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The Bishop–Phelps–Bollobás Property: a Finite-Dimensional Approach

by

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Abstract

Our goal is to study the Bishop–Phelps–Bollobás property for operators from c_0 into a Banach space. We first characterize those Banach spaces Y for which the Bishop–Phelps–Bollobás property holds for (ℓ_∞^3, Y). Examples of spaces satisfying this condition are provided.

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§1. Introduction

The Bishop–Phelps theorem [5] states that every continuous linear functional on a Banach space can be approximated by norm attaining functionals. Shortly after this assertion was proved, Bollobás gave the following "quantitative version" of that result which is called the Bishop–Phelps–Bollobás theorem [6].

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Theorem 1.1 (Bishop–Phelps–Bollobás Theorem, [7, Theorem 16.1]). *Let X be a Banach space and $0 < \varepsilon < 1$. Given $x \in B_X$ and $x^* \in S_{X^*}$ with $|1 - x^*(x)| < \varepsilon^2/4$, there are elements $y \in S_X$ and $y^* \in S_{X^*}$ such that $y^*(y) = 1$, $\|y - x\| < \varepsilon$ and $\|y^* - x^*\| < \varepsilon$.*

After the Bishop–Phelps theorem was proved, a lot of attention was devoted to extending it to operators. More recently the study of extensions of Bishop–Phelps–Bollobás theorem to operators was initiated by Acosta, Aron, García and Maestre [2].

Definition 1.2 ([2]). Let X and Y be either real or complex Banach spaces. The pair (X, Y) is said to have *the Bishop–Phelps–Bollobás property for operators* (BPBp) if for every $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that for every $T \in S_{\mathcal{L}(X,Y)}$, if $x_0 \in S_X$ is such that

$$\|Tx_0\| > 1 - \eta(\varepsilon),$$

then there exist an element u_0 in S_X and an operator S in $S_{\mathcal{L}(X,Y)}$ satisfying the following conditions:

$$\|Su_0\| = 1, \quad \|u_0 - x_0\| < \varepsilon \quad \text{and} \quad \|S - T\| < \varepsilon.$$

Acosta, Aron, García and Maestre [2] showed that the pair (X, Y) has the BPBp whenever X and Y are finite-dimensional spaces. They also proved that if Y has a certain geometric property (property β of Lindenstrauss), then (X, Y) has the BPBp for every Banach space X. They also characterized the Banach spaces Y such that (ℓ_1, Y) has the BPBp.

However, the case of $X = c_0$ is quite different and seems to be much more difficult. Acosta et al. [2] showed that (ℓ_∞^n, Y) has the BPBp for every positive integer n whenever Y is uniformly convex. Recently Kim [8] proved that (c_0, Y) also has this property under the same assumption. Aron at al. [3] studied this question for $Y = \mathcal{C}_0(L)$, where L is a locally compact Hausdorff topological space. They showed that the couple $(X, \mathcal{C}_0(L))$ has the BPBp if X is Asplund. This result was extended by Cascales, Guirao and Kadets [4] to uniform algebras. Since c_0 is Asplund it follows that $(c_0, \mathcal{C}_0(L))$ has the BPBp. For more results where the domain space is some space $\mathcal{C}(K)$ see also [1] and [9]. However until now there is no characterization of the spaces Y such that (c_0, Y) has the BPBp. In this paper, we approach this problem by using appropriate finite-dimensional spaces. From now on, we only consider **real** normed spaces.

In Section 2, we characterize the Banach spaces Y such that (ℓ_∞^3, Y) has the BPBp. We also prove that uniformly convex spaces, $\mathcal{C}(K)$ and $L_1(\mu)$ always have this property. We show that $(\ell_\infty^3, \mathcal{C}_0(L, X))$ has this property if and only if (ℓ_∞^3, X) does, whenever L is any nonempty locally compact Hausdorff space.

§2. The Bishop–Phelps–Bollobás property for operators from ℓ_∞^3 into any Banach space

As usual, we denote by ℓ_∞^n the space $\mathbb{R}^n$ endowed with the supremum norm. Since ℓ_∞^2 is isometrically isomorphic to ℓ_1^2, it is an easy consequence of Acosta et al. [2, Theorem 4.1] that (ℓ_∞^2, X) has the BPBp if and only if X has the *approximate hyperplane series property* for convex combinations of two elements. However, ℓ_∞^3 is not isometric to ℓ_1^3, and so we cannot get a parallel result.

In order to characterize the Banach spaces X such that (ℓ_∞^3, X) has the BPBp we introduce the following property. This condition resembles the restricted 3-ball property which is used to characterize when a subspace of a Banach space is an M-ideal.

Definition 2.1. A Banach space X has the *approximate hyperplane sum property for* ℓ_∞^3 (AHSP-ℓ_∞^3) if for every $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ satisfying the following.

For every subset $\{x_i : i \leq 3\} \subset B_X$ with $\|x_1 + x_2 - x_3\| \leq 1$, if there exist a nonempty subset C of $\{1, 2, 3\}$ and $x^* \in S_{X^*}$ such that $x^*(x_i) > 1 - \delta(\varepsilon)$ for every $i \in C$, then there exists $\{z_i : i \leq 3\} \subset B_X$ with $\|z_1 + z_2 - z_3\| \leq 1$ satisfying $\|z_i - x_i\| < \varepsilon$ for every $i \leq 3$ and $\|\sum_{i \in C} z_i\| = |C|$.

We try to explain the idea behind the property defined above. It is trivially satisfied that the unit ball of ℓ_∞^3 is the absolutely convex hull of four vectors. Indeed one of these vectors can be written as a linear combination of the rest with scalars $\{1, 1, -1\}$. In this way, the set $\{x_i : i \leq 3\}$ in Definition 2.1 can be identified with an operator T from ℓ_∞^3 into X whose norm is close to 1. Moreover the assumption implies that T is close to its norm at some convex combination c_1 of the elements $\{x_i : i \leq 3\}$ that belong to the unit sphere. The elements $\{z_i : i \leq 3\}$ will be associated to a new operator S close to T and attaining its norm at some convex combination c_2 of $\{z_i : i \leq 3\}$, which is close to c_1. This is why we assume that several elements of the set $\{z_i : i \leq 3\}$ have the same supporting hyperplane. We also notice that condition (3) in Proposition 2.2 below resembles the definition of the approximate hyperplane series property (see [2, Definition 1.1]).

It is very easy to check directly that $\mathbb{R}$ has the above property. In any case, later we will show more general results and provide several examples.

