The Spaces of Analytic Functions on Open Subsets of $\mathbb{R}^{\mathbb{N}}$ and $\mathbb{C}^{\mathbb{N}}$

by

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Abstract

This paper is devoted to studying the space $\mathcal{A}(U)$ of all analytic functions on an open subset U of $\mathbb{R}^{\mathbb{N}}$ or $\mathbb{C}^{\mathbb{N}}$. It is proved that if U satisfies a weak condition (that will be called the 0-property), then every $f \in \mathcal{A}(U)$ depends only on a finite number of variables. Several topologies on $\mathcal{A}(U)$ are then studied: the compact-open topology, the τ_δ topology (already known in spaces of holomorphic functions) and a new one, defined by the inductive limit of the subspaces of analytic functions which only depend on a finite number of variables.

§1. Introduction

If E is a locally convex space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and $\mathcal{A}(U)$ denotes the space of all analytic functions from an open subset $U \subset E$ into $\mathbb{K}$, perhaps the most natural topology on $\mathcal{A}(U)$ is the compact-open topology τ_0, although other natural topologies can also be considered. One is the so called τ_δ topology, introduced independently by Cœuré [8] and Nachbin [18] in the seventies for spaces of holomorphic functions. Let us recall its definition. If $\mathcal{V} = (V_j)_{j=1}^{\infty}$ is an increasing countable open cover of U, let

$$(1) \qquad \mathcal{A}_{\mathcal{V}}(U) = \Big\{ f \in \mathcal{A}(U) : \sup_{x \in V_j} |f(x)| < \infty \text{ for all } j \in \mathbb{N} \Big\}.$$

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When $\mathbb{K} = \mathbb{C}$, we sometimes write $\mathcal{H}(U)$ and $\mathcal{H}_{\mathcal{V}}(U)$ instead of $\mathcal{A}(U)$ and $\mathcal{A}_{\mathcal{V}}(U)$. The subspace $\mathcal{A}_{\mathcal{V}}(U)$ is always endowed with the topology of uniform convergence on each V_j. It is known that $\mathcal{A}(U) = \bigcup_{\mathcal{V}} \mathcal{A}_{\mathcal{V}}(U)$, where $\mathcal{V}$ ranges over the family of all increasing countable open covers of U. The proof when E is complex appears in Dineen [9, Proposition 3.18], but it is also valid for real spaces because it only depends on the continuity of analytic functions. Then τ_{δ} denotes the topology on $\mathcal{A}(U)$ defined by the inductive limit of all subspaces $\mathcal{A}_{\mathcal{V}}(U)$ in the category of locally convex spaces:

$$(2) \qquad (\mathcal{A}(U), \tau_{\delta}) = \varinjlim_{\mathcal{V}} \mathcal{A}_{\mathcal{V}}(U).$$

That is, τ_{δ} is the strongest locally convex topology on $\mathcal{A}(U)$ such that the inclusion $\mathcal{A}_{\mathcal{V}}(U) \hookrightarrow \mathcal{A}(U)$ is continuous for every $\mathcal{V}$.

In several recent papers, the present authors and Richard Aron have proved that if U is an open subset of an infinite-dimensional Banach space, then the inductive limit in (2) is not countable. In fact, it is not possible to represent $\mathcal{A}(U)$ as a countable union of such spaces $\mathcal{A}_{\mathcal{V}}(U)$ (see [2], [3, Theorem 2] and [14, Theorem 9]). On the contrary, we prove in this paper that if $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$ and U belongs to a large class of open subsets of $\mathbb{K}^{\mathbb{N}}$, then $\mathcal{A}(U)$ can be written as a countable union of subspaces of type $\mathcal{A}_{\mathcal{V}}(U)$. In the complex case, we even see that $(\mathcal{H}(U), \tau_{\delta}) = \varinjlim_{k \in \mathbb{N}} \mathcal{H}_{\mathcal{V}_k}(U)$ for a specific sequence $\{\mathcal{V}_k\}_{k=1}^{\infty}$ of covers of U.

In addition to τ_{δ}, a new topology on $\mathcal{A}(U)$ is studied in this article. It is denoted by τ_{ℓ} and is defined by the inductive limit of all subspaces of analytic functions which only depend on a finite number of variables. That new topology is strictly stronger than τ_{δ} in the real case, but they coincide in the complex case and also on the spaces of polynomials on $\mathbb{R}^{\mathbb{N}}$. The metrizability and completeness of $(\mathcal{A}(U), \tau_{\ell})$ are also studied.

The reader is referred to the article of Bochnak and Siciak [7] for the definitions and main properties of polynomials and analytic functions on locally convex spaces. For a deep study of the τ_0 and the τ_{δ} topologies on spaces of holomorphic functions, see the book of Dineen [9].

§2. Open subsets of $\mathbb{K}^{\mathbb{N}}$ with the 0-property

Given $k \in \mathbb{N}$, let $\pi_k : \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^k$ and $\pi_k^* : \mathcal{A}(\pi_k(U)) \to \mathcal{A}(U)$ represent the mappings

$$\begin{aligned} \pi_k((x_n)_{n=1}^{\infty}) &= (x_1, \ldots, x_k) && \text{for } (x_n)_{n=1}^{\infty} \in \mathbb{K}^{\mathbb{N}}, \\ \pi_k^*(g) &= g \circ \pi_k && \text{for } g \in \mathcal{A}(\pi_k(U)). \end{aligned}$$

Nachbin proved that a holomorphic function on a balanced subset of $\mathbb{C}^{\mathbb{N}}$ depends only on a finite number of variables:

Theorem 2.1 (Nachbin). *If U is a balanced open subset of $\mathbb{C}^{\mathbb{N}}$ and $f \in \mathcal{H}(U)$, then there are $k \in \mathbb{N}$ and $\widehat{f} \in \mathcal{H}(\pi_k(U))$ such that $f = \widehat{f} \circ \pi_k$ on U.*

For the proof of Theorem 2.1, see Dineen [9, p. 162]. We recall that a subset U of a vector space over the field $\mathbb{K}$ is said to be *balanced* if it has the following property: if $x \in U$, $\lambda \in \mathbb{K}$ and $|\lambda| \leq 1$, then $\lambda x \in U$. The conclusion of Theorem 2.1 does not hold in general if U does not have that property: counter-examples are due to Hirschowitz [10, p. 222] and Nachbin [19, Example 10].

In this paper, we will prove Nachbin's Theorem for a new class of open subsets of $\mathbb{K}^{\mathbb{N}}$, not necessarily balanced.

