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Classification of Finite-Dimensional Irreducible Representations of Generalized Quantum Groups via Weyl Groupoids

Dedicated to Professor Jun Morita on the occasion of his 60th birthday

by

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Abstract

Let χ be a bi-homomorphism over an algebraically closed field of characteristic zero. Let $U(\chi)$ be a generalized quantum group, associated with χ, such that $\dim U^+(\chi) = \infty$, $|R^+(\chi)| < \infty$, and $R^+(\chi)$ is irreducible, where $U^+(\chi)$ is the positive part of $U(\chi)$, and $R^+(\chi)$ is the Kharchenko positive root system of $U^+(\chi)$. In this paper, we give a list of finite-dimensional irreducible $U(\chi)$-modules, relying on a *special* reduced expression of the longest element of the Weyl groupoid of $R(\chi) := R^+(\chi) \cup (-R^+(\chi))$. From the list, we explicitly obtain lists of finite-dimensional irreducible modules for simple Lie superalgebras $\mathfrak{g}$ of types A–G and the (standard) quantum superalgebras $U_q(\mathfrak{g})$. An intrinsic gap appears between the lists for $\mathfrak{g}$ and $U_q(\mathfrak{g})$, e.g, if $\mathfrak{g}$ is $\mathrm{B}(m, n)$ or $\mathrm{D}(m, n)$.

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Introduction

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero (see also (1.1)). Let χ be a bi-homomorphism over $\mathbb{K}$.

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In this paper, we give a list of finite-dimensional irreducible modules of a generalized quantum group $U(\chi)$ over $\mathbb{K}$ whose positive part $U^+(\chi)$ is infinite-dimensional and has a Kharchenko PBW-basis with a finite irreducible positive root system. We call such a $U(\chi)$ a generalized quantum group of *finite-and-infinite-dimensional type* (FID-type, for short). From the list, we explicitly obtain lists of finite-dimensional irreducible modules for simple Lie superalgebras $\mathfrak{g}$ of types A–G and the (standard) quantum superalgebras $U_q(\mathfrak{g})$.

We begin by recalling some facts about Lie superalgebras. The class of *contragredient Lie superalgebras* [17, Subsection 2.5.1] is defined in a way similar to that for Kac–Moody Lie algebras. Kac classified the finite-dimensional simple Lie superalgebras [17, Theorem 5], where finite-dimensional irreducible contragredient Lie superalgebras played crucial roles; those are

(1) simple Lie algebras of type X_N, where $\mathrm{X} = \mathrm{A}, \ldots, \mathrm{G}$,

(2) $\mathrm{sl}(m+1\,|\,n+1)$ $(m+n \geq 1,\ m,n \geq 0)$,

(3) $\mathrm{B}(m,n)$ $(m \geq 0,\ n \geq 1)$, $\mathrm{C}(n)$ $(n \geq 3)$, $\mathrm{D}(m,n)$ $(m \geq 2,\ n \geq 1)$, $\mathrm{D}(2,1;x)$ $(x \neq 0, -1)$, $\mathrm{F}(4)$, $\mathrm{G}(3)$.

The ones in (1) and (3) are simple. The simple Lie superalgebras $\mathrm{A}(m,n)$ are defined by $\mathrm{sl}(m+1\,|\,n+1)$ if $m \neq n$, and otherwise $\mathrm{A}(n,n) := \mathrm{sl}(n+1\,|\,n+1)/\mathfrak{i}$, where $\mathfrak{i}$ is its unique one-dimensional ideal.

Bases of the root systems of the Lie superalgebras of (2)–(3) are not conjugate under the action of their Weyl groups. However any two of them can be transformed to each other by the action of their *Weyl groupoids* W, axiomatically treated by Heckenberger and the second author [13]. Kac [17, Theorem 8(c)] gave a list of finite-dimensional irreducible modules of the Lie superalgebras in (2)–(3) above. After reading the main part of this paper, a reader familiar with Lie superalgebras will realize that our approach can also be applied to recover Kac's list; indeed we can also obtain it by a specialization argument (see Subsection 7.6). Our idea is to use a specially good reduced expression of the longest element (with a 'standard' end domain) of the Weyl groupoid W (see also Remark 7.17).

Let $\mathfrak{g} := \mathrm{sl}(m+1\,|\,n+1)$ $(m \neq n)$ or $\mathrm{C}(n)$ for example. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ such that the Dynkin diagram of $(\mathfrak{g},\mathfrak{h})$ is a standard one. Let $\Pi = \{\alpha_i \mid 1 \leq i \leq \dim\mathfrak{h}\}$ be the set of simple roots α_i corresponding to $\mathfrak{h}$. Let w_0 be the longest element of the Weyl groupoid W of $\mathfrak{g}$ whose end domain corresponds to $\mathfrak{h}$. Then the length $\ell(w_0)$ of w_0 is equal to the number of positive roots of $\mathfrak{g}$. Let k be the number of even positive roots of $\mathfrak{g}$. The key fact used in this paper is that there exists a reduced expression $s_{i_1}\cdots s_{i_{\ell(w_0)}}$ of w_0 such that $s_{i_1}\cdots s_{i_{x-1}}(\alpha_{i_x})$, $1 \leq x \leq k$, are even positive roots, and $s_{i_1}\cdots s_{i_{y-1}}(\alpha_{i_y})$, $k+1 \leq y \leq \ell(w_0)$, are odd positive roots. This is essential to showing that an irreducible highest weight $\mathfrak{g}$-module of highest weight Λ is finite-dimensional if and only if $2\langle\Lambda,\alpha_i\rangle/\langle\alpha_i,\alpha_i\rangle \in \mathbb{Z}_{\geq 0}$ for all even simple roots α_i, where $\langle\,,\,\rangle$ is the bilinear form coming from the Killing form of $\mathfrak{g}$.

Motivated by Andruskiewitsch and Schneider's theory [2], [3] toward the classification of pointed Hopf algebras, Heckenberger [10] classified the Nichols algebras of diagonal type. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Let $U(\chi)$ be the $\mathbb{K}$-algebra defined as in Lusztig's book [19, 3.1.1(a)–(e)] for any bi-homomorphism $\chi : \mathbb{Z}\Pi \times \mathbb{Z}\Pi \to \mathbb{K}^\times$, where $\Pi = \{\alpha_i \mid i \in I\}$ is the set of simple roots of the Kharchenko positive root system $R^+(\chi)$ associated with χ. We call $U(\chi)$ a *generalized quantum group*. We say that χ (or $U(\chi)$) is of *finite type* if $R^+(\chi)$ is finite and irreducible. We say that χ (or $U(\chi)$) is of *finite-and-infinite-dimensional type* (FID-type, for short) if χ is of finite type and $\dim U^+(\chi) = \infty$. A Nichols algebra of diagonal type is isomorphic to the positive part $U^+(\chi)$ of $U(\chi)$ for some χ of finite type. If $U(\chi)$ is of FID-type, then it is a multi-parameter quantum algebra of a simple Lie algebra in (1), a multi-parameter quantum superalgebra of a simple Lie superalgebra in (2) or (3), or one of the two algebras in [10, Table 1, Row 5, Table 3, Row 14]. We show that every finite-dimensional irreducible $U(\chi)$-module is a highest weight module (see Lemma 4.23). Our main results, Theorems 7.1 (rank one cases), 7.2 (simple Lie algebra cases), 7.4 (A$(m-1, N-m)$ and C(N) cases), 7.6 (B$(m, N-m)$ cases), 7.7 (D$(m, N-m)$ cases) and 7.8 (F(4), G(3), D$(2,1;x)$ cases and extra cases) give a list of finite-dimensional irreducible modules of $U(\chi)$ of FID-type in the way mentioned above. From it, we explicitly obtain lists of finite-dimensional irreducible modules for the standard quantum superalgebra $U_q(\mathfrak{g})$ (see Lemma 7.12) and the simple Lie superalgebra $\mathfrak{g}$ (see Lemma 7.15) corresponding to χ.

Studying the representation theory of $U(\chi)$ is interesting and fruitful since the factorization formula of Shapovalov determinants of any $U(\chi)$ of finite type has been obtained by Heckenberger and the second author [14]. We believe that it would help us to find a new approach to Lusztig's conjecture [20].

This paper is organized as follows.

In Section 1, we collect general facts about Weyl groupoids. In Section 2, we give examples of reduced expressions of longest elements of Weyl groups, which will be used in Section 3. In Section 3, we give reduced expressions of the Weyl groupoids associated to Lie superalgebras of ABCD types. In Section 4, we give the definition of generalized quantum groups $U(\chi)$ associated with any bi-homomorphism χ, explain Kharchenko's PBW theorem for $U(\chi)$, and introduce the Weyl groupoids associated with χ. In Section 5, we discuss the properties of Weyl groupoids associated with $U(\chi)$ of finite type, and Heckenberger's classification of $U(\chi)$'s of FID type. In Section 6, we give a key criterion for determining when an irreducible highest weight $U(\chi)$-module is finite-dimensional (see Lemma 6.6). In Section 7, we give a list of finite-dimensional irreducible $U(\chi)$-modules for $U(\chi)$ having a standard Dynkin diagram, and we also show that from it, we can explicitly obtain lists of such modules for the standard quantum superalgebra and the simple Lie superalgebra corresponding to χ.

In [29], the second author has given a result similar to Theorem 7.8 for the case (FGE-4) of that theorem.

Contents

§1. Weyl groupoids

§1.1. Basic terminology

For a set $\mathfrak{s}$, let $|\mathfrak{s}|$ denote the cardinality of $\mathfrak{s}$. Let $\mathbb{N}$ denote the set of positive integers, $\mathbb{Z}$ the ring of integers, and $\mathbb{Q}$ denote the field of rational numbers. For $x, y \in \mathbb{Q}$, let $J_{x,y} := \{n \in \mathbb{Z} \mid x \leq n \leq y\}$. Note that $J_{x,y}$ is empty if $x > y$ or if $n < x \leq y < n+1$ for some $n \in \mathbb{Z}$. For $x \in \mathbb{Q}$, let $J_{x,\infty} := \{m \in \mathbb{Z} \mid m \geq x\}$ and $J_{-\infty,x} := \{m \in \mathbb{Z} \mid m \leq x\}$. Thus $\mathbb{N} = J_{1,\infty}$. Let $\mathbb{Z}_{\geq 0} := J_{0,\infty}$. Let $\mathbb{R}$ denote the field of real numbers.

For a field Y, a Y-linear space V, a non-empty subset X of V, and a subset Z of Y, let $\mathrm{Span}_Z(X) := \bigcup_{t=1}^{\infty}\{\sum_{y=1}^{t} z_y x_y \in V \mid z_y \in Z,\, x_y \in X\, (y \in J_{1,t})\}$; let $\mathrm{Span}_Z(\emptyset) := \{0\}$.

Throughout this paper, we use the fixed notation below:

$$
\text{(1.1)}\qquad
\begin{array}{l}
N \in \mathbb{N} \text{ is a fixed positive integer and } I := J_{1,N},\\
\mathbb{V} \text{ is a fixed } N\text{-dimensional } \mathbb{R}\text{-linear space},\\
\langle \Pi \rangle = (\alpha_i \,|\, i \in I) \text{ is a fixed ordered } \mathbb{R}\text{-basis of } \mathbb{V},\\
\Pi := \{\alpha_i \,|\, i \in I\}, \text{ so } \Pi \text{ is a (set) } \mathbb{R}\text{-basis of } \mathbb{V},\\
\mathbb{K} \text{ is an algebraically closed field of characteristic zero},\\
\mathbb{K}^{\times} := \mathbb{K} \setminus \{0\}.
\end{array}
$$

Let $\mathbb{Z}\Pi := \mathrm{Span}_{\mathbb{Z}}(\Pi)$ $(= \bigoplus_{i\in I} \mathbb{Z}\alpha_i \subsetneq \mathbb{V})$, i.e., $\mathbb{Z}\Pi$ is the free $\mathbb{Z}$-module with basis Π. Then $\mathrm{rank}_{\mathbb{Z}}\, \mathbb{Z}\Pi = N$. Let $\mathbb{Z}_{\geq 0}\Pi := \mathrm{Span}_{\mathbb{Z}_{\geq 0}}(\Pi)$ $(= \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i \subsetneq \mathbb{Z}\Pi)$. For an $\mathbb{R}$-linear space V, an $\mathbb{R}$-bilinear map $\eta : V \times V \to \mathbb{R}$ and an element $X = (x_i \mid i \in I)$ of $V \times \cdots \times V$ (N times) with $\dim_{\mathbb{R}} \mathrm{Span}_{\mathbb{R}}(\{x_i \mid i \in I\}) = N$, define an $\mathbb{R}$-linear map $\xi_X : \mathrm{Span}_{\mathbb{R}}(\{x_i \mid i \in I\}) \to \mathbb{V}$ by $\xi_X(x_i) := \alpha_i$ $(i \in I)$ and an $\mathbb{R}$-bilinear map $\eta_X : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ by $\eta_X := \eta \circ (\xi_X^{-1} \times \xi_X^{-1})$. For a map $\theta : I \to J_{0,1}$, define a map $[\theta] : \mathbb{Z}\Pi \to J_{0,1}$ by $[\theta](\sum_{i\in I} n_i\alpha_i) - \sum_{i\in I} n_i\theta(i) \in 2\mathbb{Z}$ $(n_i \in \mathbb{Z})$.

For $n \in \mathbb{N} \cup \{\infty\}$, let Map_n^I be the set of maps from $J_{1,n}$ to I. Let Map_0^I be the set composed of a unique element ϕ, i.e., $|\mathrm{Map}_0^I| = 1$ and $\phi \in \mathrm{Map}_0^I$.

For a unital $\mathbb{K}$-algebra G, let $\mathrm{Ch}(G)$ denote the set of $\mathbb{K}$-algebra homomorphisms from G to $\mathbb{K}$.

Let $t \in \mathbb{K}$. For $m \in \mathbb{Z}_{\geq 0}$, let $(m)_t := \sum_{j \in J_{0,m-1}} t^j$ and $(m)_t! := \prod_{j \in J_{1,m}} (j)_t$, where $(0)_t := 0$ and $(0)_t! := 1$. For $m \in \mathbb{Z}_{\geq 0}$ and $n \in J_{1,m-1}$, let $\binom{m}{0}_t := \binom{m}{m}_t := 1$ and $\binom{m}{n}_t := \binom{m-1}{n-1}_t + t^n \binom{m-1}{n}_t$. Then $\binom{m}{n}_t (n)_t! (m-n)_t! = (m)_t!$ and $\binom{m}{n}_t = t^{m-n} \binom{m-1}{n-1}_t + t^n \binom{m-1}{n}_t$.

For $m \in \mathbb{N}$, let $\mathbb{K}_m^\times := \{ r \in \mathbb{K}^\times \mid r^m = 1, r^t \neq 1 \ (t \in J_{1,m-1}) \}$. Let $\mathbb{K}_\infty^\times := \mathbb{K}^\times \setminus \bigcup_{m \in \mathbb{N}} \mathbb{K}_m^\times$.

For an associative $\mathbb{K}$-algebra $\mathfrak{a}$ and $X, Y \in \mathfrak{a}$, let $[X, Y] := XY - YX$.

Let $\uplus$ mean disjoint union of sets.

For $\mathbb{Z}$-modules $\mathfrak{b}$ and $\mathfrak{c}$, let $\mathrm{Hom}_{\mathbb{Z}}(\mathfrak{b}, \mathfrak{c})$ be the $\mathbb{Z}$-module formed by the $\mathbb{Z}$-module homomorphisms from $\mathfrak{b}$ to $\mathfrak{c}$.

The symbols δ_{ij}, $\delta_{i,j}$, and $\delta(i,j)$ denote Kronecker's delta.

§1.2. Modification of axioms of generalized root systems

Recall (1.1). We call an $N \times N$-matrix $C = [c_{ij}]_{i,j \in I}$ over $\mathbb{Z}$ a *generalized Cartan matrix* if:

(M1) $c_{ii} = 2$ $(i \in I)$.
(M2) $c_{jk} \leq 0$, $\delta(c_{jk}, 0) = \delta(c_{kj}, 0)$ $(j, k \in I,\ j \neq k)$.

Let $\mathcal{A}$ be a non-empty set. Let $\tau_i : \mathcal{A} \to \mathcal{A}$ be maps $(i \in I)$. Let $C^a = [c_{ij}^a]_{i,j \in I}$ be generalized Cartan matrices $(a \in \mathcal{A})$. We call the data

$$\mathcal{C} = \mathcal{C}(I, \mathcal{A}, (\tau_i)_{i \in I}, (C^a)_{a \in \mathcal{A}})$$

a (rank-N) Cartan scheme if:

(C1) $\tau_i^2 = \mathrm{id}_{\mathcal{A}}$ $(i \in I)$.
(C2) $c_{ij}^{\tau_i(a)} = c_{ij}^a$ $(i \in I)$.

Let $\mathcal{C} = \mathcal{C}(I, \mathcal{A}, (\tau_i)_{i \in I}, (C^a)_{a \in \mathcal{A}})$ be a Cartan scheme. Define $s_i^a \in \mathrm{GL}(\mathbb{V})$ $(a \in \mathcal{A},\ i \in I)$ by

$$s_i^a(\alpha_j) = \alpha_j - c_{ij}^a \alpha_i \quad (j \in I). \tag{1.2}$$

Then

$$(s_i^a)^2 = s_i^{\tau_i(a)} s_i^a = \mathrm{id}_{\mathbb{V}} \quad (a \in \mathcal{A},\ i \in I). \tag{1.3}$$

Notation 1.1. Let $\mathcal{C} = \mathcal{C}(I, \mathcal{A}, (\tau_i)_{i\in I}, (C^a)_{a\in\mathcal{A}})$ be a Cartan scheme.

(1) For $a \in \mathcal{A}$ and $f \in \mathrm{Map}^I_n$ for some $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, let

$$\begin{aligned} & a_{f,0} := a, && 1^a s_{f,0} := \mathrm{id}_{\mathbb{V}}, \\ & a_{f,t} := \tau_{f(t)}(a_{f,t-1}), && 1^a s_{f,t} := 1^a s_{f,t-1} s^{a_{f,t}}_{f(t)} \quad (t \in J_{1,n}). \end{aligned} \tag{1.4}$$

(2) For $a, a' \in \mathcal{A}$, let

$$\mathcal{H}(a, a') := \{1^a s_{f,t} \mid f \in \mathrm{Map}^I_\infty,\, t \in \mathbb{Z}_{\geq 0},\, a_{f,t} = a'\} \subset \mathrm{GL}(\mathbb{V}). \tag{1.5}$$

We say that a Cartan scheme $\mathcal{C} = \mathcal{C}(I, \mathcal{A}, (\tau_i)_{i\in I}, (C^a)_{a\in\mathcal{A}})$ is *connected* if $|\mathcal{H}(a, a')| \geq 1$ for all $a,\, a' \in \mathcal{A}$.

Definition 1.2. Let $\mathcal{C} = \mathcal{C}(I, \mathcal{A}, (\tau_i)_{i\in I}, (C^a)_{a\in\mathcal{A}})$ be a Cartan scheme. For each $a \in \mathcal{A}$, let $R(a)$ be a subset of $\mathbb{V} = \bigoplus_{i\in I} \mathbb{R}\alpha_i$, and $R^+(a) := R(a) \cap \mathbb{Z}_{\geq 0}\Pi$. We call the data

$$\mathcal{R} = \mathcal{R}(\mathcal{C}, (R(a))_{a\in\mathcal{A}})$$

a *generalized root system of type* $\mathcal{C}$ if:

(R1) $R(a) = R^+(a) \cup -R^+(a)$ $(a \in \mathcal{A})$.
(R2) $R(a) \cap \mathbb{Z}\alpha_i = \{\alpha_i,\, -\alpha_i\}$ $(a \in \mathcal{A},\, i \in I)$.
(R3) $s^a_i(R(a)) = R(\tau_i(a))$ $(a \in \mathcal{A},\, i \in I)$.
(R4) For $a, a' \in \mathcal{A}$, if $\mathrm{id}_{\mathbb{V}} \in \mathcal{H}(a, a')$, then $a = a'$.

Let $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R(a))_{a\in\mathcal{A}})$ be a generalized root system of type $\mathcal{C}$. By (R1)–(R3) and the definition of s^a_i, we have

$$s^a_i(R^+(a) \setminus \{\alpha_i\}) = R^+(\tau_i(a)) \setminus \{\alpha_i\}, \tag{1.6}$$

and

$$-c^a_{ij} = \max\{k \in \mathbb{Z}_{\geq 0} \mid \alpha_j + k\alpha_i \in R^+(a)\} \quad (i, j \in I,\, i \neq j). \tag{1.7}$$

If $\mathcal{C}$ is connected, we say that $\mathcal{R}$ is *connected*.

Lemma 1.3. *Let $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R(a))_{a\in\mathcal{A}})$ and $\mathcal{R}' = \mathcal{R}(\mathcal{C}', (R'(a'))_{a'\in\mathcal{A}'})$ be generalized root systems of types $\mathcal{C}$ and $\mathcal{C}'$ respectively. Let $\check{a} \in \mathcal{A}$ and $\check{a}' \in \mathcal{A}'$. Assume that $R(\check{a}) = R'(\check{a}')$. Then*

$$R(\check{a}_{f,n}) = R'(\check{a}'_{f,n}), \quad s^{\check{a}_{f,n}}_i = s^{\check{a}'_{f,n}}_i \quad (n \in \mathbb{N},\, f \in \mathrm{Map}^I_n,\, i \in I).$$

Proof. This lemma follows easily from (1.3), (1.7) and (R3). □

Definition 1.4. Let $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R(a))_{a\in\mathcal{A}})$ and $\mathcal{R}' = \mathcal{R}(\mathcal{C}', (R'(a'))_{a'\in\mathcal{A}'})$ be connected generalized root systems of types $\mathcal{C}$ and $\mathcal{C}'$ respectively. Let $\check{a} \in \mathcal{A}$ and $\check{a}' \in \mathcal{A}'$.

(1) We say that the pair $(\mathcal{R}, \check{a})$ is *quasi-isomorphic* to $(\mathcal{R}', \check{a}')$ if $R(\check{a}) = R'(\check{a}')$.

(2) We say that $(\mathcal{R}, \check{a})$ is *isomorphic* to $(\mathcal{R}', \check{a}')$ if $R(\check{a}) = R'(\check{a}')$ and for any $n \in \mathbb{N}$ and any $f \in \mathrm{Map}^I_n$, we have $\check{a}_{f,n} = \check{a}$ if and only if $\check{a}'_{f,n} = \check{a}'$.

Lemma 1.5. *Let $\mathcal{C} = \mathcal{C}(I, \mathcal{A}, (\tau_i)_{i\in I}, (C^a)_{a\in\mathcal{A}})$ be a Cartan scheme. Let $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R(a))_{a\in\mathcal{A}})$ be a generalized root system of type $\mathcal{C}$. Let $a \in \mathcal{A}$ and $i, j \in I$ with $i \neq j$. Let $m := |R^+(a) \cap (\mathbb{R}\alpha_i \oplus \mathbb{R}\alpha_j)| \in J_{2,\infty} \cup \{\infty\}$. Assume $m < \infty$. Define $f \in \mathrm{Map}^I_{2m}$ by $f(2x-1) := i$ and $f(2x) := j$ $(x \in J_{1,m})$. Then*

(R4)′ $a_{f,2m} = a$ *and* $1^a s_{f,2m} = \mathrm{id}_{\mathbb{V}}$.

Proof. For $x \in J_{1,m}$, let $\beta_x := 1^a s_{f,x-1}(\alpha_{f(x)})$. For $x \in J_{0,m}$, let $Z_x := R^+(a_{f,x}) \cap (\mathbb{R}\alpha_i \oplus \mathbb{R}\alpha_j)$ and $Y_x := Z_0 \cap -1^a s_{f,x}(Z_x)$. By (1.2) and (1.6),

$$|Z_x| = m \qquad (x \in J_{0,m}). \tag{1.8}$$

We show that for $x \in J_{1,m}$,

$(*)_x$ $|Y_x| = x$ and $Y_x = \{\beta_y \,|\, y \in J_{1,x}\}$.

Then $(*)_1$ follows from (1.6). Assume that $x \in J_{2,m}$ and $(*)_{x-1}$ holds. Then

$$\begin{aligned} Y_x &= Z_0 \cap (-1^a s_{f,x}(Z_x \setminus \{\alpha_{f(x)}\}) \cup \{-1^a s_{f,x}(\alpha_{f(x)})\}) \\ &= Z_0 \cap (-1^a s_{f,x-1}(Z_{x-1} \setminus \{\alpha_{f(x)}\}) \cup \{\beta_x\}) \quad \text{(by (1.2) and (1.6))} \\ &= (Y_{x-1} \setminus \{-\beta_x\}) \cup (Z_0 \cap \{\beta_x\}). \end{aligned} \tag{1.9}$$

Since $-1^a s_{f,x}(Z_x \setminus \{\alpha_{f(x)}\}) \cup \{-1^a s_{f,x}(\alpha_{f(x)})\} = \emptyset$, we have $(Y_{x-1} \setminus \{-\beta_x\}) \cap (Z_0 \cap \{\beta_x\}) = \emptyset$. Hence, by (1.9),

$$Y_x = (Y_{x-1} \setminus \{-\beta_x\}) \uplus (Z_0 \cap \{\beta_x\}). \tag{1.10}$$

Assume that $\beta_x \notin Z_0$. Then $\beta_x \in -Z_0$, so $1^a s_{f,x-1}(\alpha_{f(x)}) = \beta_x \in -Z_0$. Since $\beta_{x-1} \in Z_0$, $1^a s_{f,x-1}(\alpha_{f(x-1)}) = -1^a s_{f,x-2}(\alpha_{f(x-1)}) = -\beta_{x-1} \in -Z_0$. Since $\{f(x-1), f(x)\} = \{i, j\}$, we have $Y_{x-1} = Z_0$. Hence $x - 1 = m$, a contradiction. So $\beta_x \in Z_0$. From (1.10), we obtain $(*)_x$.

By (1.8) and $(*)_m$, $1^a s_{f,m}(Z_m) = -Z_0$. Hence we have $1^a s_{f,m}(\{\alpha_i, \alpha_j\}) = \{-\alpha_i, -\alpha_j\}$. By the same argument, letting $f' \in \mathrm{Map}^I_m$ by $f'(y) := f(m+y)$, we have $1^{a_{f,m}} s_{f',m}(\{\alpha_i, \alpha_j\}) = \{-\alpha_i, -\alpha_j\}$. Hence $1^a s_{f,2m}(\{\alpha_i, \alpha_j\}) = \{\alpha_i, \alpha_j\}$. By (1.2), the determinant of the 2×2-matrix $(s^{a_{f,x}}_{f(x)})_{|\mathbb{R}\alpha_i \oplus \mathbb{R}\alpha_j}$ is -1 for every $x \in J_{1,2m}$. So $1^a s_{f,2m}(\alpha_k) = \alpha_k$ for $k \in \{i, j\}$. By (1.2), for $k \in I \setminus \{i, j\}$,

$1^a s_{f,2m}(\alpha_k) \in \alpha_k + (\mathbb{Z}_{\geq 0}\alpha_i \oplus \mathbb{Z}_{\geq 0}\alpha_j)$. From (R1), we obtain the second claim of (R4)′. From (R4), we obtain the first claim of (R4)′. □

Remark 1.6. The original definition of generalized root systems was given in terms of (R1)–(R3), (R4)′ (see [13], [6]). From [13, Lemma 8(iii)] and Lemma 1.5, it follows that the definition based on (R1)–(R4) is equivalent to the one in terms of (R1)–(R3), (R4)′.

Definition 1.7. Let $\mathcal{C} = \mathcal{C}(I, \mathcal{A}, (\tau_i)_{i\in I}, (C^a)_{a\in\mathcal{A}})$ be a Cartan scheme. Let $\mathcal{W}(\mathcal{C})$ be the category defined by:

(cat1) $\mathrm{Ob}(\mathcal{W}(\mathcal{C})) = \mathcal{A}$.

(cat2) For $a, a' \in \mathcal{A}$,

$$\mathrm{Hom}_{\mathcal{W}(\mathcal{C})}(a, a') := \{(a, w, a') \mid w \in \mathcal{H}(a, a')\} \subset \mathcal{A} \times \mathrm{GL}(\mathbb{V}) \times \mathcal{A}.$$

(cat3) For $a, a', a'' \in \mathcal{A}$, the composition

$$\mathrm{Hom}_{\mathcal{W}(\mathcal{C})}(a, a') \times \mathrm{Hom}_{\mathcal{W}(\mathcal{C})}(a', a'') \to \mathrm{Hom}_{\mathcal{W}(\mathcal{C})}(a, a'')$$

is defined by

$$(a, w, a') \circ (a', w', a'') := (a, ww', a''),$$

where ww' means the product in the group $\mathrm{GL}(\mathbb{V})$.

We call $\mathcal{W}(\mathcal{C})$ the *Weyl groupoid* of $\mathcal{C}$. If $\mathcal{R}$ is a generalized root system of type $\mathcal{C}$, we let $\mathcal{W}(\mathcal{R}) := \mathcal{W}(\mathcal{C})$ and call it the *Weyl groupoid* of $\mathcal{R}$.

§1.3. Length function of a Weyl groupoid

In Subsections 1.3–1.5, we fix a Cartan scheme $\mathcal{C} = \mathcal{C}(I, \mathcal{A}, (\tau_i)_{i\in I}, (C^{a'})_{a'\in\mathcal{A}})$, a generalized root system $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R(a'))_{a'\in\mathcal{A}})$ of type $\mathcal{C}$, and $a \in \mathcal{A}$.

Let $\mathcal{H}(a, -) := \bigcup_{a'\in\mathcal{A}} \mathcal{H}(a, a')$. For $w \in \mathcal{H}(a, -)$, let

$$\mathbb{L}_a(w) := \{\beta \in R^+(a) \mid w^{-1}(\beta) \in -\mathbb{Z}_{\geq 0}\Pi\}. \tag{1.11}$$

Define a map $\ell_a : \mathcal{H}(a, -) \to \mathbb{Z}_{\geq 0}$ by

$$\ell_a(w) := |\mathbb{L}_a(w)|. \tag{1.12}$$

Lemma 1.8. (1) *Let $w \in \mathcal{H}(a, -)$. Then the following conditions are equivalent:*

(1-i) $\mathbb{L}_a(w) = R^+(a)$.

(1-ii) $w^{-1}(\Pi) \subseteq -\mathbb{Z}_{\geq 0}\Pi \setminus \{0\}$, *i.e.*, $\Pi \subseteq \mathbb{L}_a(w)$.

(1-iii) $w(\Pi) = -\Pi$.

(2) *For* $a' \in \mathcal{A}$ *and* $w \in \mathcal{H}(a,a')$*, we have*

$$\mathbb{L}_a(w) = \{\beta \in R^+(a) \,|\, w^{-1}(\beta) \in -R^+(a')\}.$$

(3) *Let* $a', a'' \in \mathcal{A}$*,* $w \in \mathcal{H}(a,a')$ *and* $w' \in \mathcal{H}(a,a'')$*. If* $w = w'$*, then* $a' = a''$*.*

(4) *For* $w \in \mathcal{H}(a,-)$*, we have*

$$\ell_a(w) = \min\{l \in \mathbb{Z}_{\geq 0} \mid \exists f \in \mathrm{Map}_l^I,\ 1^a s_{f,l} = w\}. \tag{1.13}$$

(5) *Let* $a' \in \mathcal{A}$*. For* $w \in \mathcal{H}(a,a')$ *and* $i \in I$*, we have*

$$\ell_a(w s_i^{\tau_i a'}) = \begin{cases} \ell_a(w)+1 & \text{if } w(\alpha_i) \in R^+(a), \\ \ell_a(w)-1 & \text{if } w(\alpha_i) \in -R^+(a). \end{cases} \tag{1.14}$$

(6) *Let* $w \in \mathcal{H}(a,-)$ *and* $l := \ell_a(w)$*. Let* $f \in \mathrm{Map}_l^I$ *be such that* $w = 1^a s_{f,l}$*. Then*

$$\mathbb{L}_a(w) = \{1^a s_{f,r-1}(\alpha_{f(r)}) \,|\, r \in J_{1,l}\}. \tag{1.15}$$

Proof. (1) is clear from (R1), (R2) of Definition 1.2 and (1.11); (2) follows from (R1); (3) follows from (R4); (4) (resp. (5), (6)) follows from Definition 1.2, Lemma 1.5 and [13, Lemma 8(iii)] (resp. [13, Corollary 3], [13, Corollary 2]). □

§1.4. Longest elements of a finite Weyl groupoid

Lemma 1.9 ([13, Corollary 5]). *Assume* $|R^+(a)| < \infty$*, and set* $n := |R^+(a)|$*. Then:*

(1) *There exists a unique* $1^a w_0 \in \mathcal{H}(a,-)$ *such that* $\ell_a(1^a w_0) = n$*.*

(2) *For* $w \in \mathcal{H}(a,-)$*,*

$$w = 1^a w_0 \text{ if and only if } w(\Pi) = -\Pi. \tag{1.16}$$

(3) *For* $a' \in \mathcal{A}$ *and* $w \in \mathcal{H}(a,a')$*, we have* $n = \ell_a(w) + \ell_{a'}(w^{-1}1^a w_0)$*.*

Proof. Let $a \in \mathcal{A}$. Let $w \in \mathcal{H}(a,a')$. Assume $\ell_a(w) < n$. By (1.13), we have $\ell_{a'}(w^{-1}) = \ell_a(w)$. By (R3) of Definition 1.2 and Lemma 1.8(1), there exists $i \in I$ such that $w(\alpha_i) \in R^+(a)$. By (1.14), $\ell_a(w s_i^{\tau_i a'}) = \ell_a(w)+1$. Thus we get the existence of $1^a w_0$. Let $w', w'' \in \mathcal{H}(a,-)$ be such that $\ell_a(w') = \ell_a(w'') = n$. By Lemma 1.8(1), we see that $w'(\Pi) = w''(\Pi) = -\Pi$. Since $(w'')^{-1}w'(\Pi) = \Pi$, by (1.13), we have $(w'')^{-1}w' = \mathrm{id}_{\mathbb{V}}$. Hence $w' = w''$. This yields the uniqueness of $1^a w_0$. Thus we obtain claims (1) and (2).

Let $w_1 \in \mathcal{H}(a,a')$. By (1.11), $\ell_a(w_1) \leq n$. Assume $w_1 \neq 1^a w_0$. By claim (1), $\ell_a(w_1) < n$. By an argument as above, there exists $w_2 \in \mathcal{H}(a',-)$ such that $\ell_a(w_1 w_2) = n$ and $\ell_{a'}(w_2) \leq n - \ell_a(w_1)$. By (1.13), $\ell_{a'}(w_2) = n - \ell_a(w_1)$ since $\ell_a(w_1 w_2) = n$. By (1), $w_1 w_2 = 1^a w_0$. Thus we obtain (3). □

If $|R^+(a)| < \infty$, we call the only element $1^a w_0$ (or more precisely, the pair $(a, 1^a w_0)$) as in Lemma 1.9(1) the *longest element ending with* a.

Lemma 1.10 (see [6, Proposition 2.12]). *Assume* $|R^+(a)| < \infty$*, and set* $n := |R^+(a)|$*. Let* $f \in \mathrm{Map}_n^I$ *be such that* $1^a s_{f,n} = 1^a w_0$*. Then*

$$R^+(a) = \{1^a s_{f,r-1}(\alpha_{f(r)}) \,|\, r \in J_{1,n}\}. \tag{1.17}$$

In particular,

$$R(a) = \bigcup_{k=0}^{\infty} \bigcup_{f' \in \mathrm{Map}_k^I} 1^a s_{f',k}(\Pi)\,. \tag{1.18}$$

Proof. Equation (1.17) follows from (1.12), Lemma 1.9(1), and (1.15). Equation (1.18) is clear from (1.17). □

Lemma 1.11. *Let* $n \in \mathbb{N}$*,* $f \in \mathrm{Map}_n^I$ *and* $X := \{1^a s_{f,r-1}(\alpha_{f(r)}) \mid r \in J_{1,n}\}$ $(\subset R(a))$*. Assume*

$$\Pi \subseteq X \subseteq \mathbb{Z}_{\geq 0}\Pi. \tag{1.19}$$

Then $n = |R^+(a)|$*,* $1^a s_{f,n} = 1^a w_0$ *and* $R^+(a) = X$*.*

Proof. Let $w := 1^a s_{f,n}$. It follows from (1.12), (R3) of Definition 1.2 and (1.14) that $\ell_a(w) = n$. By (1.15), $X = \mathbb{L}_a(w)$. Hence $\Pi \subseteq \mathbb{L}_a(w)$. By Lemma 1.8(1), $X = R^+(a)$. Since $X = \mathbb{L}_a(w)$, we have $|X| = \ell_a(w) = n$ by (1.12). By Lemma 1.9(1), we have $w = 1^a w_0$. □

§1.5. A technical fact

By Lemmas 1.3 and 1.11, we have

Lemma 1.12. *Keep the notation of Definition 1.4. Assume* $|R^+(\check{a})| < \infty$*. Then* $(\mathcal{R}, \check{a})$ *is quasi-isomorphic to* $(\mathcal{R}', \check{a}')$ *if and only if*

$$s_i^{\check{a}_{f,n}} = s_i^{\check{a}'_{f,n}} \qquad (n \in \mathbb{Z}_{\geq 0},\, f \in \mathrm{Map}_n^I,\, i \in I).$$

In particular, $1^{\check{a}} w_0 = 1^{\check{a}'} w_0$ *and* $R^+(\check{a}) = (R')^+(\check{a}')$*.*

§2. Longest elements of finite Weyl groups

§2.1. Root systems of types A–G

In this section, we consider some longest elements of finite Weyl groups, or crystallographic finite Coxeter groups, which will be used to study χ treated in Theorem 5.10(2) below.

Let $\hat{N} \in \mathbb{N}$. Let $\mathbb{R}^{\hat{N}}$ denote the $\hat{N}$-dimensional $\mathbb{R}$-linear space of $\hat{N}$-tuple column vectors, that is, $\mathbb{R}^{\hat{N}} = \{{}^t[x_1, \ldots, x_{\hat{N}}] \,|\, x_i \in \mathbb{R}\,(i \in J_{1,\hat{N}})\}$. Let $\{e_i \mid i \in J_{1,\hat{N}}\}$ be the standard $\mathbb{R}$-basis of $\mathbb{R}^{\hat{N}}$. For $m \in J_{1,\hat{N}}$, we regard $\mathbb{R}^m$ as the $\mathbb{R}$-linear subspace $\bigoplus_{r=1}^m \mathbb{R}e_r$ of $\mathbb{R}^{\hat{N}}$. For a subset X of $J_{1,\hat{N}}$, define an $\mathbb{R}$-linear map $P_X : \mathbb{R}^{\hat{N}} \to \mathbb{R}^{\hat{N}}$ by $P_X(e_i) := e_i$ $(i \in X)$ and $P_X(e_j) := 0$ $(j \in J_{1,\hat{N}} \setminus X)$. Let $\mathrm{M}_{\hat{N}}(\mathbb{R})$ be the $\mathbb{R}$-algebra of $\hat{N} \times \hat{N}$-matrices. Let $\mathrm{GL}_{\hat{N}}(\mathbb{R})$ be the group of invertible $\hat{N} \times \hat{N}$-matrices. Let $\hat{\eta} : \mathbb{R}^{\hat{N}} \times \mathbb{R}^{\hat{N}} \to \mathbb{R}$ be the $\mathbb{R}$-bilinear map defined by $\hat{\eta}(e_k, e_r) := \delta_{kr}$. For $v \in \mathbb{R}^{\hat{N}} \setminus \{0\}$, define $\hat{s}_v \in \mathrm{GL}_{\hat{N}}(\mathbb{R})$ by $\hat{s}_v(u) := u - \frac{2\hat{\eta}(u,v)}{\hat{\eta}(v,v)}v$ $(u \in \mathbb{R}^{\hat{N}})$, that is, $\hat{s}_v$ is the *reflection* with respect to v. Note that

$$\hat{s}_v^2 = \mathrm{id}_{\mathbb{R}^{\hat{N}}} \quad (v \in \mathbb{R}^{\hat{N}} \setminus \{0\}), \tag{2.1}$$

and

$$\hat{\eta}(\hat{s}_v(u), \hat{s}_v(u')) = \hat{\eta}(u, u') \quad (v \in \mathbb{R}^{\hat{N}} \setminus \{0\},\, u, u' \in \mathbb{R}^{\hat{N}}). \tag{2.2}$$

Using (2.1) and (2.2), we have

$$\hat{s}_v \hat{s}_{v'} \hat{s}_v = \hat{s}_{\hat{s}_v(v')} \quad (v, v' \in \mathbb{R}^{\hat{N}} \setminus \{0\}). \tag{2.3}$$

We say that a finite subset $\hat{R}$ of $\mathbb{R}^{\hat{N}} \setminus \{0\}$ is a *crystallographic root system* (in $\mathbb{R}^{\hat{N}}$) if $|\hat{R}| < \infty$, $\hat{s}_v(\hat{R}) = \hat{R}$, $\mathbb{R}v \cap \hat{R} = \{v, -v\}$ for all $v \in \hat{R}$, and $2\hat{\eta}(v', v'')/\hat{\eta}(v', v') \in \mathbb{Z}$ for all $v', v'' \in \hat{R}$ (see [15, 1.2, 2.9]).

Let $\hat{R}$ be a crystallographic root system in $\mathbb{R}^{\hat{N}}$. We call $\hat{R}$ *irreducible* if for all $\hat{\beta}, \hat{\beta}' \in \hat{R}$, there exist $r \in \mathbb{N}$ and $\hat{\beta}_t \in \hat{R}$ $(t \in J_{1,r})$ such that $\hat{\eta}(\hat{\beta}, \hat{\beta}_1) \neq 0$, $\hat{\eta}(\hat{\beta}_t, \hat{\beta}_{t+1}) \neq 0$ $(t \in J_{1,r-1})$ and $\hat{\eta}(\hat{\beta}_r, \hat{\beta}') \neq 0$ (see [15, 2.2] and (2.4)). We say that a subset $\hat{\Pi}$ of $\hat{R}$ is a *root basis* of $\hat{R}$ if $\hat{\Pi}$ is a (set) $\mathbb{R}$-basis of $\mathrm{Span}_{\mathbb{R}}(\hat{\Pi})$ and $\hat{R} \subset \mathrm{Span}_{\mathbb{Z}_{\geq 0}}(\hat{\Pi}) \cup -\mathrm{Span}_{\mathbb{Z}_{\geq 0}}(\hat{\Pi})$ (this is called a *simple system* in [15, 1.3, 2.9]).