First, recall that a subset $B \subset B_{X^*}$ is 1-*norming* if $\|x\| = \sup_{x^* \in B} |x^*(x)|$ for every $x \in X$.

The following characterization will be useful.

Proposition 2.2. *Let X be a Banach space. The following conditions are equivalent:*

(1) *X has the approximate hyperplane sum property for ℓ_∞^3.*

(2) *There is a 1-norming subset $B \subset S_{X^*}$ such that the condition stated in Definition 2.1 is satisfied for every $x^* \in B$.*

(3) *For every $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that for every subset $\{x_i : i \le 3\} \subset B_X$ with $\|x_1 + x_2 - x_3\| \le 1$ and every convex combination $\sum_{i=1}^3 \alpha_i x_i$ satisfying*

$$\Big\|\sum_{i=1}^3 \alpha_i x_i\Big\| > 1 - \eta(\varepsilon),$$

there exist $A \subset \{1,2,3\}$, $\{z_i : i \le 3\} \subset B_X$ and $x^* \in S_{X^*}$ such that

(i) $\sum_{i\in A} \alpha_i > 1 - \varepsilon$,

(ii) $\|z_i - x_i\| < \varepsilon$ *for all* $i \le 3$,

(iii) $x^*(z_i) = 1$ *for all* $i \in A$,

(iv) $\|z_1 + z_2 - z_3\| \le 1$.

Proof. Clearly (1) implies (2). Assume that X satisfies (2). Given $0 < \varepsilon < 1$, let $0 < \delta(\varepsilon) < 1$ satisfy the condition of Definition 2.1 for all $x^* \in B$. Consider a subset $\{x_i : i \le 3\} \subset B_X$ with $\|x_1 + x_2 - x_3\| \le 1$ and three nonnegative real numbers α_i for $1 \le i \le 3$ with $\sum_{i=1}^3 \alpha_i = 1$ also satisfying

$$\Big\|\sum_{i=1}^3 \alpha_i x_i\Big\| > 1 - \varepsilon\delta(\varepsilon).$$

Then there exists $x^* \in B$ such that

$$\sum_{i=1}^3 \alpha_i x^*(x_i) > 1 - \varepsilon\delta(\varepsilon).$$

By [2, Lemma 3.3] the set $A := \{i \le 3 : x^*(x_i) > 1 - \delta(\varepsilon)\}$ satisfies the estimate

$$\sum_{i\in A} \alpha_i > 1 - \varepsilon.$$

By assumption there exists $\{z_i : i \le 3\} \subset B_X$ satisfying

(ii) $\|z_i - x_i\| < \varepsilon$ for all $i \le 3$,

(iii) $\|\sum_{i\in A} z_i\| = |A|$,

(iv) $\|z_1 + z_2 - z_3\| \le 1$.

Hence X satisfies the conditions stated in (3) for $\eta(\varepsilon) = \varepsilon\delta(\varepsilon)$.

Now we assume that X has the property in (3). Given $0 < \varepsilon < 1/3$, let $\eta(\varepsilon) > 0$ satisfy conditions (i) to (iv). We are going to check that $\delta = \eta(\varepsilon)$ satisfies the condition of Definition 2.1. Let $\{x_i : i \le 3\} \subset B_X$ satisfy $\|x_1 + x_2 - x_3\| \le 1$, let $x^* \in S_{X^*}$ and let $C \subset \{1,2,3\}$ be such that $x^*(x_i) > 1 - \eta(\varepsilon)$ for every $i \in C$. We distinguish several cases.

Case 1: $C = \{1,2,3\}$. We take $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$. Obviously $\sum_{i=1}^3 \alpha_i = 1$ and

$$\Big\|\sum_{i=1}^{3} \alpha_i x_i\Big\| \geq \frac{1}{3}\sum_{i=1}^{3} x^*(x_i) > 1 - \eta(\varepsilon).$$

Thus there exist $A \subset \{1,2,3\}$, $\{z_i : i \leq 3\} \subset S_X$ and $z^* \in S_{X^*}$ such that

(i) $\sum_{i\in A} \alpha_i > 1 - \varepsilon > 2/3$,

(ii) $\|z_i - x_i\| < \varepsilon$ for all i,

(iii) $z^*(z_i) = 1$ for all $i \in A$,

(iv) $\|z_1 + z_2 - z_3\| \leq 1$.

In view of condition (i) we have $A = \{1,2,3\} = C$ and so X satisfies the AHSP-ℓ_∞^3.

Case 2: C has two elements. For instance, if $C = \{1,2\}$, we take $\alpha_1 = \alpha_2 = 1/2$ and $\alpha_3 = 0$. We have

$$\Big\|\sum_{i=1}^{3} \alpha_i x_i\Big\| \geq \frac{1}{2}\sum_{i=1}^{2} x^*(x_i) > 1 - \eta(\varepsilon).$$

Therefore, there exist $A \subset \{1,2,3\}$, $(z_i)_{i=1}^3 \subset S_X$ and $y^* \in S_{X^*}$ such that

(i) $\sum_{i\in A} \alpha_i > 1 - \varepsilon > 2/3$,

(ii) $\|z_i - x_i\| < \varepsilon$ for all i,

(iii) $y^*(z_i) = 1$ for all $i \in A$,

(iv) $\|z_1 + z_2 - z_3\| \leq 1$.

Again, by (i) we have $A \supset \{1,2\} = C$ and the condition in Definition 2.1 is satisfied.

If $C = \{1,3\}$ or $C = \{2,3\}$ we can argue in a similar way.

Case 3: $C = \{k\}$. We take $\alpha_k = 1$ and $\alpha_i = 0$ for $i \in \{1,2,3\} \setminus \{k\}$. It is straightforward that analogous arguments to those above show that X has the approximate hyperplane sum property for ℓ_∞^3.