Definition 2.2. A subset U of $\mathbb{K}^{\mathbb{N}}$ is said to have the 0-*property* for some $d \in \mathbb{N}$ if $0 \in U$ and $(x_1, \ldots, x_k, 0, 0, \ldots) \in U$ for all $(x_n)_{n=1}^{\infty} \in U$ and all $k \geq d$.

For example, if V is an open neighborhood of 0 in $\mathbb{K}^d$, then the set

$$(3) \qquad\qquad U = V \times \mathbb{K} \times \mathbb{K} \times \cdots$$

is open in $\mathbb{K}^{\mathbb{N}}$ and has the 0-property for d. Moreover, U is balanced if and only if V is balanced. There are open subsets with the 0-property that are not a product as in (3). For instance, the set

$$U = \bigcup_{m=1}^{\infty} \{(x_n)_{n=1}^{\infty} \in \mathbb{K}^{\mathbb{N}} : |x_m| < 1\}$$

is open, balanced and has the 0-property for every $d \in \mathbb{N}$. The set

$$U = \bigcup_{m=1}^{\infty} \{(x_n)_{n=1}^{\infty} \in \mathbb{K}^{\mathbb{N}} : \mathrm{Re}(x_m) < 1\}$$

is open and has the 0-property for every $d \in \mathbb{N}$, but it is not balanced.

If U is any open subset with the 0-property for some $d \in \mathbb{N}$, and if $k \in \mathbb{N}$ and $k \geq d$, then the function

$$T : (x_1, \ldots, x_k) \in \mathbb{K}^k \mapsto (x_1, \ldots, x_k, 0, 0, \ldots) \in \mathbb{K}^{\mathbb{N}}$$

is a continuous linear mapping such that $T(\pi_k(U)) \subset U$. Hence if $f \in \mathcal{A}(U)$, then the function

$$\widehat{f}(x_1, \ldots, x_k) = f(x_1, \ldots, x_k, 0, 0, \ldots)$$

is well defined and it is analytic on $\pi_k(U)$. This fact will be used several times in what follows.

Throughout the paper, we will frequently apply the Identity Theorem. This result is well known for holomorphic functions and it was also explicitly stated by Bochnak and Siciak [7, Proposition 6.6] for real or complex analytic functions on locally convex spaces. However, the proof that they present is not clear for us and we have not found any other precise reference for real analytic functions on non-normed spaces. For completeness, we present a proof which is analogous to the one given by Mujica [17, Proposition 5.7].

Lemma 2.3. *Let E be a locally convex space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let U be a connected open subset of E. If $f \in \mathcal{A}(U)$, V is a non-empty open subset of U and $f = 0$ on V, then $f = 0$ on U.*

Proof. We first assume that U is convex. Given $x \in U$, we consider a point $a \in V$ and define

$$\Lambda = \{\lambda \in \mathbb{K} : a + \lambda(x - a) \in U\}, \quad \Omega = \{\lambda \in \mathbb{K} : a + \lambda(x - a) \in V\}.$$

Since U is open and convex and $x \in U$, the set Λ is also open and convex and $1 \in \Lambda$. Moreover, Ω is open and non-empty because $0 \in \Omega$. The function

$$g(\lambda) = f(a + \lambda(x - a))$$

is analytic on the connected domain Λ and $g = 0$ on $\Omega \subset \Lambda$. By the Identity Theorem for analytic functions of one variable, we deduce that $g = 0$ on Λ. Therefore, $f(x) = g(1) = 0$. This shows that $f = 0$ on U.

In the general case, let us define

$$A = \{a \in U : f = 0 \text{ on a neighborhood of } a\}.$$

The set A is open and non-empty because $V \subset A$. To deduce that $A = U$, we only have to show that A is closed in U. Let (a_i) be a net in A which converges to $b \in U$. Let U_b be a convex open neighborhood of b such that $U_b \subset U$. There is an index i such that $a_i \in U_b$. As $a_i \in A$, there is a neighborhood V_{a_i} of a_i such that $V_{a_i} \subset U_b$ and $f = 0$ on V_{a_i}. As has been proved in the convex case, it follows that $f = 0$ on U_b and thus $b \in A$. Therefore, A is closed in U and so $A = U$. $\square$

Lemma 2.4. *Let U be a connected open subset of $\mathbb{C}^{\mathbb{N}}$ with the 0-property for some $d \in \mathbb{N}$. Let $\mathcal{F}$ be a subset of $\mathcal{H}(U)$ with the following property: there is a neighborhood of zero V contained in U such that*

$$(4) \qquad\qquad \sup\{|f(z)| : z \in V, f \in \mathcal{F}\} < \infty.$$

Then there is $k \in \mathbb{N}$, $k \geq d$, such that $\mathcal{F} \subset \pi_k^*(\mathcal{H}(\pi_k(U)))$.

Proof. We can assume that

$$V = \{(z_n)_{n=1}^{\infty} \in \mathbb{C}^{\mathbb{N}} : |z_1| < \varepsilon, \dots, |z_k| < \varepsilon\}$$

for some $k \geq d$ and $\varepsilon > 0$. Let $f \in \mathcal{F}$ and $z = (z_n)_{n=1}^{\infty} \in V$. Let

$$w = (\underbrace{0, \dots, 0}_{k}, z_{k+1}, z_{k+2}, \dots).$$

Then $z + \lambda w \in V \subset U$ for all $\lambda \in \mathbb{C}$, so the function of one complex variable

$$h(\lambda) = f(z + \lambda w)$$

is entire. By (4), h is bounded, so by Liouville's Theorem it is constant. Then

$$(5) \qquad\qquad f(z) = h(0) = h(-1) = f(z_1, \dots, z_k, 0, 0, \dots)$$

for every $z \in V$. As U has the 0-property for d and $k \geq d$, we can define the following holomorphic function on $\pi_k(U)$:

$$\widehat{f}(z_1, \dots, z_k) = f(z_1, \dots, z_k, 0, 0, \dots).$$

By (5), we know that $f = \widehat{f} \circ \pi_k$ on V and hence $f = \widehat{f} \circ \pi_k$ on U by the Identity Theorem. This proves that $\mathcal{F} \subset \pi_k^*(\mathcal{H}(\pi_k(U)))$. $\qquad\square$

Theorem 2.5. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let U be a connected open subset of $\mathbb{K}^{\mathbb{N}}$ with the 0-property for some $d \in \mathbb{N}$. If $f \in \mathcal{A}(U)$, then there are $k \in \mathbb{N}$, $k \geq d$, and $\widehat{f} \in \mathcal{A}(\pi_k(U))$ such that $f = \widehat{f} \circ \pi_k$.*

