

Power Operations and Coactions in Highly Commutative Homology Theories

by

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*This paper is dedicated to the memory of my friend **Goro Nishida** (1943–2014), whose pioneering work on power operations inspired it.*

Abstract

Power operations in the homology of infinite loop spaces, and H_∞ or E_∞ ring spectra have a long history in algebraic topology. In the case of ordinary mod p homology for a prime p , the power operations of Kudo, Araki, Dyer and Lashof interact with Steenrod operations via the Nishida relations, but for many purposes this leads to complicated calculations once iterated applications of these functions are required. On the other hand, the homology coaction turns out to provide tractable formulae better suited to exploiting multiplicative structure.

We show how to derive suitable formulae for the interaction between power operations and homology coactions in a wide class of examples; our approach makes crucial use of modern frameworks for spectra with well behaved smash products. In the case of mod p homology, our formulae extend those of Bisson and Joyal to odd primes. We also show how to exploit our results in sample calculations, and produce some apparently new formulae for the Dyer–Lashof action on the dual Steenrod algebra.

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Introduction

In this note we study the interaction between coactions over homology Hopf algebras (such as the Steenrod algebra for a prime p) and power operations (such as Dyer–Lashof operations). Some of our results are surely known, but we are only aware of partial references such as [4, 5] which only deal with the case of ordinary

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mod 2 homology. In any case, our approach to understanding this relationship involves a modern perspective based on a symmetric monoidal category of spectra with good properties such as that of [10].

The examples we discuss in detail are based on ordinary mod p homology for a prime p and the power operations originally introduced by Kudo, Araki, Dyer and Lashof, then generalised by May et al. [9, 12, 7], usually rather unhistorically referred to as Dyer–Lashof operations. Studying the interaction between the coaction and the Dyer–Lashof operations amounts to studying dual versions of the classical Nishida relations [17]. We use knowledge of the coaction of the dual Steenrod algebra $\mathcal{A}(p)_*$ to investigate the homology of commutative S -algebras R where $\pi_0(R)$ has characteristic p . Of course such questions were studied by Steinberger [7, Chapter III]. However, our approach offers some clarification of the algebra involved in the Dyer–Lashof action on the dual Steenrod algebra itself, relating it to work of Kochman [11] (see also [20]); our detailed knowledge of the homology also allows us to give a refined version of Steinberger’s splitting result, giving more information on the multiplicative structure.

The results of this paper have been used in joint work with Rolf Hoyer and some were an outcome of discussions with him.

Notation, etc.

We will use the *floor* and *ceiling* functions $\lfloor - \rfloor, \lceil - \rceil : \mathbb{R} \rightarrow \mathbb{Z}$ taking values

$$\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}, \quad \lceil x \rceil = \min\{n \in \mathbb{Z} : n \geq x\}.$$

In particular, for $x \in \mathbb{Z}$ we have $\lfloor x \rfloor = \lceil x \rceil = x$, while if $x \notin \mathbb{Z}$, then $\lceil x \rceil = \lfloor x \rfloor + 1$.

When working with power series $f(t)$ in an indeterminate t , $[f(t)]_{t^n}$ will denote the coefficient of t^n in $f(t)$.

Bimodules

We will often consider bimodules. If R, R', R'' are three rings, M is an R - R' -bimodule, and N is an R' - R'' -bimodule, then we will denote the tensor product over R' by $M \boxtimes_{R'} N$. We will reserve \otimes_R for the situation where R is commutative and U, V are two left R -modules and denote their tensor product by $U \otimes_R V$. We will sometimes consider a *left* R_* -module M_* over a graded commutative ring R_* as having a canonical *right* R_* -module structure given by

$$m \cdot r = (-1)^{|r| \cdot |m|} r m$$

for homogeneous elements $r \in R_{|r|}$ and $m \in M_{|m|}$.

Bigebrids and comodules

Suppose that A, B, H are commutative (graded) rings and that

$$A \xrightarrow{\eta_A} H \xleftarrow{\eta_B} B$$

are ring homomorphisms. We use these to define a left A -module structure and a right B -module structure on H . Given a right A -module M and a left B -module N , we can define the bimodule tensor products

$$M \boxtimes_A H, \quad H \boxtimes_B N, \quad M \boxtimes_A H \boxtimes_B N.$$

If R is a commutative graded ring, then its *opposite ring* has as its underlying set $R^{\text{op}} = R$ and multiplication of homogeneous elements given by

$$x^{\text{op}}y^{\text{op}} = (\pm)(yx)^{\text{op}},$$

where the sign is determined in the usual way in terms of the degrees of x, y . The opposite ring H^{op} admits a right A^{op} -module structure and a left B^{op} -module structure and there is a ring isomorphism

$$R \xrightarrow{\cong} R^{\text{op}}, \quad x \leftrightarrow (\pm)x^{\text{op}},$$

which interchanges the two pairs of module structures.

Part I
Power operations and coactions

§1. Extended powers and power operations

In this section we give some general observations on extended powers. We will work in the category \mathcal{M}_S of S -modules of [10] and write \wedge for \wedge_S . For an S -module M ,

$$M^{\wedge n} = M \wedge \cdots \wedge M.$$

For an S -module N with a left Σ_n -action we will denote the half-smash product by $E\Sigma_n \rtimes_{\Sigma_n} N$. In particular we will write

$$D_n M = E\Sigma_n \rtimes_{\Sigma_n} M^{\wedge n}$$

for the extended power, and when $G \leq \Sigma_n$, we will sometimes set

$$D_G M = E\Sigma_n \rtimes_G M^{\wedge n}.$$

If M is cofibrant then by [10, Theorem III.5.1], the projection of $E\Sigma_n$ to a point induces a weak equivalence

$$D_n M = E\Sigma_n \rtimes_{\Sigma_n} M^{\wedge n} \xrightarrow{\simeq} M^{\wedge n} / \Sigma_n.$$

More generally, if R is a cofibrant commutative S -algebra, then in the category \mathcal{M}_R of R -modules, for an R -module N we can define

$$D_n^R N = E\Sigma_n \times_{\Sigma_n} N^{\wedge R^n},$$

and if N is a cofibrant R -module, the natural map gives a weak equivalence

$$D_n^R N = E\Sigma_n \times_{\Sigma_n} N^{\wedge R^n} \xrightarrow{\sim} N^{\wedge R^n} / \Sigma_n.$$

If $M \in \mathcal{M}_S$, there is an isomorphism

$$(1.1) \quad R \wedge D_n M \cong D_n^R (R \wedge M).$$

Now we recall the definition of power operations. We will do this in a general setting, for three commutative S -algebras A, B, E (actually, it is enough to assume that E is an H_∞ ring spectrum). There is a map $\mu_n : D_n E \rightarrow E$ which induces a diagram of A -module morphisms.

$$\begin{array}{ccc} A \wedge D_n E & \xrightarrow{I \wedge \mu_n} & A \wedge E \\ & \searrow & \nearrow \\ & D_n^A (A \wedge E) & \end{array}$$

If $x : S^m \rightarrow A \wedge E$, then the composition of solid arrows in the commutative diagram

$$(1.2) \quad \begin{array}{ccccccc} & & e & \cdots & S^k & \cdots & \Theta^e(x) \\ & & \swarrow & & \searrow & & \\ A \wedge D_n S^m & \xrightarrow{\text{id} \wedge D_n x} & A \wedge D_n (A \wedge E) & \xrightarrow{I \wedge \mu_n} & A \wedge A \wedge E & \xrightarrow{\text{mult} \wedge \text{id}} & A \wedge E \\ & \searrow & & \swarrow & & & \\ & & \tilde{x} & & & & \end{array}$$

defines a power operation

$$(1.3) \quad \Theta^e : A_m(E) \rightarrow A_k(E), \quad \Theta^e(x) = \tilde{x}_* e,$$

for each element $e \in A_k(D_n S^m) = \pi_k(A \wedge D_n S^m)$.

§2. Generalised coactions

For any S -module X , we can use the unit $S \rightarrow B$ and switch maps to induce the horizontal morphisms in the following commutative diagram:

$$(2.1) \quad \begin{array}{ccccc} A \wedge X & \xrightarrow{\cong} & A \wedge S \wedge X & \xrightarrow{\text{unit}} & A \wedge B \wedge X \\ \text{switch} \downarrow & & \text{switch} \downarrow & & \downarrow \text{switch} \\ X \wedge A & \xrightarrow{\cong} & S \wedge X \wedge A & \xrightarrow{\text{unit}} & B \wedge X \wedge A \end{array}$$

On applying homotopy $\pi_*(-)$ we obtain maps similar to coactions, at least when suitable flatness conditions hold.

We make an algebraic assumption: the left B_* -module $B_*(A) = \pi_*(B \wedge A)$ is flat. Then on passing to homotopy groups we find that there is an isomorphism of left B_* -modules

$$B_*(X \wedge A) \xrightarrow{\cong} B_*(X) \otimes_{B_*} B_*(A),$$

and an isomorphism

$$A_*(B \wedge X) \xrightarrow{\cong} A_*(B) \boxtimes_{B_*} B_*(X).$$

The rightmost switch map induces an isomorphism

$$A_*(B \wedge X) \xrightarrow{\cong} B_*(X) \otimes_{B_*} B_*(A)$$

which converts the left A_* -module structure to a right module structure. These ingredients give the following commutative diagram:

$$\begin{array}{ccc} & A_*(X) & \\ \swarrow & & \searrow \\ A_*(B) \boxtimes_{B_*} B_*(X) & \xleftarrow{\cong} & A_*(B \wedge X) \xrightarrow{\cong} B_*(X) \otimes_{B_*} B_*(A) \end{array}$$

If $A = B$, then in

$$(2.2) \quad \begin{array}{ccc} & A_*(X) & \\ \swarrow \psi & & \searrow \tilde{\psi} \\ A_*(A) \boxtimes_{A_*} A_*(X) & \xleftarrow{\cong} & A_*(A \wedge X) \xrightarrow{\cong} A_*(X) \otimes_{A_*} A_*(A) \end{array}$$

the homomorphism ψ is the usual left $A_*(A)$ -coaction on $A_*(X)$, while $\tilde{\psi}$ is obtained by composing ψ with the antipode of the Hopf algebroid $A_*(A)$ and a switch map. In fact $\tilde{\psi}$ is a right coaction making $A_*(X)$ into a right $A_*(A)$ -comodule. If we also take $E = A$, then for each $e \in A_k(D_n S^m)$ there is a power operation Θ^e as in (1.3), but also another obtained by interchanging the rôles of the two factors of A ,

$$(2.3) \quad \tilde{\Theta}^e = \chi \Theta^e \chi,$$

where $\chi: A_*(A) \rightarrow A_*(A)$ is the antipode induced by the switch map on $A \wedge A$.

The unit $S \rightarrow B$ induces the downward morphisms in the following commutative diagram:

$$(2.4) \quad \begin{array}{ccccccc} A \wedge S \wedge D_n(A \wedge E) & \xrightarrow{\cong} & A \wedge D_n(S \wedge A \wedge E) & \longrightarrow & A \wedge (S \wedge A \wedge E) & \longrightarrow & A \wedge S \wedge E \\ \uparrow \text{id} \wedge \text{id} \wedge D_n x & & \downarrow & & \downarrow \cong & & \downarrow \cong \\ A \wedge S \wedge D_n S^m & & & & S \wedge (A \wedge A) \wedge E & \longrightarrow & S \wedge A \wedge E \\ \downarrow & & & & \downarrow & & \downarrow \\ A \wedge B \wedge D_n S^m & & & & B \wedge (A \wedge A) \wedge E & \longrightarrow & B \wedge A \wedge E \\ \downarrow \text{id} \wedge \text{id} \wedge D_n x & & & & \uparrow \cong & & \downarrow \cong \\ A \wedge B \wedge D_n(A \wedge E) & \xrightarrow{\cong} & A \wedge D_n^B(B \wedge (A \wedge E)) & \longrightarrow & A \wedge B \wedge (A \wedge E) & \longrightarrow & A \wedge B \wedge E \end{array}$$

On applying $\pi_*(-)$ to this diagram we obtain an algebraic analogue:

$$(2.5) \quad \begin{array}{ccccccc} A_*(S \wedge D_n(A \wedge E)) & \xrightarrow{\cong} & A_*(D_n(S \wedge A \wedge E)) & \longrightarrow & A_*(S \wedge A \wedge E) & \longrightarrow & A_*(S \wedge E) \\ \uparrow (\text{id} \wedge D_n x)_* & & \downarrow & & \downarrow \cong & & \downarrow \cong \\ A_*(S \wedge D_n S^m) & & & & S_*((A \wedge A) \wedge E) & \longrightarrow & S_*(A \wedge E) \\ \downarrow & & & & \downarrow & & \downarrow \\ A_*(B \wedge D_n S^m) & & & & B_*((A \wedge A) \wedge E) & \longrightarrow & B_*(A \wedge E) \\ \downarrow (\text{id} \wedge D_n x)_* & & & & \uparrow \cong & & \downarrow \cong \\ A_*(B \wedge D_n(A \wedge E)) & \xrightarrow{\cong} & A_*(D_n^B(B \wedge (A \wedge E))) & \longrightarrow & A_*(B \wedge (A \wedge E)) & \longrightarrow & A_*(B \wedge E) \end{array}$$