Let $\hat{\Pi}$ be a root basis of $\hat{R}$. We call $\dim_{\mathbb{R}} \mathrm{Span}_{\mathbb{R}}(\hat{\Pi}) = |\hat{\Pi}|$ the *rank* of $\hat{R}$. Let $\hat{W}(\hat{\Pi})$ be the subgroup of $\mathrm{GL}_{\hat{N}}(\mathbb{R})$ generated by all $\hat{s}_v$ with $v \in \hat{\Pi}$. By [15, Corollary 1.5], we have

$$\hat{R} = \hat{W}(\hat{\Pi}) \cdot \hat{\Pi}. \tag{2.4}$$

We call $\hat{W}(\hat{\Pi})$ the *Coxeter group associated with* $(\hat{R}, \hat{\Pi})$. Let $\hat{S}(\hat{\Pi}) := \{\hat{s}_v \in \hat{W}(\hat{\Pi}) \,|\, v \in \hat{\Pi}\}$. We call $(\hat{W}(\hat{\Pi}), \hat{S}(\hat{\Pi}))$ the *Coxeter system associated with* $(\hat{R}, \hat{\Pi})$ (see [15, 1.9 and Theorem 1.5]). Let $\hat{\Pi}$ be a root basis of $\hat{R}$. Let $\hat{R}^+(\hat{\Pi}) := \hat{R} \cap \mathrm{Span}_{\mathbb{Z}_{\geq 0}}(\hat{\Pi})$. We call $\hat{R}^+(\hat{\Pi})$ a *positive root system of* $\hat{R}$ *associated with* $\hat{\Pi}$ (this is called a *positive system* in [15, 1.3]).

Definition 2.1 (see [15, 2.10]). Recall N and $I = J_{1,N}$ from (1.1). Let $\hat{N} \in J_{N,\infty}$. Let $\hat{R}$ be a rank-N crystallographic root system in $\mathbb{R}^{\hat{N}}$. Let $\hat{\Pi} = \{\hat{\alpha}_i \mid i \in I\}$ be a root basis of $\hat{R}$. Let $\langle\hat{\Pi}\rangle := (\hat{\alpha}_1, \ldots, \hat{\alpha}_N) \in \mathbb{R}^{\hat{N}} \times \cdots \times \mathbb{R}^{\hat{N}}$ (N times), so $\langle\hat{\Pi}\rangle$ is an ordered $\mathbb{R}$-basis of $\mathrm{Span}_{\mathbb{R}}(\hat{R})$.

(1) Assume that $N \geq 1$ and $\hat{N} = N+1$. We call $\hat{R}$ the A_N-*type standard root system* if

$$\hat{R} = \{e_x - e_y \mid x, y \in J_{1,N+1},\, x \neq y\}. \tag{2.5}$$

We call $\langle\hat{\Pi}\rangle$ the A_N-*data* if $\hat{\alpha}_i = e_i - e_{i+1}$ $(i \in I)$.

(2) Assume $N = \hat{N} \geq 2$. We call $\hat{R}$ the B_N-*type standard root system* if

$$\begin{aligned} \hat{R} = {} & \{ce_x + c'e_y \mid x, y \in J_{1,N},\, x < y,\, c, c' \in \{1, -1\}\} \\ & \cup \{c''e_z \mid z \in J_{1,N},\, c'' \in \{1, -1\}\}. \end{aligned} \tag{2.6}$$

We call $\langle\hat{\Pi}\rangle$ the B_N-*data* if $\hat{\alpha}_i = e_i - e_{i+1}$ $(i \in J_{1,N-1})$ and $\hat{\alpha}_N = e_N$.

(3) Assume $N = \hat{N} \geq 3$. We call $\hat{R}$ the C_N-*type standard root system* if

$$\begin{aligned} \hat{R} = {} & \{ce_x + c'e_y \mid x, y \in J_{1,N},\, x < y,\, c, c' \in \{1, -1\}\} \\ & \cup \{2c''e_z \mid z \in J_{1,N},\, c'' \in \{1, -1\}\}. \end{aligned} \tag{2.7}$$

We call $\langle\hat{\Pi}\rangle$ the C_N-*data* if $\hat{\alpha}_i = e_i - e_{i+1}$ $(i \in J_{1,N-1})$ and $\hat{\alpha}_N = 2e_N$.

(4) Assume $N = \hat{N} \geq 4$. We call $\hat{R}$ the D_N-*type standard root system* if

$$\hat{R} = \{ce_x + c'e_y \mid x, y \in J_{1,N},\, x < y,\, c, c' \in \{1, -1\}\}. \tag{2.8}$$

We call $\langle\hat{\Pi}\rangle$ the D_N-*data* if $\hat{\alpha}_i = e_i - e_{i+1}$ $(i \in J_{1,N-1})$ and $\hat{\alpha}_N = e_{N-1} + e_N$.

(5) Assume that $N = 6$ and $\hat{N} = 8$. We call $\hat{R}$ the E_6-*type standard root system* if

$$\begin{aligned} \hat{R} = {} & \{ce_x + c'e_y \mid x, y \in J_{1,5},\, x < y,\, c, c' \in \{1, -1\}\} \\ & \cup \{\tfrac{1}{2}((\textstyle\sum_{r=1}^{5} c_r e_r) + (\prod_{k=1}^{5} c_k)(e_6 - e_7 - e_8)) \mid c_r \in \{1, -1\}\ (r \in J_{1,5})\}. \end{aligned}$$

We call $\langle\hat{\Pi}\rangle$ the E_6-*data* if $\hat{\alpha}_1 = \frac{1}{2}(e_1 + e_8 - \sum_{r=2}^{7} e_r))$, $\hat{\alpha}_2 = e_1 + e_2$ and $\hat{\alpha}_i = e_{i-1} - e_{i-2}$ $(i \in J_{3,6})$.

(6) Assume that $N = 7$ and $\hat{N} = 8$. We call $\hat{R}$ the E_7-*data* if

$$\begin{aligned} \hat{R} = {} & \{ce_x + c'e_y \mid x, y \in J_{1,6},\, x < y,\, c, c' \in \{1, -1\}\} \\ & \cup \{c''(e_7 - e_8) \mid c'' \in \{1, -1\}\} \\ & \cup \{\tfrac{1}{2}((\textstyle\sum_{r=1}^{6} c_r e_r) - (\prod_{k=1}^{6} c_k)(e_7 - e_8)) \mid c_r \in \{1, -1\}\, (r \in J_{1,6})\}. \end{aligned}$$

We call $\langle\hat{\Pi}\rangle$ the E_7-*data* if $\hat{\alpha}_1 = \frac{1}{2}(e_1 + e_8 - \sum_{r=2}^{7} e_r)$, $\hat{\alpha}_2 = e_1 + e_2$ and $\hat{\alpha}_i = e_{i-1} - e_{i-2}$ $(i \in J_{3,7})$.

(7) Assume $N = \hat{N} = 8$. We call $\hat{R}$ the E_8-*type standard root system* if

$$\hat{R} = \{ce_x + c'e_y \mid x, y \in J_{1,8},\ x < y,\ c, c' \in \{1, -1\}\} \\ \cup \{\tfrac{1}{2}((\textstyle\sum_{r=1}^{7} c_r e_r) + (\prod_{k=1}^{7} c_k)e_8) \mid c_r \in \{1, -1\}\,(r \in J_{1,7})\}.$$

We call $\langle\hat{\Pi}\rangle$ the E_8-*data* if $\hat{\alpha}_1 = \frac{1}{2}(e_1 + e_8 - \sum_{r=2}^{7} e_r))$, $\hat{\alpha}_2 = e_1 + e_2$ and $\hat{\alpha}_i = e_{i-1} - e_{i-2}$ $(i \in J_{3,8})$.

(8) Assume $N = \hat{N} = 4$. We call $\hat{R}$ the F_4-*type standard root system* if

$$\hat{R} = \{ce_x + c'e_y \mid x, y \in J_{1,4},\ x < y,\ c, c' \in \{1, -1\}\} \\ \cup \{c''e_z \mid z \in J_{1,4},\ c'' \in \{1, -1\}\} \\ \cup \{\tfrac{1}{2}\textstyle\sum_{r=1}^{4} c_r e_r \mid c_r \in \{1, -1\}\,(r \in J_{1,4})\}.$$

We call $\langle\hat{\Pi}\rangle$ the F_4-*data* if $\hat{\alpha}_1 = e_2 - e_3$, $\hat{\alpha}_2 = e_3 - e_4$, $\hat{\alpha}_3 = e_4$ and $\hat{\alpha}_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$.

(9) Assume that $N = 2$ and $\hat{N} = 3$. We call $\hat{R}$ the G_2-*type standard root system* if

$$\hat{R} = \{c(e_x - e_y) \mid x, y \in J_{1,3},\ x < y,\ c \in \{1, -1\}\} \\ \cup \{c'(2e_{z_1} - e_{z_2} - e_{z_3}) \mid \{z_1, z_2, z_3\} = J_{1,3},\ c' \in \{1, -1\}\}.$$

We call $\langle\hat{\Pi}\rangle$ the G_2-*data* if $\hat{\alpha}_1 = e_1 - e_2$ and $\hat{\alpha}_2 = -2e_1 + e_2 + e_3$.

(10) Let $\hat{R}$ and $\hat{\Pi}$ be any of those in (1)–(9). We call $\hat{R}$ a *rank-N standard irreducible root system*. We call $\langle\hat{\Pi}\rangle$ a *rank-N Cartan data.*

It is well-known that rank-N irreducible crystallographic root systems are isomorphic to rank-N standard irreducible root systems (cf. [15, 2.10]).

Definition 2.2. Let $\langle\hat{\Pi}\rangle$ be a rank-N Cartan data. Fix a set $\mathcal{A}_{\langle\hat{\Pi}\rangle}$ with $|\mathcal{A}_{\langle\hat{\Pi}\rangle}| = 1$. Let $\mathcal{A}_{\langle\hat{\Pi}\rangle} = \{a\}$. Let $R(a) := \xi_{\langle\hat{\Pi}\rangle}(\hat{R})$, where $\hat{R}$ is the rank-N root system corresponding to $\langle\hat{\Pi}\rangle$. Let $\hat{c}^a_{ij} := 2\hat{\eta}(\hat{\alpha}_i, \hat{\alpha}_j)/\hat{\eta}(\hat{\alpha}_i, \hat{\alpha}_i)$, and $\hat{C}^a := [\hat{c}^a_{ij}]_{ij \in I}$. Let $\hat{\tau}_i := \mathrm{id}_I$ $(i \in I)$. Let $\mathcal{C}_{\langle\hat{\Pi}\rangle} := \mathcal{C}(I, \mathcal{A}_{\langle\hat{\Pi}\rangle}, (\hat{\tau}_i)_{i \in I}, (\hat{C}^a)_{a \in \mathcal{A}_{\langle\hat{\Pi}\rangle}})$. Then $\mathcal{C}_{\langle\hat{\Pi}\rangle}$ is a Cartan scheme. Let $\mathcal{R}_{\langle\hat{\Pi}\rangle} := \mathcal{R}(\mathcal{C}_{\langle\hat{\Pi}\rangle}, (R(a))_{a \in \mathcal{A}_{\langle\hat{\Pi}\rangle}})$. We can see that $\mathcal{R}_{\langle\hat{\Pi}\rangle}$ is a generalized root system of type $\mathcal{C}_{\langle\hat{\Pi}\rangle}$, by (1.18), (2.4) and the equations

$$s^a_i = \xi_{\langle\hat{\Pi}\rangle} \circ \hat{s}_{\hat{\alpha}_i} \circ \xi^{-1}_{\langle\hat{\Pi}\rangle} \quad (a \in \mathcal{A}_{\langle\hat{\Pi}\rangle},\ i \in I). \tag{2.9}$$

Proposition 2.3. *The correspondence $\langle\hat{\Pi}\rangle \mapsto \mathcal{R}_{\langle\hat{\Pi}\rangle}$, from the set of all rank-N Cartan data to the family of all connected generalized root systems $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R(a))_{a \in \mathcal{A}})$ with $|\mathcal{A}| = 1$ and $|R(a)| < \infty$ $(a \in \mathcal{A})$ (and $|I| = N$), is injective. Moreover it is surjective up to isomorphisms in the sense of Definition 1.4(2).*

Proof. See [16, Proposition 4.9]. □

§2.2. Longest elements of irreducible Weyl groups

In this subsection, let $\langle\hat{\Pi}\rangle = (\hat{\alpha}_i \mid i \in I)$ be a rank-N Cartan data, and consider $\mathcal{R}_{\langle\hat{\Pi}\rangle} = \mathcal{R}(\mathcal{C}_{\langle\hat{\Pi}\rangle}, (R(a))_{a\in\mathcal{A}_{\langle\hat{\Pi}\rangle}})$ (see (2.9)). Let $a \in \mathcal{A}_{\langle\hat{\Pi}\rangle}$. Define a map $\hat{\ell} : \hat{W}(\hat{\Pi}) \to \mathbb{Z}_{\geq 0}$ in the following way (see [15, 1.6]). Let $\hat{\ell}(1) := 0$, where 1 is a unit of $\hat{W}(\hat{\Pi})$. Note that every $\hat{w} \in \hat{W}(\hat{\Pi})$ can be written as a product of finitely many $\hat{s}_{\hat{\beta}}$'s with $\hat{\beta} \in \hat{\Pi}$, say $\hat{w} = \hat{s}_{\hat{\beta}_1}\cdots\hat{s}_{\hat{\beta}_r}$ for some $r \in \mathbb{N}$ and some $\hat{\beta}_x \in \hat{\Pi}$ $(x \in J_{1,r})$. If $\hat{w} \neq 1$, let $\hat{\ell}(\hat{w})$ be the smallest r for which such an expression exists, and then call the expression *reduced*. By (1.13) and (2.9), we have $\hat{\ell}(\hat{w}) = \ell_a(\xi_{\langle\hat{\Pi}\rangle} \circ \hat{w} \circ \xi_{\langle\hat{\Pi}\rangle}^{-1})$. We call $\hat{\ell}(\hat{w})$ the *length of* $\hat{w}$. Let

$$\widehat{\mathbb{L}}(\hat{w}) := \{\hat{\beta} \in \hat{R}^+(\hat{\Pi}) \mid \hat{w}(\hat{\beta}) \in -\hat{R}^+(\hat{\Pi})\} \qquad (\hat{w} \in \hat{W}(\hat{\Pi})),$$

so $\widehat{\mathbb{L}}(\hat{w}) = \mathbb{L}_a(\xi_{\langle\hat{\Pi}\rangle} \circ \hat{w} \circ \xi_{\langle\hat{\Pi}\rangle}^{-1})$ by (1.11) and (2.9). By (1.12) and (2.9),

$$\hat{\ell}(\hat{w}) = |\widehat{\mathbb{L}}(\hat{w})| \tag{2.10}$$

(see also [15, Corollary 1.7]). By (1.6) and (2.9),

$$\hat{s}_{\hat{\alpha}}(\hat{R}^+(\hat{\Pi}) \setminus \{\hat{\alpha}\}) = \hat{R}^+(\hat{\Pi}) \setminus \{\hat{\alpha}\} \qquad (\hat{\alpha} \in \hat{\Pi}) \tag{2.11}$$

(see also [15, Proposition 1.4]). By (1.14) and (2.9),

$$\hat{\ell}(\hat{w}\hat{s}_{\hat{\alpha}}) = \begin{cases} \hat{\ell}(\hat{w}) + 1 & \text{if } \hat{w}(\hat{\alpha}) \in \hat{R}^+(\hat{\Pi}), \\ \hat{\ell}(\hat{w}) - 1 & \text{if } \hat{w}(\hat{\alpha}) \in -\hat{R}^+(\hat{\Pi}), \end{cases} \tag{2.12}$$

for $\hat{\alpha} \in \hat{\Pi}$ (see also [15, Lemma 1.6 and Corollary 1.7]).

Assume that $|\hat{R}| < \infty$. By the above properties, we can see that there exists a unique $\hat{w}_0 \in \hat{W}(\hat{\Pi})$ such that $\hat{w}_0(\hat{\Pi}) = -\hat{\Pi}$ (see [15, 1.8]). It is well-known that $\hat{\ell}(\hat{w}_0) = |\hat{R}^+(\hat{\Pi})|$, that $\hat{w}_0$ is the only element of $\hat{W}(\hat{\Pi})$ such that $\hat{\ell}(\hat{w}) \leq \hat{\ell}(\hat{w}_0)$ for all $\hat{w} \in \hat{W}(\hat{\Pi})$, and that

$$\hat{\ell}(\hat{w}) = \hat{\ell}(\hat{w}_0) - \hat{\ell}(\hat{w}_0\hat{w}^{-1}) \qquad \text{for all } \hat{w} \in \hat{W}(\hat{\Pi}). \tag{2.13}$$

We call $\hat{w}_0$ the *longest element of the Coxeter system* of $(\hat{W}(\hat{\Pi}), \hat{S}(\hat{\Pi}))$. Note that

$$\hat{w}_0 = \xi_{\langle\hat{\Pi}\rangle}^{-1} \circ 1^a w_0 \circ \xi_{\langle\hat{\Pi}\rangle} \qquad \text{and} \qquad \hat{\ell}(\hat{w}_0) = \ell_a(1^a w_0). \tag{2.14}$$

It is well-known that

$$\hat{\ell}(\hat{w}_0) = |\hat{R}^+(\hat{\Pi})| \tag{2.15}$$

(see also Lemma 1.9). Let $n := \hat{\ell}(\hat{w}_0)$, and let $\hat{s}_{\hat{\beta}_1}\cdots\hat{s}_{\hat{\beta}_n}$ be the reduced expression of $\hat{w}_0$, where $\hat{\beta}_k$'s are some elements of $\hat{\Pi}$. Then

$$\hat{R}^+(\hat{\Pi}) = \{\hat{s}_{\hat{\beta}_1}\cdots\hat{s}_{\hat{\beta}_{k-1}}(\hat{\beta}_k) \mid k \in J_{1,n}\} \tag{2.16}$$

(see also (1.17)).

Proposition 2.4. *Let* $\langle\hat{\Pi}\rangle = (\hat{\alpha}_1, \ldots, \hat{\alpha}_N)$ *be a rank-*N *Cartan data. Let* $\hat{\Pi} = \{\hat{\alpha}_i \mid i \in I\}$*. Let* $\hat{s}_i := \hat{s}_{\hat{\alpha}_i}$ $(i \in I)$*. Let* $\hat{w}_0$ *be the longest element of the Coxeter system* $(\hat{W}(\hat{\Pi}), \hat{S}(\hat{\Pi}))$*. Let* $h := 2|\hat{R}^+(\hat{\Pi})|/N$*. Let* $V := \mathrm{Span}_{\mathbb{R}}(\hat{\Pi})$*. Then:*

(1) $h \in \mathbb{N}$.

(2) *Assume that* $\langle\hat{\Pi}\rangle$ *is neither the* A_N*-data, the* D_N*-data, nor the* E_6*-data. Then there is no bijection* $u : I \to I$ *satisfying the condition that* $u \neq \mathrm{id}_I$ *and* $\hat{\eta}(\hat{\alpha}_i, \hat{\alpha}_j) = \hat{\eta}(\hat{\alpha}_{u(i)}, \hat{\alpha}_{u(j)})$ $(i, j \in I)$*. Moreover* $(\hat{w}_0)_{|V} = -\mathrm{id}_V$*. Furthermore* $h \in 2\mathbb{N}$*, and* $(\hat{s}_1 \cdots \hat{s}_N)^{h/2}$ *is a reduced expression of* $\hat{w}_0$*.*

(3) *Assume that* $\langle\hat{\Pi}\rangle$ *is the* A_N*-data. Then* $\hat{w}_0(e_x) = e_{\hat{N}-x+1}$ $(x \in J_{1,\hat{N}})$*, so* $\hat{w}_0(\hat{\alpha}_i) = -\hat{\alpha}_{N-i+1}$ $(i \in I)$*. Moreover*

$$\hat{w}_0 = (\hat{s}_1 \cdots \hat{s}_N)(\hat{s}_1 \cdots \hat{s}_{N-1}) \cdots (\hat{s}_1 \hat{s}_2)\hat{s}_1, \tag{2.17}$$

and the RHS of (2.17) *is a reduced expression of* $\hat{w}_0$*.*

(4) *Assume that* $\langle\hat{\Pi}\rangle$ *is the* D_N*-data. If* $N \in 2\mathbb{N}$*, then* $\hat{w}_0 = -\mathrm{id}_{\mathbb{R}^{\hat{N}}}$*. If* $N \in 2\mathbb{N}-1$*, then* $\hat{w}_0(\hat{\alpha}_i) = -\hat{\alpha}_i$ $(i \in J_{1,N-2})$*,* $\hat{w}_0(\hat{\alpha}_{N-1}) = -\hat{\alpha}_N$ *and* $\hat{w}_0(\hat{\alpha}_N) = -\hat{\alpha}_{N-1}$*. Moreover* $(\hat{s}_1 \cdots \hat{s}_N)^{N-1}$ *is a reduced expression of* $\hat{w}_0$*. Furthermore, for* $r \in J_{1,N-1}$*, we have*

$$(\hat{s}_r \hat{s}_{r+1} \cdots \hat{s}_N)^{N-r} = P_{J_{1,r-1}} - P_{J_{r,N-1}} + (-1)^{N-r} P_{J_{N,N}}. \tag{2.18}$$

(5) *If* $N = 6$*, and* $\langle\hat{\Pi}\rangle$ *is the* E_6*-data, then* $h = 12$ *and* $(\hat{s}_1\hat{s}_3\hat{s}_5\hat{s}_2\hat{s}_4\hat{s}_6)^6$ *is a reduced expression of* $\hat{w}_0$*.*

Proof. Let $\hat{b} := \hat{s}_1 \cdots \hat{s}_N$. Let h' be the order of $\hat{b}$. Then $\hat{b}$ and h' are called a *Coxeter element* and the *Coxeter number* respectively (see [15, Exercise 3.19]). By [15, Proposition 3.18], we have

$$h' = h. \tag{2.19}$$

Fix $\zeta \in \mathbb{C}^{\times}_{h'}$. It is clear from (2.2) that $\hat{b}$ acts on the N-dimensional $\mathbb{C}$-linear space $V \otimes_{\mathbb{R}} \mathbb{C}$ as a diagonalizable linear map whose eigenvalues are ζ^m for some $m \in J_{0,h'-1}$; these integers m are called the *exponents* (see [15, 3.16]).

(1) This is clear from (2.19).

(2) The first claim is clear. By (2.10) and (2.15), $\hat{w}_0(\hat{R}^+(\hat{\Pi})) = -\hat{R}^+(\hat{\Pi})$, so $\hat{w}_0(\hat{\Pi}) = -\hat{\Pi}$. Then the second claim follows from the first and (2.2). The third claim follows from (2.10), (2.15), (2.19) and the fact that h is even and all the exponents are odd (see [15, Tables 3.1 and 3.2, Theorem 3.19]).

(3) Let $\hat{b}_i := \hat{s}_1 \cdots \hat{s}_i$ $(i \in I)$. Let $\hat{w}'_0 := \hat{b}_N \hat{b}_{N-1} \cdots \hat{b}_1$. Then $\hat{b}_i(e_x) = e_{x+1}$ $(x \in J_{1,i})$, $\hat{b}_i(e_{i+1}) = e_1$, and $\hat{b}_i(e_y) = e_y$ $(y \in J_{i+1,\hat{N}})$. Hence $\hat{w}'_0(e_x) = e_{\hat{N}-x+1}$

($x \in J_{1,\hat{N}}$). In particular, $\hat{w}_0'(\hat{\Pi}) = -\hat{\Pi}$. By (2.5), $|\hat{R}^+(\hat{\Pi})| = N(N-1)/2$. Hence (3) follows from (2.15).

(4) Let $r \in J_{1,N-1}$, and $\hat{b} := \hat{s}_r\hat{s}_{r+1}\cdots\hat{s}_N$. Then $\hat{b}(e_x) = e_x$ ($x \in J_{1,r-2}$), $\hat{b}(e_y) = e_{y+1}$ ($y \in J_{1,N-2}$), $\hat{b}(e_{\hat{N}-1}) = -e_r$ and $\hat{b}(e_{\hat{N}}) = -e_{\hat{N}}$. Then we obtain (4) in a way similar to that for (3).

(5) This can be proved directly. □

§2.3. Longest element of a type-A classical Weyl group

Proposition 2.5. *Let $\langle\hat{\Pi}\rangle = (\hat{\alpha}_1, \ldots, \hat{\alpha}_N)$ be the A_N-data, and $\hat{\Pi} := \{\alpha_i \mid i \in I\}$. Let $\hat{w}_0$ be the longest element of the Coxeter system $(\hat{W}(\hat{\Pi}), \hat{S}(\hat{\Pi}))$. Let $\hat{s}_i := \hat{s}_{\hat{\alpha}_i} \in \hat{S}(\hat{\Pi})$ ($i \in I$). Let $n := N(N+1)/2$. Let $m \in J_{1,N}$ and $r := n - m(N-m+1)$. Then*

$$\hat{\ell}(\hat{w}_0) = n. \tag{2.20}$$

Moreover there exists $f \in \mathrm{Map}_n^I$ such that $\{f(t) \mid t \in J_{1,r}\} = I \setminus \{m\}$, $f(r+1) = m$ and $\hat{s}_{f(1)}\cdots\hat{s}_{f(n)}$ is a reduced expression of $\hat{w}_0$. Furthermore for such f, we have

$$\begin{aligned}&\{\hat{s}_{f(1)}\cdots\hat{s}_{f(k-1)}(\hat{\alpha}_{f(k)}) \mid k \in J_{1,r}\}\\ &\quad = \{e_x - e_{x'} \mid x, x' \in J_{1,m},\ x < x'\} \cup \{e_y - e_{y'} \mid y, y' \in J_{m+1,N+1},\ y < y'\},\end{aligned} \tag{2.21}$$

and

$$\begin{aligned}&\{\hat{s}_{f(1)}\hat{s}_{f(2)}\cdots\hat{s}_{f(t-1)}(\hat{\alpha}_{f(t)}) \mid t \in J_{r+1,N}\}\\ &\qquad = \{e_z - e_{z'} \mid z \in J_{1,m},\ z' \in J_{m+1,N+1}\}.\end{aligned} \tag{2.22}$$

Proof. The claim (2.20) follows from (2.15) and (2.5). Let $\hat{R}'$ be the crystallographic root system in $\mathbb{R}^{\hat{N}}$ defined as the RHS of (2.21). Let $\hat{\Pi}' := \hat{\Pi} \setminus \{\hat{\alpha}_m\}$. Then $\hat{\Pi}'$ is a root basis of $\hat{R}'$, Note that the Coxeter system $(\hat{W}(\hat{\Pi}'), \hat{S}(\hat{\Pi}'))$ is isomorphic to the product of the Coxeter systems of types A_{m-1} and A_{N-m} (resp. the Coxeter system of type A_{N-1}) if $m \in J_{2,N-1}$ (resp. $m \in \{1, N\}$). Note that r equals the length of the longest element of $(\hat{W}(\hat{\Pi}'), \hat{S}(\hat{\Pi}'))$. The remaining claims follow from these facts, (2.5), (2.13) and (2.16). □

Remark 2.6. A reduced expression as in Proposition 2.5 is given by

$$\begin{aligned}\hat{w}_0 &= (\hat{s}_1\cdots\hat{s}_m)(\hat{s}_1\cdots\hat{s}_{m-1})\cdots(\hat{s}_1\hat{s}_2)\hat{s}_1\\ &\quad\cdot(\hat{s}_{m+2}\hat{s}_{m+3}\cdots\hat{s}_N)(\hat{s}_{m+2}\hat{s}_{m+3}\cdots\hat{s}_{N-1})\cdots(\hat{s}_{m+2}\hat{s}_{m+3})\hat{s}_{m+2}\\ &\quad\cdot(\hat{s}_{m+1}\hat{s}_{m+2}\cdots\hat{s}_N)(\hat{s}_m\hat{s}_{m+1}\cdots\hat{s}_{N-1})\cdots(\hat{s}_1\cdots\hat{s}_{N-m}).\end{aligned} \tag{2.23}$$

This can be proved in a way similar to that for Proposition 2.4(3).

§2.4. Longest element of a type-B classical Weyl group

Lemma 2.7. *Let* $\langle\hat{\Pi}\rangle = (\hat{\alpha}_1, \ldots, \hat{\alpha}_N)$ *be the* B_N*-data, and* $\hat{\Pi} := \{\hat{\alpha}_i \mid i \in I\}$*. Let* $\hat{w}_0$ *be the longest element of* $(\hat{W}(\hat{\Pi}), \hat{S}(\hat{\Pi}))$*. Let* $\hat{s}_i := \hat{s}_{\hat{\alpha}_i} \in \hat{S}(\hat{\Pi})$ $(i \in I)$*.*

(1) *Let* $k, r \in J_{1,N}$ *with* $k \leq r$*. Let* $\hat{b} := -P_{J_{k,r}} + P_{J_{1,\hat{N}} \setminus J_{k,r}} \in \mathrm{GL}_N(\mathbb{R})$*. Then* $\hat{b} \in \hat{W}(\hat{\Pi})$ *and*

$$(\hat{s}_k \hat{s}_{k+1} \cdots \hat{s}_{N-1} \hat{s}_N \hat{s}_{N-1} \cdots \hat{s}_{r+1} \hat{s}_r)^{r-k+1} = \hat{b}. \tag{2.24}$$

Moreover the LHS of (2.24) *is a reduced expression of* $\hat{b}$*.*

(2) *Let* $k, t, r \in J_{1,\hat{N}}$ *be such that* $k \leq t < r$*. Let* $\hat{b} \in \hat{W}(\hat{\Pi})$ *be as in* (1)*. Define* $\hat{b}_1 \in \hat{W}(\hat{\Pi})$ (*resp.* $\hat{b}_2 \in \hat{W}(\hat{\Pi})$) *in the same way as* $\hat{b}$ *with* k *and* t (*resp.* $t+1$ *and* r) *in place of* k *and* r *respectively. Then* $\hat{b} = \hat{b}_1\hat{b}_2 = \hat{b}_2\hat{b}_1$ *and* $\hat{\ell}(\hat{b}) = \hat{\ell}(\hat{b}_1) + \hat{\ell}(\hat{b}_2)$*.*

(3) *Let* $m \in J_{1,N-1}$*. Then*

$$\hat{w}_0 = (\hat{s}_{N-m+1} \hat{s}_{N-m+2} \cdots \hat{s}_N)^m (\hat{s}_1 \cdots \hat{s}_{N-1} \hat{s}_N \hat{s}_{N-1} \cdots \hat{s}_{N-m})^{N-m}. \tag{2.25}$$

Moreover the RHS of (2.25) *is a reduced expression of* $\hat{w}_0$*. In particular,*

$$\hat{\ell}(\hat{w}_0) = N^2. \tag{2.26}$$

Proof. (1) Let $\hat{b}' \in \hat{W}(\hat{\Pi})$ be the LHS of (2.24). By (2.3), we have

$$\hat{s}_r \hat{s}_{r+1} \cdots \hat{s}_{N-1} \hat{s}_N \hat{s}_{N-1} \cdots \hat{s}_r = \hat{s}_{e_r}.$$

Hence

$$\hat{b}' = (\hat{s}_k \hat{s}_{k+1} \cdots \hat{s}_{r-1} \hat{s}_{e_r})^{r-k+1}.$$

Then by the same claim as in Proposition 2.4(2) for the B_N-data with $r-k$ in place of $\hat{N}$, we have $\hat{b} = \hat{b}' \in \hat{W}(\hat{\Pi})$. We see that

$$\widehat{\mathbb{L}}(\hat{b}) = \{e_t \mid t \in J_{k,r}\} \cup \{e_t + ce_{t'} \mid c \in \{-1, 1\},\, t \in J_{k,r},\, t' \in J_{t',N}\}.$$

Therefore by (2.10), we have

$$\begin{aligned} \hat{\ell}(\hat{b}) &= (r-k+1) + 2\sum_{t=k}^{r}(N-t) \\ &= (r-k+1) + 2N(r-k+1) - 2(r(r+1)/2 - k(k-1)/2) \\ &= (r-k+1)(1+2N-(r+k)) = (2N-k-r+1)(r-k+1). \end{aligned} \tag{2.27}$$

Hence we obtain the last statement of (1).

(2) The first statement is clear. The second follows from (1) and the calculation

$$\begin{aligned}
\hat{\ell}(\hat{b}_1)+\hat{\ell}(\hat{b}_2) &= (2N-k-t+1)(t-k+1)+(2N-t-r)(r-t) \\
&= 2N(r-k+1)-(k+t-1)(t-k+1)-(t+r)(r-t) \\
&= 2N(r-k+1)-(-k^2+t^2+2k-1)-(r^2-t^2) \\
&= 2N(r-k+1)+(k^2-r^2-2k+1) \\
&= 2N(r-k+1)+(k-1+r)(k-1-r) \\
&= (2N-r-k+1)(r-k+1)=\hat{\ell}(\hat{b}).
\end{aligned} \tag{2.28}$$

(3) This follows immediately from claims (1), (2) and (2.15). □

§3. Longest elements of Weyl groupoids of a simple Lie superalgebra of type ABCD

§3.1. Super-data

Let $\hat{N}\in\mathbb{N}$. Let $\{e_i \mid i\in J_{1,\hat{N}}\}$ be the standard $\mathbb{R}$-basis of $\mathbb{R}^{\hat{N}}$. Let $m\in J_{0,\hat{N}}$. Let $\mathcal{A}_{m|\hat{N}-m}$ be the set of all maps $p: J_{1,\hat{N}}\to J_{0,1}$ with $\sum_{i=1}^{\hat{N}} p(i)=m$. For $p\in\mathcal{A}_{m|\hat{N}-m}$, define an $\mathbb{R}$-bilinear map $\bar{\eta}^p:\mathbb{R}^{\hat{N}}\times\mathbb{R}^{\hat{N}}\to\mathbb{R}$ by $\bar{\eta}^p(e_i,e_j)=\delta(p(i),p(j))(-1)^{p(i)}$. Define $p^+_{m|\hat{N}-m}\in\mathcal{A}_{m|\hat{N}-m}$ by $p^+_{m|\hat{N}-m}(i)=0$ $(i\in J_{1,m})$ and $p^+_{m|\hat{N}-m}(j)=1$ $(j\in J_{m+1,\hat{N}})$. Define $p^-_{m|\hat{N}-m}\in\mathcal{A}_{m|\hat{N}-m}$ by $p^-_{m|\hat{N}-m}(i)=1$ $(i\in J_{1,m})$ and $p^-_{m|\hat{N}-m}(j)=0$ $(j\in J_{m+1,\hat{N}})$.

The sets $\bar{R}$ given in Definition 3.1 below are *almost* the sets of roots of finite-dimensional contragredient Lie superalgebras whose quotients by their centers are simple Lie superalgebras; in this paper, we impose a technical assumption on x for Definition 3.1(7), i.e. for $\mathrm{D}(2,1;x)$: instead of letting x be any element of $\mathbb{C}\setminus\{0,-1\}$, in this paper we assume $x\in\mathbb{Z}\setminus\{0,-1\}$.

See also Theorem 3.4.

Definition 3.1. Keep the notation as above. We also use the terminology of Definition 2.1. Let

$$\langle\bar{\Pi}\rangle=(\bar{\alpha}_i\mid i\in I)=(\bar{\alpha}_1,,\ldots,\bar{\alpha}_N)\in\mathbb{R}^{\hat{N}}\times\cdots\times\mathbb{R}^{\hat{N}}\ (N\text{ times}).$$

Let $\bar{R}$ be a subset of $\mathbb{R}^{\hat{N}}$. Let $\theta: I\to J_{0,1}$ be a map.

(1) Assume that $\hat{N}-1=N\geq 2$ and $m\in J_{1,N}$. We call $(\bar{\eta}^{p^+_{m|N+1-m}},\langle\bar{\Pi}\rangle)$ the $\mathrm{A}(m-1,N-m)$*-data* if $\langle\bar{\Pi}\rangle$ is the A_N-data. We call $\bar{R}$ the $\mathrm{A}(m-1,N-m)$*-type standard root system* if $\bar{R}$ is the A_N-type standard root system (see (2.5)). We call θ the $\mathrm{A}(m-1,N-m)$*-type parity map* if $\theta(m):=1$ and $\theta(i):=0$ $(i\in I\setminus\{m\})$.

(2) Assume that $\hat{N} = N \geq 1$ and $m \in J_{0,N-1}$. We call $(\bar{\eta}^{p^-_{N-m|m}}, \langle\bar{\Pi}\rangle)$ the B$(m, N-m)$-*data* if $\langle\bar{\Pi}\rangle$ is the B_N-data. Let $\hat{R}$ be the B_N-type standard root system (see (2.6)); if $N = 1$, let $\hat{R} := \{e_1, -e_1\}$. Assume that $\bar{R} = \hat{R} \cup \{2ce_i \mid i \in J_{1,N-m}, c \in \{-1, 1\}\}$. We call $\bar{R}$ the B$(m, N-m)$-*type standard root system*. Note that $\bar{R} \setminus 2\bar{R} = \hat{R}$. We call θ the B$(m, N-m)$-*type parity map* if $\theta(N-m) := 1$ and $\theta(i) := 0$ $(i \in I \setminus \{N-m\})$.

(3) Assume that $\hat{N} = N \geq 3$. We call $(\bar{\eta}^{p^-_{1|N-1}}, \langle\bar{\Pi}\rangle)$ the C(N)-*data* if $\langle\bar{\Pi}\rangle$ is the C_N-data. Let $\hat{R}$ be the C_N-type standard root system (see (2.7)). Assume that $\bar{R} = \hat{R} \setminus \{2e_1, -2e_1\}$. We call $\bar{R}$ the C(N)-*type standard root system*. We call θ the C(N)-*type parity map* if $\theta(1) := 1$ and $\theta(i) := 0$ $(i \in I \setminus \{1\})$.

(4) Assume that $\hat{N} = N \geq 3$ and $m \in J_{2,N-1}$. We call $(\bar{\eta}^{p^-_{\hat{N}-m|m}}, \langle\bar{\Pi}\rangle)$ the D$(m, N-m)$-*data* if $\langle\bar{\Pi}\rangle$ is the D_N-data. Let $\hat{R}$ be the D_N-type standard root system (see (2.8)). Assume that $\bar{R} = \hat{R} \cup \{2ce_i \mid i \in J_{1,N-m}, c \in \{-1, 1\}\}$. We call $\bar{R}$ the D$(m, N-m)$-*type standard root system*. We call θ the D$(m, N-m)$-*type parity map* if $\theta(N-m) := 1$ and $\theta(i) := 0$ $(i \in I \setminus \{N-m\})$.

(5) Assume $\hat{N} = N = 4$. We call $(\bar{\eta}^{p^-_{1|3}}, \langle\bar{\Pi}\rangle)$ the F(4)-*data* if $\bar{\alpha}_1 = \frac{1}{\sqrt{2}}(\sqrt{3}e_1 - e_2 - e_3 - e_4)$, $\bar{\alpha}_2 = \sqrt{2}e_2$, $\bar{\alpha}_3 = \sqrt{2}(-e_2 + e_3)$, and $\bar{\alpha}_4 = \sqrt{2}(-e_3 + e_4)$. Assume that $\bar{R} = \bar{R}^+ \cup (-\bar{R}^+)$, where $\bar{R}^+ := \{\bar{\alpha}_4, \bar{\alpha}_3 + \bar{\alpha}_4, \bar{\alpha}_2 + \bar{\alpha}_3 + \bar{\alpha}_4, 2\bar{\alpha}_2 + \bar{\alpha}_3 + \bar{\alpha}_4, \bar{\alpha}_1 + \bar{\alpha}_2 + \bar{\alpha}_3 + \bar{\alpha}_4, 2\bar{\alpha}_2 + 2\bar{\alpha}_3 + \bar{\alpha}_4, \bar{\alpha}_1 + 2\bar{\alpha}_2 + \bar{\alpha}_3 + \bar{\alpha}_4, \bar{\alpha}_1 + 2\bar{\alpha}_2 + 2\bar{\alpha}_3 + \bar{\alpha}_4, \bar{\alpha}_1 + 3\bar{\alpha}_2 + 2\bar{\alpha}_3 + \bar{\alpha}_4, 2\bar{\alpha}_1 + 3\bar{\alpha}_2 + 2\bar{\alpha}_3 + \bar{\alpha}_4, \bar{\alpha}_1, \bar{\alpha}_1 + \bar{\alpha}_2, \bar{\alpha}_1 + \bar{\alpha}_2 + \bar{\alpha}_3, \bar{\alpha}_1 + 2\bar{\alpha}_2 + \bar{\alpha}_3, \bar{\alpha}_2, 2\bar{\alpha}_2 + \bar{\alpha}_3, \bar{\alpha}_2 + \bar{\alpha}_3, \bar{\alpha}_3\}$. We call $\bar{R}$ the F(4)-*type standard root system*. We call θ the F(4)-*type parity map* if $\theta(1) := 1$ and $\theta(i) := 0$ $(i \in I \setminus \{1\})$. (Let $\langle\hat{\Pi}\rangle = (\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4)$ and $\hat{R}$ be the F_4-data and the F_4-type standard root system respectively. Define an $\mathbb{R}$-linear isomorphism $\xi^\flat : \mathbb{R}^4 \to \mathbb{R}^4$ by $\xi^\flat(\hat{\alpha}_i) := \bar{\alpha}_{5-i}$. Then $\bar{R} = \xi^\flat(\hat{R} \setminus \{ce_1 + c'e_x \mid x \in J_{2,4},\ c, c' \in \{1, -1\}\})$.)