Proof. If $\mathbb{K} = \mathbb{C}$, the result is a particular case of Lemma 2.4 with $\mathcal{F} = \{f\}$. Therefore, we will suppose that $\mathbb{K} = \mathbb{R}$. If f is an analytic function on U, then there are an open subset $U_{\mathbb{C}} \subset \mathbb{C}^{\mathbb{N}}$ and a holomorphic function $f_{\mathbb{C}} : U_{\mathbb{C}} \to \mathbb{C}$ such that $U \subset U_{\mathbb{C}}$ and $f_{\mathbb{C}} = f$ on U (see Bochnak and Siciak [7, Theorem 7.1]). Since $0 \in U$, there are $m \in \mathbb{N}$ and $\varepsilon > 0$ such that

$$W = \{(x_n)_{n=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} : |x_1| < \varepsilon, \dots, |x_m| < \varepsilon\} \subset U,$$
$$V = \{(z_n)_{n=1}^{\infty} \in \mathbb{C}^{\mathbb{N}} : |z_1| < \varepsilon, \dots, |z_m| < \varepsilon\} \subset U_{\mathbb{C}}.$$

The set V has the 0-property for d. As has been proved in the complex case, there are $k \in \mathbb{N}$, $k \geq d$, and $\widehat{f_{\mathbb{C}}} \in \mathcal{H}(\pi_k(V))$ such that $f_{\mathbb{C}} = \widehat{f_{\mathbb{C}}} \circ \pi_k$ on V. Let $\widehat{f} \in \mathcal{A}(\pi_k(U))$ be the function

$$\widehat{f}(x_1, \dots, x_k) = f(x_1, \dots, x_k, 0, 0, \dots).$$

If $(x_n)_{n=1}^\infty \in W \subset V$, then also $(x_1, \ldots, x_k, 0, 0, \ldots) \in W \subset V$, so

$$
\begin{aligned}
f((x_n)_{n=1}^\infty) = f_{\mathbb{C}}((x_n)_{n=1}^\infty) &= \widehat{f_{\mathbb{C}}}(x_1, \ldots, x_k) \\
&= f_{\mathbb{C}}(x_1, \ldots, x_k, 0, 0, \ldots) = f(x_1, \ldots, x_k, 0, 0, \ldots) \\
&= \widehat{f}(x_1, \ldots, x_k) = (\widehat{f} \circ \pi_k)((x_n)_{n=1}^\infty).
\end{aligned}
$$

Hence the analytic functions f and $\widehat{f} \circ \pi_k$ agree on the open subset W of U. Then $f = \widehat{f} \circ \pi_k$ on U by the Identity Theorem. $\qquad\square$

§3. Topologies on $\mathcal{A}(U)$

It was proved in Theorem 2.5 that if U is a connected open subset of $\mathbb{K}^{\mathbb{N}}$ with the 0-property for some $d \in \mathbb{N}$, then

$$
\mathcal{A}(U) = \bigcup_{k=d}^\infty \pi_k^*(\mathcal{A}(\pi_k(U))).
$$

Thus, we can introduce a new topology on $\mathcal{A}(U)$. Let τ_ℓ denote the topology on $\mathcal{A}(U)$ defined by the inductive limit of the subspaces $(\pi_k^*(\mathcal{A}(\pi_k(U))), \tau_0)$ in the category of locally convex spaces:

$$
(\mathcal{A}(U), \tau_\ell) = \varinjlim_{k \geq d} (\pi_k^*(\mathcal{A}(\pi_k(U))), \tau_0).
$$

That is, τ_ℓ is the strongest locally convex topology on $\mathcal{A}(U)$ such that the inclusion

$$
(\pi_k^*(\mathcal{A}(\pi_k(U))), \tau_0) \hookrightarrow \mathcal{A}(U)
$$

is continuous for every $k \geq d$.

In order to study the properties of τ_ℓ, we first have to introduce some notation and to prove a preliminary lemma. If U is an open subset of $\mathbb{K}^{\mathbb{N}}$ with the 0-property for some $d \in \mathbb{N}$, $k, j \in \mathbb{N}$ and $k \geq d$, we define the following sets:

(6) $\quad V_{k,j} = \{x \in U : |x_1| < j, \ldots, |x_k| < j, \operatorname{dist}(\pi_k(x), \mathbb{K}^k \setminus \pi_k(U)) > 1/j\}$,

(7) $\quad C_{k,j} = \{x \in \mathbb{K}^k : |x_1| \leq j, \ldots, |x_k| \leq j, \operatorname{dist}(x, \mathbb{K}^k \setminus \pi_k(U)) \geq 1/j\}$,

(8) $\quad L_{k,j} = \{x \in \mathbb{K}^{\mathbb{N}} : (x_1, \ldots, x_k) \in C_{k,j} \text{ and } x_n = 0 \text{ for all } n > k\}$.

Note that there is $j_k \in \mathbb{N}$ such that $V_{k,j} \neq \emptyset$, $C_{k,j} \neq \emptyset$ and $L_{k,j} \neq \emptyset$ for every $j \geq j_k$.

The sequence $\mathcal{V}_k = (V_{k,j})_{j=1}^\infty$ is an increasing countable open cover of U. Moreover, every $C_{k,j}$ is a compact subset of $\pi_k(U)$. Finally, $L_{k,j}$ is a compact subset of U. Indeed, if $(x_1, \ldots, x_k, 0, 0, \ldots) \in L_{k,j}$, then $(x_1, \ldots, x_k) \in C_{k,j} \subset \pi_k(U)$. Hence there is $x \in U$ such that $\pi_k(x) = (x_1, \ldots, x_k)$. Since U has the 0-property for d and $k \geq d$, we deduce that $(x_1, \ldots, x_k, 0, 0, \ldots) \in U$, that is, $L_{k,j} \subset U$.