When $B = A$ and $A_*(A)$ is A_* -flat, $(A_*, A_*(A))$ has the structure of a Hopf algebroid. For any spectrum X , the unit $S \rightarrow A$ induces a map

$$A \wedge X \xrightarrow{\cong} A \wedge S \wedge X \rightarrow A \wedge A \wedge X,$$

and there is a left coaction

$$\psi: A_*(X) \rightarrow A_*(A) \boxtimes_{A_*} A_*(X)$$

which fits into the following commutative diagram:

$$\begin{array}{ccc} & & A_*(A \wedge X) \\ & \searrow & \downarrow \cong \\ A_*(X) \cong A_*(S \wedge X) & & \\ & \searrow \psi & A_*(A) \boxtimes_{A_*} A_*(X) \end{array}$$

In this situation, (2.5) can be used to study the $A_*(A)$ -coaction and its relationship with power operations defined above. Taking an element

$$e \in A_*(D_n S^m) \cong A_*(S \wedge D_n S^m)$$

and chasing it upwards to the right and downwards to

$$A_*(A \wedge E) \cong A_*(A) \boxtimes_{A_*} A_*(E)$$

and then comparing the result with that obtained by going downwards to the right,

$$\begin{array}{ccccc}
 & & & A_*(E) & \xrightarrow{\cong} & A_*(S \wedge E) \\
 & & & \downarrow \psi & & \\
 & & & A_*(A) \boxtimes_{A_*} A_*(E) & \xrightarrow{\cong} & A_*(A \wedge E) \\
 & & & \downarrow \cong & & \\
 & & & A_*(E) \otimes_{A_*} A_*(A) & \xrightarrow{\cong} & A_*(E \wedge A) \\
 & & & \uparrow \tilde{\psi} & & \\
 A_*(D_n S^m) & \xrightarrow{\tilde{\psi}} & A_*(D_n S^m) \otimes_{A_*} A_*(A) & \xrightarrow{\quad} & A_*(E) \otimes_{A_*} A_*(A) & \xrightarrow{\cong} & A_*(E \wedge A)
 \end{array}$$

we obtain the following important formula:

$$(2.6) \quad \tilde{\psi}(\Theta^e(x)) = \sum_i (1 \otimes \chi(\theta_i)) \Theta^{e_i}(\tilde{\psi}x),$$

where $\psi(e) = \sum_i \theta_i \otimes e_i$.

§3. Further generalisations

The situation of the previous sections can be generalised somewhat. Suppose that M is a right A -module. Then we can replace the element of $A_k(D_n S^m)$ with $e \in M_k(D_n S^m)$ and use the composition

$$(3.1) \quad \begin{array}{ccccccc}
 & & & & \tilde{x} & & \\
 & & & & \curvearrowright & & \\
 M \wedge D_n S^m & \xrightarrow{\text{id} \wedge D_n x} & M \wedge D_n(A \wedge E) & \xrightarrow{I \wedge \mu_n} & M \wedge A \wedge E & \xrightarrow{\text{mult} \wedge \text{id}} & M \wedge E
 \end{array}$$

to define a power operation

$$(3.2) \quad \Theta^e: A_m(E) \rightarrow M_k(E), \quad \Theta^e(x) = \tilde{x}_* e,$$

analogous to that of (1.3).

In order to get a sensible notion of left coaction $M_*(X) \rightarrow M_*(B) \boxtimes_{B_*} B_*(X)$ leading to analogues of the formulae above in good situations, it is necessary to assume that $B_*(A)$ is B_* -flat, and also that one of the following conditions holds:

- $A_*(X)$ is A_* -flat;
- M_* is A_* -flat as a right A_* -module.

When $B = A$, the assumptions that $A_*(A)$ is flat as a left or right A_* -module are equivalent, and in the most important cases in algebraic topology this holds for any M . We leave the interested reader to work out the details. Such operations are likely to be hard to work with unless M has suitable multiplicative structure (e.g., it is a commutative A -algebra).

One important class of examples is that where $A = B = E_n$, the n -th Lubin–Tate spectrum for a prime p , and $M = K_n$, the n -th Morava K -theory. In this case, K_n is an E_n ring spectrum (non-homotopy commutative if $p = 2$); more generally, we could take $M = E_n \wedge W$, where W is a generalised Moore spectrum as in [8]. The work of the latter suggests defining power operations using pro-systems of such operations; this is presumably related to the work of McClure [7, Chapter IX] on power operations in K -theory.

Part II

Eilenberg–Mac Lane spectra and Dyer–Lashof operations

§4. Eilenberg–Mac Lane spectra and the dual Steenrod algebra

In this section we discuss the important case of the Eilenberg–Mac Lane spectrum for a prime p and take $A = B = H = H\mathbb{F}_p$. The dual Steenrod algebra $\mathcal{A}_* = \mathcal{A}(p)_* = H_*(H)$ is actually a Hopf algebra over $\pi_*(H) = \mathbb{F}_p$ since the two unit homomorphisms coincide. We will usually write $\otimes = \otimes_{\mathbb{F}_p}$ in place of $\boxtimes_{\mathbb{F}_p}$ as there is no danger of confusion. The above isomorphism

$$H_*(H) \boxtimes_{\mathbb{F}_p} H_*(X) \xrightarrow{\cong} H_*(X) \otimes_{\mathbb{F}_p} H_*(H)$$

coincides with the composition switch $\circ (\chi \otimes I)$, and

$$\tilde{\psi} = \text{switch} \circ (\chi \otimes I) \circ \psi.$$

On a basic tensor $\alpha \otimes x \in H_*(H) \otimes H_*(X)$ this gives

$$\alpha \otimes x \leftrightarrow (-1)^{|\alpha| \cdot |x|} x \otimes \chi(\alpha).$$

The Steenrod algebra \mathcal{A}^* is the \mathbb{F}_p -linear dual of \mathcal{A}_* with associated dual pairing

$$\langle - \mid - \rangle : \mathcal{A}^* \otimes \mathcal{A}_* \rightarrow \mathbb{F}_p.$$

This gives rise to a *right* action of \mathcal{A}^* on a *left* \mathcal{A}_* -comodule M_* by

$$a_* x = x \cdot a = (-1)^{|\alpha| \cdot |x|} \sum_i \langle a \mid \gamma_i \rangle x_i,$$

where $a \in \mathcal{A}^*$, $x \in M_*$ and $\psi x = \sum_i \gamma_i \otimes x_i$. There is also a dual pairing

$$\langle - | - \rangle : \mathcal{A}_* \otimes \mathcal{A}^* \rightarrow \mathbb{F}_p \quad \text{defined by} \quad \langle \alpha | a \rangle = (-1)^{|\alpha| \cdot |a|} \langle a | \chi \alpha \rangle,$$

giving an alternative formulation of the right action as

$$a_* x = \sum_i \langle \gamma'_i | a \rangle x_i, \quad \text{where} \quad \tilde{\psi} x = \sum_i x_i \otimes \gamma'_i.$$

§4.1. The case $p = 2$

When $A = B = H = H\mathbb{F}_2$, the dual Steenrod algebra is

$$\mathcal{A}_* = \mathbb{F}_2[\xi_r : r \geq 1] = \mathbb{F}_2[\zeta_r : r \geq 1],$$

where the Milnor generator $\xi_r \in \mathcal{A}_{2^r-1}$ is defined to be the image of the generator of $H_{2^r-1}(\mathbb{R}P^\infty)$ under the homomorphism induced by the canonical map $\mathbb{R}P^\infty \rightarrow \Sigma H\mathbb{F}_2$, and $\zeta_r = \chi(\xi_r)$ is its conjugate; by convention $\xi_0 = \zeta_0 = 1$. We will make use of the generating series

$$\xi(t) = t + \sum_{r \geq 1} \xi_r t^{2^r}, \quad \zeta(t) = t + \sum_{r \geq 1} \zeta_r t^{2^r}$$

which are composition inverses, i.e., $\zeta(\xi(t)) = t = \xi(\zeta(t))$.

We have

$$H_{2m+r}(D_2 S^m) = \begin{cases} \mathbb{F}_2 & \text{if } r \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and the generator in degree $r + 2m$ gives the operation $Q_r = Q^{r+m}$. We write $\tilde{Q}_r = \tilde{Q}^{r+m}$ for the twisted version of these as in (2.3), so

$$\tilde{Q}_r = \chi Q_r \chi = \chi Q^{r+m} \chi = \tilde{Q}^{r+m}.$$

Theorem 4.1. *Let $x \in H_m(E)$ and $\psi(x) = \sum_i \alpha_i \otimes x_i$. Then*

$$\sum_{m \leq r} \psi(Q^r x) t^r = \sum_{m \leq k} \sum_{0 \leq j \leq k} \sum_i \xi(t)^k \tilde{Q}^j \alpha_i \otimes Q^{k-j} x_i,$$

or equivalently

$$\psi(Q^r x) = \sum_{m \leq k} \sum_{0 \leq j \leq k} \sum_i [\xi(t)^k]_{t^r} \tilde{Q}^j \alpha_i \otimes Q^{k-j} x_i.$$

Proof. We recall that for $m \in \mathbb{Z}$, there is a weak equivalence

$$D_2 S^m \xrightarrow{\sim} \Sigma^m \mathbb{R}P_m^\infty,$$

where $\mathbb{R}P_m^\infty$ is the Thom spectrum of the virtual bundle $m\lambda$, and $\lambda \downarrow \mathbb{R}P^\infty$ is the canonical real line bundle associated to the real sign representation of Σ_2 . When $m \geq 0$,

$$\mathbb{R}P_m^\infty = \mathbb{R}P^\infty / \mathbb{R}P^{m-1}.$$

Writing \bar{e}_{r+m} ($r \geq 0$) for the image of the generator $e_r \in H_r(\mathbb{R}P^\infty)$ under the Thom isomorphism

$$H_*(\mathbb{R}P^\infty) \cong H_{*+m}(\mathbb{R}P_m^\infty),$$

the coaction is given by

$$\sum_{r \geq 0} t^{r+m} \psi \bar{e}_{r+m} = \sum_{s \geq 0} \xi(t)^{s+m} \otimes \bar{e}_{s+m}.$$

Under the composition of the isomorphisms

$$H_*(D_2S^m) \xrightarrow{\cong} H_*(\Sigma^m \mathbb{R}P_m^\infty) \xrightarrow{\cong} H_{*-m}(\mathbb{R}P_m^\infty)$$

induced by the above equivalence, the following elements correspond:

$$e_r \otimes x_m^{\otimes 2} \leftrightarrow \bar{e}_{r+m},$$

where $x_m \in H_m(S^m)$ is the generator. Now the result follows from (2.6), which gives the following in terms of the right coaction $\tilde{\psi}$:

$$\sum_{m \leq r} \tilde{\psi}(Q^r x) t^r = \sum_{m \leq k} Q^k(\tilde{\psi} x)(1 \otimes \zeta(t)^k). \quad \square$$

We will sometimes use generating functions to express such formulae. For example, we have the series

$$Q_t = \sum_{r \in \mathbb{Z}} Q^r t^r,$$

and on substituting $\zeta(t)$ for t ,

$$Q_{\zeta(t)} = \sum_{r \in \mathbb{Z}} Q^r \zeta(t)^r.$$

Then

$$(4.1) \quad \tilde{\psi} Q_t x = \sum_{|x| \leq r} \tilde{\psi}(Q^r x) t^r = \sum_{|x| \leq r} Q^r(\tilde{\psi} x) \zeta(t)^r = Q_{\zeta(t)}(\tilde{\psi} x).$$

The following formulae for Dyer–Lashof operations at the prime 2 are due to Steinberger [7, Theorem III.2.2].

Theorem 4.2. For $r, s \geq 1$,

$$Q^{2^s-2}\zeta_1 = \zeta_s, \quad Q^r\zeta_1 \neq 0, \quad Q^r\zeta_s = \begin{cases} Q^{r+2^s-2}\zeta_1 & \text{if } r \equiv 0, -1 \pmod{2^s}, \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 4.3. For $s, t \geq 1$,

$$\tilde{Q}^{2^s-2}\xi_1 = \xi_s, \quad \tilde{Q}^r\xi_1 \neq 0, \quad \tilde{Q}^r\xi_s = \begin{cases} \tilde{Q}^{r+2^s-2}\xi_1 & \text{if } r \equiv 0, -1 \pmod{2^s}, \\ 0 & \text{otherwise.} \end{cases}$$

For later use we record a result that may be known but we know of no reference.

