group of G . Let $V \in {}^H_H\mathcal{YD}$ be an H -Yetter–Drinfel’d module of dimension N . It admits a decomposition into linear subspaces $V = \bigoplus_{g \in G} V_g$ where $V_g = \{v \in V \mid \delta(v) = g \otimes v\}$ and $\delta : V \rightarrow H \otimes V$ is the comodule structure map. Moreover, there exist a basis $\{v_1, \dots, v_N\}$ of V , elements $g_1, \dots, g_N \in G$ and characters $\chi_1, \dots, \chi_N \in \widehat{G}$ such that $v_i \in V_{g_i}$ and for any $g \in G$,

$$g \cdot v_i = \chi_i(g) v_i.$$

In this case the braiding $\sigma_{V,V}$ has the following explicit form: for $1 \leq i, j \leq N$,

$$\sigma_{V,V}(v_i \otimes v_j) = \chi_j(g_i) v_j \otimes v_i.$$

Therefore the Nichols algebra associated to $(V, \sigma_{V,V})$ is of diagonal type with braiding matrix $(q_{ij})_{1 \leq i, j \leq N} = (\chi_j(g_i))_{1 \leq i, j \leq N} \in M_N(\mathbb{K}^\times)$.

For an arbitrary matrix $A = (q_{ij}) \in M_N(\mathbb{K}^\times)$, we let $\mathfrak{B}(V_A)$ denote the Nichols algebra associated to the H -Yetter–Drinfel’d module V of diagonal type with braiding matrix A . If the matrix A under consideration is fixed, we denote it by $\mathfrak{B}(V)$ for short.

From now on let $I = \{1, \dots, N\}$ denote the index set.

§2.3. Differential operators

Let $V \in {}^H_H\mathcal{YD}$ be an H -Yetter–Drinfel’d module of diagonal type and $\{v_1, \dots, v_N\}$ be the basis of V as fixed in the last subsection.

Definition 2.5 ([18]). Let A and B be two Hopf algebras with invertible antipodes. A *generalized Hopf pairing* between A and B is a bilinear form $\varphi : A \times B \rightarrow \mathbb{K}$ such that:

1. For any $a \in A$ and $b, b' \in B$, $\varphi(a, bb') = \sum \varphi(a_{(1)}, b) \varphi(a_{(2)}, b')$.
2. For any $a, a' \in A$ and $b \in B$, $\varphi(aa', b) = \sum \varphi(a, b_{(2)}) \varphi(a', b_{(1)})$.
3. For any $a \in A$ and $b \in B$, $\varphi(a, 1) = \varepsilon(a)$ and $\varphi(1, b) = \varepsilon(b)$.

Let φ be a generalized Hopf pairing on $T(V)$ such that $\varphi(v_i, v_j) = \delta_{ij}$ (Kronecker delta). This pairing is not necessarily non-degenerate, the radical is the defining ideal $\mathfrak{J}(V)$; passing to the quotient gives a non-degenerate generalized Hopf pairing on $\mathfrak{B}(V)$ (see, for example, [2, Section 3.2] for details).

Definition 2.6 ([22, Proposition 2.4]; [3, Section 2.1]; [8, Definition 14]). The *left* and the *right differential operators* associated to the element $a \in T(V)$ are defined by

$$\begin{aligned} \partial_a^L : T(V) &\rightarrow T(V), & \partial_a^L(x) &= \sum \varphi(a, x_{(1)})x_{(2)}, \\ \partial_a^R : T(V) &\rightarrow T(V), & \partial_a^R(x) &= \sum x_{(1)}\varphi(a, x_{(2)}). \end{aligned}$$

If $a = v_i$ for some $i \in I$, they will be denoted by ∂_i^L and ∂_i^R , respectively.

These differential operators descend to endomorphisms of $\mathfrak{B}(V)$, which will also be denoted by ∂_a^L and ∂_a^R .

The following lemma, whose proof is trivial, will be useful. It also holds with $T(V)$ replaced by $\mathfrak{B}(V)$.

Lemma 2.7. (1) For any $a, x \in T(V)$,

$$\Delta(\partial_a^L(x)) = \sum \partial_a^L(x_{(1)}) \otimes x_{(2)}, \quad \Delta(\partial_a^R(x)) = \sum x_{(1)} \otimes \partial_a^R(x_{(2)}).$$

(2) For any $a, b \in T(V)$, $\partial_a^L \partial_b^R = \partial_b^R \partial_a^L$.

§3. Identities in braid groups

§3.1. Braid groups

Let $n \geq 2$ be an integer, and \mathfrak{S}_n be the symmetric group; it acts on an alphabet of n letters by permuting their positions and is generated by the set of transpositions $\{s_i = (i, i + 1) \mid 1 \leq i \leq n - 1\}$.

Let \mathfrak{B}_n be the braid group of n strands. It is defined by generators σ_i for $1 \leq i \leq n - 1$ and relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for } |i - j| \geq 2; \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for } 1 \leq i \leq n - 2.$$

Let $\sigma = s_{i_1} \cdots s_{i_r} \in \mathfrak{S}_n$ be a reduced expression of σ . We denote the corresponding lifted element by $T_\sigma = \sigma_{i_1} \cdots \sigma_{i_r} \in \mathfrak{B}_n$. This gives a linear map $T : \mathbb{K}[\mathfrak{S}_n] \rightarrow \mathbb{K}[\mathfrak{B}_n]$ called the *Matsumoto-Tits section*.

§3.2. Defining ideals

The *total symmetrization* operator in $\mathbb{K}[\mathfrak{B}_n]$ is defined by

$$S_n = \sum_{\sigma \in \mathfrak{S}_n} T_\sigma \in \mathbb{K}[\mathfrak{B}_n].$$

Since $V \in {}^H_H\mathcal{YD}$ admits a braiding σ , \mathfrak{B}_n acts naturally on $V^{\otimes n}$ via $\sigma_i \mapsto \text{id}^{\otimes(i-1)} \otimes \sigma \otimes \text{id}^{\otimes(n-i-1)}$, which allows us to view S_n as an endomorphism of $V^{\otimes n}$.

Proposition 3.1 ([24, Proposition 9]; [25, Corollary 2.4 and Theorem 2.7]; [2, Proposition 3.2.12]). *Let V be an H -Yetter–Drinfel’d module. Then*

$$\mathfrak{B}(V) = \bigoplus_{n \geq 0} (V^{\otimes n} / \ker(S_n)).$$

Details of this proposition and some different characterizations of the defining ideal can be found in [2].

By this proposition, $\mathfrak{B}(V)$ can be viewed as imposing relations in $T(V)$ which are annihilated by the total symmetrization map, so finding defining relations of $\mathfrak{B}(V)$ can be reduced to the study of the subspaces $\ker S_n$.

§3.3. Particular elements in braid groups and their relations

We start by introducing some particular elements in the group algebra of braid groups.

Definition 3.2. Let $n \geq 2$ be an integer. We define the following elements in $\mathbb{K}[\mathfrak{B}_n]$:

Garside element: $\Delta_n = (\sigma_1 \cdots \sigma_{n-1})(\sigma_1 \cdots \sigma_{n-2}) \cdots (\sigma_1 \sigma_2) \sigma_1$;

Central element: $\theta_n = \Delta_n^2$;

Right differential element: $T_n = 1 + \sigma_{n-1} + \sigma_{n-1}\sigma_{n-2} + \cdots + \sigma_{n-1}\sigma_{n-2} \cdots \sigma_1$;

Right Dynkin element:

$$P_n = (1 - \sigma_{n-1}\sigma_{n-2} \cdots \sigma_1)(1 - \sigma_{n-1}\sigma_{n-2} \cdots \sigma_2) \cdots (1 - \sigma_{n-1});$$

$$T'_n = (1 - \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1)(1 - \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_2) \cdots (1 - \sigma_{n-1}^2);$$

Left differential element: $U_n = 1 + \sigma_1 + \sigma_1\sigma_2 + \cdots + \sigma_1\sigma_2 \cdots \sigma_{n-1}$;

Left Dynkin element: $Q_n = (1 - \sigma_1\sigma_2 \cdots \sigma_{n-1})(1 - \sigma_1\sigma_2 \cdots \sigma_{n-2}) \cdots (1 - \sigma_1)$;

$$U'_n = (1 - \sigma_1^2 \sigma_2 \cdots \sigma_{n-1})(1 - \sigma_1^2 \sigma_2 \cdots \sigma_{n-2}) \cdots (1 - \sigma_1^2).$$

We give a summary of some known results on relations between them:

Proposition 3.3 ([8], [19]). *The following identities hold:*

- (1) For $n \geq 3$, $Z(\mathfrak{B}_n)$, the centre of \mathfrak{B}_n , is generated by θ_n .
- (2) For any $1 \leq i \leq n-1$, $\sigma_i \Delta_n = \Delta_n \sigma_{n-i}$.
- (3) $\theta_n = \Delta_n^2 = (\sigma_{n-1} \sigma_{n-2} \cdots \sigma_1)^n = (\sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1)^{n-1}$.
- (4) $(\sum_{k=0}^{n-2} (\sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1)^k) (1 - \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1) = 1 - \Delta_n^2 = 1 - \theta_n$.

(5) $S_n = T_2 T_3 \cdots T_n = U_2 U_3 \cdots U_n$.

(6) $T_n P_n = T'_n, U_n Q_n = U'_n$.

Proof. (1) and (2) are proved in [19, Theorem 1.24]. (3) is [8, Lemma 4 and Proposition 4]. (4) is [8, Corollary 2]. For the first identities of (5) and (6), see [8, Propositions 5 and 6]. Applying Δ_n to these identities gives $S_n = U_2 \cdots U_n$ and $U_n Q_n = U'_n$. □

We fix the notation for the embedding of braid groups at a fixed position.

Definition 3.4. For $m \geq 3$ and $2 \leq k \leq m - 1$, we let $\iota_k^m : \mathfrak{B}_k \rightarrow \mathfrak{B}_m$ denote the group homomorphism defined by $\iota_k^m(\sigma_i) = \sigma_{m-k+i}$.

§3.4. Relations to differential operators

The following lemma explains the relation between the operator T_n and the differential operators ∂_i^R defined in Section 2.3.

Lemma 3.5. *Let $x \in T^n(V)$. The following statements are equivalent:*

- (1) $T_n x = 0$.
- (2) $\partial_i^R(x) = 0$ for any $i \in I$.

Proof. This comes from the following identity, which is clear from definition: for any $x \in T^n(V)$,

$$(3.1) \quad T_n x = \sum_{i \in I} \partial_i^R(x) v_i. \quad \square$$

Remark 3.6. The same result holds for left operators: $U_n x = 0$ if and only if $\partial_i^L(x) = 0$ for any $i \in I$.