Lemma 3.1. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let U be a connected open subset of $\mathbb{K}^{\mathbb{N}}$ with the 0-property for some $d \in \mathbb{N}$. Let $k \in \mathbb{N}$, $k \geq d$.*

(a) *If $\mathcal{V}_k = (V_{k,j})_{j=1}^{\infty}$ is the increasing countable open cover of U defined in (6), then $\pi_k^*(\mathcal{A}(\pi_k(U))) \subset \mathcal{A}_{\mathcal{V}_k}(U)$.*

(b) *The space $(\pi_k^*(\mathcal{A}(\pi_k(U))), \tau_0)$ is metrizable.*

(c) *$\pi_k^*(\mathcal{A}(\pi_k(U))) \subset \pi_{k+1}^*(\mathcal{A}(\pi_{k+1}(U)))$.*

(d) *$\pi_k^*(\mathcal{A}(\pi_k(U)))$ is closed in $(\mathcal{A}(U), \tau_0)$ and hence in $(\pi_{k+1}^*(\mathcal{A}(\pi_{k+1}(U))), \tau_0)$.*

Proof. If $f \in \pi_k^*(\mathcal{A}(\pi_k(U)))$, there is $\widehat{f} \in \mathcal{A}(\pi_k(U))$ such that $f = \widehat{f} \circ \pi_k$. Then

$$\sup_{x \in V_{k,j}} |f(x)| = \sup\{|\widehat{f}(x_1, \ldots, x_k)| : (x_1, \ldots, x_k) \in C_{k,j}\} = \sup_{x \in L_{k,j}} |f(x)| < \infty$$

for every $j \in \mathbb{N}$ (note that $C_{k,j} = \overline{\pi_k(V_{k,j})}$ and $L_{k,j}$ is a compact subset of U). Therefore, $f \in \mathcal{A}_{\mathcal{V}_k}(U)$, which proves that $\pi_k^*(\mathcal{A}(\pi_k(U))) \subset \mathcal{A}_{\mathcal{V}_k}(U)$.

If L is a compact subset of U, then $\pi_k(L)$ is compact in $\pi_k(U)$ and there is $j \in \mathbb{N}$ such that $\pi_k(L) \subset C_{k,j}$. Therefore, if $f \in \pi_k^*(\mathcal{A}(\pi_k(U)))$, then

$$\sup_{x \in L} |f(x)| = \sup\{|\widehat{f}(x_1, \ldots, x_k)| : (x_1, \ldots, x_k) \in \pi_k(L)\}$$

$$\leq \sup\{|\widehat{f}(x_1, \ldots, x_k)| : (x_1, \ldots, x_k) \in C_{k,j}\} = \sup_{x \in L_{k,j}} |f(x)|.$$

Consequently, the compact-open topology on $\pi_k^*(\mathcal{A}(\pi_k(U)))$ is defined by the sequence of seminorms $\sup_{x \in L_{k,j}} |f(x)|$, where $j \in \mathbb{N}$. This implies that the space $(\pi_k^*(\mathcal{A}(\pi_k(U))), \tau_0)$ is metrizable.

If $f \in \pi_k^*(\mathcal{A}(\pi_k(U)))$, $\widehat{f} \in \mathcal{A}(\pi_k(U))$ and $f = \widehat{f} \circ \pi_k$, let $g \in \mathcal{A}(\pi_{k+1}(U))$ be the following function:

$$g(x_1, \ldots, x_{k+1}) = \widehat{f}(x_1, \ldots, x_k).$$

If $(x_n)_{n=1}^{\infty} \in U$, then

$$f((x_n)_{n=1}^{\infty}) = \widehat{f}(x_1, \ldots, x_k) = g(x_1, \ldots, x_{k+1}) = g \circ \pi_{k+1}((x_n)_{n=1}^{\infty}).$$

Therefore, $f = g \circ \pi_{k+1} \in \pi_{k+1}^*(\mathcal{A}(\pi_{k+1}(U)))$. This proves that $\pi_k^*(\mathcal{A}(\pi_k(U))) \subset \pi_{k+1}^*(\mathcal{A}(\pi_{k+1}(U)))$.

Let us see that each $\pi_k^*(\mathcal{A}(\pi_k(U)))$ is closed in $(\mathcal{A}(U), \tau_0)$. Let (f_α) be a net in $\pi_k^*(\mathcal{A}(\pi_k(U)))$ that converges to a function $f \in \mathcal{A}(U)$ uniformly on compact subsets of U. For each index α there is $\widehat{f_\alpha} \in \mathcal{A}(\pi_k(U))$ such that $f_\alpha = \widehat{f_\alpha} \circ \pi_k$. The function

$$\widehat{f}(x_1, \ldots, x_k) = f(x_1, \ldots, x_k, 0, 0, \ldots)$$

is analytic on $\pi_k(U)$ because U has the 0-property. If $(x_n)_{n=1}^\infty \in U$, then also $(x_1, \ldots, x_k, 0, 0, \ldots) \in U$ and

$$\begin{aligned}
f((x_n)_{n=1}^\infty) &= \lim_\alpha f_\alpha((x_n)_{n=1}^\infty) = \lim_\alpha \widehat{f_\alpha}(x_1, \ldots, x_k) \\
&= \lim_\alpha f_\alpha(x_1, \ldots, x_k, 0, 0, \ldots) = f(x_1, \ldots, x_k, 0, 0, \ldots) \\
&= \widehat{f}(x_1, \ldots, x_k) = \widehat{f} \circ \pi_k((x_n)_{n=1}^\infty).
\end{aligned}$$

Hence, $f = \widehat{f} \circ \pi_k \in \pi_k^*(\mathcal{A}(\pi_k(U)))$. Thus, $\pi_k^*(\mathcal{A}(\pi_k(U)))$ is closed in $(\mathcal{A}(U), \tau_0)$. $\quad\square$

Theorem 3.2. *If U is a connected open subset of $\mathbb{K}^\mathbb{N}$ with the 0-property for some $d \in \mathbb{N}$, then $\tau_0 \leq \tau_\delta \leq \tau_\ell$ on $\mathcal{A}(U)$.*

Proof. The fact that $\tau_0 \leq \tau_\delta$ is well known in the case of spaces of holomorphic functions. If $\mathcal{W} = (W_j)_{j=1}^\infty$ is an increasing countable open cover of U and K is a compact subset of U, there is $j \in \mathbb{N}$ such that $K \subset W_j$ and so $\sup_{x \in K} |f(x)| \leq \sup_{x \in W_j} |f(x)|$ for every $f \in \mathcal{A}(U)$. This implies that the inclusion

$$\mathcal{A}_\mathcal{W}(U) \hookrightarrow (\mathcal{A}(U), \tau_0)$$

is continuous for every increasing countable open cover $\mathcal{W}$ of U. By the definition of τ_δ, we have $\tau_0 \leq \tau_\delta$.

