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Non-local Operators, Non-Archimedean Parabolic-type Equations with Variable Coefficients and Markov Processes

by

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Abstract

In this article, we introduce a new class of parabolic-type pseudodifferential equations with variable coefficients over the p-adics. We establish the existence and uniqueness of solutions for the Cauchy problem associated with these equations. The fundamental solutions of these equations are connected with Markov processes. Some of these equations are related to new models of complex systems.

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§1. Introduction

In [3]–[4], Avetisov et al. introduced a new class of models for complex systems based on p-adic analysis. In these models the time-evolution of a complex system is described by a p-adic master equation (a parabolic-type pseudodifferential equation) which controls the time-evolution of a transition function of a Markov process on an ultrametric space, and this stochastic process is used to describe the dynamics of the system in the space of configuration states which is approximated by an ultrametric space ($\mathbb{Q}_p$).

In [7], the present authors introduced a new type of non-local operators which are naturally connected with parabolic-type pseudodifferential equations. Build-

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ing up on [7] and [14], in this article, we introduce a new class of parabolic-type pseudodifferential equations with variable coefficients, which contains the one-dimensional p-adic heat equation of [17], the equations studied by Kochubei [14], and the equations studied by Rodríguez-Vega [15]. Our theory is not applicable to the equations studied in [6], [18]. We establish the existence and uniqueness of solutions for the Cauchy problem for such equations (see Theorems 4.1, 5.5, 6.3). We show that the fundamental solutions of these equations are transition density functions of Markov processes (see Theorem 7.4). Finally, we study the well-posedness of the Cauchy problem (see Theorem 8.1).

On the other hand, stochastic processes on p-adic spaces, or more generally on ultrametric spaces, have been studied extensively (see e.g. [1], [3], [4], [7], [9], [12], [13], [14], [17], [18], and the references therein). In [1] and [13] a very large class of stochastic processes was constructed on treelike graphs, which includes the p-adics. This type of constructions does not need algebraic, topological or analytical properties of the p-adic numbers, it uses only the hierarchical structure of the p-adics. The construction of our stochastic processes does not rely on the hierarchical structure of the p-adics, but on some strong analytical properties of and methods for the p-adics. There are some recent developments on analysis of jump processes (that correspond to non-local operators) using Dirichlet forms on metric measure spaces, not necessarily on p-adic spaces (see e.g. [5], [8]). The last reference discusses heat kernel estimates for "mixtures of stable-like processes" (see Example 2.3 there) on some metric measure spaces, similar to what is discussed in Section 4 of this article.

§2. Preliminaries

In this section we fix the notation and collect some basic results of p-adic analysis that we will use throughout the article. For a detailed exposition the reader may consult [2], [17].

§2.1. The field of p-adic numbers

Along this article p will denote a prime number. The field $\mathbb{Q}_p$ of p-adic numbers is defined as the completion of the field $\mathbb{Q}$ of rational numbers with respect to the p-adic norm $|\cdot|_p$, which is defined as

$$|x|_p = \begin{cases} 0 & \text{if } x = 0, \\ p^{-\gamma} & \text{if } x = p^{\gamma}\dfrac{a}{b}, \end{cases}$$

where a and b are integers coprime with p. The integer $\gamma := \operatorname{ord}(x)$, with $\operatorname{ord}(0) := +\infty$, is called the *$p$-adic order* of x. We extend the p-adic norm to $\mathbb{Q}_p^n$ by taking

$$\|x\|_p := \max_{1\leq i\leq n} |x_i|_p \qquad \text{for } x = (x_1, \dots, x_n) \in \mathbb{Q}_p^n.$$

We define $\mathrm{ord}(x) = \min_{1\le i\le n} \mathrm{ord}(x_i)$; then $\|x\|_p = p^{-\mathrm{ord}(x)}$. The set $(\mathbb{Q}_p^n, \|\cdot\|_p)$ is a complete ultrametric space. As a topological space, $\mathbb{Q}_p$ is homeomorphic to a Cantor-like subset of the real line.

Any p-adic number $x \neq 0$ has a unique expansion $x = p^{\mathrm{ord}(x)} \sum_{j=0}^{\infty} x_j p^j$, where $x_j \in \{0, 1, \dots, p-1\}$ and $x_0 \neq 0$. By using this expansion, we define the *fractional part* of $x \in \mathbb{Q}_p$, denoted $\{x\}_p$, as the rational number

$$\{x\}_p = \begin{cases} 0 & \text{if } x = 0 \text{ or } \mathrm{ord}(x) \ge 0, \\ p^{\mathrm{ord}(x)} \sum_{j=0}^{-\mathrm{ord}(x)-1} x_j p^j & \text{if } \mathrm{ord}(x) < 0. \end{cases}$$

For $\gamma \in \mathbb{Z}$, denote by $B_\gamma^n(a) = \{x \in \mathbb{Q}_p^n : \|x - a\|_p \le p^\gamma\}$ the *ball of radius* p^γ *with center at* $a = (a_1, \dots, a_n) \in \mathbb{Q}_p^n$, and take $B_\gamma^n(0) := B_\gamma^n$. Notice that $B_\gamma^n(a) = B_\gamma(a_1) \times \cdots \times B_\gamma(a_n)$, where $B_\gamma(a_i) := \{x \in \mathbb{Q}_p : |x - a_i|_p \le p^\gamma\}$ is the one-dimensional ball of radius p^γ with center at $a_i \in \mathbb{Q}_p$. The ball B_0^n equals the product of n copies of $B_0 := \mathbb{Z}_p$, the *ring of* p*-adic integers*. We denote by $\Omega(\|x\|_p)$ the characteristic function of B_0^n. For more general sets, say Borel sets, we use $1_A(x)$ to denote the characteristic function of A.

§2.2. The Bruhat–Schwartz space

A complex-valued function φ defined on $\mathbb{Q}_p^n$ is called *locally constant* if for any $x \in \mathbb{Q}_p^n$ there exists an integer $l = l(x) \in \mathbb{Z}$ such that

$$\varphi(x + x') = \varphi(x) \quad \text{for } x' \in B_l^n. \tag{2.1}$$

The set of all locally constant functions φ for which the integer $l(x)$ is independent of x form a $\mathbb{C}$-vector space denoted by $\widetilde{\mathcal{E}}(\mathbb{Q}_p^n) =: \widetilde{\mathcal{E}}$. Given $\varphi \in \widetilde{\mathcal{E}}$, we call the largest possible $l = l(\varphi)$ the *parameter of local constancy of* φ. A function $\varphi : \mathbb{Q}_p^n \to \mathbb{C}$ is called a *Bruhat–Schwartz function* (or a *test function*) if it is locally constant with compact support. The $\mathbb{C}$-vector space of Bruhat–Schwartz functions is denoted by $S(\mathbb{Q}_p^n) =: S$. Notice that $S \subset \widetilde{\mathcal{E}}$.

Let $S'(\mathbb{Q}_p^n) := S'$ denote the set of all functionals (distributions) on $S(\mathbb{Q}_p^n)$. All functionals on $S(\mathbb{Q}_p^n)$ are continuous.

Set $\Psi(y) = \exp(2\pi i \{y\}_p)$ for $y \in \mathbb{Q}_p$. The map $\Psi(\cdot)$ is an additive character on $\mathbb{Q}_p$, i.e. a continuous map from $\mathbb{Q}_p$ into the unit circle satisfying $\Psi(y_0 + y_1) = \Psi(y_0)\Psi(y_1)$ for all $y_0, y_1 \in \mathbb{Q}_p$.

§2.3. Fourier transform

Given $\xi = (\xi_1, \dots, \xi_n), x = (x_1, \dots, x_n) \in \mathbb{Q}_p^n$, we set $\xi \cdot x := \sum_{j=1}^n \xi_j x_j$. The *Fourier transform* of $\varphi \in S(\mathbb{Q}_p^n)$ is defined as

$$(\mathcal{F}\varphi)(\xi) = \int_{\mathbb{Q}_p^n} \Psi(-\xi \cdot x)\varphi(x)\, d^n x \quad \text{for } \xi \in \mathbb{Q}_p^n,$$

where $d^n x$ is the Haar measure on $\mathbb{Q}_p^n$ normalized by the condition $\mathrm{vol}(B_0^n) = 1$. The Fourier transform is a linear isomorphism from $S(\mathbb{Q}_p^n)$ onto itself satisfying $(\mathcal{F}(\mathcal{F}\varphi))(\xi) = \varphi(-\xi)$. We will also use the notation $\mathcal{F}_{x\to\xi}\varphi$ and $\widehat{\varphi}$ for the Fourier transform of φ.

The Fourier transform $\mathcal{F}[T]$ of a distribution $T \in S'(\mathbb{Q}_p^n)$ is defined by

$$(\mathcal{F}[T], \varphi) = (T, \mathcal{F}[\varphi]) \qquad \text{for all } \varphi \in S(\mathbb{Q}_p^n).$$

The Fourier transform $T \mapsto \mathcal{F}[T]$ is a linear isomorphism from $S'(\mathbb{Q}_p^n)$ onto $S'(\mathbb{Q}_p^n)$. Furthermore, $T = \mathcal{F}[\mathcal{F}[T](-\xi)]$.

§3. A class of non-local operators

Denote by $\mathfrak{M}_\lambda$, with $\lambda \geq 0$, the $\mathbb{C}$-vector space of all functions $\varphi \in \widetilde{\mathcal{E}}$ satisfying $|\varphi(x)| \leq C(1 + \|x\|_p^\lambda)$. If φ depends also on a parameter t, we shall say that φ belongs to $\mathfrak{M}_\lambda$ *uniformly with respect to* t if its constant C and its parameter of local constancy do not depend on t. Notice that if $0 \leq \lambda_1 \leq \lambda_2$, then $\mathfrak{M}_0 \subseteq \mathfrak{M}_{\lambda_1} \subseteq \mathfrak{M}_{\lambda_2}$, and $S(\mathbb{Q}_p^n) \subseteq \mathfrak{M}_0$.

Take $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$, and fix a function $w_\alpha : \mathbb{Q}_p^n \to \mathbb{R}_+$ with the following properties:

(i) $w_\alpha(y)$ is a radial (i.e. $w_\alpha(y) = w_\alpha(\|y\|_p)$) and continuous function, increasing in $\|y\|_p$;

(ii) $w_\alpha(y) = 0$ if and only if $y = 0$;

(iii) there exist constants $C_0, C_1 > 0$ and $\alpha > n$ such that

$$C_0\|y\|_p^\alpha \leq w_\alpha(\|y\|_p) \leq C_1\|y\|_p^\alpha \qquad \text{for any } y \in \mathbb{Q}_p^n.$$

Set

$$A_{w_\alpha}(\xi) := \int_{\mathbb{Q}_p^n} \frac{1 - \Psi(-y \cdot \xi)}{w_\alpha(\|y\|_p)}\, d^n y.$$

In [7], we established that the function A_{w_α} is radial, positive, continuous, $A_{w_\alpha}(0) = 0$, and $A_{w_\alpha}(\xi) = A_{w_\alpha}(\|\xi\|_p) = A_{w_\alpha}(p^{-\operatorname{ord}(\xi)})$ is a decreasing function of $\operatorname{ord}(\xi)$ (cf. [7, Lemma 3.2]; the condition that $w_\alpha(\|y\|_p)$ be increasing was omitted there by error, but it is necessary). In addition, we introduce the operator

$$(3.1) \qquad (\mathbf{W}_\alpha\varphi)(x) = \int_{\mathbb{Q}_p^n} \frac{\varphi(x-y) - \varphi(x)}{w_\alpha(\|y\|_p)}\, d^n y, \qquad \varphi \in S(\mathbb{Q}_p^n).$$

Lemma 3.1. *If $\alpha - n > \lambda$, then $\mathbf{W}_\alpha$ can be extended to $\mathfrak{M}_\lambda$ and formula (3.1) holds. Furthermore, $\mathbf{W}_\alpha : \mathfrak{M}_\lambda \to \mathfrak{M}_\lambda$.*

Proof. Notice that if $\varphi \in \mathfrak{M}_\lambda$, there exists a constant $l = l(\varphi) \in \mathbb{Z}$ such that

$$(\mathbf{W}_\alpha\varphi)(x) = \int_{\|y\|_p \geq p^l} \frac{\varphi(x-y) - \varphi(x)}{w_\alpha(\|y\|_p)}\, d^n y. \tag{3.2}$$