Let us remark that we have checked that the function $\delta(\varepsilon) = \eta(\varepsilon)$ satisfies the condition of Definition 2.1, where η is the function appearing in condition (3). □

Recall that a Banach space X is *uniformly convex* if for every $\varepsilon > 0$ there is $0 < \delta < 1$ such that

$$u, v \in B_X,\ \frac{\|u+v\|}{2} > 1 - \delta \ \Rightarrow\ \|u - v\| < \varepsilon.$$

In that case, the *modulus of convexity* of X is the function defined by

$$\delta(\varepsilon) := \inf\{1 - \|u+v\|/2 : u, v \in B_X, \|u - v\| \geq \varepsilon\}.$$

Proposition 2.3. *Every uniformly convex space has the approximate hyperplane sum property for ℓ_∞^3.*

Proof. Let X be uniformly convex with modulus of convexity δ. Given $0<\varepsilon<1$, take $0<t=\min\{\delta(\varepsilon)/2, 2\varepsilon/3\}$. Assume that $\{x_i : i\le 3\}\subset B_X$ with $\|x_1+x_2-x_3\|\le 1$, $\emptyset\ne C\subset\{1,2,3\}$ and $x^*\in S_{X^*}$ are such that $x^*(x_i)>1-t$ for every $i\in C$. Then for all $i,j\in C$ we have

$$1-\frac{\delta(\varepsilon)}{2}\le 1-t<x^*\left(\frac{x_i+x_j}{2}\right)\le\left\|\frac{x_i+x_j}{2}\right\|.$$

As a consequence $\|x_i-x_j\|<\varepsilon$ for all $i,j\in C$.

If $C=\{1,2,3\}$, choose $i_0\in C$ and take $z_i=x_{i_0}/\|x_{i_0}\|$ for every $i\in C$. By the definition of the modulus of convexity it follows that $\|z_i-x_i\|<\varepsilon$ for every $i\in C$.

Assume that $C=\{1,2\}$; then

$$2-2t-x^*(x_3)<x^*(x_1+x_2-x_3)\le\|x_1+x_2-x_3\|\le 1.$$

Thus

$$1-\delta(\varepsilon)\le 1-2t<x^*(x_3).$$

By using again the fact that X is uniformly convex, if we take $z_3=x_{i_0}/\|x_{i_0}\|$, all the requirements in Definition 2.1 are satisfied.

If $C=\{1,3\}$, then it suffices to take $z_2=x_2$ and $z_1=z_3=x_1/\|x_1\|$. The case $C=\{2,3\}$ is analogous. Finally, consider the case where C contains only one element, say $C=\{1\}$. In that case we take $z_1=x_1/\|x_1\|$. Hence

$$\|z_1+x_2-x_3\|\le 1+\|x_1\|\left(\frac{1}{\|x_1\|}-1\right)<1+t.$$

If $\|z_1+x_2-x_3\|\le 1$ it suffices to take $z_i=x_i$ for $i=2,3$. Otherwise, we write $s=\|z_1+x_2-x_3\|-1$ and we know that $0<s<t\le 2\varepsilon/3$. We take $a=\frac{s}{1+s}, b=\frac{s}{2(1+s)}$ and $z_2=(1-a)x_2-bz_1, z_3=(1-a)x_3+bz_1$. We have

$$\|z_i\|\le 1-a+b=1-\frac{s}{1+s}+\frac{s}{2(1+s)}\le 1,\qquad i=2,3.$$

Moreover

$$\begin{aligned}\|z_1+z_2-z_3\|&=\|(1-2b)z_1+(1-a)(x_2-x_3)\|=(1-a)\|z_1+x_2-x_3\|\\&=\frac{1}{1+s}(1+s)=1.\end{aligned}$$

Finally, for $i=2,3$ we obtain

$$\|z_i-x_i\|\le a+b=\frac{s}{1+s}+\frac{s}{2(1+s)}=\frac{3s}{2(1+s)}<\frac{3t}{2}\le\varepsilon.$$

It is also clear that

$$\|z_1 - x_1\| = \left\| \frac{x_1}{\|x_1\|} - x_1 \right\| = 1 - \|x_1\| < 1 - t < \varepsilon.$$

Thus in each case we have proved all the conditions in Definition 2.1. □

Since $\mathbb{R}$ has the AHSP-ℓ_∞^3, from the next proposition it follows immediately that the space $\mathcal{C}_0(L)$ also has this property.

Proposition 2.4. *Let L be a nonempty locally compact Hausdorff topological space and X a Banach space. Then $\mathcal{C}_0(L, X)$ has the AHSP-ℓ_∞^3 if and only if X does.*

Proof. (1) Assume that X has the AHSP-ℓ_∞^3. Given $\varepsilon > 0$ assume that δ satisfies the condition of Definition 2.1 for $\varepsilon/2$. It suffices to show that (2) in Proposition 2.2 is satisfied for the set $B = \{x^* \circ \delta_t : t \in L,\ x^* \in S_{X^*}\} \subset S_{(\mathcal{C}_0(L,X))^*}$.

Assume that $\{f_i : i \leq 3\} \subset B_{\mathcal{C}_0(L,X)}$, $C \subset \{1, 2, 3\}$, $z^* \in S_{X^*}$ and $t_1 \in L$ satisfy

$$\|f_1 + f_2 - f_3\| \leq 1 \quad \text{and} \quad z^*(f_i(t_1)) > 1 - \delta \quad \forall i \in C.$$

By assumption there exist $\{z_i : i \leq 3\} \subset B_X$ and $x^* \in S_{X^*}$ satisfying

$$\|z_1 + z_2 - z_3\| \leq 1, \qquad \|z_i - f_i(t_1)\| < \varepsilon/2 \quad \forall i \leq 3, \qquad x^*(z_i) = 1 \quad \forall i \in C.$$

Consider the open neighborhood U of t_1 given by

$$U = \bigcap_{i=1}^{3} \{t \in L : \|f_i(t) - f_i(t_1)\| < \varepsilon/2\}.$$

By the Urysohn Lemma, there exists a function ϕ in $\mathcal{C}_0(L)$ whose support is contained in U, satisfying also $0 \leq \phi \leq 1$ and $\phi(t_1) = 1$. Now we define three elements in $\mathcal{C}_0(L, X)$ by

$$g_i = \phi z_i + (\mathbf{1} - \phi) f_i \quad (i \leq 3).$$

It is clear that $g_i(t_1) = z_i$ for every $i \leq 3$, and for every $t \in L$ we have

$$\|g_i(t)\| \leq |\phi(t)| + |1 - \phi(t)| = 1.$$

Moreover $g_i(t) = f_i(t)$ for any $t \in L \setminus U$ and $i \leq 3$. If $t \in U$, for every $i \leq 3$ we have

$$\begin{aligned}\|g_i(t) - f_i(t)\| &= \|(z_i - f_i(t))\phi(t)\| \leq \|z_i - f_i(t_1)\| + \|f_i(t_1) - f_i(t)\| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

Hence $\|g_i - f_i\| < \varepsilon$ for every $i \leq 3$.