(6) Assume that $N = 3$ and $\hat{N} = 4$. We call $(\bar{\eta}^{p^-_{1|2}}, \langle\bar{\Pi}\rangle)$ the G(3)-*data* if $\bar{\alpha}_1 = \sqrt{2}e_1 + e_3 - e_4$, $\bar{\alpha}_2 = e_2 - e_3$, $\bar{\alpha}_3 = -2e_2 + e_3 + e_4$. Assume that $\bar{R} = \bar{R}^+ \cup -\bar{R}^+$, where $\bar{R}^+ := \{\bar{\alpha}_1, \bar{\alpha}_1 + \bar{\alpha}_2, \bar{\alpha}_1 + \bar{\alpha}_2 + \bar{\alpha}_3, \bar{\alpha}_1 + 2\bar{\alpha}_2 + \bar{\alpha}_3, \bar{\alpha}_1 + 3\bar{\alpha}_2 + \bar{\alpha}_3, \bar{\alpha}_1 + 3\bar{\alpha}_2 + 2\bar{\alpha}_3, \bar{\alpha}_1 + 4\bar{\alpha}_2 + 2\bar{\alpha}_3, \bar{\alpha}_2, 3\bar{\alpha}_2 + \bar{\alpha}_3, 3\bar{\alpha}_2 + 2\bar{\alpha}_3, 2\bar{\alpha}_2 + \bar{\alpha}_3, \bar{\alpha}_2 + \bar{\alpha}_3, \bar{\alpha}_3, 2\bar{\alpha}_1 + 4\bar{\alpha}_2 + 2\bar{\alpha}_3\}$. We call $\bar{R}$ the G(3)-*type standard root system*. Note that $\bar{R} \cap 2\bar{R} = \{2\bar{\alpha}_1 + 4\bar{\alpha}_2 + 2\bar{\alpha}_3\}$. We call θ the G(3)-*type parity map* if $\theta(1) := 1$ and $\theta(i) := 0$ $(i \in I \setminus \{1\})$.

(7) Assume that $N = 3$ and $\hat{N} = 4$. Let $x \in \mathbb{Z} \setminus \{0, -1\}$. We call $(\bar{\eta}^{p^+_{2|2}}, \langle\bar{\Pi}\rangle)$ the D$(2, 1; x)$-*data* if $\bar{\alpha}_1 = e_3 - e_4$, $\bar{\alpha}_2 = e_2 - e_3$, $\bar{\alpha}_3 = x(e_3 + e_4) + \sqrt{2x^2 - 2x}e_1$. Assume that $\bar{R} = \bar{R}^+ \cup -\bar{R}^+$, where $\bar{R}^+ := \{\bar{\alpha}_1, \bar{\alpha}_1 + \bar{\alpha}_2, \bar{\alpha}_2, 2\bar{\alpha}_1 + \bar{\alpha}_2 + \bar{\alpha}_3, \bar{\alpha}_1 + \bar{\alpha}_3, \bar{\alpha}_3\}$. We call $\bar{R}$ the D$(2, 1; x)$-*type standard root system*. We call θ the D$(2, 1; x)$-*type parity map* if $\theta(1) := 1$ and $\theta(i) := 0$ $(i \in I \setminus \{1\})$.

(8) Let $(\bar{\eta}, \langle\bar{\Pi}\rangle)$ be a data as in (1)–(7) above, i.e., $(\bar{\eta}, \langle\bar{\Pi}\rangle)$ is the X-data with X being A$(m-1, N-m)$ (for some $m \in J_{1,N}$ if $N \geq 2$), B$(m, N-m)$ (for some

$m \in J_{0,N-1}$ if $N \geq 1$), C(N) (if $N \geq 3$), D($m, N-m$) (for some $m \in J_{2,N-1}$ if $N \geq 3$), F(4) (if $N = 4$), G(3) (if $N = 3$), or D($2, 1; x$) (for some $x \in \mathbb{Z} \setminus \{0, -1\}$ if $N = 3$). (Note that $\mathrm{X} \neq \mathrm{A}(0,0)$.) We then call $(\bar{\eta}, \langle\bar{\Pi}\rangle)$ a *rank-N standard super-data*. If $\bar{R}$ is the X-type standard root system, we call $\bar{R}$ the *standard root system associated with* $(\bar{\eta}, \langle\bar{\Pi}\rangle)$. If θ is the X-type parity map, we call θ the *parity map associated with* $(\bar{\eta}, \langle\bar{\Pi}\rangle)$.

We can directly see

Lemma 3.2. *Let $(\bar{\eta}, \langle\bar{\Pi}\rangle)$ be a rank-N standard super-data. Let $\bar{R}$ be the standard root system associated with $(\bar{\eta}, \langle\bar{\Pi}\rangle)$. Let θ be the parity map associated with $(\bar{\eta}, \langle\bar{\Pi}\rangle)$. Let $\bar{\alpha} \in \bar{R}$. Then $[\theta](\xi_{\langle\bar{\Pi}\rangle}(\bar{\alpha})) = 1$ if and only if $\bar{\eta}(\bar{\alpha}, \bar{\alpha}) = 0$ or $2\bar{\alpha} \in \bar{R}$.*

Definition 3.3. (1) Let $\mathfrak{g}$ be a $\mathbb{C}$-linear space. Let $\mathfrak{g}(t)$ ($t \in J_{0,1}$) be subspaces of $\mathfrak{g}$ with $\mathfrak{g} = \mathfrak{g}(0) \oplus \mathfrak{g}(1)$. Define subspaces $\mathfrak{g}(t)$ ($t \in \mathbb{Z} \setminus J_{0,1}$) of $\mathfrak{g}$ by $\mathfrak{g}(t) = \mathfrak{g}(t \pm 2)$. Let $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ be a $\mathbb{C}$-bilinear map. We call $\mathfrak{g}$ a *Lie superalgebra* if:

(Su1) $[x, y] \in \mathfrak{g}(t + t')$ and $[x, y] = -(-1)^{tt'}[y, x]$ ($t, t' \in \mathbb{Z}$, $x \in \mathfrak{g}(t)$, $y \in \mathfrak{g}(t')$).

(Su2) $[x, [y, z]] = [[x, y], z] + (-1)^{tt'}[y, [x, z]]$ ($t, t' \in \mathbb{Z}$, $x \in \mathfrak{g}(t)$, $y \in \mathfrak{g}(t')$, $z \in \mathfrak{g}$).

(2) Let $\mathfrak{g}$ be a Lie superalgebra. Let Y be a non-empty subset of $\mathfrak{g}$. Let $\langle Y\rangle_{\mathfrak{g}}^{(1)} := \mathrm{Span}_{\mathbb{C}}(Y)$. Let $\langle Y\rangle_{\mathfrak{g}}^{(t)} := \mathrm{Span}_{\mathbb{C}}(\{[y, z] \mid y \in Y, z \in \langle Y\rangle_{\mathfrak{g}}^{(t-1)}\})$ for $t \in J_{2,\infty}$. Let $\langle Y\rangle_{\mathfrak{g}} := \mathrm{Span}_{\mathbb{C}}(\bigcup_{t=1}^{\infty} \langle Y\rangle_{\mathfrak{g}}^{(t)})$. We call $\langle Y\rangle_{\mathfrak{g}}$ the *sub Lie superalgebra of $\mathfrak{g}$ generated by Y*.

(3) Let $\mathcal{Y}_N$ be the set of all pairs $a^\sharp = (\eta^\sharp, \theta^\sharp)$ of symmetric $\mathbb{R}$-bilinear maps $\eta^\sharp : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ and maps $\theta^\sharp : I \to J_{0,1}$. In a standard way, for $a^\sharp = (\eta^\sharp, \theta^\sharp) \in \mathcal{Y}_N$, we have the Lie superalgebra $\mathfrak{g}(a^\sharp)$ over $\mathbb{C}$ (unique up to isomorphism) satisfying the following conditions, where we let $\mathfrak{g} := \mathfrak{g}(a^\sharp)$:

(CoSu1) There exist $3N$ linearly independent elements $H_i^\sharp$, $E_i^\sharp$, $F_i^\sharp$ ($i \in I$) of $\mathfrak{g}$ such that $\langle\{H_i^\sharp, E_i^\sharp, F_i^\sharp \mid i \in I\}\rangle_{\mathfrak{g}} = \mathfrak{g}$.

(CoSu2) $H_i^\sharp \in \mathfrak{g}(0)$, $E_i^\sharp$, $F_i^\sharp \in \mathfrak{g}(\theta^\sharp(i))$ ($i \in I$).

(CoSu3) $[H_i^\sharp, H_j^\sharp] = 0$, $[H_i^\sharp, E_j^\sharp] = \eta^\sharp(\alpha_i, \alpha_j)E_j^\sharp$, $[H_i^\sharp, F_j^\sharp] = -\eta^\sharp(\alpha_i, \alpha_j)F_j^\sharp$, $[E_i^\sharp, F_j^\sharp] = \delta_{ij}H_i^\sharp$ ($i, j \in I$).

(CoSu4) There exist subspaces $\mathfrak{g}_\alpha$ ($\alpha \in \mathbb{Z}\Pi$) of $\mathfrak{g}(\eta, \theta)$ such that $\mathfrak{g} = \bigoplus_{\alpha \in \mathbb{Z}\Pi} \mathfrak{g}_\alpha$, $[\mathfrak{g}_\beta, \mathfrak{g}_{\beta'}] \subseteq \mathfrak{g}_{\beta+\beta'}$ ($\beta, \beta' \in \mathbb{Z}\Pi$), $\mathfrak{g}_0 = \bigoplus_{i \in I} \mathbb{C}H_i^\sharp$, and $\mathfrak{g}_{\alpha_j} = \mathbb{C}E_j^\sharp$, $\mathfrak{g}_{-\alpha_j} = \mathbb{C}F_j^\sharp$ ($j \in I$). We also assume that $\dim_{\mathbb{C}} \mathfrak{g}_0 = N$ and $\dim_{\mathbb{C}} \mathfrak{g}_{\alpha_i} = \dim_{\mathbb{C}} \mathfrak{g}_{-\alpha_i} = 1$ ($i \in I$).

(CoSu5) Let $\alpha \in \mathbb{Z}_{\geq 0}\Pi \setminus (\Pi \cup \{0\})$. For $X \in \mathfrak{g}_\alpha$, if $[X, F_i^\sharp] = 0$ for all $i \in I$, then $X = 0$. For $Y \in \mathfrak{g}_{-\alpha}$, if $[E_i^\sharp, Y] = 0$ for all $i \in I$, then $Y = 0$.

(Note that the conditions (CoSu1)–(CoSu4) imply that $\mathfrak{g}_\beta = \{0\}$ for $\beta \in \mathbb{Z}\Pi \setminus (\mathbb{Z}_{\geq 0}\Pi \cup -\mathbb{Z}_{\geq 0}\Pi)$.)

Let $R^\sharp(a^\sharp) := \{\beta \in \mathbb{Z}\Pi \setminus \{0\} \mid \mathfrak{g}(a^\sharp)_\beta \neq \{0\}\}$. We call $\mathfrak{g}(a^\sharp)$ the *contragredient Lie superalgebra*.

The following is well-known:

Theorem 3.4 (cf. [17, Proposition 2.5.5]). *Define $a^\sharp = (\eta^\sharp, \theta^\sharp) \in \mathcal{Y}_N$ and a subset X of $\mathbb{Z}\Pi$ by* (i) *or* (ii) *below.*

(i) *Let $\langle\hat{\Pi}\rangle$ be a rank-N Cartan data. Let $\eta^\sharp := \hat{\eta}_{\langle\hat{\Pi}\rangle}$. Define $\theta^\sharp$ by $\theta^\sharp(i) := 0$ $(i \in I)$. Let $\hat{R}$ be the rank-N standard irreducible root system associated with $\langle\hat{\Pi}\rangle$. Let $X := \xi_{\langle\hat{\Pi}\rangle}(\hat{R})$.*

(ii) *Let $(\bar{\eta}, \langle\bar{\Pi}\rangle)$ be a rank-N standard super-data. Let $\eta^\sharp := \bar{\eta}_{\langle\hat{\Pi}\rangle}$. Let $\bar{R}$ and θ be the standard root system and the parity map associated with $(\bar{\eta}, \langle\bar{\Pi}\rangle)$ respectively. Let $\theta^\sharp := \theta$ and $X := \xi_{\langle\bar{\Pi}\rangle}(\bar{R})$.*

Then for $\alpha \in \mathbb{Z}\Pi$, $\dim \mathfrak{g}(a^\sharp)_\alpha \geq 1$ if and only if $\alpha \in X \cup \{0\}$. Moreover, for $\alpha \in X$, we have $\dim \mathfrak{g}(a^\sharp)_\alpha = 1$. In particular, $|R^\sharp(a^\sharp)| < \infty$.

Let $(\bar{\eta}, \langle\bar{\Pi}\rangle)$ be a rank-N standard super-data. Let θ be the parity map associated with $(\bar{\eta}, \langle\bar{\Pi}\rangle)$. Let $a^\sharp := (\bar{\eta}_{\langle\bar{\Pi}\rangle}, \theta)$ and $\mathfrak{g} := \mathfrak{g}(a^\sharp)$. Following [17], we introduce the following terminology. Assume that $(\bar{\eta}, \langle\bar{\Pi}\rangle)$ is the X-data. If X $=$ A$(m-1, N-m)$ and $2m - 1 = N \geq 2$, then $\mathfrak{g}$ has a one-dimensional center $\mathfrak{c}$, $\mathfrak{g}/\mathfrak{c}$ is a simple Lie superalgebra and $\mathfrak{g}/\mathfrak{c}$ is called A$(m-1, m-1)$. Otherwise $\mathfrak{g}$ is a simple Lie superalgebra and called X. If X is A$(m-1, N-m)$ (resp. B$(m, N-m)$, C(N), D$(m, N-m)$), then $\mathfrak{g}$ is also called sl$(m \mid N-m+1)$ (resp. osp$(2m+1 \mid 2(N-m))$, osp$(2 \mid 2(N-1))$, osp$(2m \mid 2(N-m))$).

§3.2. The Weyl groupoids of Lie superalgebras

For $a^\sharp = (\eta^\sharp, \theta^\sharp) \in \mathcal{Y}_N$, $i, j \in I$ with $i \neq j$, and $k \in \mathbb{N}$, let

$$t^{a^\sharp}_{i,j;k} := \begin{cases} \frac{k(k-1)}{2}\eta^\sharp(\alpha_i, \alpha_i) + k\eta^\sharp(\alpha_i, \alpha_j) & \text{if } \theta^\sharp(i) = 0, \\ \frac{k-1}{2}\eta^\sharp(\alpha_i, \alpha_i) + \eta^\sharp(\alpha_i, \alpha_j) & \text{if } \theta^\sharp(i) = 1 \text{ and } k \in 2\mathbb{N} - 1, \\ \frac{k}{2}\eta^\sharp(\alpha_i, \alpha_i) & \text{if } \theta^\sharp(i) = 1 \text{ and } k \in 2\mathbb{N}, \end{cases}$$

and let $t^{a^\sharp}_{i,j;m}! := \prod_{k=1}^m t^{a^\sharp}_{i,j;k}$ for $m \in \mathbb{Z}_{\geq 0}$.

For $a^\sharp \in \mathcal{Y}_N$ and $i, j \in I$, define $c^{a^\sharp}_{ij} \in \{2\} \cup J_{-\infty,0} \cup \{-\infty\}$ by

$$c^{a^\sharp}_{ij} := \begin{cases} 2 & \text{if } i = j, \\ -\infty & \text{if } i \neq j \text{ and } t^{a^\sharp}_{i,j;m}! \neq 0 \text{ for all } m \in \mathbb{N}, \\ -\max\{m \in \mathbb{Z}_{\geq 0} \mid t^{a^\sharp}_{i,j;m}! \neq 0\} & \text{otherwise.} \end{cases} \tag{3.1}$$

For $i\in I$, let $(\mathcal{Y}_N^{\prime\prime,\mathrm{fin}})_i := \{a^\sharp\in\mathcal{Y}_N \mid \forall j\in I,\, c_{ij}^{a^\sharp}\neq -\infty\}$. For $i\in I$ and $a^\sharp=(\eta^\sharp,\theta^\sharp)\in(\mathcal{Y}_N^{\prime\prime,\mathrm{fin}})_i$, define $s_i^{a^\sharp}\in\mathrm{GL}(\mathbb{Z}\Pi)$ by $s_i^{a^\sharp}(\alpha_j)=\alpha_j-c_{ij}^{a^\sharp}\alpha_i$, and define $\tau_i^\sharp a^\sharp=(\tau_i^\sharp\eta^\sharp,\tau_i^\sharp\theta^\sharp)\in\mathcal{Y}_N$ by $\tau_i^\sharp\eta^\sharp(\alpha_j,\alpha_k):=\eta^\sharp(s_i^{a^\sharp}(\alpha_j),s_i^{a^\sharp}(\alpha_k))$ and $\tau_i^\sharp\theta^\sharp(j):=[\theta^\sharp](s_i^{a^\sharp}(\alpha_j))$.

We can directly see that for $i\in I$ and $a^\sharp=(\eta^\sharp,\theta^\sharp)\in(\mathcal{Y}_N^{\prime\prime,\mathrm{fin}})_i$,

$$\tau_i^\sharp a^\sharp\in(\mathcal{Y}_N^{\prime\prime,\mathrm{fin}})_i,\ \tau_i^\sharp\tau_i^\sharp a^\sharp=a^\sharp,\ s_i^{a^\sharp}=(s_i^{a^\sharp})^{-1}=s_i^{\tau_i^\sharp a^\sharp}. \tag{3.2}$$

We can also see that for $i\in I$ and $a^\sharp=(\eta^\sharp,\theta^\sharp)\in(\mathcal{Y}_N^{\prime\prime,\mathrm{fin}})_i$, if $\eta^\sharp(\alpha_i,\alpha_i)\neq 0$, then $\tau_i^\sharp a^\sharp=a^\sharp$.

We can see that

$$\{r\in\mathbb{Z}\mid \alpha_i+r\alpha_j\in R^\sharp(a^\sharp)\}=J_{0,-c_{ij}}\quad\text{for } i,j\in I \text{ with } i\neq j, \tag{3.3}$$

where $c_{ij}:=c_{ij}^{a^\sharp}$.

For $i\in I$ and $a^\sharp=(\eta^\sharp,\theta^\sharp)\in(\mathcal{Y}_N^{\prime\prime,\mathrm{fin}})_i$, we have the Lie superalgebra isomorphism

$$T_i^\sharp=T_i^{\sharp,\tau_i^\sharp a^\sharp}:\mathfrak{g}(\tau_i^\sharp a^\sharp)\to\mathfrak{g}(a^\sharp) \tag{3.4}$$

defined by

$$\begin{aligned}
&T_i^\sharp(H_j^\sharp):=H_j^\sharp-c_{ij}H_i^\sharp,\\
&T_i^\sharp(E_j^\sharp):=\begin{cases}F_j^\sharp & \text{if } i=j,\\ (\mathrm{ad}\,E_i^\sharp)^{-c_{ij}}E_j^\sharp & \text{if } i\neq j,\end{cases}\\
&T_i^\sharp(F_j^\sharp):=\begin{cases}(-1)^{\theta^\sharp(i)}E_j^\sharp & \text{if } i=j,\\ \dfrac{(-1)^{\theta^\sharp(i)((-c_{ij})\theta^\sharp(j)+|2\mathbb{N}\cap J_{1,-c_{ij}}|)}}{t_{i,j;-c_{ij}}^{a^\sharp}!}(\mathrm{ad}\,F_i^\sharp)^{-c_{ij}}F_j^\sharp & \text{if } i\neq j,\end{cases}
\end{aligned}$$

where $c_{ij}:=c_{ij}^{a^\sharp}$, so we have $s_i^{\tau_i^\sharp a^\sharp}(R^\sharp(\tau_i^\sharp a^\sharp))=R^\sharp(a^\sharp)$.

Let $\mathcal{Y}_N^{\prime\prime,\mathrm{fin}}:=\bigcap_{i\in I}(\mathcal{Y}_N^{\prime\prime,\mathrm{fin}})_i$. Let $\mathcal{Y}_N^{\prime,\mathrm{fin}}:=\{a^\sharp\in\mathcal{Y}_N^{\prime\prime,\mathrm{fin}}\mid\forall i\in I,\,\tau_i^\sharp a^\sharp\in\mathcal{Y}_N^{\prime\prime,\mathrm{fin}}\}$. For $a^\sharp\in\mathcal{Y}_N^{\prime,\mathrm{fin}}$, let $\mathcal{G}^\sharp(a^\sharp):=\{\tau_{i_1}^\sharp\cdots\tau_{i_r}^\sharp a^\sharp\mid r\in\mathbb{N},\,i_t\in I\,(t\in J_{1,r})\}$. Let $\mathcal{Y}_N^{\mathrm{fin}}:=\{a^\sharp\in\mathcal{Y}_N\mid |R^\sharp(a^\sharp)|<\infty\}$. By (3.3), $\mathcal{Y}_N^{\mathrm{fin}}\subset\mathcal{Y}_N^{\prime,\mathrm{fin}}$.

Definition 3.5. Let $a^\sharp\in\mathcal{Y}_N^{\prime,\mathrm{fin}}$.

(1) For $i\in I$, define a map $\tau_i^{\mathcal{G}^\sharp(a^\sharp)}:\mathcal{G}^\sharp(a^\sharp)\to\mathcal{G}^\sharp(a^\sharp)$ by $\tau_i^{\mathcal{G}^\sharp(a^\sharp)}(a^\sharp):=\tau_i^\sharp a^\sharp$. For $a^{\sharp,\prime}\in\mathcal{G}^\sharp(a^\sharp)$, let $C^{a^{\sharp,\prime}}$ be the $N\times N$-matrix $[c_{ij}^{a^{\sharp,\prime}}]$. We call the quadruple $\mathcal{C}_{a^\sharp}=\mathcal{C}_{a^\sharp}(I,\mathcal{G}^\sharp(a^\sharp),(\tau_i^{\mathcal{G}^\sharp(a^\sharp)})_{i\in I},(C^{a^{\sharp,\prime}})_{a^{\sharp,\prime}\in\mathcal{G}(\chi)})$ the *Cartan scheme associated with* $a^\sharp$. Indeed, by (3.2), $\mathcal{C}_{a^\sharp}$ is a connected Cartan scheme.

(2) We call the data $\mathcal{R}_{a^\sharp} = \mathcal{R}_{a^\sharp}(\mathcal{C}_{a^\sharp}, (R^\sharp(a^{\sharp,\prime}) \setminus 2R^\sharp(a^{\sharp,\prime}))_{a^{\sharp,\prime}\in\mathcal{G}^\sharp(a^\sharp)})$ the *generalized root system associated with* $a^\sharp$. Indeed, by Definition 1.1, $\mathcal{R}_{a^\sharp}$ is a root system of type $\mathcal{C}_{a^\sharp}$.

Remark 3.6. The Weyl groupoids $\mathcal{W}(\mathcal{R}_{a^\sharp})$ seem to be closely related to the finite groups $\widetilde{W}$ introduced in [22, Section 6] (see also [13, Remark 4]). See [31] for the root systems of extended affine Lie superalgebras.

§3.3. Longest element of the simple Lie superalgebra A$(m-1, N-m)$

Let $\hat{N} \in J_{2,\infty}$. For $i \in J_{1,\hat{N}-1}$, define a bijection $\wp_i^{(\hat{N})} : J_{1,\hat{N}} \to J_{1,\hat{N}}$ by

$$\wp_i^{(\hat{N})}(j) := \begin{cases} i+1 & \text{if } j = i, \\ i & \text{if } j = i+1, \\ j & \text{if } j \in J_{1,\hat{N}} \setminus J_{i,i+1}. \end{cases} \tag{3.5}$$

Let $m \in J_{1,N}$. For $i \in I$, define a bijection $\breve{\tau}_i : \mathcal{A}_{m|N+1-m} \to \mathcal{A}_{m|N+1-m}$ by $\breve{\tau}_i(p) := p \circ \wp_i^{(N+1)}$. Let $\langle\hat{\Pi}\rangle = (\hat{\alpha}_i \mid i \in I)$ and $\hat{R}$ be the A$_N$-data and the A$_N$-type standard root system respectively. For any $p \in \mathcal{A}_{m|N+1-m}$, let $\breve{R}(p) := \xi_{\langle\hat{\Pi}\rangle}(\hat{R})$, and define a generalized Cartan matrix $\breve{C}^p = (\breve{c}_{ij}^p)_{i,j\in I}$ by $\breve{c}_{ij}^p = 2\hat{\eta}(\hat{\alpha}_i, \hat{\alpha}_j)/\hat{\eta}(\hat{\alpha}_i, \hat{\alpha}_i)$. We easily see that

$$\breve{\mathcal{C}}_{m|N+1-m} := \mathcal{C}(I, \mathcal{A}_{m|N+1-m}, (\breve{\tau}_i)_{i\in I}, (\breve{C}^p)_{p\in\mathcal{A}_{m|N+1-m}})$$

is a Cartan scheme, and $\breve{\mathcal{R}}_{m|N+1-m} := \mathcal{R}(\breve{\mathcal{C}}, (\breve{R}(p))_{p\in\mathcal{A}_{m|N+1-m}})$ is a generalized root system of type $\breve{\mathcal{C}}_{m|N+1-m}$. In particular, we have

Lemma 3.7. *Let $\langle\hat{\Pi}\rangle = (\hat{\alpha}_i \mid i \in I)$ be the* A$_N$*-data. Let a be the only element of $\mathcal{A}_{\langle\hat{\Pi}\rangle}$. Then for any $m \in J_{1,N}$ and any $p \in \mathcal{A}_{m|N+1-m}$, $(\breve{\mathcal{R}}_{m|N+1-m}, p)$ and $(\mathcal{R}_{\langle\hat{\Pi}\rangle}, a)$ are quasi-isomorphic.*

Note that for the generalized root system $\breve{\mathcal{R}}_{m|N+1-m}$, we have

$$\bar{\eta}^p_{\langle\hat{\Pi}\rangle}(1^p s_{f,r}(x), 1^p s_{f,r}(y)) = \bar{\eta}^{p_{f,r}}_{\langle\hat{\Pi}\rangle}(x,y) \quad (x, y \in \mathbb{V}) \tag{3.6}$$

for $p \in \mathcal{A}_{m|N+1-m}$, $f \in \mathrm{Map}^I_\infty$ and $r \in \mathbb{N}$.

Recall that if $p := p^+_{m|N+1-m} \in \mathcal{A}_{m|N+1-m}$, then $(\bar{\eta}^p, \langle\hat{\Pi}\rangle)$ is the A$(m-1, N-m)$-data and $\hat{R}$ is the standard root system associated with $(\bar{\eta}^p, \langle\hat{\Pi}\rangle)$.

We have

Lemma 3.8. *Let $m \in I$, and consider the Weyl groupoid $\mathcal{W}(\breve{\mathcal{R}}_{m|N+1-m})$. Let $p := p^+_{m|N+1-m} \in \mathcal{A}_{m|N+1-m}$. Let θ be the* A$(m-1, N-m)$*-type parity map.*

(1) *Let* $a^\sharp := (\bar{\eta}^p_{\langle\hat{\Pi}\rangle}, \theta) \in \mathcal{Y}^{\rm fin}_N$. *Then* $(\breve{\mathcal{R}}_{m|N+1-m}, p)$ *and* $(\mathcal{R}_{a^\sharp}, a^\sharp)$ *are isomorphic. Moreover for* $f \in \mathrm{Map}^I_\infty$ *and* $r \in \mathbb{N}$, $a^\sharp_{f,r} = (\bar{\eta}^{p_{f,r}}_{\langle\hat{\Pi}\rangle}, \theta_{f,r})$, *where* $\theta_{f,r}(i) := [\theta](1^p s_{f,r}(\alpha_i))$ $(i \in I)$.

(2) *Let* $n := N(N+1)/2$ *and* $r := n - m(N-m+1)$. *Let* $f \in \mathrm{Map}^I_n$ *be as in Proposition* 2.5 (*or the one obtained from* (2.23)). *Let* $\beta_x := 1^p s_{f,x-1}(\alpha_x)$ $(x \in J_{1,n})$. *Then* $\ell^p(1^p w_0) = n$, $1^p w_0 = 1^p s_{f,n}$ *and* $\breve{R}^+(p) = \{\beta_x \mid x \in J_{1,n}\}$. *Moreover*

$$\begin{aligned} &\bar{\eta}^p_{\langle\hat{\Pi}\rangle}(\beta_k, \beta_k) \subseteq \{-2, 2\}, \quad [\theta](\beta_k) = 0 \quad (k \in J_{1,r}), \\ &\bar{\eta}^p_{\langle\hat{\Pi}\rangle}(\beta_t, \beta_t) = 0, \quad [\theta](\beta_t) = 1 \quad (t \in J_{r+1,n}). \end{aligned} \tag{3.7}$$

Proof. (1) can be proved directly, and (2) follows from Lemmas 1.12 and 3.7 and (1.17), (2.14), (2.16), (2.21) and (2.22). □

§3.4. Longest element of the Weyl groupoid of the simple Lie superalgebra B$(m, N-m)$ with $m \geq 1$

Let $m \in J_{1,N}$. For $i \in I$, define a bijection $\dot{\tau}_i : \mathcal{A}_{m|N-m} \to \mathcal{A}_{m|N-m}$ by

$$\dot{\tau}_i(p) := \begin{cases} p \circ \wp^{(N)}_i & \text{if } i \in J_{1,N-1}, \\ p & \text{if } i = N. \end{cases}$$

Let $\langle\hat{\Pi}\rangle = (\hat{\alpha}_i \mid i \in I)$ and $\hat{R}$ be the B$_N$-data and the B$_N$-type standard root system respectively. For any $p \in \mathcal{A}_{m|N-m}$, let $\dot{R}(p) := \xi_{\langle\hat{\Pi}\rangle}(\hat{R})$, and define a generalized Cartan matrix $\dot{C}^p = (\dot{c}^p_{ij})_{i,j\in I}$ by $\dot{c}^p_{ij} = 2\hat{\eta}(\hat{\alpha}_i, \hat{\alpha}_j)/\hat{\eta}(\hat{\alpha}_i, \hat{\alpha}_i)$. We see that $\dot{\mathcal{C}}_{m|N-m} := \mathcal{C}(I, \mathcal{A}_{m|N-m}, (\dot{\tau}_i)_{i\in I}, (\dot{C}^p)_{p\in\mathcal{A}_{m|N-m}})$ is a Cartan scheme, and $\dot{\mathcal{R}}_{m|N-m} := \mathcal{R}(\dot{\mathcal{C}}, (\dot{R}(p))_{p\in\mathcal{A}_{m|N-m}})$ is a generalized root system of type $\dot{\mathcal{C}}_{m|N-m}$. In particular, we have

Lemma 3.9. *Let* $\langle\hat{\Pi}\rangle = (\hat{\alpha}_i \mid i \in I)$ *be the* B$_N$*-data. Let* a *be the only element of* $\mathcal{A}_{\langle\hat{\Pi}\rangle}$. *Then for any* $m \in J_{1,N}$ *and any* $p \in \mathcal{A}_{m|N-m}$, $(\dot{\mathcal{R}}_{m|N-m}, p)$ *and* $(\mathcal{R}_{\langle\hat{\Pi}\rangle}, a)$ *are quasi-isomorphic.*

Note that for the generalized root system $\dot{\mathcal{R}}_{m|N+1-m}$, we have

$$\bar{\eta}^p_{\langle\hat{\Pi}\rangle}(1^p s_{f,r}(x), 1^p s_{f,r}(y)) = \bar{\eta}^{p_{f,r}}_{\langle\hat{\Pi}\rangle}(x, y) \quad (x, y \in \mathbb{V}) \tag{3.8}$$

for $p \in \mathcal{A}_{m|N-m}$, $f \in \mathrm{Map}^I_\infty$ and $r \in \mathbb{N}$.

Define $\dot{f}_{m|N-m} \in \mathrm{Map}^I_{N^2}$ by

$$(3.9)\qquad \dot{f}_{m|N-m}(t) := \begin{cases} N-m+t & \text{if } t \in J_{1,m}, \\ \dot{f}_{m|N-m}(t-m) & \text{if } t \in J_{m+1,m^2}, \\ t-m^2 & \text{if } t \in J_{m^2+1,m^2+N}, \\ N-(t-(m^2+N)) & \text{if } t \in J_{m^2+N+1,m^2+N+m}, \\ \dot{f}_{m|N-m}(t-(N+m)) & \text{if } t \in J_{m^2+N+m+1,N^2}. \end{cases}$$

Let $\dot{p}_{m|N-m} := p^-_{N-m|m} \in \mathcal{A}_{m|N-m}$. Note that if $\dot{p} = \dot{p}_{m|N-m}$, then $(\bar{\eta}^{\dot{p}}, \langle\hat{\Pi}\rangle)$ is the $\mathrm{B}(m, N-m)$-data. Moreover if $\dot{p} = \dot{p}_{m|N-m}$ and $\bar{R}$ is the standard root system associated with $(\bar{\eta}^{\dot{p}}, \langle\hat{\Pi}\rangle)$, then $\hat{R} = \bar{R} \setminus 2\bar{R}$.

Let $\dot{p}^{(0)}_{m|N-m} := \dot{p}_{m|N-m}$. For $k \in J_{1,m}$, let $\dot{p}^{(k)}_{m|N-m} := \dot{p}^{(k-1)}_{m|N-m} \circ \wp^{(N)}_{N-m+k-1}$. Then for $t \in J_{1,N^2}$, we have

$$(3.10)\qquad (\dot{p}_{m|N-m})_{\dot{f}_{m|N-m},t} = \begin{cases} \dot{p}_{m|N-m} & \text{if } t \in J_{1,m^2+N-m-1}, \\ \dot{p}^{(t-(m^2+N-m-1))}_{m|N-m} & \text{if } t \in J_{m^2+N-m,m^2+N-1}, \\ \dot{p}^{(m-(t-(m^2+N)))}_{m|N-m} & \text{if } t \in J_{m^2+N,m^2+N+m}, \\ \dot{p}^{(t-(N+m))}_{m|N-m} & \text{if } t \in J_{m^2+N+m+1,N^2}. \end{cases}$$

Lemma 3.10. *Let $m \in I$, and consider the Weyl groupoid $\mathcal{W}(\dot{\mathcal{R}}_{m|N-m})$. Let $\dot{p} := \dot{p}_{m|N-m} \in \mathcal{A}_{m|N-m}$.*

(1) *Let θ be the $\mathrm{B}(m, N-m)$-type parity map. Let $a^\sharp := (\bar{\eta}^{\dot{p}}, \theta) \in \mathcal{Y}^{\mathrm{fin}}_N$. Then $(\dot{\mathcal{R}}_{m|N-m}, \dot{p})$ and $(\mathcal{R}_{a^\sharp}, a^\sharp)$ are isomorphic. Moreover for $f \in \mathrm{Map}^I_\infty$ and $r \in \mathbb{N}$, $a^\sharp_{f,r} = (\bar{\eta}^{\dot{p}_{f,r}}, \theta_{f,r})$, where $\theta_{f,r}(i) := [\theta](1^{\dot{p}} s_{f,r}(\alpha_i))$ $(i \in I)$.*

(2) *Let $n := N^2$ and $\dot{f} := \dot{f}_{m|N-m} \in \mathrm{Map}^I_n$. Then*

$$(3.11)\qquad \ell^{\dot{p}}(1^{\dot{p}} w_0) = n \quad \textit{and} \quad 1^{\dot{p}} w_0 = 1^{\dot{p}} s_{\dot{f},n}.$$

Moreover for $t \in J_{1,n}$, letting $\beta_t := 1^{\dot{p}} s_{\dot{f},t-1}(\alpha_t)$, we have $\dot{R}^+ = \{\beta_t \mid t \in J_{1,n}\}$,

$$(3.12)\qquad \bar{\eta}^{\dot{p}}_{\langle\hat{\Pi}\rangle}(\beta_t, \beta_t) = \begin{cases} 2 & \text{if } t \in J_{1,m-1}, \\ 1 & \text{if } t = m, \\ \bar{\eta}^{\dot{p}}_{\langle\hat{\Pi}\rangle}(\beta_{t-m}, \beta_{t-m}) & \text{if } t \in J_{m+1,m^2}, \\ -2 & \text{if } t \in J_{m^2+1,m^2+N-m-1}, \\ 0 & \text{if } t \in J_{m^2+N-m,m^2+N-1}, \\ -1 & \text{if } t = m^2+N, \\ 0 & \text{if } t \in J_{m^2+N+1,m^2+N+m}, \\ \bar{\eta}^{\dot{p}}_{\langle\hat{\Pi}\rangle}(\beta_{t-(N+m)}, \beta_{t-(N+m)}) & \text{if } t \in J_{m^2+N+m+1,N^2}, \end{cases}$$

$$(3.13)\qquad [\theta](\beta_t) = \delta(\bar{\eta}^{\dot{p}}_{\langle\hat{\Pi}\rangle}(\beta_t, \beta_t), 0) + \delta(\bar{\eta}^{\dot{p}}_{\langle\hat{\Pi}\rangle}(\beta_t, \beta_t), -1).$$

Proof. (1) can be proved directly.

(2) We have (3.11) by Lemmas 1.12 and 3.9 and (2.14), (2.16), (2.25) and (2.26). Then (3.12) follows directly from (3.11), (1.17), (3.8), (3.9) and (3.10). We can also prove (3.13) directly. □

§3.5. Longest element of the Weyl groupoid of the simple Lie superalgebra $\mathrm{D}(m, N-m)$

In this subsection, assume $N \geq 3$. Let $\ddot{I} := I \cup \{N+1\} = J_{1,N+1}$.

Let $m \in J_{1,N-1}$. Let $\ddot{\mathcal{A}}_{m|N-m}$ be the set of all maps $\ddot{p} : \ddot{I} \to J_{0,1}$ satisfying:

($p1$) $\sum_{i\in I} \ddot{p}(i) = m$.
($p2$) If $\ddot{p}(N) = 0$, then $\ddot{p}(N+1) = 0$.

For example, $|\ddot{\mathcal{A}}_{2|2}| = 9$ (see Figure 1 in Section 5).

For $i \in I$, define a bijection

$$\ddot{\tau}_i : \ddot{\mathcal{A}}_{m|N-m} \to \ddot{\mathcal{A}}_{m|N-m}$$

by the following conditions (recall $\wp_i^{(N+1)}$ from (3.5)):

($\ddot{\tau}$-1) For $i \in J_{1,N-2}$, let $\ddot{\tau}_i(\ddot{p}) := \ddot{p} \circ \wp_i^{(N+1)}$.
($\ddot{\tau}$-2) For $i = N-1$, define $\ddot{\tau}_i(\ddot{p}) := \ddot{p} \circ \wp_{N-1}^{(N+1)}$ if $\ddot{p}(N+1) = 0$, and define $\ddot{\tau}_i(\ddot{p}) := \ddot{p}$ if $\ddot{p}(N+1) = 1$.
($\ddot{\tau}$-3) For $i = N$, define $\ddot{\tau}_i(\ddot{p}) := \ddot{p}$ if $\ddot{p}(N-1) = \ddot{p}(N)$, and define $\ddot{\tau}_i(\ddot{p})$ by $\ddot{\tau}_i(\ddot{p})_{|J_{1,N-2}} := \ddot{p}_{|J_{1,N-2}}$ and $(\ddot{\tau}_i(\ddot{p}(N-1)), \ddot{\tau}_i(\ddot{p}(N)), \ddot{\tau}_i(\ddot{p}(N+1))) := (0,1,0)$ (resp. $(1,0,0)$, $(0,1,1)$) if $(\ddot{p}(N-1), \ddot{p}(N), \ddot{p}(N+1))$ equals $(0,1,0)$ (resp. $(0,1,1)$, $(1,0,0)$).

All $\ddot{p} \in \ddot{\mathcal{A}}_{m|N-m}$ with $N = 4$ and $m = 2$ are given in Figure 1.