We now show that $|(\mathbf{W}_\alpha\varphi)(x)| \leq A(1 + \|x\|_p^\lambda)$. Since $\varphi \in \mathfrak{M}_\lambda$ and $\alpha > n$,

$$|(\mathbf{W}_\alpha\varphi)(x)| \leq C \int_{\|y\|_p \geq p^l} \frac{1 + \|x-y\|_p^\lambda}{\|y\|_p^\alpha}\, d^n y + C'(1 + \|x\|_p^\lambda).$$

Hence, it is sufficient to show that the above integral is bounded by $A(1 + \|x\|_p^\lambda)$ for some positive constant A. If $\|x\|_p > \|y\|_p$, then

$$\int_{\|y\|_p \geq p^l} \frac{1 + \|x-y\|_p^\lambda}{\|y\|_p^\alpha}\, d^n y \leq (1 + \|x\|_p^\lambda) \int_{\|y\|_p \geq p^l} \frac{1}{\|y\|_p^\alpha}\, d^n y = B(1 + \|x\|_p^\lambda),$$

where B is a positive constant. If $\|x\|_p < \|y\|_p$, then $\alpha - n > \lambda$ implies

$$\int_{\|y\|_p \geq p^l} \frac{1 + \|x-y\|_p^\lambda}{\|y\|_p^\alpha}\, d^n y \leq \int_{\|y\|_p \geq p^l} \frac{1 + \|y\|_p^\lambda}{\|y\|_p^\alpha}\, d^n y < \infty.$$

If $\|x\|_p = \|y\|_p \geq p^l$, write $x = p^L u$, $y = p^L v$, with $\|v\|_p = \|u\|_p = 1$, $L \in \mathbb{Z}$; then

$$\begin{aligned}\int_{\|y\|_p = \|x\|_p} \frac{1 + \|x-y\|_p^\lambda}{\|y\|_p^\alpha}\, d^n y &= p^{-L(n-\alpha)} \int_{\|v\|_p = 1} (1 + p^{-L\lambda}\|u - v\|_p^\lambda)\, d^n v \\ &\leq A(\|x\|_p^{-(\alpha-n)} + \|x\|_p^{-(\alpha-n-\lambda)}) \leq A'(p, l, \alpha, n, \lambda),\end{aligned}$$

where A, A' are positive constants. Finally, by (3.2), $\mathbf{W}_\alpha\varphi$ is locally constant. □

§4. Parabolic-type equations with constant coefficients

Consider the following Cauchy problem:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) - \kappa \cdot (\mathbf{W}_\alpha u)(x,t) = f(x,t), & x \in \mathbb{Q}_p^n,\ t \in (0,T], \\ u(x,0) = \varphi(x), \end{cases} \tag{4.1}$$

where $\alpha > n$, κ, T are positive constants, $\varphi \in \mathrm{Dom}(\mathbf{W}_\alpha) := \mathfrak{M}_\lambda$ with $\alpha - n > \lambda$, f is continuous in (x,t) and belongs to $\mathfrak{M}_\lambda$ uniformly with respect to t, and $u : \mathbb{Q}_p^n \times [0,T] \to \mathbb{C}$ is an unknown function. We say that $u(x,t)$ is a *solution* of (4.1) if $u(x,t)$ is continuous in (x,t), $u(\cdot,t) \in \mathrm{Dom}(\mathbf{W}_\alpha)$ for $t \in [0,T]$, $u(x,\cdot)$ is continuously differentiable on $(0,T]$, $u(\cdot,t) \in \mathfrak{M}_\lambda$ uniformly in t, and u satisfies (4.1) for all $t > 0$.

The Cauchy problem (4.1) was studied in [7] using semigroup theory. In this article, we study this problem in the space $\mathfrak{M}_\lambda$, which is not contained in L^ρ for any $\rho \in [1,\infty]$, and thus we cannot use semigroup theory (see e.g. [7, Theorem 6.5]).

We define

$$Z(x,t) := \int_{\mathbb{Q}_p^n} e^{-\kappa t A_{w_\alpha}(\|\xi\|_p)} \Psi(x \cdot \xi)\, d^n\xi, \tag{4.2}$$

for $t > 0$ and $x \in \mathbb{Q}_p^n$. Notice that $Z(x,t) = \mathcal{F}^{-1}_{\xi \to x}[e^{-\kappa t A_{w_\alpha}(\|\xi\|_p)}] \in L^1 \cap L^2$ for $t > 0$, since $C'\|\xi\|_p^{\alpha-n} \leq A_{w_\alpha}(\|\xi\|_p) \leq C''\|\xi\|_p^{\alpha-n}$ (cf. [7, Lemma 3.4]). Furthermore, $Z(x,t) \geq 0$ for $t > 0$ and $x \in \mathbb{Q}_p^n$ (cf. [7, Theorem 4.3(i)]). These functions are called *heat kernels*. When considering $Z(x,t)$ as a function of x for t fixed we will write $Z_t(x)$.

We set

$$u_1(x,t) := \int_{\mathbb{Q}_p^n} Z(x-y,t)\varphi(y)\, d^n y, \quad u_2(x,t) := \int_0^t \int_{\mathbb{Q}_p^n} Z(x-y,t-\theta) f(y,\theta)\, d^n y\, d\theta,$$

for φ, $f \in \mathfrak{M}_\lambda$ with $\alpha - n > \lambda$, $0 \leq t \leq T$, and $x \in \mathbb{Q}_p^n$. The main result of this section is the following:

Theorem 4.1. *The function $u(x,t) = u_1(x,t) + u_2(x,t)$ is a solution of the Cauchy problem (4.1).*

The proof requires several steps.

§4.1. Claim: $u(x,t) \in \mathfrak{M}_\lambda$

In order to prove this claim, we need some preliminary results.

Remark 4.2. The function $Z_t(x)$ is radial since it is the inverse Fourier transform of the radial function $e^{-\kappa t A_{w_\alpha}(\|\xi\|_p)}$. Then $Z_t(x)$ is locally constant in $\mathbb{Q}_p^n \setminus \{0\}$. Furthermore, if $y \in \mathbb{Q}_p^n$, $x \in \mathbb{Q}_p^n \setminus \{0\}$, $\|y\|_p < \|x\|_p$ and $t > 0$, then $Z_t(x+y) = Z_t(x)$.

Lemma 4.3. *There exist positive constants C_1, C_2 such that:*

(i) $Z(x,t) \leq C_1 t^{-n/(\alpha-n)}$ *for* $t > 0$ *and* $x \in \mathbb{Q}_p^n$*;*

(ii) $Z(x,t) \leq C_2 t \|x\|_p^{-\alpha}$ *for* $t > 0$ *and* $x \in \mathbb{Q}_p^n \setminus \{0\}$*;*

(iii) $Z(x,t) \leq \max\{2^\alpha C_1, 2^\alpha C_2\} t (\|x\|_p + t^{1/(\alpha-n)})^{-\alpha}$ *for* $t > 0$ *and* $x \in \mathbb{Q}_p^n$*;*

(iv) $\int_{\mathbb{Q}_p^n} Z(x,t)\, d^n x = 1$ *for* $t > 0$*.*

Proof. (i) By (4.2) and [7, Lemma 3.4],

$$Z(x,t) \leq \int_{\mathbb{Q}_p^n} e^{-\kappa t A_{w_\alpha}(\|\xi\|_p)}\, d^n\xi \leq \int_{\mathbb{Q}_p^n} e^{-C_0 t \|\xi\|_p^{\alpha-n}}\, d^n\xi.$$

Let m be an integer such that $p^{m-1} \leq t^{1/(\alpha-n)} \leq p^m$. Then

$$Z(x,t) \leq \int_{\mathbb{Q}_p^n} e^{-C_0 \|p^{-(m-1)}\xi\|_p^{\alpha-n}}\, d^n\xi.$$

By changing variables via $z = p^{-(m-1)}\xi$, we have

$$Z(x,t) \le p^{-(m-1)n} \int_{\mathbb{Q}_p^n} e^{-C_0\|z\|_p^{\alpha-n}}\, d^n z \le C_1 t^{-n/(\alpha-n)}.$$

(ii) follows from [7, Lemma 4.1].

(iii) The result is obtained from the following two inequalities. If $\|x\|_p \ge t^{1/(\alpha-n)}$, then $\|x\|_p \ge \|x\|_p/2 + t^{1/(\alpha-n)}/2$ and $\|x\|_p^{-\alpha} \le 2^\alpha(\|x\|_p + t^{1/(\alpha-n)})^{-\alpha}$; multiplying by $C_2 t$ and using (ii), we obtain

$$Z(x,t) \le 2^\alpha C_2 t(\|x\|_p + t^{1/(\alpha-n)})^{-\alpha}.$$

If $\|x\|_p \le t^{1/(\alpha-n)}$, then $\|x\|_p/2 + t^{1/(\alpha-n)}/2 \le t^{1/(\alpha-n)}$ and $(\|x\|_p + t^{1/(\alpha-n)})^{-\alpha} \ge 2^{-\alpha}t^{-\alpha/(\alpha-n)} = 2^{-\alpha}t^{-1-n/(\alpha-n)}$; multiplying by C_1 and using (i) gives

$$Z(x,t) \le 2^\alpha C_1 t(\|x\|_p + t^{1/(\alpha-n)})^{-\alpha}.$$

(iv) By (iii), $Z_t(\cdot) \in L^1(\mathbb{Q}_p^n)$ for $t > 0$. Now, the announced identity follows by applying the Fourier inversion formula. □

Proposition 4.4 ([16, Proposition 2]). *If $b > 0$, $0 \le \lambda < \alpha$, and $x \in \mathbb{Q}_p^n$, then*

$$\int_{\mathbb{Q}_p^n} (b + \|x-\xi\|_p)^{-\alpha-n}\|\xi\|_p^\lambda\, d^n\xi \le Cb^{-\alpha}(1 + \|x\|_p^\lambda),$$

where the constant C does not depend on b or x.

Lemma 4.5. *The functions u_1, u_2 belong to $\mathfrak{M}_\lambda$ uniformly in t for $\lambda + n < \alpha$.*

Proof. By Lemma 4.3(iii) and Proposition 4.4,

$$\begin{aligned} |u_1(x,t)| &\le \int_{\mathbb{Q}_p^n} Z(x-y,t)|\varphi(y)|\, d^n y \\ &\le C\int_{\mathbb{Q}_p^n} t(t^{1/(\alpha-n)} + \|x-y\|_p)^{-\alpha}(1 + \|y\|_p^\lambda)\, d^n y \le C'(1 + \|x\|_p^\lambda). \end{aligned}$$

On the other hand, since $u_1(x,t) = \int_{\mathbb{Q}_p^n} Z(w,t)\varphi(x-w)\, d^n w$, u_1 is locally constant and $l(u_1) = l(\varphi)$ uniformly in t. The proof for u_2 is similar. □

Remark 4.6. Notice that u_1, u_2, $\mathbf{W}_\gamma u_1$, $\mathbf{W}_\gamma u_2 \in \mathfrak{M}_\lambda$ for any γ satisfying $\lambda + n < \gamma \le \alpha$.

§4.2. Claim: $u(x,t)$ satisfies the initial condition

This claim follows from Lemma 4.5 by using the following result.

Lemma 4.7. *If $\varphi \in \mathfrak{M}_\lambda$ with $\alpha > \lambda + n$, then*

$$\lim_{t\to 0^+} \int_{\mathbb{Q}_p^n} Z(x-\xi,t)\varphi(\xi)\, d^n\xi = \varphi(x).$$

Proof. By Lemma 4.3(iv),

$$\int_{\mathbb{Q}_p^n} Z(x-\xi,t)\varphi(\xi)\, d^n\xi = \int_{\mathbb{Q}_p^n} Z(x-\xi,t)[\varphi(\xi)-\varphi(x)]\, d^n\xi + \varphi(x). \tag{4.3}$$

Now, by Lemma 4.3(iii) and the local constancy of φ,

$$\begin{aligned}\int_{\mathbb{Q}_p^n} & Z(x-\xi,t)[\varphi(\xi)-\varphi(x)]\, d^n\xi \\ &\le Ct\int_{\|x-\xi\|_p\ge p^l} (t^{1/(\alpha-n)}+\|x-\xi\|_p)^{-\alpha}|\varphi(\xi)-\varphi(x)|\, d^n\xi \\ &\le Ct\int_{\|z\|_p\ge p^l} (t^{1/(\alpha-n)}+\|z\|_p)^{-\alpha}|\varphi(x-z)-\varphi(x)|\, d^n z \\ &\le Ct\int_{\|z\|_p\ge p^l} \|z\|_p^{-\alpha}(1+\|x-z\|^\lambda)\, d^n z + C't|\varphi(x)| \le th(x).\end{aligned}$$

The desired formula is obtained by letting $t\to 0^+$ in (4.3). □

§4.3. Claim: $u(x,t)$ is a solution of the Cauchy problem (4.1)

This claim is a consequence of Corollary 4.10 and Lemmas 4.11 and 4.12. Several preliminary results are required.