Moreover,

$$\|g_1 + g_2 - g_3\| = \|\phi(z_1 + z_2 - z_3) + (\mathbf{1} - \phi)(f_1 + f_2 - f_3)\| \leq 1.$$

Since $(x^* \circ \delta_{t_1})(g_i) = 1$ for every $i \in C$, we have proved that $\mathcal{C}_0(L, X)$ has the AHSP-ℓ_∞^3.

(2) Assume now that $\mathcal{C}_0(L, X)$ has the AHSP-ℓ_∞^3. Given $\varepsilon > 0$, let δ satisfy the condition of Definition 2.1.

Suppose that $\{x_i : i \leq 3\} \subset B_X$ and for some $z^* \in S_{X^*}$ and $\emptyset \neq C \subset \{1, 2, 3\}$ we have

$$\|x_1 + x_2 - x_3\| \leq 1 \quad \text{and} \quad z^*(x_i) > 1 - \delta \quad \forall i \in C.$$

Since L is nonempty, we can choose $t_0 \in L$. Again, by the Urysohn Lemma, we can find a continuous function $\phi : L \to [0, 1]$ with compact support and $\phi(t_0) = 1$. Observe that if L is compact we can take the constant function $\mathbf{1}$ as ϕ.

Now we define $f_i \in B_{\mathcal{C}_0(L,X)}$ by $f_i(t) = \phi(t)x_i$ for any $i \leq 3$ and $t \in L$. It is clear that $\|f_1 + f_2 - f_3\| \leq 1$ and

$$(z^* \circ \delta_{t_0})(f_i) > 1 - \delta \qquad \forall i \in C.$$

By assumption there exists $\{g_i : i \leq 3\} \subset B_{\mathcal{C}_0(L,X)}$ such that

$$\|g_i - f_i\| < \varepsilon \text{ for every } i \leq 3, \qquad \|g_1 + g_2 - g_3\| \leq 1, \quad \text{and} \quad \Big\|\sum_{i \in C} g_i\Big\| = |C|.$$

Hence there is $t_1 \in L$ such that $|C| = \|\sum_{i \in C} g_i\| = \|\sum_{i \in C} g_i(t_1)\|$. Clearly $\|g_i(t_1)\| = 1$ for every i in C. On the other hand

$$\|f_i(t_1) - g_i(t_1)\| \leq \|f_i - g_i\| < \varepsilon \qquad \text{for every } i \leq 3.$$

For any $i \in C$, we have

$$1 - \varepsilon < \|f_i(t_1)\| = \phi(t_1)\|x_i\| \leq \phi(t_1).$$

Finally, we define $z_i = g_i(t_1)$ for $i \leq 3$, and so $\{z_i : i \leq 3\} \subset B_X$. For every $i \leq 3$ it is clear that

$$\begin{aligned}\|z_i - x_i\| &\leq \|(g_i - f_i)(t_1)\| + \|f_i(t_1) - x_i\| \\ &< \varepsilon + \|\phi(t_1)x_i - x_i\| \leq \varepsilon + 1 - \phi(t_1) < 2\varepsilon.\end{aligned}$$

Also

$$\|z_1 + z_2 - z_3\| \leq \|g_1 + g_2 - g_3\| \leq 1,$$

and

$$\Big\|\sum_{i \in C} z_i\Big\| = \Big\|\sum_{i \in C} g_i(t_1)\Big\| = |C|.$$

We have proved that X has the AHSP-ℓ_∞^3. $\square$

Remark 2.5. We do not know whether AHSP-ℓ_∞^3 is stable under injective tensor products. If it were, this would extend Proposition 2.4.

Our goal now is to show that every space $L_1(\mu)$ has the AHSP-ℓ_∞^3.

Proposition 2.6. *The space ℓ_1^n has the approximate hyperplane sum property for ℓ_∞^3 for every positive integer n. Actually, for every positive real $\varepsilon < 1$, the number $\delta(\varepsilon) = \varepsilon/48$ satisfies condition (2) in Proposition 2.2 for the set $\operatorname{Ext}(B_{(\ell_1^n)^*})$.*

Proof. We check the last statement. Let x_1, x_2, x_3 be elements in the unit ball of ℓ_1^n with $\|x_1 + x_2 - x_3\| \le 1$ and such that there is $x^* \in \operatorname{Ext}(B_{\ell_\infty^n})$ fulfilling for some subset C of $\{1,2,3\}$ the condition $x^*(x_i) > 1 - \delta$ for every $i \in C$. By applying an isometry in ℓ_1^n, we can always assume that $x^*(e_k) = 1$ for every $k \le n$. Hence

$$\sum_{k=1}^{n} x_i(k) > 1 - \delta \quad \text{for every } i \in C, \quad \text{and} \quad \|x_1 + x_2 - x_3\| \le 1.$$

We are going to discuss the possible cases.

Case 1: $C = \{1,2,3\}$. First, additionally assume that every x_i is a nonnegative element in $B_{\ell_1^n}$ for $i \le 3$. We write

$$D = \big\{k \le n : x_3(k) \le x_1(k) + x_2(k)\big\} \quad \text{and} \quad E = \{1, \dots, n\} \setminus D.$$

Since

$$1 - 2\delta < \sum_{k=1}^{n} (x_1 + x_2 - x_3)(k) \le \sum_{k \in D} (x_1 + x_2 - x_3)(k) \le \|x_1 + x_2 - x_3\| \le 1,$$

we have $\|(x_1 + x_2 - x_3)\chi_E\| < 2\delta$.

Define $u_3 = x_3\chi_D + (x_1 + x_2)\chi_E$. Clearly $0 \le u_3 \le x_3$ and so $\|u_3\| \le 1$. Moreover

$$\|u_3 - x_3\| = \|(x_1 + x_2 - x_3)\chi_E\| < 2\delta \tag{2.1}$$

and

$$\|x_1 + x_2 - u_3\| = \|(x_1 + x_2 - x_3)\chi_D\| \le 1.$$

From the definition of u_3 it follows that

$$0 \le u_3 \le x_1 + x_2.$$

We write $a = \|x_1\|$, $b = \|x_2\|$ and $c = \|u_3\|$. Since $\|x_1 + x_2 - u_3\| \le 1$ and $x_1 + x_2 - u_3 \ge 0$ we have $a + b - c \le 1$, that is,

$$1 - c \le (1 - a) + (1 - b). \tag{2.2}$$

We define

$$z_1 = x_1 + (1 - a)e_1, \quad z_2 = x_2 + (1 - b)e_1, \quad z_3 = u_3 + (1 - c)e_1.$$