Lemma 4.4. For $s \geq 1$,

$$(4.2) \quad Q^{2^s}\zeta_s = \zeta_{s+1} + \zeta_1\xi_s^2.$$

Proof. Before proving this we note that if $1 \leq r \leq s$, then for degree reasons

$$Q^{2^s}(\zeta_r\xi_{s-r}^{2^r}) = (Q^{2^r-1}\zeta_r)Q^{2^s-2^r+1}(\xi_{s-r}^{2^r}) + (Q^{2^r}\zeta_r)Q^{2^s-2^r}(\xi_{s-r}^{2^r}) = \zeta_{r+1}\xi_{s-r}^{2^{r+1}}.$$

Suppose that (4.2) is true for $s < n$. By definition of the antipode χ , and using Theorem 4.2 we obtain

$$\begin{aligned} Q^{2^n}\xi_n &= Q^{2^n}(\zeta_n + \zeta_{n-1}\xi_1^{2^{n-1}} + \cdots + \zeta_1\xi_{n-1}^2) \\ &= \zeta_{n+1} + (Q^{2^{n-1}}\zeta_{n-1})\xi_1^{2^n} + \cdots + (Q^2\zeta_1)\xi_{n-1}^{2^2} \\ &= \zeta_{n+1} + \zeta_n\xi_1^{2^n} + \cdots + \zeta_2\xi_{n-1}^{2^2} \\ &= (\zeta_{n+1} + \zeta_n\xi_1^{2^n} + \cdots + \zeta_2\xi_{n-1}^{2^2} + \zeta_1\xi_n^2) + \zeta_1\xi_n^2 \\ &= \xi_{n+1} + \xi_1\xi_n^2. \end{aligned} \quad \square$$

§4.2. The case of an odd prime

Suppose that p is an odd prime and $A = H = H\mathbb{F}_p$. The dual Steenrod algebra is

$$\mathcal{A}_* = \mathbb{F}_p[\xi_r : r \geq 1] \otimes \Lambda(\tau_s : s \geq 0) = \mathbb{F}_p[\zeta_r : r \geq 1] \otimes \Lambda(\bar{\tau}_s : s \geq 0)$$

where the Milnor generators $\xi_r \in \mathcal{A}_{2(p^r-1)}$ and $\tau_r \in \mathcal{A}_{2p^r-1}$ are the images of generators of $H_{2p^r-1}(BC_p)$ and $H_{2p^r}(BC_p)$ under the homomorphism induced by the canonical map $BC_p \rightarrow \Sigma H\mathbb{F}_p$, and $\zeta_r = \chi(\xi_r)$, $\bar{\tau}_r = \chi(\tau_r)$ are their conjugates; by convention $\xi_0 = \zeta_0 = 1$.

For $m \in \mathbb{Z}$,

$$H_{2mp+2r(p-1)-\varepsilon}(D_pS^{2m}) = \begin{cases} \mathbb{F}_p & \text{if } r \geq 0 \text{ and } \varepsilon = 0, \\ \mathbb{F}_p & \text{if } r \geq 1 \text{ and } \varepsilon = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and a suitably chosen generator in degree $2mp + 2r(p - 1) - \varepsilon$ gives rise to the operation $\beta^\varepsilon Q_r = \beta^\varepsilon Q^{r+m}$.

In order to give a similar discussion to that for the case $p = 2$, we follow the outline of [7, Section V.2]. Let

$$W = \{(x_1, \dots, x_p) \in \mathbb{R}^p : x_1 + \dots + x_p = 0\}$$

be the reduced real permutation representation of Σ_p in which not all elements act orientably, although $C_p \leq \Sigma_p$ does act by preserving orientations. Given any finite-dimensional real vector space U , we can view $U^p = U \oplus \dots \oplus U$ with the permutation action of Σ_p as equivalent to

$$(\mathbb{R} \oplus W) \otimes_{\mathbb{R}} U \cong U \oplus (W \otimes_{\mathbb{R}} U)$$

with Σ_p acting only on the left hand factor and second summands respectively. As C_p -representations,

$$W \cong W_1 \oplus \dots \oplus W_{(p-1)/2},$$

where $W_r = \mathbb{R}^2$ with the generator of C_p acting as the matrix

$$\begin{bmatrix} \cos(2\pi r/p) & -\sin(2\pi r/p) \\ \sin(2\pi r/p) & \cos(2\pi r/p) \end{bmatrix},$$

which commutes with $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Therefore each W_r together with W itself has a natural complex structure compatible with the C_p -action, so $W \otimes_{\mathbb{R}} U$ can be viewed as a complex C_p -representation. In particular, for any n , as a C_p -representation,

$$(\mathbb{R}^n)^p \cong \mathbb{R}^n \oplus (W \otimes_{\mathbb{R}} \mathbb{R}^n) \cong \mathbb{R}^n \oplus W^n,$$

where W^n has the componentwise action, and it follows that

$$D_{C_p} S^n \cong S^n \wedge E\Sigma_p \times_{C_p} (W^n)^\dagger,$$

where $(-)^{\dagger}$ denotes one-point compactification of a vector space. Here $E\Sigma_p \times_{C_p} (W^n)^\dagger$ is the Thom spectrum of the bundle

$$E\Sigma_p \times_{C_p} W^n \downarrow BC_p.$$

As explained in [7, Section V.2], this spectrum can be interpreted as the suspension spectrum of a truncated lens space, but the orientability of this bundle suffices for our purposes since there is a Thom isomorphism in mod p homology

$$H_*(BC_p) \cong H_{*+n(p-1)}(E\Sigma_p \times_{C_p} (W^n)^\dagger).$$

We remark that this viewpoint is likely to be useful in investigating the kind of operations mentioned in Section 3 associated with Lubin–Tate spectra and Morava K -theory.

Let $z \in H^1(BC_p)$ and let $y = \beta z \in H^2(BC_p)$ be generators of

$$H^*(BC_p) = \mathbb{F}_p[y] \otimes \Lambda(z).$$

Let $a_n \in H_n(BC_p)$ be dual to $z^{\varepsilon(n)}y^{\lfloor n/2 \rfloor}$, where $\varepsilon(n) = (1 - (-1)^n)/2$ and $\lfloor - \rfloor$ is the floor function. We will use a formula for the left coaction $\psi: H_*(BC_p) \rightarrow \mathcal{A}_* \otimes H_*(BC_p)$, originally due to Milnor [16] (see also Boardman’s account [6]).

We introduce two formal variables t_+, t_- in degrees $-2, -1$ respectively (so the usual graded commutativity rules apply), and defining generating series

$$\begin{aligned} a(t) &= a(t_+, t_-) = \sum_{\substack{n \geq 1 \\ r=0,1}} a_{2n-r} t_+^{n-r} t_-^r, \\ \xi(t) &= \xi(t_+) = \sum_{r \geq 0} \xi_r t_+^{p^r}, & \zeta(t) &= \zeta(t_+) = \sum_{r \geq 0} \zeta_r t_+^{p^r}, \\ \tau(t) &= \tau(t_+, t_-) = t_- + \sum_{r \geq 0} \tau_r t_+^{p^r}, & \bar{\tau}(t) &= \bar{\tau}(t_+, t_-) = t_- + \sum_{r \geq 0} \bar{\tau}_r t_+^{p^r}. \end{aligned}$$

Notice that $\tau(t)$ and $\bar{\tau}(t)$ have odd degree and so $\tau(t)^2 = 0 = \bar{\tau}(t)^2$. The left coaction is given by

$$\begin{aligned} (4.3) \quad \psi a(t) &= \sum_{\substack{n \geq 1 \\ r=0,1}} \psi a_{2n-r} t_+^{n-r} t_-^r = a(\xi(t_+), \tau(t_+, t_-)) \\ &= \sum_{k \geq 1} (\xi(t_+)^k \otimes a_{2k} - \tau(t_+, t_-) \xi(t_+)^{k-1} \otimes a_{2k-1}) \\ &= \sum_{k \geq 1} \left(\xi(t_+)^k \otimes a_{2k} - \sum_{r \geq 0} \tau_r t_+^{p^r} \xi(t_+)^{k-1} \otimes a_{2k-1} + \xi(t_+)^{k-1} \otimes a_{2k-1} t_- \right) \\ &= \sum_{k \geq 1} \left(\xi(t_+)^k \otimes a_{2k} - \sum_{r \geq 0} \tau_r t_+^{p^r} \xi(t_+)^{k-1} \otimes a_{2k-1} - \xi(t_+)^{k-1} t_- \otimes a_{2k-1} \right). \end{aligned}$$

Notice the effect of interchanging t_- and a_{2k-1} which disappears when we instead take the right coaction:

$$\begin{aligned} (4.4) \quad \tilde{\psi} a(t) &= a(\zeta(t_+), \bar{\tau}(t_+, t_-)) \\ &= \sum_{k \geq 1} (a_{2k} \otimes \zeta(t_+)^k + a_{2k-1} \otimes \bar{\tau}(t_+, t_-) \zeta(t_+)^{k-1}) \\ &= \sum_{k \geq 1} \left(a_{2k} \otimes \zeta(t_+)^k + \sum_{r \geq 0} a_{2k-1} \otimes \bar{\tau}_r t_+^{p^r} \zeta(t_+)^{k-1} + a_{2k-1} \otimes \zeta(t_+)^{k-1} t_- \right). \end{aligned}$$

By comparing coefficients of monomials in t_+ and t_- we obtain explicit formulae for $n \geq 1$:

$$(4.5) \quad \psi(a_{2n}) = \sum_{k=1}^n [\xi(t_+)^k]_{t_+^n} \otimes a_{2k} - \sum_{\substack{0 \leq p^r \leq n \\ 1 \leq k \leq n}} [\tau_r t_+^{p^r} \xi(t_+)^{k-1}]_{t_+^n} \otimes a_{2k-1},$$

$$(4.6) \quad \psi(a_{2n-1}) = \sum_{k=1}^n [\xi(t_+)^{k-1}]_{t_+^{n-1}} \otimes a_{2k-1},$$

or equivalently

$$(4.7) \quad \tilde{\psi}(a_{2n}) = \sum_{k=1}^n a_{2k} \otimes [\zeta(t_+)^k]_{t_+^n} + \sum_{\substack{0 \leq p^r \leq n \\ 1 \leq k \leq n}} a_{2k-1} \otimes [\bar{\tau}_r t_+^{p^r} \zeta(t_+)^{k-1}]_{t_+^n},$$

$$(4.8) \quad \tilde{\psi}(a_{2n-1}) = \sum_{k=1}^n a_{2k-1} \otimes [\zeta(t_+)^{k-1}]_{t_+^{n-1}}.$$

We recall from [7, Chapter III] the following definitions of Dyer–Lashof operations on $x \in H_m(E)$. As for the prime 2, $Q^r x$ and $\beta Q^r x$ originate on elements $H_*(D_{C_p} S^m)$, namely whenever $2r \geq m$,

$$\begin{aligned} (-1)^r \nu(m) a_{(2r-m)(p-1)} \otimes x_m^{\otimes p} &\mapsto Q^r x, \\ (-1)^r \nu(m) a_{(2r-m)(p-1)-1} \otimes x_m^{\otimes p} &\mapsto \beta Q^r x. \end{aligned}$$

where

$$\nu(m) = (-1)^{m(m-1)(p-1)/4} ((p-1)/2!)^m.$$

Notice that this factor does not depend on r .

We will use generating series in the indeterminates t_+, t_- for encoding actions of Dyer–Lashof operations. We set

$$\begin{aligned} Q_t(-) &= Q_{t_+, t_-}(-) = \sum_r Q^r(-) t_+^{r(p-1)}, \\ \beta Q_t(-) &= \beta Q_{t_+, t_-}(-) = \sum_r \beta Q^r(-) t_+^{r(p-1)-1} t_-, \end{aligned}$$

where the coefficients are operators that can be applied to homology elements.

We obtain the following result on the coaction and Dyer–Lashof operations in the homology of an H_∞ ring spectrum E . To ease the notation, we state it in terms of the right coaction $\hat{\psi}$, omitting \otimes when no confusion seems likely.

Choose $\omega \in \mathbb{F}_{p^2}^\times$ to be a primitive $(p-1)$ -th root of -1 ; although not uniquely determined, ω gives a well defined element of the cyclic group $\mathbb{F}_{p^2}^\times / \mathbb{F}_p^\times$. We make use of the ceiling function $\lceil - \rceil$.

Theorem 4.5. For $x \in H_m(E)$,

$$\begin{aligned} \tilde{\psi}(Q_t x) &= \sum_{\lceil m/2 \rceil \leq r} \tilde{\psi}(Q^r x) t_+^{r(p-1)} \\ &= Q_{\omega\zeta(\omega^{-1}t_+)}(\tilde{\psi}x) \\ &\quad + (-1)^m (\beta Q_{\omega\zeta(\omega^{-1}t_+), \bar{\tau}(\omega\zeta(\omega^{-1}t_+), \omega^{-1}t_-)}(\tilde{\psi}x) - \beta Q_{\omega\zeta(\omega^{-1}t_+), t_-}(\tilde{\psi}x)) \\ &= \sum_{\lceil m/2 \rceil \leq k} Q^k(\tilde{\psi}x) (1 \otimes (\omega\zeta(\omega^{-1}t_+))^{k(p-1)}) \\ &\quad + \sum_{r \geq 0} \sum_{\lceil m/2 \rceil \leq \ell} (-1)^{m+r} \beta Q^\ell(\tilde{\psi}x) (1 \otimes \bar{\tau}_r t_+^{p^r} (\omega\zeta(\omega^{-1}t_+))^{\ell(p-1)-1}), \\ \tilde{\psi}(\beta Q_t(x)) &= \sum_{\lceil m/2 \rceil \leq r} \tilde{\psi}(\beta Q^r x) t_+^{r(p-1)-1} t_- \\ &= \beta Q_{\omega\zeta(\omega^{-1}t_+), t_-}(\tilde{\psi}x) = \sum_{\lceil m/2 \rceil \leq s} \beta Q^s(\tilde{\psi}x) (1 \otimes (\omega\zeta(\omega^{-1}t_+))^{s(p-1)-1} t_-). \end{aligned}$$

Outline of proof for $m \geq 0$. Using the description of an extended power $D_{C_p} S^m$ as a suspension of a truncated lens space, we can pull back to BC_p . The origins of $Q^r x, \beta Q^r x$ then lie in the m -fold suspension of the elements

$$(-1)^r a_{2(p-1)r} = (\omega^{-1})^{r(p-1)} a_{2(p-1)r}, \quad (-1)^r a_{2(p-1)r-1} = (\omega^{-1})^{r(p-1)} a_{2(p-1)r-1}.$$

As the map defining the Dyer–Lashof operations factors through $D_{C_p} S^m \rightarrow D_p S^m$, the formulae follow from (4.7) and (4.8).