Lemma 4.8. *There exist positive constants C_3, C_4 such that:*

(i) $\frac{\partial Z(x,t)}{\partial t} = -\kappa\int_{\mathbb{Q}_p^n} A_{w_\alpha}(\|\xi\|_p)e^{-\kappa t A_{w_\alpha}(\|\xi\|_p)}\Psi(x\cdot\xi)\, d^n\xi$ *for $t>0$ and $x\in\mathbb{Q}_p^n$;*

(ii) $\left|\frac{\partial Z(x,t)}{\partial t}\right| \le C_3 t^{-\alpha/(\alpha-n)}$ *for $t>0$ and $x\in\mathbb{Q}_p^n$;*

(iii) $\left|\frac{\partial Z(x,t)}{\partial t}\right| \le C_4 t\|x\|_p^{n-2\alpha}$ *for $t>0$ and $x\in\mathbb{Q}_p^n\setminus\{0\}$;*

(iv) $\left|\frac{\partial Z(x,t)}{\partial t}\right| \le 2^\alpha C_3(\|x\|_p + t^{1/(\alpha-n)})^{-\alpha}$ *for $t>0$ and $x\in\mathbb{Q}_p^n\setminus\{0\}$.*

Proof. (i) The formula is obtained by applying the Lebesgue Dominated Convergence Theorem and the fact that $-\kappa A_{w_\alpha}(\|\xi\|_p)e^{-\kappa\tau A_{w_\alpha}(\|\xi\|_p)}\Psi(x\cdot\xi)\in L^1(\mathbb{Q}_p^n)$ for $\tau>0$ fixed (cf. [7, Lemma 3.4]).

(ii) By using (i) and [7, Lemma 3.4],

$$\left|\frac{\partial Z(x,t)}{\partial t}\right| \le \int_{\mathbb{Q}_p^n} C_1\|\xi\|_p^{\alpha-n}e^{-\kappa C_2 t\|\xi\|_p^{\alpha-n}}\, d^n\xi.$$

We now pick an integer m such that $p^{m-1} \leq t^{1/(\alpha-n)} \leq p^m$, and proceed as in the proof of Lemma 4.3(i) to obtain

$$\left|\frac{\partial Z(x,t)}{\partial t}\right| \leq C_1 p^{-(m-1)n-(m-1)(\alpha-n)} \int_{\mathbb{Q}_p^n} \|z\|_p^{\alpha-n} e^{-\kappa C_2 \|z\|_p^{\alpha-n}}\, d^n z \leq C_3 t^{-\alpha/(\alpha-n)}.$$

(iii) Set $\|x\|_p = p^\beta$. Now, since $A_{w_\alpha}(\|\xi\|_p) e^{-\kappa t A_{w_\alpha}(\|\xi\|_p)} \in L^1 \cap L^2$ for $t > 0$, it follows that $\partial Z(x,t)/\partial t \in L^1 \cap L^2$ for $t > 0$, and by applying the formula for the Fourier transform of a radial function, we get

$$\frac{\partial Z(x,t)}{\partial t} = \|x\|_p^{-n}$$
$$\times \Big((1-p^{-n}) \sum_{j=0}^{\infty} A_{w_\alpha}(p^{-\beta-j}) e^{-\kappa t A_{w_\alpha}(p^{-\beta-j})} p^{-nj} - A_{w_\alpha}(p^{-\beta+1}) e^{-\kappa t A_{w_\alpha}(p^{-\beta+1})}\Big).$$

Now, since $A_{w_\alpha}(\xi)$ is a decreasing function of $\operatorname{ord}(\xi)$,

$$\begin{aligned}\left|\frac{\partial Z(x,t)}{\partial t}\right| &\leq \|x\|_p^{-n} A_{w_\alpha}(p^{-\beta+1}) \left|(1-p^{-n}) \sum_{j=0}^{\infty} p^{-nj} - e^{-\kappa t A_{w_\alpha}(p^{-\beta+1})}\right| \\ &\leq \|x\|_p^{-n} A_{w_\alpha}(p^{-\beta+1})(1 - e^{-\kappa t A_{w_\alpha}(p^{-\beta+1})}).\end{aligned}$$

By using the Mean Value Theorem and [7, Lemma 3.4], we obtain

$$\left|\frac{\partial Z(x,t)}{\partial t}\right| \leq C_4 \|x\|_p^{n-2\alpha} t.$$

(iv) If $\|x\|_p \leq t^{1/(\alpha-n)}$, then $\|x\|_p/2 + t^{1/(\alpha-n)}/2 \leq t^{1/(\alpha-n)}$ and hence $t^{-\alpha/(\alpha-n)} \leq 2^\alpha (\|x\|_p + t^{1/(\alpha-n)})^{-\alpha}$; multiplying by C_3 and using (ii), we get

$$\left|\frac{\partial Z(x,t)}{\partial t}\right| \leq 2^\alpha C_3 (\|x\|_p + t^{1/(\alpha-n)})^{-\alpha}.$$

Now, if $\|x\|_p \geq t^{1/(\alpha-n)}$, then by using (iii),

$$\left|\frac{\partial Z(x,t)}{\partial t}\right| \leq C_3 \|x\|_p^{-\alpha}, \tag{4.4}$$

and since $\|x\|_p \geq t^{1/(\alpha-n)}$, we find that $\|x\|_p \geq \|x\|_p/2 + t^{1/(\alpha-n)}/2$ and so $2^\alpha(\|x\|_p + t^{1/(\alpha-n)})^{-\alpha} \geq \|x\|_p^{-\alpha}$; multiplying by C_3 and using (4.4), we have

$$\left|\frac{\partial Z(x,t)}{\partial t}\right| \leq 2^\alpha C_3 (\|x\|_p + t^{1/(\alpha-n)})^{-\alpha}.$$ □

Lemma 4.9. $(\mathbf{W}_\gamma Z_t)(x)$ *with* $\gamma \leq \alpha$ *satisfies the following conditions:*

(i) $(\mathbf{W}_\gamma Z_t)(x) = -\int_{\mathbb{Q}_p^n} A_{w_\gamma}(\|\xi\|_p) e^{-\kappa t A_{w_\alpha}(\|\xi\|_p)} \Psi(x \cdot \xi)\, d^n\xi$ *for* $t > 0$ *and* $x \in \mathbb{Q}_p^n$*;*

(ii) $|(\mathbf{W}_\gamma Z_t)(x)| \leq 2^\gamma C(\|x\|_p + t^{1/(\alpha-n)})^{-\gamma}$ *for* $t > 0$ *and* $x \in \mathbb{Q}_p^n$ *and some positive constant* C*;*

(iii) $\int_{\mathbb{Q}_p^n} (\mathbf{W}_\gamma Z_t)(x)\, d^n x = 0$.

Proof. (i) Define

$$Z_t^{(M)}(x) = \int_{\|\eta\|_p \leq p^M} \Psi(x \cdot \eta) e^{-\kappa t A_{w_\alpha}(\|\eta\|_p)}\, d^n\eta \quad \text{for } M \in \mathbb{N}. \tag{4.5}$$

This function is locally constant on $\mathbb{Q}_p^n$. Indeed, if $\|\xi\|_p \leq p^{-M}$, then $Z_t^{(M)}(x+\xi) = Z_t^{(M)}(x)$. Furthermore, $Z_t^{(M)}(x)$ is bounded, and thus $Z_t^{(M)} \in \mathfrak{M}_0 \subset \mathrm{Dom}(\mathbf{W}_\gamma)$. We now use formula (3.1) and Fubini's Theorem to compute

$$\begin{aligned}
(\mathbf{W}_\gamma Z_t^{(M)})(x) &= \int_{\mathbb{Q}_p^n} \frac{Z_t^{(M)}(x-\xi) - Z_t^{(M)}(x)}{w_\gamma(\|\xi\|_p)}\, d^n\xi \\
&= \int_{\|\xi\|_p > p^{-M}} \int_{\|\eta\|_p \leq p^M} e^{-\kappa t A_{w_\alpha}(\|\eta\|_p)} \Psi(x \cdot \eta) \frac{\Psi(\xi \cdot \eta) - 1}{w_\gamma(\|\xi\|_p)}\, d^n\eta\, d^n\xi \\
&= \int_{\|\eta\|_p \leq p^M} e^{-\kappa t A_{w_\alpha}(\|\eta\|_p)} \Psi(x \cdot \eta) \int_{\|\xi\|_p > p^{-M}} \frac{\Psi(\xi \cdot \eta) - 1}{w_\gamma(\|\xi\|_p)}\, d^n\xi\, d^n\eta \\
&= -\int_{\|\eta\|_p \leq p^M} e^{-\kappa t A_{w_\alpha}(\|\eta\|_p)} \Psi(x \cdot \eta) A_{w_\gamma}(\|\eta\|_p)\, d^n\eta.
\end{aligned}$$

By using the fact that $e^{-\kappa t A_{w_\alpha}(\|\cdot\|_p)} A_{w_\gamma}(\|\cdot\|_p) \in L^1(\mathbb{Q}_p^n)$ for $t > 0$ (cf. [7, Lemma 3.4]) and the Dominated Convergence Theorem, we obtain

$$\lim_{M \to \infty} (\mathbf{W}_\gamma Z_t^{(M)})(x) = -\int_{\mathbb{Q}_p^n} A_{w_\gamma}(\|\eta\|_p) e^{-\kappa t A_{w_\alpha}(\|\eta\|_p)} \Psi(x \cdot \eta)\, d^n\eta. \tag{4.6}$$

On the other hand, by fixing $x \neq 0$ and for $t > 0$, $Z_t(x-\xi) - Z_t(x)$ is locally constant (cf. Remark 4.2), and bounded (cf. Lemma 4.3(iii)), so $(\mathbf{W}_\gamma Z_t)(x)$ is well-defined, and since $Z_t^{(M)}(x)$ is radial,

$$(\mathbf{W}_\gamma Z_t^{(M)})(x) = \int_{\|\xi\|_p > \|x\|_p} \frac{Z_t^{(M)}(x-\xi) - Z_t^{(M)}(x)}{w_\gamma(\|\xi\|_p)}\, d^n\xi,$$

and by the Dominated Convergence Theorem, $\lim_{M\to\infty} (\mathbf{W}_\gamma Z_t^{(M)})(x) = (\mathbf{W}_\gamma Z_t)(x)$. Therefore by (4.6), we have

$$(\mathbf{W}_\gamma Z_t)(x) = -\int_{\mathbb{Q}_p^n} A_{w_\gamma}(\|\eta\|_p) e^{-\kappa t A_{w_\alpha}(\|\eta\|_p)} \Psi(x \cdot \eta)\, d^n\eta.$$

Finally, we note the right-hand side in the above formula is continuous at $x = 0$.