Let $\ddot{p} \in \ddot{\mathcal{A}}_{m|N-m}$. Define an $\mathbb{R}$-bilinear map $\ddot{\eta}^{\ddot{p}} : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ by

$$\ddot{\eta}^{\ddot{p}}(e_i, e_j) := \delta(\ddot{p}(i), \ddot{p}(j))(-1)^{\ddot{p}(i)} \qquad (i, j \in I). \tag{3.14}$$

Define a subset $\ddot{R}(\ddot{p})$ of $\mathbb{R}^N$ by

$$\begin{aligned} \ddot{R}(\ddot{p}) := &\{ce_i + c'e_j \mid i, j \in I,\, i \neq j,\, c, c' \in \{1, -1\}\} \\ &\cup \{c''e_i \mid i \in I,\, \ddot{p}(i) = 1,\, c'' \in \{2, -2\}\}. \end{aligned} \tag{3.15}$$

Let $\langle \ddot{\Pi}^{\ddot{p}} \rangle := \langle \ddot{\alpha}_i^{\ddot{p}} \mid i \in I \rangle \in \mathbb{R}^N \times \cdots \times \mathbb{R}^N$ (N times). For $i \in I$, define $\ddot{\alpha}_i^{\ddot{p}} \in \mathbb{R}^N$ by

$$\ddot{\alpha}_i^{\ddot{p}} := \begin{cases} e_i - e_{i+1} & \text{if } i \in J_{1,N-2}, \\ e_{N-1} - e_N & \text{if } i = N-1 \text{ and } \ddot{p}(N+1) = 0, \\ -2e_N & \text{if } i = N-1 \text{ and } \ddot{p}(N+1) = 1, \\ e_{N-1} + e_N & \text{if } i = N \text{ and } \ddot{p}(N) = \ddot{p}(N+1), \\ 2e_N & \text{if } i = N \text{ and } \ddot{p}(N) = 1,\, \ddot{p}(N+1) = 0. \end{cases} \tag{3.16}$$

We can directly see that

$$\ddot{\eta}^{\ddot{p}}_{\langle\ddot{\Pi}^{\ddot{p}}\rangle}(x,y)=\ddot{\eta}^{\ddot{\tau}_i\ddot{p}}_{\langle\ddot{\Pi}'\rangle}(\hat{s}_{\ddot{\alpha}_i^{\ddot{p}}}(x),\hat{s}_{\ddot{\alpha}_i^{\ddot{p}}}(y))\quad (i\in I,\, x,y\in\mathbb{R}^N). \tag{3.17}$$

where $\langle\ddot{\Pi}'\rangle := \langle\ddot{\Pi}^{\ddot{\tau}_i\ddot{p}}\rangle$. Define the generalized Cartan matrix $\ddot{C}^{\ddot{p}} = [\ddot{c}^{\ddot{p}}_{ij}]_{i,j\in I} \in \mathrm{M}_N(\mathbb{Z})$ by

$$\ddot{c}^{\ddot{p}}_{ij} := \begin{cases} 2 & \text{if } i=j, \\ \dfrac{2\ddot{\eta}^{\ddot{p}}(\ddot{\alpha}^{\ddot{p}}_i,\ddot{\alpha}^{\ddot{p}}_j)}{\ddot{\eta}^{\ddot{p}}(\ddot{\alpha}^{\ddot{p}}_i,\ddot{\alpha}^{\ddot{p}}_i)} & \text{if } i\neq j \text{ and } \ddot{\eta}^{\ddot{p}}(\ddot{\alpha}^{\ddot{p}}_i,\ddot{\alpha}^{\ddot{p}}_i)\neq 0, \\ 0 & \text{if } i\neq j \text{ and } \ddot{\eta}^{\ddot{p}}(\ddot{\alpha}^{\ddot{p}}_i,\ddot{\alpha}^{\ddot{p}}_i)=\ddot{\eta}^{\ddot{p}}(\ddot{\alpha}^{\ddot{p}}_i,\ddot{\alpha}^{\ddot{p}}_j)=0, \\ -1 & \text{if } i\neq j,\ \ddot{\eta}^{\ddot{p}}(\ddot{\alpha}^{\ddot{p}}_i,\ddot{\alpha}^{\ddot{p}}_i)=0 \text{ and } \ddot{\eta}^{\ddot{p}}(\ddot{\alpha}^{\ddot{p}}_i,\ddot{\alpha}^{\ddot{p}}_j)\neq 0. \end{cases}$$

We can directly see the following:

Proposition 3.11. (1) $\ddot{\mathcal{C}}_{m|N-m} := \mathcal{C}(I,\ddot{\mathcal{A}}_{m|N-m},(\ddot{\tau}_i)_{i\in I},(\ddot{C}^{\ddot{p}})_{\ddot{p}\in\ddot{\mathcal{A}}_{m|N-m}})$ *is a Cartan scheme.*

(2) *There exists a unique generalized root system*

$$\ddot{\mathcal{R}}_{m|N-m} := \mathcal{R}(\ddot{\mathcal{C}}_{m|N-m},(R(\ddot{p}))_{\ddot{p}\in\ddot{\mathcal{A}}_{m|N-m}})$$

of type $\ddot{\mathcal{C}}_{m|N-m}$ such that

$$\ddot{\xi}^{\ddot{p}}_{\langle\ddot{\Pi}^{\ddot{p}}\rangle}(\ddot{R}(\ddot{p}))=R(\ddot{p})\quad (\ddot{p}\in\ddot{\mathcal{A}}_{m|N-m}). \tag{3.18}$$

Moreover,

$$\ddot{\xi}^{\ddot{\tau}_i\ddot{p}}_{\langle\ddot{\Pi}'\rangle}\circ\hat{s}_{\ddot{\alpha}_i^{\ddot{p}}}=s_i^{\ddot{p}}\circ\ddot{\xi}^{\ddot{p}}_{\langle\ddot{\Pi}^{\ddot{p}}\rangle}\quad (\ddot{p}\in\ddot{\mathcal{A}}_{m|N-m},\, i\in I), \tag{3.19}$$

where $\langle\ddot{\Pi}'\rangle := \langle\ddot{\Pi}^{\ddot{\tau}_i\ddot{p}}\rangle$.

Define $\ddot{p}_{m|N-m}\in\ddot{\mathcal{A}}_{m|N-m}$ by

$$\ddot{p}_{m|N-m}(i) := \begin{cases} 1 & \text{if } i\in J_{1,N-m}, \\ 0 & \text{if } i\in J_{N-m+1,N+1}. \end{cases}$$

Note that for $m\in J_{1,N-1}$, $(\ddot{\eta}^{\ddot{p}_{m|N-m}},\langle\ddot{\Pi}^{\ddot{p}_{m|N-m}}\rangle)$ is the $\mathrm{D}(m,N-m)$-data and $\ddot{R}(\ddot{\eta}^{\ddot{p}_{m|N-m}})$ is the $\mathrm{D}(m,N-m)$ standard root system associated with $(\ddot{\eta}^{\ddot{p}_{m|N-m}}, \langle\ddot{\Pi}^{\ddot{p}_{m|N-m}}\rangle)$.

For $k\in J_{0,m}$, define $\ddot{p}^{(k)}_{m|N-m}\in\ddot{\mathcal{A}}_{m|N-m}$ by

$$\ddot{p}^{(k)}_{m|N-m}(i) := \begin{cases} 0 & \text{if } i\in J_{1,N-m-1}\cup\{N-m+k\}\cup\{N+1\}, \\ 1 & \text{if } i\in J_{N-m,N-m+k-1}\cup J_{N-m+k+1,N}, \end{cases}$$

so

$$\ddot{p}_{m|N-m} = \ddot{p}^{(0)}_{m|N-m} \quad \text{and} \quad \ddot{p}^{(k)}_{m|N-m} = \ddot{p}^{(k-1)}_{m|N-m} \circ \wp^{(N+1)}_{N-m-1+k} \ (k \in J_{1,m}). \tag{3.20}$$

By Lemma 1.9(1), (3.15) and (3.18),

$$\ell_{\ddot{p}}(1^{\ddot{p}} w_0) = |R^+(\ddot{p})| = N^2 - m \qquad (\ddot{p} \in \ddot{\mathcal{A}}_{m|N-m}). \tag{3.21}$$

Define $\ddot{f}_{m|N-m} \in \mathrm{Map}^I_{N^2-m}$ by

$$\begin{aligned} &\ddot{f}_{m|N-m}(t) \\ &:= \begin{cases} N-m+t & \text{if } t \in J_{1,m}, \\ \ddot{f}_{m|N-m}(t-m) & \text{if } t \in J_{m+1,m(m-1)}, \\ t-m(m-1) & \text{if } t \in J_{m(m-1)+1,m(m-1)+N}, \\ N-(t-(m(m-1)+N)) & \text{if } t \in J_{m(m-1)+N+1,m(m-1)+N+m}, \\ \ddot{f}_{m|N-m}(t-(N+m)) & \text{if } t \in J_{N+m^2+1,N^2-m}. \end{cases} \end{aligned} \tag{3.22}$$

Using (3.20), we can see that for $t \in J_{1,N^2-m}$,

$$\begin{aligned} &(\ddot{p}_{m|N-m})_{\ddot{f}_{m|N-m},t} \\ &= \begin{cases} \ddot{p}_{m|N-m} & \text{if } t \in J_{1,m(m-1)+N-m-1}, \\ \ddot{p}^{(t-(m(m-1)+N-m-1))}_{m|N-m} & \text{if } t \in J_{m(m-1)+N-m,m(m-1)+N-1}, \\ \ddot{p}^{(m-(t-(m(m-1)+N))}_{m|N-m} & \text{if } t \in J_{m(m-1)+N,m(m-1)+N+m}, \\ \ddot{p}^{(t-(N+m))}_{m|N-m} & \text{if } t \in J_{N+m^2+1,N^2-m}. \end{cases} \end{aligned} \tag{3.23}$$

Note that if $m \in J_{2,N-1}$, $\ddot{p} = \ddot{p}_{m|N-m}$, then $(\ddot{\eta}^{\ddot{p}}, \langle \ddot{\Pi}^{\ddot{p}} \rangle)$ is the $\mathrm{D}(m, N-m)$-data and $\ddot{R}(\ddot{p})$ is the standard root system associated with $(\ddot{\eta}^{\ddot{p}}, \langle \ddot{\Pi}^{\ddot{p}} \rangle)$.

Proposition 3.12. *Let $m \in I$, and consider the Weyl groupoid $\mathcal{W}(\ddot{\mathcal{R}}_{m|N-m})$. Let $\ddot{p} := \ddot{p}_{m|N-m} \in \ddot{\mathcal{A}}_{m|N-m}$.*

(1) *Assume $m \in J_{2,N-1}$. Let θ be the $\mathrm{D}(m, N-m)$-type parity map. Let $a^\sharp := (\ddot{\eta}^{\ddot{p}}_{\langle \ddot{\Pi}^{\ddot{p}} \rangle}, \theta) \in \mathcal{Y}^{\mathrm{fin}}_N$. Then $(\ddot{\mathcal{R}}_{m|N-m}, \ddot{p})$ and $(\mathcal{R}_{a^\sharp}, a^\sharp)$ are isomorphic. Moreover for $f \in \mathrm{Map}^I_\infty$ and $r \in \mathbb{N}$, $a^\sharp_{f,r} = (\ddot{\eta}^{\ddot{p}_{f,r}}_{\langle \ddot{\Pi}' \rangle}, \theta_{f,r})$, where $\langle \ddot{\Pi}' \rangle := \langle \ddot{\Pi}^{\ddot{p}_{f,r}} \rangle$, and $\theta_{f,r}$ is defined by $\theta_{f,r}(i) := [\theta](1^p s_{f,r}(\alpha_i))$ $(i \in I)$.*

(2) *Let $n := N^2 - m$ and $\ddot{f} := \ddot{f}_{m|N-m} \in \mathrm{Map}^I_n$. Then*

$$\ell_{\ddot{p}}(1^{\ddot{p}} w_0) = N^2 - m, \qquad 1^{\ddot{p}} w_0 = 1^{\ddot{p}} s_{\ddot{f},n}, \tag{3.24}$$

$$\ddot{p}_{\ddot{f},n} = \ddot{p}. \tag{3.25}$$

Moreover, letting $r := m(m-1)$, we have

$$\ddot{\eta}^{\ddot{p}_{\ddot{f},t-1}}(\ddot{\alpha}_t,\ddot{\alpha}_t)\ (=\ddot{\eta}^{\ddot{p}}_{\langle\ddot{\Pi}^{\ddot{p}}\rangle}(1^{\ddot{p}}s_{\ddot{f},t-1}(\alpha_t),1^{\ddot{p}}s_{\ddot{f},t-1}(\alpha_t)))$$
$$=\begin{cases} 2 & \text{if } t\in J_{1,m}, \\ \ddot{\eta}^{\ddot{p}_{\ddot{f},t-m-1}}(\ddot{\alpha}_{t-m},\ddot{\alpha}_{t-m}) & \text{if } t\in J_{m+1,r}, \\ 2 & \text{if } t\in J_{r+1,r+N-m-1}, \\ 0 & \text{if } t\in J_{r+N-m,r+N-1}, \\ -4 & \text{if } t=m^2+N, \\ 0 & \text{if } t\in J_{r+N+1,r+N+m}, \\ \ddot{\eta}^{\ddot{p}_{\ddot{f},t-(N+m)-1}}(\ddot{\alpha}_{t-(N+m)},\ddot{\alpha}_{t-(N+m)}) & \text{if } t\in J_{r+N+m+1,n}, \end{cases} \tag{3.26}$$

and

$$[\theta](1^{\ddot{p}}s_{\ddot{f},t-1}(\alpha_t))=\delta(\ddot{\eta}^{p}_{\langle\ddot{\Pi}^{\ddot{p}}\rangle}(1^{\ddot{p}}s_{\ddot{f},t-1}(\alpha_t),1^{\ddot{p}}s_{\ddot{f},t-1}(\alpha_t)),0)\qquad(t\in J_{1,N}). \tag{3.27}$$

Proof. (1) can be proved directly.

(2) Let $r := m(m-1)$ and $r' := n-r$. Define $\ddot{f}'\in \mathrm{Map}^I_{r'}$ by $\ddot{f}'(y) := \ddot{f}(y+r)$ $(y\in J_{1,r'})$. By (3.23), $\ddot{p}=\ddot{p}_{\ddot{f},r}=\ddot{p}_{\ddot{f},n}$, since $r'=(N+m)(N-m)$. Hence we have (3.25) and

$$1^{\ddot{p}}s_{\ddot{f},n}=1^{\ddot{p}}s_{\ddot{f},r}1^{\ddot{p}}s_{\ddot{f}',r'}. \tag{3.28}$$

For $t\in J_{1,r}$, since $\ddot{f}(t)\in J_{N-m+1,N}$ for $t\in J_{1,r}$, by (3.16) and (3.23), we have

$$\ddot{\alpha}^{\ddot{p}_{f,t}}_{\ddot{f}(t)}=\begin{cases} e_{\ddot{f}(t)}-e_{\ddot{f}(t)+1} & \text{if } \ddot{f}(t)\in J_{N-m+1,N-1}, \\ e_{\ddot{f}(t)-1}+e_{\ddot{f}(t)} & \text{if } \ddot{f}(t)=N. \end{cases}$$

Hence by (2.18) and (3.19), we have

$$(\xi_{\langle\ddot{\Pi}^{\ddot{p}}\rangle})^{-1}\circ 1^{\ddot{p}}s_{\ddot{f},r}\circ\xi_{\langle\ddot{\Pi}^{\ddot{p}}\rangle}=P_{J_{1,N-m}}-P_{J_{N-m+1,N-1}}-(-1)^m P_{J_{N,N}}. \tag{3.29}$$

For $t'\in J_{1,r'}$, we can directly see

$$\ddot{\alpha}^{\ddot{p}_{\ddot{f}',t'}}_{\ddot{f}'(t')}=\begin{cases} e_{\ddot{f}'(t')}-e_{\ddot{f}'(t')+1} & \text{if } \ddot{f}'(t')\in J_{1,N-1}, \\ 2e_N & \text{if } \ddot{f}'(t')=N. \end{cases} \tag{3.30}$$

By (2.24), (3.19) and (3.30), we have

$$(\xi_{\langle\ddot{\Pi}^{\ddot{p}}\rangle})^{-1}\circ 1^{\ddot{p}}s_{\ddot{f}',r'}\circ\xi_{\langle\ddot{\Pi}^{\ddot{p}}\rangle}=-P_{J_{1,N-m}}+P_{J_{N-m+1,N}}. \tag{3.31}$$

By (3.28), (3.29) and (3.31), we have

$$(\xi_{\langle\ddot{\Pi}^{\ddot{p}}\rangle})^{-1}\circ 1^{\ddot{p}}s_{f,n}\circ\xi_{\langle\ddot{\Pi}^{\ddot{p}}\rangle}=-P_{J_{1,N-1}}-(-1)^m P_{J_{N,N}}. \tag{3.32}$$

By (3.16) and (3.32), we have

$$1^{\ddot{p}}s_{\ddot{f},n}(\alpha_i) = \begin{cases} -\alpha_i & \text{if } i \in J_{1,N-2}, \\ -\alpha_i & \text{if } m \in 2\mathbb{N} \text{ and } i \in J_{N-1,N}, \\ -\alpha_N & \text{if } m \in 2\mathbb{N}-1 \text{ and } i = N-1, \\ -\alpha_{N-1} & \text{if } m \in 2\mathbb{N}-1 \text{ and } i = N. \end{cases}$$

Hence $1^{\ddot{p}}s_{\ddot{f},n}(\Pi) = -\Pi$. By (1.16) and (3.21), we have (3.24). We can prove (3.26) and (3.27) directly.

□

§3.6. Longest element of the Weyl groupoid of the simple Lie superalgebra C(N)

In this subsection, assume $N \geq 3$. Define $\acute{p}_N \in \ddot{\mathcal{A}}_{1|N-1}$ by $\acute{p}_N(i) := 1-\delta_{i,1}-\delta_{i,N+1}$ $(i \in \ddot{I})$. Note that if $\acute{p} = \acute{p}_N$ and $\langle\acute{\Pi}\rangle = \langle\ddot{\Pi}^{\acute{p}}\rangle$, then $(-\ddot{\eta}^{\acute{p}}, \langle\acute{\Pi}\rangle)$ is the C(N)-data and $\ddot{R}(\acute{p})$ is the standard root system associated with $(-\ddot{\eta}^{\acute{p}}, \langle\acute{\Pi}\rangle)$, where $\ddot{\eta}^{\acute{p}}$ is defined by (3.14). Define $\acute{f}_N \in \mathrm{Map}^I_{N^2-1}$ by

$$\acute{f}_N(t) := \begin{cases} t+1 & \text{if } t \in J_{1,N-1}, \\ \acute{f}_N(t-(N-1)) & \text{if } t \in J_{N,(N-1)^2}, \\ t-(N-1)^2 & \text{if } t \in J_{(N-1)^2+1,(N-1)^2+N}, \\ N-1-(t-((N-1)^2+N)) & \text{if } t \in J_{(N-1)^2+N+1,N^2-1}. \end{cases}$$

Proposition 3.13. *Consider the Weyl groupoid $\mathcal{W}(\ddot{\mathcal{R}}_{1|N-1})$. Let $\acute{p} := \acute{p}_N$ and $\langle\acute{\Pi}\rangle := \langle\ddot{\Pi}^{\acute{p}}\rangle$.*

(1) *Let θ be the C(N)-type parity map. Let $a^\sharp := (-\ddot{\eta}^{\acute{p}}_{\langle\acute{\Pi}\rangle}, \theta) \in \mathcal{Y}^{\mathrm{fin}}_N$. Then $(\ddot{\mathcal{R}}_{1|N-1}, \acute{p})$ and $(\mathcal{R}_{a^\sharp}, a^\sharp)$ are isomorphic. Moreover for $f \in \mathrm{Map}^I_\infty$ and $r \in \mathbb{N}$, $a^\sharp_{f,r} = (-\ddot{\eta}^{\acute{p}_{f,r}}_{\langle\acute{\Pi}'\rangle}, \theta_{f,r})$, where $\langle\acute{\Pi}'\rangle := \langle\ddot{\Pi}^{\acute{p}_{f,r}}\rangle$, and $\theta_{f,r}$ is defined by $\theta_{f,r}(i) := [\theta](1^p s_{f,r}(\alpha_i))$ $(i \in I)$.*

(2) *We have*

$$\ell_{\acute{p}}(1^{\acute{p}}w_0) = N^2-1, \qquad 1^{\acute{p}}w_0 = 1^{\acute{p}}s_{\acute{f}_N,N^2-1}, \tag{3.33}$$

$$\acute{p}_{\acute{f}_N,N^2-1}(i) = 1-\delta_{i,1} \qquad (i \in \ddot{I}). \tag{3.34}$$

Moreover letting $m_\alpha := -\ddot{\eta}^{\acute{p}}_{\langle\acute{\Pi}\rangle}(\alpha,\alpha)$ $(\alpha \in R^+(p))$, we have

$$\begin{aligned} &\{1^{\acute{p}}s_{\acute{f}_N,t-1}(\alpha_{\acute{f}_N(t)}) \mid t \in J_{1,(N-1)^2}\} \\ &\qquad = \{\alpha \in R^+(\acute{p}) \mid m_\alpha \in \{2,4\}\} = \{\alpha \in R^+(\acute{p}) \mid [\theta](\alpha) = 0\}, \end{aligned} \tag{3.35}$$

$$\begin{aligned} &\{1^{\acute{p}}s_{\acute{f}_N,t-1}(\alpha_{\acute{f}_N(t)}) \mid t \in J_{(N-1)^2+1,N^2-1}\} \\ &\qquad = \{\alpha \in R^+(\acute{p}) \mid m_\alpha = 0\} = \{\alpha \in R^+(\acute{p}) \mid [\theta](\alpha) = 1\}. \end{aligned} \tag{3.36}$$

Proof. (1) This can be proved directly.

(2) Let $\acute{f} := \acute{f}_N$ and $r := (N-1)^2$. For $t \in J_{1,r}$, we have $\acute{p}_{\acute{f},t} = p$, so (3.16) implies

$$\ddot{\alpha}^{\acute{p}_{\acute{f},t}}_{\acute{f}(t)} = \begin{cases} e_{\acute{f}(t)} - e_{\acute{f}(t)+1} & \text{if } \acute{f}(t) \in J_{2,N-1}, \\ 2e_N & \text{if } \acute{f}(t) = N. \end{cases} \tag{3.37}$$

In particular,

$$\acute{p}_{\acute{f},r} = \acute{p}. \tag{3.38}$$

By (2.24), (3.19) and (3.37), we have

$$(\xi_{\langle\acute{\Pi}\rangle})^{-1} \circ 1^{\acute{p}} s_{\acute{f},r} \circ \xi_{\langle\acute{\Pi}\rangle} = P_{J_{1,1}} - P_{J_{2,N}}. \tag{3.39}$$

By (3.38), we directly have (3.34) and

$$\ddot{\alpha}^{\acute{p}_{\acute{f},r+t}}_{\acute{f}(r+t)} = \begin{cases} e_t - e_{t+1} & \text{if } t \in J_{1,N-1}, \\ e_{N-1} + e_N & \text{if } t = N, \\ e_{2N-1-t} - e_{2N-t} & \text{if } t \in J_{N+1,2N-2}, \end{cases} \tag{3.40}$$

for $t \in J_{1,2N-2}$. Since $\hat{s}_{e_{N-1}-e_N}\hat{s}_{e_{N-1}+e_N} = \hat{s}_{e_{N-1}}\hat{s}_{e_N}$, by (3.40) we have

$$\hat{s}_{\acute{f}(r+1)} \cdots \hat{s}_{\acute{f}(N^2-1)} = \hat{s}_{e_1}\hat{s}_{e_N} = -P_{J_{1,1}} + P_{J_{2,N-1}} - P_{J_{N,N}}. \tag{3.41}$$

By (2.24), (3.19), (3.39) and (3.41),

$$(\xi_{\langle\acute{\Pi}\rangle})^{-1} \circ 1^{\acute{p}} s_{\acute{f},n} \circ \xi_{\langle\acute{\Pi}'\rangle} = -P_{J_{1,N-1}} + P_{J_{N,N}}, \tag{3.42}$$

where $n := N^2 - 1$ and $\langle\acute{\Pi}'\rangle := \langle\ddot{\Pi}^{\acute{f},n}\rangle$. By (3.16) and (3.42),

$$1^{\acute{p}} s_{\acute{f},N^2-1}(\alpha_i) = \begin{cases} -\alpha_i & \text{if } i \in J_{1,N-2}, \\ -\alpha_N & \text{if } i = N-1, \\ -\alpha_{N-1} & \text{if } i = N. \end{cases}$$

Hence $1^{\acute{p}} s_{\acute{f},N^2-1}(\Pi) = -\Pi$. Hence by (1.16) and (3.21), we have (3.33). By (1.15), (3.15), (3.16), (3.18) and (3.39), the LHS of (3.35) is $R^+(\acute{p}) \cap \ddot{\xi}^{\acute{p}}(\bigoplus_{x=2}^N \mathbb{R}e_x)$. Hence we obtain (3.35) and (3.36). □

§4. Generalized quantum groups

§4.1. Bi-homomorphism χ and Dynkin diagram of χ

We say that a map $\chi : \mathbb{Z}\Pi \times \mathbb{Z}\Pi \to \mathbb{K}^\times$ is a *bi-homomorphism on* Π if

$$\chi(\alpha, \beta + \gamma) = \chi(\alpha, \beta)\chi(\alpha, \gamma), \qquad \chi(\alpha + \beta, \gamma) = \chi(\alpha, \gamma)\chi(\beta, \gamma)$$

for all $\alpha, \beta, \gamma \in \mathbb{Z}\Pi$. Let $\mathcal{X}_N$ be the set of bi-homomorphisms on Π.

Let $\chi \in \mathcal{X}_N$ and let $q_{ij} := \chi(\alpha_i, \alpha_j)$ for $i, j \in I$. By the *Dynkin diagram* of χ, we mean the unoriented graph with N dots such that the i-th dot is labeled α_i and q_{ii}, and the j-th and k-th dots with $j \neq k$ and $q_{jk}q_{kj} \neq 1$ are joined by a single edge labeled $q_{jk}q_{kj}$. For example, if $N = 3$ and $q_{11} = -1$, $q_{22} = \hat{q}^2$, $q_{33} = \hat{q}^6$, $q_{12}q_{21} = \hat{q}^{-2}$, $q_{23}q_{32} = \hat{q}^{-6}$ and $q_{13}q_{31} = 1$ for some $\hat{q} \in \mathbb{K}^\times_\infty$, then the Dynkin diagram of χ is the leftmost diagram of Figure 9. Note that the Dynkin diagram of χ does not determine χ. In fact, it determines the $\equiv$ equivalence class of χ, which will be introduced in (4.18) below.

§4.2. The quantum group $U = U(\chi)$ associated with $\chi \in \mathcal{X}_N$

From now on until the end of Subsection 4.4, we fix $\chi \in \mathcal{X}_N$, and let $q_{ij} := \chi(\alpha_i, \alpha_j)$ for $i, j \in I$.

Let $\tilde{U} := \tilde{U}(\chi)$ be the unital associative $\mathbb{K}$-algebra defined by the generators

$$\tilde{K}_\alpha, \tilde{L}_\alpha \ (\alpha \in \mathbb{Z}\Pi), \qquad \tilde{E}_i, \tilde{F}_i \ (i \in I)$$

and relations

$$\begin{array}{l} \tilde{K}_0 = \tilde{L}_0 = 1, \quad \tilde{K}_\alpha \tilde{K}_\beta = \tilde{K}_{\alpha+\beta}, \quad \tilde{L}_\alpha \tilde{L}_\beta = \tilde{L}_{\alpha+\beta}, \quad \tilde{K}_\alpha \tilde{L}_\beta = \tilde{L}_\beta \tilde{K}_\alpha, \\ \tilde{K}_\alpha \tilde{E}_i = \chi(\alpha, \alpha_i)\tilde{E}_i \tilde{K}_\alpha, \quad \tilde{L}_\alpha \tilde{E}_i = \chi(-\alpha_i, \alpha)\tilde{E}_i \tilde{L}_\alpha, \\ \tilde{K}_\alpha \tilde{F}_i = \chi(\alpha, -\alpha_i)\tilde{F}_i \tilde{K}_\alpha, \quad \tilde{L}_\alpha \tilde{F}_i = \chi(\alpha_i, \alpha)\tilde{F}_i \tilde{L}_\alpha, \\ \tilde{E}_i \tilde{F}_j - \tilde{F}_j \tilde{E}_i = \delta_{ij}(-\tilde{K}_{\alpha_i} + \tilde{L}_{\alpha_i}), \end{array} \tag{4.1}$$

for all $\alpha, \beta \in \mathbb{Z}\Pi$ and all $i, j \in I$.

Remark 4.1. $\tilde{U}$ has a Hopf algebra structure with coproduct $\Delta : \tilde{U} \to \tilde{U} \oplus \tilde{U}$ such that $\Delta(\tilde{K}_\alpha) = \tilde{K}_\alpha \otimes \tilde{K}_\alpha$, $\Delta(\tilde{L}_\alpha) = \tilde{L}_\alpha \otimes \tilde{L}_\alpha$ $(\alpha \in \mathbb{Z}\Pi)$, and $\Delta(\tilde{E}_i) = \tilde{E}_i \otimes 1 + \tilde{K}_{\alpha_i} \otimes \tilde{E}_i$, $\Delta(\tilde{F}_i) = \tilde{F}_i \otimes \tilde{L}_{\alpha_i} + 1 \otimes \tilde{F}_i$ $(i \in I)$.

Define a $\mathbb{K}$-algebra automorphism $\tilde{\Omega} : \tilde{U} \to \tilde{U}$ by $\tilde{\Omega}(\tilde{K}_\alpha) := \tilde{K}_{-\alpha}$, $\tilde{\Omega}(\tilde{L}_\alpha) := \tilde{L}_{-\alpha}$, $\tilde{\Omega}(\tilde{E}_i) := \tilde{F}_i \tilde{L}_{-\alpha_i}$, and $\tilde{\Omega}(\tilde{F}_i) := \tilde{K}_{-\alpha_i}\tilde{E}_i$. Define $\chi^{\mathrm{op}} \in \mathcal{X}_N$ by $\chi^{\mathrm{op}}(\alpha, \beta) := \chi(\beta, \alpha)$ $(\alpha, \beta \in \mathbb{Z}\Pi)$. Define a $\mathbb{K}$-algebra automorphism $\tilde{\Upsilon} : \tilde{U}(\chi^{\mathrm{op}}) \to \tilde{U}(\chi)$ by $\tilde{\Upsilon}(\tilde{K}_\alpha) := \tilde{L}_\alpha$, $\tilde{\Upsilon}(\tilde{L}_\alpha) := \tilde{K}_\alpha$, $\tilde{\Upsilon}(\tilde{E}_i) := \tilde{F}_i$, and $\tilde{\Upsilon}(\tilde{F}_i) := \tilde{E}_i$.

Let $\tilde{U}^0 := \tilde{U}^0(\chi)$ (resp. $\tilde{U}^+ := \tilde{U}^+(\chi)$, $\tilde{U}^- := \tilde{U}^-(\chi)$) be the unital subalgebra of $\tilde{U}$ generated by $\tilde{K}_\alpha$, $\tilde{L}_\alpha$ $(\alpha \in \mathbb{Z}\Pi)$ (resp. $\tilde{E}_i$ $(i \in I)$, $\tilde{F}_i$ $(i \in I)$).

Lemma 4.2. *The elements*

$$\tilde{F}_{j_1} \cdots \tilde{F}_{j_m} \tilde{K}_\alpha \tilde{L}_\beta \tilde{E}_{i_1} \cdots \tilde{E}_{i_r} \tag{4.2}$$

with $m, r \in \mathbb{Z}_{\geq 0}$, $i_x \in I$ $(x \in J_{1,r})$, $j_y \in I$ $(y \in J_{1,m})$, $\alpha, \beta \in \mathbb{Z}\Pi$ *form a* $\mathbb{K}$*-basis of* $\tilde{U}$, *where we use the convention that if* $r = 0$ (*resp.* $m = 0$), *then* $\tilde{E}_{i_1} \cdots \tilde{E}_{i_r}$ (*resp.* $\tilde{F}_{j_1} \cdots \tilde{F}_{j_m}$) *means* 1.

Proof. This can be proved in a standard way as in [24, Lemma 2.2]. □

Define the $\mathbb{Z}\Pi$-grading $\tilde{U} = \bigoplus_{\alpha\in\mathbb{Z}\Pi} \tilde{U}_\alpha$ on $\tilde{U}$ by $\tilde{K}_\alpha \in \tilde{U}_0$, $\tilde{L}_\alpha \in \tilde{U}_0$, $\tilde{E}_i \in \tilde{U}_{\alpha_i}$, $\tilde{F}_i \in \tilde{U}_{-\alpha_i}$, and $\tilde{U}_\alpha\tilde{U}_\beta \subseteq \tilde{U}_{\alpha+\beta}$. For $\alpha \in \mathbb{Z}\Pi$, let $\tilde{U}^\pm_\alpha := \tilde{U}^\pm \cap \tilde{U}_\alpha$. Then $\tilde{U}^\pm = \bigoplus_{\alpha\in\pm\mathbb{Z}_{\geq 0}\Pi} \tilde{U}^\pm_\alpha$.

For $m \in \mathbb{Z}_{\geq 0}$, and $t_1, t_2 \in \mathbb{K}^\times$, let

$$(m; t_1, t_2) := 1 - t_1^{m-1}t_2 \quad \text{and} \quad (m; t_1, t_2)! := \prod_{j\in J_{1,m}} (j; t_1, t_2).$$

For $m \in \mathbb{Z}_{\geq 0}$ and $i, j \in I$ with $i \neq j$, define $\tilde{E}_{m,\alpha_i,\alpha_j} \in \tilde{U}^+_{m\alpha_i+\alpha_j}$, and $\tilde{F}_{m,\alpha_i,\alpha_j} \in \tilde{U}^-_{-m\alpha_i-\alpha_j}$ inductively by $\tilde{E}_{0,\alpha_i,\alpha_j} := \tilde{E}_j$, $\tilde{F}_{0,\alpha_i,\alpha_j} := \tilde{F}_j$, and

$$\begin{aligned} \tilde{E}_{m+1,\alpha_i,\alpha_j} &:= \tilde{E}_i\tilde{E}_{m,\alpha_i,\alpha_j} - q_{ii}^m q_{ij}\tilde{E}_{m,\alpha_i,\alpha_j}\tilde{E}_i, \\ \tilde{F}_{m+1,\alpha_i,\alpha_j} &:= \tilde{F}_i\tilde{F}_{m,\alpha_i,\alpha_j} - q_{ii}^m q_{ji}\tilde{F}_{m,\alpha_i,\alpha_j}\tilde{F}_i. \end{aligned} \tag{4.3}$$

Remark 4.3. The elements in (4.3) appear naturally when considering the twisting of the Hopf algebra structure of quantum groups. For example, see [27, Section 7]. Here *twisting* refers to the method given by [27, Proposition 7.2.3] in order to obtain another Hopf algebra structure.

We have

$$\tilde{\Upsilon}(\tilde{F}_{m,\alpha_i,\alpha_j}) = \tilde{E}_{m,\alpha_i,\alpha_j} \quad \text{and} \quad \tilde{\Upsilon}(\tilde{E}_{m,\alpha_i,\alpha_j}) = \tilde{F}_{m,\alpha_i,\alpha_j}.$$

Lemma 4.4. (1) *For $m \in \mathbb{N}$ and $i, j \in I$ with $i \neq j$, we have*

$$[\tilde{E}_i, \tilde{F}_i^m] = (m)_{q_{ii}}(-\tilde{K}_{\alpha_i} + q_{ii}^{-m+1}\tilde{L}_{\alpha_i})\tilde{F}_i^{m-1}. \tag{4.4}$$

(2) *For $m \in \mathbb{N}$ and $i, j \in I$ with $i \neq j$, we have*

$$[\tilde{E}_i, \tilde{F}_{m,\alpha_i,\alpha_j}] = -(m)_{q_{ii}}(m; q_{ii}, q_{ij}q_{ji})\tilde{K}_{\alpha_i}\tilde{F}_{m-1,\alpha_i,\alpha_j}. \tag{4.5}$$

(3) *Let $n, m \in \mathbb{Z}_{\geq 0}$ with $n < m$ and $i, j \in I$ with $i \neq j$. Then*

$$[\tilde{E}_{n,\alpha_i,\alpha_j}, \tilde{F}_{m,\alpha_i,\alpha_j}] = (n)_{q_{ii}}!\binom{m}{n}_{q_{ii}}(m; q_{ii}, q_{ij}q_{ji})!\tilde{F}_i^{m-n}\tilde{L}_{n\alpha_i+\alpha_j}.$$

In particular,

$$[\tilde{E}_j, \tilde{F}_{m,\alpha_i,\alpha_j}] = (m; q_{ii}, q_{ij}q_{ji})!\tilde{F}_i^m\tilde{L}_{\alpha_j}. \tag{4.6}$$

(4) *For $m \in \mathbb{Z}_{\geq 0}$ and $i, j \in I$ with $i \neq j$, we have*

$$[\tilde{E}_{m,\alpha_i,\alpha_j}, \tilde{F}_{m,\alpha_i,\alpha_j}] = (m)_{q_{ii}}!(m; q_{ii}, q_{ij}q_{ji})!(-\tilde{K}_{m\alpha_i+\alpha_j} + \tilde{L}_{m\alpha_i+\alpha_j}). \tag{4.7}$$

(5) *For $m, n \in \mathbb{Z}_{\geq 0}$ and $i, j, k \in I$ with $i \neq j \neq k \neq i$, we have*

$$[\tilde{E}_{m,\alpha_i,\alpha_j}, \tilde{F}_{n,\alpha_i,\alpha_k}] = 0.$$

Proof. These equations are obtained directly as in [11, Corollary 4.25, Lemma 4.27]. □

For $\alpha = \sum_{i\in I} n_i\alpha_i \in \mathbb{Z}_{\geq 0}\Pi$ with $n_i \in \mathbb{Z}_{\geq 0}$, using induction on $\sum_{i\in I} n_i$, we define a $\mathbb{K}$-subspace $\tilde{\mathcal{I}}^-_{-\alpha}$ of $\tilde{U}^-_{-\alpha}$ as follows. Let $\tilde{\mathcal{I}}^-_0 := \{0\}$. For $\alpha \in \mathbb{Z}_{\geq 0}\Pi \setminus \{0\}$, let $\tilde{\mathcal{I}}^-_{-\alpha}$ be the $\mathbb{K}$-subspace of $\tilde{U}^-_{-\alpha}$ formed by the elements $\tilde{Y} \in \tilde{U}^-_{-\alpha}$ with $[\tilde{E}_i, \tilde{Y}] \in \tilde{\mathcal{I}}^-_{-\alpha+\alpha_i}\tilde{K}_{\alpha_i} + \tilde{\mathcal{I}}^-_{-\alpha+\alpha_i}\tilde{L}_{\alpha_i}$ for all $i \in I$ with $\alpha - \alpha_i \in \mathbb{Z}_{\geq 0}\Pi$. Note that $\tilde{\mathcal{I}}^-_{-\alpha_i} = \{0\}$ for all $i \in I$.

Let $\tilde{\mathcal{I}}^- := \bigoplus_{\alpha\in\mathbb{Z}_{\geq 0}\Pi} \tilde{\mathcal{I}}^-_{-\alpha}$ and set $\tilde{\mathcal{J}}^- := \mathrm{Span}_{\mathbb{K}}(\tilde{\mathcal{I}}^-\tilde{U}^0\tilde{U}^+)$. Then $\tilde{\mathcal{J}}^-$ is an ideal of $\tilde{U}$. We define a unital $\mathbb{K}$-algebra $U = U(\chi)$ by

$$U := \tilde{U}/(\tilde{\mathcal{J}}^- + \tilde{\Omega}(\tilde{\mathcal{J}}^-)) = \tilde{U}/(\tilde{\mathcal{J}}^- + \tilde{\Upsilon}(\tilde{\mathcal{J}}^-)) \tag{4.8}$$

(the quotient algebra).

The $\mathbb{K}$-algebra $U(\chi)$ defined by (4.8) is isomorphic to the one given by [14, (3.14)]. By [14, Proposition 3.5(iv)] (see also [11, Proposition 5.4, Theorem 5.8]), $U(\chi)$ can also be defined in a way similar to that given by Lusztig [19, 3.1.1(a)–(e)]. See also Remarks 5.12 and 7.11 below.

Let $\pi_\chi := \pi : \tilde{U} \to U$ be the canonical map. We denote $\pi(\tilde{K}_\alpha)$, $\pi(\tilde{L}_\alpha)$, $\pi(\tilde{E}_i)$, $\pi(\tilde{F}_i)$ by K_α, L_α, E_i, F_i respectively. Let

$$E_{m,\alpha_i,\alpha_j} := \pi(\tilde{E}_{m,\alpha_i,\alpha_j}), \qquad F_{m,\alpha_i,\alpha_j} := \pi(\tilde{E}_{m,\alpha_i,\alpha_j}) \qquad (m \in \mathbb{Z}_{\geq 0},\ i,j \in I,\ i \neq j). \tag{4.9}$$

Let $U^0 := U^0(\chi) := \pi(\tilde{U}^0)$ and $U^\pm := U^\pm(\chi) := \pi(\tilde{U}^\pm)$. For $\alpha \in \mathbb{Z}\Pi$, let $U_\alpha := U(\chi)_\alpha := \pi(\tilde{U}_\alpha)$ and $U^\pm_\alpha := U^\pm(\chi)_\alpha := \pi(\tilde{U}^\pm_\alpha)$.

We have a $\mathbb{K}$-algebra automorphism $\Omega : U \to U$ with $\Omega \circ \pi = \pi \circ \tilde{\Omega}$. We have a $\mathbb{K}$-algebra isomorphism $\Upsilon : U(\chi^{\mathrm{op}}) \to U(\chi)$ with $\Upsilon \circ \pi_{\chi^{\mathrm{op}}} = \pi_\chi \circ \tilde{\Upsilon}$.

Using Ω, we have

Lemma 4.5. *There exists a unique $\mathbb{K}$-linear isomorphism from $U^- \otimes U^0 \otimes U^+$ to U sending $Y \otimes Z \otimes X$ to YZX ($X \in U^+$, $Z \in U^0$, $Y \in U^-$). Moreover, $U^0 = \bigoplus_{\alpha,\beta\in\mathbb{Z}\Pi} \mathbb{K}K_\alpha L_\beta$, $U = \bigoplus_{\alpha\in\mathbb{Z}\Pi} U_\alpha$, $U^\pm = \bigoplus_{\alpha\in\mathbb{Z}_{\geq 0}\Pi} U^\pm_{\pm\alpha}$, and $\dim U^+_\alpha = \dim U^-_{-\alpha}$ for all $\alpha \in \mathbb{Z}_{\geq 0}\Pi$.*

Remark 4.6. By Lemma 4.2, we have the same results as Lemma 4.5 with $\tilde{U}$ in place of U. By Lemma 4.5, the structure of $U^0 = U^0(\chi)$ as a unital $\mathbb{K}$-algebra is independent of the choice of $\chi \in \mathcal{X}_N$.

Remark 4.7. U can be regarded as a Hopf algebra such that π is a Hopf algebra homomorphism.

§4.3. Kharchenko PBW theorem

Define $h^\chi : \mathbb{Z}\Pi \to \mathbb{N} \cup \{\infty\}$ by

$$h^\chi(\alpha) := \begin{cases} \infty & \text{if } (m)_{\chi(\alpha,\alpha)}! \neq 0 \text{ for all } m \in \mathbb{N}, \\ \max\{m \in \mathbb{N} \mid (m)_{\chi(\alpha,\alpha)}! \neq 0\} & \text{otherwise.} \end{cases}$$

For $i, j \in I$, define $c_{ij} := c^\chi_{ij} \in \{2\} \cup J_{-\infty,0} \cup \{-\infty\}$ by
(4.10)

$$c_{ij} := c^\chi_{ij} := \begin{cases} 2 & \text{if } i = j, \\ -\infty & \text{if } i \neq j \text{ and } (m)_{q_{ii}}!(m; q_{ii}, q_{ij}q_{ji})! \neq 0 \text{ for all } m \in \mathbb{N}, \\ -\max\{m \in \mathbb{Z}_{\geq 0} \mid (m)_{q_{ii}}!(m; q_{ii}, q_{ij}q_{ji})! \neq 0\} & \text{otherwise.} \end{cases}$$

By (4.4) and Lemma 4.5, the following lemma holds.

Lemma 4.8. *Let $i \in I$ and $m \in \mathbb{Z}_{\geq 0}$. Then $F_i^m = 0$ if and only if $m > h^\chi(\alpha_i)$. In particular, if $m \in J_{0,h^\chi(\alpha_i)}$, then $\dim U^-_{-m\alpha_i} = 1$, and if $m > h^\chi(\alpha_i)$, then $\dim U^-_{-m\alpha_i} = 0$.*

We also have the following result.