The case where $m < 0$ can be proved using Thom spectra of virtual bundles. □

The odd primary part of Steinberger’s [7, Theorem III.2.2] gives the following result.

Theorem 4.6. For $r, s \geq 1$,

$$\begin{aligned} Q^{(p^s-1)/(p-1)} \tau_0 &= (-1)^s \bar{\tau}_s, \\ \beta Q^{(p^s-1)/(p-1)} \tau_0 &= (-1)^s \chi \zeta_s = (-1)^s \zeta_s, \quad \beta Q^r \tau_0 \neq 0, \\ Q^r \zeta_s &= \begin{cases} (-1)^s \beta Q^{r+(p^s-1)/(p-1)} \tau_0 & \text{if } r \equiv -1 \pmod{p^s}, \\ (-1)^{s+1} \beta Q^{r+(p^s-1)/(p-1)} \tau_0 & \text{if } r \equiv 0 \pmod{p^s}, \\ 0 & \text{otherwise,} \end{cases} \\ Q^r \bar{\tau}_s &= \begin{cases} (-1)^{s+1} Q^{r+(p^s-1)/(p-1)} \tau_0 & \text{if } r \equiv 0 \pmod{p^s}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

In particular,

$$Q^{p^s} \zeta_s = \zeta_{s+1}, \quad Q^{p^s} \bar{\tau}_s = \bar{\tau}_{s+1}.$$

Our next result is analogous to Lemma 4.4.

Lemma 4.7. *For $r \geq 0$ and $s \geq 1$,*

$$(4.9) \quad Q^{p^r} \tau_r = \tau_{r+1} - \tau_0 \xi_{r+1},$$

$$(4.10) \quad \beta Q^{p^r} \tau_r = \xi_{r+1},$$

$$(4.11) \quad Q^{p^s} \xi_s = \xi_{s+1} - \xi_1 \xi_s^p.$$

Proof. Conjugation is defined by the recursive formulae

$$\begin{aligned} \bar{\tau}_s + \bar{\tau}_{s-1} \xi_1^{p^{s-1}} + \bar{\tau}_{s-2} \xi_2^{p^{s-2}} + \cdots + \bar{\tau}_0 \xi_s + \tau_s &= 0, \\ \zeta_s + \zeta_{s-1} \xi_1^{p^{s-1}} + \zeta_{s-2} \xi_2^{p^{s-2}} + \cdots + \zeta_1 \xi_{s-1}^p + \xi_s &= 0. \end{aligned}$$

Applying Q^{p^s} to the first equation, using the Cartan formula and considering degrees carefully, we obtain

$$\begin{aligned} Q^{p^s} \tau_s &= -(Q^{p^s} \bar{\tau}_s + (Q^{p^{s-1}} \bar{\tau}_{s-1}) \xi_1^{p^s} + (Q^{p^{s-2}} \bar{\tau}_{s-2}) \xi_2^{p^{s-1}} + \cdots + (Q^1 \bar{\tau}_0) \xi_s^p) \\ &= -(\bar{\tau}_{s+1} + \bar{\tau}_s \xi_1^{p^s} + \bar{\tau}_{s-1} \xi_2^{p^{s-1}} + \cdots + \bar{\tau}_1 \xi_s^p) \\ &= -(\bar{\tau}_{s+1} + \bar{\tau}_s \xi_1^{p^s} + \bar{\tau}_{s-1} \xi_2^{p^{s-1}} + \cdots + \bar{\tau}_1 \xi_s^p + \bar{\tau}_0 \xi_{s+1} + \tau_{s+1}) + \bar{\tau}_0 \xi_{s+1} + \tau_{s+1} \\ &= \tau_{s+1} - \tau_0 \xi_{s+1}, \end{aligned}$$

and

$$\begin{aligned} Q^{p^s} \xi_s &= -(Q^{p^s} \zeta_s + (Q^{p^{s-1}} \zeta_{s-1}) \xi_1^{p^s} + (Q^{p^{s-2}} \zeta_{s-2}) \xi_2^{p^{s-1}} + \cdots + (Q^p \zeta_1) \xi_s^{p^2}) \\ &= -(\zeta_{s+1} + \zeta_s \xi_1^{p^s} + \zeta_{s-1} \xi_2^{p^{s-1}} + \cdots + \zeta_2 \xi_s^{p^2}) \\ &= -(\zeta_{s+1} + \zeta_s \xi_1^{p^s} + \zeta_{s-1} \xi_2^{p^{s-1}} + \cdots + \zeta_2 \xi_s^{p^2} + \zeta_1 \xi_s^p + \xi_{s+1}) + \zeta_1 \xi_s^p + \xi_{s+1} \\ &= \xi_{s+1} - \xi_1 \xi_s^p. \end{aligned}$$

We also have

$$\beta Q^{p^s} \tau_s = \beta \tau_{s+1} - \beta(\tau_0 \xi_{s+1}) = 0 - (\beta \tau_0) \xi_{s+1} + \tau_0 (\beta \xi_{s+1}) = \xi_{s+1},$$

since the Bockstein β acts on \mathcal{A}_* by the left action of \mathcal{A}^* , i.e., if $a \in \mathcal{A}_*$ and $\psi(a) = \sum_i a'_i \otimes a''_i$, then

$$\beta a = \sum_i \langle \beta, \chi(a'_i) \rangle a''_i,$$

where $\langle -, - \rangle$ is the dual pairing between \mathcal{A}^* and \mathcal{A}_* . This gives $\beta \tau_0 = -1$ as used above. \square

§5. Dyer–Lashof operations on the dual Steenrod algebra

We will require more information on the action of Dyer–Lashof operations in $\mathcal{A}(p)_*$. In the discussion following we make use of Kochman [11] and Steinberger [7].

Let R be a commutative ring. Define the *Newton polynomials*

$$N_n(t) = N_n(t_1, \dots, t_n) \in R[t_1, \dots, t_n]$$

recursively by setting $N_1(t) = t_1$ and

$$N_n(t) = t_1 N_{n-1}(t) - t_2 N_{n-2}(t) + \dots + (-1)^{n-2} t_{n-1} N_1(t) + (-1)^{n-1} n t_n.$$

It is well known that for a prime p ,

$$N_{pn}(t) \equiv N_n(t)^p \pmod{p},$$

In $\mathcal{A}(p)_*$, we can consider the values of these obtained by setting

$$t_n = \begin{cases} \xi_r & \text{if } n = p^r - 1, \\ 0 & \text{otherwise,} \end{cases}$$

and we denote these elements by $N_n(\xi)$. They satisfy recurrence relations of the form

$$N_n(\xi) = -\xi_1 N_{n-p+1}(\xi) - \xi_2 N_{n-p^2+1}(\xi) + \dots$$

and in particular

$$\begin{aligned} N_{p^s-1}(\xi) &= -\xi_1 N_{p^s-p}(\xi) - \xi_2 N_{p^s-p^2}(\xi) + \dots - \xi_{s-1} N_{p^s-p^{s-1}}(\xi) - (p^s - 1)\xi_s \\ &= -\xi_1 N_{p^{s-1}-1}(\xi)^p - \xi_2 N_{p^{s-2}-1}(\xi)^{p^2} + \dots - \xi_{s-1} N_{p-1}(\xi)^{p^{s-1}} + \xi_s. \end{aligned}$$

Since $N_{p-1}(\xi) = \xi_1 = -\zeta_1$, it follows that the negatives $-N_{p^r-1}(\xi)$ satisfy the same recurrence relation as the conjugates $\zeta_r = \chi(\xi_r)$, hence for each $s \geq 1$,

$$(5.1) \quad N_{p^s-1}(\xi) = -\zeta_s.$$

See [1, Lemma 2.8] for a closely related result which also implies this one. We also mention another easy consequence of the recursion formula which can be verified by working modulo the ideal $(\xi_i : i \geq 2) \triangleleft \mathcal{A}(p)_*$.

Lemma 5.1. *For any prime p and any $k \geq 1$,*

$$N_{k(p-1)}(\xi) \neq 0.$$

The generating series for the $(-1)^n N_n(\xi)$ satisfies the relation

$$\left(1 + \sum_{r \geq 1} \xi_r t^{p^r-1}\right) \left(\sum_{n \geq 1} (-1)^n N_n(\xi) t^n\right) = \sum_{r \geq 1} \xi_r t^{p^r-1},$$

hence

$$(5.2) \quad \sum_{n \geq 1} (-1)^n N_n(\xi) t^n = 1 - \left(1 + \sum_{r \geq 1} \xi_r t^{p^r - 1} \right)^{-1} = 1 - \frac{t}{\xi(t)}.$$

We will give formulae for the $N_n(\xi)$ modulo the ideal $(\zeta_j : j \neq s) \triangleleft \mathcal{A}(p)_*$, for some fixed $s \geq 1$. The recursive formula for the antipode of $\mathcal{A}(p)_*$ gives

$$\xi_{ns} \equiv -\zeta_s \xi_{(n-1)s}^{p^s} \pmod{(\zeta_j : j \neq s)},$$

and an induction shows that

$$(5.3) \quad \xi_{ns} \equiv (-1)^n \zeta_s^{(p^{ns} - 1)/(p^s - 1)} \pmod{(\zeta_j : j \neq s)}.$$

Combining this with (5.2) we obtain

$$(5.4) \quad \sum_{n \geq 1} (-1)^n N_n(\xi) t^n \equiv 1 - \left(1 + \sum_{r \geq 1} (-1)^r \zeta_s^{(p^{rs} - 1)/(p^s - 1)} t^{p^{rs} - 1} \right)^{-1} \pmod{(\zeta_j : j \neq s)}.$$

Our next result is a number-theoretic observation.

Lemma 5.2. *Let p be a prime and let $s \geq 1$. Suppose that the natural number n has p -adic expansion*

$$n = n_k p^k + n_{k+1} p^{k+1} + \dots + n_{k+\ell} p^{k+\ell}$$

where $\ell \geq 0$ and $n_k, n_{k+\ell} \not\equiv 0 \pmod{p}$. Then

$$\binom{np^s - 1}{n} \not\equiv 0 \pmod{p} \Leftrightarrow n_{s+k} \leq n_k - 1, n_{s+k+1} \leq n_{k+1}, \dots, n_{k+\ell} \leq n_{k+\ell-s}.$$

Proof. The p -adic expansion of $np^s - 1$ is

$$np^s - 1 = (p - 1) + (p - 1)p + \dots + (p - 1)p^{s+k-1} + (n_k - 1)p^{s+k} + n_{k+1}p^{s+k+1} + \dots + n_{k+\ell}p^{s+k+\ell}$$

so

$$\binom{np^s - 1}{n} \equiv \binom{p-1}{n_k} \dots \binom{p-1}{n_{s+k-1}} \binom{n_k-1}{n_{s+k}} \binom{n_{k+1}}{n_{s+k+1}} \dots \binom{n_{k+\ell-s}}{n_{k+\ell}} \pmod{p}.$$

This does not vanish mod p exactly when the stated conditions hold. □

For example, when $p = 2$ and $s = 1$,

$$(5.5) \quad \binom{2n-1}{n} \not\equiv 0 \pmod{2} \Leftrightarrow n = 2^k \text{ for some } k \geq 0.$$

We will use this in proving our next result.

Lemma 5.3. *Let p be a prime and let $s \geq 1$. Then*

$$\begin{aligned} & \left(1 + \sum_{m \geq 1} (-1)^m \zeta_s^{(p^{ms}-1)/(p^s-1)} t^{p^{ms}-1}\right)^{-1} \\ & \equiv 1 + \sum_{\substack{n \in \mathbb{N} \\ \binom{np^s-1}{n} \not\equiv 0 \pmod{p}}} (-1)^{n+1} \binom{np^s-1}{n} \zeta_s^n t^{n(p^s-1)} \pmod{(\zeta_j : j \neq s)}. \end{aligned}$$

Hence

$$N_{n(p^s-1)}(\xi) \equiv \binom{np^s-1}{n} \zeta_s^n \pmod{(\zeta_j : j \neq s)},$$

and this is non-zero precisely when the coefficients in the p -adic expansion

$$n = n_k p^k + \dots + n_{k+l} p^{k+l}$$

satisfy the inequalities

$$n_{s+k} \leq n_k - 1, \quad n_{s+k+1} \leq n_{k+1}, \dots, \quad n_{k+l} \leq n_{k+l-s}.$$

Proof. We will use residue calculus to determine the coefficient of t^n of positive degree, and will denote by

$$\oint f(z) dz = c_{-1}$$

the coefficient of z^{-1} in a meromorphic Laurent series

$$f(z) = \sum_{k_0 \leq k \in \mathbb{Z}} c_k z^k \in \mathbb{R}[[z]][z^{-1}],$$

where \mathbb{R} is any commutative ring and $k_0 \in \mathbb{Z}$. We may apply standard rules of calculus for manipulating such expressions. For example, on changing variable by setting $z = h(w) \in \mathbb{R}[[w]][w^{-1}]$, we obtain

$$\oint f(z) dz = \oint f(h(w)) h'(w) dw.$$

We may determine the coefficient of $t^{n(p^s-1)}$ in $t/\xi(t)$ by calculating

$$\begin{aligned} \oint \frac{t}{\xi(t)} \frac{dt}{t^{n(p^s-1)+1}} &= \oint \frac{\zeta(u)}{u} \frac{du}{\zeta(u)^{n(p^s-1)+1}} = \oint \left(\frac{\zeta(u)}{u}\right)^{-n(p^s-1)} \frac{du}{u^{n(p^s-1)+1}} \\ &\equiv \oint (1 + \zeta_s u^{p^s-1})^{-n(p^s-1)} \frac{du}{u^{n(p^s-1)+1}} \equiv \oint (1 + \zeta_s v)^{-n(p^s-1)} \frac{(-1)dv}{v^{n+1}} \\ &\equiv -\binom{-n(p^s-1)}{n} \zeta_s^n \equiv (-1)^{n+1} \binom{n(p^s-1) + (n-1)}{n} \zeta_s^n \\ &\equiv (-1)^{n+1} \binom{np^s-1}{n} \zeta_s^n \pmod{(\zeta_j : j \neq s)}. \end{aligned}$$

Now we can use Lemma 5.2 to complete the analysis of these coefficients. □

To determine the Dyer–Lashof operations on the dual Steenrod algebra \mathcal{A}_* we use results of Kochman [11], where the operations are calculated in $H_*(BO; \mathbb{F}_2)$ and $H_*(BU; \mathbb{F}_p)$. We remark that in each case the Thom isomorphism is known to respect the Dyer–Lashof operations, so this determines the Dyer–Lashof actions in $H_*(MO; \mathbb{F}_2)$ and $H_*(MU; \mathbb{F}_p)$.