For each $k \in \mathbb{N}$, $k \geq d$, let $\mathcal{V}_k = (V_{k,j})_{j=1}^\infty$ be the increasing countable open cover of U defined in (6). By Lemma 3.1, it is known that

$$\pi_k^*(\mathcal{A}(\pi_k(U))) \subset \mathcal{A}_{\mathcal{V}_k}(U).$$

The topology on $\mathcal{A}_{\mathcal{V}_k}(U)$ is defined by the sequence of seminorms

$$\sup_{x \in V_{k,j}} |f(x)| = \sup\{|\widehat{f}(x_1, \ldots, x_k)| : (x_1, \ldots, x_k) \in C_{k,j}\} = \sup_{x \in L_{k,j}} |f(x)|,$$

where $j \in \mathbb{N}$. Since every $L_{k,j}$ is compact in U, the inclusion

$$(\pi_k^*(\mathcal{A}(\pi_k(U))), \tau_0) \hookrightarrow \mathcal{A}_{\mathcal{V}_k}(U)$$

is continuous. Moreover, the mapping

$$\mathcal{A}_{\mathcal{V}_k}(U) \hookrightarrow (\mathcal{A}(U), \tau_\delta)$$

is also continuous by the definition of τ_δ. Then the inclusion

$$(\pi_k^*(\mathcal{A}(\pi_k(U))), \tau_0) \hookrightarrow (\mathcal{A}(U), \tau_\delta)$$

is continuous for every $k \geq d$. By the definition of τ_ℓ, we deduce $\tau_\delta \leq \tau_\ell$ on $\mathcal{A}(U)$.
$\quad\square$

We now recall the Dieudonné–Schwartz Theorem, whose proof can be found in Horváth [11, p. 161]. It will be applied to prove that the τ_ℓ topology is defined by a regular inductive limit.

Theorem 3.3 (Dieudonné–Schwartz). *Let $X = \varinjlim_{k \in \mathbb{N}} (X_k, \tau_k)$ be a countable inductive limit of locally convex spaces with the following properties:*

(a) $X_k \subset X_{k+1}$ *for all $k \in \mathbb{N}$.*
(b) *The topologies τ_k and τ_{k+1} are equal on X_k.*
(c) X_k *is closed in (X_{k+1}, τ_{k+1}) for every k.*

Then the inductive limit is regular. That is, a subset $B \subset X$ is bounded if and only if there is $k \in \mathbb{N}$ such that B is a bounded subset of (X_k, τ_k).

Theorem 3.4. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. If U is a connected open subset of $\mathbb{K}^{\mathbb{N}}$ with the 0-property for some $d \in \mathbb{N}$, then the inductive limit*

$$(\mathcal{A}(U), \tau_\ell) = \varinjlim_{k \geq d} (\pi_k^*(\mathcal{A}(\pi_k(U))), \tau_0)$$

is regular.

Proof. This follows directly from Lemma 3.1 and the Dieudonné–Schwartz Theorem. □

Theorem 3.5. *If U is a connected open subset of $\mathbb{C}^{\mathbb{N}}$ with the 0-property for some $d \in \mathbb{N}$, then $\tau_\delta = \tau_\ell$ on $\mathcal{H}(U)$.*

Proof. It was seen in Lemma 3.1 that every subspace $(\pi_k^*(\mathcal{A}(\pi_k(U))), \tau_0)$ is metrizable and hence it is bornological. Then the inductive limit $(\mathcal{A}(U), \tau_\ell) = \varinjlim_{k \geq d}(\pi_k^*(\mathcal{A}(\pi_k(U))), \tau_0)$ is also bornological (see Horváth [11, p. 222]).

By Theorem 3.2 we know that $\tau_0 \leq \tau_\delta \leq \tau_\ell$. We prove now that τ_0 and τ_ℓ define the same bounded subsets of $\mathcal{H}(U)$. Let $\mathcal{F}$ be a bounded subset of $(\mathcal{H}(U), \tau_0)$. As $\mathbb{C}^{\mathbb{N}}$ is metrizable, the set $\mathcal{F}$ is locally bounded at zero; that is, there is a neighborhood of zero $V \subset U$ such that

(9) $$\sup\{|f(z)| : z \in V, \, f \in \mathcal{F}\} < \infty$$

(see Dineen [9, Example 3.20a]). By Lemma 2.4, there is $k \in \mathbb{N}$, $k \geq d$, such that $\mathcal{F} \subset \pi_k^*(\mathcal{H}(\pi_k(U)))$. As $\mathcal{F}$ is τ_0-bounded and the inclusion

$$(\pi_k^*(\mathcal{H}(\pi_k(U))), \tau_0) \hookrightarrow (\mathcal{H}(U), \tau_\ell)$$

is continuous by the definition of τ_ℓ, we see that $\mathcal{F}$ is also τ_ℓ-bounded. Therefore, τ_0 and τ_ℓ define the same bounded subsets of $\mathcal{H}(U)$ and τ_ℓ is the bornological topology associated with τ_0. Finally, as $\mathbb{C}^\mathbb{N}$ is metrizable, it is known that τ_δ is the bornological topology associated with τ_0 (see Dineen [9, Example 3.20a]). Thus $\tau_\ell = \tau_\delta$ on $\mathcal{H}(U)$. $\qquad\square$

The fact that $\tau_\delta = \tau_\ell$ on $\mathcal{H}(\mathbb{C}^\mathbb{N})$ was already proved in [1, Proposition 1.3]. It is also known that $\tau_0 < \tau_\delta$ on $\mathcal{H}(\mathbb{C}^\mathbb{N})$ (see Barroso and Nachbin [5]).

As mentioned above, if U is an open subset of a locally convex space E, then $\mathcal{A}(U)$ can be written as the union of all the subspaces of type $\mathcal{A}_\mathcal{V}(U)$ defined in (1). However, it was proved in [2] that if E is any infinite-dimensional complex Banach space with a Schauder basis, then $\mathcal{H}(E)$ is not a countable union of subspaces of type $\mathcal{H}_\mathcal{V}(E)$. That result was generalized in [14, Theorem 9] to the case of open subsets of a complex Banach space without basis and finally in [3, Theorem 2] to the case of real Banach spaces. Therefore, the inductive limit $(\mathcal{A}(U), \tau_\delta) = \varinjlim_\mathcal{V} \mathcal{A}_\mathcal{V}(U)$ is not countable if U is an open subset of an infinite-dimensional Banach space. On the contrary, the situation can be different for non-normed spaces, as the next theorem states.