§3.5. Tensor space representation of \mathfrak{B}_n

An n -uplet $\underline{i} = (i_1, \dots, i_m) \in \mathbb{N}^m$ is called a *partition* of n , denoted by $\underline{i} \vdash n$, if $i_1 + \cdots + i_m = n$. Suppose from now on that $V \in {}^H_H \mathcal{YD}$ is of diagonal type with braiding matrix (q_{ij}) . The braid group \mathfrak{B}_n acts on $V^{\otimes n}$, making it a $\mathbb{K}[\mathfrak{B}_n]$ -module. Since the braiding is of diagonal type, the $\mathbb{K}[\mathfrak{B}_n]$ -module $V^{\otimes n}$ admits a decomposition into submodules:

$$(3.2) \quad V^{\otimes n} = \bigoplus_{\underline{i} \in \mathcal{P}_n} \mathbb{K}[\mathfrak{B}_n].v_1^{i_1} \cdots v_m^{i_m},$$

where $\mathcal{P}_n = \{\underline{i} = (i_1, \dots, i_m) \in \mathbb{N}^m \mid \underline{i} \vdash n\}$.

To simplify notation, for $\underline{i} = (i_1, \dots, i_m)$, we denote $v_{\underline{i}} := v_1^{i_1} \cdots v_m^{i_m}$ and the $\mathbb{K}[\mathfrak{B}_n]$ -module $\mathbb{K}[X_{\underline{i}}] := \mathbb{K}[\mathfrak{B}_n].v_{\underline{i}}$.

We let \mathcal{H} denote the set $(V^{\otimes n})^{\theta_n}$ of invariants under the action of the central element θ_n . As $\theta_n \in Z(\mathfrak{B}_n)$, $\theta_n v_{\underline{i}} = v_{\underline{i}}$ implies that $\mathbb{K}[X_{\underline{i}}] \subset \mathcal{H}$. Moreover, there exists some subset $J \subset \mathcal{P}_n$ such that

$$\mathcal{H} = \bigoplus_{\underline{i} \in J} \mathbb{K}[X_{\underline{i}}]$$

(see the argument in [8, Section 6.1]).

We finish this subsection with the following remark, which will frequently appear in the following discussions.

Remark 3.7. Suppose that $\underline{i} \vdash n$, $x \in \mathbb{K}[X_{\underline{i}}]$ and $v \in V$. Then $(\text{id} - \sigma_n \cdots \sigma_1)(vx)$ is in the ideal generated by x .

To show this, notice that the coefficient λ such that $\sigma_n \cdots \sigma_1(vx) = \lambda xv$ only depends on the chosen partition \underline{i} , it is therefore a constant for any $x \in \mathbb{K}[X_{\underline{i}}]$.

§3.6. Defining ideals of degree 2

Elements of degree two in the defining ideal can be easily computed. They are characterized by the following proposition:

Proposition 3.8. *For $i \neq j$, the following statements are equivalent:*

- (1) $q_{ij}q_{ji} = 1$.
- (2) $v_i v_j - q_{ij} v_j v_i \in \ker S_2$.
- (3) $\theta_2(v_i v_j) = v_i v_j$.

Proof. It suffices to prove that (2) is equivalent to (3). Notice that $v_i v_j - q_{ij} v_j v_i = P_2(v_i v_j)$ and $S_2 = T_2$. Then $T_2 P_2(v_i v_j) = 0$ if and only if $T'_2(v_i v_j) = 0$ if and only if $\theta_2(v_i v_j) = v_i v_j$. □

§4. Another characterization of $\mathfrak{J}(V)$

In this section, we give a characterization of a generating set of the defining ideal $\mathfrak{J}(V)$ using the kernels of the operators T_n , which is motivated by the work of Frønsdal and Galindo [9]. In fact, we could use the left or right differential operators to give a full characterization of the generators of the defining ideal.

The following definition is due to Frønsdal and Galindo [9]:

Definition 4.1. An element $x \in T^n(V)$ is called a *right* (resp. *left*) *constant* of degree n if $T_n x = 0$ (resp. $U_n x = 0$). We let Con_n^R (resp. Con_n^L) denote the vector space generated by all right (resp. left) constants of degree n and for any $m \geq 2$, we define

$$\begin{aligned} \text{Con}_{\leq m}^R &= \text{span}\left(\bigcup_{2 \leq n \leq m} \text{Con}_n^R\right), & \text{Con}^R &= \text{span}\left(\bigcup_{n \geq 2} \text{Con}_n^R\right), \\ \text{Con}_{\leq m}^L &= \text{span}\left(\bigcup_{2 \leq n \leq m} \text{Con}_n^L\right), & \text{Con}^L &= \text{span}\left(\bigcup_{n \geq 2} \text{Con}_n^L\right), \end{aligned}$$

where $\text{span}(X)$ stands for the \mathbb{K} -vector space generated by the set X .

The main technical tool is the following non-commutative version of the Taylor lemma for the diagonal braiding.

Lemma 4.2 (Taylor Lemma, [9]). (1) (Left version) *For any integer $l \geq 1$ and $\underline{i} = (i_1, \dots, i_l) \in \{1, \dots, l\}^l$, there exists*

$$A^{\underline{i}} = \sum_{\sigma \in \mathfrak{S}_l} A^\sigma v_{i_{\sigma(1)}} \cdots v_{i_{\sigma(l)}} \in T^l(V)$$

with $A^\sigma \in \mathbb{K}$ such that for any $x \in T^m(V)$,

$$x = c(x) + \sum_{l \geq 1} \sum_{\underline{i} \in \{1, \dots, l\}^l} A^{\underline{i}} \partial_{i_1}^L \cdots \partial_{i_l}^L(x),$$

where $c(x) \in T^m(V)$ satisfies $\partial_i^L(c(x)) = 0$ for any $i \in I$.

(2) (Right version) *For any integer $l \geq 1$ and $\underline{i} = (i_1, \dots, i_l) \in \{1, \dots, l\}^l$, there exists*

$$B^{\underline{i}} = \sum_{\sigma \in \mathfrak{S}_l} B^\sigma v_{i_{\sigma(1)}} \cdots v_{i_{\sigma(l)}} \in T^l(V)$$

with $B^\sigma \in \mathbb{K}$ such that for any $x \in T^m(V)$,

$$x = d(x) + \sum_{l \geq 1} \sum_{\underline{i} \in \{1, \dots, l\}^l} \partial_{i_1}^R \cdots \partial_{i_l}^R(x) B^{\underline{i}},$$

where $d(x) \in T^m(V)$ satisfies $\partial_i^R(d(x)) = 0$ for any $i \in I$.

Lemma 4.3. *For any $m \geq 2$, $\text{Con}_{\leq m}^L$ and $\text{Con}_{\leq m}^R$ are coideals in the coalgebra $T^{\leq m}(V)$.*

Proof. We prove the statement for $\text{Con}_{\leq m}^R$: if $x \in \ker T_n$ for some $n \leq m$, then for any $i \in I$, $\partial_i^R(x) = 0$, which implies that (see Lemma 2.7)

$$0 = \Delta(\partial_i^R(x)) = \sum x_{(1)} \otimes \partial_i^R(x_{(2)})$$

and therefore $\partial_i^R(x_{(2)}) = 0$ for any $i \in I$. This shows $\Delta(x) - x \otimes 1 \in T^{\leq m}(V) \otimes \text{Con}_{\leq m}^R$ and

$$\Delta(x) \in \text{Con}_{\leq m}^R \otimes T^{\leq m}(V) + T^{\leq m}(V) \otimes \text{Con}_{\leq m}^R. \quad \square$$

For a ring R and a subset $X \subset R$, $\langle X \rangle_{\text{ideal}}$ denotes the ideal in R generated by X .

Theorem 4.4. *For any $m \geq 2$, let*

$$R^m = \langle \text{Con}_{\leq m}^R \rangle_{\text{ideal}} \cap T^m(V).$$

Then $R^m = \ker S_m$.

Proof. Since $S_r = T_2 T_3 \cdots T_r$ and $\ker(T_r : T^r(V) \rightarrow T^r(V)) \subset \ker S_r \subset \mathfrak{J}(V)$, the inclusion $R^m \subset \ker S_m$ comes from the fact that $\mathfrak{J}(V) = \bigoplus_{m \geq 2} \ker S_m$ is an ideal. It suffices to show the other inclusion. Take $x \in \ker S_m$; we prove that $x \in R^m$ by induction on m . The case $m = 2$ is clear as $T_2 = S_2$.

Suppose that for any $2 \leq k \leq m - 1$, $R^k = \ker S_k$. It suffices to show the following statement: if $\partial_i^R(x) \in \ker S_{m-1}$ for all $i \in I$, then $x \in R^m$. Indeed, for $x \in \ker S_m$, there are two cases:

1. $T_m x = 0$. In this case, $x \in R^m$ is clear by definition.
2. $T_m x \neq 0$. From the decomposition of S_n , $T_m x \in \ker S_{m-1}$. According to (3.1), this implies that $\partial_i^R(x) \in \ker S_{m-1}$ for any $i \in I$. The proof will be terminated if the above statement is proved.

We proceed to show the above statement. The following lemma is needed.

Lemma 4.5. *For any $k \geq 3$, if $x \in R^k$, then $\partial_i^R(x) \in R^{k-1}$ for any $i \in I$.*

We continue the proof of the theorem. Let $x \in T^m(V)$ be such that for any $i \in I$, $\partial_i^R(x) \in \ker S_{m-1} = R^{m-1}$. From the right version of the Taylor lemma,

$$x = d(x) + \sum_{l \geq 1} \sum_{\underline{i} \in \{1, \dots, l\}^l} \partial_{i_1}^R \cdots \partial_{i_l}^R(x) B^{\underline{i}}.$$

The first term $d(x)$ on the right hand side satisfies $T_m(d(x)) = 0$ so is in R^m . Moreover, the hypothesis on $\partial_i^R(x)$ and the lemma above force $\partial_{i_1}^R \cdots \partial_{i_n}^R(x)$ to be in R^{m-n} , so the second term is in R^m . □

Now it remains to prove the lemma.

Proof of Lemma 4.5. It suffices to deal with the case where $x = urw \in R^k$ is such that $r \in \ker T_s \cap \mathbb{K}[X_{\underline{i}}]$ for some $\underline{i} \vdash s$, $u \in T^p(V)$ and $w \in T^q(V)$ satisfying $k = s + p + q$.