It is clear that $z_i \geq 0$ and $z_i \in S_{\ell_1^n}$ for every $i \leq 3$. Since $a, b > 1-\delta$ and $c > 1-2\delta$, we obtain

$$\|z_i - x_i\| < \delta \quad \text{for } i = 1, 2 \quad \text{and} \quad \|z_3 - u_3\| < 2\delta. \tag{2.3}$$

Since $u_3 \leq x_1 + x_2$, in view of (2.2) we obtain $z_3 \leq z_1 + z_2$. Hence

$$\|z_1 + z_2 - z_3\| = \sum_{k=1}^{n} (z_1 + z_2 - z_3)(k) = 1.$$

We also know that $\|z_i - x_i\| < \delta$ for $i = 1, 2$ and by using (2.3) and (2.1) we get

$$\|z_3 - x_3\| \leq \|z_3 - u_3\| + \|u_3 - x_3\| < 2\delta + 2\delta = 4\delta.$$

Thus we have proved the desired fact for nonnegative elements in ℓ_1^n.

Now we will prove that the condition in Definition 2.1 is satisfied in the case $C = \{1, 2, 3\}$ for any elements in ℓ_1^n. Assume that x_i for $i \leq 3$ are elements in $B_{\ell_1^n}$ satisfying

$$\sum_{k=1}^{n} x_i(k) > 1 - \delta \quad \text{and} \quad \|x_1 + x_2 - x_3\| \leq 1.$$

For $i \leq 3$ we denote

$$P_i = \big\{k \leq n : x_i(k) \geq 0\big\} \quad \text{and} \quad N_i = \{1, \ldots, n\} \setminus P_i.$$

By assumption, for each $i \leq 3$ we have

$$1 - \delta < \sum_{k=1}^{n} x_i(k) \leq \sum_{k \in P_i} x_i(k) \leq \|x_i\| \leq 1.$$

So $u_i = x_i \chi_{P_i}$ is nonnegative and satisfies $1 - \delta < \|u_i\| \leq 1$ and

$$\|u_i - x_i\| = \|x_i \chi_{N_i}\| \leq 1 - \|u_i\| < \delta \qquad \forall i \leq 3. \tag{2.4}$$

Hence

$$\|u_1 + u_2 - u_3\| \leq \|x_1 + x_2 - x_3\| + 3\delta \leq 1 + 3\delta.$$

Since $1 - 2\delta < \sum_{k=1}^{n} (u_1 + u_2 - u_3)(k) \leq \|u_1 + u_2 - u_3\| \leq 1 + 3\delta$, if we denote

$$D = \big\{k \leq n : u_3(k) < (u_1 + u_2)(k)\big\} \quad \text{and} \quad E = \{1, \ldots, n\} \setminus D,$$

then $\sum_{k \in D} (u_1 + u_2 - u_3)(k) > 1 - 2\delta$.

If $\|u_1 + u_2 - u_3\| \leq 1$, then we do not change the elements u_1, u_2 and finish the proof using the fact proved for positive elements. If $\|u_1 + u_2 - u_3\| > 1$ we write $d = \|u_1 + u_2 - u_3\| - 1$. We clearly have

$$d \leq (1 + 3\delta) - 1 = 3\delta < 1 - 2\delta < \sum_{k \in D} (u_1 + u_2 - u_3)(k). \tag{2.5}$$

As a consequence, we can find $v_i \in B_{\ell_1^n}$ for $i = 1, 2$ such that

$$
\begin{aligned}
&0 \le v_i\chi_D \le u_i\chi_D, \quad v_i\chi_E = u_i\chi_E, \\
&\|v_i - u_i\| \le d < 3\delta, \quad u_3\chi_D \le (v_1 + v_2)\chi_D
\end{aligned}
\tag{2.6}
$$

and also

$$\sum_{k\in D} |(v_1 + v_2 - u_3)(k)| = \sum_{k\in D} (v_1 + v_2 - u_3)(k) = \sum_{k\in D} (u_1 + u_2 - u_3)(k) - d.$$

From this equality we obtain

$$
\begin{aligned}
\|v_1 + v_2 - u_3\| &= \|(v_1 + v_2 - u_3)\chi_D\| + \|(v_1 + v_2 - u_3)\chi_E\| \\
&= \|(u_1 + u_2 - u_3)\chi_D\| - d + \|(u_1 + u_2 - u_3)\chi_E\| = \|u_1 + u_2 - u_3\| - d = 1.
\end{aligned}
$$

In view of (2.4) and (2.6) we can also obtain

$$\|v_1 - x_1\| \le \|v_1 - u_1\| + \|u_1 - x_1\| < 3\delta + \delta = 4\delta$$

and analogously $\|v_2 - x_2\| < 4\delta$. So we have

$$
\begin{aligned}
&\sum_{k=1}^n u_3(k) \ge \sum_{k=1}^n x_3(k) > 1 - \delta, \\
&\sum_{k=1}^n v_i(k) > \sum_{k=1}^n x_i(k) - \|v_i - x_i\| > 1 - 5\delta \qquad \text{for } i = 1, 2.
\end{aligned}
$$

Let us remark that v_1, v_2 and u_3 are nonnegative elements in $B_{\ell_1^n}$. By the fact proved for positive elements, there are $z_i \in S_{\ell_1^n}$ such that

$$
\begin{aligned}
&\|z_3 - u_3\| < 20\delta, \qquad \|z_1 + z_2 - z_3\| \le 1, \qquad \|z_1 + z_2 + z_3\| = 3, \\
&\|z_i - v_i\| < 20\delta \qquad \text{for } i = 1, 2.
\end{aligned}
$$

Finally, for $i = 1, 2$ we obtain

$$\|z_i - x_i\| \le \|z_i - v_i\| + \|v_i - x_i\| < 20\delta + 4\delta = 24\delta < \varepsilon. \tag{2.7}$$

By (2.4) we have

$$\|z_3 - x_3\| \le \|z_3 - u_3\| + \|u_3 - x_3\| < 20\delta + \delta = 21\delta < \varepsilon. \tag{2.8}$$

Hence the statement is proved if $C = \{1, 2, 3\}$.

Case 2: $|C| = 2$. We have either $C = \{1, 2\}$, $C = \{1, 3\}$ or $C = \{2, 3\}$. The last two cases are equivalent to each other. Hence we have only to check the first two cases.