Theorem 3.6. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let U be a connected open subset of $\mathbb{K}^\mathbb{N}$ with the 0-property for some $d \in \mathbb{N}$. For each $k \in \mathbb{N}$, $k \geq d$, let $\mathcal{V}_k = (V_{k,j})_{j=1}^\infty$ be the increasing countable open cover of U defined in (6). Then*

$$\mathcal{A}(U) = \bigcup_{k=d}^\infty \mathcal{A}_{\mathcal{V}_k}(U).$$

Moreover, if $\mathbb{K} = \mathbb{C}$, then

$$(\mathcal{H}(U), \tau_\delta) = \varinjlim_{k \geq d} \mathcal{H}_{\mathcal{V}_k}(U).$$

Proof. By Lemma 3.1, we know that $\pi_k^*(\mathcal{A}(\pi_k(U))) \subset \mathcal{A}_{\mathcal{V}_k}(U)$. By Theorem 2.5, it is known that

$$\mathcal{A}(U) = \bigcup_{k=d}^\infty \pi_k^*(\mathcal{A}(\pi_k(U))).$$

Hence

$$\mathcal{A}(U) = \bigcup_{k=d}^\infty \mathcal{A}_{\mathcal{V}_k}(U).$$

We now assume that $\mathbb{K} = \mathbb{C}$ and prove that $\mathcal{H}_{\mathcal{V}_k}(U) = \pi_k^*(\mathcal{H}(\pi_k(U)))$ if $k \geq d$. Let us suppose that $f \in \mathcal{H}_{\mathcal{V}_k}(U)$. The function

$$g(z_1, \ldots, z_k) = f(z_1, \ldots, z_k, 0, 0, \ldots)$$

is holomorphic on $\pi_k(U)$. Let $j \in \mathbb{N}$ be such that $V_{k,j} \neq \emptyset$. As in Lemma 2.4, if $z = (z_n)_{n=1}^\infty \in V_{k,j}$ and

$$w = (\underbrace{0, \ldots, 0}_{k}, z_{k+1}, z_{k+2}, \ldots) \in \mathbb{C}^\mathbb{N},$$

then $z + \lambda w \in V_{k,j}$ for all $\lambda \in \mathbb{C}$. The function $h(\lambda) = f(z + \lambda w)$ is holomorphic on $\mathbb{C}$ and is bounded because $f \in \mathcal{H}_{\mathcal{V}_k}(U)$:

$$\sup_{\lambda \in \mathbb{C}} |h(\lambda)| = \sup_{\lambda \in \mathbb{C}} |f(z + \lambda w)| \leq \sup_{v \in V_{k,j}} |f(v)| < \infty.$$

Therefore, h is constant. Then

$$f(z) = h(0) = h(-1) = f(z_1, \ldots, z_k, 0, 0, \ldots)$$
$$= g(z_1, \ldots, z_k) = g \circ \pi_k(z).$$

Hence $f(z) = g \circ \pi_k(z)$ for all $z \in V_{k,j}$. By the Identity Theorem, $f = g \circ \pi_k$ on U. Therefore, $\mathcal{H}_{\mathcal{V}_k}(U) = \pi_k^*(\mathcal{H}(\pi_k(U)))$ if $k \geq d$.

Let $C_{k,j}$ and $L_{k,j}$ be the compact subsets of $\pi_k(U)$ and U respectively defined in (7) and (8). The topology on $\mathcal{H}_{\mathcal{V}_k}(U)$ is defined by the sequence of seminorms

$$\sup_{z \in V_{k,j}} |f(z)| = \sup\{|\widehat{f}(z_1, \ldots, z_k)| : (z_1, \ldots, z_k) \in C_{k,j}\} = \sup_{z \in L_{k,j}} |f(z)|,$$

where $j \in \mathbb{N}$. Thus we deduce that the identity mapping

$$(\pi_k^*(\mathcal{H}(\pi_k(U))), \tau_0) \to \mathcal{H}_{\mathcal{V}_k}(U)$$

is continuous. Since every compact subset of U is contained in some $V_{k,j}$, the identity mapping

$$\mathcal{H}_{\mathcal{V}_k}(U) \to (\pi_k^*(\mathcal{H}(\pi_k(U))), \tau_0)$$

is also continuous. Hence $\mathcal{H}_{\mathcal{V}_k}(U) = (\pi_k^*(\mathcal{H}(\pi_k(U))), \tau_0)$ topologically if $k \geq d$. Finally, by Theorem 3.5,

$$(\mathcal{H}(U), \tau_\delta) = (\mathcal{H}(U), \tau_\ell) = \varinjlim_{k \geq d} (\pi_k^*(\mathcal{H}(\pi_k(U))), \tau_0) = \varinjlim_{k \geq d} \mathcal{H}_{\mathcal{V}_k}(\mathbb{C}^\mathbb{N}). \qquad \square$$

Let us remark that, in contrast to the complex case, the subspaces $\mathcal{A}_{\mathcal{V}_k}(U)$ and $\pi_k^*(\mathcal{H}(\pi_k(U)))$ never agree if U is an open subset of $\mathbb{R}^\mathbb{N}$. For example, the function

$$f((x_n)_{n=1}^\infty) = \sin(x_{k+1})$$

is bounded on $\mathbb{R}^\mathbb{N}$, so $f \in \mathcal{A}_{\mathcal{V}_k}(U)$. However, f depends on the $(k+1)$th variable and so $f \notin \pi_k^*(\mathcal{A}(\pi_k(U)))$.

Next we will prove that, in the real case, $(\mathcal{A}(U), \tau_\delta)$ is not the inductive limit of all the subspaces $(\pi_k^*(\mathcal{A}(\pi_k(U))), \tau_0)$, in contrast to the complex case of Theorem 3.5.

Theorem 3.7. *If U is a connected open subset of $\mathbb{R}^\mathbb{N}$ with the 0-property for some $d \in \mathbb{N}$, then $\tau_\delta < \tau_\ell$ on $\mathcal{A}(U)$.*