We have the decomposition $T_k = T_k^1 + T_k^2 + T_k^3$ where

$$\begin{aligned} T_k^1 &= 1 + \sigma_{k-1} + \sigma_{k-1}\sigma_{k-2} + \cdots + \sigma_{k-1} \cdots \sigma_{p+s+1}, \\ T_k^2 &= \sigma_{k-1} \cdots \sigma_{p+s}(\iota_s^{p+s}(T_s)), \\ T_k^3 &= \sigma_{k-1} \cdots \sigma_{p-1} + \cdots + \sigma_{k-1} \cdots \sigma_1. \end{aligned}$$

It is clear that $T_k^2 x = 0$. By Remark 3.7, both $T_k^1 x$ and $T_k^3 x$ are contained in R^k since they are in the ideal generated by r . Moreover, it should be remarked that in $T_k x$, r is always contained in the first $k - 1$ tensorands, by the definitions of T_k^1 and T_k^3 .

Finally, we have shown that $T_k x \in R^k$, so $\partial_i^R(x) \in R^{k-1}$ for any $i \in I$, by the formula (3.1). □

In the proof of the above theorem, we have shown as a byproduct the following proposition, which can be viewed as a kind of “invariance under integration”.

Proposition 4.6. *For $x \in T^m(V)$ where $m \geq 3$, the following statements are equivalent:*

- (1) $x \in R^m$.
- (2) $\partial_i^R(x) \in R^{m-1}$ for any $i \in I$.

The above results are correct with “right” replaced by “left”. The proof above can be adapted by using the left version of the Taylor lemma. We omit these statements, but give the following corollary.

Corollary 4.7. *Let*

$$L^m = \langle \text{Con}_{\leq m}^L \rangle_{\text{ideal}} \cap T^m(V).$$

Then $R^m = L^m = \ker S_m$.

In conclusion, to find the generating relations, it suffices to consider those in the intersection of $\ker \partial_i^R$ for all $i \in I$, or the intersection of $\ker \partial_i^L$ for all $i \in I$. In the next section we will establish a refined result giving more constraints.

Remark 4.8. Globally, when passing to the generating ideal, there is no difference between the left and right cases. But an element annihilated by all right differentials is not necessarily contained in the kernel of all ∂_i^L . We will return to this problem in Section 5.2.

The following lemma is a direct consequence of Lemma 2.7.

Lemma 4.9. *For any $i \in I$ and $m \geq 3$, ∂_i^R (resp. ∂_i^L) sends $\text{Con}_{\leq m}^L$ (resp. $\text{Con}_{\leq m}^R$) to $\text{Con}_{\leq m-1}^L$ (resp. $\text{Con}_{\leq m-1}^R$).*

§5. Defining relations in the diagonal type

§5.1. More constraints: pre-relations

In this subsection, we propose a smaller set of generators in $\mathfrak{J}(V)$ by posing more constraints on the left and right constants. These constraints give a restriction

on the degrees where this new set of generators may appear. We start by some motivations for the main definition.

Let H be a Hopf algebra and $X \subset H$ be a subset. The Hopf ideal generated by X is the smallest Hopf ideal containing X .

Proposition 5.1 ([8]). *The Hopf ideal in ${}^H_H\mathcal{YD}$ generated by $\bigoplus_{n \geq 2}(\ker(S_n) \cap \text{im}(P_n))$ is $\mathfrak{I}(V)$.*

This proposition, combined with Theorem 4.4, gives more constraints.

Corollary 5.2. *The Hopf ideal in ${}^H_H\mathcal{YD}$ generated by $\bigoplus_{n \geq 2}(\ker(T_n) \cap \text{im}(P_n))$ is $\mathfrak{I}(V)$.*

Thanks to this corollary, to find relations imposed in $\mathfrak{B}(V)$, it suffices to concentrate on elements in $\text{im}(P_n)$ annihilated by all right differentials ∂_i^R .

According to Proposition 3.3(6), to find a generating set of $\mathfrak{I}(V)$, it suffices to consider the solution of the equation $T'_n x = T_n P_n x = 0$ in $T^n(V)$. This observation motivates the following definition:

Definition 5.3. Let $n \geq 2$ be an integer. We call a non-zero element $v \in T^n(V)$ a *right pre-relation* of degree n if:

1. $T_n v = 0$ and $v = P_n w$ for some $w \in T^n(V)$.
2. $\iota_{n-1}^n(T'_{n-1})w \neq 0$.

Let Rel_r^n denote the vector space generated by all right pre-relations of degree n and Rel_r denote the vector space generated by $\bigcup_{n \geq 2} \text{Rel}_r^n$. Elements in Rel_r are called *right pre-relations*.

We can similarly define left pre-relations of degree n by replacing T_n by U_n , T'_{n-1} by U'_{n-1} and P_n by Q_n in the above definition. Let Rel_l^n denote the \mathbb{K} -vector space generated by all left pre-relations of degree n , and let Rel_l be the \mathbb{K} -vector space generated by $\bigcup_{n \geq 2} \text{Rel}_l^n$. Elements in Rel_l are called *left pre-relations*.

Remark 5.4. They are called “pre-relations” as they may be redundant.

We establish some properties of pre-relations. Recall the definition of T'_n :

$$T'_n = (1 - \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1)(1 - \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_2) \cdots (1 - \sigma_{n-1}^2 \sigma_{n-2})(1 - \sigma_{n-1}^2).$$

We define the following elements in $\mathbb{K}[\mathfrak{B}_n]$ for $1 \leq m \leq n-1$:

$$X_{m,n} = (1 - \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_{n-m}) \cdots (1 - \sigma_{n-1}^2 \sigma_{n-2})(1 - \sigma_{n-1}^2) = \iota_{m+1}^n(T'_{m+1}).$$

Then $X_{1,n} = (1 - \sigma_{n-1}^2)$, $X_{n-1,n} = T'_n$ and $X_{n-2,n} = \iota_{n-1}^n(T'_{n-1})$.

Proposition 5.5. *If $T'_n w = 0$ and $X_{n-2,n} w \neq 0$ for some $w \in T^n(V)$, then $\theta_n w = w$.*

Proof. From the definition, $T'_n w = (1 - \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1) X_{n-2,n} w$. If $X_{n-2,n} w \neq 0$, it will be a solution of the equation $(1 - \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1) x = 0$. By Proposition 3.3, multiplying both sides by $\sum_{k=0}^{n-2} (\sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1)^k$ gives $\theta_n X_{n-2,n} w = X_{n-2,n} w$. This implies $\theta_n w = w$ by the argument at the beginning of Section 3.5. \square

Corollary 5.6. *Let $v \in T^n(V)$ be a right pre-relation. Then $\theta_n v = v$.*

Proof. By Proposition 5.5, if $v = P_n w$ for some $w \in T^n(V)$, then $\theta_n w = w$. Therefore

$$\theta_n v = \theta_n P_n w = P_n \theta_n w = P_n w = v. \quad \square$$

In conclusion, to solve the equation $T_n x = 0$ with the aim of finding defining relations, it suffices to work inside the $\mathbb{K}[\mathfrak{B}_n]$ -module $\mathbb{K}[X_i]$ such that $\theta_n(v_i) = v_i$.

Corollary 5.7. *If $w \in T^n(V)$ is such that $T'_n w = 0$ and for any $2 \leq k \leq n - 1$, $\iota_k^n(\theta_k) w \neq w$, then $\theta_n w = w$.*

Proof. By Proposition 5.5, it suffices to show that $X_{n-2,n} w \neq 0$. Otherwise, take the smallest k such that $X_{k-1,n} w \neq 0$ but $X_{k,n} w = 0$. As $X_{k,n} = \iota_{k+1}^n(T'_{k+1})$, Proposition 5.5 can be applied to this case to give $\iota_{k+1}^n(\theta_{k+1}) w = w$. This contradicts the hypothesis. \square

The main result of this section is the following theorem.

Theorem 5.8. *The Hopf ideal generated by Rel_r is $\mathfrak{I}(V)$.*

Proof. Let $w \in T^n(V)$ be a solution of the equation $T'_n x = X_{n-1,n} x = 0$. There are two possibilities:

1. $X_{n-2,n} w \neq 0$. It is clear that $P_n w$ is a right pre-relation.
2. $X_{n-2,n} w = 0$. Then there exists a smallest k such that $X_{k-1,n} v \neq 0$ but $X_{k,n} v = 0$.

We would like to show that only relations falling into the first case are interesting. To be more precise, if w falls into the second case, then $P_n w$ can be generated by lower degree elements in the first case. This is stated in the following lemma.

Lemma 5.9. *If $w \in T^n(V)$ is an element such that $T'_n w = 0$ and $X_{n-2,n} w = 0$, then $P_n w$ is in the ideal generated by right pre-relations of lower degrees.*

Proof. The proof is by induction on n . There is nothing to prove for $n = 2$.

Let $w \in T^n(V)$ be such that $T'_n w = 0$ and $X_{n-2,n} w = 0$. Let k be the smallest integer such that $X_{k-1,n} w \neq 0$ but $X_{k,n} w = 0$. By definition of P_n ,

$$P_n w = (1 - \sigma_{n-1} \cdots \sigma_1) \cdots (1 - \sigma_{n-1} \cdots \sigma_{k+1}) \iota_{k+1}^n (P_{k+1}) w.$$

We write

$$w = \sum_i \sum_{\underline{j} \vdash k+1} u_i \otimes w_{i,\underline{j}},$$

where $u_i \in T^{n-k-1}(V)$ are linearly independent and $w_{i,\underline{j}} \in T^{k+1}(V) \cap \mathbb{K}[X_{\underline{j}}]$. Recall that $X_{k,n} = \iota_{k+1}^n(T'_{k+1})$; then $X_{k,n} w = 0$ implies that

$$\sum_{\underline{j} \vdash k+1} T'_{k+1} w_{i,\underline{j}} = 0.$$

As these $\mathbb{K}[X_{\underline{j}}]$ have trivial intersection, $X_{k,n} w_{i,\underline{j}} = 0$ for any \underline{j} .

There are two cases:

1. $T'_k w_{i,\underline{j}} = 0$. In this case, by applying the induction hypothesis to $w_{i,\underline{j}}$, we see that $P_k w_{i,\underline{j}}$ is generated by right pre-relations of lower degrees. So $P_n(u_i w_{i,\underline{j}})$ is generated by right pre-relations of lower degrees by Remark 3.7.
2. $T'_k w_{i,\underline{j}} \neq 0$. Then $P_k w_{i,\underline{j}}$ is a right pre-relation of degree k and $P_n(u_i w_{i,\underline{j}})$ is generated by right pre-relations of lower degrees by Remark 3.7.

In summary, for any i and \underline{j} , $P_n(u_i w_{i,\underline{j}})$ is generated by right pre-relations of lower degree, hence so is $P_n w$. □

By Corollary 5.2, to terminate the proof of the theorem, it suffices to show that the Hopf ideal generated by $\bigoplus_{n \geq 2} (\ker(T_n) \cap \text{im}(P_n))$ coincides with the Hopf ideal generated by Rel_r .