If $C = \{1, 2\}$, we have $x^*(x_1) > 1 - \delta$ and $x^*(x_2) > 1 - \delta$ too. In this case, since $\|x_1 + x_2 - x_3\| \le 1$, we obtain $x^*(x_3) > 1 - 2\delta$. By the fact proved in Case 1, the inequalities (2.7) and (2.8) imply that there are $z_i \in B_X$ $(1 \le i \le 3)$ satisfying $\|z_i - x_i\| < 48\delta \le \varepsilon$ for every $i \le 3$, $\|\sum_{i=1}^3 z_i\| = 3$ and $\|z_1 + z_2 - z_3\| \le 1$.

If $C = \{1,3\}$, then $x^*(x_j) > 1 - \delta$ and so $\|x_j\| > 1 - \delta$ for $j = 1,3$. If we denote

$$P_j = \{k \le n : x_j(k) \ge 0\} \qquad (j = 1,3)$$

then $u_j = x_j \chi_{P_j}$ is nonnegative and satisfies $1 - \delta < \|u_j\| \le 1$ and $\|u_j - x_j\| < \delta$ for $j = 1,3$. For $j \in C$, set $v_j = u_j + (1 - \|u_j\|)e_1$. We have $\|v_j - x_j\| < 2\delta < \varepsilon$, $\|v_j\| = 1$ for $j = 1,3$ and $x^*(v_1 + v_3) = 2$. If $\|v_1 + x_2 - v_3\| \le 1$ we are done simply by taking $z_j = v_j$ for $j = 1,3$ and $z_2 = x_2$.

If $a = \|v_1 + x_2 - v_3\| > 1$, then

$$\left\| \frac{v_1}{a} + \frac{x_2}{a} - \frac{v_3}{a} \right\| = 1,$$

and

$$a = \|v_1 + x_2 - v_3\| \le \|x_1 + x_2 - x_3\| + \|x_1 - v_1\| + \|x_3 - v_3\| < 1 + 4\delta.$$

Let $z_j = v_j/a + (1 - 1/a)e_1$ for $j = 1,3$, and $z_2 = x_2/a$. Clearly $\|z_j\| = 1$ for $j = 1,3$, $x^*(z_1 + z_3) = 2$ and $\|z_2\| \le 1$. Moreover

$$\|z_1 + z_2 - z_3\| = \left\| \frac{v_1}{a} + \frac{x_2}{a} - \frac{v_3}{a} \right\| = 1.$$

For $j = 1,3$ we have $\|v_j/a - v_j\| = 1 - 1/a$ and so

$$\|z_j - x_j\| \le \left\| z_j - \frac{v_j}{a} \right\| + \left\| \frac{v_j}{a} - v_j \right\| + \|v_j - x_j\| < 2\left(1 - \frac{1}{a}\right) + 2\delta < 10\delta < \varepsilon,$$

and the conclusion follows in this case too.

Case 3: C is a singleton; we can clearly assume $C = \{1\}$. We will argue as in Case 2. If we assume $x^*(x_1) > 1 - \delta$ we take $v_1 = x_1\chi_{P_1} + (1 - \|x_1\chi_{P_1}\|)e_1$. Then v_1 is a nonnegative element of the sphere of ℓ_1^n such that $\|v_1 - x_1\| < 2\delta < \varepsilon$. If $\|v_1 + x_2 - x_3\| \le 1$ we get the conclusion. On the other hand, if $a = \|v_1 + x_2 - x_3\| > 1$, we have $a < 1 + 2\delta$, and so $1 - 1/a < 2\delta$. Then by taking $z_1 = v_1/a + (1 - 1/a)e_1$, $z_2 = x_2/a$ and $z_3 = x_3/a + (1 - 1/a)e_1$, we find that $z_i \in B_{\ell_1^n}$ for every $i \le 3$, $x^*(z_1) = 1$,

$$\|z_1 + z_2 - z_3\| = \frac{1}{a}\|v_1 + x_2 - x_3\| = 1$$

and also

$$\|z_1 - x_1\| < 6\delta < \varepsilon, \qquad \|z_2 - x_2\| < 2\delta < \varepsilon, \qquad \|z_3 - x_3\| < 4\delta < \varepsilon.$$

We have proved that ℓ_1^n satisfies the AHSP-ℓ_∞^3 with $\delta(\varepsilon) = \varepsilon/48$. $\square$

Theorem 2.7. *Let X be a Banach space such that $X = \overline{\bigcup\{Y_\alpha : \alpha \in \Lambda\}}$, where $\{Y_\alpha : \alpha \in \Lambda\}$ is a nested family of subspaces of X satisfying the AHSP-ℓ_∞^3 with the same function δ. Then X has the AHSP-ℓ_∞^3.*

Proof. By assumption and the proof of Proposition 2.2, we can assume that there is a function η satisfying (3) in Proposition 2.2 for each subspace Y_α.

Assume that $\{x_i : i \leq 3\} \subset B_X$ satisfies $\|x_1 + x_2 - x_3\| \leq 1$ and also for some element in the convex hull of $\{x_i : i \leq 3\}$ we have

$$\Big\|\sum_{i=1}^{3} \alpha_i x_i\Big\| > 1 - \eta\Big(\frac{\varepsilon}{2}\Big).$$

Choose $t > 0$ such that

$$t < \frac{1}{4}\min\bigg\{\frac{\varepsilon}{2}, \Big\|\sum_{i=1}^{3} \alpha_i x_i\Big\| - 1 + \eta\Big(\frac{\varepsilon}{2}\Big)\bigg\}.$$

Since X is the closure of a nested family of subspaces, there exist $Y = Y_\alpha \subset X$ and $y_i \in B_Y$ for $i \leq 3$ satisfying

$$\|y_i - x_i\| < t \qquad \text{for all } i \leq 3.$$

Hence we have

$$\Big\|\sum_{i=1}^{3} \alpha_i y_i\Big\| > \Big\|\sum_{i=1}^{3} \alpha_i x_i\Big\| - \max\{\|x_i - y_i\| : i \leq 3\} > 1 - \eta\Big(\frac{\varepsilon}{2}\Big).$$

If additionally $\|y_1 + y_2 - y_3\| \leq 1$, then we are done since Y satisfies the AHSP-ℓ_∞^3.