For $p = 2$, $H_*(MO) = H_*(MO; \mathbb{F}_2)$ is the polynomial algebra on generators $a_n \in H_n(MO)$ which correspond to generators of $H_*(BO)$ coming from those in $H_*(BO(1)) = H_*(\mathbb{R}P^\infty)$. Under the homomorphism $H_*(MO) \rightarrow \mathcal{A}_*$ induced by the orientation $MO \rightarrow H\mathbb{F}_2$,

$$a_n \mapsto \begin{cases} \xi_s & \text{if } n = 2^s - 1, \\ 0 & \text{otherwise.} \end{cases}$$

The Newton polynomial $N_n(a) = N_n(a_1, \dots, a_n) \in H_n(MO)$ corresponds to the Hopf algebra primitive generator in $H_n(BO)$, so [11, Corollary 35] gives

$$Q^r N_n(a) = \binom{r-1}{n-1} N_{n+r}(a).$$

This yields the following formula in \mathcal{A}_* :

$$(5.6) \quad Q^r N_n(\xi) = \binom{r-1}{n-1} N_{n+r}(\xi).$$

Using (5.1) for $p = 2$, we obtain

$$(5.7) \quad Q^r \zeta_s = \binom{r-1}{2^s-2} N_{2^s-1+r}(\xi),$$

and it is easy to see that

$$(5.8) \quad \binom{r-1}{2^s-2} \equiv 1 \pmod{2} \Leftrightarrow r \equiv 0, -1 \pmod{2^s}.$$

Using Lemma 5.1, this recovers part of Steinberger’s result (see our Theorem 4.2).

For an odd prime p , $H_*(MU)$ is polynomial on generators $b_n \in H_{2n}(MU)$ coinciding under the Thom isomorphism with generators of $H_*(BU)$ coming from $H_*(BU(1)) = H_*(\mathbb{C}P^\infty)$. Under the homomorphism induced by the orientation $MU \rightarrow H\mathbb{F}_p$,

$$b_n \mapsto \begin{cases} \xi_s & \text{if } n = p^s - 1, \\ 0 & \text{otherwise,} \end{cases}$$

and so by [11, Theorem 5], the Newton polynomial $N_n(\xi)$ satisfies

$$(5.9) \quad Q^r N_n(\xi) = (-1)^{r+n} \binom{r-1}{n-1} N_{n+r(p-1)}(\xi),$$

and using (5.1) we obtain

$$(5.10) \quad Q^r \zeta_s = (-1)^{r+1} \binom{r-1}{p^s-2} N_{p^s-1+r(p-1)}(\xi).$$

It is easy to see that

$$(5.11) \quad \binom{r-1}{p^s-2} \not\equiv 0 \pmod{p} \Leftrightarrow r \equiv 0, -1 \pmod{p^s},$$

thus recovering part of Steinberger's result (see Theorem 4.6).

§6. Verification of the Nishida relations

For completeness we show how the usual Nishida relations are consequences of coaction formulae.

§6.1. The case $p = 2$

First we recall that with respect to the monomial basis for $\mathcal{A}_* = \mathcal{A}(2)_*$, the dual element of $\xi_1^{r_1} \cdots \xi_\ell^{r_\ell}$ is $\text{Sq}^{(r_1, \dots, r_\ell)} \in \mathcal{A}^*$. The dual of ξ_1^n is the Steenrod operation $\text{Sq}^{(n)} = \text{Sq}^n$, i.e.,

$$\langle \text{Sq}^n \mid \xi_1^{r_1} \cdots \xi_\ell^{r_\ell} \rangle = \begin{cases} 1 & \text{if } \xi_1^{r_1} \cdots \xi_\ell^{r_\ell} = \xi_1^n, \\ 0 & \text{otherwise.} \end{cases}$$

In terms of the right pairing, this becomes

$$\langle \zeta_1^{r_1} \cdots \zeta_\ell^{r_\ell} \mid \text{Sq}^n \rangle = \begin{cases} 1 & \text{if } \zeta_1^{r_1} \cdots \zeta_\ell^{r_\ell} = \zeta_1^n, \\ 0 & \text{otherwise.} \end{cases}$$

We will work with the right coaction so the latter formulae will often be used.

Notice that for a left \mathcal{A}_* -comodule M_* and $x \in M_*$, we have

$$(6.1) \quad \psi(x) = \sum_{(r_1, \dots, r_\ell)} \xi_1^{r_1} \cdots \xi_\ell^{r_\ell} \otimes \text{Sq}_*^{(r_1, \dots, r_\ell)} x,$$

$$(6.2) \quad \tilde{\psi}(x) = \sum_{(r_1, \dots, r_\ell)} \text{Sq}_*^{(r_1, \dots, r_\ell)} x \otimes \zeta_1^{r_1} \cdots \zeta_\ell^{r_\ell}.$$

In particular

$$\text{Sq}_*^n x = \langle \text{Sq}^n \otimes 1 \mid \psi x \rangle = \langle \tilde{\psi} x \mid 1 \otimes \text{Sq}^n \rangle.$$

We want to determine expressions of the form $\text{Sq}_*^r \text{Q}^s x$ where $x \in H_*(E)$ for a commutative S -algebra E . We have

$$\text{Sq}_*^r \text{Q}^s x = \langle \tilde{\psi} \text{Q}^s x \mid 1 \otimes \text{Sq}^r \rangle,$$

and combining these for all values of s we obtain

$$\begin{aligned} \text{Sq}_*^r \text{Q}_t x &= \sum_s (\text{Sq}_*^r \text{Q}^s x) t^s = \left\langle \sum_s \tilde{\psi} \text{Q}^s x t^s \mid 1 \otimes \text{Sq}^r \right\rangle \\ &= \left\langle \sum_k \text{Q}^k(\tilde{\psi} x) \zeta(t)^k \mid 1 \otimes \text{Sq}^r \right\rangle = \langle \text{Q}_{\zeta(t)}(\tilde{\psi} x) \mid 1 \otimes \text{Sq}^r \rangle. \end{aligned}$$

In the expression (6.2), applying $\text{Q}_{\zeta(t)}$ to a term yields

$$\begin{aligned} \text{Q}_{\zeta(t)}(\text{Sq}_*^{(r_1, \dots, r_\ell)} x \otimes \zeta_1^{r_1} \cdots \zeta_\ell^{r_\ell}) &= \text{Q}_{\zeta(t)}(\text{Sq}_*^{(r_1, \dots, r_\ell)} x) \otimes \text{Q}_{\zeta(t)}(\zeta_1^{r_1} \cdots \zeta_\ell^{r_\ell}) \\ &= \text{Q}_{\zeta(t)}(\text{Sq}_*^{(r_1, \dots, r_\ell)} x) \otimes (\text{Q}_{\zeta(t)} \zeta_1)^{r_1} \cdots (\text{Q}_{\zeta(t)} \zeta_\ell)^{r_\ell}, \end{aligned}$$

so we need to investigate the terms $\text{Q}_{\zeta(t)} \zeta_s$. In fact for our purposes it is sufficient to know these mod $(\zeta_j : j > 1)$.

Lemma 6.1. *For $s \geq 2$,*

$$\text{Q}_t \zeta_s \equiv 0 \pmod{(\zeta_j : j > 1)}.$$

Proof. By (5.8) and (5.7), $\text{Q}^r \zeta_s \neq 0$ only when $r \equiv 0, -1 \pmod{2^s}$, and then

$$\text{Q}^r \zeta_s = \begin{cases} N_{1+2+\dots+2^{s-1}+2^k+r_{k+1}2^{k+1}+\dots+2^\ell}(\xi) & \text{if } r \equiv 0 \pmod{2^s}, \\ N_{2+\dots+2^{s-1}+2^k+r_{k+1}2^{k+1}+\dots+2^\ell}(\xi) & \text{if } r \equiv -1 \pmod{2^s}, \end{cases}$$

for some k, ℓ with $s \leq k \leq \ell$. In either case we find that $\text{Q}^r \zeta_s \equiv 0 \pmod{(\zeta_j : j > 1)}$ by using Lemma 5.3 (with $s = 1$). \square

For the case $s = 1$, we have

$$\text{Q}_t \zeta_1 = \frac{1}{t} - \frac{1}{\xi(t)} + \zeta_1,$$

hence

$$\begin{aligned} Q_{\zeta(t)}\zeta_1 &= \frac{1}{\zeta(t)} - \frac{1}{t} + \zeta_1 \equiv \frac{1 - (1 + \zeta_1 t) + \zeta_1(t + \zeta_1 t^2)}{(t + \zeta_1 t^2)} \\ &\equiv \frac{\zeta_1^2 t}{(1 + \zeta_1 t)} \equiv \zeta_1^2 t(1 + \zeta_1 t)^{-1} \pmod{(\zeta_j : j > 1)}. \end{aligned}$$

So we have

$$\begin{aligned} \text{Sq}_*^r Q_t x &= \sum_{j \geq 0} \langle Q_{\zeta(t)}(\text{Sq}_*^j x)(Q_{\zeta(t)}\zeta_1)^j \mid 1 \otimes \text{Sq}^r \rangle \\ &= \sum_{j \geq 0} \sum_k \langle \zeta(t)^k (Q_{\zeta(t)}\zeta_1)^j \mid 1 \otimes \text{Sq}^r \rangle Q^k \text{Sq}_*^j x \\ &= \sum_{j \geq 0} \sum_k \langle \zeta_1^{2j} t^{j+k} (1 + \zeta_1 t)^{k-j} \mid 1 \otimes \text{Sq}^r \rangle Q^k \text{Sq}_*^j x \\ &= \sum_{j \geq 0} \sum_k \binom{k-j}{r-2j} Q^k \text{Sq}_*^j x t^{r+k-j}, \end{aligned}$$

or equivalently

$$\text{Sq}_*^r Q^n x = \sum_{j \geq 0} \binom{n-r}{r-2j} Q^{n-r+j} \text{Sq}_*^j x,$$

which is the usual form of the Nishida relations.

§6.2. The case p odd

We begin by determining formulae for Dyer–Lashof operations in $\mathcal{A}_* = \mathcal{A}(p)_* \pmod{(\zeta_j : j \geq 2)}$. By (5.10) and (5.11) we find that for an indeterminate t of degree -2 ,

$$\begin{aligned} Q_t \zeta_s &= \sum_{k \geq 1} \left((-1)^{k+1} \binom{kp^s - 1}{p^s - 2} N_{(kp^s + p^{s-1} + \dots + p+1)(p-1)}(\xi) t^{kp^s(p-1)} \right. \\ &\quad \left. + (-1)^k \binom{kp^s - 2}{p^s - 2} N_{(kp^s + p^{s-1} + \dots + p)(p-1)}(\xi) t^{(kp^s - 1)(p-1)} \right) \\ &\equiv \sum_{k \geq 1} (-1)^k (N_{(kp^{s-1} + p^{s-2} + \dots + p+1)(p-1)}(\xi))^p t^{(kp^s - 1)(p-1)} \pmod{(\zeta_j : j \geq 2)}, \end{aligned}$$

where the first term in each summand vanishes thanks to Lemma 5.3. Also, when $s \geq 2$, Lemma 5.3 implies that

$$(6.3) \quad Q_t \zeta_s \equiv 0 \pmod{(\zeta_j : j \geq 2)}.$$

When $s = 1$,

$$\begin{aligned} Q_t \zeta_1 &\equiv \sum_{k \geq 1} (-1)^k (N_{k(p-1)}(\xi))^p t^{(kp-1)(p-1)} \equiv t^{-(p-1)} \left(\sum_{k \geq 1} N_{k(p-1)}(\xi) (-t^{(p-1)})^k \right)^p \\ &\equiv t^{-(p-1)} \left(\sum_{k \geq 1} N_{k(p-1)}(\xi) (\omega^{-1}t)^{k(p-1)} \right)^p \pmod{(\zeta_j : j \geq 2)}, \end{aligned}$$

where $\omega \in \mathbb{F}_{p^2}^\times$ is a primitive $(p-1)$ -th root of -1 as introduced earlier. Using (5.2), we obtain

$$(6.4) \quad \mathcal{Q}_t \zeta_1 \equiv -\frac{1}{(\omega^{-1}t)^{p-1}} \left(1 - \frac{\omega^{-1}t}{\xi(\omega^{-1}t)}\right)^p \pmod{(\zeta_j : j \geq 2)}.$$

Replacing t by $\omega\zeta(\omega^{-1}t)$ gives another useful formula:

$$(6.5) \quad \begin{aligned} \mathcal{Q}_{\omega\zeta(\omega^{-1}t)} \zeta_1 &\equiv -\frac{1}{\zeta(\omega^{-1}t)^{p-1}} \left(1 - \frac{\zeta(\omega^{-1}t)}{\omega^{-1}t}\right)^p \equiv \zeta_1^p t^{(p-1)^2} (1 - \zeta_1 t^{p-1})^{1-p} \\ &\equiv \sum_{k \geq 0} \binom{p-2+k}{k} \zeta_1^{p+k} t^{(p+k-1)(p-1)} \pmod{(\zeta_j : j \geq 2)}. \end{aligned}$$

Now we follow a similar line of argument to that for the case $p = 2$ above. We recall that $\mathcal{A}(p)_*$ has a basis consisting of monomials

$$\xi_1^{r_1} \cdots \xi_k^{r_k} \tau_0^{e_0} \cdots \tau_\ell^{e_\ell}$$

where $e_i = 0, 1$ and $r_i \geq 0$. In $\mathcal{A}(p)^*$, the dual basis element is $\mathcal{P}^{(r_1, \dots, r_k; e_0, \dots, e_\ell)}$. In particular, $\mathcal{P}^{(r)} = \mathcal{P}^r$ is the reduced power operation.