Take $x \in \ker(T_n) \cap \text{im}(P_n)$. There exists w such that $P_n w = x$ and $T'_n w = P_n T_n x = 0$. By the above argument, if $X_{n-2,n} w \neq 0$, then by definition $x \in \text{Rel}_r$; if not, by Lemma 5.9, $x = P_n w$ is contained in the Hopf ideal generated by Rel_r . □

Example 5.10. We compute pre-relations of degree 2. Since we have $P_2 = Q_2$ and $T_2 = U_2$, Rel_r^2 coincides with Rel_l^2 . It suffices to consider each $\mathbb{K}[X_{\underline{i}}]$ where $\underline{i} = (s, t)$. The following facts are clear by Proposition 3.8:

1. $T_2 P_2 = 1 - \theta_2$ acts as zero on Rel_r^2 , so it suffices to consider the fixed points of θ_2 .
2. $\theta_2 v_{\underline{i}} = v_{\underline{i}}$ if and only if $q_{st} q_{ts} = 1$.

These observations give the following characterization of Rel_r^2 :

$$\text{Rel}_r^2 = \text{span}\{v_s v_t - q_{st} v_t v_s \mid s < t \text{ such that } q_{ts} q_{st} = 1 \text{ and } s = t, q_{ss} = -1\}.$$

There are no redundant relations in this list and it coincides with the set of constants of degree 2.

§5.2. Balancing left and right

The sets of left and of right constants or pre-relations may not coincide; in this subsection we study symmetries between them.

The following lemma is clear by Proposition 3.3.

Lemma 5.11. *For any $n \geq 2$, $\Delta_n T_n = U_n \Delta_n$ and $\Delta_n P_n = Q_n \Delta_n$.*

The Garside element gives a symmetry between the left and right pre-relations.

Corollary 5.12. *The Garside element Δ_n induces a linear isomorphism*

$$\text{Rel}_r^n \cong \text{Rel}_l^n.$$

Proof. According to Proposition 5.5, $\Delta_n^2 = \theta_n$ acts as the identity on Rel_r^n , thus Δ_n is a linear isomorphism. It suffices to show that the image of Δ_n is contained in Rel_l^n .

We verify that $\Delta_n w \in \text{Rel}_l^n$ for $w \in \text{Rel}_r^n$. The first condition of Definition 5.3 holds by the above lemma and the other point comes from the injectivity of Δ_n . If we write $w = P_n v$, then by the above lemma again, $\Delta_n w = \Delta_n P_n v = Q_n \Delta_n v$ implies $\Delta_n w$ is in the image of Q_n . □

A similar result holds when pre-relations are replaced by constants.

Corollary 5.13. *The Garside element Δ_n induces a linear isomorphism*

$$\text{Con}_r^n \cong \text{Con}_l^n.$$

Proof. It is clear that Δ_n sends Con_r^n to Con_l^n , so it suffices to show that Δ_n is an isomorphism.

Thanks to the decomposition (3.2) and notation therein, we can decompose Con_r^n and Con_l^n into direct sums of $\mathbb{K}[\mathfrak{B}_n]$ -modules $\mathbb{K}[X_{\underline{i}}]$ for $\underline{i} \vdash n$ such that the action of θ_n on each summand is given by an invertible scalar. So Δ_n induces a linear isomorphism

$$\mathbb{K}[X_{\underline{i}}] \cap \text{Con}_r^n \cong \mathbb{K}[X_{\underline{i}}] \cap \text{Con}_l^n,$$

and therefore a linear isomorphism between Con_r^n and Con_l^n . □

§6. Generalized quantum groups

§6.1. Generalized quantum groups

For a Nichols algebra $\mathfrak{B}(V)$ associated to a Yetter–Drinfel’d module $V \in {}^H_H\mathcal{YD}$, the bosonization $\mathfrak{B}(V) \# H$ is a true Hopf algebra [4]. This construction, once applied to the Nichols algebra of diagonal type associated to the data of a symmetrizable Kac–Moody Lie algebra, gives the positive or negative part of the corresponding quantum group. But here, we would like to define them in a more direct way.

Let $\mathbb{K}(q)$, the field of rational functions in one variable over \mathbb{K} , be the base field in this subsection.

Definition 6.1. Let $A = (q_{ij})_{1 \leq i, j \leq N}$ be a braiding matrix in $M_N(\mathbb{K}(q))$ such that $q_{ij} = q^{n_{ij}}$ for some $n_{ij} \in \mathbb{Z}$.

1. $T^{\leq 0}(A)$ is defined as the Hopf algebra generated by $F_i, K_i^{\pm 1}$ for $i \in I$ with relations

$$K_i F_j K_i^{-1} = q_{ij}^{-1} F_j, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1,$$

$$\Delta(F_i) = K_i \otimes F_i + F_i \otimes 1, \quad \Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1}, \quad \varepsilon(F_i) = 0, \quad \varepsilon(K_i) = 1.$$

2. $T^{\geq 0}(A)$ is defined as the Hopf algebra generated by $E_i, K_i^{\pm 1}$ for $i \in I$ with relations

$$K'_i E_j K_i'^{-1} = q_{ij} E_j, \quad K'_i K_i'^{-1} = K_i'^{-1} K'_i = 1,$$

$$\Delta(E_i) = 1 \otimes E_i + E_i \otimes K_i'^{-1}, \quad \Delta(K_i'^{\pm 1}) = K_i'^{\pm 1} \otimes K_i'^{\pm 1}, \quad \varepsilon(E_i) = 0, \quad \varepsilon(K'_i) = 1.$$

3. $D^{\leq 0}(A)$ (resp. $D^{\geq 0}(A)$) is defined as the quotient of $T^{\leq 0}(A)$ (resp. $T^{\geq 0}(A)$) by the biideal generated by the right (resp. left) pre-relations.

We define a generalized Hopf pairing $\varphi : T^{\geq 0}(A) \times T^{\leq 0}(A) \rightarrow \mathbb{K}(q)$ such that for any $i, j \in I$,

$$\varphi(E_i, F_j) = -\frac{\delta_{ij}}{q - q^{-1}}, \quad \varphi(K'_i, K_j) = q_{ij}, \quad \varphi(E_i, K_j^{\pm 1}) = \varphi(K_i'^{\pm 1}, F_j) = 0.$$

By Theorem 5.8, the pre-relations generate the defining ideal. It is shown in [2, Theorem 3.2.29] that radicals of the generalized Hopf pairing coincide with the defining ideal in $T^{\geq 0}(A)$ and $T^{\leq 0}(A)$, hence φ induces a non-degenerate generalized Hopf pairing $\varphi : D^{\geq 0}(A) \times D^{\leq 0}(A) \rightarrow \mathbb{K}(q)$.

The following quantum double construction allows us to define the generalized quantum group.

Definition 6.2 ([18, Theorem 3.2]). Let A, B be two Hopf algebras with invertible antipodes and φ be a generalized Hopf pairing between them. Their *quantum double* $D_\varphi(A, B)$ is defined by:

1. As a vector space, it is $A \otimes B$.
2. As a coalgebra, it is the tensor product of the coalgebras A and B .
3. As an algebra, it has multiplication given by, for $a, a' \in A$ and $b, b' \in B$,

$$(a \otimes b)(a' \otimes b') = \sum \varphi(S^{-1}(a'_{(1)}), b_{(1)})\varphi(a'_{(3)}, b_{(3)})aa'_{(2)} \otimes b_{(2)}b'.$$

Definition 6.3. Suppose moreover that A is a symmetric matrix. The *generalized quantum group* $D_q(A)$ associated to the braiding matrix A is defined by

$$D_q(A) = D_\varphi(D^{\geq 0}(A), D^{\leq 0}(A))/(K_i - K'_i \mid i \in I),$$

where $(K_i - K'_i \mid i \in I)$ is the Hopf ideal generated by $K_i - K'_i$ for $i \in I$.

We can similarly define the Hopf algebra $T_q(A)$ by replacing $D^{\geq 0}(A)$ and $D^{\leq 0}(A)$ by $T^{\geq 0}(A)$ and $T^{\leq 0}(A)$. Then $D_q(A)$ is the quotient of $T_q(A)$ by the Hopf ideal generated by the defining ideals.

A routine computation gives the commutation relation between E_i and F_j :

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}.$$

Remark 6.4. We use the notation $D_q(A)$ instead of $U_q(A)$ as it may not be related to the universal enveloping algebra associated to a Kac–Moody Lie algebra. This phenomenon will be examined in Example 7.3.

§6.2. Averaged quantum group

We will be interested in a particular case of the above construction where the braiding matrix arises from a generalized Cartan matrix.

Let $C = (c_{ij})_{1 \leq i, j \leq N}$ be a *generalized Cartan matrix* in $M_N(\mathbb{Z})$, i.e., a matrix of integral entries satisfying

1. $c_{ii} = 2$.
2. $c_{ij} \leq 0$ for any $i \neq j$.
3. $c_{ij} = 0$ implies $c_{ji} = 0$.

In the following discussion, we take $K = \mathbb{K}(q^{1/2})$ as the ground field since elements in our matrices may be in the additive group $\frac{1}{2}\mathbb{Z}$.

Definition 6.5. Let $C \in M_N(\mathbb{Z})$ be a generalized Cartan matrix.

1. The *averaged matrix* associated to C is defined by $\bar{C} = (\bar{c}_{ij})_{1 \leq i, j \leq N} \in M_N(\mathbb{Q})$ where $\bar{c}_{ij} = \frac{1}{2}(c_{ij} + c_{ji})$. We denote $A = (q^{\bar{c}_{ij}})_{1 \leq i, j \leq N}$; it is a symmetric matrix.
2. The *q -enveloping algebra* $U_q(C)$ associated to C is defined by:
 - (a) If C is symmetrizable, we keep the original definition of the quantized enveloping algebra associated to the Kac–Moody Lie algebra $\mathfrak{g}(C)$ as a $\mathbb{K}(q^{1/2})$ -algebra.
 - (b) If C is non-symmetrizable, it is $D_q(A)$ as a $\mathbb{K}(q^{1/2})$ -algebra.

We let $D^{>0}(A)$ (resp. $D^{<0}(A)$) denote the subalgebra of $D^{\geq 0}(A)$ (resp. $D^{\leq 0}(A)$) generated by E_i (resp. F_i) for $i \in I$. They are Nichols algebras associated to the braiding matrix $A = (q^{\bar{c}_{ij}})$ (resp. $A' = (q^{-\bar{c}_{ij}})$). The following result due to Andruskiewitsch and Schneider shows that passing to the averaged matrix will not lose too much information.