(ii) By (i) and [7, Lemma 3.4],

$$|(\mathbf{W}_\gamma Z_t)(x)| \leq C_0 \int_{\mathbb{Q}_p^n} \|\xi\|_p^{\gamma-n} e^{-\kappa C_1 t \|\xi\|_p^{\alpha-n}} \, d^n\xi.$$

We now pick an integer m such that $p^{m-1} \leq t^{1/(\alpha-n)} \leq p^m$, and proceed as in the proof of Lemma 4.3(i) to obtain

$$|(\mathbf{W}_\gamma Z_t)(x)| \leq C t^{-\gamma/(\alpha-n)}. \tag{4.7}$$

Now, if $\|x\|_p \leq t^{1/(\alpha-n)}$, then $\|x\|_p/2 + t^{1/(\alpha-n)}/2 \leq t^{1/(\alpha-n)}$ and consequently $t^{-\gamma/(\alpha-n)} \leq 2^\gamma (\|x\|_p + t^{1/(\alpha-n)})^{-\gamma}$; multiplying by C and using (4.7), we have

$$|(\mathbf{W}_\gamma Z_t)(x)| \leq 2^\gamma C (\|x\|_p + t^{1/(\alpha-n)})^{-\gamma}.$$

On the other hand, let $\|x\|_p = p^\beta$, since $A_{w_\gamma}(\|\xi\|_p) e^{-\kappa t A_{w_\alpha}(\|\xi\|_p)} \in L^1 \cap L^2$ for $t > 0$, we have $\mathbf{W}_\gamma Z_t \in L^1 \cap L^2$ for $t > 0$, by proceeding as in the proof of Lemma 4.8(iii), we obtain

$$|(\mathbf{W}_\gamma Z_t)(x)| \leq C t \|x\|_p^{n-\alpha-\gamma}.$$

Now, if $\|x\|_p \geq t^{1/(\alpha-n)}$, then

$$|(\mathbf{W}_\gamma Z_t)(x)| \leq C \|x\|_p^{-\gamma}. \tag{4.8}$$

If $\|x\|_p \geq t^{1/(\alpha-n)}$, then $\|x\|_p \geq \|x\|_p/2 + t^{1/(\alpha-n)}/2$ and $2^\gamma (\|x\|_p + t^{1/(\alpha-n)})^{-\gamma} \geq \|x\|_p^{-\gamma}$; multiplying by C and using (4.8), we conclude that

$$|(\mathbf{W}_\gamma Z_t)(x)| \leq 2^\gamma C (\|x\|_p + t^{1/(\alpha-n)})^{-\gamma}.$$

(iii) follows from (i) by the inversion formula for the Fourier transform. □

Corollary 4.10. *$\frac{\partial Z(x,t)}{\partial t} = \kappa \cdot (\mathbf{W}_\alpha Z_t)(x)$ for $t > 0$ and $x \in \mathbb{Q}_p^n$.*

Proof. The formula follows from Lemmas 4.8(i) and 4.9(i). □

Proposition 4.11. *Assume that $\varphi \in \mathfrak{M}_\lambda$. Then:*

(i) *$\frac{\partial u_1}{\partial t}(x,t) = \int_{\mathbb{Q}_p^n} \frac{\partial Z(x-y,t)}{\partial t} \varphi(y) \, d^n y$ for $t > 0$ and $x \in \mathbb{Q}_p^n \setminus \{0\}$;*

(ii) *$(\mathbf{W}_\gamma u_1)(x,t) = \int_{\mathbb{Q}_p^n} (\mathbf{W}_\gamma Z_t)(x-y) \varphi(y) \, d^n y$ for $n + \lambda < \gamma \leq \alpha$, $t > 0$ and $x \in \mathbb{Q}_p^n \setminus \{0\}$.*

Proof. (i) By the Mean Value Theorem, $\frac{\partial u_1}{\partial t}(x,t)$ equals

$$\lim_{h \to 0} \int_{\mathbb{Q}_p^n} \frac{Z(x-y,t+h) - Z(x-y,t)}{h} \varphi(y) \, d^n y = \lim_{h \to 0} \int_{\mathbb{Q}_p^n} \frac{\partial Z(x-y,\tau)}{\partial t} \varphi(y) \, d^n y,$$

where τ is between t and $t+h$. Now, the result follows by applying the Dominated Convergence Theorem and Lemma 4.8(iv).

(ii) By Remark 4.6, if $n+\lambda<\gamma$, then $u_1\in \mathrm{Dom}(\mathbf{W}_\gamma)$ for $t>0$. Then for any $L\in\mathbb{N}$, by Fubini's Theorem (cf. Lemma 4.3(iii)), we get

$$\int_{\|y\|_p>p^{-L}}\frac{u_1(x-y,t)-u_1(x,t)}{w_\gamma(\|y\|_p)}\,d^ny = \int_{\mathbb{Q}_p^n}\varphi(\xi)\int_{\|y\|_p>p^{-L}}\frac{Z_t(x-\xi-y)-Z_t(x-\xi)}{w_\gamma(\|y\|_p)}\,d^ny\,d^n\xi.$$

We now fix a positive integer M such that $\|y\|_p<p^{-L}<p^{-M}<\|x-\xi\|_p$. By Remark 4.2,

$$\int_{\|x-\xi\|_p>p^{-M}}\varphi(\xi)\int_{\|y\|_p\le p^{-L}}\frac{Z_t(x-\xi-y)-Z_t(x-\xi)}{w_\gamma(\|y\|_p)}\,d^ny\,d^n\xi=0;$$

then

$$\begin{aligned}(\mathbf{W}_\gamma u_1)(x,t) &= \lim_{L\to\infty}\int_{\|y\|_p>p^{-L}}\frac{u_1(x-y,t)-u_1(x,t)}{w_\gamma(\|y\|_p)}\,d^ny\\ &= \int_{\|x-\xi\|_p>p^{-M}}\varphi(\xi)(\mathbf{W}_\gamma Z_t)(x-\xi)\,d^n\xi\\ &\quad+\lim_{L\to\infty}\int_{\|x-\xi\|_p\le p^{-M}}\varphi(\xi)\int_{\|y\|_p>p^{-L}}\frac{Z_t(x-\xi-y)-Z_t(x-\xi)}{w_\gamma(\|y\|_p)}\,d^ny\,d^n\xi\\ &= \int_{\|x-\xi\|_p>p^{-M}}\varphi(\xi)(\mathbf{W}_\gamma Z_t)(x-\xi)\,d^n\xi+\int_{\|x-\xi\|_p\le p^{-M}}\varphi(\xi)(\mathbf{W}_\gamma Z_t)(x-\xi)\,d^n\xi,\end{aligned}$$

where the limit was computed by using the Lebesgue Dominated Convergence Theorem and the fact that

$$\begin{aligned}&\int_{\|x-\xi\|_p\le p^{-M}}\varphi(\xi)\int_{\|y\|_p>p^{-L}}\frac{Z_t(x-\xi-y)-Z_t(x-\xi)}{w_\gamma(\|y\|_p)}\,d^ny\,d^n\xi\\ &\qquad=\int_{\|x-\xi\|_p\le p^{-M}}\varphi(\xi)\int_{\|y\|_p>p^{-M}}\frac{Z_t(x-\xi-y)-Z_t(x-\xi)}{w_\gamma(\|y\|_p)}\,d^ny\,d^n\xi,\end{aligned}$$

because $Z_t(x-\xi-y)=Z_t(x-\xi)$ for $\|x-\xi\|_p>\|y\|_p$ (cf. Remark 4.2). The convergence of this last integral follows from Proposition 4.4. □

Set $u_2(x,t,\tau):=\int_\tau^t\int_{\mathbb{Q}_p^n}Z(x-y,t-\theta)f(y,\theta)\,d^ny\,d\theta$ for $f\in\mathfrak{M}_\lambda$ with $\alpha-n>\lambda$, $0\le\tau\le t\le T$, and $x\in\mathbb{Q}_p^n$. By reasoning as in the proof of Lemma 4.5, we find that $u_2(\cdot,t,\tau)\in\mathfrak{M}_\lambda$ uniformly in t and τ.

Proposition 4.12. *Assume that $f \in \mathfrak{M}_\lambda$ with $\alpha - n > \lambda$. Then:*

(i) $\frac{\partial u_2}{\partial t}(x,t,\tau) = f(x,t) + \int_\tau^t \big(\int_{\mathbb{Q}_p^n} \frac{\partial Z(x-y,t-\theta)}{\partial t}[f(y,\theta) - f(x,\theta)]\, d^n y\big)\, d\theta$ *for* $t > 0$ *and* $x \in \mathbb{Q}_p^n$*;*

(ii) $(\mathbf{W}_\gamma u_2)(x,t,\tau) = \int_\tau^t \int_{\mathbb{Q}_p^n} (\mathbf{W}_\gamma Z)(x-y,t-\theta) f(y,\theta)\, d^n y\, d\theta$ *for* $n+\lambda < \gamma \le \alpha$, $t > 0$ *and* $x \in \mathbb{Q}_p^n$.

Proof. Set

$$u_{2,h}(x,t,\tau) := \int_\tau^{t-h} \int_{\mathbb{Q}_p^n} Z(x-y,t-\theta) f(y,\theta)\, d^n y\, d\theta, \qquad 0 < h < t-\tau.$$

A standard reasoning shows that

$$\frac{\partial u_{2,h}}{\partial t}(x,t,\tau) = \int_\tau^{t-h} \int_{\mathbb{Q}_p^n} \frac{\partial Z(x-y,t-\theta)}{\partial t} f(y,\theta)\, d^n y\, d\theta + \int_{\mathbb{Q}_p^n} Z(x-y,h) f(y,t-h)\, d^n y\, d\theta.$$

This formula can be rewritten as

$$\begin{aligned}\frac{\partial u_{2,h}}{\partial t}(x,t,\tau) = &\int_\tau^{t-h} \int_{\mathbb{Q}_p^n} \frac{\partial Z(x-y,t-\theta)}{\partial t}[f(y,\theta) - f(x,\theta)]\, d^n y\, d\theta \\ &+ \int_\tau^{t-h} f(x,\theta) \int_{\mathbb{Q}_p^n} \frac{\partial Z(x-y,t-\theta)}{\partial t}\, d^n y\, d\theta \\ + \int_{\mathbb{Q}_p^n} Z(x-y,h)[f(y,t-h) - f(y,t)]\, d^n y &+ \int_{\mathbb{Q}_p^n} Z(x-y,h) f(y,t)\, d^n y.\end{aligned}$$

The first integral contains no singularity at $t = \theta$ due to Lemma 4.8(iv) and the local constancy of f. By Lemma 4.3(iv), the second integral is zero. The third integral can be written as a sum of integrals over $\{y \in \mathbb{Q}_p^n : \|x-y\|_p \ge p^M\}$ and the complement of this set; one of these integrals is estimated on the basis of the uniform continuity of f, while the other contains no singularity (see Lemma 4.8(iv)). Finally, the fourth integral tends to $f(x,t)$ as $h \to 0^+$ (cf. Lemma 4.7).

(ii) By Lemma 4.5, $\mathbf{W}_\gamma u_{2,h}$ is well-defined for any γ satisfying $n+\lambda < \gamma \le \alpha$. Then, for any $L \in \mathbb{N}$, the following integral exists:

$$\begin{aligned}(4.9)\qquad &\int_{\|\xi\|_p \ge p^{-L}} \frac{u_{2,h}(x-\xi,t,\tau) - u_{2,h}(x,t,\tau)}{w_\gamma(\|\xi\|_p)}\, d^n \xi \\ &= \int_\tau^{t-h} \int_{\mathbb{Q}_p^n} \int_{\|\xi\|_p \ge p^{-L}} \frac{Z(x-\xi-y,t-\theta) - Z(x-y,t-\theta)}{w_\gamma(\|\xi\|_p)} f(y,\theta)\, d^n \xi\, d^n y\, d\theta.\end{aligned}$$

On the other hand, by Fubini's Theorem,

$$\int_{\|\xi\|_p \geq p^{-L}} \frac{Z(x-\xi-y,t-\theta)-Z(x-y,t-\theta)}{w_\gamma(\|\xi\|_p)}\,d^n\xi = \int_{\mathbb{Q}_p^n} \Psi((x-y)\cdot\eta)e^{-\kappa(t-\theta)A_{w_\gamma}(\|\eta\|_p)}P_k(\eta)\,d^n\eta,$$

where $P_k(\eta) = \int_{\|\xi\|_p \geq p^{-L}} \frac{\Psi(-\xi\cdot\eta)-1}{w_\gamma(\|\xi\|_p)}\,d^n\xi$. A simple calculation shows that $|P_k(\eta)| \leq C'\|\eta\|_p^{\gamma-n}$, and so

$$\int_{\|\xi\|_p \geq p^{-L}} \frac{Z(x-\xi-y,t-\theta)-Z(x-y,t-\theta)}{w_\gamma(\|\xi\|_p)}\,d^n\xi \leq C,$$

where the constant does not depend on x, $t \geq h+\tau$, L. Now, by expressing the right integral of (4.9) as

$$\int_\tau^{t-h}\int_{\|x-\xi\|_p > p^{-M}}\int_{\|\xi\|_p \geq p^{-L}} \frac{Z(x-\xi-y,t-\theta)-Z(x-y,t-\theta)}{w_\gamma(\|\xi\|_p)}f(y,\theta)\,d^n\xi\,d^ny\,d\theta$$
$$+\int_\tau^{t-h}\int_{\|x-\xi\|_p \leq p^{-M}}\int_{\|\xi\|_p \geq p^{-L}} \frac{Z(x-\xi-y,t-\theta)-Z(x-y,t-\theta)}{w_\gamma(\|\xi\|_p)}f(y,\theta)\,d^n\xi\,d^ny\,d\theta$$

where M is a positive integer such that $\|\xi\|_p < p^{-L} < p^{-M} < \|x-\xi\|_p$, and using the same reasoning as in the final part of the proof of Proposition 4.11(ii), we obtain

$$(\mathbf{W}_\gamma u_{2,h})(x,t) = \int_\tau^{t-h}\int_{\mathbb{Q}_p^n} (\mathbf{W}_\gamma Z)(x-\xi,t-\theta)f(y,\theta)\,d^n\xi\,d\theta. \tag{4.10}$$