We want to determine the series $\mathcal{P}_*^r \mathcal{Q}_t x$, and this turns out to be given by

$$\begin{aligned} \mathcal{P}_*^r \mathcal{Q}_t x &= \sum_{j \geq 0} \langle \mathcal{Q}_{\omega\zeta(\omega^{-1}t)} (\mathcal{P}_*^j x) (\mathcal{Q}_{\omega\zeta(\omega^{-1}t)} \zeta_1)^j \mid \mathcal{P}^r \rangle \\ &= \sum_{j \geq 0} \sum_k \langle (\omega\zeta(\omega^{-1}t))^{k(p-1)} (\mathcal{Q}_{\omega\zeta(\omega^{-1}t)} \zeta_1)^j \mid \mathcal{P}^r \rangle \mathcal{Q}^k \mathcal{P}_*^j x \\ &= \sum_{j \geq 0} \sum_k t^{(k+j(p-1))(p-1)} \langle \zeta_1^{jp} (1 - \zeta_1 t^{p-1})^{(k-j)(p-1)} \mid \mathcal{P}^r \rangle \mathcal{Q}^k \mathcal{P}_*^j x \\ &= \sum_{j \geq 0} \sum_k t^{(k+j(p-1))(p-1)} (-1)^{r-jp} \binom{(k-j)(p-1)}{r-jp} t^{(r-jp)(p-1)} \mathcal{Q}^k \mathcal{P}_*^j x \\ &= \sum_{j \geq 0} \sum_k (-1)^{j+r} t^{(k-j+r)(p-1)} \binom{(k-j)(p-1)}{r-jp} \mathcal{Q}^k \mathcal{P}_*^j x. \end{aligned}$$

Taking the coefficient of $t^{s(p-1)}$ by putting $k = s - r + j$ we obtain

$$\mathcal{P}_*^r \mathcal{Q}^s x = \sum_{j \geq 0} (-1)^{r+j} \binom{(s-r)(p-1)}{r-jp} \mathcal{Q}^{s-r+j} \mathcal{P}_*^j x,$$

which is the usual form of the Nishida relations.

We leave the interested reader to perform a similar verification of the Nishida relations for $\mathcal{P}_*^r \beta \mathcal{Q}^s$.

§7. Working modulo squares and Milnor primitives

In this section we work at the prime 2, but there are analogous results at odd primes. Let E be a commutative S -algebra.

As another example of the utility of our methods, we will investigate the induced coaction

$$\begin{array}{ccc}
 & \Psi & \\
 & \curvearrowright & \\
 H_*(E) & \xrightarrow{\psi} \mathcal{A}_* \otimes H_*(E) & \longrightarrow \mathcal{E}_* \otimes H_*(E)
 \end{array}$$

where

$$\mathcal{E}_* = \mathcal{A}_* // \mathcal{A}_*^{(2)} = \mathcal{A}_* / (\zeta_s^2 : s \geq 1)$$

is the exterior quotient Hopf algebra dual to the Hopf subalgebra of \mathcal{A}^* generated by the Milnor primitives $q^r \in \mathcal{A}^{2^{r+1}-1}$ recursively defined by setting $q^0 = Sq^1$ and for $r \geq 1$,

$$q^r = [q^{r-1}, Sq^{2^r}] = q^{r-1} Sq^{2^r} + Sq^{2^r} q^{r-1}.$$

To avoid cumbersome notation, we will write $u \doteq v$ in place of $u \equiv v \pmod{(\zeta_s^2 : s \geq 1)}$ when working with the quotient ring \mathcal{E}_* , and identify elements of \mathcal{A}_* with their residue classes.

As with ψ , there is a corresponding right coaction

$$\begin{array}{ccc}
 & \tilde{\Psi} & \\
 & \curvearrowright & \\
 H_*(E) & \xrightarrow{\tilde{\psi}} H_*(E) \otimes \mathcal{A}_* & \longrightarrow H_*(E) \otimes \mathcal{E}_*
 \end{array}$$

Our interest is in the general form of the right coaction on elements of the form $Q^J z$, or equivalently in $q_*^r Q^J z$ for $r \geq 0$. Of course it is well known that

$$q_*^0 Q^a z = Sq_*^1 Q^a z = (a + 1) Q^{a-1} z.$$

Using the monomial basis in the residue classes $\bar{\xi}_i$ of the ξ_i , q^r is dual to the residue class of

$$\xi_{r+1} = \chi(\zeta_{r+1}) = \zeta_1 \xi_r^2 + \cdots + \zeta_r \xi_1^{2^r} + \zeta_{r+1} \equiv \zeta_{r+1} \pmod{(\zeta_i^2 : i \geq 1)}.$$

Hence $\xi_{r+1} \doteq \zeta_{r+1}$ is primitive in the quotient Hopf algebra \mathcal{E}_* . To calculate $q_*^r w$ we may use the formulae

$$q_*^r w = \langle q^r \otimes 1 \mid \Psi w \rangle = \langle \tilde{\Psi} w \mid 1 \otimes q^r \rangle.$$

It is clear that the action of the Dyer–Lashof operations descends from \mathcal{A}_* to the quotient \mathcal{E}_* . We start by determining the image of $Q^r \zeta_s$ in \mathcal{E}_* .

Lemma 7.1. *For $s \geq 1$ and $r \geq s$, we have*

$$Q^r \zeta_s \doteq \begin{cases} \zeta_{s+m} & \text{if } r = 2^{s+m} - 2^s, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Using (5.7) and (5.8), it suffices to consider the cases $r = 2^s k, 2^s k - 1$. The Newton recurrence formula gives

$$\begin{aligned} Q^{2^s k} \zeta_s &= N_{2^s(k+1)-1}(\xi) = \xi_1 N_{2^s(k+1)-2}(\xi) + \xi_2 N_{2^s(k+1)-4}(\xi) + \cdots \\ &= \xi_1 N_{2^{s-1}(k+1)-1}(\xi)^2 + \xi_2 N_{2^{s-2}(k+1)-1}(\xi)^4 + \cdots, \end{aligned}$$

and this is 0 mod $(\zeta_i^2 : i \geq 1)$ unless $2^s(k+1) = 2^{s+m}$, i.e., $k = 2^m - 1$, and then

$$Q^{2^{s+m}-2^s} \zeta_s \doteq \zeta_{s+m}.$$

Also,

$$Q^{2^s k-1} \zeta_s = N_{2^s(k+1)-2}(\xi) = N_{2^{s-1}(k+1)-1}(\xi)^2 \doteq 0. \quad \square$$

Using the notation

$$\Xi(r) = \sum_{r+1 \leq k} \zeta_k t^{2^k - 2^r}, \quad \Xi(r, s) = \Xi(r) - \Xi(s) = \sum_{r+1 \leq k \leq s} \zeta_k t^{2^k - 2^r},$$

where $0 \leq r < s$, we obtain the following succinct formula:

$$(7.1) \quad Q_t \zeta_s \doteq \sum_{s+1 \leq k} \zeta_k t^{2^k - 2^s} = \Xi(s).$$

If $s \geq 1$,

$$\Xi(s)^2 \doteq 0,$$

hence when $s_1 < s_2$,

$$(7.2) \quad \Xi(s_1)\Xi(s_2) \doteq \Xi(s_1, s_2)\Xi(s_2).$$

More generally, if $s_1 < s_2 < \cdots < s_\ell$, then

$$(7.3) \quad \begin{aligned} Q_t(\zeta_{s_1} \zeta_{s_2} \cdots \zeta_{s_\ell}) &= Q_t(\zeta_{s_1})Q_t(\zeta_{s_2}) \cdots Q_t(\zeta_{s_\ell}) \\ &\doteq \Xi(s_1, s_2)\Xi(s_2, s_3) \cdots \Xi(s_{\ell-1}, s_\ell)\Xi(s_\ell). \end{aligned}$$

Now we can give a formula for the \mathcal{E}_* -coaction.

Proposition 7.2. *If $z \in H_n(E)$, then*

$$\tilde{\Psi} Q_t z \doteq Q_t(\tilde{\Psi} z) + \sum_{a \geq n} \sum_{j \geq 1} (a+1) Q^{a-2^j+1}(\tilde{\Psi} z) \zeta_j t^a.$$

Equivalently, for each $a \geq n$,

$$\tilde{\Psi} Q^a z \doteq Q^a(\tilde{\Psi} z) + (a+1) \sum_{j \geq 1} Q^{a-2^j+1}(\tilde{\Psi} z) \zeta_j.$$

Proof. This follows from the calculation

$$\begin{aligned} \tilde{\Psi}Q^a z &\doteq \sum_{n \leq k \leq a} Q^k(\tilde{\Psi}z) [(1 + \Xi(0))^k]_{t^{a-k}} \\ &\doteq Q^a(\tilde{\Psi}z) + \sum_{n \leq k \leq a-1} kQ^k(\tilde{\Psi}z) [\Xi(0)]_{t^{a-k}} \\ &\doteq Q^a(\tilde{\Psi}z) + \sum_{n \leq a-2^j+1 \leq a-1} (a - 2^j + 1)Q^{a-2^j+1}(\tilde{\Psi}z) [\Xi(0)]_{t^{2^j-1}} \\ &\doteq Q^a(\tilde{\Psi}z) + \sum_{2 \leq 2^j \leq a-n+1} (a + 1)Q^{a-2^j+1}(\tilde{\Psi}z)\zeta_j. \quad \square \end{aligned}$$

We can use this to derive formulae for the action of the Milnor primitives.

Proposition 7.3. *If $z \in H_n(E)$, $s \geq 0$ and $a > n$, then*

$$q_*^s Q^a z = (a + 1)Q^{a-2^{s+1}+1}z + \sum_{0 \leq r \leq s-1} Q^{a-2^{s+1}+2^{r+1}}(q_*^r z).$$

Proof. We can determine $q_*^s Q^a z$ using the inner product, i.e.,

$$q_*^s Q^a z = \langle \tilde{\Psi}Q^a z \mid 1 \otimes q^s \rangle.$$

By Proposition 7.2, we have

$$q_*^s Q^a z = \langle Q^a(\tilde{\Psi}z) \mid 1 \otimes q^s \rangle + (a + 1)Q^{a-2^{s+1}+1}z.$$

To analyse $\langle Q^a(\tilde{\Psi}z) \mid 1 \otimes q^s \rangle$, we note that only the term in $Q^a(\tilde{\Psi}z)$ of form $(?) \otimes \zeta_{s+1}$ can provide a non-zero contribution, while in $\tilde{\Psi}z$ any term of form $(?) \otimes \zeta_{i_1} \cdots \zeta_{i_\ell}$ with $\ell > 1$ contributes zero. Since $Q^{2^{s+1}-2^{r+1}}(\zeta_{r+1}) \doteq \zeta_{s+1}$ we must have

$$\begin{aligned} \langle Q^a(\tilde{\Psi}z) \mid 1 \otimes q^s \rangle &= \sum_r \langle Q^a(q_*^r z \otimes \zeta_r) \mid 1 \otimes q^s \rangle \\ &= \sum_r \langle Q^{a-2^{s+1}+2^{r+1}}(q_*^r z) \otimes \zeta_{s+1} \mid 1 \otimes q^s \rangle \\ &= \sum_r Q^{a-2^{s+1}+2^{r+1}}(q_*^r z). \quad \square \end{aligned}$$

This result is useful when calculating with iterated Dyer–Lashof operations. For example,

$$\begin{aligned} q_*^1 Q^a z &= (a + 1)Q^{a-3}z + Q^{a-2}(q_*^0 z), \\ q_*^1 Q^a Q^b z &= (a + 1)Q^{a-3}Q^b z + (b + 1)Q^{a-2}Q^{b-1}z. \end{aligned}$$

In general, $q_*^s Q^{a_1} \cdots Q^{a_{s+1}} z$ does not depend on the coaction on z .