Proposition 6.6 ([4]). *Let V and V' be two Yetter–Drinfel’d modules of diagonal type with braiding matrices $(q_{ij})_{1 \leq i, j \leq N}$ and $(q'_{ij})_{1 \leq i, j \leq N}$ satisfying $q_{ij}q_{ji} = q'_{ij}q'_{ji}$ for any $i, j \in I$ with respect to bases v_1, \dots, v_N of V and v'_1, \dots, v'_N of V' . Then:*

- (1) *There exists a linear isomorphism $\psi : \mathfrak{B}(V) \rightarrow \mathfrak{B}(V')$ such that for any $i \in I$, $\psi(v_i) = v'_i$.*
- (2) *This linear map ψ almost preserves the algebra structure: for any $i, j \in I$,*

$$\psi(v_i v_j) = \begin{cases} q'_{ij} q_{ij}^{-1} v'_i v'_j & \text{if } i \leq j, \\ v'_i v'_j & \text{if } i > j. \end{cases}$$

Remark 6.7. Let C be a non-symmetrizable generalized Cartan matrix. By Remark 1 and Theorem 21 in [24], in order that $D^{<0}(A)$ be of finite Gel’fand–Kirillov dimension, the matrix \bar{C} must be in $M_N(\mathbb{Z})$. This implies that a large number of algebras we are considering are of infinite Gel’fand–Kirillov dimension.

§6.3. Bar involution in the symmetric case

In this subsection, we suppose moreover that the braiding matrix is symmetric: $q_{ij} = q_{ji}$ for any $i, j \in I$. This is always the case when the q -enveloping algebra $U_q(C)$ is under consideration.

This hypothesis allows us to define the bar involution on Nichols algebras, which is fundamental in the study of quantum groups, especially for canonical (global crystal) bases.

Definition 6.8. The *bar involution* $- : T(V) \rightarrow T(V)$ is the \mathbb{K} -linear automorphism defined by $q^{1/2} \mapsto q^{-1/2}$ and $v_i \mapsto v_i$.

Definition 6.9. For any $i \in I$, we define $d_i^R, d_i^L \in \text{End}_K(T(V))$ to be the $\mathbb{K}(q^{1/2})$ -linear maps such that for any monomial $v_{i_1} \cdots v_{i_l}$ in $T(V)$,

$$d_i^R(v_{i_1} \cdots v_{i_l}) = \overline{\partial_i^R(v_{i_1} \cdots v_{i_l})}, \quad d_i^L(v_{i_1} \cdots v_{i_l}) = \overline{\partial_i^L(v_{i_1} \cdots v_{i_l})}.$$

We start by showing that bar involution descends to Nichols algebras. The following lemma will be needed.

Lemma 6.10. For any $i, j \in I$, $\partial_j^R d_i^R = q_{ij}^{-1} d_i^R \partial_j^R$.

Proof. It is easy to show that for any $i_1, \dots, i_n \in I$,

$$\begin{aligned} \partial_j^R(v_{i_1} \cdots v_{i_n}) &= q_{j,i_n} \partial_j^R(v_{i_1} \cdots v_{i_{n-1}})v_{i_n} + v_{i_1} \cdots v_{i_{n-1}} \partial_j^R(v_{i_n}), \\ d_i^R(v_{i_1} \cdots v_{i_n}) &= q_{i,i_n}^{-1} d_i^R(v_{i_1} \cdots v_{i_{n-1}})v_{i_n} + v_{i_1} \cdots v_{i_{n-1}} d_i^R(v_{i_n}). \end{aligned}$$

Since both ∂_j^R and d_i^R are $\mathbb{K}(q)$ -linear, it suffices to verify the lemma for monomials. We use induction on the degree n of the monomial. The case $n = 1$ is trivial. Taking a monomial $v_{i_1} \cdots v_{i_n}$ and applying the formulas above gives

$$\begin{aligned} \partial_j^R d_i^R(v_{i_1} \cdots v_{i_n}) &= q_{j,i_n} q_{i,i_n}^{-1} \partial_j^R d_i^R(v_{i_1} \cdots v_{i_{n-1}})v_{i_n} + q_{i,i_n}^{-1} d_i^R(v_{i_1} \cdots v_{i_{n-1}}) \partial_j^R(v_{i_n}) \\ &\quad + \partial_j^R(v_{i_1} \cdots v_{i_{n-1}}) d_i^R(v_{i_n}), \\ d_i^R \partial_j^R(v_{i_1} \cdots v_{i_n}) &= q_{j,i_n} q_{i,i_n}^{-1} d_i^R \partial_j^R(v_{i_1} \cdots v_{i_{n-1}})v_{i_n} + q_{j,i_n} \partial_j^R(v_{i_1} \cdots v_{i_{n-1}}) d_i^R(v_{i_n}) \\ &\quad + d_i^R(v_{i_1} \cdots v_{i_{n-1}}) \partial_j^R(v_{i_n}). \end{aligned}$$

Then induction hypothesis can be applied to give

$$\begin{aligned} &(\partial_j^R d_i^R - q_{ij}^{-1} d_i^R \partial_j^R)(v_{i_1} \cdots v_{i_n}) \\ &= (1 - q_{ij}^{-1} q_{j,i_n}) \partial_j^R(v_{i_1} \cdots v_{i_{n-1}}) d_i^R(v_{i_n}) + (q_{i,i_n}^{-1} - q_{ij}^{-1}) d_i^R(v_{i_1} \cdots v_{i_{n-1}}) \partial_j^R(v_{i_n}). \end{aligned}$$

Notice that if $i \neq i_n$ and $j \neq i_n$, the right hand side is zero. It remains to handle the following three cases:

1. If $i = j = i_n$, then the two coefficients on the right hand side are zero.
2. If $i \neq j, i = i_n$, then the coefficient of the second term on the right hand side vanishes and the first term is zero.
3. If $i \neq j, j = i_n$ then the coefficient of the first term on the right hand side vanishes and the second term is zero.

This terminates the proof. □

Proposition 6.11. The restriction of the bar involution induces a linear automorphism of $\mathfrak{J}(V)$.

Proof. We let $\mathfrak{J}(V)_n$ denote the set of degree n elements in $\mathfrak{J}(V)$. Since $\bar{}$ is an involution, it suffices to show $\overline{\mathfrak{J}(V)} \subset \mathfrak{J}(V)$.

Lemma 6.12. *For any $i \in I$, $d_i^R(\mathfrak{J}(V)) \subset \mathfrak{J}(V)$.*

Proof. The proof is by induction on the degree of elements in $\mathfrak{J}(V)$.

The case $n = 2$ is clear since by Proposition 3.8, $\mathfrak{J}(V)_2$ is generated as a vector space by $v_i v_j - q_{ij} v_j v_i$ for some q_{ij} satisfying $q_{ij} q_{ji} = 1$; the hypothesis on the braiding matrix forces $q_{ij} = \pm 1$, hence

$$d_k^R(v_i v_j - q_{ij} v_j v_i) = \overline{\partial_k^R(v_i v_j - q_{ij} v_j v_i)}$$

is zero since ∂_k^R annihilates $\mathfrak{J}(V)_2$.

Take $v \in \mathfrak{J}(V)_n$, by Proposition 4.6, it suffices to show that for any $j \in I$, $\partial_j^R d_i^R(v) \in \mathfrak{J}(V)$. Applying Lemma 6.10 gives

$$\partial_j^R d_i^R(v) = q_{ij}^{-1} d_i^R \partial_j^R(v);$$

by Proposition 4.6 again, $\partial_j^R(v) \in \mathfrak{J}(V)$ with a lower degree, hence $d_i^R \partial_j^R(v) \in \mathfrak{J}(V)$ by induction hypothesis and therefore $\partial_j^R d_i^R(v) \in \mathfrak{J}(V)$. \square

Returning to the proof of the proposition for $v \in \mathfrak{J}(V)_n$, we show that

$$\overline{\partial_i^R(\bar{v})} = d_i^R(v).$$

Indeed, we write $v = \sum \alpha_{i_1, \dots, i_n} v_{i_1} \cdots v_{i_n}$ for some $\alpha_{i_1, \dots, i_n} \in \mathbb{K}(q^{1/2})$ where the sum is over $i_1, \dots, i_n \in I$; then

$$\overline{\partial_i^R(\bar{v})} = \overline{\sum \alpha_{i_1, \dots, i_n} \partial_i^R(v_{i_1} \cdots v_{i_n})} = \sum \alpha_{i_1, \dots, i_n} \overline{\partial_i^R(v_{i_1} \cdots v_{i_n})} = d_i^R(v).$$

Below, by induction on the degree of $v \in \mathfrak{J}(V)_n$, we show that for any $i \in I$, $\partial_i^R(\bar{v}) \in \mathfrak{J}(V)$, and then apply Proposition 4.6.

The case $n = 2$ has been shown between the lines in the previous proof. For $n \geq 3$, to show that $\partial_i^R(\bar{v}) \in \mathfrak{J}(V)$, it suffices to see that $\overline{\partial_i^R(\bar{v})} \in \mathfrak{J}(V)$ by the induction hypothesis and the fact that the bar map is an involution. But we have shown $\overline{\partial_i^R(\bar{v})} = d_i^R(v)$, which is in $\mathfrak{J}(V)$ by the above lemma. This finishes the proof. \square

According to the above proposition, the bar involution may pass the quotient to give a \mathbb{K} -linear automorphism of the Nichols algebra $\mathfrak{B}(V)$.

The relation between the bar involution and the action of the Garside element on the image of the Dynkin operator P_n is explained in the following proposition.

Proposition 6.13. *For any $v \in T^n(V)$ satisfying $\theta_n v = v$,*

$$\Delta_n P_n v = (-1)^{n-1} \overline{P_n \bar{v}}.$$

Proof. We first assume that $v = v_{i_1} \cdots v_{i_n}$ is a monomial. In this case, the identity to be proved is $\Delta_n P_n v = (-1)^{n-1} \overline{P_n v}$.