Proof. By Theorem 3.2 we have $\tau_\delta \leq \tau_\ell$. Let us see that $\tau_\delta \neq \tau_\ell$. For every $m \in \mathbb{N}$ and each $(x_n)_{n=1}^\infty \in \mathbb{R}^\mathbb{N}$, let

$$g_m((x_n)_{n=1}^\infty) = \sin(x_m).$$

Let $\mathcal{W} = (W_j)_{j=1}^\infty$ be an increasing countable open cover of U. Since

$$\sup_{m \in \mathbb{N}} \left(\sup_{x \in W_j} |g_m(x)| \right) \leq 1$$

for all $j \in \mathbb{N}$, it follows that $\{g_m : m \in \mathbb{N}\}$ is bounded in $\mathcal{A}_\mathcal{W}(U)$. As the inclusion $\mathcal{A}_\mathcal{W}(U) \hookrightarrow (\mathcal{A}(U), \tau_\delta)$ is continuous, the set $\{g_m : m \in \mathbb{N}\}$ is also bounded in $(\mathcal{A}(U), \tau_\delta)$. By Theorem 3.4, the inductive limit

$$(\mathcal{A}(U), \tau_\ell) = \varinjlim_{k \geq d} (\pi_k^*(\mathcal{A}(\pi_k(U))), \tau_0)$$

is regular. If $\{g_m : m \in \mathbb{N}\}$ were bounded for τ_ℓ, there would be $k \in \mathbb{N}$, $k \geq d$, such that $\{g_m : m \in \mathbb{N}\} \subset \pi_k^*(\mathcal{A}(\pi_k(U)))$. However, $g_{k+1} \notin \pi_k^*(\mathcal{A}(\pi_k(U)))$. Therefore, $\{g_m : m \in \mathbb{N}\}$ is not bounded in $(\mathcal{A}(U), \tau_\ell)$ and $\tau_\delta \neq \tau_\ell$. □

If E is a locally convex space and $m \in \mathbb{N}$, the symbol $\mathcal{P}(^mE)$ denotes the space of all continuous m-homogeneous polynomials from E into $\mathbb{K}$. The following result is due to Bochnak and Siciak [6, Theorem 3 and Lemma 4] and it will be applied to prove that $\tau_0 = \tau_\delta = \tau_\ell$ on $\mathcal{P}(^m(\mathbb{K}^\mathbb{N}))$.

Theorem 3.8 (Bochnak and Siciak). *If $P \in \mathcal{P}(^m(\mathbb{R}^\mathbb{N}))$, then there is $P_\mathbb{C} \in \mathcal{P}(^m(\mathbb{C}^\mathbb{N}))$ such that $P_\mathbb{C}(x) = P(x)$ for every $x \in \mathbb{R}^\mathbb{N}$. If U is a convex subset of $\mathbb{R}^\mathbb{N}$ and $P(U) \subset (-1, 1)$, then*

$$P_\mathbb{C}\left(\frac{1}{4e}(U + iU) \right) \subset (-1, 1) + i(-1, 1).$$

Theorem 3.9. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. If $m \in \mathbb{N}$ and $\mathcal{F}$ is any equicontinuous subset of $\mathcal{P}(^m(\mathbb{K}^\mathbb{N}))$, then there is $k \in \mathbb{N}$ such that $\mathcal{F} \subset \pi_k^*(\mathcal{A}(\mathbb{K}^k))$.*

Proof. If $\mathbb{K} = \mathbb{C}$, the result is a particular case of Lemma 2.4. Therefore, we will suppose that $\mathbb{K} = \mathbb{R}$. As $\mathcal{F}$ is equicontinuous, there is a neighborhood U of zero

in $\mathbb{R}^{\mathbb{N}}$ such that $|P(x)| < 1$ for all $x \in U$ and all $P \in \mathcal{F}$. We can assume that

$$U = \{(x_n)_{n=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} : |x_1| < \varepsilon, \ldots, |x_k| < \varepsilon\}$$

for some $k \in \mathbb{N}$ and $\varepsilon > 0$. We will prove that $\mathcal{F} \subset \pi_k^*(\mathcal{A}(\mathbb{R}^k))$.

Let $P \in \mathcal{F}$ and let $\widehat{P} : \mathbb{R}^k \to \mathbb{R}$ be the mapping defined as

$$\widehat{P}(x_1, \ldots, x_k) = P(x_1, \ldots, x_k, 0, 0, \ldots).$$

Given $x = (x_n)_{n=1}^{\infty} \in U$, let

$$y = (\underbrace{0, \ldots, 0}_{k}, x_{k+1}, x_{k+2}, \ldots) \in \mathbb{R}^{\mathbb{N}}.$$

If $\lambda = a + ib \in \mathbb{C}$, then

$$x + \lambda y = (x_1, \ldots, x_k, x_{k+1} + ax_{k+1}, x_{k+2} + ax_{k+2}, \ldots)$$
$$+ i(0, \ldots, 0, bx_{k+1}, bx_{k+2}, \ldots) \in U + iU.$$

As $P(U) \subset (-1, 1)$, Theorem 3.8 implies that

$$P_{\mathbb{C}}\left(\frac{1}{4e}(x + \lambda y)\right) \in (-1, 1) + i(-1, 1)$$

for every $\lambda \in \mathbb{C}$, where $P_{\mathbb{C}}$ is the complex extension of P. Then $|P_{\mathbb{C}}(x + \lambda y)| < (4e)^m \sqrt{2}$ for all $\lambda \in \mathbb{C}$. Hence the function

$$h : \lambda \in \mathbb{C} \mapsto h(\lambda) = P_{\mathbb{C}}(x + \lambda y)$$

is holomorphic and bounded on $\mathbb{C}$, hence constant. Therefore,

$$P(x) = P_{\mathbb{C}}(x) = h(0) = h(-1) = P_{\mathbb{C}}(x - y) = P(x - y)$$
$$= P(x_1, \ldots, x_k, 0, 0, \ldots) = \widehat{P} \circ \pi_k(x).$$

This proves that $P(x) = \widehat{P} \circ \pi_k(x)$ for all $x \in U$. By the Identity Theorem, $P = \widehat{P} \circ \pi_k$ on $\mathbb{R}^{\mathbb{N}}$. $\square$

Theorem 3.10 (Mujica). *Let E be a metrizable locally convex space. Let $m \in \mathbb{N}$ and let τ_p be the topology on $\mathcal{P}(^mE)$ of pointwise convergence. Then τ_0 is the finest locally convex topology on $\mathcal{P}(^mE)$ that agrees with τ_p on equicontinuous subsets of $\mathcal{P}(^mE)$.*

Proof. The result is due to Mujica [16, Theorem 2.1], who states it for complex spaces. However, the same proof is also valid for real spaces. $\square$

It is known that $\tau_0 = \tau_\delta$ on $\mathcal{P}(^m(\mathbb{C}^{\mathbb{N}}))$ for every $m \in \mathbb{N}$ (see Barroso and Nachbin [5, Proposition 10] and Dineen [9, Proposition 3.22b]). Now we prove that these topologies and τ_ℓ also agree on spaces of polynomials on $\mathbb{R}^{\mathbb{N}}$.

Theorem 3.11. *Let* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{K} = \mathbb{C}$. *If* $m \in \mathbb{N}$, *then* $\tau_0 = \tau_\delta = \tau_\ell$ *on* $\mathcal{P}(^m(\mathbb{K}^\mathbb{N}))$.