Lemma 4.9. *Let $i, j \in I$ with $i \neq j$, and $m \in \mathbb{Z}_{\geq 0}$.*

(1) *$F_{m,\alpha_i,\alpha_j} \neq 0$ if and only if $m \in J_{0,-c_{ij}}$.*
(2) *If $m > h^\chi(\alpha_i) - c_{ij}$, then $\dim U^-_{-m\alpha_i-\alpha_j} = 0$. Also if $m \leq h^\chi(\alpha_i) - c_{ij}$, then the elements $F_{r,\alpha_i,\alpha_j} F_i^{m-r}$ with $r \in J_{\max\{0,m-h^\chi(\alpha_i)\},\min\{m,-c_{ij}\}}$ form a $\mathbb{K}$-basis of $U^-_{-m\alpha_i-\alpha_j}$.*
(3) *$\dim U^-_{-\alpha_i-\alpha_j} \in J_{1,2}$. Moreover, $q_{ij}q_{ji} = 1 \Leftrightarrow c_{ij} = 0 \Leftrightarrow \dim U^-_{-\alpha_i-\alpha_j} = 1 \Leftrightarrow F_iF_j = q_{ij}F_jF_i$.*

Proof. (1) follows from Lemmas 4.5 and 4.8, together with (4.5)–(4.7).

(2) If $m = 0$, this follows from Lemma 4.8. Assume $m \geq 1$. Note that $-c_{ij} \leq h^\chi(\alpha_i)$. By (1) and (4.3), $U^-_{-m\alpha_i-\alpha_j}$ is spanned by elements as in the statement.

Assume $m \leq h^\chi(\alpha_i) - c_{ij}$. Let $Z := J_{\max\{0,m-h^\chi(\alpha_i)\},\min\{m,-c_{ij}\}}$ and $X := \sum_{r\in Z} y_r F_{r,\alpha_i,\alpha_j} F_i^{m-r}$ with $y_r \in \mathbb{K}$. Assume $X = 0$. Observing the coefficients $F_{r,\alpha_i,\alpha_j} F_i^{m-r-1} L_{\alpha_i}$ ($r \in Z \cap J_{0,m-1}$) of $[E_i, X]$, by Lemma 4.5 and (4.4), (4.5) and induction, we see $y_r = 0$ for $r \in Z \cap J_{0,m-1}$. In a similar way, we see $y_m = 0$ (if $m \leq -c_{ij}$) by (4.5).

(3) follows from (1) and (2). □

By the Kharchenko PBW theorem, we have

Theorem 4.10 ([18], see also [9, Section 3, (P)], and [14, Theorem 4.9]).

(1) *There exists a unique pair*

$$(R^+ = R^+(\chi), \varphi = \varphi_\chi),$$

where $R^+ \subset \mathbb{Z}_{\geq 0}\Pi \setminus \{0\}$ *and* $\varphi : R^+ \to \mathbb{N}$ *satisfy the condition that there exist* $k \in \mathbb{N} \cup \{\infty\}$, *a surjective map* $\psi : J_{1,k} \to R^+$ *with* $|\psi^{-1}(\{\alpha\})| = \varphi(\alpha)$ $(\alpha \in R^+)$, *and* $F[r] \in U^-_{-\psi(r)} \setminus \{0\}$ $(r \in J_{1,k})$ *such that the elements*

$$F[1]^{x_1} \cdots F[m]^{x_m} \qquad (m \in J_{1,k},\, x_y \in J_{0,h^\chi(\psi(y))}\, (y \in J_{1,m})) \tag{4.11}$$

form a $\mathbb{K}$*-basis of* U^- *(where we mean that for* $m \leq \bar{m}$, $F[1]^{x_1} \cdots F[m]^{x_m} = F[1]^{\bar{x}_1} \cdots F[\bar{m}]^{\bar{x}_{\bar{m}}}$ *if and only if* $x_y = \bar{x}_y$ *for all* $y \in J_{1,m}$ *and* $\bar{x}_{\bar{y}} = 0$ *for all* $\bar{y} \in J_{m+1,\bar{m}}$*).*

(2) *Assume* $|R^+| < \infty$. *Then* $\varphi(R^+) = \{1\}$, *and there exist* $F_\alpha \in U^-_{-\alpha} \setminus \{0\}$ $(\alpha \in R^+)$ *satisfying*

(i) *Let* $n := |R^+|$. *Then for any bijection* $\psi : J_{1,n} \to R^+$, *the elements* $F^{z_1}_{\psi(1)} \cdots F^{z_n}_{\psi(n)}$ *with* $z_t \in J_{0,h^\chi(\psi(t))}$ $(t \in J_{1,n})$ *form a* $\mathbb{K}$*-basis of* U^-.

(ii) *For* $\beta \in R^+$ *with* $h^\chi(\beta) < \infty$, $F^{h^\chi(\beta)+1}_\beta = 0$.

(*See also Remarks 6.2 and 4.11(1) below.*)

Remark 4.11. (1) By Lemma 4.5, using Υ, we can easily see that $(R^+(\chi^{\mathrm{op}}), \varphi_{\chi^{\mathrm{op}}}) = (R^+(\chi), \varphi_\chi)$. It is clear that $\varphi_\chi(\alpha_i) = 1$ for $i \in I$. Note that we have the $\mathbb{K}$-algebra isomorphism $\Upsilon_{|U^-(\chi^{\mathrm{op}})} : U^-(\chi^{\mathrm{op}}) \to U^+(\chi)$.

(2) The following facts are well-known (see [19, 33.3, Corollary 33.1.5] and [12, Section 2.4] for example). Let $\hat{q} \in \mathbb{K}^\times_\infty$. Let $A = [a_{ij}]_{i,j \in I}$ be a symmetrizable generalized Cartan matrix. Let $d_i \in \mathbb{N}$ $(i \in I)$ be such that $d_i a_{ij} = d_j a_{ji}$ $(i, j \in I)$. Let $\mathfrak{g}$ be the Kac–Moody Lie algebra defined for A. Let $\mathfrak{n}^-$ be the negative part of $\mathfrak{g}$, and $\mathcal{U}(\mathfrak{n}^-)$ be the universal enveloping algebra of $\mathfrak{n}^-$. Let $\chi \in \mathcal{X}_N$ be such that $\chi(\alpha_i, \alpha_j) = \hat{q}^{d_i a_{ij}}$ $(i, j \in I)$. Then the ideal $\tilde{\mathcal{I}}^-$ of $\tilde{U}^-(\chi)$ is generated by $\tilde{F}_{1-a_{ij},\alpha_i,\alpha_j}$ $(i \neq j)$, and $\dim U^-(\chi)_{-\alpha} = \dim \mathcal{U}(\mathfrak{n}^-)_{-\alpha}$ for all $\alpha \in \mathbb{Z}_{\geq 0}\Pi$, where $\mathcal{U}(\mathfrak{n}^-)_{-\alpha}$ is the weight subspace of $\mathcal{U}(\mathfrak{n}^-)$ corresponding to $-\alpha$. In particular, $R^+(\chi)$ can be identified with the set of positive roots of $\mathfrak{g}$ and $\varphi_\chi(\alpha) = \dim \mathfrak{g}_\alpha$ $(= \dim \mathfrak{g}_{-\alpha})$ for all $\alpha \in R^+(\chi)$.

Once we know Theorem 4.10, the following lemma is clear from Lemmas 4.8 and 4.9.

Lemma 4.12. (1) $\Pi \subseteq R^+$.

(2) *For $i, j \in I$ with $i \neq j$, we have*

$$R^+ \cap (\alpha_j + \mathbb{Z}_{\geq 0}\alpha_i) = \{\alpha_j + n\alpha_i \mid n \in J_{0,-c^{\chi}_{ij}}\}, \tag{4.12}$$

and $\varphi(\alpha_j + r\alpha_i) = 1$ for all $r \in J_{0,-c_{ij}}$.

(3) *Let I' be a non-empty proper subset of I. Let $I'' = I \setminus I'$. Then $R^+ = (R^+ \cap \bigoplus_{i\in I'} \mathbb{Z}_{\geq 0}\alpha_i) \uplus (R^+ \cap \bigoplus_{j\in I''} \mathbb{Z}_{\geq 0}\alpha_j)$ if and only if $q_{ij}q_{ji} = 1$ for all $i \in I'$ and all $j \in I''$.*

For $\chi \in \mathcal{X}_N$, we say that χ is *irreducible* if its Dynkin diagram is connected, that is, for any $i, j \in I$ with $i \neq j$, there exist $m \in \mathbb{Z}_{\geq 0}$, $i_r \in I$ ($r \in J_{1,m}$) such that $q_{i_t,i_{t+1}}q_{i_{t+1},i_t} \neq 1$ for all $t \in J_{0,m}$, where we let $i_0 := i$ and $i_{m+1} := j$. Let $\mathcal{X}^{\mathrm{irr}}_N := \{\chi \in \mathcal{X}_N \mid \ \chi \text{ is irreducible}\}$.

§4.4. Irreducible highest weight modules

Let $\Lambda \in \mathrm{Ch}(U^0)$. By Lemma 4.5, we have a unique left U-module $\mathcal{M}_\chi(\Lambda)$ satisfying the following conditions:

(i) There exists $\tilde{v}_\lambda \in \mathcal{M}_\chi(\Lambda) \setminus \{0\}$ such that $Z\tilde{v}_\Lambda = \Lambda(Z)\tilde{v}_\Lambda$ for all $Z \in U^0$ and $E_i\tilde{v}_\Lambda = 0$ for all $i \in I$.

(ii) The $\mathbb{K}$-linear map $U^- \to \mathcal{M}_\chi(\Lambda)$, $Y \mapsto Y\tilde{v}_\Lambda$, is bijective.

For $i \in I$ and $m \in \mathbb{Z}_{\geq 0}$, by (4.4), we have

$$E_iF_i^m\tilde{v}_\Lambda = \begin{cases} 0 & \text{if } m = 0, \\ (m)_{q_{ii}}(-q_{ii}^{1-m}\Lambda(K_{\alpha_i}) + \Lambda(L_{\alpha_i}))F_i^{m-1}\tilde{v}_\Lambda & \text{otherwise.} \end{cases} \tag{4.13}$$

For $\alpha \in \mathbb{Z}\Pi$, let $\mathcal{M}_\chi(\Lambda)_\alpha := U^-_\alpha\tilde{v}_\Lambda$. We say that a $\mathbb{K}$-subspace $\mathcal{V}$ of $\mathcal{M}_\chi(\Lambda)$ is $\mathbb{Z}\Pi$-*graded* if $\mathcal{V} = \bigoplus_{\alpha\in\mathbb{Z}_{\geq 0}\Pi}(\mathcal{V} \cap \mathcal{M}_\chi(\Lambda)_\alpha)$. If $\mathcal{V}$ is a $\mathbb{Z}\Pi$-graded $U(\chi)$-submodule of $\mathcal{M}_\chi(\Lambda)$, then $\mathcal{V} \neq \mathcal{M}_\chi(\Lambda)$ if and only if $\tilde{v}_\Lambda \notin \mathcal{V}$.

Let $\mathcal{N} := \mathcal{N}_\chi(\Lambda)$ be the maximal proper $\mathbb{Z}\Pi$-graded $U(\chi)$-submodule of $\mathcal{M}_\chi(\Lambda)$. Note $\mathcal{N} \cap \mathbb{K}\tilde{v}_\Lambda = \{0\}$. Let $\mathcal{L}_\chi(\Lambda)$ be the quotient left U-module defined by

$$\mathcal{L}_\chi(\Lambda) := \mathcal{M}_\chi(\Lambda)/\mathcal{N}.$$

We denote the element $\tilde{v}_\Lambda + \mathcal{N}$ of $\mathcal{L}_\chi(\Lambda)$ by v_Λ. For $\alpha \in \mathbb{Z}\Pi$, let $\mathcal{L}_\chi(\Lambda)_\alpha := U^-_\alpha v_\Lambda$. Then $\mathcal{L}_\chi(\Lambda) = \bigoplus_{\alpha\in\mathbb{Z}_{\geq 0}\Pi}\mathcal{L}_\chi(\Lambda)_{-\alpha}$. Also $\mathcal{L}_\chi(\Lambda)_0 = \mathbb{K}v_\Lambda$, and $\dim \mathcal{L}_\chi(\Lambda)_0 = 1$. We say that a $\mathbb{K}$-subspace $\mathcal{V}'$ of $\mathcal{L}_\chi(\Lambda)$ is $\mathbb{Z}\Pi$-*graded* if $\mathcal{V}' = \bigoplus_{\alpha\in\mathbb{Z}_{\geq 0}\Pi}(\mathcal{V}' \cap \mathcal{L}_\chi(\Lambda)_\alpha)$. There exists no non-zero proper $\mathbb{Z}\Pi$-graded $U(\chi)$-submodule of $\mathcal{L}_\chi(\Lambda)$.

By (4.13), for $m \in \mathbb{N}$, we have

$$F_i^m v_\Lambda = 0 \ \Leftrightarrow \ (m)_{q_{ii}}!(m; q_{ii}^{-1}, \Lambda(K_{\alpha_i}L_{-\alpha_i}))! = 0. \tag{4.14}$$

Let

$$h_{\chi,\Lambda,i} := \begin{cases} \infty \quad \text{if } (m)_{q_{ii}}!(m; q_{ii}^{-1}, \Lambda(K_{\alpha_i} L_{-\alpha_i}))! \neq 0 \text{ for all } m \in \mathbb{N}, \\ \max\{m \in \mathbb{Z}_{\geq 0} \mid (m)_{q_{ii}}!(m; q_{ii}^{-1}, \Lambda(K_{\alpha_i} L_{-\alpha_i}))! \neq 0\} \\ \quad \text{otherwise.} \end{cases} \tag{4.15}$$

By (4.14), since $\mathcal{L}_\chi(\Lambda)$ is $\mathbb{Z}\Pi$-graded, we have

$$\dim \mathcal{L}_\chi(\Lambda) < \infty \;\Rightarrow\; \forall i \in I,\, h_{\chi,\Lambda,i} < \infty. \tag{4.16}$$

Lemma 4.13. (1) *For any* $\Lambda \in \mathrm{Ch}(U^0(\chi))$, $\mathcal{L}_\chi(\Lambda)$ *is an irreducible* $U(\chi)$*-module.*
(2) *Let* Λ, $\Lambda' \in \mathrm{Ch}(U^0(\chi))$. *Let* $f : \mathcal{L}_\chi(\Lambda') \to \mathcal{L}_\chi(\Lambda)$ *be a* $U(\chi)$*-module homomorphism. Then* $\Lambda = \Lambda'$, *and* $f = c \cdot \mathrm{id}$ *for some* $c \in \mathbb{K}$. *In particular,*

$$\text{if } \Lambda \neq \Lambda', \text{ then } \mathcal{L}_\chi(\Lambda) \text{ and } \mathcal{L}_\chi(\Lambda') \text{ are not isomorphic as } U(\chi)\text{-modules.} \tag{4.17}$$

Proof. These claims are clear from the following fact:

$$\forall \Lambda \in \mathrm{Ch}(U^0(\chi)), \forall v \in \mathcal{L}_\chi(\Lambda) \setminus \{0\}, \exists X \in U^+(\chi), \quad Xv = v_\Lambda. \qquad \square$$

§4.5. Notation ≡

Notation 4.14. Let $\chi, \chi' \in \mathcal{X}_N$. Let $q_{ij} := \chi(\alpha_i, \alpha_j)$ and $q'_{ij} := \chi'(\alpha_i, \alpha_j)$. We write

$$\chi \equiv \chi' \quad \text{if } q_{ii} = q'_{ii} \text{ for all } i \in I \text{ and } q_{jk} q_{kj} = q'_{jk} q'_{kj} \text{ for all } j, k \in I. \tag{4.18}$$

By (4.10),

$$\chi \equiv \chi' \;\Rightarrow\; \forall i, j \in I,\, c^\chi_{ij} = c^{\chi'}_{ij}. \tag{4.19}$$

Let $\Lambda \in \mathrm{Ch}(U^0(\chi))$ and $\Lambda' \in \mathrm{Ch}(U^0(\chi'))$. We write

$$(\chi, \Lambda) \equiv (\chi', \Lambda') \quad \text{if } \chi \equiv \chi' \text{ and } \Lambda(K_{\alpha_i} L_{-\alpha_i}) = \Lambda'(K_{\alpha_i} L_{-\alpha_i}) \text{ for all } i \in I.$$

Note that

$$(\chi, \Lambda) \equiv (\chi', \Lambda') \;\Rightarrow\; \forall i \in I,\, h_{\chi,\Lambda,i} = h_{\chi',\Lambda',i}. \tag{4.20}$$

§4.6. Weyl groupoids of bi-homomorphisms

Let $\mathcal{X}^{\mathrm{fin}}_N := \{\chi \in \mathcal{X}_N \mid |R^+(\chi)| < \infty\}$. If $\chi \in \mathcal{X}^{\mathrm{fin}}_N \cap \mathcal{X}^{\mathrm{irr}}_N$, we say that χ is *of finite type.*

Let $i \in I$, and set $(\mathcal{X}''^{,\mathrm{fin}}_N)_i := \{\chi \in \mathcal{X}_N \mid \forall j \in I,\, c^\chi_{ij} \neq -\infty\}$. For $\chi \in (\mathcal{X}''^{,\mathrm{fin}}_N)_i$, define $s^\chi_i \in \mathrm{GL}(\mathbb{Z}\Pi)$ by $s^\chi_i(\alpha_j) = \alpha_j - c^\chi_{ij}\alpha_i$. Note that $s^\chi_i(\alpha_i) = -\alpha_i$ and $(s^\chi_i)^2 = \mathrm{id}_{\mathbb{V}}$ $(i \in I)$. For $\chi \in (\mathcal{X}''^{,\mathrm{fin}}_N)_i$, define $\tau_i\chi \in \mathcal{X}_N$ by

$$\tau_i\chi(\alpha, \beta) := \chi(s^\chi_i(\alpha), s^\chi_i(\beta)) \quad \text{for all } \alpha, \beta \in \mathbb{Z}\Pi. \tag{4.21}$$

Lemma 4.15 (see also [14, (2.10)–(2.11)]). *Let* $\chi \in (\mathcal{X}_N^{\prime\prime,\mathrm{fin}})_i$.

(1) *If* $\chi(\alpha_i,\alpha_i)=1$, *then* $\chi(\alpha_i,\alpha_j)\chi(\alpha_j,\alpha_i)=1$ *for all* $j\in I$.

(2) *We have*

$$c_{ij}^{\tau_i\chi}=c_{ij}^{\chi}\ (j\in I),\qquad \tau_i\chi\in(\mathcal{X}_N^{\prime\prime,\mathrm{fin}})_i,\qquad s_i^{\tau_i\chi}=s_i^{\chi},\qquad \tau_i\tau_i\chi=\chi. \tag{4.22}$$

(3) *If* $\chi\in\mathcal{X}_N^{\mathrm{irr}}$, *then* $\tau_i\chi\in\mathcal{X}_N^{\mathrm{irr}}$.

Proof. (1) is clear from (4.10) since $\chi\in(\mathcal{X}_N^{\prime\prime,\mathrm{fin}})_i$.

(2) Let $c_{ij}:=c_{ij}^{\chi}$ $(i,j\in I)$. Let $q_{xy}:=\chi(\alpha_x,\alpha_y)$, and $q'_{xy}:=\tau_i\chi(\alpha_x,\alpha_y)$ $(x,y\in I)$. We have $q'_{ii}=q_{ii}$, and $(q'_{ii})^m q'_{ij}q'_{ji}=(q_{ii}^{2(-c_{ij})-m}(q_{ij}q_{ji}))^{-1}$ for $j\in I$ and $m\in\mathbb{Z}$. If $q_{ii}=1$, the statement is clear from (1) since $c_{ij}=0$ for all $j\in I\setminus\{i\}$. Assume $q_{ii}\neq 1$. Let $j\in I\setminus\{i\}$. Assume $c_{ij}=0$. Then $q_{ij}q_{ji}=1$. Hence $q'_{ij}q'_{ji}=1$, so $c_{ij}^{\tau_i\chi}=0$. Assume $-c_{ij}\geq 1$. Assume $q_{ii}^{-c_{ij}}q_{ij}q_{ji}=1$. Then $(q'_{ii})^{-c_{ij}}q'_{ij}q'_{ji}=1$. Moreover, we have $(q'_{ii})^{m+1}=q_{ii}^{m+1}\neq 1$ and $(q'_{ii})^m q'_{ij}q'_{ji}=q_{ii}^{-(-c_{ij}-m)}\neq 1$ for $m\in J_{0,-c_{ij}-1}$. Hence $c_{ij}^{\tau_i\chi}=c_{ij}$. Assume $q_{ii}^{-c_{ij}}q_{ij}q_{ji}\neq 1$. Then q_{ii} is a primitive $(1-c_{ij})$-th root of unity. Since $q_{ii}^m q_{ij}q_{ji}\neq 1$ for $m\in J_{0,-c_{ij}}$, we have $q_{ii}^n q_{ij}q_{ji}\neq 1$ for all $n\in\mathbb{Z}$. Hence $c_{ij}^{\tau_i\chi}=c_{ij}$.

(3) Let q_{xy} and q'_{xy} be as above. By (4.10), for $x,y\in I$ with $c_{ix}=c_{iy}=0$, we have $q_{xy}=q'_{xy}$. Then (3) follows from (2). □

Let $\mathcal{X}_N^{\prime\prime,\mathrm{fin}}:=\bigcap_{i\in I}(\mathcal{X}_N^{\prime\prime,\mathrm{fin}})_i$. By (4.12), we have

$$\mathcal{X}_N^{\mathrm{fin}}\subseteq\mathcal{X}_N^{\prime\prime,\mathrm{fin}}. \tag{4.23}$$

Notation 4.16. Let $\chi,\chi'\in\mathcal{X}_N$. We write $\chi\sim\chi'$ if $\chi=\chi'$ or there exist $m\in\mathbb{N}$, $i_t\in I$ $(t\in J_{1,m})$ and $\chi_r\in\mathcal{X}_N$ $(r\in J_{1,m+1})$ such that $\chi=\chi_1$, $\chi'=\chi_{m+1}$ and $\chi_t\in(\mathcal{X}_N^{\prime\prime,\mathrm{fin}})_{i_t}$, $\tau_{i_t}\chi_t=\chi_{t+1}$ $(t\in J_{1,m})$.

For $\chi\in\mathcal{X}_N$, let $\mathcal{G}(\chi):=\{\chi'\in\mathcal{X}_N\mid\chi'\sim\chi\}$.
Let $\mathcal{X}_N^{\prime,\mathrm{fin}}:=\{\chi\in\mathcal{X}_N^{\prime\prime,\mathrm{fin}}\mid\tau_i\chi'\in\mathcal{X}_N^{\prime\prime,\mathrm{fin}}\,(\chi'\in\mathcal{G}(\chi),\,i\in I)\}$.

Definition 4.17. Let $\chi\in\mathcal{X}_N^{\prime,\mathrm{fin}}$. For $i\in I$, define a map $\tau_i^{\mathcal{G}(\chi)}:\mathcal{G}(\chi)\to\mathcal{G}(\chi)$ by $\tau_i^{\mathcal{G}(\chi)}(\chi'):=\tau_i\chi'$. For $\chi'\in\mathcal{G}(\chi)$, let $C^{\chi'}$ be the $N\times N$-matrix $[c_{ij}^{\chi'}]$ over $\mathbb{Z}$, where by Lemma 4.9(3), $C^{\chi'}$ is a generalized Cartan matrix (see (M1), (M2) of Subsection 1.2). We call the quadruple

$$\mathcal{C}_\chi=\mathcal{C}_\chi(I,\mathcal{G}(\chi),(\tau_i^{\mathcal{G}(\chi)})_{i\in I},(C^{\chi'})_{\chi'\in\mathcal{G}(\chi)})$$

the *Cartan scheme associated with* χ. Indeed, by (4.22), $\mathcal{C}_\chi$ is a Cartan scheme (see (C1), (C2) of Subsection 1.2).

Let $\chi \in \mathcal{X}_N^{\prime,\mathrm{fin}}$. Recall Notation 1.1. Since $\mathcal{C}_\chi$ is a Cartan scheme, by (4.21) we have

$$\chi_{f,t-1}(\alpha_{f(t)},\alpha_{f(t)}) = \chi(1^\chi s_{f,t-1}(\alpha_{f(t)}), 1^\chi s_{f,t-1}(\alpha_{f(t)})) \qquad (n \in \mathbb{Z}_{\geq 0}\cup\{\infty\},\ f \in \mathrm{Map}_n^I,\ t\in\mathbb{N}). \tag{4.24}$$

For $\chi \in \mathcal{X}_N$, let $R(\chi) := R^+(\chi)\cup(-R^+(\chi))$, and extend the initial domain of φ_χ to $R(\chi)$ by $\varphi_\chi(-\alpha) = \varphi_\chi(\alpha)$.

It is well-known that

Theorem 4.18 ([9, Proposition 1], see also [13, Example 4]). (1) *For $i \in I$ and $\chi \in (\mathcal{X}_N^{\prime\prime,\mathrm{fin}})_i$, we have*

$$\begin{aligned} &\tau_i\chi \in (\mathcal{X}_N^{\prime\prime,\mathrm{fin}})_i, \quad s_i^\chi(R^+(\chi)\setminus\{\alpha_i\}) = R^+(\tau_i\chi)\setminus\{\alpha_i\}, \\ &\varphi_{\tau_i\chi}(s_i^\chi(\beta)) = \varphi_\chi(\beta) \quad (\beta\in R(\chi)). \end{aligned} \tag{4.25}$$

In particular, $s_i^\chi(R(\chi)) = R(\tau_i\chi)$.

(2) *Let $\chi\in\mathcal{X}_N^{\mathrm{fin}}$. Then $\tau_i\chi\in\mathcal{X}_N^{\mathrm{fin}}$ and $|R^+(\chi)| = |R^+(\tau_i\chi)|$.*

We have obtained the first property in (4.25) by (4.22).

The following theorem is also well-known.

Theorem 4.19 ([11, *Theorem* 3.14]). *Let $\chi\in\mathcal{X}_N^{\prime,\mathrm{fin}}$. Then the data*

$$\mathcal{R}_\chi = \mathcal{R}_\chi(\mathcal{C}_\chi,(R(\chi'))_{\chi'\in\mathcal{G}(\chi)})$$

is a root system of type $\mathcal{C}_\chi$ (see Definition 1.2). In particular, for $\chi',\chi''\in\mathcal{G}(\chi)$ and $w\in\mathcal{H}(\chi',\chi'')$, we have

$$w(R(\chi'')) = R(\chi'). \tag{4.26}$$

Proof. This can easily be shown by using Theorem 4.18 and Definition 1.2. □

Corollary 4.20. *Let $\chi,\chi'\in\mathcal{X}_N^{\prime,\mathrm{fin}}$ be such that $R(\chi) = R(\chi')$ (as a subset of $\mathbb{V}$). Then $1^\chi s_{f,t} = 1^{\chi'} s_{f,t}$ (as an element of $\mathrm{GL}(\mathbb{V})$) for all $f\in\mathrm{Map}_\infty^I$ and $t\in\mathbb{Z}_{\geq 0}$.*

Proof. This follows from Lemma 4.12 and (4.26). (See also Lemma 1.12.) □

By (4.23) and Theorem 4.18(2), we have

$$\mathcal{X}_N^{\mathrm{fin}} \subseteq \mathcal{X}_N^{\prime,\mathrm{fin}}. \tag{4.27}$$

Lemma 4.21. *Let $i\in I$ and $\chi\in(\mathcal{X}_N^{\prime\prime,\mathrm{fin}})_i$. Assume that $\chi(\alpha_i,\alpha_i)\in\mathbb{K}_\infty^\times$. Then $\tau_i\chi\equiv\chi$.*

Proof. Let $q_{xy} := \chi(\alpha_x, \alpha_y)$ and $q'_{xy} := \tau_i\chi(\alpha_x, \alpha_y)$ for $x, y \in I$. Let $j, k \in I$. Assume $j \neq i \neq k$. By (4.10), $q_{ii}^{-c_{ij}} q_{ij}q_{ji} = q_{ii}^{-c_{ik}} q_{ik}q_{ki} = 1$ since $q_{ii} \in \mathbb{K}^{\times}_{\infty}$. Hence

$$\begin{aligned} q'_{jk}q'_{kj} &= \chi(\alpha_j - c_{ij}\alpha_i, \alpha_k - c_{ik}\alpha_i)\chi(\alpha_k - c_{ik}\alpha_i, \alpha_j - c_{ij}\alpha_i) \\ &= q_{jk}q_{kj}(q_{ij}q_{ji})^{-c_{ik}}(q_{ik}q_{ki})^{-c_{ij}} q_{ii}^{2c_{ij}c_{ik}} = q_{jk}q_{kj}, \end{aligned}$$

as desired. The other cases can be treated similarly. □

Let $i \in I$. Let $\chi \in (\mathcal{X}_N^{\prime\prime,\mathrm{fin}})_i$ and $\chi' \in \mathcal{X}_N$. By (4.19), we have

$$\chi \equiv \chi' \;\Rightarrow\; \chi' \in (\mathcal{X}_N^{\prime\prime,\mathrm{fin}})_i,\ s_i^{\chi} = s_i^{\chi'},\ \tau_i\chi \equiv \tau_i\chi'. \tag{4.28}$$

Lemma 4.22. *Let $\chi \in \mathcal{X}_N^{\mathrm{fin}}$. Let $\chi' \in \mathcal{X}_N$ be such that $\chi' \equiv \chi$. Then $\chi' \in \mathcal{X}_N^{\mathrm{fin}}$, $1^{\chi}w_0 = 1^{\chi'}w_0$ and $R(\chi') = R(\chi)$.*

Proof. This follows from (4.28) and Lemma 1.12. □

§4.7. Finite-dimensional irreducible $U(\chi)$-modules

Here as an application of Theorem 4.19, we show that if $\chi \in \mathcal{X}_N^{\mathrm{fin}}$ satisfies an extra condition, every finite-dimensional irreducible $U(\chi)$-module is isomorphic to $\mathcal{L}_\chi(\Lambda)$ for some $\Lambda \in \mathrm{Ch}(U^0(\chi))$.

Lemma 4.23. *Let $\chi \in \mathcal{X}_N^{\mathrm{fin}} \cap \mathcal{X}_N^{\mathrm{irr}}$. Assume that if $N = 1$, then $\chi(\alpha_1, \alpha_1) \neq 1$. Then $\{\mathcal{L}_\chi(\Lambda) \mid \Lambda \in \mathrm{Ch}(U^0(\chi)), \dim \mathcal{L}_\chi(\Lambda) < \infty\}$ is a complete set of pairwise non-isomorphic finite-dimensional irreducible $U(\chi)$-modules (see also (4.17)).*

Proof. Let $\mathcal{V}$ be a non-zero finite-dimensional irreducible $U(\chi)$-module. For $\Lambda \in \mathrm{Ch}(U^0(\chi))$, let $\mathcal{V}_\Lambda = \{v \in \mathcal{V} \mid \forall Z \in U^0(\chi), Zv = \Lambda(Z)v\}$, and $\mathcal{V}^\Lambda = \{v \in \mathcal{V}_\Lambda \mid \forall i \in I, E_iv = 0\}$. Let $\hat{O} := \{\Lambda \in \mathrm{Ch}(U^0(\chi)) \mid \mathcal{V}_\Lambda \neq \{0\}\}$. Note $|\hat{O}| < \infty$, since $\dim \mathcal{V} < \infty$. Since $\mathbb{K}$ is an algebraically closed field, $\hat{O} \neq \emptyset$. Let $\check{O} := \{\Lambda \in \hat{O} \mid \mathcal{V}^\Lambda \neq \{0\}\}$. Since $\mathcal{V}$ is irreducible, we have $\hat{\mathcal{V}} = \mathcal{V} = \bigoplus_{\Lambda\in\hat{O}} \mathcal{V}_\Lambda$ ($\neq \{0\}$). Let $U^+(\chi)' := \bigoplus_{\beta\in\mathbb{Z}_{\geq 0}\setminus\{0\}} U^+(\chi)_\beta$ and $U^-(\chi)' := \bigoplus_{\beta\in\mathbb{Z}_{\geq 0}\setminus\{0\}} U^-(\chi)_{-\beta}$.

Assume for a moment that

$$(*) \qquad \exists r \in \mathbb{N},\ \forall v \in \mathcal{V},\ \forall X_t \in U^+(\chi)'\ (t \in J_{1,r}), \quad X_1 \cdots X_rv = 0,$$

and that

$$(**) \qquad \exists r \in \mathbb{N},\ \forall v \in \mathcal{V},\ \forall Y_t \in U^-(\chi)'\ (t \in J_{1,r}), \quad Y_1 \cdots Y_rv = 0.$$

By $(*)$, there exist $\Lambda \in \mathrm{Ch}(U^0(\chi))$ and $v \in \mathcal{V}^\Lambda \neq \{0\}$. Then there exists a non-zero $U(\chi)$-module epimorphism $f : \mathcal{M}_\chi(\Lambda) \to \mathcal{V}$ with $f(\tilde{v}_\Lambda) = v$. Assume that $\mathcal{N}_\chi(\Lambda) \not\subset \ker f$. Then there exists $Y' \in U^-(\chi)' \setminus \{0\}$ such that $\tilde{v}_\Lambda + Y'\tilde{v}_\Lambda \in \ker f$. Then $v = (-Y')v$, which contradicts $(**)$. Hence $\mathcal{N}_\chi(\Lambda) \subset \ker f$, so $\mathcal{N}_\chi(\Lambda) = \ker f$ since

$\mathcal{V}$ is irreducible. Thus f induces the $U(\chi)$-module isomorphism $f' : \mathcal{L}_\chi(\Lambda) \to \mathcal{V}$ with $f'(v_\Lambda) = v$.

Let $F_\alpha \in U^-(\chi)_{-\alpha}$ ($\alpha \in R^+(\chi)$) be as in Theorem 4.10(2). Since $|\hat{O}| < \infty$, for $\alpha \in R^+(\chi)$, if $\alpha \in \mathbb{K}^\times_\infty$, F_α acts nilpotently on $\mathcal{V}$. By Lemma 4.15(1), Theorem 4.19 and (1.18), since $\chi \in \mathcal{X}^{\rm irr}_N$, we have $\chi(\alpha,\alpha) \neq 1$ for all $\alpha \in R^+(\chi)$. Hence Theorem 4.10(2) implies $(**)$. Similarly we also have $(*)$. □

Remark 4.24. Let $\chi \in \mathcal{X}^{\rm fin}_N \cap \mathcal{X}^{\rm irr}_N$. Assume $\dim U^-(\chi) < \infty$ (see also Lemma 5.9 below). By Lemma 4.23, $\{\mathcal{L}_\chi(\Lambda) \mid \Lambda \in {\rm Ch}(U^0(\chi))\}$ is a complete set of pairwise non-isomorphic finite-dimensional irreducible $U(\chi)$-modules (see also (4.17)).

§5. FID-type bi-homomorphisms

§5.1. Some bi-homomorphisms

Let

(5.1) $$\mathcal{Y}'^{\flat,{\rm fin}}_N := \{a^\sharp = (\eta^\sharp, \theta^\sharp) \in \mathcal{Y}'^{,{\rm fin}}_N \mid \eta^\sharp(\alpha_i,\alpha_j) \in \mathbb{Z}\ (i,j \in I)\}.$$

Lemma 5.1. *Let $\hat{q} \in \mathbb{K}^\times_\infty$.*

(1) *We have an injection $\varpi_{\hat{q}} : \mathcal{Y}'^{\flat,{\rm fin}}_N \to \mathcal{X}'^{,{\rm fin}}_N$ defined by*

(5.2) $$\varpi_{\hat{q}}(a^\sharp)(\alpha_i,\alpha_j) := (-1)^{\theta^\sharp(\alpha_i)\theta^\sharp(\alpha_j)} \hat{q}^{\eta^\sharp(\alpha_i,\alpha_j)} \qquad (i,j \in I)$$

for $a^\sharp = (\eta^\sharp, \theta^\sharp) \in \mathcal{Y}'^{\flat,{\rm fin}}_N$.

(2) *Let $a^\sharp \in \mathcal{Y}'^{\flat,{\rm fin}}_N$. Then $(\mathcal{R}_{a^\sharp}, a^\sharp)$ and $(\mathcal{R}_{\varpi_{\hat{q}}(a^\sharp)}, \varpi_{\hat{q}}(a^\sharp))$ are isomorphic.*

(3) *$\varpi_{\hat{q}}(\mathcal{Y}^{\rm fin}_N) \subset \mathcal{X}^{\rm fin}_N$. Moreover $R(a^\sharp) = R(\varpi_{\hat{q}}(a^\sharp))$ for $a^\sharp \in \mathcal{Y}^{\rm fin}_N$.*

Proof. Claims (1) and (2) can be proved directly. Claim (3) follows from (2) and Lemma 1.12. □

Definition 5.2. Recall Definition 2.1. Define $\theta : I \to I_{0,1}$ by $\theta(i) := 0$ ($i \in I$). For a rank-N Cartan data $\langle \hat{\Pi} \rangle$, let $a^\sharp_{\langle \hat{\Pi} \rangle} := (\hat{\eta}_{\langle \hat{\Pi} \rangle}, \theta) \in \mathcal{Y}^{\rm fin}_N$ (cf. Theorem 3.4). Let $\dot{\mathcal{X}}^{\rm Cartan}_N$ be the subset of $\mathcal{X}_N$ formed by $\varpi_{\hat{q}}(a^\sharp_{\langle \hat{\Pi} \rangle})$ for some $\hat{q} \in \mathbb{K}^\times_\infty$ and some rank-N Cartan data $\langle \hat{\Pi} \rangle$. Let $\dot{\mathcal{X}}^{\rm Cartan}_N({\rm X}_N)$ be the subset of $\dot{\mathcal{X}}^{\rm Cartan}_N$ formed by $\varpi_{\hat{q}}(a^\sharp_{\langle \hat{\Pi} \rangle})$ for some $\hat{q} \in \mathbb{K}^\times_\infty$ and some ${\rm X}_N$-data $\langle \hat{\Pi} \rangle$, where X is one of A, …, G.

Using Lemma 5.1, we can easily prove

Lemma 5.3. *Let $\langle \hat{\Pi} \rangle$ be a rank-N Cartan data. Let $\chi := \varpi_{\hat{q}}(a^\sharp_{\langle \hat{\Pi} \rangle}) \in \dot{\mathcal{X}}^{\rm Cartan}_N$. Then $(\mathcal{R}_\chi, \chi)$ is isomorphic to $(\mathcal{R}_{\langle \hat{\Pi} \rangle}, a)$, where $a \in \mathcal{A}_{\langle \hat{\Pi} \rangle}$ (note $|\mathcal{A}_{\langle \hat{\Pi} \rangle}| = 1$). In particular, $|\mathcal{G}(\chi)| = 1$, and identifying $\langle \Pi \rangle$ with $\langle \hat{\Pi} \rangle$, we have $R(\chi) = \hat{R}$, where $\hat{R}$ is the rank-N root system corresponding to $\langle \hat{\Pi} \rangle$.*

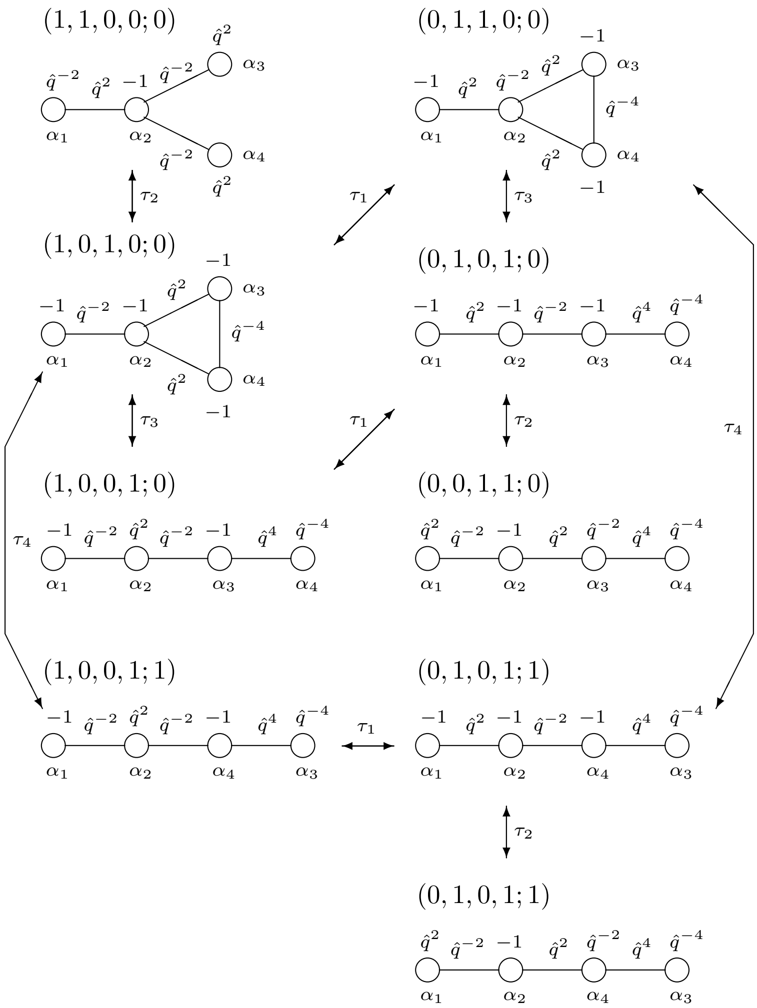

Figure 1. All $\ddot{p} \in \ddot{\mathcal{A}}_{2|2}$ and the Dynkin diagrams of all $\chi \in \dot{\mathcal{X}}_N^{\mathrm{Super}}(\mathrm{D}(2,2))$, where $(z_1, z_2, z_3, z_4; z_5)$ means $\ddot{p} \in \ddot{\mathcal{A}}_{2|2}$ with $\ddot{p}(i) = z_i$.