For $1 \leq j_1 < \cdots < j_s \leq n - 1$, we denote

$$E_{j_1, \dots, j_s} = (\sigma_{n-1} \cdots \sigma_{j_1}) \cdots (\sigma_{n-1} \cdots \sigma_{j_s}).$$

Then the Dynkin operator P_n can be written as

$$P_n = \sum_{s=0}^{n-1} (-1)^s \sum_{1 \leq j_1 < \cdots < j_s \leq n-1} E_{j_1, \dots, j_s}.$$

Lemma 6.14. *Let $1 \leq j_1 < \cdots < j_s \leq n - 1$ and $1 \leq j'_1 < \cdots < j'_t \leq n - 1$ be such that $\{j_1, \dots, j_s\}$ and $\{j'_1, \dots, j'_t\}$ form a partition of $\{1, \dots, n - 1\}$. Then for any $v = v_{i_1} \cdots v_{i_n}$,*

$$\Delta_n E_{j_1, \dots, j_s} v = \overline{E_{j'_1, \dots, j'_t} v}.$$

Proof. To simplify notation, we define

$$Q_{i_1, \dots, i_n}^{j_1, \dots, j_s} = q_{j_s, j_s+1} \cdots q_{j_s, i_n} q_{j_{s-1}, j_{s-1}+1} \cdots q_{j_{s-1}, i_n} \cdots q_{j_1, j_1+1} \cdots q_{j_1, i_n}.$$

Then the condition $\theta_n v = v$ and the fact that the braiding matrix is symmetric imply

$$Q_{i_1, \dots, i_n}^{j_1, \dots, j_s} = \overline{Q_{i_1, \dots, i_n}^{j'_1, \dots, j'_t}}.$$

With this notation,

$$\begin{aligned} \Delta_n E_{j_1, \dots, j_s} (v_{i_1} \cdots v_{i_n}) &= Q_{i_1, \dots, i_n}^{j_1, \dots, j_s} v_{j_1} \cdots v_{j_s} v_{j'_t} \cdots v_{j'_1}, \\ E_{j'_1, \dots, j'_t} (v_{i_1} \cdots v_{i_n}) &= Q_{i_1, \dots, i_n}^{j'_1, \dots, j'_t} v_{j_1} \cdots v_{j_s} v_{j'_t} \cdots v_{j'_1}. \end{aligned}$$

The lemma can be proved by combining the above formulas. □

We now compute the left hand side of the formula in Proposition 6.13 when $v = v_{i_1} \cdots v_{i_n}$:

$$\begin{aligned} \Delta_n P_n v &= \sum_{s=0}^{n-1} (-1)^s \sum_{1 \leq j_1 < \cdots < j_s \leq n-1} \Delta_n E_{j_1, \dots, j_s} v \\ &= \sum_{s=0}^{n-1} (-1)^s \sum_{1 \leq j'_1 < \cdots < j'_{n-1-s} \leq n-1} \overline{E_{j'_1, \dots, j'_{n-1-s}} v} \\ &= (-1)^{n-1} \sum_{t=0}^{n-1} (-1)^t \sum_{1 \leq j'_1 < \cdots < j'_t \leq n-1} \overline{E_{j'_1, \dots, j'_t} v} = (-1)^{n-1} \overline{P_n v}. \end{aligned}$$

In the general case, when $v = \sum a_i v_i$ where v_i are monomials, the above formula gives

$$\begin{aligned} \Delta_n P_n v &= \sum a_i \Delta_n P_n v_i = (-1)^{n-1} \sum a_i \overline{P_n v_i} \\ &= (-1)^{n-1} \sum \overline{a_i P_n v_i} = (-1)^{n-1} \overline{P_n v}. \end{aligned} \quad \square$$

Corollary 6.15. *If $v \in T(V)$ satisfies $\bar{v} = v$ and $P_n v$ is a right pre-relation, then $\overline{P_n v}$ is a left pre-relation.*

The condition $\bar{v} = v$ holds for instance when v is a monomial.

§7. On the specialization problem

Recall that the field \mathbb{K} is of characteristic 0.

§7.1. A result on Kac–Moody Lie algebras

Let C be an arbitrary matrix. The Kac–Moody Lie algebra associated to C is defined by $\mathfrak{g}(C) = \tilde{\mathfrak{g}}(C)/\mathfrak{r}$, where $\tilde{\mathfrak{g}}(C)$ is the Lie algebra with Chevalley generators e_i, f_i, h_i for $i \in I$ and relations with respect to a realization $(\mathfrak{h}, \Pi, \Pi^\vee)$ of C and \mathfrak{r} is the unique maximal ideal in $\tilde{\mathfrak{g}}(C)$ intersecting \mathfrak{h} trivially (see [15, Chapter 1] for details). Moreover, we have the following decomposition as subalgebras:

$$\tilde{\mathfrak{g}}(C) = \tilde{\mathfrak{n}}_- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_+.$$

It gives $\mathfrak{r} = \mathfrak{r}_+ \oplus \mathfrak{r}_-$ as a direct sum of ideals, where $\mathfrak{r}_+ = \mathfrak{r} \cap \tilde{\mathfrak{n}}_+$ and $\mathfrak{r}_- = \mathfrak{r} \cap \tilde{\mathfrak{n}}_-$. We denote the quotients by $\mathfrak{n}_- = \tilde{\mathfrak{n}}_-/\mathfrak{r}_-$ and $\mathfrak{n}_+ = \tilde{\mathfrak{n}}_+/\mathfrak{r}_+$.

For these Lie algebras, we let $U(\mathfrak{r}_\pm), U(\mathfrak{n}_\pm), U(\tilde{\mathfrak{n}}_\pm), U(\mathfrak{g})$ and $U(\tilde{\mathfrak{g}})$ denote the corresponding enveloping algebras. The following theorem and proposition should be known to experts in enveloping algebras. We provide their proofs for the absence of a proper reference.

Theorem 7.1. *Let $x \in U(\tilde{\mathfrak{n}}_-)$. The following conditions are equivalent:*

- (1) $[e_i, x] \in U(\mathfrak{r}_-)$ for any $i \in I$.
- (2) $x \in U(\mathfrak{r}_-)$.

The proof of this theorem occupies the rest of this subsection.

Proposition 7.2. *Let $x \in U(\mathfrak{n}_-)$ be such that $[e_i, x] \in \mathbb{K}$ for any $i \in I$. Then $x \in \mathbb{K}$ is a constant.*

Proof. Given $x \in U(\mathfrak{n}_-)$ not a constant, we will find an index $t \in I$ such that $[e_t, x] \notin \mathbb{K}$.

We start by showing that it suffices to consider those x which are homogeneous with respect to the height gradation. Write $x = x_l + \dots + x_1 + x_0$ with x_i of height i . Since $[e_k, x_s]$ is either of height $s - 1$ or of height 0, to show that $[e_t, x] \notin \mathbb{K}$, it suffices to consider $[e_t, x_l]$.

We apply induction on the height. If x is of height 1, then it is in the vector space generated by f_i for $i \in I$. This case is clear.

Suppose that a totally ordered basis $\{f_\gamma\}_{\gamma \in \Gamma}$ of \mathfrak{n}_- is chosen such that elements of smaller heights are smaller. Let f_β denote the maximal basis element among those appearing in the monomials of x . By the PBW theorem, $x = \sum_{s=0}^l r_s f_\beta^s$ where f_β does not appear in the monomials of r_s . Hence

$$[e_i, x] = \sum_{s=0}^l [e_i, r_s] f_\beta^s + \sum_{s=0}^l r_s [e_i, f_\beta^s]$$

where the term containing f_β^l is $[e_i, r_l]$. There are three cases:

1. $r_l \in \mathbb{K}$ and all r_s are zero. It suffices to apply Lemma 1.5 of [15].
2. $r_l \in \mathbb{K}$ and there exists some maximal $0 \leq k \leq l - 1$ such that $r_k \neq 0$. In this case, $r_k \notin \mathbb{K}$ since x is homogeneous and the highest power of f_β in $[e_i, x]$ is in $[e_i, r_{l-1} + l f_\beta] f_\beta^{l-1}$ if $k = l - 1$ and otherwise in $l [e_i, f_\beta] f_\beta^{l-1}$. For the former, by induction we can always find some e_t such that $[e_t, r_{l-1} + l f_\beta] \notin \mathbb{K}$; for the latter it suffices to apply Lemma 1.5 of [15].
3. $r_l \notin \mathbb{K}$. Since r_s is homogeneous and of a smaller height, the induction hypothesis can be applied to give some e_t such that $[e_t, r_l] \neq 0$, therefore $[e_t, x] \neq 0$. \square

Proof of Theorem 7.1. (2) implies (1) is clear since $[e_i, \cdot]$ is a derivation and \mathfrak{r}_- is an ideal in $\tilde{\mathfrak{n}}_-$.

We suppose that (1) holds and introduce another \mathbb{N} -gradation on $U(\tilde{\mathfrak{n}}_-)$: by the PBW theorem, $U(\tilde{\mathfrak{n}}_-)$ is a free $U(\mathfrak{r}_-)$ -module (see [6, Proposition 2.2.7]). We define the partial degree on $U(\tilde{\mathfrak{n}}_-)$ by taking the height gradation on $U(\mathfrak{r}_-)$ and letting elements in $U(\mathfrak{n}_-)$ be of degree 0. Then $x \in U(\tilde{\mathfrak{n}}_-)$ is of partial degree 0 if and only if $x \in U(\mathfrak{n}_-)$.

The proof will be by induction on the largest partial degree l among the components of x .

If $l = 0$, then x is in $U(\mathfrak{n}_-)$, and $[e_i, x] \in U(\mathfrak{r}_-)$ implies that $[e_i, x] \in \mathbb{K}$. By Proposition 7.2, x is a constant, and therefore in $U(\mathfrak{r}_-)$.

In general, let l be the maximal partial degree among components of x ; we write $x = x_l + x_{l-1} + \dots + x_1 + x_0$ where $x_i \in U(\tilde{\mathfrak{n}}_-)$ is of partial degree i . We write $x_l = \sum r_k n_k$ for some $r_k \in U(\mathfrak{r}_-)$ and $n_k \in U(\mathfrak{n}_-)$ such that the r_k are linearly independent. In $[e_i, x]$, since the partial degree of $[e_i, r_k]$ is less than

that of r_k , the component of maximal partial degree is given by $\sum r_k [e_i, n_k]$; as $[e_i, x] \in U(\mathfrak{r}_-)$, this forces $[e_i, n_k] \in U(\mathfrak{r}_-)$ and hence $[e_i, n_k] \in \mathbb{K}$. This shows that for any $i \in I$ and any k , $[e_i, n_k] \in \mathbb{K}$; by Proposition 7.2, n_k are constants and $x_l \in U(\mathfrak{r}_-)$. Finally we consider $x - x_l$; it has lower partial degree and satisfies $[e_i, x - x_l] \in U(\mathfrak{r}_-)$. By induction hypothesis, $x - x_l \in U(\mathfrak{r}_-)$, hence $x \in U(\mathfrak{r}_-)$. \square

If moreover C is a generalized Cartan matrix, some elements in \mathfrak{r} have been discovered in Section 3.3 of [15]: in $\mathfrak{g}(C)$, for $i \neq j$,

$$(7.1) \quad (\text{ad } e_i)^{1-c_{ij}}(e_j) = 0, \quad (\text{ad } f_i)^{1-c_{ij}}(f_j) = 0.$$

If the matrix C is not symmetrizable, the ideal generated by these relations may not exhaust \mathfrak{r} .