Part III
Free commutative S -algebras

§8. Free commutative S -algebras and their homology

Following [10] we work in the model categories of left S -modules $\mathcal{M} = \mathcal{M}_S$ and commutative S -algebras $\mathcal{C} = \mathcal{C}_S$. The latter are the commutative monoids in \mathcal{M} . The forgetful functor $\mathbb{U}: \mathcal{C} \rightarrow \mathcal{M}$ has a left adjoint $\mathbb{P}: \mathcal{M} \rightarrow \mathcal{C}$, the free commutative S -algebra functor, giving a Quillen adjunction:

$$\begin{array}{ccc} & \mathbb{P} & \\ \mathcal{C} & \xleftarrow{\quad} & \mathcal{M} \\ & \mathbb{U} & \end{array}$$

For an S -module X ,

$$\mathbb{P}X = \bigvee_{j \geq 0} X^{(j)}/\Sigma_j,$$

where $X^{(j)} = X \wedge \cdots \wedge X$ is the j -fold smash power with its evident Σ_j -action, and $X^{(j)}/\Sigma_j$ is the orbit spectrum. When X is cofibrant, the natural map $D_j X \rightarrow X^{(j)}/\Sigma_j$ is a weak equivalence, hence there is a weak equivalence

$$\bigvee_{j \geq 0} D_j X \xrightarrow{\sim} \mathbb{P}X.$$

The mod p homology of extended powers $D_n X$ has been studied extensively, and the answer is expressible in terms of a free algebra construction. Recently, Kuhn & McCarty [14] gave an explicit description for the prime 2, and we adopt a similar viewpoint. Older references of relevance are May [15], McClure [7, Theorem IX.2.1], and Kuhn [13]. In keeping with our emphasis on coactions and comodule structures, we phrase this in terms of the dual Steenrod algebra, thus avoiding the local finiteness condition for actions of the Steenrod algebra.

Fix a prime p and let $\mathcal{A}_* = \mathcal{A}(p)_*$. We adopt the following notation.

- **Comod** $_{\mathcal{A}_*}$ is the category of \mathbb{Z} -graded right \mathcal{A}_* -comodules, where we denote the coaction by $\Psi: M_* \rightarrow M_* \otimes \mathcal{A}_*$.
- **Vect**^{DL} is the category of graded \mathbb{F}_p -vector spaces V_* equipped with actions of Dyer–Lashof operations $Q^r: V_* \rightarrow V_{*+2(p-1)r}$ and $\beta Q^r: V_* \rightarrow V_{*+2(p-1)r-1}$ (when $p = 2$, $Q^r: V_* \rightarrow V_{*+r}$) subject to the Adem relations and the unstable condition $Q^r v = 0$ if $2r < |v|$ (when $p = 2$, $Q^r v = 0$ if $r < |v|$).
- **Comod** $_{\mathcal{A}_*}^{\text{DL}}$ is the full subcategory of **Comod** $_{\mathcal{A}_*} \cap \mathbf{Vect}^{\text{DL}}$ which consists of right \mathcal{A}_* -comodules with Dyer–Lashof action that satisfies the formulae of Theorem 4.5 when p is odd, or Theorem 4.1 when $p = 2$.

The free algebra $\mathbb{P}X$ has a natural homotopy coproduct $\Delta: \mathbb{P}X \rightarrow \mathbb{P}X \wedge \mathbb{P}X$ induced by the pinch map $X \rightarrow X \vee X$. The induced homomorphism

$$\Delta_*: H_*(\mathbb{P}X) \rightarrow H_*(\mathbb{P}X) \otimes H_*(\mathbb{P}X)$$

turns $H_*(\mathbb{P}X) = H_*(\mathbb{P}X; \mathbb{F}_p)$ into a cocommutative coalgebra, and so $H_*(\mathbb{P}X)$ is a bicommutative Hopf algebra. This structure is discussed in detail in [14, Section 2.3] at least for the prime 2. The component maps of Δ are transfers associated to inclusions of block subgroups $\Sigma_r \times \Sigma_s \leq \Sigma_{r+s}$ and the Dyer–Lashof operations on $H_*(\mathbb{P}X)$ satisfy a Cartan formula making it a bicommutative \mathcal{A}_* -comodule Hopf algebra with Dyer–Lashof action satisfying the restriction condition $Q^{|x|/2}x = x^p$ if $|x|$ is even (and $Q^{|x|}x = x^2$ if $p = 2$). We denote the category of all such bicommutative Hopf algebras by $\mathbf{HopfAlg}^{\mathcal{A}_*, \text{DL}}$.

There are two algebraic free functors that are relevant here.

- The left adjoint

$$\mathbf{R}: \mathbf{Comod}_{\mathcal{A}_*} \rightarrow \mathbf{Comod}_{\mathcal{A}_*}^{\text{DL}}$$

of the forgetful functor $\mathbf{Comod}_{\mathcal{A}_*}^{\text{DL}} \rightarrow \mathbf{Comod}_{\mathcal{A}_*}$; this is a coproduct $\mathbf{R} = \bigoplus_s \mathbf{R}_s$ where the summand \mathbf{R}_s is expressed in terms of Dyer–Lashof words of length s .

- The left adjoint

$$\mathbf{U}: \mathbf{Comod}_{\mathcal{A}_*}^{\text{DL}} \rightarrow \mathbf{HopfAlg}^{\mathcal{A}_*, \text{DL}}$$

of the coalgebra primitives functor $\mathbf{Pr}: \mathbf{HopfAlg}^{\mathcal{A}_*, \text{DL}} \rightarrow \mathbf{Comod}_{\mathcal{A}_*}^{\text{DL}}$; this involves the free graded commutative algebra functor with additional relations coming from the restriction condition.

The structure of $H_*(\mathbb{P}X)$ is given by the next result.

Theorem 8.1. *If X is cofibrant, then in $\mathbf{HopfAlg}^{\mathcal{A}_*, \text{DL}}$ there is a natural isomorphism*

$$\mathbf{U}(\mathbf{R}(H_*(X))) \cong H_*(\mathbb{P}X).$$

Of course this is an abstract version of a description in terms of a free algebra on admissible Dyer–Lashof monomials applied to elements of $H_*(X)$ with suitable excess conditions; see [2] for details.

§9. Sample calculations for $p = 2$

In this section we take $p = 2$, and assume that all spectra are localised at 2.

Consider the commutative S -algebra $S//2$ obtained as the pushout in the diagram of commutative S -algebras

$$\begin{array}{ccc}
 \mathbb{P}S^0 & \longrightarrow & \mathbb{P}D^1 \\
 \downarrow & \lrcorner & \downarrow \\
 S & \xleftarrow{\sim} \tilde{S} \longrightarrow & S//2
 \end{array}$$

in which $S^0 \xrightarrow{\sim} S$ is the functorial cofibrant replacement of S as an S -module, $S^0 \rightarrow D^1 \sim *$ is the functorial cofibrant replacement for the collapse map $S^0 \rightarrow *$, the diagonal map is induced from a realisation of the degree 2 map $S^0 \rightarrow S$, and \tilde{S} is defined using the functorial factorisation in the model category \mathcal{C}_S . It follows that $S//2$ is cofibrant in \mathcal{C}_S , and furthermore there is an isomorphism of commutative S -algebras

$$S//2 \cong \tilde{S} \wedge_{\mathbb{P}S^0} \mathbb{P}D^1.$$

This description allows a calculation of homology using the Künneth spectral sequence. Since the degree 2 map induces the trivial map in mod 2 ordinary homology, we can determine $H_*(S//2) = H_*(S//2; \mathbb{F}_2)$ with the aid of [3, Theorem 1.7]. The answer is

$$H_*(S//2) = \mathbb{F}_2[Q^I x_1 : I \text{ admissible, } \text{exc}(I) > 1],$$

where $x_1 \in H_1(S//2)$ satisfies $\text{Sq}_*^1 x_1 = 1$.

Our formulae for the right coaction give

$$\begin{aligned}
 \tilde{\psi}Q^r x_1 &= \sum_{1 \leq k \leq r} Q^k(x_1 \otimes 1 + 1 \otimes \zeta_1)[1 \otimes \zeta(t)^k]_{tr} \\
 &= \sum_{1 \leq k \leq r} [Q^k x_1 \otimes \zeta(t)^k + 1 \otimes (Q^k \zeta_1)\zeta(t)^k]_{tr}.
 \end{aligned}$$

For example

$$\begin{aligned}
 \tilde{\psi}Q^2 x_1 &= Q^2 x_1 \otimes 1 + x_1^2 \otimes \zeta_1 + 1 \otimes (\zeta_1^3 + \zeta_2) = Q^2 x_1 \otimes 1 + x_1^2 \otimes \zeta_1 + 1 \otimes \zeta_2, \\
 \tilde{\psi}Q^3 x_1 &= Q^3 x_1 \otimes 1 + 1 \otimes Q^3 \zeta_1 = Q^3 x_1 \otimes 1 + 1 \otimes \zeta_1^4,
 \end{aligned}$$

which give

$$\psi Q^2 x_1 = 1 \otimes Q^2 x_1 + \zeta_1 \otimes x_1^2 + \zeta_2 \otimes 1, \quad \psi Q^3 x_1 = 1 \otimes Q^3 x_1 + \zeta_1^4 \otimes 1.$$

Using ideas of [18, 19] we will give a description of $H_*(S//2)$ as an extended \mathcal{A}_* -comodule algebra, which then gives an explicit description of $S//2$ as a wedge of suspensions of $H\mathbb{F}_2$.

First we specify some elements, namely for $s \geq 1$,

$$X_s = Q^{2^s} X_{s-1} = Q^{2^s} Q^{2^{s-1}} \cdots Q^2 x_1,$$

where $X_0 = x_1$. Notice that the degree of X_s is $|X_s| = 2^{s+1} - 1$.

Definition 9.1. When considering an element of the form $Q^I x$, we will say that it is *strictly allowable* if I is admissible and $\text{exc}(I) > |x|$.

This is more stringent than the usual notion of allowable where only $\text{exc}(I) \geq |x|$ is required.

Lemma 9.2. *The Dyer–Lashof monomial $Q^r X_s$ is only strictly allowable if $r = 2^{s+1}$.*

Proof. If $Q^r X_s$ is admissible then $r \leq 2^{s+1}$, while the excess condition holds only if $r > 2^{s+1} - 1$. \square

Next we consider the coaction on these elements.

Proposition 9.3. *In the ring $H_*(S//2)$, the sequence X_0, X_1, X_2, \dots is regular, and generates an ideal $J = (X_s : s \geq 0) \triangleleft H_*(S//2)$. The left coaction on X_s is given by*

$$(9.1) \quad \begin{aligned} \psi X_s &= 1 \otimes X_s + \zeta_1 \otimes X_{s-1}^2 + \zeta_2 \otimes X_{s-2}^{2^2} + \cdots + \zeta_s \otimes X_0^{2^s} + \zeta_{s+1} \otimes 1 \\ &\equiv \zeta_{s+1} \otimes 1 \pmod{J}. \end{aligned}$$

Proof. We begin with the formula

$$\tilde{\psi}(X_0) = X_0 \otimes 1 + 1 \otimes \zeta_1 = X_0 \otimes 1 + 1 \otimes \xi_1.$$

We will verify by induction on s that

$$\tilde{\psi} X_s = X_s \otimes 1 + X_{s-1}^2 \otimes \xi_1 + X_{s-2}^{2^2} \otimes \xi_2 + \cdots + X_0^{2^s} \otimes \xi_s + 1 \otimes \xi_{s+1}.$$