Definition 5.4. Recall Definition 3.1. For a rank-N standard super-data $(\bar{\eta}, \langle\bar{\Pi}\rangle)$, let $a^{\sharp}_{\langle\bar{\Pi}\rangle} := (\bar{\eta}_{\langle\bar{\Pi}\rangle}, \theta) \in \mathcal{Y}_N^{\mathrm{fin}}$ (cf. Theorem 3.4), where θ is the parity map associated with $(\bar{\eta}, \langle\bar{\Pi}\rangle)$. Let $\dot{\mathcal{X}}_N^{\mathrm{Super}}$ be the subset of $\mathcal{X}_N$ formed by $\varpi_{\hat{q}}(a^{\sharp}_{\langle\bar{\Pi}\rangle})$ for some $\hat{q} \in \mathbb{K}_\infty^\times$ and some rank-N standard super-data $(\bar{\eta}, \langle\bar{\Pi}\rangle = (\bar{\alpha}_i \mid i \in I))$. Let $\dot{\mathcal{X}}_N^{\mathrm{Super}}(\mathrm{X})$ be the subset of $\dot{\mathcal{X}}_N^{\mathrm{Super}}$ formed by $\varpi_{\hat{q}}(a^{\sharp}_{\langle\bar{\Pi}\rangle})$ for some $\hat{q} \in \mathbb{K}_\infty^\times$ and some X-data.

Using Lemma 5.1 and facts mentioned in Section 3, we can prove

Lemma 5.5. *Let* $\chi \in \dot{\mathcal{X}}_N^{\mathrm{Super}}$. *Assume that* χ *belongs to* $\dot{\mathcal{X}}_N^{\mathrm{Super}}(\mathrm{A}(m-1, N-m))$ *with* $N \geq 2$ *and* $m \in J_{1,N}$, $\dot{\mathcal{X}}_N^{\mathrm{Super}}(\mathrm{B}(m, N-m))$ *with* $N \geq 1$ *and* $m \in J_{1,N}$, $\dot{\mathcal{X}}_N^{\mathrm{Super}}(\mathrm{C}(N))$ *with* $N \geq 3$, *or* $\dot{\mathcal{X}}_N^{\mathrm{Super}}(\mathrm{D}(m, N-m))$ *with* $N \geq 3$ *and* $m \in J_{2,N-1}$. *Define* $\hat{q} \in \mathbb{K}_\infty^\times$, *a Cartan scheme* $\mathcal{C} = \mathcal{C}(I, \mathcal{A}, (\tau_i)_{i\in I}, (C^a)_{a\in\mathcal{A}})$, *and a generalized root system* $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R(a))_{a\in\mathcal{A}})$ *of type* $\mathcal{C}$, $p \in \mathcal{A}$, *and symmetric bilinear maps* $\eta^a : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ $(a \in \mathcal{A})$ *by Tables 1 and 2.*

	$\hat{q}$	$\mathcal{C}$	$\mathcal{R}$
$\mathrm{A}(m-1, N-m)$	$\chi(\alpha_1, \alpha_2)^{-1}$	$\check{\mathcal{C}}_{m\mid N+1-m}$	$\check{\mathcal{R}}_{m\mid N+1-m}$
$\mathrm{B}(m, N-m)$	$\chi(\alpha_N, \alpha_N)$	$\dot{\mathcal{C}}_{m\mid N-m}$	$\dot{\mathcal{R}}_{m\mid N-m}$
$\mathrm{C}(N)$	$\chi(\alpha_1, \alpha_2)^{-1}$	$\breve{\mathcal{C}}_{1\mid N-1}$	$\breve{\mathcal{R}}_{1\mid N-1}$
$\mathrm{D}(m, N-m)$	$\chi(\alpha_{N-2}, \alpha_{N-1})^{-1}$	$\ddot{\mathcal{C}}_{m\mid N-m}$	$\ddot{\mathcal{R}}_{m\mid N-m}$

Table 1. Generalized root systems of ABCD-type quantum superalgebras.

	p	η^a
$\mathrm{A}(m-1, N-m)$	$p^+_{m\mid N+1-m}$	$\bar{\eta}^a_{\langle\bar{\Pi}\rangle}$
$\mathrm{B}(m, N-m)$	$p^-_{N-m\mid m}$	$\bar{\eta}^a_{\langle\bar{\Pi}\rangle}$
$\mathrm{C}(N)$	$\acute{p}_N$	$-\ddot{\eta}^a_{\langle\ddot{\Pi}^a\rangle}$
$\mathrm{D}(m, N-m)$	$\ddot{p}_{m\mid N-m}$	$\ddot{\eta}^a_{\langle\ddot{\Pi}^a\rangle}$

Table 2. Bilinear forms of ABCD-type quantum superalgebras.

For $a \in \mathcal{A}$, *define the map* $\theta^a : I \to \{0, 1\}$ *by*

$$\theta^a(i) := \begin{cases} 1 & \text{if } \eta^a(\alpha_i, \alpha_i) \in \{0, -1\}, \\ 0 & \text{if } \eta^a(\alpha_i, \alpha_i) \in \{1, 2, -2, 4, -4\}. \end{cases}$$

($\eta^a(\alpha_i, \alpha_i) \in \{1, -1\}$ *if and only if* $\chi \in \dot{\mathcal{X}}_N^{\mathrm{Super}}(\mathrm{B}(m, N-m))$ *and* $i = N$.) *Let* $a^\sharp := (\eta^p, \theta^p) \in \mathcal{Y}_N$. *Then* $a^\sharp \in \mathcal{Y}_N^{\mathrm{fin}}$. *Then for* $f \in \mathrm{Map}_\infty^I$ *and* $t \in \mathbb{Z}_{\geq 0}$, *we have* $a^\sharp_{f,t} \in \mathcal{Y}_N^{\mathrm{fin}}$ (*in* $\mathcal{C}_{a^\sharp}$) *and*

$$\chi_{f,t} = \varpi_{\hat{q}}(a^\sharp_{f,t}) \in \mathcal{X}_N^{\mathrm{fin}} \tag{5.3}$$

(*in* $\mathcal{C}_\chi$). *In particular,* $(\mathcal{R}_\chi, \chi)$ *is isomorphic to* $(\mathcal{R}, p)$.

Lemma 5.6. *Let* $\chi := \varpi_{\hat{q}}(a^\sharp_{\langle\bar{\Pi}\rangle}) \in \dot{\mathcal{X}}_N^{\mathrm{Super}}$ *be as in Definition 5.4. Let* $\bar{R}$ *be the standard root system associated with* $(\bar{\eta}, \langle\bar{\Pi}\rangle)$. *Then*

$$R(\chi) = \xi_{\langle\bar{\Pi}\rangle}(\bar{R} \setminus 2\bar{R}). \tag{5.4}$$

In particular, $\chi \in \mathcal{X}_N^{\mathrm{fin}}$.

Proof. This follows from (1.18), Theorem 3.4, Definition 3.5(2) and Lemma 5.1. □

Definition 5.7. If $N \in \mathbb{N} \setminus J_{2,4}$, let $\dot{\mathcal{X}}_N^{\mathrm{Extra}} := \emptyset$. If $N = 2$ (resp. $N = 3$, $N = 4$), let $\dot{\mathcal{X}}_N^{\mathrm{Extra}}$ be the set of bi-homomorphisms $\chi \in \mathcal{X}_N$ satisfying condition (5.5) (resp. (5.6), (5.7)) below. In the following, let $q_{ij} := \chi(\alpha_i, \alpha_j)$.

$$q_{11}^2 + q_{11} + 1 = 0, \quad q_{12} = q_{21} \in \mathbb{K}_\infty^\times, \quad q_{12}q_{21}q_{22} = 1. \tag{5.5}$$

$$\begin{aligned} &q_{12} = q_{21} \in \mathbb{K}_\infty^\times, \quad q_{11}q_{12}q_{21} = 1, \quad q_{22} = -1, \quad q_{13} = q_{31} = 1, \\ &q_{23} = q_{32}, \quad q_{23}q_{32}q_{33} = 1, \quad q_{33} \neq 1, \quad q_{11}q_{33} \neq 1. \end{aligned} \tag{5.6}$$

$$\begin{aligned} &q_{12} = q_{21} = q_{23} = q_{32} \in \mathbb{K}_\infty^\times, \quad q_{34} = q_{43}, \\ &q_{13} = q_{31} = q_{14} = q_{41} = q_{24} = q_{42} = 1, \\ &q_{33} = q_{11}q_{44} = -1, \quad q_{11}q_{12}q_{21} = q_{12}q_{21}q_{22} = 1, \quad q_{34}q_{43}q_{44} = 1. \end{aligned} \tag{5.7}$$

Remark 5.8. Note that

$$\text{if } N = 3, \text{ then } \textstyle\bigcup_{x \in \mathbb{Z} \setminus \{0,-1\}} \dot{\mathcal{X}}_N^{\mathrm{Super}}(\mathrm{D}(2,1;x)) \subset \dot{\mathcal{X}}_N^{\mathrm{Extra}}. \tag{5.8}$$

Let $\chi \in \dot{\mathcal{X}}_N^{\mathrm{Extra}}$. If $N = 2$, then $R^+(\chi) = \{\alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, \alpha_1\}$. If $N = 4$, then $R^+(\chi) = \{\alpha_1, \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3, \alpha_1 + \alpha_2, \alpha_2\}$. If $N = 3$, then $R^+(\chi) = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_2, \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4, \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \alpha_3 + \alpha_4, \alpha_2 + 2\alpha_3 + \alpha_4, \alpha_2 + \alpha_3, \alpha_3\}$. We can directly prove these facts using Lemma 1.11 and (7.37) below.

§5.2. Heckenberger's classification of FID-type bi-homomorphisms

Lemma 5.9. *Let $\chi \in \mathcal{X}_N^{\mathrm{irr}}$. Let $q_{ij} := \chi(\alpha_i, \alpha_j)$ for $i, j \in I$. Then $\dim U^-(\chi) < \infty$ if and only if $\chi \in \mathcal{X}_N^{\mathrm{fin}}$ and either* (x) *or* (y) *below holds.*

(x) *$N = 1$ and $q_{11} \notin \mathbb{K}_\infty^\times \cup \mathbb{K}_1^\times$.*

(y) *$N \geq 2$ and $q_{ij}q_{ji} \notin \mathbb{K}_\infty^\times$ for all $i, j \in I$.*

Proof. By Theorem 4.10, we see that

$$\dim U^-(\chi) < \infty \;\Leftrightarrow\; \chi \in \mathcal{X}_N^{\mathrm{fin}} \text{ and } \forall \alpha \in R^+(\chi),\ \chi(\alpha, \alpha) \notin \mathbb{K}_\infty^\times \cup \mathbb{K}_1^\times. \tag{5.9}$$

Hence, if $N = 1$, the statement is true.

Assume $N \geq 2$ and $\chi \in \mathcal{X}_N^{\mathrm{fin}}$. Since $\chi \in \mathcal{X}_N^{\mathrm{irr}}$, by (4.23) and Lemma 4.15(1), we have $q_{ii} \neq 1$ for all $i \in I$.

Assume that (y) is not true. If $q_{ii} \in \mathbb{K}_\infty^\times$ for some $i \in I$, then $\dim U^-(\chi) = \infty$ by (5.9). Assume that there exist $i, j \in I$ with $i \neq j$ such that $q_{ii}, q_{jj} \notin \mathbb{K}_\infty^\times$ and $q_{ij}q_{ji} \in \mathbb{K}_\infty^\times$. Then $-c_{ij}^\chi \in \mathbb{N}$ by (4.10) and (4.23). Let $\beta := \alpha_i + \alpha_j$. By (4.12), $\beta \in R^+(\chi)$. Since $\chi(\beta, \beta) \in \mathbb{K}_\infty^\times$, we have $\dim U^-(\chi) = \infty$ by (5.9).

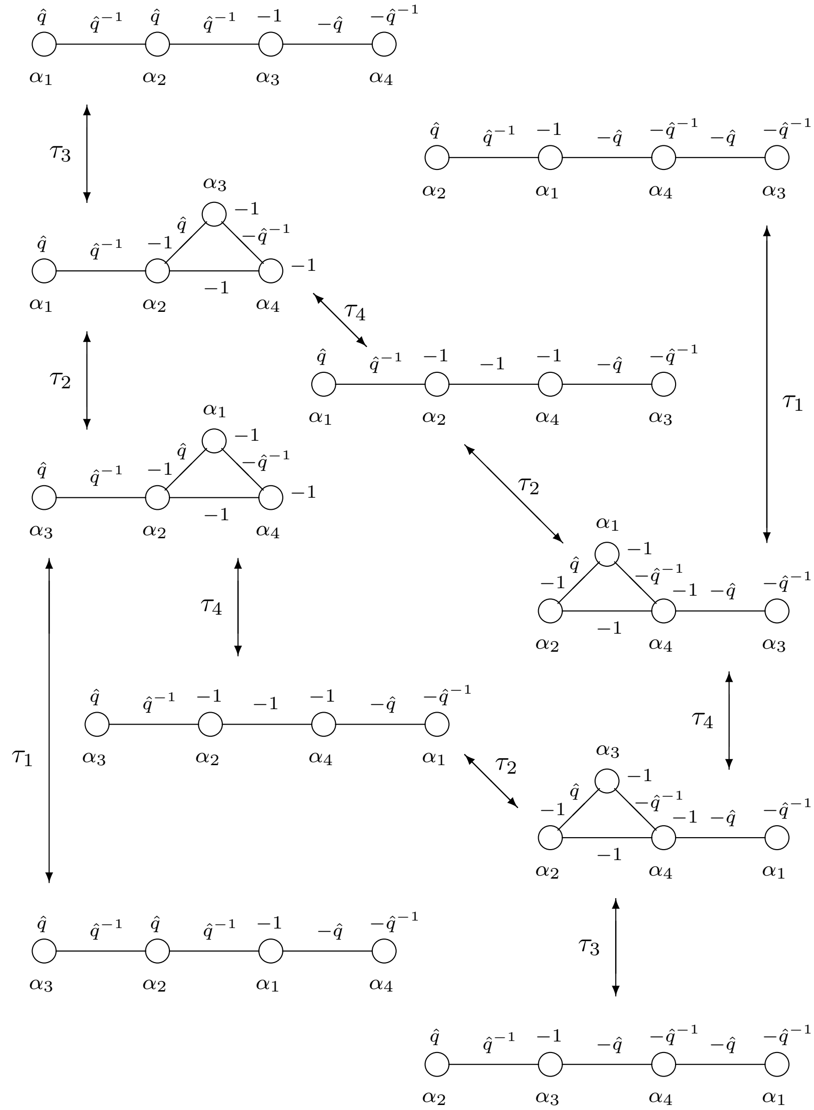

Figure 2. Dynkin diagrams of $\dot{\mathcal{X}}_4^{\mathrm{Extra}}$.

Assume that (y) is true. Let $\alpha \in R^+(\chi)$. It is clear that $\chi(\alpha,\alpha) \notin \mathbb{K}^\times_\infty$. By (1.17), Theorem 4.18(2) and Lemma 4.15(3), there exist $\chi' \in \mathcal{X}_N^{\mathrm{fin}} \cap \mathcal{X}_N^{\mathrm{irr}}$ and $i \in I$ such that $\chi' \sim \chi$ and $\chi'(\alpha_i,\alpha_i) = \chi(\alpha,\alpha)$. By (4.23) and Lemma 4.15(1), since $\chi' \in \mathcal{X}_N^{\mathrm{fin}} \cap \mathcal{X}_N^{\mathrm{irr}}$, we have $\chi(\alpha,\alpha) \neq 1$. By (5.9), we have $\dim U^-(\chi) < \infty$, as desired. $\square$

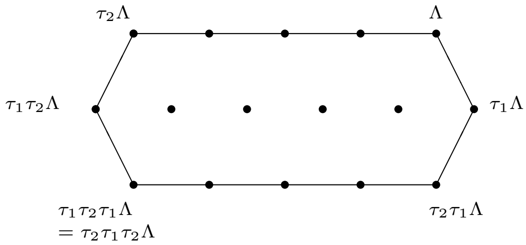

Figure 3. $\mathcal{L}_\chi(\Lambda)$ for $\chi' \in \dot{\mathcal{X}}_N^{\mathrm{Super}}(\mathrm{A}(m-1, N-m))$ with $N = 2$, $m = 1$, $\lambda_1 \neq -1$, $\lambda_2 = \hat{q}^8$, $\lambda_1\lambda_2\hat{q}^2 \neq -1$, where $\lambda_i := \Lambda(K_{\alpha_i} L_{-\alpha_i})$.

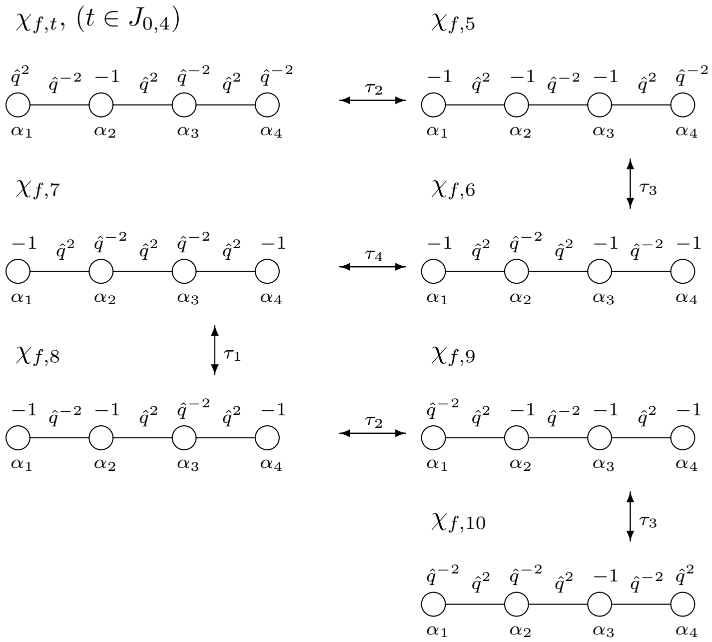

Figure 4. Dynkin diagrams of $\chi = \chi_{f,0} \equiv \chi' \in \dot{\mathcal{X}}_N^{\mathrm{Super}}(\mathrm{A}(m-1, N-m))$ with $N = 4$ and $m = 1$, and $\chi_{f,u}$, where $1^\chi w_0 = s_1^{\chi_{f,1}} s_3^{\chi_{f,2}} s_4^{\chi_{f,3}} s_3^{\chi_{f,4}} s_2^{\chi_{f,5}} s_3^{\chi_{f,6}} s_4^{\chi_{f,7}} s_1^{\chi_{f,8}} s_2^{\chi_{f,9}} s_3^{\chi_{f,10}}$.

For $\chi, \chi' \in \mathcal{X}_N$, we write $\chi \approx \chi'$ if there exist $\chi_1, \chi_2 \in \mathcal{X}_N$ and a bijection $f : I \to I$ such that $\chi \sim \chi_1 \equiv \chi_2$ and $\chi'(\alpha_i, \alpha_j) = \chi_2(\alpha_{f(i)}, \alpha_{f(j)})$ $(i, j \in I)$.

By Heckenberger's classification [10, Tables 1–4, Theorems 17, 22] and Lemma 5.9, we have

Theorem 5.10 ([10]). (1) *Assume* $N = 1$. *Let* $\chi \in \mathcal{X}_N$. *Then* $\dim U^-(\chi) = \infty$ *if and only if* $\chi(\alpha_1, \alpha_1) \in \mathbb{K}^\times_\infty$ *or* $\chi(\alpha_1, \alpha_1) = 1$.

(2) *Assume* $N \geq 2$. *Then*

$$\begin{aligned}(5.10) \qquad & \{\chi \in \mathcal{X}_N^{\mathrm{irr}} \cap \mathcal{X}_N^{\mathrm{fin}} \mid \dim U^-(\chi) = \infty\} \\ & \quad = \{\chi \in \mathcal{X}_N \mid \exists \chi' \in \dot{\mathcal{X}}_N^{\mathrm{Cartan}} \cup \dot{\mathcal{X}}_N^{\mathrm{Super}} \cup \dot{\mathcal{X}}_N^{\mathrm{Extra}},\ \chi \approx \chi'\}.\end{aligned}$$

As in Introduction, if χ is as in (5.10) or is the one with $N = 1$ and $\chi(\alpha_1, \alpha_1)$ in $\mathbb{K}_\infty^\times \cup \{1\}$, we say that χ (or $U(\chi)$) is of *finite-and-infinite-dimensional type* (FID-type, for short).

Remark 5.11. We have $\dot{\mathcal{X}}_N^{\mathrm{Cartan}}(\mathrm{B}_N) = \dot{\mathcal{X}}_N^{\mathrm{Super}}(\mathrm{B}(0, N))$. We have a bijection $\dot{\mathcal{X}}_N^{\mathrm{Super}}(\mathrm{A}(m-1, N-m)) \to \dot{\mathcal{X}}_N^{\mathrm{Super}}(\mathrm{A}(N-m, m-1))$, $\chi \mapsto \chi'$, defined by $\chi'(\alpha_i, \alpha_j) := \chi(\alpha_{N-i+1}, \alpha_{N-j+1})$ $(i, j \in I)$, where we also have $\chi \approx \chi'$ (see Figure 4). If $N = 3$, then we have (5.8) and $\dot{\mathcal{X}}_N^{\mathrm{Super}}(\mathrm{D}(2,1)) = \dot{\mathcal{X}}_N^{\mathrm{Super}}(\mathrm{D}(2,1;1))$, and we see that for χ, $\chi' \in \dot{\mathcal{X}}_N^{\mathrm{Extra}}$, $\chi \approx \chi'$ if and only if $\{q_{11}, q_{33}, (q_{11}q_{33})^{-1}\} = \{q'_{11}, q'_{33}, (q'_{11}q'_{33})^{-1}\}$, where $q_{ij} := \chi(\alpha_i, \alpha_j)$ and $q'_{ij} := \chi'(\alpha_i, \alpha_j)$ (see also Figure 10).

Remark 5.12. For the defining relations of $U(\chi)$ with $\chi \approx \chi'$ for some $\chi' \in \dot{\mathcal{X}}_N^{\mathrm{Super}}$, see [4], [1], [25], [26], [27], [28], [30]. See also Remark 7.11 below.

§6. Lusztig isomorphisms

§6.1. Lusztig isomorphisms of generalized quantum groups

In this section, fix $i \in I$ and $\chi \in (\mathcal{X}_N^{\prime\prime,\mathrm{fin}})_i$, and let $q_{ij} := \chi(\alpha_i, \alpha_j)$ $(j \in I)$.

Theorem 6.1 ([11, Theorem 6.11]). *Assume* $\chi \in (\mathcal{X}_N^{\prime\prime,\mathrm{fin}})_i$. *Then there exists a unique* $\mathbb{K}$*-algebra isomorphism*

$$T_i := T_i^{\tau_i\chi} : U(\tau_i\chi) \to U(\chi)$$

such that

$$\begin{aligned} T_i(K_\alpha) &= K_{s_i^{\tau_i\chi}(\alpha)}, \quad T_i(L_\alpha) = L_{s_i^{\tau_i\chi}(\alpha)}, \\ T_i(E_i) &= F_i L_{-\alpha_i}, \qquad T_i(F_i) = K_{-\alpha_i} E_i, \\ T_i(E_j) &= E_{-c_{ij}^\chi, \alpha_i, \alpha_j}, \\ T_i(F_j) &= \frac{1}{(-c_{ij}^\chi)_{q_{ii}}!(-c_{ij}^\chi; q_{ii}, q_{ij}q_{ji})!} F_{-c_{ij}^\chi, \alpha_i, \alpha_j} \end{aligned}$$

for $\alpha \in \mathbb{Z}\Pi$ *and* $j \in I \setminus \{i\}$. *In particular,*

$$(6.1) \qquad T_i(U(\tau_i\chi)_\alpha) = U(\chi)_{s_i^\chi(\alpha)} \quad (\alpha \in \mathbb{Z}\Pi).$$

Assume $\chi \in \mathcal{X}_N^{\prime,\mathrm{fin}}$. Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and $f \in \mathrm{Map}_r^I$. For $t \in J_{0,r}$, define the $\mathbb{K}$-algebra isomorphism $1^\chi T_{f,t} : U(\chi_{f,t}) \to U(\chi)$ in the same way as in (1.4) with T_j's of Theorem 6.1 in place of s_j's.

Remark 6.2. Assume $\chi \in \mathcal{X}_N^{\mathrm{fin}}$. Let $n := |R^+(\chi)|$. Let $f \in \mathrm{Map}_n^I$ be such that $1^\chi s_{f,n} = 1^\chi w_0$. For $\beta \in R^+(\chi)$, letting $t \in J_{1,n}$ be such that $\beta = 1^\chi s_{f,t-1}(\alpha_{f(t)})$ (see (1.17) and Theorem 4.19), let $F_\beta := 1^\chi T_{f,t-1}(F_{f(t)})$. Then the elements F_β ($\beta \in R^+(\chi)$) have the properties of Theorem 4.10(2) (see [14, Theorem 4.9]).

§6.2. Lusztig isomorphisms between irreducible modules

In this subsection, fix $\Lambda \in \mathrm{Ch}(U^0(\chi))$. Let $h := h_{\chi,\Lambda,i}$. Assume that $\chi \in (\mathcal{X}_N^{\prime\prime,\mathrm{fin}})_i$ and $h \neq \infty$. Define $\tau_i\Lambda := \tau_i^\chi\Lambda \in \mathrm{Ch}(U^0(\tau_i\chi))$ by

$$(6.2) \qquad \tau_i^\chi\Lambda(K_\alpha L_\beta) := \Lambda(K_{s_i^{\tau_i\chi}(\alpha)} L_{s_i^{\tau_i\chi}(\beta)}) \frac{\chi(\alpha_i, s_i^{\tau_i\chi}(\beta))^h}{\chi(s_i^{\tau_i\chi}(\alpha), \alpha_i)^h} \qquad (\alpha, \beta \in \mathbb{Z}\Pi).$$

By (4.15) and (4.22), we have

$$(6.3) \qquad h_{\tau_i\chi,\tau_i\Lambda,i} = h \quad \text{and} \quad \tau_i^{\tau_i\chi}\tau_i^\chi\Lambda = \Lambda.$$

By (4.20), (4.28) and (6.2), for $\chi' \in \mathcal{X}_N$ and $\Lambda' \in \mathrm{Ch}(U^0(\chi'))$, we have

$$(6.4) \qquad (\chi, \Lambda) \equiv (\chi', \Lambda') \ \Rightarrow\ (\tau_i\chi, \tau_i^\chi\Lambda) \equiv (\tau_i\chi', \tau_i^{\chi'}\Lambda'),$$

where $\chi' \in (\mathcal{X}_N^{\prime\prime,\mathrm{fin}})_i$.

Lemma 6.3. *Assume $h < \infty$. There exists a unique $\mathbb{K}$-linear isomorphism*

$$\hat{T}_i := \hat{T}_i^{\tau_i\chi,\tau_i\Lambda} : \mathcal{L}_{\tau_i\chi}(\tau_i^\chi\Lambda) \to \mathcal{L}_\chi(\Lambda)$$

such that

$$(6.5) \qquad \hat{T}_i(Xv_{\tau_i\chi}) = T_i(X)F_i^h v_\Lambda \qquad (X \in U(\tau_i\chi)).$$

Proof. We can regard $\mathcal{L}_\chi(\Lambda)$ as a left $U(\tau_i\chi)$-module defined by $X \cdot u := T_i(X)u$ ($X \in U(\tau_i\chi)$, $u \in \mathcal{L}_\chi(\Lambda)$). Let $v' := F_i^h v_\Lambda \in \mathcal{L}_\chi(\Lambda)$. Note that a $U(\chi)$-submodule of $\mathcal{L}_\chi(\Lambda)$ is a $U(\tau_i\chi)$-submodule, and vice versa. By (4.14), $v' \neq 0$ and $E_i \cdot v' = 0$, so we also have $E_j \cdot v' = 0$ for $j \in I \setminus \{i\}$. Then we have a $U(\tau_i\chi)$-module homomorphism $z : \mathcal{M}_{\tau_i\chi}(\tau_i\Lambda) \to \mathcal{L}_\chi(\Lambda)$ such that $z(X\tilde{v}_{\tau_i\Lambda}) = X \cdot v'$ for $X \in U(\tau_i\chi)$. By (4.13), z is surjective, and we also have $z(\mathcal{M}_{\tau_i\chi}(\tau_i\Lambda)_\alpha) = \mathcal{L}_\chi(\Lambda)_{s_i(\alpha)-h\alpha_i}$ for $\alpha \in \mathbb{Z}_{\geq 0}\Pi$. In particular, $\ker z$ is a proper $\mathbb{Z}\Pi$-graded left $U(\tau_i\chi)$-submodule of $\mathcal{M}_{\tau_i\chi}(\tau_i\Lambda)$, so $\ker z \subseteq \mathcal{N}_{\tau_i\chi}(\tau_i\Lambda)$. By (4.14) and (6.3), we have $\mathcal{M}_{\tau_i\chi}(\tau_i\Lambda)_{-h\alpha_i} \cap \mathcal{N}_{\tau_i\chi}(\tau_i\Lambda) = \{0\}$ since $\mathcal{M}_{\tau_i\chi}(\tau_i\Lambda)_{-h\alpha_i} = F_i^h\tilde{v}_{\tau_i\Lambda}$. Therefore, $z(\mathcal{N}_{\tau_i\chi}(\tau_i\Lambda))$ is a proper $\mathbb{Z}\Pi$-graded $U(\chi)$-submodule of $\mathcal{L}_\chi(\Lambda)$. Hence $z(\mathcal{N}_{\tau_i\chi}(\tau_i\Lambda)) = \{0\}$, which implies $\mathcal{N}_{\tau_i\chi}(\tau_i\Lambda) = \ker z$. Hence z induces a $U(\tau_i\chi)$-module isomorphism $\hat{T}_i$, as desired. ∎

Using Lemma 4.21, together with (4.20) and (4.27), the following lemma is an easy exercise for the reader.

Lemma 6.4. *Assume* $\chi \in \mathcal{X}_N^{\mathrm{fin}}$. *Assume that* $q_{ii} \in \mathbb{K}_\infty^\times$ *and* $h_{\chi,\Lambda,i} \neq \infty$. *Then* $(\tau_i\chi, \tau_i^\chi\Lambda) \equiv (\chi,\Lambda)$, *and* $h_{\tau_i\chi,\tau_i^\chi\Lambda,j} = h_{\chi,\Lambda,j}$ *for* $j \in I$.

Definition 6.5 (Definition of $H(\chi,\Lambda,f)$). Assume $\chi \in \mathcal{X}_N^{\prime,\mathrm{fin}}$. Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and $f \in \mathrm{Map}_r^I$. Recall Notation 1.1. Let $\Lambda_{\chi,f,0} := \Lambda$. If $t \in J_{1,r}$, we define $\Lambda_{\chi,f,t} := \tau_{f(t)}^{\chi_{f,t-1}}\Lambda_{\chi,f,t-1}$ if $\Lambda_{\chi,f,t-1}$ can be defined and $h_{\chi_{f,t-1},\Lambda_{\chi,f,t-1},f(t)} < \infty$. Define $H(\chi,\Lambda,f) \in J_{0,r}$ as follows. If $r=0$, let $H(\chi,\Lambda,f) := 0$. If there exists $t \in J_{0,r-1}$ such that $h_{\chi_{f,k-1},\Lambda_{\chi,f,k-1},f(k)} < \infty$ for all $k \in J_{1,t}$ and $h_{\chi_{f,t},\Lambda_{\chi,f,t},f(t+1)} = \infty$, let $H(\chi,\Lambda,f) := t$. If $h_{\chi_{f,k-1},\Lambda_{\chi,f,k-1},f(k)} < \infty$ for all $k \in J_{1,r}$, let $H(\chi,\Lambda,f) := r$. For $t \in J_{0,H(\chi,\Lambda,f)}$, define the $\mathbb{K}$-linear isomorphism $1^{\chi,\Lambda}\hat{T}_{f,t} : \mathcal{L}_{\chi_{f,t}}(\Lambda_{\chi,f,t}) \to \mathcal{L}_\chi(\Lambda)$ as in (1.4) with $\hat{T}_j$'s of Lemma 6.3 in place of s_j's.

The following lemma is crucial to this paper.

Lemma 6.6. *Assume* $\chi \in \mathcal{X}_N^{\mathrm{fin}}$. *Let* $n := |R^+(\chi)|$. *Let* $f \in \mathrm{Map}_n^I$ *be such that* $1^\chi s_{f,n} = 1^\chi w_0$ *(see also Lemma 1.9(2)). Then* $\dim \mathcal{L}_\chi(\Lambda) < \infty$ *if and only if* $H(\chi,\Lambda,f) = n$.

Proof. Assume $\dim \mathcal{L}_\chi(\Lambda) < \infty$. Then $h_{\chi_{f,0},\Lambda_{f,0},f(1)} < \infty$. By Lemma 6.3, we have $\dim \mathcal{L}_{\chi_{f,1}}(\Lambda_{f,1}) < \infty$. Repeating this argument, we find $H(\chi,\Lambda,f) = n$.

Assume $H(\chi,\Lambda,f) = n$. Let $\gamma := \sum_{t\in J_{1,n}} h_{\chi_{f,t-1},\Lambda_{f,t-1},f(t)} 1^\chi s_{f,t-1}(\alpha_{f(t)})$. By (1.17), $\gamma \in \mathbb{Z}_{\geq 0}\Pi$. By (6.5), $1^{\chi,\Lambda}\hat{T}_{f,n}(v_{\Lambda_{\chi,f,n}}) \in \mathcal{L}_\chi(\Lambda)_{-\gamma}$. Let

$$X := \{\beta \in \mathbb{Z}_{\geq 0}\Pi \mid \gamma - \beta \in \mathbb{Z}_{\geq 0}\Pi\}.$$

Then $|X| < \infty$. We have

$$\begin{aligned}
\mathcal{L}_\chi(\Lambda) &= 1^{\chi,\Lambda}\hat{T}_{f,n}(\mathcal{L}_{\chi_{f,n}}(\Lambda_{\chi,f,n})) = 1^{\chi,\Lambda}\hat{T}_{f,n}(U^-(\chi_{f,n})v_{\Lambda_{\chi,f,n}}) \\
&= \bigoplus_{\alpha\in\mathbb{Z}_{\geq 0}\Pi} 1^{\chi,\Lambda}\hat{T}_{f,n}(U^-(\chi_{f,n})_{-\alpha}v_{\Lambda_{\chi,f,n}}) \\
&= \bigoplus_{\alpha\in\mathbb{Z}_{\geq 0}\Pi} 1^{\chi,\Lambda}\hat{T}_{f,n}(U(\chi_{f,n})_{-\alpha}v_{\Lambda_{\chi,f,n}}) \\
&= \bigoplus_{\alpha\in\mathbb{Z}_{\geq 0}\Pi} 1^{\chi}T_{f,n}(U(\chi_{f,n})_{-\alpha})1^{\chi,\Lambda}\hat{T}_{f,n}(v_{\Lambda_{\chi,f,n}}) \quad \text{(by (6.5))} \\
&= \bigoplus_{\beta\in\mathbb{Z}_{\geq 0}\Pi} U(\chi)_\beta 1^{\chi,\Lambda}\hat{T}_{f,n}(v_{\Lambda_{\chi,f,n}}) \quad \text{(by (1.16) and (6.1))} \\
&= \bigoplus_{\beta\in\mathbb{Z}_{\geq 0}\Pi} \mathcal{L}_\chi(\Lambda)_{\beta-\gamma}
\end{aligned}$$

$$= \bigoplus_{\beta\in X} \mathcal{L}_\chi(\Lambda)_{\beta-\gamma} \quad (\text{since } \mathcal{L}_\chi(\Lambda) = \textstyle\bigoplus_{\alpha\in\mathbb{Z}_{\geq 0}\Pi} \mathcal{L}_\chi(\Lambda)_{-\alpha}).$$

Hence $\dim \mathcal{L}_\chi(\Lambda) = \sum_{\beta\in X} \dim \mathcal{L}_\chi(\Lambda)_{\beta-\gamma} < \infty$, as desired. □

The following lemma follows from (4.20), (6.4), and Lemmas 6.6 and 4.22.

Lemma 6.7. *Assume $\chi \in \mathcal{X}_N^{\rm fin}$. Let $\chi' \in \mathcal{X}_N^{\rm fin}$ and $\Lambda' \in \mathrm{Ch}(U^0)$ be such that $(\chi', \Lambda') \equiv (\chi, \Lambda)$. Then $\dim \mathcal{L}_{\chi'}(\Lambda') < \infty$ if and only if $\dim \mathcal{L}_\chi(\Lambda) < \infty$.*

§7. Main theorems

§7.1. Irreducible modules for Cartan and super-AC cases

For $\chi \in \mathcal{X}_N$ and $i \in I$, let

$$\mathbb{S}_i(\chi) := \{\Lambda \in \mathrm{Ch}(U^0(\chi)) \mid \exists r \in \mathbb{Z}_{\geq 0},\ \Lambda(K_{\alpha_i} L_{-\alpha_i}) = \chi(\alpha_i, \alpha_i)^r\}.$$

Theorem 7.1. *Assume $N = 1$. Let $\chi \in \mathcal{X}_N$. Let $q_{11} := \chi(\alpha_1, \alpha_1)$.*

(1) *If $q_{11} \in \mathbb{K}^\times_\infty$ (resp. $q_{11} \in \mathbb{K}^\times \setminus (\mathbb{K}^\times_\infty \cup \{1\})$). then $\{\mathcal{L}_\chi(\Lambda) \mid \Lambda \in \mathbb{S}_1(\chi)\}$ (resp. $\{\mathcal{L}_\chi(\Lambda) \mid \Lambda \in \mathrm{Ch}(U^0(\chi))\}$) is a complete set of pairwise non-isomorphic finite-dimensional irreducible $U(\chi)$-modules (see also (4.17)).*

(2) *Assume $q_{11} = 1$. Let $\mathcal{V}$ be a finite-dimensional $U(\chi)$-module. Then $\mathcal{V}$ is irreducible if and only if $\dim \mathcal{V} = 1$. If this is the case, $K_{\alpha_1} L_{-\alpha_1} v = v$ for all $v \in \mathcal{V}$.*

Proof. (1) This follows from (4.14) and Lemma 4.23. (See also Remark 4.24.)

(2) Assume that $\mathcal{V}$ is irreducible. Let $f : U(\chi) \to \mathrm{End}_{\mathbb{K}}(\mathcal{V})$ be the $\mathbb{K}$-algebra homomorphism obtained from $\mathcal{V}$. Then $f(K_{\alpha_1}) = x \cdot \mathrm{id}_{\mathcal{V}}$ for some $x \in \mathbb{K}^\times$, and $f(L_{\alpha_1}) = y \cdot \mathrm{id}_{\mathcal{V}}$ for some $y \in \mathbb{K}^\times$. Since the trace of $f(E_1F_1 - F_1E_1)$ is zero, we have $x = y$, so $f(E_1F_1 - F_1E_1) = 0$. Then $\dim \mathcal{V} = 1$, since $\mathbb{K}$ is algebraically closed. □

Theorem 7.2. *Assume $N \geq 2$. Let $\chi \in \mathcal{X}_N$ be such that $\chi \approx \chi'$ for some $\chi' \in \dot{\mathcal{X}}_N^{\rm Cartan}$. Then $\{\mathcal{L}_\chi(\Lambda) \mid \Lambda \in \bigcap_{i\in I} \mathbb{S}_i(\chi)\}$ is a complete set of pairwise non-isomorphic finite-dimensional irreducible $U(\chi)$-modules (see also (4.17)).*

Proof. By Theorem 5.10, $\chi \in \mathcal{X}_N^{\rm fin}$. Note that $\chi(\alpha_i, \alpha_i) \in \mathbb{K}^\times_\infty$ for all $i \in I$. Now the conclusion follows from (4.15), (4.16) and Lemmas 4.23, 6.4 and 6.6. □

Remark 7.3. See [21] for some related results concerning Theorem 7.2.

Theorem 7.4. (1) *Assume $N \geq 2$. Let $m \in I$. Let $\chi \in \mathcal{X}_N$ be such that $\chi \equiv \chi'$ for some $\chi' \in \dot{\mathcal{X}}_N^{\rm Super}(\mathrm{A}(m-1, N-m))$. Then $\{\mathcal{L}_\chi(\Lambda) \mid \Lambda \in \bigcap_{i\in I\setminus\{m\}} \mathbb{S}_i(\chi)\}$ is* *a complete set of pairwise non-isomorphic finite-dimensional irreducible* $U(\chi)$*-modules* (*see also* (4.17)).

(2) *Assume* $N \geq 3$. *Let* $\chi \in \mathcal{X}_N$, *and assume that* $\chi \equiv \chi'$ *for some* $\chi' \in \dot{\mathcal{X}}_N^{\mathrm{Super}}(\mathrm{C}(N))$. *Then* $\{\mathcal{L}_\chi(\Lambda) \mid \Lambda \in \bigcap_{i\in I\setminus\{1\}} \mathbb{S}_i(\chi)\}$ *is a complete set of pairwise non-isomorphic finite-dimensional irreducible* $U(\chi)$*-modules* (*see also* (4.17)).

Proof. (1) By Theorem 5.10, we see that $\chi, \chi' \in \mathcal{X}_N^{\mathrm{fin}}$. By Lemma 6.7, we may assume $\chi = \chi'$. Let $\Lambda \in \mathrm{Ch}(U^0(\chi))$. By Lemma 4.23, we only need to show that $\dim \mathcal{L}_\chi(\Lambda) < \infty$ if and only if $\Lambda \in \bigcap_{i\in I\setminus\{m\}} \mathbb{S}_i(\chi)$. By (4.15) and (4.16), we see that the 'only if' part holds.

We show the 'if' part. Let $n, r \in \mathbb{N}$ and $f \in \mathrm{Map}_n^I$ be as in Proposition 2.5. By Lemmas 1.12, 3.8(2) and 5.5, we have $1^\chi s_{f,n} = 1^\chi w_0$. By Lemmas 3.2 and 5.5 and (4.24), (3.7), (5.2), we see that

$$\chi_{f,k-1}(\alpha_k, \alpha_k) \in \mathbb{K}^\times_\infty \qquad (k \in J_{1,r}), \tag{7.1}$$

$$\chi_{f,t-1}(\alpha_t, \alpha_t) = -1 \qquad (t \in J_{r+1,n}). \tag{7.2}$$

By (4.15), (7.1) and Lemma 6.4, we also see that $\Lambda \in \bigcap_{i\in I\setminus\{m\}} \mathbb{S}_i(\chi)$ implies $H(\chi, \Lambda, f) \geq r$. By (4.15), (7.2) and Lemma 6.4, $H(\chi, \Lambda, f) \geq r$ must be $H(\chi, \Lambda, f) = n$. Thus the 'if' part follows from Lemma 6.6. This completes the proof of (1).