Now, by Lemma 4.9(ii), the fact that $f \in \mathfrak{M}_\lambda$, Proposition 4.4, and the Dominated Convergence Theorem, we can take the limit as $h \to 0^+$, which completes the proof when $\gamma < \alpha$. If $\gamma = \alpha$, formula (4.10) remains valid. By using Lemma 4.9(iii), formula (4.10) can be rewritten as

$$(\mathbf{W}_\gamma u_{2,h})(x,t) = \int_\tau^{t-h}\int_{\mathbb{Q}_p^n} (\mathbf{W}_\gamma Z)(x-\xi,t-\theta)[f(y,\theta)-f(x,\theta)]\,d^n\xi\,d\theta.$$

Now, by using the local constancy of f, we can justify the passage to the limit as $h \to 0^+$, which completes the proof. □

Remark 4.13. By Lemmas 4.3(iv) and 4.8(i), $\int_{\mathbb{Q}_p^n} \frac{\partial Z(x-y,t-\theta)}{\partial t}\,d^ny = 0$, and so

$$\frac{\partial u_2}{\partial t}(x,t,\tau) = f(x,t) + \int_\tau^t \left(\int_{\mathbb{Q}_p^n} \frac{\partial Z(x-y,t-\theta)}{\partial t} f(y,\theta)\,d^ny\right)d\theta$$

for $t > 0$ and $x \in \mathbb{Q}_p^n$.

§5. Parabolic-type equations with variable coefficients

First, we fix the notation that will be used throughout this section. We assume that $\alpha > n+1$. We fix $N+1$ positive real numbers satisfying $n < \alpha_1 < \cdots < \alpha_N < \alpha$. We fix $N+2$ functions $a_k(x,t)$, $k = 0,\dots,N$, and $b(x,t)$ from $\mathbb{Q}_p^n \times [0,T]$ to $\mathbb{R}$, where T is a positive constant. We assume that: (i) $b(x,t)$ and $a_k(x,t)$, for $k = 0,\dots,N$, belong (with respect to x) to $\mathfrak{M}_0$ uniformly with respect to $t \in [0,T]$; (ii) $a_0(x,t)$ satisfies the Hölder condition in t with exponent $v \in (0,1)$ uniformly in x. We also assume the uniform parabolicity condition $a_0(x,t) \geq \mu > 0$ and that $\alpha_{N+1} := n + (\alpha - n)(1-v) > \alpha_N$. Notice that $\alpha_{N+1} < \alpha$.

Set $\widetilde{\mathbf{W}} := \sum_{k=1}^{N} a_k(x,t)\mathbf{W}_{\alpha_k} - b(x,t)\mathbf{I}$ with domain $\mathfrak{M}_\lambda$ and $0 \leq \lambda + n < \alpha_1$. Notice that $\widetilde{\mathbf{W}} : \mathfrak{M}_\lambda \to \mathfrak{M}_\lambda$.

In this section we construct a solution for the initial value problem

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) - a_0(x,t)(\mathbf{W}_\alpha u)(x,t) - (\widetilde{\mathbf{W}}u)(x,t) = f(x,t), \\ u(x,0) = \varphi(x), \end{cases} \tag{5.1}$$

where $x \in \mathbb{Q}_p^n$, $t \in (0,T]$, $\varphi \in \mathfrak{M}_\lambda$, $f(\cdot,t) \in \mathfrak{M}_\lambda$ uniformly with respect to t with $0 \leq \lambda < \alpha_1 - n$, and $f(x,t)$ is continuous in (x,t) (if $a_1(x,t) = \cdots = a_N(x,t) \equiv 0$ then we shall assume that $0 \leq \lambda < \alpha - n$).

§5.1. Parametrized Cauchy problem

We first study the following Cauchy problem:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) - a_0(y,\theta)(\mathbf{W}_\alpha u)(x,t) = 0, \quad x \in \mathbb{Q}_p^n,\ t \in (0,T], \\ u(x,0) = \varphi(x), \end{cases} \tag{5.2}$$

where $y \in \mathbb{Q}_p^n$ and $\theta > 0$ are parameters. By applying the results of Section 4 with $\kappa = a_0(y,\theta) \geq \mu > 0$, the Cauchy problem (5.2) has a fundamental solution given by

$$Z(x,t;y,\theta) := \int_{\mathbb{Q}_p^n} \Psi(x \cdot \xi) e^{-a_0(y,\theta)tA_{w_\alpha}(\|\xi\|_p)}\, d^n\xi$$

for $t > 0$ and $x \in \mathbb{Q}_p^n$.

Remark 5.1. All statements from Lemmas 4.3, 4.8, 4.9 hold for $Z(x,t;y,\theta)$ and the constants involved do not depend on y or θ. Thus, we have the estimates

$$Z(x,t;y,\theta) \leq C_1 t(\|x\|_p + t^{1/(\alpha-n)})^{-\alpha} \quad \text{for } t > 0, \tag{5.3}$$

$$\left|\frac{\partial Z(x,t;y,\theta)}{\partial t}\right| \leq C_2(\|x\|_p + t^{1/(\alpha-n)})^{-\alpha} \quad \text{for } t > 0, \tag{5.4}$$

(5.5) $$|(\mathbf{W}_{\gamma}Z)(x,t;y,\theta)| \leq C_3(\|x\|_p + t^{1/(\alpha-n)})^{-\gamma} \qquad \text{for } t>0 \text{ and } \gamma \leq \alpha,$$

and the identities

(5.6) $$\int_{\mathbb{Q}_p^n} Z(x,t;y,\theta)\, d^n x = 1 \quad \text{ for } t>0,$$

(5.7) $$\frac{\partial Z(x,t;y,\theta)}{\partial t} = -a_0(y,\theta)\int_{\mathbb{Q}_p^n} A_{w_\alpha}(\|\xi\|_p) e^{-a_0(y,\theta)tA_{w_\alpha}(\|\xi\|_p)}\Psi(x\cdot\xi)\, d^n\xi \quad \text{ for } t>0,$$

(5.8) $$(\mathbf{W}_{\gamma}Z)(x,t;y,\theta) = -\int_{\mathbb{Q}_p^n} A_{w_\gamma}(\|\xi\|_p) e^{-a_0(y,\theta)tA_{w_\alpha}(\|\xi\|_p)}\Psi(x\cdot\xi)\, d^n\xi$$

for $t>0$ and $\gamma \leq \alpha$, and

(5.9) $$\int_{\mathbb{Q}_p^n} (\mathbf{W}_{\gamma}Z_t)(x,t;y,\theta)\, d^n x = 0 \quad \text{ for } t>0 \text{ and } \gamma\leq\alpha.$$

Lemma 5.2. *There exists a positive constant C such that*

(5.10) $$\left|\int_{\mathbb{Q}_p^n} \frac{\partial Z(x-y;t,y,\theta)}{\partial t}\, d^n y\right| \leq C.$$

Proof. See [14, proof of Lemma 4.5]. □

§5.2. Heat potentials

We define the *parameterized heat potentials* as follows:

$$u(x,t,\tau) := \int_{\tau}^{t}\int_{\mathbb{Q}_p^n} Z(x-y,t-\theta;y,\theta) f(y,\theta)\, d^n y\, d\theta,$$

where $f\in\mathfrak{M}_\lambda$, $0\leq\lambda<\alpha-n$, f is continuous in (y,θ). By using the same argument given to prove Lemma 4.5, one proves that $u\in\mathfrak{M}_\lambda$ uniformly in t and τ.

We now calculate the derivative with respect to t and the action of the operator $\mathbf{W}_\gamma$ on $u(x,t,\tau)$ for $n+\lambda<\gamma\leq\alpha$.

Proposition 5.3. *Assume that $f\in\mathfrak{M}_\lambda$, $0\leq\lambda<\alpha-n$, and f is continuous in (y,θ). Then*

(i) $\frac{\partial u(x,t,\tau)}{\partial t} = f(x,t) + \int_\tau^t\int_{\mathbb{Q}_p^n}\frac{\partial Z(x-y,t-\theta;y,\theta)}{\partial t} f(y,\theta)\, d^n y\, d\theta$;

(ii) $(\mathbf{W}_\gamma u)(x,t,\tau) = \int_\tau^t\int_{\mathbb{Q}_p^n}(\mathbf{W}_\gamma Z)(x-y,t-\theta;y,\theta) f(y,\theta)\, d^n y\, d\theta$, $\gamma\leq\alpha$.

Proof. It is a simple variation of the proof of Proposition 4.12. □

The following technical result will be used later on.

Lemma 5.4 ([14, Lemma 4.6]). *Let*

$$J(x,\xi,t,\tau)=\int_\tau^t (t-\theta)^{-\rho/\beta}(\theta-\tau)^{-\sigma/\beta} \times\left(\int_{\mathbb{Q}_p^n}[(t-\theta)^{1/\beta}+\|x-\eta\|_p]^{-n-b_1}[(\theta-\tau)^{1/\beta}+\|\eta-\xi\|_p]^{-n-b_2}\,d^n\eta\right)d\theta,$$

where $x,\xi\in\mathbb{Q}_p^n$, $0\le\tau<t$, $b_1,b_2>0$ *and* $\rho+b_1,\sigma+b_2<\beta$, $\beta>1$. *Then*

$$\begin{aligned}J(x,\xi,t,\tau)&\\ \le C\Big\{&B\Big(1-\frac{\rho}{\beta},1-\frac{\sigma+b_2}{\beta}\Big)[(t-\tau)^{1/\beta}+\|x-\xi\|_p]^{-n-b_1}(t-\tau)^{-(\rho+\sigma+b_2-\beta)/\beta}\\ &+B\Big(1-\frac{\rho+b_1}{\beta},1-\frac{\sigma}{\beta}\Big)[(t-\tau)^{1/\beta}+\|x-\xi\|_p]^{-n-b_2}(t-\tau)^{-(\rho+\sigma+b_1-\beta)/\beta}\Big\},\end{aligned}$$

where C *is a positive constant depending only on* b_1,b_2, *and* $B(\cdot,\cdot)$ *denotes the Archimedean Beta function.*

The proof is a simple variation of that given by Kochubei for [14, Lemma 4.6].

§5.3. Construction of a solution

Theorem 5.5. *The Cauchy problem* (5.1) *has a solution which can be represented in the form*

$$u(x,t)=\int_0^t\int_{\mathbb{Q}_p^n}\Lambda(x,t,\xi,\tau)f(\xi,\tau)\,d^n\xi\,d\tau+\int_{\mathbb{Q}_p^n}\Lambda(x,t,\xi,0)\varphi(\xi)\,d^n\xi, \tag{5.11}$$

where the fundamental solution $\Lambda(x,t,\xi,\tau)$, $x,\xi\in\mathbb{Q}_p^n$, $0\le\tau<t\le T$, *has the form*

$$\Lambda(x,t,\xi,\tau)=Z(x-\xi,t-\tau;\xi,\tau)+\mathcal{W}(x,t,\xi,\tau), \tag{5.12}$$

with

$$\begin{aligned}|\mathcal{W}(x,t,\xi,\tau)|\le C\Big\{&(t-\tau)^{2-\lambda/(\alpha-n)}[(t-\tau)^{1/(\alpha-n)}+\|x-\xi\|_p]^{-\alpha}\\ &+(t-\tau)\sum_{k=1}^{N+1}[(t-\tau)^{1/(\alpha-n)}+\|x-\xi\|_p]^{-\alpha_k}\Big\}.\end{aligned} \tag{5.13}$$

Furthermore $Z(x,t;y,\theta)$ *satisfies the estimates* (5.3)–(5.5) *and* (5.10).