So assume that this holds for some $s \geq 0$. We have

$$\begin{aligned} \tilde{\psi} X_{s+1} &= \tilde{\psi} Q^{2^{s+1}} X_s = (Q^{2^{s+1}-1} \tilde{\psi} X_s)(1 \otimes \xi_1) + Q^{2^{s+1}} \tilde{\psi} X_s \\ &= X_s^2 \otimes \xi_1 + X_{s-1}^{2^2} \otimes \xi_1^3 + X_{s-2}^{2^3} \otimes \xi_2^2 \xi_1 \\ &\quad + \cdots + X_0^{2^{s+1}} \otimes \xi_s^2 \xi_1 + 1 \otimes \xi_{s+1}^2 \xi_1 + X_{s+1} \otimes 1 \\ &\quad + Q^{2^{s+1}-2} (X_{s-1}^{2^2}) \otimes Q^2 \xi_1 + Q^{2^{s+1}-2^2} (X_{s-2}^{2^2}) \otimes Q^{2^2} \xi_2 \\ &\quad + \cdots + Q^{2^{s+1}-2^s} (X_0^{2^s}) \otimes Q^{2^s} \xi_2 + 1 \otimes Q^{2^{s+1}} \xi_{s+1} \end{aligned}$$

$$\begin{aligned}
 &= X_s^2 \otimes \xi_1 + X_{s-1}^{2^2} \otimes \xi_1^3 + X_{s-2}^{2^3} \otimes \xi_2^2 \xi_1 + \cdots + X_0^{2^{s+1}} \otimes \xi_s^2 \xi_1 \\
 &\quad + 1 \otimes \xi_{s+1}^2 \xi_1 + X_{s+1} \otimes 1 + X_{s-1}^{2^2} \otimes Q^2 \xi_1 + X_{s-2}^{2^3} \otimes Q^2 \xi_2 \\
 &\quad + \cdots + X_0^{2^{s+1}} \otimes Q^{2^s} \xi_s + 1 \otimes Q^{2^{s+1}} \xi_{s+1} \\
 &= X_{s+1} \otimes 1 + X_s^2 \otimes \xi_1 + X_{s-1}^{2^2} \otimes (\xi_1^3 + Q^2 \xi_1) + X_{s-2}^{2^3} \otimes (\xi_2^2 \xi_1 + Q^{2^2} \xi_2) \\
 &\quad + \cdots + X_0^{2^{s+1}} \otimes (\xi_s^2 \xi_1 + Q^{2^s} \xi_s) + 1 \otimes (\xi_{s+1}^2 \xi_1 + Q^{2^{s+1}} \xi_{s+1}) \\
 &= X_{s+1} \otimes 1 + X_s^2 \otimes \xi_1 + X_{s-1}^{2^2} \otimes \xi_2 + X_{s-2}^{2^3} \otimes \xi_3 \\
 &\quad + \cdots + X_0^{2^{s+1}} \otimes \xi_{s+1} + 1 \otimes \xi_{s+2},
 \end{aligned}$$

where we make use of Lemma 4.4 in the last step. □

Now consider the following composition of left \mathcal{A}_* -comodule algebra homomorphisms:

$$\begin{array}{ccc}
 & \xrightarrow{\quad \bar{\psi} \quad} & \\
 H_*(S//2) & \xrightarrow{\quad \psi \quad} \mathcal{A}_* \otimes H_*(S//2) & \xrightarrow{\quad \text{quo} \quad} \mathcal{A}_* \otimes H_*(S//2)/J
 \end{array}$$

where the second and third terms are extended left comodules. By Proposition 9.3 this composition is an isomorphism of comodule algebras

$$H_*(S//2) \xrightarrow[\cong]{\bar{\psi}} \mathcal{A}_* \otimes H_*(S//2)/J$$

and there is a polynomial subalgebra $P_* \subseteq H_*(S//2)$ with $\bar{\psi}P_* = \mathbb{F}_2 \otimes H_*(S//2)/J$. A standard argument shows that

$$\pi_*(S//2) \cong P_* = \text{Ext}_{\mathcal{A}_*}^{0,*}(\mathbb{F}_2, H_*(S//2)) \subseteq H_*(S//2),$$

and in fact as a spectrum $S//2$ is weakly equivalent to a wedge of suspensions of $H\mathbb{F}_2$, and a choice of basis for P_* determines such a splitting.

We remark that any connective commutative S -algebra E for which $0 = 2 \in \pi_0(E)$ admits a morphism of commutative S -algebras $u: S//2 \rightarrow E$. Using the commutative diagram of \mathbb{F}_2 -algebras

$$\begin{array}{ccc}
 H_*(S//2) & \xrightarrow{\quad \psi \quad} & \mathcal{A}_* \otimes H_*(S//2) \\
 u_* \downarrow & & \downarrow I \otimes u_* \\
 H_*(E) & \xrightarrow{\quad \psi \quad} & \mathcal{A}_* \otimes H_*(E)
 \end{array}$$

we see that

$$\psi(u_* X_s) = 1 \otimes u_* X_s + \zeta_1 \otimes u_* X_{s-1}^2 + \zeta_2 \otimes u_* X_{s-2}^{2^2} + \cdots + \zeta_s \otimes u_* X_1^{2^s} + \zeta_{s+1} \otimes 1$$

so the u_*X_s is a sequence of algebraically independent elements. It follows that there is an isomorphism of \mathcal{A}_* -comodule algebras

$$H_*(E) \cong \mathcal{A}_* \otimes H_*(E)/(u_*X_s : s \geq 0),$$

so $H_*(E)$ is also an extended comodule and E is weakly equivalent to a wedge of suspensions of $H\mathbb{F}_2$. This gives a different approach to proving Steinberger's result [7, Theorem III.4.1], which potentially contains more information on the multiplicative structure of the splitting.

Rolf Hoyer has pointed out some explicit formulae for primitives in $H_*(S//2)$ and thus for families of polynomial generators for $\pi_*(S//2)$.

§10. Sample calculations for odd primes

Now we assume that p is an odd prime and that all spectra are localised at p . There are similarities to the 2-primary case, although some of the details are slightly more complicated.

Consider the commutative S -algebra $S//p$ which is the pushout in the diagram of commutative S -algebras

$$\begin{array}{ccc} \mathbb{P}S^0 & \xrightarrow{\quad} & \mathbb{P}D^1 \\ \swarrow & \downarrow \Gamma & \downarrow \\ S & \xleftarrow{\sim} \tilde{S} \xrightarrow{\quad} & S//p \end{array}$$

where the notation is similar to that in the case $p = 2$. Then $S//p$ is cofibrant in \mathcal{C}_S , and there is an isomorphism of commutative S -algebras

$$S//p \cong \tilde{S} \wedge_{\mathbb{P}S^0} \mathbb{P}D^1.$$

Since the degree p map induces the trivial map in mod p ordinary homology, $H_*(S//p) = H_*(S//p; \mathbb{F}_p)$ can be determined by methods of [3, Theorem 1.7]. The answer is a free graded commutative algebra

$$H_*(S//p) = \mathbb{F}_p \langle Q^I x_1 : I \text{ admissible, } \text{exc}(I) > 1 \rangle,$$

where $x_1 \in H_1(S//p)$ satisfies $\beta x_1 = 1$.

We define two sequences of elements, beginning with $X_0 = x_1$ and $Y_0 = 1$,

$$\begin{aligned} X_s &= Q^{p^{s-1}} X_{s-1} = Q^{p^{s-1}} Q^{p^{s-2}} \cdots Q^p Q^1 x_1, \\ Y_s &= \beta Q^{p^{s-1}} X_{s-1} = \beta Q^{p^{s-1}} Q^{p^{s-2}} \cdots Q^p Q^1 x_1. \end{aligned}$$

Notice that the degrees of these elements are $|X_s| = 2p^s - 1$ and $|Y_s| = 2(p^s - 1)$.

We will again use the terminology of strictly allowable introduced in Definition 9.1.

Lemma 10.1. *Let $r \geq 1$. Then the Dyer–Lashof monomial $Q^r X_s$ is strictly allowable only if $r = p^s$, while $Q^r Y_s$ is never strictly allowable.*

Proof. If $Q^r X_s = Q^r Q^{p^{s-1}} X_{s-1}$ is admissible then $r \leq p^s$, while the required excess condition is $2r > 2p^s - 1$.

If $Q^r Y_s = Q^r \beta Q^{p^{s-1}} X_{s-1}$ is admissible then $r < p^s$, while the excess condition required for it to be strictly allowable is $2r > 2(p^s - 1)$. Clearly these conditions are contradictory. \square

Proposition 10.2. *The left coaction on X_s and Y_s is given by*

$$(10.1) \quad \psi X_s = 1 \otimes X_s + \bar{\tau}_0 \otimes Y_s + \bar{\tau}_1 \otimes Y_{s-1}^p + \bar{\tau}_2 \otimes Y_{s-2}^{p^2} + \cdots + \bar{\tau}_{s-1} \otimes Y_1^{p^{s-1}} + \bar{\tau}_s \otimes 1,$$

$$(10.2) \quad \psi Y_s = 1 \otimes Y_s + \zeta_1 \otimes Y_{s-1}^p + \zeta_2 \otimes Y_{s-2}^{p^2} + \cdots + \zeta_{s-1} \otimes Y_1^{p^{s-1}} + \zeta_s \otimes 1.$$

Proof. Translating the formulae into statements about the right coaction we must prove that the following equations are satisfied for every s :

$$\begin{aligned} \tilde{\psi} X_s &= X_s \otimes 1 + Y_s \otimes \tau_0 + Y_{s-1}^p \otimes \tau_1 + Y_{s-2}^{p^2} \otimes \tau_2 + \cdots + Y_1^{p^{s-1}} \otimes \tau_{s-1} + 1 \otimes \tau_s, \\ \tilde{\psi} Y_s &= Y_s \otimes 1 + Y_{s-1}^p \otimes \xi_1 + Y_{s-2}^{p^2} \otimes \xi_2 + \cdots + Y_1^{p^{s-1}} \otimes \xi_{s-1} + 1 \otimes \xi_s. \end{aligned}$$

Assuming these are true for some s , we have

$$\begin{aligned} \tilde{\psi} X_{s+1} &= \tilde{\psi} Q^{p^s} X_s = Q^{p^s} (\tilde{\psi} X_s) - \beta Q^{p^s} (\tilde{\psi} X_s) \bar{\tau}_0 \\ &= Q^{p^s} (X_s \otimes 1 + Y_s \otimes \tau_0 + Y_{s-1}^p \otimes \tau_1 + Y_{s-2}^{p^2} \otimes \tau_2 + \cdots + Y_1^{p^{s-1}} \otimes \tau_{s-1} + 1 \otimes \tau_s) \\ &\quad + \beta Q^{p^s} (X_s \otimes 1 + Y_s \otimes \tau_0 + Y_{s-1}^p \otimes \tau_1 + Y_{s-2}^{p^2} \otimes \tau_2 + \cdots + Y_1^{p^{s-1}} \otimes \tau_{s-1} + 1 \otimes \tau_s) \tau_0 \\ &= (Q^{p^s} X_s \otimes 1 + (Q^{p^s-1} Y_s) \otimes Q^1 \tau_0 + (Q^{p^s-1-1} Y_{s-1})^p \otimes Q^p \tau_1 \\ &\quad + (Q^{p^s-2-1} Y_{s-2})^{p^2} \otimes Q^{p^2} \tau_2 + \cdots + (Q^{p-1} Y_1)^{p^{s-1}} \otimes Q^{p^{s-1}} \tau_{s-1} + 1 \otimes Q^{p^s} \tau_s) \\ &\quad + (\beta Q^{p^s} X_s \otimes 1 + (Q^{p^s-1} Y_s) \otimes \beta Q^1 \tau_0 + (Q^{p^s-1-1} Y_{s-1})^p \otimes \beta Q^p \tau_1 \\ &\quad + (Q^{p^s-2-1} Y_{s-2})^{p^2} \otimes \beta Q^{p^2} \tau_2 + \cdots + (Q^{p-1} Y_1)^{p^{s-1}} \otimes \beta Q^{p^{s-1}} \tau_{s-1} + 1 \otimes \beta Q^{p^s} \tau_s) \tau_0 \\ &= (X_{s+1} \otimes 1 + Y_s^p \otimes (\tau_1 - \tau_0 \xi_1) + Y_{s-1}^{p^2} \otimes (\tau_2 - \tau_0 \xi_2) + Y_{s-2}^{p^3} \otimes (\tau_3 - \tau_0 \xi_3) \\ &\quad + \cdots + Y_1^{p^s} \otimes (\tau_s - \tau_0 \xi_s) + 1 \otimes (\tau_{s+1} - \tau_0 \xi_{s+1})) \\ &\quad + (Y_{s+1} \otimes 1 + Y_s^p \otimes \xi_1 + Y_{s-1}^{p^2} \otimes \xi_2 + Y_{s-2}^{p^3} \otimes \xi_3 + \cdots + Y_1^{p^s} \otimes \xi_s + 1 \otimes \xi_{s+1}) \tau_0 \\ &= X_{s+1} \otimes 1 + Y_{s+1} \otimes \tau_0 + Y_s^p \otimes \tau_1 + Y_{s-1}^{p^2} \otimes \tau_2 + \cdots + Y_1^{p^s} \otimes \tau_s + 1 \otimes \tau_{s+1}. \end{aligned}$$

A similar calculation shows that

$$\begin{aligned} \tilde{\psi}Y_{s+1} &= Y_{s+1} \otimes 1 + Y_s^p \otimes \xi_1 + Y_{s-1}^{p^2} \otimes \xi_2 \\ &\quad + \cdots + Y_2^{p^{s-1}} \otimes \xi_{s-1} + Y_1^{p^s} \otimes \xi_s + 1 \otimes \xi_{s+1}. \end{aligned}$$

The result follows by induction. \square

Let $J = (X_s, Y_{s+1} : s \geq 0) \triangleleft H_*(S//p)$ be the ideal generated by the elements X_s, Y_s . As happens for the prime 2, the following composition of left \mathcal{A}_* -comodule algebra homomorphisms

$$H_*(S//p) \xrightarrow{\psi} \mathcal{A}_* \otimes H_*(S//p) \xrightarrow{\text{quo}} \mathcal{A}_* \otimes H_*(S//p)/J$$

$\bar{\psi}$

is an isomorphism, where the second and third terms are extended left comodules. Here $H_*(S//p)/J$ is a free graded commutative algebra since the generators X_s and Y_s are amongst the generators of the free graded commutative algebra $H_*(S//p)$. There is a subalgebra $P_* \subseteq H_*(S//p)$ which is identified with $\mathbb{F}_p \otimes H_*(S//p)/J$ under the isomorphism $\bar{\psi}$, i.e., $\bar{\psi}P_* = \mathbb{F}_p \otimes H_*(S//p)/J$. A standard argument shows the spectrum $S//p$ is equivalent to a wedge of suspensions of $H\mathbb{F}_p$. As we saw in the 2-primary case, this leads to a proof of Steinberger's result [7, Theorem III.4.1].

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