§7.2. Specialization (I): general definition and a counterexample

We follow the specialization procedure in [13]. Let C be a generalized Cartan matrix, $\mathcal{A} = \mathbb{K}[q^{1/2}, q^{-1/2}]$ and \mathcal{A}_1 the localization of $\mathbb{K}[q^{1/2}]$ at $(q^{1/2} - 1)$.

When the braiding matrix A is of the form $(q^{c_{ij}})_{1 \leq i, j \leq N}$ for a generalized Cartan matrix $C = (c_{ij})_{1 \leq i, j \leq N}$, we will denote the Hopf algebra $T_q(A)$ by $T_q(C)$ and $D_q(A)$ by $D_q(C)$.

We start by defining an \mathcal{A}_1 -form of $T_q(C)$. Let $T_{\mathcal{A}_1}(C)$ be the \mathcal{A}_1 -subalgebra of $T_q(C)$ generated by

$$E_i, \quad F_i, \quad K_i^{\pm 1} \quad \text{and} \quad [K_i; 0] = \frac{K_i - K_i^{-1}}{q - q^{-1}}$$

for any $i \in I$. It inherits a Hopf algebra structure from that of $T_q(C)$. We let $T_{\mathcal{A}_1}^{<0}(C)$ (resp. $T_{\mathcal{A}_1}^{>0}(C)$) denote the subalgebra of $T_{\mathcal{A}_1}(C)$ generated by F_i (resp. E_i) for $i \in I$.

Since $(q^{1/2} - 1)$ is a maximal ideal in \mathcal{A}_1 , \mathbb{K} admits an \mathcal{A}_1 -module structure via $\mathcal{A}_1 / (q^{1/2} - 1) \cong \mathbb{K}$, given by evaluating $q^{1/2}$ to 1. We define $T_1(C) = T_{\mathcal{A}_1}(C) \otimes_{\mathcal{A}_1} \mathbb{K}$. There exists a natural algebra morphism $\tilde{\sigma} : T_{\mathcal{A}_1}(C) \rightarrow T_1(C)$, which is called the *specialization map*.

For $i \in I$, we let e_i, f_i and h_i denote the images of E_i, F_i and $\frac{K_i - 1}{q - 1}$ under the map $\tilde{\sigma}$. Then $K_i^{\pm 1}$ are sent to 1 and $[K_i; 0]$ has image h_i under $\tilde{\sigma}$. Relations in $T_{\mathcal{A}_1}(C)$ are specialized to relations in $T_1(C)$;

$$\begin{aligned} [e_i, f_j] &= \tilde{\sigma}([E_i, F_j]) = \delta_{ij} \tilde{\sigma}([K_i; 0]) = \delta_{ij} h_i, \\ [h_i, e_j] &= \tilde{\sigma}([[K_i; 0], E_j]) = \tilde{\sigma} \left(\frac{(1 - q^{-c_{ij}})K_i - (1 - q^{c_{ij}})K_i^{-1}}{q - q^{-1}} E_j \right) = c_{ij} e_j, \end{aligned}$$

and similarly $[h_i, f_j] = -c_{ij} f_j$ and $[h_i, h_j] = 0$.

The following facts hold:

1. The specialization map $\tilde{\sigma} : T_{\mathcal{A}_1}(C) \rightarrow T_1(C) \cong U(\tilde{\mathfrak{g}}(C))$ is a Hopf algebra morphism. When composed with the projection $U(\tilde{\mathfrak{g}}(C)) \rightarrow U(\mathfrak{g}(C))$, it gives a Hopf algebra morphism $\sigma : T_{\mathcal{A}_1}(C) \rightarrow U(\mathfrak{g}(C))$, which is also called the specialization map.
2. The restrictions of σ give the specialization maps $T_{\mathcal{A}_1}^{<0}(C) \rightarrow U(\mathfrak{n}_-(C))$ and $T_{\mathcal{A}_1}^{>0}(C) \rightarrow U(\mathfrak{n}_+(C))$.

To obtain a true specialization map of the quantum group, the morphism σ should pass through the quotient by defining ideals.

Example 7.3. We consider the non-symmetrizable generalized Cartan matrix

$$C = \begin{bmatrix} 2 & -2 & -1 \\ -1 & 2 & -1 \\ -3 & -1 & 2 \end{bmatrix}.$$

In the braided tensor Hopf algebra of diagonal type associated to this matrix, we want to find some particular pre-relations. It is easy to show that $\theta_4(F_3^3 F_1) = F_3^3 F_1$. Recall that $T'_4 = (1 - \sigma_3^2 \sigma_2 \sigma_1)(1 - \sigma_3^2 \sigma_2)(1 - \sigma_3^2)$; since $1 - \sigma_3^2 \sigma_2$ and $1 - \sigma_3^2$ act as non-zero scalars on $F_3^3 F_1$ and $1 - \sigma_3^2 \sigma_2 \sigma_1$ acts as 0 on it,

$$T_4 P_4(F_3^3 F_1) = T'_4(F_3^3 F_1) = 0.$$

Moreover, since $\iota_3^4(T'_3) = (1 - \sigma_3^2 \sigma_2)(1 - \sigma_3^2)$ acts as a non-zero scalar on $F_3^3 F_1$, by definition, $P_4(F_3^3 F_1)$ is a right pre-relation of degree 4 where

$$P_4(F_3^3 F_1) = F_3^3 F_1 - (q^{-3} + q^{-1} + q)F_3^2 F_1 F_3 + (q^{-4} + q^{-2} + 1)F_3 F_1 F_3^2 - q^{-3} F_1 F_3^3.$$

If the specialization map to the enveloping algebra of the Kac–Moody Lie algebra associated to C were well-defined, this element would be specialized to

$$[f_3, [f_3, [f_3, f_1]]] = f_3^3 f_1 - 3f_3^2 f_1 f_3 + 3f_3 f_1 f_3^2 - f_1 f_3^3$$

in $U(\tilde{\mathfrak{n}}_-)$. We show that it is not contained in $U(\mathfrak{r}_-)$ so does not give 0, contradicting the definition of the Kac–Moody Lie algebra.

The successive adjoint actions of e_3 give

$$\begin{aligned} [e_3, f_3^3 f_1 - 3f_3^2 f_1 f_3 + 3f_3 f_1 f_3^2 - f_1 f_3^3] &= 3(f_3^2 f_1 - 2f_3 f_1 f_3 + f_1 f_3^2), \\ [e_3, f_3^2 f_1 - 2f_3 f_1 f_3 + f_1 f_3^2] &= 4(f_3 f_1 - f_1 f_3), \\ [e_3, f_3 f_1 - f_1 f_3] &= 3f_1. \end{aligned}$$

If $[f_3, [f_3, [f_3, f_1]]]$ were in $U(\mathfrak{t}_-)$, so would be f_1 , hence $[e_1, f_1] = h_1$ according to Theorem 7.1. This is impossible since by definition of the Kac–Moody Lie algebra, $\mathfrak{t}_- \cap \mathfrak{h} = \{0\}$.

In conclusion, this example shows that the specialization map may not be well-defined if the matrix is not symmetric.

§7.3. Specialization (II): the quantum group case

Let C be a generalized Cartan matrix and \bar{C} be the associated averaged matrix. To have a well-defined specialization map, we need to pass to the q -enveloping algebra associated to this averaged matrix. We suppose moreover that the matrix C is non-symmetrizable as otherwise there would be no problem.

Recall that $U_q(C) := D_q(\bar{C})$ is the quotient of $T_q(\bar{C})$ by its defining ideals and $U_q^{<0}(C)$ is the subalgebra of $U_q(C)$ generated by F_i for $i \in I$. $U_q(C)$ admits an \mathcal{A}_1 -form since the defining ideals are given by the kernel of the total symmetrization map S_n , which preserves both $T_{\mathcal{A}_1}^{<0}(\bar{C})$ and $T_{\mathcal{A}_1}^{>0}(\bar{C})$. This \mathcal{A}_1 -form of $U_q(C)$ is generated as an \mathcal{A}_1 -module by $E_i, F_i, K_i^{\pm 1}$ and $[K_i; 0]$ for $i \in I$; we denote it by $U_{\mathcal{A}_1}(C)$. Since the operators $T_n, P_n, T'_n, U_n, Q_n, U'_n, \Delta_n$ and θ_n preserve the subalgebra $T_{\mathcal{A}_1}(\bar{C})$ of $T_q(\bar{C})$, left and right pre-relations, left and right constants are well-defined over \mathcal{A}_1 .

Theorem 7.4. *The specialization map $\sigma : T_{\mathcal{A}_1}(\bar{C}) \rightarrow U(\mathfrak{g}(\bar{C}))$ passes to the quotient to give a surjective map $\sigma : U_{\mathcal{A}_1}(C) \rightarrow U(\mathfrak{g}(\bar{C}))$.*

The proof of this theorem will occupy the rest of this subsection. We start by the following lemma (see also [12, Lemma 4.15]).

Lemma 7.5. *For any $w \in T_{\mathcal{A}_1}^{<0}(\bar{C})$ and any $i \in I$,*

$$[E_i, w] = \frac{K_i \partial_i^L(w) - \partial_i^R(w) K_i^{-1}}{q - q^{-1}} = \frac{d_i^R(w) K_i - \partial_i^R(w) K_i^{-1}}{q - q^{-1}} \in T_{\mathcal{A}_1}^{<0}(\bar{C}).$$

This formula can be proved either by induction or by verifying directly on a monomial; notice that the symmetry of the braiding matrix is necessary.

Recall that $\tilde{\sigma} : T_q(\bar{C}) \rightarrow U(\tilde{\mathfrak{g}}(\bar{C}))$ is the specialization map.

Lemma 7.6. *Let $w \in T_{\mathcal{A}_1}^{<0}(\bar{C})$ be a right constant of degree n . Then $\tilde{\sigma}(w)$ is in $U(\mathfrak{t}_-)$.*

Proof. By Theorem 7.1, it suffices to verify that for any $i \in I$,

$$\tilde{\sigma}([E_i, w]) = [e_i, \tilde{\sigma}(w)] \in U(\mathfrak{t}_-).$$

We apply induction on the degree n of the right constant w .