Proof. We apply the usual parametrix method (see e.g. [11], [14]). Our proof is essentially self-contained. We look for a fundamental solution of (5.1) having the form (5.12) with

$$\mathcal{W}(x,t,\xi,\tau)=\int_{\tau}^{t}\int_{\mathbb{Q}_p^n} Z(x-\eta,t-\theta;\eta,\theta)\Phi(\eta,\theta,\xi,\tau)\,d^n\eta\,d\theta,$$

and satisfying

$$\begin{aligned}(5.14)\qquad \frac{\partial\Lambda}{\partial t}(x,t,\xi,\tau)-a_0(x,t)(\mathbf{W}_\alpha\Lambda)(x,t,\xi,\tau)&\\ -\sum_{k=1}^{N}a_k(x,t)(\mathbf{W}_{\alpha_k}\Lambda)(x,t,\xi,\tau)+b(x,t)\Lambda(x,t,\xi,\tau)&=0\end{aligned}$$

for $x\neq 0$, $t>0$. Now by using (5.12), (5.1)–(5.8) and Proposition 5.3, we have formally

$$\begin{aligned}&\frac{\partial Z}{\partial t}(x-\xi,t-\tau,\xi,\tau)+\Phi(x,t,\xi,\tau)+\int_{\tau}^{t}\int_{\mathbb{Q}_p^n}\frac{\partial Z(x-\eta,t-\theta;\eta,\theta)}{\partial t}\Phi(\eta,\theta,\xi,\tau)\,d^n\eta\,d\theta\\ &-a_0(x,t)\Big\{(\mathbf{W}_\alpha Z)(x-\xi,t-\tau;\xi,\tau)\\ &+\int_{\tau}^{t}\int_{\mathbb{Q}_p^n}(\mathbf{W}_\alpha Z)(x-\eta,t-\theta;\eta,\theta)\Phi(\eta,\theta,\xi,\tau)\,d^n\eta\,d\theta\Big\}\\ &-\sum_{k=1}^{N}a_k(x,t)\Big\{(\mathbf{W}_{\alpha_k} Z)(x-\xi,t-\tau;\xi,\tau)\\ &+\int_{\tau}^{t}\int_{\mathbb{Q}_p^n}(\mathbf{W}_{\alpha_k} Z)(x-\eta,t-\theta;\eta,\theta)\Phi(\eta,\theta,\xi,\tau)\,d^n\eta\,d\theta\Big\}\\ &+b(x,t)\Big\{Z(x-\xi,t-\tau;\xi,\tau)+\int_{\tau}^{t}\int_{\mathbb{Q}_p^n}Z(x-\eta,t-\theta;\eta,\theta)\Phi(\eta,\theta,\xi,\tau)\,d^n\eta\,d\theta\Big\}=0.\end{aligned}$$

By taking

$$\begin{aligned}R(x,t,\xi,\tau):=&\,(a_0(x,t)-a_0(\xi,\tau))(\mathbf{W}_\alpha Z)(x-\xi,t-\tau;\xi,\tau)\\ &+\sum_{k=1}^{N}a_k(x,t)(\mathbf{W}_{\alpha_k}Z)(x-\xi,t-\tau;\xi,\tau)-b(x,t)Z(x-\xi,t-\tau;\xi,\tau),\end{aligned}$$

one finds that $\Phi(x,t,\xi,\tau)$ satisfies the integral equation

$$(5.15)\qquad \Phi(x,t,\xi,\tau)=R(x,t,\xi,\tau)+\int_{\tau}^{t}\int_{\mathbb{Q}_p^n}R(x,t,\eta,\theta)\Phi(\eta,\theta,\xi,\tau)\,d^n\eta\,d\theta.$$

Now, by using (5.5) and (5.3), we obtain

$$|R(x,t,\xi,\tau)| \leq C_0\Big(|a_0(x,t)-a_0(\xi,\tau)|[(t-\tau)^{1/(\alpha-n)}+\|x-\xi\|_p]^{-\alpha} + \sum_{k=1}^{N}[(t-\tau)^{1/(\alpha-n)}+\|x-\xi\|_p]^{-\alpha_k} + [(t-\tau)^{1/(\alpha-n)}+\|x-\xi\|_p]^{-\alpha}(t-\tau)\Big). \tag{5.16}$$

Claim A.

$$|a_0(x,t)-a_0(\xi,\tau)|[(t-\tau)^{1/(\alpha-n)}+\|x-\xi\|_p]^{-\alpha} \leq C_1'[(t-\tau)^{1/(\alpha-n)}+\|x-\xi\|_p]^{-\alpha_{N+1}},$$

where $\alpha_{N+1} = n + (\alpha-n)(1-v) > \alpha_N$.

The proof is based on the Hölder condition for $a_0(x,t)$:

$$|a_0(x,t)-a_0(\xi,\tau)| \leq C_1(t-\tau)^v + |a_0(x,\tau)-a_0(\xi,\tau)|.$$

Let $l(a_0)$ be the parameter of local constancy of a_0. Thus, if $\|x-\xi\|_p \leq p^{l(a_0)}$, then $|a_0(x,t)-a_0(\xi,\tau)| \leq C_1(t-\tau)^v$. In the case $\|x-\xi\|_p \leq p^{l(a_0)}$, the inequality of the claim follows from the fact that $(t-\tau)^v[(t-\tau)^{1/(\alpha-n)}+\|x-\xi\|_p]^{-\alpha+\alpha_{N+1}}$ is bounded, which in turn follows from $\lim_{t\to\tau}(t-\tau)^{v+\frac{-\alpha+\alpha_{N+1}}{\alpha-n}} = 1$. In the case $\|x-\xi\|_p > p^{l(a_0)}$, by using $|a_0(x,t)-a_0(\xi,\tau)| \leq C_0$, the inequality follows from

$$[(t-\tau)^{1/(\alpha-n)}+\|x-\xi\|_p]^{-\alpha+\alpha_{N+1}} \leq \|x-\xi\|_p^{-\alpha+\alpha_{N+1}} \leq p^{(-\alpha+\alpha_{N+1})l(a_0)}.$$

Claim B.

$$(t-\tau)[(t-\tau)^{1/(\alpha-n)}+\|x-\xi\|_p]^{-\alpha} \leq C_2[(t-\tau)^{1/(\alpha-n)}+\|x-\xi\|_p]^{-\alpha_{N+1}}.$$

This is a consequence of the fact that $\lim_{t\to\tau}(t-\tau)^{1+\frac{-\alpha+\alpha_{N+1}}{\alpha-n}} = 0$.

Now from (5.16), and Claims A–B, we have

$$|R(x,t,\xi,\tau)| \leq C\sum_{k=1}^{N+1}[(t-\tau)^{1/(\alpha-n)}+\|x-\xi\|_p]^{-\alpha_k}. \tag{5.17}$$

We solve the integral equation (5.15) by the method of successive approximations:

$$\Phi(x,t,\xi,\tau) = \sum_{m=1}^{\infty} R_m(x,t,\eta,\theta), \tag{5.18}$$

where $R_1 \equiv R$ and

$$R_{m+1}(x,t,\xi,\tau) = \int_\tau^t \int_{\mathbb{Q}_p^n} R(x,t,\eta,\theta)R_m(\eta,\theta,\xi,\tau)\, d^n\eta\, d\theta \quad \text{for } m \geq 1.$$

Claim C.

$$|R_{m+1}(x,t,\xi,\tau)| \leq C(2N+2)^m (t-\tau)^{mv} \frac{(\Gamma(v))^{m+1}}{\Gamma((m+1)v)} \times \sum_{j=1}^{N+1} [(t-\tau)^{1/(\alpha-n)} + \|x-\xi\|_p]^{-\alpha_j},$$

for $m \geq 0$*, where* $\Gamma(\cdot)$ *denotes the Archimedean Gamma function.*

The proof of this assertion will be given later.

It follows from Claim A, by the Stirling formula, that series (5.18) is convergent and that

$$|\Phi(x,t,\xi,\tau)| \leq C_0 \sum_{k=1}^{N+1} [(t-\tau)^{1/(\alpha-n)} + \|x-\xi\|_p]^{-\alpha_k}. \tag{5.19}$$

Now (5.13) follows from (5.19) and Lemma 5.4.

Denote by $u_1(x,t)$ and $u_2(x,t)$ the first and second terms on the right-hand side of (5.11). Substituting (5.12) into (5.11), we find that

$$u_1(x,t) = \int_0^t \int_{\mathbb{Q}_p^n} Z(x-\xi, t-\tau; \xi, \tau) f(\xi,\tau)\, d^n\xi\, d\tau + \int_0^t \int_{\mathbb{Q}_p^n} Z(x-\eta, t-\theta; \eta, \theta) F(\eta,\theta)\, d^n\eta\, d\theta,$$

$$u_2(x,t) = \int_{\mathbb{Q}_p^n} Z(x-\xi, t; \xi, 0)\varphi(\xi)\, d^n\xi + \int_0^t \int_{\mathbb{Q}_p^n} Z(x-\eta, t-\theta; \eta, \theta) G(\eta,\theta)\, d^n\eta\, d\theta,$$

where

$$F(\eta,\theta) = \int_0^\theta \int_{\mathbb{Q}_p^n} \Phi(\eta,\theta,\xi,\tau) f(\xi,\tau)\, d^n\xi\, d\tau, \tag{5.20}$$

$$G(\eta,\theta) = \int_{\mathbb{Q}_p^n} \Phi(\eta,\theta,\xi,0)\varphi(\xi)\, d^n\xi. \tag{5.21}$$

Now, by Proposition 4.4 and (5.19), it follows that

$$|F(\eta,\theta)| \leq C_0(1+\|\eta\|_p^\lambda), \quad |G(\eta,\theta)| \leq C_1(1+\|\eta\|_p^\lambda),$$

for all $\eta \in \mathbb{Q}_p^n$ and $\theta \in (0,T]$.

Claim D. *The functions* F *and* G *belong to* $\widetilde{\mathcal{E}}$*, and their parameters of local constancy do not depend on* θ*.*

We first note that by (5.20)–(5.21), it is sufficient to show that $\Phi(\cdot,\theta,\star,\tau)$ is a locally constant function on $(\mathbb{Q}_p^\times)^n \times \mathbb{Q}_p^n$ and that its parameter of local constancy does not depend on θ or τ. Now, by the recursive definition of the function Φ we see that if L is the parameter of local constancy for all the functions $a_k(\cdot,t)$, $b(\cdot,t)$, $(\mathbf{W}_{\alpha_k}Z)(\cdot,t-\tau;\star,\tau)$ and $Z(\cdot,t-\tau;\star,\tau)$ on $(\mathbb{Q}_p^\times)^n\times\mathbb{Q}_p^n$, and if $\|\delta\|_p\le p^{-L}$, we have $R(x+\delta,t,\xi+\delta,\tau)=R(x,t,\xi,\tau)$. Furthermore, we successively obtain

$$\begin{aligned}R_{m+1}(x+\delta,t,\xi+\delta,\tau)&=\int_\tau^t\int_{\mathbb{Q}_p^n}R(x+\delta,t,\eta,\theta)R_m(\eta,\theta,\xi+\delta,\tau)\,d^n\eta\,d\theta\\&=\int_\tau^t\int_{\mathbb{Q}_p^n}R(x+\delta,t,\zeta+\delta,\theta)R_m(\zeta+\delta,\theta,\xi+\delta,\tau)\,d^n\zeta\,d\theta\\&=R_{m+1}(x,t,\xi,\tau),\end{aligned}$$

so that $\Phi(x+\delta,t,\xi+\delta,\tau)=\Phi(x,t,\xi,\tau)$, and hence

$$\begin{aligned}F(\eta+\delta,\theta)&=\int_0^\theta\int_{\mathbb{Q}_p^n}\Phi(\eta+\delta,\theta,\xi,\tau)f(\xi,\tau)\,d^n\xi\,d\tau\\&=\int_0^\theta\int_{\mathbb{Q}_p^n}\Phi(\eta+\delta,\theta,\xi+\delta,\tau)f(\xi+\delta,\tau)\,d^n\xi\,d\tau=F(\eta,\theta).\end{aligned}$$

Similarly, $G(\eta+\delta,\theta)=G(\eta,\theta)$ when $|\delta|_p\le p^{-L}$. Thus $u_1(\cdot,t),u_2(\cdot,t)\in\mathfrak{M}_\lambda$ uniformly in t. Thus the potentials in the expressions for $u_1(x,t)$, $u_2(x,t)$ satisfy the conditions to use the differentiation formulas given in Proposition 5.3. By using these formulas along with Proposition 5.3, (5.1)–(5.8) and (5.15), one verifies after simple transformations that $u(x,t)$ is a solution of the Cauchy problem (5.1).