Otherwise,

$$1 < a := \|y_1 + y_2 - y_3\| \leq \|x_1 + x_2 - x_3\| + \sum_{i=1}^{3} \|x_i - y_i\| < 1 + 3t$$

and so

$$\Big\|\frac{y_1}{a} + \frac{y_2}{a} - \frac{y_3}{a}\Big\| = 1.$$

Since for every $i \leq 3$ we have

$$\Big\|y_i - \frac{y_i}{a}\Big\| \leq 1 - \frac{1}{a} < 3t,$$

it follows that

$$\Big\|x_i - \frac{y_i}{a}\Big\| \leq \|x_i - y_i\| + \Big\|y_i - \frac{y_i}{a}\Big\| < t + 3t = 4t < \frac{\varepsilon}{2}.$$

We also have

$$\left\|\sum_{i=1}^{3}\alpha_i\frac{y_i}{a}\right\| \geq \left\|\sum_{i=1}^{3}\alpha_i x_i\right\| - \max\left\{\left\|x_i-\frac{y_i}{a}\right\| : i\leq 3\right\} > 1-4t > 1-\eta\left(\frac{\varepsilon}{2}\right).$$

By Proposition 2.2 there are $A\subset\{1,2,3\}$, $z_i\in B_Y$ and $y^*\in S_{Y^*}$ such that

(i) $\sum_{i\in A}\alpha_i > 1-\varepsilon/2$,
(ii) $\|z_i - y_i/a\| < \varepsilon/2$ for every $i\leq 3$,
(iii) $y^*(z_i)=1$ for all $i\in A$,
(iv) $\|z_1+z_2-z_3\|\leq 1$.

Hence for every $i\leq 3$ we also obtain

$$\|z_i-x_i\| \leq \left\|z_i-\frac{y_i}{a}\right\| + \left\|\frac{y_i}{a}-x_i\right\| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

If we consider a Hahn–Banach extension x^* of the functional y^* to X, our conclusion follows. By Proposition 2.2, X has the AHSP-ℓ_∞^3. □

Let us remark that the subspace of $L_1(\mu)$ generated by a finite number of characteristic functions is linearly isometric to some space ℓ_1^n. Since the space of simple functions is dense in $L_1(\mu)$, in view of Proposition 2.6, we conclude that $L_1(\mu)$ satisfies the assumption of the previous result. Hence we deduce the following assertion.

Proposition 2.8. *The space $L_1(\mu)$ has the approximate hyperplane sum property for ℓ_∞^3 for any positive measure μ.*

If Y has the AHSP-ℓ_∞^3, then it is immediate that Y has the *approximate hyperplane series property* for convex combinations of two elements. By the proof of [2, Theorem 4.1] the pair (ℓ_1^2, Y) has the BPBp. Hence so does (ℓ_∞^2, Y). Now we prove the main result.

Theorem 2.9. *(ℓ_∞^3, Y) has the BPBp if and only if Y has the AHSP-ℓ_∞^3.*

Proof. Given $0<\varepsilon<1$, assume that (ℓ_∞^3, Y) has the BPBp with a function $\eta(\varepsilon)$. We choose $0<\delta<\varepsilon/6$.

Let $\{y_k : k\leq 3\}$ be a subset of B_Y and consider nonnegative real numbers $\{\alpha_k : 1\leq k\leq 3\}$ with $\sum_{k=1}^{3}\alpha_k=1$ satisfying $\|y_1+y_2-y_3\|\leq 1$ and

$$\left\|\sum_{k=1}^{3}\alpha_k y_k\right\| > 1-\min\{\delta,\eta(\delta)\}.$$

We write $u_1 = (-1,1,1)$, $u_2 = (1,-1,1)$, $u_3 = (1,1,1)$ and $u_4 = (-1,-1,1) = u_1 + u_2 - u_3$ and

$$x = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = (-\alpha_1 + \alpha_2 + \alpha_3, \alpha_1 - \alpha_2 + \alpha_3, 1).$$

Consider the linear operator $T : \ell_\infty^3 \to Y$ that satisfies $T(u_i) = y_i$ for every $i \leq 3$. Since the unit ball of ℓ_∞^3 is the absolutely convex hull of $\{u_i : i \leq 4\}$, by assumption we know that $0 < \|T\| \leq 1$, and also

$$\left\| \frac{Tx}{\|T\|} \right\| \geq \|Tx\| > 1 - \min\{\delta, \eta(\delta)\}.$$

Hence there exist $S : \ell_\infty^3 \to Y$, $z \in S_{\ell_\infty^3}$ and $y^* \in S_{Y^*}$ such that

$$y^* S z = \|S\| = 1, \qquad \left\| S - \frac{T}{\|T\|} \right\| < \delta \quad \text{and} \quad \|z - x\| < \delta$$

and so

$$\|z - x\| < 3\delta. \tag{2.9}$$

We now show that there exists a subset $A \subset \{1,2,3\}$ such that the functional y^* and the elements $v_i = S(u_i)$ for $i \leq 3$ satisfy condition (3) in Proposition 2.2. First, notice that for any $i \leq 3$ we have

$$\begin{aligned} \|v_i - y_i\| = \|S(u_i) - T(u_i)\| &\leq \left\| S(u_i) - \frac{T(u_i)}{\|T\|} \right\| + \left\| \frac{T(u_i)}{\|T\|} - T(u_i) \right\| \\ &< \delta + 1 - \|T\| < 2\delta < \varepsilon, \end{aligned}$$

so assertion (ii) of (3) in Proposition 2.2 is satisfied.

We will use the functionals $\{u_i^* : 1 \leq i \leq 3\} \subset S_{(\ell_\infty^3)^*}$ given by

$$u_1^* = \tfrac{1}{2}(-e_1^* + e_3^*), \qquad u_2^* = \tfrac{1}{2}(-e_2^* + e_3^*), \qquad u_3^* = \tfrac{1}{2}(e_1^* + e_2^*).$$

Clearly, $u_i^*(u_j) = \delta_i^j$ for any $i, j \leq 3$.

We claim that we may assume z is the convex combination $\sum_{i=1}^3 \beta_i u_i$.

Indeed, since we know that $z = (z_1, z_2, z_3)$ satisfies $z_3 \neq -1$, z is a convex combination of $(z_1, z_2, 1)$ and $(z_1, z_2, -1)$ and $\|(z_1, z_2, 1) - x\| \leq \|z - x\|$, so we may assume $z = (z_1, z_2, 1)$. Hence, we can write z as a convex combination $\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3$ or $z = \beta_1 u_1 + \beta_2 u_2 + \beta_4 u_4$, where $\beta_4 > 0$.