(2) can be proved in the same way by using Propositions 3.13(2) and Lemma 5.5. □

§7.2. Some technical maps

In Section 7, for $\lambda \in (\mathbb{K}^\times)^N$ and $i \in I$, let λ_i be the i-th component of λ, that is, $\lambda = (\lambda_1, \ldots, \lambda_N)$.

In Subsection 7.2, assume $N \geq 2$ and let $\hat{q} \in \mathbb{K}^\times_\infty$ and $m \in J_{1,N-1}$. Let $\mathcal{Q}_{\hat{q}} := \{\hat{q}^x \mid x \in \mathbb{Z}_{\geq 0}\}$. Let

$$\mathcal{K}_{\hat{q}}^{(m)} := \{\lambda \in (\mathbb{K}^\times)^N \mid \lambda_i \in \mathcal{Q}_{\hat{q}} \ (i \in J_{N-m+1,N})\}. \tag{7.3}$$

Define maps $\nabla_k^{(\hat{q},m)} : \mathcal{K}_{\hat{q}}^{(m)} \to \mathbb{K}^\times$ $(k \in J_{N-m,N})$ and $\tilde{\nabla}_N^{(\hat{q},m)} : \mathcal{K}_{\hat{q}}^{(m)} \to \mathbb{K}^\times$ by

$$\begin{aligned} &\nabla_{N-m}^{(\hat{q},m)}(\lambda) := \lambda_{N-m}, \\ &\nabla_k^{(\hat{q},m)}(\lambda) := \lambda_k \nabla_{k-1}^{(\hat{q},m)}(\lambda) \hat{q}^{2(1-\delta(1,\nabla_{k-1}^{(\hat{q},m)}(\lambda)))} \qquad (k \in J_{N-m+1,N}), \\ &\tilde{\nabla}_N^{(\hat{q},m)}(\lambda) := \frac{\lambda_N}{\lambda_{N-1}} \nabla_{N-1}^{(\hat{q},m)}(\lambda)^2 \hat{q}^{4(1-\delta(1,\nabla_{N-1}^{(\hat{q},m)}(\lambda)))}. \end{aligned} \tag{7.4}$$

The following lemma is used in the proofs of Theorems 7.6 and 7.7 below.

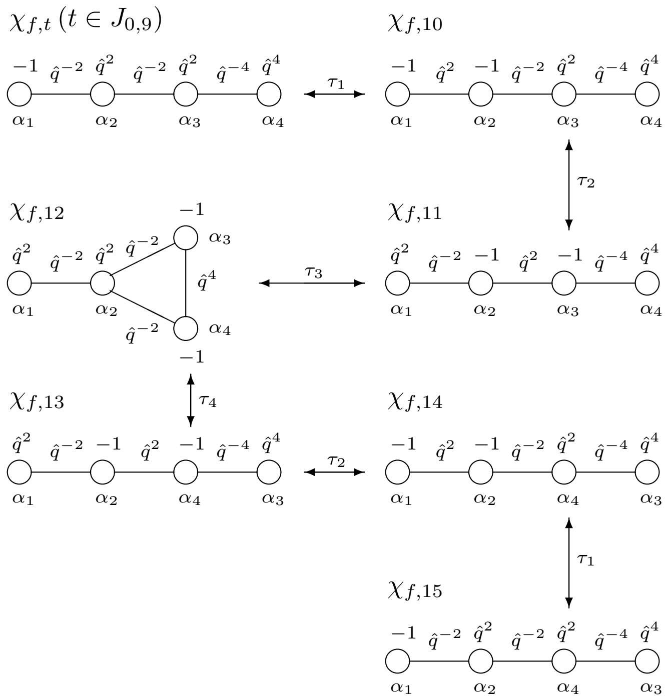

Figure 5. Dynkin diagrams of $\chi = \chi_{f,0} \equiv \chi' \in \dot{\mathcal{X}}_N^{\mathrm{Super}}(\mathrm{C}(N))$ with $N = 4$, and $\chi_{f,u}$ with $f = \acute{f}_N$, where $1^\chi w_0 = s_2^{\chi_{f,1}} s_3^{\chi_{f,2}} s_4^{\chi_{f,3}} s_2^{\chi_{f,4}} s_3^{\chi_{f,5}} s_4^{\chi_{f,6}} s_2^{\chi_{f,7}} s_3^{\chi_{f,8}} s_4^{\chi_{f,9}} s_1^{\chi_{f,10}}$ $ds_2^{\chi_{f,11}} s_3^{\chi_{f,12}} s_4^{\chi_{f,13}} s_2^{\chi_{f,14}} s_1^{\chi_{f,15}}$.

Lemma 7.5. (1) *Let $z \in \mathbb{K}^\times \setminus \mathcal{Q}_{\hat{q}}$. Then*

$$(\nabla_N^{(\hat{q},m)})^{-1}(\{z\}) = \Big\{\lambda \in \mathcal{K}_{\hat{q}}^{(m)} \;\Big|\; \prod_{i=N-m}^{N} \lambda_i = \hat{q}^{-2m} z\Big\}. \tag{7.5}$$

(2) *Let $z \in \mathbb{K}^\times \setminus \mathcal{Q}_{\hat{q}}$. Then*

$$\begin{aligned}(\tilde{\nabla}_N^{(\hat{q},m)})^{-1}(\{z\}) &= \Big\{\lambda \in \mathcal{K}_{\hat{q}}^{(m)} \;\Big|\; \prod_{j=N-m}^{N-1} \lambda_j \neq \hat{q}^{-2(m-1)},\ \lambda_{N-1}\lambda_N \prod_{i=N-m}^{N-2} \lambda_i^2 = \hat{q}^{-4m} z\Big\} \\ &\cup \Big\{\lambda \in \mathcal{K}_{\hat{q}}^{(m)} \;\Big|\; \prod_{j=N-m}^{N-1} \lambda_j = \hat{q}^{-2(m-1)},\ \lambda_N = \lambda_{N-1} z\Big\}.\end{aligned} \tag{7.6}$$

(3) *We have*

$$(7.7)\qquad (\nabla_N^{(\hat{q},m)})^{-1}(\{1\}) = \bigcup_{t=0}^{m} \Big\{ \lambda \in \mathcal{K}_{\hat{q}}^{(m)} \,\Big|\, \prod_{i=N-m}^{N-m+t} \lambda_i = \hat{q}^{-2t},\ \lambda_j = 1\,(j \in J_{N-m+t+1,N}) \Big\}.$$

(4) *We have*

$$(7.8)\qquad \begin{aligned}(\tilde{\nabla}_N^{(\hat{q},m)})^{-1}(\{1\}) &= \{\lambda \in \mathcal{K}_{\hat{q}}^{(m)} \mid \lambda_i = 1 \ (i \in J_{N-m,N})\} \\ &\cup \bigcup_{t=1}^{m-2} \Big\{ \lambda \in \mathcal{K}_{\hat{q}}^{(m)} \,\Big|\, \prod_{j=N-m}^{N-m+t} \lambda_j = \hat{q}^{-2t},\ \lambda_i = 1 \ (i \in J_{N-m+t+1,N}) \Big\} \\ &\cup \Big\{ \lambda \in \mathcal{K}_{\hat{q}}^{(m)} \,\Big|\, \prod_{j=N-m}^{N-1} \lambda_j = \hat{q}^{-2(m-1)},\ \lambda_{N-1} = \lambda_N \Big\} \\ &\cup \Big\{ \lambda \in \mathcal{K}_{\hat{q}}^{(m)} \,\Big|\, \prod_{j=N-m}^{N-1} \lambda_j \neq \hat{q}^{-2(m-1)},\ \lambda_{N-1}\lambda_N \prod_{i=N-m}^{N-2} \lambda_i^2 = \hat{q}^{-4m} \Big\}.\end{aligned}$$

Proof. In this proof, we fix $\lambda \in \mathcal{K}_{\hat{q}}^{(m)}$, and use the following notation. For $k \in J_{N-m-1,N+1}$, let

$$c_k(\lambda) := \begin{cases} 0 & \text{if } k = N-m-1, \\ |\{r \in J_{N-m,k} \mid \nabla_r^{(\hat{q},m)}(\lambda) \neq 1\}| & \text{if } k \in J_{N-m,N}, \\ c_N(\lambda) & \text{if } k = N+1. \end{cases}$$

Let

$$g_k(\lambda) := \prod_{j=N-m}^{k} \lambda_j \quad (k \in J_{N-m,N}) \quad \text{and} \quad \tilde{g}_N(\lambda) := \lambda_{N-1}\lambda_N \prod_{j=N-m}^{N-2} \lambda_j^2.$$

Let $t(\lambda) := \delta(g_{N-1}(\lambda), \hat{q}^{-2(m-1)}) \in J_{0,1}$. Let

$$r(\lambda) := \begin{cases} \min\{x \in J_{N-m,N} \mid \nabla_x^{(\hat{q},m)}(\lambda) = 1\} \\ \qquad\qquad \text{if } \nabla_y^{(\hat{q},m)}(\lambda) = 1 \text{ for some } y \in J_{N-m,N}, \\ N+1 \quad \text{otherwise.} \end{cases}$$

Note that

$$(7.9)\qquad \begin{aligned} &r(\lambda) - (N-m) = c_{r(\lambda)-1}(\lambda) = c_{r(\lambda)}(\lambda), \\ &c_{r(\lambda)}(\lambda) \leq c_y(\lambda) \quad \text{for } y \in J_{r(\lambda)+1,N+1}, \\ &c_{r(\lambda)}(\lambda) = c_y(\lambda) \quad \text{if and only if} \quad \lambda_z = 1 \ (z \in J_{r(\lambda)+1,y}). \end{aligned}$$

We have

$$\text{for } i \in J_{N-m+1,N}, \quad \lambda_i = \hat{q}^{l_i} \quad \text{for some } l_i \in \mathbb{Z}_{\geq 0}, \tag{7.10}$$

$$\text{for } k \in J_{N-m+1,N}, \quad \nabla_k^{(\hat{q},m)}(\lambda) = \hat{q}^{2c_{k-1}(\lambda)} g_k(\lambda). \tag{7.11}$$

By (7.10), we can easily see that

$$\text{for } k \in J_{N-m,N}, \quad g_k(\lambda) = \hat{q}^{-2(k-(N-m))} \quad \text{if and only if} \quad r(\lambda) = k. \tag{7.12}$$

From (7.10)–(7.12), we easily deduce claims (1) and (3).

Since $c_{N-1}(\lambda) = c_{N-2}(\lambda) + \delta(1, \nabla_{N-1}^{(\hat{q},m)}(\lambda))$, by (7.11), we have

$$\tilde{\nabla}_N^{(\hat{q},m)}(\lambda) = \hat{q}^{4c_{N-1}(\lambda)} \tilde{g}_N(\lambda). \tag{7.13}$$

By (7.9) and (7.12), if $r(\lambda) \in J_{N-m,N-1}$, then

$$\tilde{g}_N(\lambda) = \begin{cases} \hat{q}^{-4c_{r(\lambda)}(\lambda)} \lambda_{N-1}\lambda_N \prod_{i=r(\lambda)+1}^{N-2} \lambda_i^2 & \text{if } r(\lambda) \in J_{N-m,N-2}, \\ \hat{q}^{-4c_{r(\lambda)}(\lambda)} \dfrac{\lambda_N}{\lambda_{N-1}} & \text{if } r(\lambda) = N-1. \end{cases} \tag{7.14}$$

We now prove claim (2). Let Y_1 and Y_2 be the LHS and RHS of (7.6).

Let $\lambda \in Y_2$. Then

$$z = \hat{q}^{4(m-t(\lambda))} \tilde{g}_N(\lambda). \tag{7.15}$$

Assume $r(\lambda) \in J_{N-m,N-2}$. By (7.9), (7.14) and (7.15),

$$z = \hat{q}^{4(N-t(\lambda)-r(\lambda))} \lambda_{N-1}\lambda_N \Big(\prod_{i=r(\lambda)+1}^{N-2} \lambda_i^2 \Big). \tag{7.16}$$

Since $t(\lambda) \in J_{0,1}$, by (7.10) and (7.16), $z \in \mathcal{Q}_{\hat{q}} \setminus \{1\}$, a contradiction. Hence $r(\lambda) \in J_{N-1,N+1}$. Hence $c_{N-2}(\lambda) = m-1$. By (7.12), $c_{N-1}(\lambda) = m - t(\lambda)$. By (7.13) and (7.15), we have $\lambda \in Y_1$. Hence $Y_2 \subseteq Y_1$. Let $\lambda \in Y_1$, i.e., $\tilde{\nabla}_N^{(\hat{q},m)}(\lambda) = z$. Assume $r(\lambda) \in J_{N,N+1}$. By (7.13), $\tilde{g}_N(\lambda) = \hat{q}^{-4m} z$. Since $r(\lambda) \neq N-1$, by (7.12), $\hat{q}^{2(m-1)} g_{N-1}(\lambda) \neq 1$. Hence $\lambda \in Y_2$. Assume $r(\lambda) \in J_{N-m,N-2}$. By (7.13) and (7.14), we have

$$z = \hat{q}^{4(c_{N-1}(\lambda)-c_{r(\lambda)}(\lambda))} \lambda_{N-1}\lambda_N \prod_{i=r(\lambda)+1}^{N-2} \lambda_i^2.$$

Hence by (7.9) and (7.10), $z \in \mathcal{Q}_{\hat{q}}$, a contradiction. Assume $r(\lambda) = N-1$. By (7.12), $g_{N-1}(\lambda) = \hat{q}^{-2(m-1)}$. By (7.9), $c_{N-1}(\lambda) = m-1$. By (7.14) and (7.13), $z = \lambda_N/\lambda_{N-1}$. Hence $\lambda \in Y_2$, so $Y_1 \subseteq Y_2$, and finally $Y_1 = Y_2$, proving (2).

Finally, we prove claim (4). Let Y_3 and Y_4 be the LHS and RHS of (7.8) respectively. For $t \in J_{N-m,N}$, let

$$Y_{4,t} := \begin{cases} \{\lambda \in \mathcal{K}_{\hat{q}}^{(m)} \mid r(\lambda) = t,\ \lambda_i = 1\ (i \in J_{t+1,N})\} & \text{if } t \in J_{N-m,N-2}, \\ \{\lambda \in \mathcal{K}_{\hat{q}}^{(m)} \mid r(\lambda) = N-1,\ \lambda_{N-1} = \lambda_N\} & \text{if } t = N-1, \\ \{\lambda \in \mathcal{K}_{\hat{q}}^{(m)} \mid r(\lambda) \in J_{N,N+1},\ \tilde{g}_N(\lambda) = \hat{q}^{-4m}\} & \text{if } t = N. \end{cases}$$

By (7.10), (7.12) and (7.14), we have $Y_4 = \bigcup_{t=N-m}^{N} Y_{4,t}$. Then, by (7.10), (7.13) and (7.14), we can easily see that $Y_4 \subseteq Y_3$. Let $\lambda \in Y_3$. By (7.13),

$$\tilde{g}_N(\lambda) = \hat{q}^{-4c_{N-1}(\lambda)}. \tag{7.17}$$

Assume $r(\lambda) \in J_{N,N+1}$, that is, $c_{N-1}(\lambda) = m$. By (7.17), $\lambda \in Y_{4,N}$. Assume $r(\lambda) = N-1$. By (7.14) and (7.17), $\lambda_N/\lambda_{N-1} = 1$. Hence $\lambda \in Y_{4,N-1}$. Assume $r(\lambda) \in J_{N-m,N-2}$. By (7.9), $c_{r(\lambda)} \leq c_{N-1}$. By (7.10), (7.14) and (7.17), we have $\lambda \in Y_{4,r(\lambda)}$. Thus we have $Y_3 \subseteq Y_4$. Consequently, $Y_3 = Y_4$, as desired. □

§7.3. Irreducible modules for super-BD cases

Theorem 7.6. *Assume $N \geq 1$. Let $m \in J_{0,N-1}$. Let $\chi \in \mathcal{X}_N$ be such that $\chi \equiv \chi'$ for some $\chi' \in \dot{\mathcal{X}}_N^{\mathrm{Super}}(\mathrm{B}(m, N-m))$. If $m = 0$, let $\mathbb{S}(\mathrm{B}(m, N-m)) := \bigcap_{i\in I} \mathbb{S}_i(\chi)$. If $m \in J_{1,N-1}$, then letting $\hat{q} := \chi(\alpha_N, \alpha_N) \in \mathbb{K}_\infty^\times$ and letting $g(\Lambda') := \prod_{i=N-m}^{N} \Lambda'(K_{\alpha_i} L_{-\alpha_i})$, for $\Lambda' \in \mathrm{Ch}(U^0(\chi))$, let $\mathbb{S}(\mathrm{B}(m, N-m))$ be the subset of $\mathrm{Ch}(U^0(\chi))$ formed by Λ satisfying*

(irrB-m-1) $\Lambda \in \bigcap_{i\in I\setminus\{N-m\}} \mathbb{S}_i(\chi)$.

(irrB-m-2) $g(\Lambda) \in \{\hat{q}^{-2x} \mid x \in J_{0,m-1}\} \cup \{(-\hat{q})^{-(y+2m)} \mid y \in \mathbb{Z}_{\geq 0}\}$.

(irrB-m-3) *If $g(\Lambda) = \hat{q}^{-2x}$ for some $x \in J_{0,m-1}$, then $\Lambda(K_{\alpha_i} L_{-\alpha_i}) = 1$ for all $i \in J_{N-m+x+1,N}$.*

Then $\{\mathcal{L}_\chi(\Lambda) \mid \Lambda \in \mathbb{S}(\mathrm{B}(m, N-m))\}$ is a complete set of pairwise non-isomorphic finite-dimensional irreducible $U(\chi)$-modules (see also (4.17)).

Proof (see also Figure 6). If $m = 0$, the claim follows from Theorem 7.2. Assume $m \in J_{1,N-1}$.

Let $\Lambda \in \mathrm{Ch}(U^0(\chi))$. By Theorem 5.10, we see that $\chi, \chi' \in \mathcal{X}_N^{\mathrm{fin}}$. Hence by Lemma 6.7, we may assume $\chi = \chi'$. Let $n := N^2$. Let $f \in \mathrm{Map}_n^I$ be $\dot{f}_{m|N-m}$ (see (3.9)). By Lemmas 1.12, 3.10(2), 4.22 and 5.5, we have $n = |R^+(\chi)|$ and $1^\chi s_{f,n} = 1^\chi w_0$. By (3.8), (3.12) and (5.3), we find that

$$\chi_{f,t-1}(\alpha_{f(t)}, \alpha_{f(t)}) \in \mathbb{K}_\infty^\times \qquad (t \in J_{1,m^2+N-m-1}), \tag{7.18}$$

$$\chi_{f,t-1}(\alpha_{f(t)}, \alpha_{f(t)}) = -1 \qquad (t \in J_{m^2+N-m,m^2+N+m} \setminus \{m^2+N\}), \tag{7.19}$$

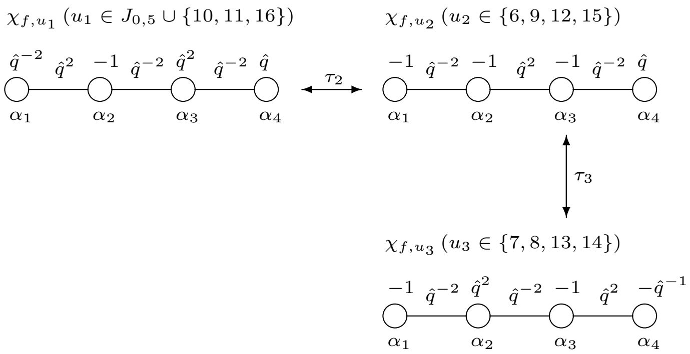

Figure 6. Dynkin diagrams of $\chi = \chi_{f,0} \equiv \chi' \in \dot{\mathcal{X}}_N^{\mathrm{Super}}(\mathrm{B}(m, N-m))$ with $N = 4$ and $m = 2$, and $\chi_{f,u}$ with $f = \dot{f}_{m|N-m}$, where $1^\chi w_0 = s_3^{\chi_{f,1}} s_4^{\chi_{f,2}} s_3^{\chi_{f,3}} s_4^{\chi_{f,4}} s_1^{\chi_{f,5}} s_2^{\chi_{f,6}} s_3^{\chi_{f,7}} s_4^{\chi_{f,8}} s_3^{\chi_{f,9}} s_2^{\chi_{f,10}} s_1^{\chi_{f,11}} s_2^{\chi_{f,12}} s_3^{\chi_{f,13}} s_4^{\chi_{f,14}} s_3^{\chi_{f,15}} s_2^{\chi_{f,16}}$.

(7.20) $$\chi_{f,t-1}(\alpha_{f(t)}, \alpha_{f(t)}) = -\hat{q}^{-1} \quad \text{if } t = m^2 + N.$$

By (3.9),

(7.21) $$f(J_{1,m^2+N-m-1}) = I \setminus \{N-m\}.$$

By (4.15), (4.20), (7.18), (7.21) and Lemma 6.4, we see that

(7.22) $$H(\chi, \Lambda, f) \geq m^2 + N - m - 1 \text{ if and only if (irrB-}m\text{-1) holds.}$$

By (4.15) and (7.19), $H(\chi, \Lambda, f) \geq m^2 + N - m - 1$ can be replaced by $H(\chi, \Lambda, f) \geq m^2 + N - 1$. Hence

(7.23) $$H(\chi, \Lambda, f) \geq m^2 + N - 1 \text{ if and only if (irrB-}m\text{-1) holds.}$$

Assume $H(\chi, \Lambda, f) \geq m^2 + N - 1$. (By (3.10) and (5.3), $\chi_{f,m^2+N-m-1} = \chi$.) By (7.18) and Lemma 6.4, we have

(7.24) $$(\chi_{f,m^2+N-m-1}, \Lambda_{\chi,f,m^2+N-m-1}) \equiv (\chi, \Lambda).$$

Let $\lambda := (\Lambda(K_{\alpha_i} L_{-\alpha_i}) | i \in I) \in (\mathbb{K}^\times)^N$. By (7.23), $\lambda \in \mathcal{K}_{\hat{q}}^{(m)}$ (see also (7.3)). Let $t \in J_{m^2+N-m, m^2+N-1}$. Let $\bar{h}_t := h_{\chi_{f,t-1}, \Lambda_{\chi,f,t-1}, f(t)}$. By (3.9),

(7.25) $$f(t) = t - m^2 \quad \text{and} \quad f(t+1) = f(t) + 1.$$

By (7.25) and Lemmas 3.9 and 5.5,

(7.26) $$s_{f(t)}^{\tau_{f(t)}\chi_{f,t-1}}(\alpha_{f(t)+1}) = \alpha_{f(t)} + \alpha_{f(t)+1}.$$

Using induction, we have

$$\begin{aligned}
(7.27)\qquad & \Lambda_{\chi,f,t}(K_{\alpha_{f(t+1)}}L_{-\alpha_{f(t+1)}}) \\
&= \tau_{f(t)}^{\chi_{f,t-1}}\Lambda_{\chi,f,t-1}(K_{\alpha_{f(t)+1}}L_{-\alpha_{f(t)+1}}) \quad \text{(by (7.25) and Definition 6.5)} \\
&= \Lambda_{\chi,f,t-1}(K_{\alpha_{f(t)}+\alpha_{f(t)+1}}L_{-(\alpha_{f(t)}+\alpha_{f(t)+1})})\frac{\chi_{f,t-1}(\alpha_{f(t)},-(\alpha_{f(t)}+\alpha_{f(t)+1}))^{\bar{h}_t}}{\chi_{f,t-1}(\alpha_{f(t)}+\alpha_{f(t)+1},\alpha_{f(t)})^{\bar{h}_t}} \\
&\quad \text{(by (6.2) and (7.26))} \\
&= \Lambda_{\chi,f,t-1}(K_{\alpha_{f(t)}+\alpha_{f(t)+1}}L_{-(\alpha_{f(t)}+\alpha_{f(t)+1})})\hat{q}^{2\bar{h}_t} \\
&\quad \text{(by (7.19), since } \chi_{f,t-1}(\alpha_{f(t)},\alpha_{f(t)+1})\chi_{f,t-1}(\alpha_{f(t)+1},\alpha_{f(t)}) = \hat{q}^{-2} \\
&\quad \text{by (3.10) and (5.3))} \\
&= \Lambda_{\chi,f,t-1}(K_{\alpha_{f(t)}+\alpha_{f(t)+1}}L_{-(\alpha_{f(t)}+\alpha_{f(t)+1})})\hat{q}^{2(1-\delta(\Lambda_{\chi,f,t-1}(K_{\alpha_{f(t)}}L_{-\alpha_{f(t)}}),1))} \\
&\quad \text{(by (4.15) and (7.19))} \\
&= \nabla_{f(t)+1}^{(\hat{q},m)}(\lambda) \quad \text{(by (7.4) and (7.28) below).}
\end{aligned}$$

As above, we have

$$(7.28)\qquad \Lambda_{\chi,f,t-1}(K_{\alpha_i}L_{-\alpha_i}) = \Lambda(K_{\alpha_i}L_{-\alpha_i}) \quad (i \in J_{f(t)+1,N}),$$

where we use (7.24) if $t = m^2+N-m$.

Since $\hat{q} \in \mathbb{K}_\infty^\times$, by (4.15), (7.20), (7.25) and (7.27) for $t = m^2+N-1$, we see that

(7.29) $H(\chi,\Lambda,f) \geq m^2+N$ if and only if (irrB-m-1) holds and there exists $x \in \mathbb{Z}_{\geq 0}$ with $\nabla_N^{(\hat{q},m)}(\lambda) = (-\hat{q}^{-1})^x$.

By (7.5), (7.7) and (7.29), we can see that

(7.30) $H(\chi,\Lambda,f) \geq m^2+N$ if and only if (irrB-m-1)–(irrB-m-3) hold.

Assume $H(\chi,\Lambda,f) \geq m^2+N$. By (4.15) and (7.19), we see that

$$(7.31)\qquad H(\chi,\Lambda,f) \geq m^2+N+m.$$

(By (3.10) and (5.3), $\chi_{f,m^2+N-1} = \chi_{f,m^2+N}$.) By (7.19) and Lemma 6.4,

$$(7.32)\qquad (\chi_{f,m^2+N-1},\Lambda_{\chi,f,m^2+N-1}) \equiv (\chi_{f,m^2+N},\Lambda_{\chi,f,m^2+N}).$$

(By (3.10) and (5.3), $\chi = \chi_{f,m^2+N+m}$.) By (3.9),

$$(7.33)\qquad f(t) = f(2(m^2+N)-t) \quad (t \in J_{m^2+N+1,m^2+N+m}).$$

For $t \in J_{m^2+N+1,m^2+N+m}$, we inductively see

$$
\begin{aligned}
&(\chi_{f,t}, \Lambda_{\chi,f,t}) \\
&\quad = (\tau_{f(t)}\chi_{f,t-1}, \tau_{f(t)}^{\chi_{f,t-1}}\Lambda_{\chi,f,t-1}) \\
&\qquad \text{(by Notation 1.1(1) and Definition 6.5)} \\
&\quad \equiv (\tau_{f(t)}\chi_{f,2(m^2+N)-t}, \tau_{f(t)}^{\chi_{f,2(m^2+N)-t}}\Lambda_{\chi,f,2(m^2+N)-t}) \\
&\qquad \text{(by induction and (6.4); use (7.32) if } t = m^2+N+1) \\
&\quad \equiv (\tau_{f(t)}\tau_{f(t)}\chi_{f,2(m^2+N)-t-1}, \tau_{f(t)}^{\chi_{f,2(m^2+N)-t}}\tau_{f(t)}^{\chi_{f,2(m^2+N)-t-1}}\Lambda_{\chi,f,2(m^2+N)-t-1}) \\
&\qquad \text{(by Notation 1.1(1), Definition 6.5 and (7.33))} \\
&\quad = (\chi_{f,2(m^2+N)-t-1}, \Lambda_{\chi,f,2(m^2+N)-t-1}) \quad \text{(by (4.22) and (6.3)).}
\end{aligned}
$$

In particular,

$$
(\chi_{f,m^2+N+m}, \Lambda_{\chi,f,m^2+N+m}) \equiv (\chi_{f,m^2+N-m-1}, \Lambda_{\chi,f,m^2+N-m-1}). \tag{7.34}
$$

By (7.30) and (7.31), we see that

$$
H(\chi,\Lambda,f) \geq m^2+N+m \text{ if and only if (irrB-}m\text{-1)–(irrB-}m\text{-3) hold.} \tag{7.35}
$$

By (3.9), $f(t) = f(t-(N+m))$ ($t \in J_{m^2+N+m+1,n}$). Hence by (7.34) and (7.35),

$$
H(\chi,\Lambda,f) = n \text{ if and only if (irrB-}m\text{-1)–(irrB-}m\text{-3) hold.} \tag{7.36}
$$

By (7.36) and Lemmas 4.23 and 6.6, Theorem 7.6 is proved. □

Theorem 7.7. *Assume $N \geq 3$. Let $m \in J_{2,N-1}$. Let $\chi \in \mathcal{X}_N$ be such that $\chi \equiv \chi'$ for some $\chi' \in \dot{\mathcal{X}}_N^{\mathrm{Super}}(\mathrm{D}(m,N-m))$. Let $\hat{q} \in \mathbb{K}_\infty^\times$ be such that $\hat{q}^2 = \chi(\alpha_N,\alpha_N)$. For $\Lambda' \in \mathrm{Ch}(U^0(\chi))$, let*

$$
\tilde{g}(\Lambda') := \Big(\prod_{i=N-m}^{N-2}\Lambda'(K_{\alpha_i}L_{-\alpha_i})\Big)^2\Lambda'(K_{\alpha_{N-1}}L_{-\alpha_{N-1}})\Lambda'(K_{\alpha_N}L_{-\alpha_N}).
$$

Let $\mathbb{S}(\mathrm{D}(m,N-m))$ be the subset of $\mathrm{Ch}(U^0(\chi))$ formed by all Λ satisfying:

(irrD-m-1) $\Lambda \in \bigcap_{i\in I\setminus\{N-m\}}\mathbb{S}_i(\chi)$.

(irrD-m-2) $\tilde{g}(\Lambda) = \hat{q}^{-4x}$ *for some* $x \in \mathbb{Z}_{\geq 0}$.

(irrD-m-3) *If* $\tilde{g}(\Lambda) = \hat{q}^{-4y}$ *for some* $y \in J_{0,m-2}$, *then* $\prod_{i=N-m}^{N-m+y}\Lambda(K_{\alpha_i}L_{-\alpha_i}) = \hat{q}^{-2y}$ *and* $\Lambda(K_{\alpha_j}L_{-\alpha_j}) = 1$ *for all* $j \in J_{N-m+y+1,N}$.

(irrD-m-4) *If* $\tilde{g}(\Lambda) = \hat{q}^{-4(m-1)}$, *then* $\prod_{i=N-m}^{N-1}\Lambda(K_{\alpha_i}L_{-\alpha_i}) = \hat{q}^{-2(m-1)}$.

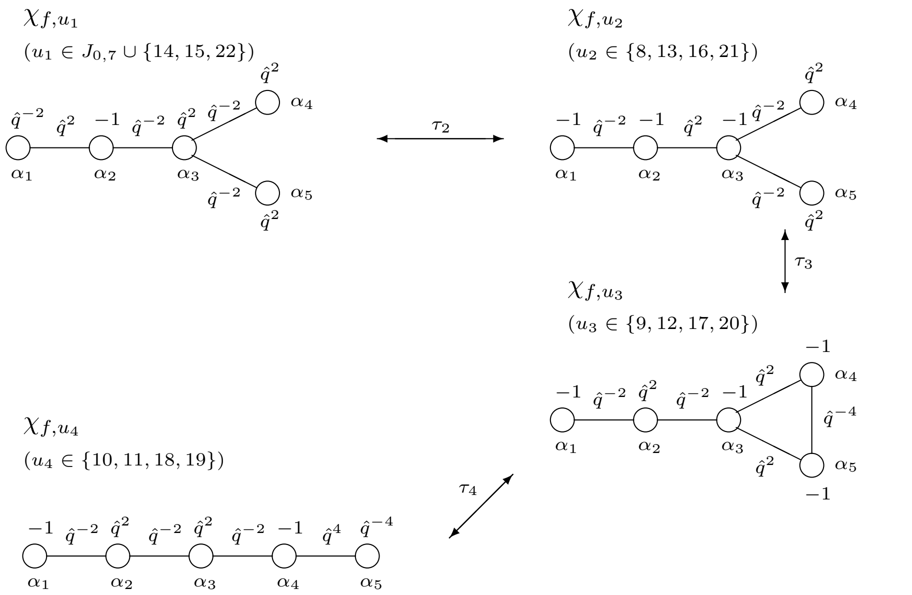

Figure 7. Dynkin diagrams of $\chi = \chi_{f,0} \equiv \chi' \in \dot{\mathcal{X}}_N^{\mathrm{Super}}(\mathrm{D}(m, N-m))$ with $N = 5$ and $m = 3$, and $\chi_{f,u}$ with $f = \ddot{f}_{m|N-m}$, where $1^\chi w_0 = s_3^{\chi_{f,1}} s_4^{\chi_{f,2}} s_5^{\chi_{f,3}} s_3^{\chi_{f,4}} s_4^{\chi_{f,5}} s_5^{\chi_{f,6}} s_1^{\chi_{f,7}} s_2^{\chi_{f,8}} s_3^{\chi_{f,9}} s_4^{\chi_{f,10}} s_5^{\chi_{f,11}} s_4^{\chi_{f,12}} s_3^{\chi_{f,13}} s_2^{\chi_{f,14}} s_1^{\chi_{f,15}} s_2^{\chi_{f,16}} s_3^{\chi_{f,17}} s_4^{\chi_{f,18}} s_5^{\chi_{f,19}} s_4^{\chi_{f,20}} s_3^{\chi_{f,21}} s_2^{\chi_{f,22}}$.

Then $\{\mathcal{L}_\chi(\Lambda) \mid \Lambda \in \mathbb{S}(\mathrm{D}(m, N-m))\}$ *is a complete set of pairwise non-isomorphic finite-dimensional irreducible* $U(\chi)$*-modules* (*see also* (4.17)).

Proof (see also Figure 7). Let $\Lambda \in \mathrm{Ch}(U^0(\chi))$. Let $n := N^2 - m$. Let $f \in \mathrm{Map}_n^I$ be $\ddot{f}_{m|N-m}$ defined by (3.22). Using this f, the theorem can be proved in a way similar to that for Theorem 7.6. Here we only mention the following facts. By Lemmas 1.9(1) and 5.5 and (3.24), $\ell_\chi(1^\chi w_0) = |R^+(\chi)| = n$ and $1^\chi s_{f,n} = 1^\chi w_0$. Let $r := m(m-1) + N$. By (3.26) and (5.3), we have $\chi_{f,t_1-1}(\alpha_{f(t_1)}, \alpha_{f(t_1)}) \in \mathbb{K}_\infty^\times$ $(t_1 \in J_{1,r-m-1})$ and $\chi_{f,t_2-1}(\alpha_{f(t_2)}, \alpha_{f(t_2)}) = -1$ $(t_2 \in J_{r-m,r-1})$. Then, similarly to (7.23), we can see that $H(\chi, \Lambda, f) \geq r-1$ if and only if (irrD-m-1) holds. Note that $f(r) = N$. By (3.26) and (5.3), $\chi_{f,r-1}(\alpha_N, \alpha_N) = \hat{q}^{-4} \in \mathbb{K}_\infty^\times$. Similarly to (7.27), letting $\lambda := (\Lambda(K_{\alpha_i} L_{-\alpha_i}) \mid i \in I)$, we have $\Lambda_{\chi,f,r-1}(K_{\alpha_N} L_{-\alpha_N}) = \tilde{\nabla}_N^{(\hat{q},m)}(\lambda)$. Similarly to (7.27), by (7.6) and (7.8), we can see that $H(\chi, \Lambda, f) \geq r-1$ if and only if (irrD-m-1)–(irrD-m-4) hold. Similarly to (7.36), we can see that $H(\chi, \Lambda, f) = n$ if and only if (irrD-m-1)–(irrD-m-4) hold. Hence, by Lemmas 4.23 and 6.6, the conclusion follows. $\square$

§7.4. Irreducible modules for super-FG and extra cases

Theorem 7.8. *Let $\chi \in \mathcal{X}_N$. Assume $N \in J_{2,4}$. Assume that one of the following conditions is satisfied:*

(FGE-1) *$N = 4$ and $\chi \equiv \chi'$ for some $\chi' \in \dot{\mathcal{X}}_N^{\mathrm{Super}}(\mathrm{F}(4))$.*

(FGE-2) *$N = 3$ and $\chi \equiv \chi'$ for some $\chi' \in \dot{\mathcal{X}}_N^{\mathrm{Super}}(\mathrm{G}(3))$.*

(FGE-3) *$N = 3$ and $\chi \equiv \chi'$ for some $\chi' \in \dot{\mathcal{X}}_N^{\mathrm{Extra}}$.*

(FGE-4) *$N = 2$ and $\chi \equiv \chi'$ for some $\chi' \in \dot{\mathcal{X}}_N^{\mathrm{Extra}}$.*

(FGE-5) *$N = 4$ and $\chi \equiv \chi'$ for some $\chi' \in \dot{\mathcal{X}}_N^{\mathrm{Extra}}$.*

For $k \in J_{1,5}$, if χ satisfies (FGE-k)*, let $\mathbb{S}$ be the subset of $\mathrm{Ch}(U^0(\chi))$ defined in* (cL-k) *below. Then $\{\mathcal{L}_\chi(\Lambda) \mid \Lambda \in \mathbb{S}\}$ is a complete set of pairwise non-isomorphic finite-dimensional irreducible $U(\chi)$-modules (see also* (4.17)*).*

In the following, $\lambda_i := \Lambda(K_{\alpha_i} L_{-\alpha_i})$ $(i \in I)$.

(cL-1) *Let $\hat{q} \in \mathbb{K}_\infty^\times$ be such that $\hat{q}^2 = \chi(\alpha_2, \alpha_2)$. Let $\mathbb{S}$ be the subset of $\mathrm{Ch}(U^0(\chi))$ formed by all Λ satisfying one of the following conditions:*

(irrF-1) *$\Lambda \in \bigcap_{i=2}^4 \mathbb{S}_i(\chi)$ and $\lambda_1^2\lambda_2^3\lambda_3^2\lambda_4 = \hat{q}^{-6(x+4)}$ for some $x \in \mathbb{Z}_{\geq 0}$.*

(irrF-2) *$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1$.*

(irrF-3) *$\Lambda \in \mathbb{S}_3(\chi)$, $\lambda_2 = \lambda_4 = 1$ and $\lambda_1\lambda_3 = \hat{q}^{-6}$.*

(irrF-4) *$\Lambda \in \bigcap_{i=3}^4 \mathbb{S}_i(\chi)$, $\lambda_1\lambda_3\lambda_4^2 = \hat{q}^{-12}$ and $\lambda_2 = \hat{q}^2\lambda_4$.*

(cL-2) *Let $\hat{q} \in \mathbb{K}_\infty^\times$ be such that $\hat{q}^2 = \chi(\alpha_2, \alpha_2)$. Let $\mathbb{S}$ be the subset of $\mathrm{Ch}(U^0(\chi))$ formed by all Λ satisfying one of the following conditions:*

(irrG-1) *$\Lambda \in \bigcap_{i=2}^3 \mathbb{S}_i(\chi)$ and $\lambda_1\lambda_2^2\lambda_3 = (-\hat{q}^{-2})^{x+6}$ for some $x \in \mathbb{Z}_{\geq 0}$.*

(irrG-2) *$\lambda_1 = \lambda_2 = \lambda_3 = 1$.*

(irrG-3) *$\Lambda \in \mathbb{S}_3(\chi)$, $\lambda_1\lambda_3 = \hat{q}^{-8}$ and $\lambda_2 = 1$.*

(cL-3) *Let $\hat{q} := \chi(\alpha_1, \alpha_1) \in \mathbb{K}_\infty^\times$ and $\hat{r} := \chi(\alpha_3, \alpha_3) \in \mathbb{K}^\times \setminus \{1, \hat{q}^{-1}\}$. Let $\mathbb{S}' := \mathbb{S}_1(\chi) \cap \mathbb{S}_3(\chi)$ (resp. $\mathbb{S}' := \mathbb{S}_1(\chi)$) if $\hat{r} \in \mathbb{K}_\infty^\times$ (resp. $\hat{r} \notin \mathbb{K}_\infty^\times$). If $\hat{q}\hat{r} \notin \mathbb{K}_\infty^\times$, let $\mathbb{S} := \mathbb{S}'$. If $\hat{q}\hat{r} \in \mathbb{K}_\infty^\times$, let $\mathbb{S}$ be the subset of $\mathrm{Ch}(U^0(\chi))$ formed by all Λ satisfying one of the following conditions:*

(irrEx3-1) *$\Lambda \in \mathbb{S}'$ and $\lambda_1\lambda_2^2\lambda_3 = (\hat{q}\hat{r})^{-(x+2)}$ for some $x \in \mathbb{Z}_{\geq 0}$.*

(irrEx3-2) *$\lambda_1 = \lambda_2 = \lambda_3 = 1$.*

(irrEx3-3) *$\Lambda \in \mathbb{S}'$ and $\lambda_2 = 1$, $\lambda_1\lambda_3 = (\hat{q}\hat{r})^{-1}$.*

(irrEx3-4) *$\Lambda \in \mathbb{S}'$ and $\lambda_1\lambda_2 = \hat{q}^{-1}$, $\lambda_2\lambda_3 = \hat{r}^{-1}$.*

(cL-4) *Let $\hat{q} := \chi(\alpha_2, \alpha_2) \in \mathbb{K}_\infty^\times$ and $\hat{\zeta} := \chi(\alpha_1, \alpha_1) \in \mathbb{K}_3^\times$. Let $\mathbb{S}$ be the subset of $\mathrm{Ch}(U^0(\chi))$ formed by all Λ satisfying one of the following conditions:*

(irrEx2-1) $\Lambda \in \mathbb{S}_2(\chi)$ *and* $\lambda_1^2\lambda_2 = (\hat{\zeta}\hat{q}^{-1})^{x+2}$ *for some* $x \in \mathbb{Z}_{\geq 0}$.