Let us show that $u(x,t)\to\varphi(x)$ as $t\to0^+$. Due to (5.12) and (5.13), it is sufficient to verify that

$$v(x,t):=\int_{\mathbb{Q}_p^n}Z(x-\xi,t;\xi,0)\varphi(\xi)\,d^n\xi\to\varphi(x)\qquad\text{as }t\to0^+.$$

By (5.6), we have

$$\begin{aligned}v(x,t)&=\int_{\mathbb{Q}_p^n}[Z(x-\xi,t;\xi,0)-Z(x-\xi,t;x,0)]\varphi(\xi)\,d^n\xi\\&\quad+\int_{\mathbb{Q}_p^n}Z(x-\xi,t;x,0)[\varphi(\xi)-\varphi(x)]\,d^n\xi+\varphi(x).\end{aligned}$$

Now, since $Z(x-\xi,t;\cdot,0)$ and $\varphi(\cdot)$ are locally constant functions, it follows that both integrals are actually over the set

$$\{\xi\in\mathbb{Q}_p^n:\|\xi-x\|_p\ge p^{-L}\}.$$

By applying (5.3) on this set, we see that both integrals tend to zero as $t\to0^+$.

Proof of Claim C. We use induction on m. The case $m=0$ is (5.17). We assume the case m holds. Then

$$\begin{aligned}|R_{m+1}(x,t,\xi,\tau)| &\leq \int_\tau^t \int_{\mathbb{Q}_p^n} |R(x,t,\eta,\theta)|\,|R_m(\eta,\theta,\xi,\tau)|\,d^n\eta\,d\theta \\ &\leq C_0(2N+2)^{m-1}\frac{(\Gamma(v))^m}{\Gamma(mv)}\sum_{j,k=1}^{N+1}\int_\tau^t(\theta-\tau)^{(m-1)v} \\ &\quad\times\int_{\mathbb{Q}_p^n}[(\theta-\tau)^{1/(\alpha-n)}+\|\eta-\xi\|_p]^{-\alpha_j}[(t-\theta)^{1/(\alpha-n)}+\|x-\eta\|_p]^{-\alpha_k}\,d^n\eta\,d\theta.\end{aligned}$$

Now by Lemma 5.4 with $-\sigma=(m-1)(\alpha-n)v$, $\rho=0$, $-n-b_2=-\alpha_j$, $-n-b_1=-\alpha_k$, $\beta=\alpha-n$ (notice that the condition $\alpha>n+1$ implies $\beta>1$), we have

$$\begin{aligned}&|R_{m+1}(x,t,\xi,\tau)| \\ &\quad\leq C_0\Bigg\{(2N+2)^{m-1}\frac{(\Gamma(v))^m}{\Gamma(mv)}\sum_{j,k=1}^{N+1}B\left(1,\frac{\alpha+(m-1)(\alpha-n)v-\alpha_j}{\alpha-n}\right) \\ &\qquad\qquad\times[(t-\tau)^{1/(\alpha-n)}+\|x-\xi\|_p]^{-\alpha_k}(t-\tau)^{\frac{(m-1)(\alpha-n)v-\alpha_j+\alpha)}{\alpha-n}} \\ &\qquad+B\left(\frac{\alpha-\alpha_k}{\alpha-n},\frac{\alpha-n+(m-1)(\alpha-n)v}{\alpha-n}\right)[(t-\tau)^{1/(\alpha-n)}+\|x-\xi\|_p]^{-\alpha_j} \\ &\qquad\qquad\times(t-\tau)^{\frac{(m-1)(\alpha-n)v-\alpha_k+\alpha)}{\alpha-n}}\Bigg\},\end{aligned}$$

where $B(\cdot,\cdot)$ denotes the Archimedean Beta function.

By using $B(z_1+\epsilon,z_2+\delta)\leq B(z_1,z_2)$ for $\epsilon,\delta\geq 0$,

$$B\left(1,\frac{\alpha+(m-1)(\alpha-n)v-\alpha_j}{\alpha-n}\right)\leq B(v,mv),$$
$$B\left(\frac{\alpha-\alpha_k}{\alpha-n},\frac{\alpha-n+(m-1)(\alpha-n)v}{\alpha-n}\right)\leq B(v,mv),$$

and

$$(t-\tau)^{\frac{(m-1)(\alpha-n)v-\alpha_r+\alpha)}{\alpha-n}}=(t-\tau)^{mv-\frac{(\alpha-n)v+\alpha_r-\alpha}{\alpha-n}}\leq C(t-\tau)^{mv}$$

for $1\leq r\leq N+1$, we get

$$\begin{aligned}|R_{m+1}(x,t,\xi,\tau)| &\leq C(2N+2)^m\frac{(\Gamma(v))^m}{\Gamma((m+1)v)}(t-\tau)^{mv} \\ &\quad\times\sum_{k=1}^{N+1}[(t-\tau)^{1/(\alpha-n)}+\|x-\xi\|_p]^{-\alpha_k}.\end{aligned}$$

□

§6. Uniqueness of the solution

We recall that $\widetilde{\mathcal{E}}$ is the $\mathbb{C}$-vector space of all functions $\varphi : \mathbb{Q}_p^n \to \mathbb{C}$ such that there exists a ball B_l^n, with l depending only on φ, such that $\varphi(x + x') = \varphi(x)$ for any $x \in \mathbb{Q}_p^n$ and $x' \in B_l^n$. Notice that $\mathfrak{M}_\lambda \subset \widetilde{\mathcal{E}}$ for any λ. We identify each element of $\widetilde{\mathcal{E}}$ with a distribution on $\mathbb{Q}_p^n$. We now recall the following fact: $T \in S'$ with $\operatorname{supp}(T) \subset B_N^n$ if and only if $\widehat{T} \in \widetilde{\mathcal{E}}$ and its parameter of local constancy is greater than $-N$ (cf. [17, p. 109]).

Lemma 6.1. $\mathbf{W}_\alpha : \widetilde{\mathcal{E}} \to \widetilde{\mathcal{E}}$ *is a well-defined linear operator. Furthermore,*

$$(\mathbf{W}_\alpha\varphi)(x) = -\mathcal{F}^{-1}_{\xi\to x}(A_{w_\alpha}(\|\xi\|_p)\mathcal{F}_{x\to\xi}\varphi).$$

Proof. Let l be a parameter of local constancy of φ. Then

$$\begin{aligned}(\mathbf{W}_\alpha\varphi)(x) &= \int_{\|y\|_p\ge p^l} \frac{\varphi(x-y)-\varphi(x)}{w_\alpha(\|y\|_p)}\, d^n y \\ &= \frac{1_{\mathbb{Q}_p^n\setminus B_l^n}(x)}{w_\alpha(\|x\|_p)} * \varphi(x) - \varphi(x)\int_{\|y\|_p\ge p^l} \frac{d^n y}{w_\alpha(\|y\|_p)}.\end{aligned}$$

Then by taking the Fourier transform in S' we get

$$\mathcal{F}(\mathbf{W}_\alpha\varphi)(\xi) = \left(\int_{\mathbb{Q}_p^n} 1_{\mathbb{Q}_p^n\setminus B_l^n}(x)\frac{\Psi(x\cdot\xi)-1}{w_\alpha(\|x\|_p)}\, d^n x\right)(\mathcal{F}\varphi)(\xi),$$

and since $\mathcal{F}\varphi \in S'$ with $\operatorname{supp}(\mathcal{F}\varphi) \subset B_{-l}^n$,

$$\mathcal{F}(\mathbf{W}_\alpha\varphi)(\xi) = \left(\int_{\mathbb{Q}_p^n} \frac{\Psi(x\cdot\xi)-1}{w_\alpha(\|x\|_p)}\, d^n x\right)(\mathcal{F}\varphi)(\xi).$$

Therefore,

$$(\mathbf{W}_\alpha\varphi)(x) = -\mathcal{F}^{-1}_{\xi\to x}(A_{w_\alpha}(\|\xi\|_p)\mathcal{F}_{x\to\xi}\varphi) \in \widetilde{\mathcal{E}}. \qquad \square$$

Take γ to be a real number such that $\lambda < \gamma < \alpha_1 - n < \cdots < \alpha_N - n < \alpha - n$, fix an integer L, and set $\psi(x) := p^{Ln}\Omega(p^L\|x\|_p) * \|x\|_p^\gamma$. Then

$$\psi(x) = \begin{cases} \|x\|_p^\gamma & \text{if } \|x\|_p > p^{-L}, \\ C & \text{if } \|x\|_p \le p^{-L}, \end{cases} \tag{6.1}$$

and thus $\psi \in \widetilde{\mathcal{E}}$.

Lemma 6.2. *With the above notation, there exist positive constants C_1 and C_2 such that*

(i) $|(\mathbf{W}_\alpha\psi)(x)| \le C_1\|x\|_p^{\alpha-\gamma+n}$,
(ii) $|(\mathbf{W}_{\alpha_k}\psi)(x)| \le C_2\|x\|_p^{\alpha_k-\gamma+n}$ *for $k = 1, \ldots, N$.*

Proof. By Lemma 6.1,

$$(\mathbf{W}_\alpha\psi)(x) = -\mathcal{F}^{-1}_{\xi\to x}\left(A_{w_\alpha}(\|\xi\|_p)\Omega(p^{-L}\|\xi\|_p)\frac{\|\xi\|_p^{-\gamma-n}}{\Gamma_n(n+\gamma)}\right) \quad \text{in } S',$$

where $\Gamma_n(n+\gamma) = \frac{1-p^{\gamma}}{1-p^{-\gamma-n}}$. Now, since $A_{w_\alpha}(\|\xi\|_p)\Omega(p^{-L}\|\xi\|_p)\|\xi\|_p^{-\gamma-n}/\Gamma_n(n+\gamma)$ is radial and locally integrable, by applying the formula for the Fourier transform of a radial function (see e.g. [17, Example 8, p. 43]) we get

$$\begin{aligned}(\mathbf{W}_\alpha\varphi)(x) \\ = \frac{-\|x\|_p^{-n}}{\Gamma_n(n+\gamma)}\Big[(1-p^{-n})\|x\|_p^{\gamma+n}\sum_{j=0}^{\infty}A_{w_\alpha}(\|x\|_p^{-1}p^{-j})\Omega(\|x\|_p^{-1-j}p^{-j})p^{j(\gamma+n)-jn} \\ - A_{w_\alpha}(\|x\|_p p^{-j})\Omega(\|x\|_p^{-1}p^{-L+1})\|x\|_p^{\gamma+n}\Big],\end{aligned}$$

as a distribution on $\mathbb{Q}_p^n\setminus\{0\}$. Now by using [7, Lemma 3.4],

$$|(\mathbf{W}_\alpha\varphi)(x)| \le C'\Big[(1-p^{-n})\sum_{j=0}^{\infty}p^{-j(\alpha-n)+j\gamma} - p^{(-L+1)(\alpha-n)}\Big]\|x\|_p^{-\alpha+n+\gamma}.$$

The proof of (ii) is similar. □

Theorem 6.3. *Assume that the coefficients $a_k(x,t)$, $k=0,1,\dots,N$ are non-negative bounded continuous functions, $b(x,t)$ is a bounded continuous function, $0\le\lambda<\alpha_1-n$, $\alpha>n+1$ (if $a_1(x,t)=\cdots=a_k(x,t)\equiv 0$, we suppose that $0\le\lambda<\alpha-n$) and $u(x,t)$ is a solution of the Cauchy problem (5.1) with $f(x,t)=\varphi(x)\equiv 0$ that belongs to the class $\mathfrak{M}_\lambda$. Then $u(x,t)\equiv 0$.*

Proof. We may assume that $b(x,t)\ge 0$, otherwise we take $u(x,t)e^{\lambda t}$ with $\lambda > b(x,t)$. We first prove that $u(x,t)\ge 0$. For contradiction, suppose that $u(x',t')<0$ for some $x'\in\mathbb{Q}_p^n$ and $t'\in(0,T]$. By Lemma 6.2, it follows that $(\mathbf{W}_\alpha\psi)(x)\to 0$ and $(\mathbf{W}_{\alpha_k}\psi)(x)\to 0$ as $\|x\|_p\to\infty$, and thus

$$M := \sup_{\substack{0\le t\le T\\ x\in\mathbb{Q}_p^n}}\Big\{a_0(x,t)|(\mathbf{W}_\alpha\psi)(x)| + \sum_{k=1}^{N}a_k(x,t)|(\mathbf{W}_{\alpha_k}\psi)(x)|\Big\} < \infty.$$