The case $n = 2$ is clear since all constants of degree 2 are computed in Proposition 3.8. For general $n \geq 3$, by the above lemma and the fact that $\partial_i^R(w) = 0$ for any $i \in I$,

$$[E_i, w] = K_i \partial_i^L \left(\frac{w}{q - q^{-1}} \right) \in T_{\mathcal{A}_1}^{<0}(\overline{C}).$$

Since ∂_i^L and ∂_i^R commute, $\partial_i^L(w)$ is annihilated by T_{n-1} and of degree at most $n - 1$. By induction hypothesis, $\partial_i^L \left(\frac{w}{q - q^{-1}} \right)$ is specialized to $U(\mathfrak{r}_-)$, hence

$$\tilde{\sigma}([E_i, w]) = \tilde{\sigma} \left(K_i \partial_i^L \left(\frac{w}{q - q^{-1}} \right) \right) \in U(\mathfrak{r}_-). \quad \square$$

Proof of Theorem 7.4. We have proved in the above lemma that right constants are specialized to $U(\mathfrak{r}_-)$ under $\tilde{\sigma}$. A similar argument can be applied to left constants to show that their specializations are in $U(\mathfrak{r}_+)$. We therefore obtain a well-defined algebra map $\sigma : U_{\mathcal{A}_1}(C) \rightarrow U(\mathfrak{g}(\overline{C}))$ and the surjectivity is clear. \square

§7.4. Specialization (III): the Nichols algebra case

Let C be a generalized Cartan matrix, $A = (q^{c_{ij}})_{1 \leq i, j \leq N}$ and $D^{<0}(C)$ be the Nichols algebra of the braiding matrix A with respect to a basis F_1, \dots, F_N , which is the subalgebra of $D^{\leq 0}(A)$ generated by F_i for $i \in I$.

Theorem 7.7. *There exists a surjective algebra morphism*

$$\varphi : D^{<0}(C) \rightarrow U(\mathfrak{n}_-(\overline{C}))$$

sending v_i to f_i .

Proof. We let $D^{<0}(\overline{C})$ denote the Nichols algebra of diagonal type of braiding matrix $(q^{\bar{c}_{ij}})_{1 \leq i, j \leq N}$ with respect to a basis w_1, \dots, w_N . By Proposition 6.6, there exists a linear isomorphism $\psi : D^{<0}(C) \rightarrow D^{<0}(\overline{C})$ sending F_i to w_i . Composing with the restriction of the specialization map σ to the negative part of $U_q(C)$ gives a linear surjection $\varphi : D^{<0}(C) \rightarrow U(\mathfrak{n}_-(\overline{C}))$.

It remains to show that φ is an algebra morphism: for $1 \leq i, j \leq N$,

$$\varphi(F_i F_j) = \sigma \circ \psi(F_i F_j) = \begin{cases} \sigma(q^{\frac{1}{2}(c_{ji} - c_{ij})} w_i w_j) = f_i f_j & \text{if } i \leq j, \\ \sigma(w_i w_j) = f_i f_j & \text{if } i > j. \end{cases} \quad \square$$

§8. Application

It is natural to ask for the size of Rel_r , we will relate it to the integral points of some quadratic forms.

§8.1. General calculation

Let $A = (a_{ij})_{1 \leq i, j \leq N} \in M_N(\mathbb{Z})$ be a generalized Cartan matrix. We consider the element $v_{\underline{i}}$ for $\underline{i} = (1^{m_1}, \dots, N^{m_N})$: the action of the central element θ_m where $m = m_1 + \dots + m_N$ gives

$$\theta_m(v_{\underline{i}}) = q^\lambda v_{\underline{i}}$$

where

$$\lambda = \sum_{k=1}^N 2m_k(m_k - 1) - \sum_{p=1}^N \sum_{q < p} (a_{pq} + a_{qp})m_p m_q.$$

So there exists a pre-relation in $\mathbb{K}[X_{\underline{i}}]$ only if $\lambda = 0$. To find these pre-relations, it suffices to consider the integral solutions of this quadratic form.

§8.2. Study of the quadratic form

The above computation motivates the study of the following quadratic forms:

$$Q(x_1, \dots, x_n) = \sum_{i=1}^n x_i^2 - \sum_{i < j} b_{ij} x_i x_j,$$

$$S(x_1, \dots, x_n) = \sum_{i=1}^n (x_i - 1)^2,$$

where $b_{ij} = a_{ij} + a_{ji}$ are non-negative integers as in the last subsection.

Let $m \leq n$ be an integer (not necessarily positive) and C_m be the intersection of the two varieties

$$Q(x_1, \dots, x_n) = m, \quad S(x_1, \dots, x_n) = n - m.$$

Let $E(C_m)$ be the set of integral points on C_m and $E = \bigcup_{m \leq n} E(C_m)$. Then the set of all integral solutions of $\lambda = 0$ is the same as E .

Proposition 8.1. *If the quadratic form $Q(x_1, \dots, x_n)$ is positive semi-definite, E is a finite set.*

Proof. If $Q(x_1, \dots, x_n)$ is positive semi-definite, E is a finite union of $E(C_m)$ for $0 \leq m \leq n$. For each m , as $S(x_1, \dots, x_n) = n - m$ is compact, so is its intersection with $Q(x_1, \dots, x_n) = m$. The finiteness of $E(C_m)$ and of E is clear. □

Corollary 8.2. *If the quadratic form $Q(x_1, \dots, x_n)$ is positive semi-definite, the defining ideal $\mathfrak{J}(V)$ is finitely generated.*

Proof. By the above proposition, there are only finitely many indices \underline{i} such that $\mathbb{K}[X_{\underline{i}}]$ contains right pre-relations; moreover, each $\mathbb{K}[X_{\underline{i}}]$ is finite-dimensional. □

Acknowledgements

This paper is extracted (with revision) from the author's Ph.D thesis, supervised by Professor Marc Rosso, to whom I would like to express my sincere gratitude. I would like to thank the referee for valuable comments and suggestions.

References

- [1] N. Andruskiewitsch, Some remarks on Nichols algebras, in *Hopf algebras*, Dekker, 2004, 35–45. [Zbl 1068.16052](#)
- [2] N. Andruskiewitsch and M. Graña, Braided Hopf algebras over non-abelian finite groups, in *Colloquium on operator algebras and quantum groups*, Bol. Acad. Nac. Cienc. (Córdoba) **63** (1999), 45–78. [Zbl 1007.17010](#) [MR 1714540](#)
- [3] N. Andruskiewitsch, I. Heckenberger and H.-J. Schneider, The Nichols algebra of a semi-simple Yetter–Drinfeld module, *Amer. J. Math.* **132** (2010), 1493–1547. [Zbl 1214.16024](#) [MR 2766176](#)
- [4] N. Andruskiewitsch and H.-J. Schneider, Pointed Hopf algebras, in *New directions in Hopf algebra theory*, Math. Sci. Res. Inst. Publ. 43, Cambridge Univ. Press, Cambridge, 2002, 1–68. [Zbl 1011.16025](#) [MR 1913436](#)
- [5] I. Angiono, On Nichols algebras of diagonal type, *J. Reine Angew. Math.* **683** (2013), 189–251. [Zbl 06224213](#) [MR 3181554](#)
- [6] J. Dixmier, *Enveloping algebras*, Grad. Stud. Math. 11, Amer. Math. Soc., 1996. [Zbl 0867.17001](#) [MR 1393197](#)
- [7] X. Fang, Quantum groups, q -Boson algebras and quantized Weyl algebras, *Int. J. Math.* **22** (2011), 675–694. [Zbl 1238.16029](#) [MR 2799884](#)
- [8] ———, On defining ideals and differential algebras of Nichols algebras, *J. Algebra* **346** (2011), 299–331. [Zbl 1264.16032](#) [MR 2842083](#)
- [9] C. Frønsdal and A. Galindo, The ideals of free differential algebras, *J. Algebra* **222** (1999), 708–746. [Zbl 0958.17001](#) [MR 1734224](#)
- [10] O. Gabber and V. Kac, On defining relations of certain infinite-dimensional Lie algebras, *Bull. Amer. Math. Soc. (N.S.)* **5** (1981), 185–189. [Zbl 0474.17007](#) [MR 0621889](#)
- [11] I. Heckenberger, The Weyl groupoid of a Nichols algebra of diagonal type, *Invent. Math.* **164** (2006), 175–188. [Zbl 1174.17011](#) [MR 2207786](#)
- [12] ———, Lusztig isomorphisms for Drinfel'd doubles of bosonizations of Nichols algebras of diagonal type, *J. Algebra* **323** (2010), 2130–2180. [Zbl 1238.17010](#) [MR 2596372](#)
- [13] J. Hong and S.-J. Kang, *Introduction to quantum groups and crystal bases*, Grad. Stud. Math. 42, Amer. Math. Soc., Providence, RI, 2002. [Zbl 1134.17007](#) [MR 1881971](#)
- [14] A. Joseph and P. Lamprou, A Littelmann path model for crystals of generalized Kac–Moody algebras, *Adv. Math.* **221** (2009), 2019–2058. [Zbl 1194.17006](#) [MR 2522835](#)
- [15] V. Kac, *Infinite-dimensional Lie algebras*, 3rd ed., Cambridge Univ. Press, Cambridge, 1990. [Zbl 0716.17022](#) [MR 1104219](#)
- [16] M. Kashiwara, On crystal bases of the q -analogue of universal enveloping algebras, *Duke Math. J.* **63** (1991), 465–516. [Zbl 0739.17005](#) [MR 1115118](#)
- [17] ———, On crystal bases, in *Representations of groups* (Banff, AB, 1994), CMS Conf. Proc. 16, Amer. Math. Soc., Providence, RI, 1995, 155–197. [Zbl 0851.17014](#) [MR 1357199](#)
- [18] C. Kassel, M. Rosso and V. Turaev, *Quantized enveloping algebras and knot invariants*, Panoramas Synthèses 5, Soc. Math. France, 1997.

- [19] C. Kassel and V. Turaev, *Braid groups*, Grad. Texts in Math. 247, Springer, New York, 2008. [Zbl 1208.20041](#) [MR 2435235](#)
- [20] P. Littelmann, Paths and root operators in representation theory, *Ann. of Math. (2)* **142** (1995), 499–525. [Zbl 0858.17023](#) [MR 1356780](#)
- [21] G. Lusztig, Quantum deformations of certain simple modules over enveloping algebras, *Adv. Math.* **70** (1988), 237–249. [Zbl 0651.17007](#) [MR 0954661](#)
- [22] A. Milinski and H.-J. Schneider, Pointed indecomposable Hopf algebras over Coxeter groups, in *New trends in Hopf algebra theory* (La Falda, 1999), *Contemp. Math.* 267, Amer. Math. Soc., Providence, RI, 2000, 215–236. [Zbl 1093.16504](#) [MR 1800714](#)
- [23] M. Rosso, Groupes quantiques et algèbres de battage quantiques, *C. R. Acad. Sci. Paris Sér. I Math.* **320** (1995), 145–148. [Zbl 0929.17005](#) [MR 1320345](#)
- [24] ———, Quantum groups and quantum shuffles, *Invent. Math.* **133** (1998), 399–416. [Zbl 0912.17005](#) [MR 1632802](#)
- [25] P. Schauenburg, A characterization of the Borel-like subalgebras of quantum enveloping algebras, *Comm. Algebra* **24** (1996), 2811–2823. [Zbl 0856.17017](#) [MR 1396857](#)