Proof. By Theorem 3.2, we have $\tau_0 \leq \tau_\delta \leq \tau_\ell$ on $\mathcal{A}(\mathbb{K}^\mathbb{N})$, so we only have to prove that $\tau_\ell \leq \tau_0$ on $\mathcal{P}(^m(\mathbb{K}^\mathbb{N}))$. Let us suppose that $\mathcal{F}$ is an equicontinuous subset of $\mathcal{P}(^m(\mathbb{K}^\mathbb{N}))$. By Theorem 3.9, there is $k \in \mathbb{N}$ such that $\mathcal{F} \subset \pi_k^*(\mathcal{A}(\mathbb{K}^k))$. The inclusions

$$(\mathcal{F}, \tau_0) \hookrightarrow (\pi_k^*(\mathcal{A}(\mathbb{K}^k)), \tau_0) \hookrightarrow (\mathcal{A}(\mathbb{K}^\mathbb{N}), \tau_\ell)$$

are continuous by the definition of τ_ℓ, so $\tau_\ell \leq \tau_0$ on $\mathcal{F}$ and thus $\tau_0 = \tau_\ell$ on $\mathcal{F}$.

By Theorem 3.10, it is known that $\tau_0 = \tau_p$ on $\mathcal{F}$. Hence $\tau_p = \tau_\ell$ on every equicontinuous subset $\mathcal{F} \subset \mathcal{P}(^m(\mathbb{K}^\mathbb{N}))$. Then Theorem 3.10 implies that $\tau_\ell \leq \tau_0$ on $\mathcal{P}(^m(\mathbb{K}^\mathbb{N}))$. Therefore, $\tau_0 = \tau_\ell$ on $\mathcal{P}(^m(\mathbb{K}^\mathbb{N}))$. □

Theorem 3.12. *Let* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{K} = \mathbb{C}$. *Let* U *be a connected open subset of* $\mathbb{K}^\mathbb{N}$ *with the* 0-*property for some* $d \in \mathbb{N}$. *If* τ *is any topology on* $\mathcal{A}(U)$ *such that* $\tau_0 \leq \tau \leq \tau_\ell$, *then the space* $(\mathcal{A}(U), \tau)$ *is not metrizable.*

Proof. Let β denote the topology on the dual space $(\mathbb{K}^\mathbb{N})' = \mathcal{P}(^1(\mathbb{K}^\mathbb{N}))$ of uniform convergence on bounded subsets of $\mathbb{K}^\mathbb{N}$. As every bounded subset of $\mathbb{K}^\mathbb{N}$ is relatively compact, we deduce that $\tau_0 = \beta$ on $(\mathbb{K}^\mathbb{N})'$. Then Theorem 3.11 implies that $\beta = \tau_0 = \tau = \tau_\ell$ on $(\mathbb{K}^\mathbb{N})'$.

If $(\mathcal{A}(U), \tau)$ were metrizable, then the subspace $((\mathbb{K}^\mathbb{N})', \tau) = ((\mathbb{K}^\mathbb{N})', \beta)$ would also be metrizable. Hence both $\mathbb{K}^\mathbb{N}$ and $((\mathbb{K}^\mathbb{N})', \beta)$ would be metrizable, which is not possible because $\mathbb{K}^\mathbb{N}$ is not a normed space (see Köthe [12, p. 394]). □

Let us mention that, in the complex case, the fact that $(\mathcal{H}(U), \tau_0)$ and $(\mathcal{H}(U), \tau_\delta)$ are not metrizable was already proved in [13] for every open subset U of a metrizable locally convex space of infinite dimension.

Mujica proposed the following problem [15, Problem 11.8]: find metrizable complex locally convex spaces E and open subsets $U \subset E$ such that the space $(\mathcal{H}(U), \tau_\delta)$ is complete. That happens, for example, if E is metrizable and complete and U is balanced (see Dineen [9, Corollary 3.53]). The following theorem shows the completeness of $(\mathcal{H}(U), \tau_\delta)$ for some open subsets of $\mathbb{C}^\mathbb{N}$ that may not be balanced.

Theorem 3.13. *If* U *is a connected open subset of* $\mathbb{C}^\mathbb{N}$ *with the* 0-*property for some* $d \in \mathbb{N}$, *then the space* $(\mathcal{H}(U), \tau_\delta)$ *is complete.*

Proof. As $\mathbb{C}^\mathbb{N}$ is metrizable, the space $(\mathcal{H}(U), \tau_0)$ is complete (see Barroso [4, p. 239]). By Lemma 3.1, the subspace $\pi_k^*(\mathcal{H}(\pi_k(U)))$ is closed in $(\mathcal{H}(U), \tau_0)$, so it is also complete for the compact-open topology. By Theorem 3.5, we have

$$(\mathcal{H}(U), \tau_\delta) = (\mathcal{H}(U), \tau_\ell) = \varinjlim_{k \geq d} (\pi_k^*(\mathcal{H}(\pi_k(U))), \tau_0).$$

Thus $(\mathcal{H}(U), \tau_\delta)$ is a countable inductive limit of complete spaces, which implies that $(\mathcal{H}(U), \tau_\delta)$ is also complete (see Köthe [12, p. 225]). $\square$

Theorem 3.14. *If U is a connected open subset of $\mathbb{R}^{\mathbb{N}}$ with the 0-property for some $d \in \mathbb{N}$, then the space $(\mathcal{A}(U), \tau_\ell)$ is not complete.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be the periodic continuous function defined as $f(t) = |t|$ on $[-1, 1]$ and then extended to $\mathbb{R}$. If S_m is the mth sum of the Fourier series of f, then S_m is an analytic function on $\mathbb{R}$ and the sequence $(S_m)_{m=1}^\infty$ converges to f uniformly on $\mathbb{R}$. Let $h : U \to \mathbb{R}$ be the mapping

$$h((x_n)_{n=1}^\infty) = f(x_1).$$

For each $m \in \mathbb{N}$, the mapping $h_m : U \to \mathbb{R}$ defined as

$$h_m((x_n)_{n=1}^\infty) = S_m(x_1)$$

is analytic on U and $(h_m)_{m=1}^\infty$ converges to h uniformly on U. Then $(h_m)_{m=1}^\infty$ is a Cauchy sequence in $(\pi_1^*(\mathcal{A}(\pi_1(U))), \tau_0)$, so also in $(\mathcal{A}(U), \tau_\ell)$. However, the function h, which is the unique possible limit, is not analytic on U because f is not analytic at zero in $\mathbb{R}$. $\square$

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