Assume that $\beta_4 > 0$. By using

$$\alpha_3 + \beta_4 = u_3^*(x - z) \leq \|x - z\| < \delta,$$

we obtain $\beta_4 < \delta$. Hence $u = \frac{\beta_1 u_1 + \beta_2 u_2}{\beta_1 + \beta_2}$ clearly satisfies $y^*(Su) = 1$ and also

$$\begin{aligned}\|u - x\| &\leq \|u - z\| + \|z - x\| \leq \|u - (\beta_1 u_1 + \beta_2 u_2)\| + \|\beta_4 u_4\| + \|z - x\| \\ &\leq \left(\frac{1}{\beta_1 + \beta_2} - 1\right)(\beta_1 + \beta_2) + \beta_4 + \delta = 2\beta_4 + \delta < 3\delta.\end{aligned}$$

So we can use u instead of z, and u belongs to the convex hull of $\{u_i : i \leq 2\}$ and satisfies the estimate (2.9).

Hence we may assume that $z = \sum_{i=1}^{3} \beta_i u_i$. We take

$$A = \{i \leq 3 : \beta_i \neq 0\}.$$

Since for every $i \leq 3$ we have

$$|\beta_i - \alpha_i| = |u_i^*(z - x)| \leq \|z - x\| < 3\delta,$$

we obtain $\alpha_i < 3\delta$ for every $i \in \{1, 2, 3\} \setminus A$ and so

$$\sum_{i \in A} \alpha_i > 1 - 6\delta > 1 - \varepsilon,$$

hence condition (i) in (3) of Proposition 2.2 is satisfied. Conditions (iii) and (iv) there also hold for the functional y^* and the vectors $v_k = S(u_k)$ $(1 \leq k \leq 3)$.

Assume now that Y has the AHSP-ℓ_∞^3. Assume that condition (3) in Proposition 2.2 is satisfied for a function $\eta(\varepsilon)$. Given $0 < \varepsilon < 1/2$, assume that $T \in S_{L(\ell_\infty^3, Y)}$ and $x \in S_{\ell_\infty^3}$ satisfy

$$\|Tx\| > 1 - \eta(\varepsilon).$$

Since x is in $S_{\ell_\infty^3}$, it is a convex combination of three extreme points of $B_{\ell_\infty^3}$ that belong to the same face of the unit ball. By composing with a convenient isometry, we may assume that these extreme points are $u_1 = (-1, 1, 1)$, $u_2 = (1, -1, 1)$ and $u_3 = (1, 1, 1)$. Assume that x is the convex combination

$$x = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3.$$

Since $T \in S_{L(\ell_\infty^3, Y)}$, it is clear that $y_1 = Tu_1$, $y_2 = Tu_2$ and $y_3 = Tu_3$ satisfy the assumption of condition (3) in Proposition 2.2. Hence there are $A \subset \{1, 2, 3\}$, $\{z_k : k \leq 3\} \subset B_Y$ and $y^* \in S_{Y^*}$ satisfying conditions (i) to (iv) there.

Consider the linear operator $S : \ell_\infty^3 \to Y$ satisfying $Su_1 = z_1$, $Su_2 = z_2$ and $Su_3 = z_3$. We already know that $\{z_k : 1 \leq k \leq 3\}$ is contained in B_Y. Moreover in view of condition (iv) the element $S(u_4) = z_1 + z_2 - z_3$ belongs to B_Y too. Thus, as above, $\|S\| \leq 1$. From condition (iii) we obtain

$$1 = y^*\left(S\left(\sum_{i \in A} \frac{\alpha_i}{\sum_{j \in A} \alpha_j} u_i\right)\right) \leq \left\|S\left(\sum_{i \in A} \frac{\alpha_i}{\sum_{j \in A} \alpha_j} u_i\right)\right\| \leq \|S\| \leq 1,$$

so S is in $S_{L(\ell_\infty^3,Y)}$ and attains its norm at $v = \sum_{i\in A} \frac{\alpha_i}{\sum_{j\in A}\alpha_j} u_i$. We also obtain

$$\|S - T\| < 3\varepsilon \quad \text{and} \quad \|v - x\| \leq 2\varepsilon.$$

Hence the pair (ℓ_∞^3, Y) has the Bishop–Phelps–Bollobás property for operators. □

In view of the previous characterization and known results providing pairs of Banach spaces satisfying the BPBp, we already know the following examples of spaces with the AHSP-ℓ_∞^3:

- Finite-dimensional spaces [2, Proposition 2.4].
- Spaces with the property β of Lindenstrauss [2, Theorem 2.2].

Let us remark that not every Banach space has the AHSP-ℓ_∞^3.

Proposition 2.10. *A strictly convex Banach space has the AHSP-ℓ_∞^3 if and only if it is uniformly convex.*

Proof. Assume that a Banach space X has the AHSP-ℓ_∞^3. Given $0 < \varepsilon < 1$, consider $x, y \in B_X$ such that $\|(x+y)/2\| > 1 - \eta(\varepsilon/2)$. We take $x_1 = x$, $x_2 = y$, $x_3 = 0$, $\alpha_1 = \alpha_2 = 1/2$ and $\alpha_3 = 0$. By assumption there exist $A \subset \{1,2,3\}$, $\{z_i : i \leq 3\} \subset B_X$ and $x^* \in S_{X^*}$ such that

(i) $\sum_{i\in A} \alpha_i > 1 - \varepsilon/2 > 1/2$,
(ii) $\|z_i - x_i\| < \varepsilon/2$ for all $i \leq 3$,
(iii) $x^*(z_i) = 1$ for all $i \in A$,
(iv) $\|z_1 + z_2 - z_3\| \leq 1$.

By (i) we have $A \supset \{1,2\}$. In view of (iii) we deduce that $z_1, z_2 \in S_X$ and $\|z_1 + z_2\| = 2$. As every point of S_X is an extreme point, we get $z_1 = z_2$. Hence, by (ii) we get $\|x - y\| < \varepsilon$. It follows that X is uniformly convex with modulus of convexity $\delta(\varepsilon) \geq \eta(\varepsilon/2)$. □

Very recently, in [10, Proposition 6], it has been shown that if Y is a strictly convex space without the approximation property, then there exist an infinite-dimensional subspace X of c_0 and a compact operator T from X into Y such that T cannot be approximated by norm attaining operators. This gives a wealth of couples (X, Y) failing the BPBp. Proposition 2.10 shows that it is possible to remove the hypothesis of infinite-dimensionality for X. More precisely, if Y is a strictly convex space which is not uniformly convex, then (ℓ_∞^3, Y) does not have the BPBp.

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