(irrEx2-2) $\lambda_1 = \lambda_2 = 1$.

(cL-5) *Let* $\hat{q} := \chi(\alpha_2, \alpha_2) \in \mathbb{K}_\infty^\times$. *Let* $\mathbb{S}$ *be the subset of* $\mathrm{Ch}(U^0(\chi))$ *formed by all* Λ *satisfying one of the following conditions:*

(irrEx4-1) $\Lambda \in \mathbb{S}_1(\chi) \cap \mathbb{S}_2(\chi) \cap \mathbb{S}_4(\chi)$ *and* $\lambda_1\lambda_2^2\lambda_3^3\lambda_4 = (-\hat{q}^{-1})^{x+3}$ *for some* $x \in \mathbb{Z}_{\geq 0}$.

(irrEx4-2) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1$.

(irrEx4-3) $\lambda_2 = \lambda_3 = 1$ *and* $\lambda_1 = \hat{q}^{2x}$, $\lambda_4 = (-\hat{q}^{-1})^{2x+1}$ *for some* $x \in \mathbb{Z}_{\geq 0}$.

(irrEx4-4) $\lambda_3 = 1$ *and there exist* $x, y \in \mathbb{Z}_{\geq 0}$ *such that* $\lambda_1 = \hat{q}^{2x}$, $\lambda_2 = \hat{q}^y$ *and* $\lambda_4 = \hat{q}^{-2(x+y+1)}$.

(irrEx4-5) *There exist* $x \in \mathbb{Z}_{\geq 0}$ *and* $y \in J_{0,x/2}$ *such that* $\lambda_1 = \hat{q}^{x+1}$, $\lambda_2 = \hat{q}^{x-2y}$, $\lambda_3 = \hat{q}^{-x+2y-1}$ *and* $\lambda_4 = \hat{q}^{-2y}$.

Proof. We define $n \in \mathbb{N}$, $f \in \mathrm{Map}_n^I$ and a map $z : J_{1,n} \to \mathbb{K}^\times$ as follows. Let $\hat{f} := (f(t) \mid t \in J_{1,n}) \in I^n$ and $\hat{z} := (z(t) \mid t \in J_{1,n}) \in (\mathbb{K}^\times)^n$.

If χ is as in (FGE-1), let $n := 18$, $\hat{f} := (2,3,4,2,3,4,2,3,4,1,2,3,4,1,4,3,2,1)$ and $\hat{z} := (\hat{q}^2, \hat{q}^4, \hat{q}^4, \hat{q}^2, \hat{q}^4, \hat{q}^4, \hat{q}^2, \hat{q}^4, \hat{q}^4, -1, -1, -1, -1, \hat{q}^{-6}, -1, -1, -1, -1)$.

If χ is as in (FGE-2), let $n := 13$, $\hat{f} := (2,3,2,3,2,3,1,2,3,1,3,2,1)$ and $\hat{z} := (\hat{q}^2, \hat{q}^6, \hat{q}^2, \hat{q}^6, \hat{q}^2, \hat{q}^6, -1, -1, -1, -\hat{q}^{-2}, -1, -1, -1)$.

If χ is as in (FGE-3), let $n := 7$, $\hat{f} := (1,3,2,1,3,1,2)$ and $\hat{z} := (\hat{q}, \hat{r}, -1, -1, (\hat{q}\hat{r})^{-1}, -1, -1)$.

If χ is as in (FGE-4), let $n := 4$, $\hat{f} := (2,1,2,1)$ and $\hat{z} := (\hat{q}, \hat{\zeta}, \hat{\zeta}\hat{q}^{-1}, \hat{\zeta})$.

If χ is as in (FGE-5), let $n := 15$, $\hat{f} := (1,2,1,4,3,4,2,1,4,3,1,2,4,2,1)$ and $\hat{z} := (\hat{q}, \hat{q}, \hat{q}, -\hat{q}^{-1}, -1, -1, -1, -1, -\hat{q}^{-1}, -\hat{q}^{-1}, -1, -1, -1, -1, -1)$.

Using Lemmas 1.11 and 4.22 (see also Figures 8–12), we can directly see that

$$n = |R^+(\chi)| \quad \text{and} \quad 1^\chi s_{\hat{f},n} = 1^\chi w_0. \tag{7.37}$$

We can also see that $\hat{z}(t) = \chi_{f,t-1}(\alpha_{f(t)}, \alpha_{f(t)})$ $(t \in J_{1,n})$ (see also Figures 8–12).

Define $r := \max\{t \in J_{1,n} \mid \forall t' \in J_{1,t},\ \hat{z}(t) \in \mathbb{K}_\infty^\times\}$ and $b := |\{t \in J_{b+1,n} \mid \hat{z}(t) \in \mathbb{K}_\infty^\times\}|$. Then $b \leq 2$, and $b = 2$ if and only if $N = 4$ and $\chi \equiv \chi' \in \dot{\mathcal{X}}_N^{\mathrm{Extra}}$. Then we can prove the theorem in much the same way as Theorems 7.6 and 7.7; in fact, it is easier since $b \leq 2$.

Let us explain more precisely how to prove the theorem for χ as in (FGE-5). Let $\Lambda \in \mathrm{Ch}(U^0(\chi))$. Note $r = 4$. Since $\hat{z}(t) = -1 \notin \mathbb{K}_\infty^\times$ $(t \in J_{5,8})$, by an argument similar to that for (7.22), we see that

$$H(\chi, \Lambda, f) \geq 8 \quad \text{if and only if} \quad \Lambda \in \mathbb{S}_1(\chi) \cap \mathbb{S}_2(\chi) \cap \mathbb{S}_4(\chi). \tag{7.38}$$

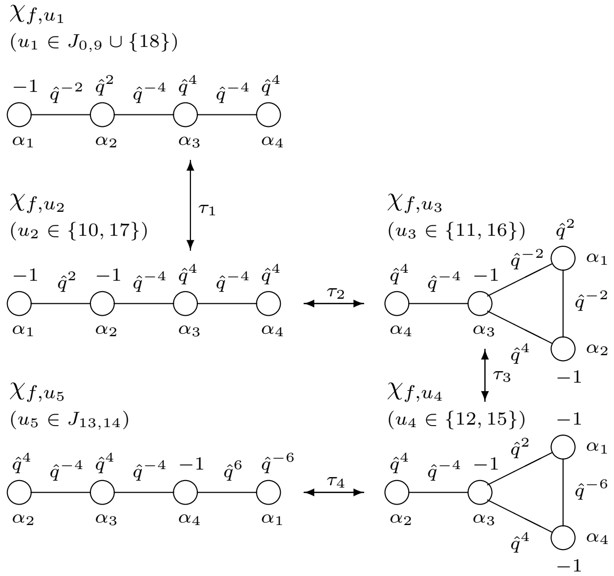

Figure 8. Dynkin diagrams of $\chi = \chi_{f,0} \equiv \chi' \in \dot{\mathcal{X}}_N^{\mathrm{Super}}(\mathrm{F}(4))$ with $N = 4$, and $\chi_{f,u}$ with $f = \hat{f}$, where $1^\chi w_0 = s_2^{\chi_{f,1}} s_3^{\chi_{f,2}} s_4^{\chi_{f,3}} s_2^{\chi_{f,4}} s_3^{\chi_{f,5}} s_4^{\chi_{f,6}} s_2^{\chi_{f,7}} s_3^{\chi_{f,8}} s_4^{\chi_{f,9}} s_1^{\chi_{f,10}} s_2^{\chi_{f,11}} s_3^{\chi_{f,12}} s_4^{\chi_{f,13}} s_1^{\chi_{f,14}} s_4^{\chi_{f,15}} s_3^{\chi_{f,16}} s_2^{\chi_{f,17}} s_1^{\chi_{f,18}}$.

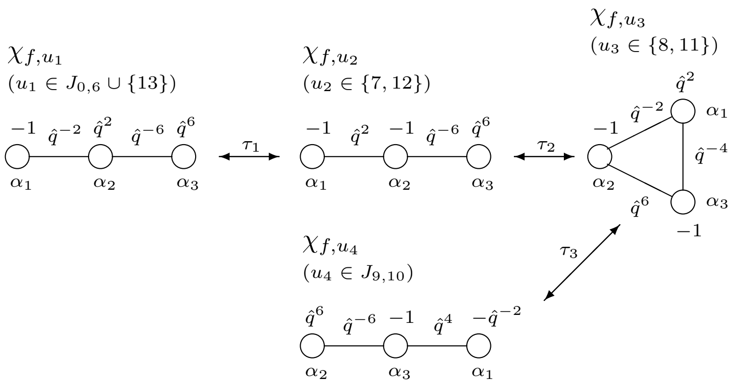

Figure 9. Dynkin diagrams of $\chi = \chi_{f,0} \equiv \chi' \in \dot{\mathcal{X}}_N^{\mathrm{Super}}(\mathrm{G}(3))$ with $N = 3$, and $\chi_{f,u}$ with $f = \hat{f}$, where $1^\chi w_0 = s_2^{\chi_{f,1}} s_3^{\chi_{f,2}} s_2^{\chi_{f,3}} s_3^{\chi_{f,4}} s_2^{\chi_{f,5}} s_3^{\chi_{f,6}} s_1^{\chi_{f,7}} s_2^{\chi_{f,8}} s_3^{\chi_{f,9}} s_1^{\chi_{f,10}} s_3^{\chi_{f,11}} s_2^{\chi_{f,12}} s_1^{\chi_{f,13}}$.

Assume $H(\chi,\Lambda,f) \geq 8$. By (7.38), $(\lambda_1,\lambda_2,\lambda_4) = (\hat{q}^{l_1}, \hat{q}^{l_2}, (-\hat{q}^{-1})^{l_4})$ for some $(l_1,l_2,l_4) \in (\mathbb{Z}_{\geq 0})^3$. Let $h_t := h_{\chi_{f,t-1},\Lambda_{f,t-1},f(t)}$ for $t \in J_{1,H(\chi,\Lambda,f)}$. By Lemma 6.4, $h_t = l_{f(t)}$ $(t \in J_{1,4})$, $h_5 = 1 - \delta(1,\lambda_3)$, $h_6 = 1 - \delta(1, (-\hat{q}^{-1})^{h_5}\lambda_3\lambda_4)$, $h_7 = 1 -$

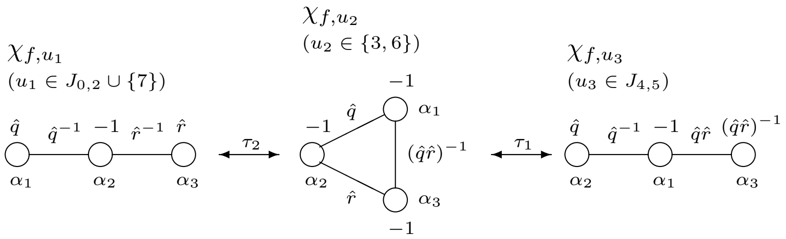

Figure 10. Dynkin diagrams of $\chi = \chi_{f,0} \equiv \chi' \in \dot{\mathcal{X}}_N^{\mathrm{Extra}}$ with $N = 3$, and $\chi_{f,u}$ with $f = \hat{f}$, where $1^\chi w_0 = s_1^{\chi_{f,1}} s_3^{\chi_{f,2}} s_2^{\chi_{f,3}} s_1^{\chi_{f,4}} s_3^{\chi_{f,5}} s_1^{\chi_{f,6}} s_2^{\chi_{f,7}}$.

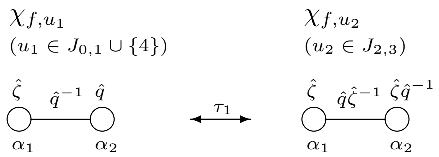

Figure 11. Dynkin diagrams of $\chi = \chi_{f,0} \equiv \chi' \in \dot{\mathcal{X}}_N^{\mathrm{Extra}}$ with $N = 2$, and $\chi_{f,u}$ with $f = \hat{f}$, where $1^\chi w_0 = s_2^{\chi_{f,1}} s_1^{\chi_{f,2}} s_2^{\chi_{f,3}} s_1^{\chi_{f,4}}$.

$\delta(1, (-1)^{h_5+h_6}\lambda_2\lambda_3^2\lambda_4)$, and $h_8 = 1 - \delta(1, (-1)^{h_5+h_6}\hat{q}^{h_7}\lambda_1\lambda_2\lambda_3^2\lambda_4)$. We see that

$$\begin{aligned} &H(\chi, \Lambda, f) \geq 9 \quad \text{if and only if} \\ &(-\hat{q}^{-1})^c = (-\hat{q}^{-1})^{-(h_5+h_7+h_8)}\lambda_1\lambda_2^2\lambda_3^3\lambda_4 \quad \text{for some } c \in \mathbb{Z}_{\geq 0}. \end{aligned} \tag{7.39}$$

We can see that if $H(\chi, \Lambda, f) \geq 9$, then

$$H(\chi, \Lambda, f) \geq 10 \ \Leftrightarrow\ \exists c' \in \mathbb{Z}_{\geq 0},\ (-\hat{q}^{-1})^{c'} = (-\hat{q}^{-1})^{h_5-h_6}\lambda_4. \tag{7.40}$$

By (7.39) and (7.40), we see that $H(\chi, \Lambda, f) \geq 9$ can be replaced by $H(\chi, \Lambda, f) \geq 10$, since if $h_5 = l_4 = 0$, then $\lambda_3 = 1$ and $h_6 = 0$. Since $\hat{z}(t) = -1 \notin \mathbb{K}_\infty^\times$ $(t \in J_{11,n})$, $H(\chi, \Lambda, f) \geq 10$ can be replaced by $H(\chi, \Lambda, f) = n$. Then using Lemma 6.6 and (7.39), by a direct argument, we can see that claim (5) holds. □

§7.5. Irreducible modules of $U_q(\mathfrak{g})$

For arguments below involving the symbol 'σ', see also [26, Subsection 1.9] and [27, Subsection 6.4].

We call $\chi \in \mathcal{X}_N$ *symmetric* if $\chi(\alpha, \beta) = \chi(\beta, \alpha)$ for all $\alpha, \beta \in \mathbb{Z}\Pi$.

Let $\chi \in \mathcal{X}_N$, and assume it is symmetric. Let $\dot{U} = \dot{U}(\chi)$ be the quotient $\mathbb{K}$-algebra of $U(\chi)$ by the two-sided ideal generated by the elements $K_\alpha L_\alpha - 1$ $(\alpha \in \mathbb{Z}\Pi)$. Let $\dot{\pi} : U(\chi) \to \dot{U}(\chi)$ be the canonical map. Let $\dot{U}^+ := \dot{U}^+(\chi) := \dot{\pi}(U^+(\chi))$, $\dot{U}^0 := \dot{U}^0(\chi) := \dot{\pi}(U^0(\chi))$ and $\dot{U}^- := \dot{U}^-(\chi) := \dot{\pi}(U^-(\chi))$. Then we

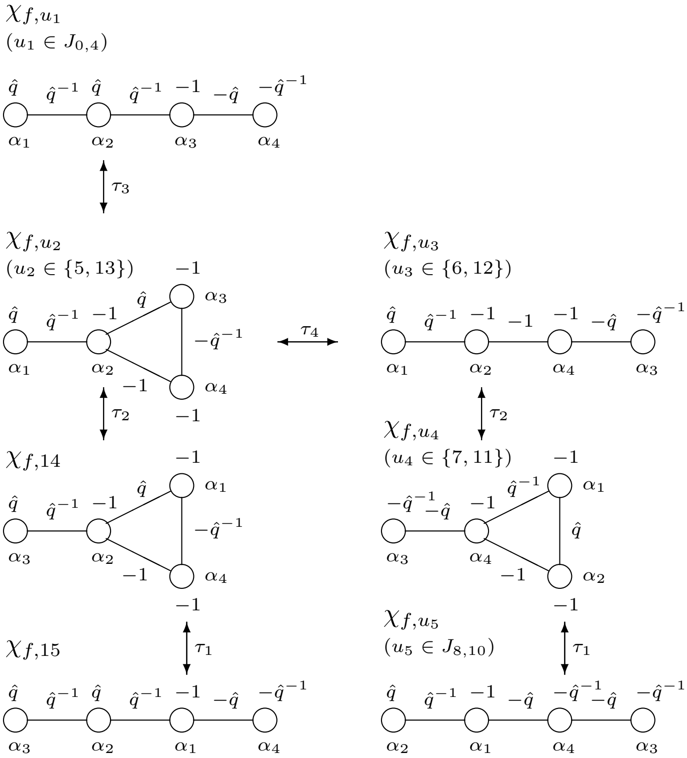

Figure 12. Dynkin diagrams of $\chi = \chi_{f,0} \equiv \chi' \in \dot{\mathcal{X}}_N^{\mathrm{Extra}}$ with $N = 4$, and $\chi_{f,u}$ with $f = \hat{f}$, where $1^\chi w_0 = s_1^{\chi_{f,1}} s_2^{\chi_{f,2}} s_1^{\chi_{f,3}} s_4^{\chi_{f,4}} s_3^{\chi_{f,5}} s_4^{\chi_{f,6}} s_2^{\chi_{f,7}} s_1^{\chi_{f,8}} s_4^{\chi_{f,9}} s_3^{\chi_{f,10}} s_1^{\chi_{f,11}} s_2^{\chi_{f,12}} s_4^{\chi_{f,13}} s_2^{\chi_{f,14}} s_1^{\chi_{f,15}}$.

have the $\mathbb{K}$-linear isomorphism $\dot{U}^- \otimes \dot{U}^0 \otimes \dot{U}^+ \to \dot{U}$ $(Y \otimes Z \otimes X \mapsto YZX)$. We also have the $\mathbb{K}$-algebra isomorphisms $\dot{\pi}_{|U^+(\chi)} : U^+(\chi) \to \dot{U}^+(\chi)$ and $\dot{\pi}_{|U^-(\chi)} : U^-(\chi) \to \dot{U}^-(\chi)$. Let $\dot{K}_\alpha := \dot{\pi}(K_\alpha)$ $(\alpha \in \mathbb{Z}\Pi)$, so $\dot{K}_{-\alpha} = \dot{\pi}(L_\alpha)$. The elements $\dot{K}_\alpha$ $(\alpha \in \mathbb{Z}\Pi)$ form a $\mathbb{K}$-basis of $\dot{U}^0$. Let $\dot{E}_i := \dot{\pi}(E_i)$ and $\dot{F}_i := \dot{\pi}(F_i)$ $(i \in I)$. For $\dot{\Lambda} \in \mathrm{Ch}(\dot{U}^0(\chi))$, letting $\Lambda := \dot{\Lambda} \circ \dot{\pi}_{|U^0(\chi)} \in \mathrm{Ch}(U^0(\chi))$, we can regard the $U(\chi)$-module $\mathcal{L}_\chi(\Lambda)$ as the $\dot{U}(\chi)$-module so that $\dot{\pi}(X') \cdot v = X'v$ $(v \in \mathcal{L}_\chi(\Lambda)$, $X' \in U(\chi))$, and, when regarding it as a $\dot{U}(\chi)$-module, we denote it by $\dot{\mathcal{L}}_\chi(\dot{\Lambda})$; we also denote v_Λ $(\in \mathcal{L}_\chi(\Lambda))$ by $v_{\dot{\Lambda}}$ $(\in \dot{\mathcal{L}}_\chi(\dot{\Lambda}))$.

By Lemma 4.23, we have

Lemma 7.9. *Let $\chi \in \mathcal{X}_N$ be as in Lemma 4.23. Assume it is symmetric. Then $\{\dot{\mathcal{L}}_\chi(\dot{\Lambda}) \mid \dot{\Lambda} \in \mathrm{Ch}(\dot{U}^0(\chi)), \dim \dot{\mathcal{L}}_\chi(\dot{\Lambda}) < \infty\}$ is a complete set of pairwise non-isomorphic finite-dimensional irreducible $\dot{U}(\chi)$-modules.*

Assume χ is symmetric. Let $\dot{U}^\sigma = \dot{U}^\sigma(\chi)$ be the $\mathbb{K}$-algebra including $\dot{U} = \dot{U}(\chi)$ as a $\mathbb{K}$-subalgebra and having an element σ with $\sigma^2 = 1$, $\sigma\dot{E}_i\sigma = (-1)^{\theta^\sharp(i)}\dot{E}_i$, $\sigma\dot{F}_i\sigma = (-1)^{\theta^\sharp(i)}\dot{F}_i$ $(i \in I)$ and $\dot{U}^\sigma = \dot{U} \oplus \dot{U}\sigma$ (as $\mathbb{K}$-linear spaces). For $\dot{\Lambda} \in \mathrm{Ch}(\dot{U}^0(\chi))$ and $x \in \{\pm 1\}$, we can regard the $\dot{U}(\chi)$-module $\dot{\mathcal{L}}_\chi(\dot{\Lambda})$ as a $\dot{U}(\chi)^\sigma$-module so that $X \cdot v = Xv$ $(v \in \dot{\mathcal{L}}_\chi(\dot{\Lambda}),\ X \in \dot{U}(\chi))$ and $\sigma v_{\dot{\Lambda}} = x v_{\dot{\Lambda}}$; when regarding it as a $\dot{U}(\chi)^\sigma$-module, we denote it by $\dot{\mathcal{L}}_\chi(\dot{\Lambda})^{(x)}$.

Definition 7.10. Let $\hat{q} \in \mathbb{K}^\times_\infty$, and $\chi := \varpi_{\hat{q}}(a^\sharp) \in \mathcal{X}'^{,\mathrm{fin}}_N$ (see (5.2)) for some $a^\sharp = (\eta^\sharp, \theta^\sharp) \in \mathcal{Y}'^{\flat,\mathrm{fin}}_N$ (see (5.1)). Then χ is symmetric. For $i \in I$, let $\bar{K}_i := \dot{K}_{\alpha_i}\sigma^{\theta^\sharp(i)}$, $\bar{E}_i := \dot{E}_i$, $\bar{F}_i := -\frac{1}{\hat{q}-\hat{q}^{-1}}\dot{F}_i\sigma^{\theta^\sharp(i)}$ $(\in \dot{U}(\chi)^\sigma)$. Recall the Lie superalgebra $\mathfrak{g} = \mathfrak{g}(a^\sharp)$ (see Definition 3.3). Let $U_{\hat{q}}(\mathfrak{g})$ be the $\mathbb{K}$-subalgebra of $\dot{U}(\chi)^\sigma$ generated by $\bar{K}_i$, $\bar{K}_i^{-1}$, $\bar{E}_i$, $\bar{F}_i$ for all $i \in I$. We call $U_{\hat{q}}(\mathfrak{g})$ the (standard) *quantum superalgebra* of $\mathfrak{g}$ (over $\mathbb{K}$) (see Remark 7.11 below for the original definition of $U_{\hat{q}}(\mathfrak{g})$.)

Remark 7.11 (Defining relations of $U_{\hat{q}}(\mathfrak{g})$). Keep the notation of Definition 7.10. The generators $\bar{K}_i$, $\bar{K}_i^{-1}$, $\bar{E}_i$, $\bar{F}_i$ $(i \in I)$ of $U_{\hat{q}}(\mathfrak{g})$ satisfy the equations (7.41) below, which are the usual ones for the quantum superalgebra:

$$
\begin{aligned}
&\bar{K}_i\bar{K}_i^{-1} = \bar{K}_i^{-1}\bar{K}_i = 1, \qquad \bar{K}_i\bar{K}_j = \bar{K}_j\bar{K}_i, \\
&\bar{K}_i\bar{E}_j\bar{K}_i^{-1} = \hat{q}^{\eta^\sharp(\alpha_i,\alpha_j)}\bar{E}_j, \qquad \bar{K}_i\bar{F}_j\bar{K}_i^{-1} = \hat{q}^{-\eta^\sharp(\alpha_i,\alpha_j)}\bar{F}_j, \\
&\bar{E}_i\bar{F}_j - (-1)^{\theta^\sharp(i)\theta^\sharp(j)}\bar{F}_j\bar{E}_i = \delta_{ij}\frac{\bar{K}_i - \bar{K}_i^{-1}}{\hat{q}-\hat{q}^{-1}} \qquad (i,j \in I).
\end{aligned}
\tag{7.41}
$$

Let $U_{\hat{q}}(\mathfrak{g})^\sigma := \dot{U}(\chi)^\sigma$. Then $U_{\hat{q}}(\mathfrak{g})^\sigma = U_{\hat{q}}(\mathfrak{g}) \oplus U_{\hat{q}}(\mathfrak{g})\sigma$ as $\mathbb{K}$-linear spaces. Historically, in [26, Theorem 2.9.4], [27, (6.4.1)] (see also [25, Section 3]), $U_{\hat{q}}(\mathfrak{g})^\sigma$ has been introduced as the Hopf (non-super) algebra defined in the same way as in (4.8), which is similar to Lusztig's well-known way [19, 3.1.1(a)–(e)] (see also [5, Subsection 2.1]), and $U_{\hat{q}}(\mathfrak{g})$ has been introduced as its subalgebra ($U_{\hat{q}}(\mathfrak{g})$ is a Hopf superalgebra); see [26, Corollary 2.9.11], [27, Subsection 6.8]. For a complete set of its defining relations, we make the same remark as in Remark 5.12. Let $\bar{E}_{m,\alpha_i,\alpha_j} := \dot{\pi}(E_{m,\alpha_i,\alpha_j})$, $\bar{F}_{m,\alpha_i,\alpha_j} := \dot{\pi}(F_{m,\alpha_i,\alpha_j})\sigma^{m\theta^\sharp(i)+\theta^\sharp(j)} \in U_{\hat{q}}(\mathfrak{g})$ (see (4.9) for E_{m,α_i,α_j} and F_{m,α_i,α_j}). Assume that $\chi \in \dot{\mathcal{X}}^{\mathrm{Super}}_N$. If $\chi \notin \dot{\mathcal{X}}^{\mathrm{Super}}_N(\mathrm{B}(0,N))$, there exists a unique $\bar{o} \in I$ with $\eta^\sharp(\alpha_{\bar{o}}, \alpha_{\bar{o}}) = 0$. If $\chi \in \dot{\mathcal{X}}^{\mathrm{Super}}_N(\mathrm{B}(0,N))$, such an $\bar{o}$ does not exist. Recall $c_{ij} := c^{a^\sharp}_{ij}$ from (3.1). By [25, Section 3], [26, Theorem 10.5.1(ii)], [27, Theorem 6.8.1], a complete set of defining relations of the $\mathbb{K}$-algebra $U_{\hat{q}}(\mathfrak{g})$ is formed by the relations in (7.41) and in (7.42) below.

$$
(7.42)\quad
\begin{aligned}
&\bar{E}_{1-c_{ij},\alpha_i,\alpha_j} = \bar{F}_{1-c_{ij},\alpha_i,\alpha_j} = 0 \quad (\bar{o} \neq i \neq j) \\
&\qquad\qquad \text{(usual quantum Serre relations)}, \\
&\bar{E}_{\bar{o}}^2 = \bar{F}_{\bar{o}}^2 = 0, \\
&\bar{E}_{1,\alpha_k,\alpha_{\bar{o}}}\bar{E}_{1,\alpha_r,\alpha_{\bar{o}}} + \bar{E}_{1,\alpha_r,\alpha_{\bar{o}}}\bar{E}_{1,\alpha_k,\alpha_{\bar{o}}} = 0, \\
&\bar{F}_{1,\alpha_k,\alpha_{\bar{o}}}\bar{F}_{1,\alpha_r,\alpha_{\bar{o}}} + \bar{F}_{1,\alpha_r,\alpha_{\bar{o}}}\bar{F}_{1,\alpha_k,\alpha_{\bar{o}}} = 0 \\
&(\bar{o} \neq k \neq r \neq \bar{o},\ \eta^\sharp(\alpha_k,\alpha_{\bar{o}}) = -\eta^\sharp(\alpha_r,\alpha_{\bar{o}}) \neq 0).
\end{aligned}
$$

Since the Dynkin diagram of $\mathfrak{g}$ corresponding to $a^\sharp$ is the *best-known* among all those of $\mathfrak{g}$ (recall the definition of $\dot{\mathcal{X}}_N^{\mathrm{Super}}$), the number of relations is rather small.

Let $\hat{q} \in \mathbb{K}_\infty^\times$ and $\chi \in \mathcal{X}_N := \varpi_{\hat{q}}(a^\sharp)$ (see (5.2)) for some $a^\sharp = (\eta^\sharp, \theta^\sharp) \in \mathcal{Y}_N^{\prime\flat,\mathrm{fin}}$. Let $\mathfrak{g} := \mathfrak{g}(a^\sharp)$. For $\kappa = (\kappa_i \mid i \in I) \in (\mathbb{K}^\times)^N$, define $\dot{\Lambda}_\kappa \in \mathrm{Ch}(\dot{U}(\chi))$ by $\dot{\Lambda}_\kappa(\dot{K}_{\alpha_i}) := \kappa_i$, and when regarding the $\dot{U}(\chi)^\sigma$-module $\dot{\mathcal{L}}_\chi(\dot{\Lambda}_\kappa)^{(1)}$ as the $U_{\hat{q}}(\mathfrak{g})$-module, we denote it by $\mathcal{L}_{\hat{q}}(\kappa)$; let $v_\kappa := v_{\dot{\Lambda}_\kappa}$ and note that

$$\bar{K}_i \cdot v_\kappa = \kappa_i v_\kappa, \quad \bar{E}_i \cdot v_\kappa = 0 \quad (i \in I), \qquad \mathcal{L}_{\hat{q}}(\kappa) = U_{\hat{q}}(\mathfrak{g}) \cdot v_\kappa,$$

where $\cdot$ means the action of $U_{\hat{q}}(\mathfrak{g})$ on $\mathcal{L}_{\hat{q}}(\kappa)$.

By using Lemma 7.9 and an argument similar to that for Lemma 4.23, we have

Lemma 7.12. *Let $a^\sharp = (\eta^\sharp, \theta^\sharp) \in \mathcal{Y}_N^{\prime\flat,\mathrm{fin}}$ be as in Theorem 3.4. Then $\{\mathcal{L}_{\hat{q}}(\kappa) \mid \kappa \in (\mathbb{K}^\times)^N, \dim \mathcal{L}_{\hat{q}}(\kappa) < \infty\}$ is a complete set of pairwise non-isomorphic finite-dimensional irreducible $U_{\hat{q}}(\mathfrak{g}(a^\sharp))$-modules.*

Example 7.13. Let $a^\sharp = (\eta^\sharp, \theta^\sharp) \in \mathcal{Y}_N^{\prime\flat,\mathrm{fin}}$ be as in Lemma 7.12. Let $\hat{q} \in \mathbb{K}_\infty^\times$. Fix $\hat{q}^{1/2} \in \mathbb{K}^\times$ so that $(\hat{q}^{1/2})^2 = \hat{q}$. For $r \in \mathbb{Z}$, let $\hat{q}^{r/2} := (\hat{q}^{1/2})^r$. Fix $\sqrt{-1} \in \mathbb{K}^\times$ so that $(\sqrt{-1})^2 = -1$. Let $\chi := \varpi_{\hat{q}}(a^\sharp) \in \mathcal{X}_N^{\prime,\mathrm{fin}}$. Recall that the $U_{\hat{q}}(\mathfrak{g}(a^\sharp))$-module $\mathcal{L}_{\hat{q}}(\kappa)$ can also be regarded as the $U(\chi)$-module $\mathcal{L}_\chi(\dot{\Lambda}_\kappa \circ \dot{\pi})$. If χ is considered in Theorems 7.1, 7.2, 7.4, 7.6, 7.7, or 7.8, since $\kappa_i^2 = (\dot{\Lambda}_\kappa \circ \dot{\pi})(K_{\alpha_i} L_{-\alpha_i})$ $(i \in I)$, we explicitly obtain $\dim \mathcal{L}_{\hat{q}}(\kappa) < \infty$ from $\dim \mathcal{L}_\chi(\dot{\Lambda}_\kappa \circ \dot{\pi}) < \infty$.

(1) (See Theorem 7.4(1).) Assume $a^\sharp$ is the $\mathrm{A}(m-1, N-m)$-data ($N \geq 2$, $m \in J_{1,N}$). Then $\dim \mathcal{L}_{\hat{q}}(\kappa) < \infty$ if and only if there exist $r_i \in \mathbb{Z}_{\geq 0}$, $x_i \in \{\pm 1\}$ $(i \in I \setminus \{m\})$ and $y \in \mathbb{K}^\times$ with $\kappa_i = x_i \hat{q}^{r_i}$ $(i \in J_{1,m-1})$, $\kappa_m = y$, and $\kappa_j = x_j \hat{q}^{-r_j}$ $(j \in J_{m+1,N})$.

(2) (See Theorem 7.6.) Assume $a^\sharp$ is the $\mathrm{B}(m, N-m)$-data ($N \geq 2$, $m \in J_{1,N-1}$). Then $\dim \mathcal{L}_{\hat{q}}(\kappa) < \infty$ if and only if there exist $r_i \in \mathbb{Z}_{\geq 0}$, $x_i \in \{\pm 1\}$ $(i \in I)$ and $c \in J_{0,1}$ with $\kappa_i = x_i \hat{q}^{-r_i}$ $(i \in J_{1,N-m-1})$, $\kappa_{N-m} = x_{N-m}(\sqrt{-1})^c \hat{q}^{-r_{N-m}/2}$, $\kappa_j = x_j \hat{q}^{r_j}$ $(j \in J_{N-m+1,N-1})$, $\kappa_N = x_N \hat{q}^{r_N/2}$ for which one of the following cases occurs, where $\bar{b} := r_{N-m} - (r_N + 2\sum_{j=N-m+1}^{N-1} r_j)$:

(B$m\hat{q}$-1) $c=0$, and $\bar{b}=2(m+z)$ for some $z\in\mathbb{Z}_{\geq 0}$.

(B$m\hat{q}$-2) $c=1$, and $\bar{b}=2(m+z)+1$ for some $z\in\mathbb{Z}_{\geq 0}$.

(B$m\hat{q}$-3) $c=0$, and there exists $k\in J_{0,m-1}$ such that $\bar{b}=2k$, $r_{N-m}=2(k+\sum_{j=N-m+1}^{N-m+k}r_j)$, and $r_t=0$ $(t\in J_{N-m+k+1,N})$.

(3) (See Theorem 7.7.) Assume $a^\sharp$ is the D$(m,N-m)$-data ($N\geq 2$, $m\in J_{2,N-1}$). Then $\dim\mathcal{L}_{\hat{q}}(\kappa)<\infty$ if and only if there exists $r_i\in\mathbb{Z}_{\geq 0}$, $x_i\in\{\pm 1\}$ $(i\in I)$ and $c\in J_{0,1}$ with $\kappa_i=x_i\hat{q}^{-r_i}$ $(i\in J_{1,N-m-1})$, $\kappa_{N-m}=x_i(\sqrt{-1})^c\hat{q}^{-r_{N-m}}$, $\kappa_j=x_j\hat{q}^{r_j}$ $(j\in J_{N-m+1,N})$ for which one of the following cases occurs, where $\bar{d}:=r_{N-m}-(r_{N-1}+r_N+2\sum_{j=N-m+1}^{N-2}r_j)$:

(D$m\hat{q}$-1) $\bar{d}=2(m+z)$ for some $z\in\mathbb{Z}_{\geq 0}$.

(D$m\hat{q}$-2) $c=0$, $\bar{d}=2(m-1)$, and $r_{N-m}=m-1+\sum_{j=N-m+1}^{N-1}r_j$. (Hence $r_{N-1}=r_N$.)

(D$m\hat{q}$-3) $c=0$, and there exists $k\in J_{0,m-2}$ such that $\bar{d}=2k$, $r_{N-m}=k+\sum_{j=N-m+1}^{N-m+k}r_j$, and $r_t=0$ $(t\in J_{N-m+k+1,N})$.

§7.6. Recovery of Kac's list as $q\to 1$

Geer [8, Theorem 1.2] showed that any irreducible highest weight module of the Lie superalgebra $\mathfrak{g}$ (treated below) allows an '$\hbar$-deformation' as a topological highest weight module of the Drinfeld-type quantized superalgebra $U_\hbar(\mathfrak{g})$ (see Remark 7.14). The argument here proceeds in the opposite direction.

Let $\mathbb{C}((\hbar))$ be the field of fractions of the formal power series ring $\mathbb{C}[[\hbar]]$ over $\mathbb{C}$. In this subsection, we assume $\mathbb{K}$ is an algebraic closure of $\mathbb{C}((\hbar))$. Let $\hat{q}:=\exp(\hbar)\in\mathbb{C}[[\hbar]]$. Let $a^\sharp=(\eta^\sharp,\theta^\sharp)\in\mathcal{Y}_N^{\prime\flat,\mathrm{fin}}$ be as in Theorem 3.4, as in Subsection 7.5. Let $\mathfrak{g}:=\mathfrak{g}(a^\sharp)$. Let $\bar{X}_i:=\frac{\bar{K}_i-\bar{K}_i^{-1}}{\hat{q}-\hat{q}^{-1}}\in U_{\hat{q}}(\mathfrak{g})$ $(i\in I)$. Let $\hat{U}_\hbar(\mathfrak{g})$ be the $\mathbb{C}[[\hbar]]$-subalgebra of $U_{\hat{q}}(\mathfrak{g})$ generated by $\bar{K}_i$, $\bar{K}_i^{-1}$, $\bar{X}_i$, $\bar{E}_i$, $\bar{F}_i$ $(i\in I)$. We can easily see that

$$T_i(\hat{U}_\hbar(\mathfrak{g}(\tau_i^\sharp a^\sharp)))=\hat{U}_\hbar(\mathfrak{g}), \tag{7.43}$$

where we recall T_i from (6.1). By [26, Theorem 10.5.1], we have a $\mathbb{C}$-algebra monomorphism $\Gamma:U(\mathfrak{g})\to\hat{U}_\hbar(\mathfrak{g})/\hbar\hat{U}_\hbar(\mathfrak{g})$ such that $\Gamma(H_i^\sharp)=\bar{X}_i+\hbar\hat{U}_\hbar(\mathfrak{g})$, $\Gamma(E_i^\sharp)=\bar{E}_i+\hbar\hat{U}_\hbar(\mathfrak{g})$, $\Gamma(F_i^\sharp)=\bar{F}_i+\hbar\hat{U}_\hbar(\mathfrak{g})$ $(i\in I)$, where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. By [26, Theorem 10.5.1], we see that for any $\mathbb{C}$-basis $\{m_z\mid z\in\mathcal{Z}\}$ of $U(\mathfrak{g})$, letting $\check{m}_z\in\hat{U}_\hbar(\mathfrak{g})$ $(z\in\mathcal{Z})$ be such that $\check{m}_z+\hbar\hat{U}_\hbar(\mathfrak{g})=\Gamma(m_z)$, the set $\{\check{m}_z\cdot\bar{K}_i^c\mid z\in\mathcal{Z},\,c\in J_{0,1}\}$ is a free $\mathbb{C}[[\hbar]]$-basis of $\hat{U}_\hbar(\mathfrak{g})$.

Remark 7.14. As in [27, Subsections 6.6, 7.1], by [26, Theorem 10.5.1], we obtain a Drinfeld-type topological quantum superalgebra $U_\hbar(\mathfrak{g})$ which includes the (non-topological) $\mathbb{C}[[\hbar]]$-subalgebra $\hat{U}_\hbar(\mathfrak{g})$ as a dense subset.

Let $x = (x_i \mid i \in I) \in \mathbb{C}^N$. Let $\mathcal{L}^{\sharp,\prime}(x) := \hat{U}_\hbar(\mathfrak{g})v_{e(x)}/\hbar\hat{U}_\hbar(\mathfrak{g})v_{e(x)}$ where $e(x) := (\exp(x_i\hbar) \mid i \in I)$, and regard it as a $U(\mathfrak{g})$-module through Γ. Let $\hat{v}_x := v_{e(x)} + \hbar\hat{U}_\hbar(\mathfrak{g})v_{e(x)} \in \mathcal{L}^{\sharp,\prime}(x)$. Then $H_i^\sharp\hat{v}_x = x_i\hat{v}_x$, $E_i^\sharp\hat{v}_x = 0$, and $\mathcal{L}^{\sharp,\prime}(x) = U(\mathfrak{g})\hat{v}_x$. Let Y be a maximal proper submodule of $\mathcal{L}^{\sharp,\prime}(x)$. (By the same argument as in the proof of Lemma 4.23, we can see that Y exists.) Let $\mathcal{L}^\sharp(x) := \mathcal{L}^{\sharp,\prime}(x)/Y$. (By [8, Theorem 1.2], $\mathcal{L}^{\sharp,\prime}(x)$ is irreducible, so $Y = \{0\}$.)

By (7.43), T_i's induce $T_i^\sharp$'s (see (3.4)), and induce results for the irreducible $\mathfrak{g}$-modules $\mathcal{L}^\sharp(x)$'s similar to those of Lemmas 6.3 and 6.6; we also have a result for $\mathfrak{g}$ similar to Lemma 4.23. Thus we have Lemma 7.15 below, from which, using Theorems 7.4, 7.6, 7.7 and 7.8, we can recover Kac's list [17, Theorem 8(c)] of irreducible $\mathfrak{g}$-modules.

Lemma 7.15. *Let $a^\sharp \in \mathcal{Y}_N^{\prime\flat,\mathrm{fin}}$ be as in Lemma 7.12. Then $\{\mathcal{L}^\sharp(x) \mid x \in \mathbb{C}^N$, $\dim \mathcal{L}_{\hat{q}}(e(x)) < \infty\}$ is a complete set of pairwise non-isomorphic finite-dimensional irreducible $\mathfrak{g}(a^\sharp)$-modules.*

Remark 7.16. If $\mathfrak{g} = \mathrm{B}(m, N-m)$ or $\mathrm{D}(m, N-m)$, an intrinsic gap appears between the list for $\mathfrak{g}$ and the one for $U_{\hat{q}}(\mathfrak{g})$; there does not exist the case for $\mathfrak{g}$ corresponding to (B$m\hat{q}$-2) or (D$m\hat{q}$-1) with $c = 1$.

Remark 7.17. Shu and Wang [23, Theorem 5.3, Remark 5.4] also recovered Kac's list for the simple Lie superalgebras $\mathrm{B}(m, N-m)$, $\mathrm{C}(N)$ and $\mathrm{D}(m, N-m)$ by using odd reflections in a way totally different from that in this paper.

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