We pick $\rho>0$ such that $u(x',t')+T\rho<0$, and then $\sigma>0$ such that

$$u(x',t')+T\rho+\sigma\psi(x')<0, \tag{6.2}$$

$$\rho-\sigma M<0. \tag{6.3}$$

We now consider the function $v(x,t) := u(x,t)+t\rho+\sigma\psi(x)$. From (6.2), it follows that $v(x',t')<0$, so that

$$\inf_{0\le t\le T,\, x\in\mathbb{Q}_p^n} v(x,t) < 0.$$

On the other hand, since $u(\cdot,t) \in \mathfrak{M}_\lambda$, we have $\lim_{\|x\|_p\to\infty} u(x,t)/\psi(x) = 0$ and thus $\lim_{\|x\|_p\to\infty} v(x,t) > 0$ for any $t > 0$. This implies that there exist $x_0 \in \mathbb{Q}_p^n$ and $t_0 \in (0,T]$ such that

$$\inf_{0\le t\le T,\, x\in\mathbb{Q}_p^n} v(x,t) = \min_{0\le t\le T,\, x\in\mathbb{Q}_p^n} v(x,t) = v(x_0,t_0) < 0,$$

and thus, by formula (3.1), $(\mathbf{W}_\alpha v)(x_0,t_0) \ge 0, (\mathbf{W}_{\alpha_k} v)(x_0,t_0) \ge 0$ for all k, and $\frac{\partial v}{\partial t}(x_0,t_0) \le 0$, hence

$$\frac{\partial v}{\partial t}(x_0,t_0) - a_0(x,t)(\mathbf{W}_\alpha v)(x_0,t_0) - \sum_{k=1}^N a_k(x,t)(\mathbf{W}_{\alpha_k} v)(x_0,t_0) + b(x,t)v(x_0,t_0) < 0.$$

Now, by (6.3),

$$\begin{aligned}\frac{\partial v}{\partial t}(x,t) - a_0(x,t)(\mathbf{W}_\alpha v)(x,t) - \sum_{k=1}^N a_k(x,t)(\mathbf{W}_{\alpha_k} v)(x,t) + b(x,t)v(x,t)\\ = \rho - \sigma[a_0(x,t)(\mathbf{W}_\alpha \psi)(x) + \sum_{k=1}^N a_k(x,t)(\mathbf{W}_{\alpha_k}\psi)(x)] + b(x,t)[\rho t + \sigma\psi(x)]\\ \ge \rho - \sigma M > 0.\end{aligned}$$

We have obtained a contradiction, thus $u(x,t) \ge 0$. Finally, taking $-u(x,t)$ instead of $u(x,t)$, we conclude that $u(x,t) \equiv 0$. □

§7. Markov processes

In this section we show that the fundamental solution $\Lambda(x,t,\xi,\tau)$ of the Cauchy problem (5.1) is the transition density of a Markov process. We need some preliminary results.

Lemma 7.1. *If the coefficients $a_k(x,t)$ and $b(x,t)$ are nonnegative, then $\Lambda(x,t,\xi,\tau) \ge 0$.*

Proof. It is sufficient to show that $u(x,t) = \int_{\mathbb{Q}_p^n} \Lambda(x,t,\xi,\tau)\varphi(\xi)\, d^n\xi \ge 0$, where $u(x,t)$ is the solution of (5.1) with $f(x,t) \equiv 0$, and initial condition $u(x,0) = \varphi(x) \ge 0$ with $\varphi \in S(\mathbb{Q}_p^n)$. From (5.12), (5.13), and Lemma 4.3(iii), it follows that

$$u(x,t) \to 0 \quad \text{as } \|x\|_p \to \infty. \tag{7.1}$$

Now, if $u(x,t) < 0$, then there exist $x_0 \in \mathbb{Q}_p^n$ and $t_0 \in (0,T]$ such that

$$\inf_{0\le t\le T,\, x\in\mathbb{Q}_p^n} u(x,t) = u(x_0,t_0) < 0. \tag{7.2}$$

This implies that $(\mathbf{W}_\alpha u)(x_0,t_0)\geq 0$, $(\mathbf{W}_{\alpha_k} u)(x_0,t_0)\geq 0$ for all k, and $\frac{\partial u}{\partial t}(x_0,t_0)$ ≤ 0. On the other hand,

$$\frac{\partial u}{\partial t}(x,t)-a_0(x,t)(\mathbf{W}_\alpha u)(x,t)-\sum_{k=1}^{N}a_k(x,t)(\mathbf{W}_{\alpha_k}u)(x,t)=0.$$

By using the uniform parabolicity condition $a_0(x,t)\geq\mu>0$, we get $(\mathbf{W}_\alpha u)(x_0,t_0)$ $=0$; then by (3.1), $u(x,t_0)$ is constant, and by (7.1), $u(x,t_0)\equiv 0$, which contradicts (7.2). □

Lemma 7.2. *If* $b(x,t)\equiv 0$, *then* $\int_{\mathbb{Q}_p^n}\Lambda(x,t,\xi,\tau)\,d^n\xi=1$.

Proof. By integrating (5.14) in the variable ξ over the whole space $\mathbb{Q}_p^n$, and by using Lemma 4.9(iii), we have $\frac{\partial}{\partial t}(\int_{\mathbb{Q}_p^n}\Lambda(x,t,\xi,\tau)\,d^n\xi)=0$, thus $\int_{\mathbb{Q}_p^n}\Lambda(x,t,\xi,\tau)\,d^n\xi$ is independent of t. Now, by integrating (5.12) over the whole space $\mathbb{Q}_p^n$ in the variable ξ and by using Lemma 4.3(iv), we have

$$\int_{\mathbb{Q}_p^n}\Lambda(x,t,\xi,\tau)\,d^n\xi=1+\int_\tau^t\int_{\mathbb{Q}_p^n}\int_{\mathbb{Q}_p^n}Z(x-\eta,t-\theta;\eta,\theta)\phi(\eta,\theta,\xi,\tau)\,d^n\eta\,d^n\xi\,d\theta.$$

The result is obtained by taking $t=\tau$ in the above formula. □

Lemma 7.3. *If* $b(x,t)\equiv 0$ *and* $f(x,t)\equiv 0$, *then*

$$\Lambda(x,t,\xi,\tau)=\int_{\mathbb{Q}_p^n}\Lambda(x,t,y,\sigma)\Lambda(y,\sigma,\xi,\tau)\,d^ny. \tag{7.3}$$

Proof. Consider the following initial value problem:

$$\begin{cases}\frac{\partial u}{\partial t}(x,t)-a_0(x,t)(\mathbf{W}_\alpha u)(x,t)-(\widetilde{\mathbf{W}}u)(x,t)=0,\\ u(x,\tau)=\varphi(x),\quad x\in\mathbb{Q}_p^n \text{ and } t\in(\tau,\sigma].\end{cases} \tag{7.4}$$

By Theorem 5.5, $u(x,\sigma)=\int_{\mathbb{Q}_p^n}\Lambda(x,\sigma,\xi,\tau)\varphi(\xi)\,d^n\xi$. Now consider

$$\begin{cases}\frac{\partial u}{\partial t}(x,t)-a_0(x,t)(\mathbf{W}_\alpha u)(x,t)-(\widetilde{\mathbf{W}}u)(x,t)=0,\\ u(x,\sigma)=\int_{\mathbb{Q}_p^n}\Lambda(x,\sigma,\xi,\tau)\varphi(\xi)\,d^n\xi,\quad x\in\mathbb{Q}_p^n,\ t\in(\sigma,T],\ \tau<\sigma<T.\end{cases} \tag{7.5}$$

By Theorem 5.5 and Fubini's Theorem, the solution of (7.5) is given by

$$u(x,t)=\int_{\mathbb{Q}_p^n}\left(\int_{\mathbb{Q}_p^n}\Lambda(x,t,y,\sigma)\Lambda(y,\sigma,\xi,\tau)\,d^ny\right)\varphi(\xi)\,d^n\xi.$$

On the other hand, (7.5) is equivalent to

$$\begin{cases}\frac{\partial u}{\partial t}(x,t)-a_0(x,t)(\mathbf{W}_\alpha u)(x,t)-(\widetilde{\mathbf{W}}u)(x,t)=0,\\ u(x,\tau)=\varphi(x),\quad x\in\mathbb{Q}_p^n\ ,\ t\in(\tau,T],\end{cases} \tag{7.6}$$

which has a solution given by $u(x,t) = \int_{\mathbb{Q}_p^n} \Lambda(x,t,\xi,\tau)\varphi(\xi)\, d^n\xi$. Now, by Theorem 6.3,

$$\int_{\mathbb{Q}_p^n} \Lambda(x,t,\xi,\tau)\varphi(\xi)\, d^n\xi = \int_{\mathbb{Q}_p^n} \left(\int_{\mathbb{Q}_p^n} \Lambda(x,t,y,\sigma)\Lambda(y,\sigma,\xi,\tau)\, d^n y \right) \varphi(\xi)\, d^n\xi$$

for any test function φ, which implies (7.3). □

Theorem 7.4. *If the coefficients $a_k(x,t)$, $k = 1,\dots,N$, are non-negative bounded continuous functions, $b(x,t) \equiv 0$, $0 \le \lambda < \alpha_1 - n$, $\alpha > n+1$ (if $a_1(x,t) = \dots = a_k(x,t) \equiv 0$, we suppose that $0 \le \lambda < \alpha - n$), and $f(x,t) \equiv 0$, then the fundamental solution $\Lambda(x,t,\xi,\tau)$ is the transition density of a bounded right-continuous Markov process without second kind discontinuities.*

Proof. The result follows from [10, Theorem 3.6] by using Lemmas 7.1–7.3 and 5.12–5.13, and Lemma 4.3(iii). □

§8. The Cauchy problem is well-posed

In this section, we study the continuity of the solution of the Cauchy problem (5.1) with respect to $\varphi(x)$ and $f(x,t)$. We assume that the coefficients $a_k(x,t)$, $k = 0,1,\dots,N$, are non-negative bounded continuous functions, $b(x,t)$ is a bounded continuous function, $0 \le \lambda < \alpha_1 - n$ (if $a_1(x,t) = \dots = a_k(x,t) \equiv 0$, we suppose that $0 \le \lambda < \alpha - n$), $\varphi \in \mathfrak{M}_\lambda$ and $f(\cdot,t) \in \mathfrak{M}_\lambda$ uniformly in t, with $0 \le \lambda < \alpha_1 - n$.

We identify $\mathfrak{M}_\lambda$ with the $\mathbb{R}$-vector space of all functions "$\phi(\cdot,t) \in \mathfrak{M}_\lambda$ uniformly in t," and introduce on $\mathfrak{M}_\lambda$ the following norm:

$$\|\phi\|_{\mathfrak{M}_\lambda} := \sup_{t\in[0,T]} \sup_{x\in\mathbb{Q}_p^n} \left| \frac{\phi(x,t)}{1+\|x\|_p^\lambda} \right|.$$

From now on, we consider $\mathfrak{M}_\lambda$ as a topological vector space with the topology induced by $\|\cdot\|_{\mathfrak{M}_\lambda}$. We also consider $\mathfrak{M}_\lambda \times \mathfrak{M}_\lambda$ as a topological vector space with the topology induced by the norm $\|\cdot\|_{\mathfrak{M}_\lambda} + \|\star\|_{\mathfrak{M}_\lambda}$.

Theorem 8.1. *With the above hypotheses, consider the following operator:*

$$\mathfrak{M}_\lambda \times \mathfrak{M}_\lambda \xrightarrow{\boldsymbol{L}} \mathfrak{M}_\lambda, \qquad (\varphi(\cdot), f(\cdot,t)) \mapsto u(\cdot,t),$$

where $u(x,t)$ is given by (5.11). Then $\|u(\cdot,t)\|_{\mathfrak{M}_\lambda} \le C(\|\varphi(\cdot)\|_{\mathfrak{M}_\lambda} + \|f(\cdot,t)\|_{\mathfrak{M}_\lambda})$, i.e. $\boldsymbol{L}$ is a continuous operator.

Proof. We write $u(x,t) = u_1(x,t) + u_2(x,t)$ where

$$u_1(x,t) = \int_0^t \int_{\mathbb{Q}_p^n} \Lambda(x,t,\xi,\tau) f(\xi,\tau)\, d^n\xi\, d\tau \quad \text{and} \quad u_2(x,t) = \int_{\mathbb{Q}_p^n} \Lambda(x,t,\xi,0)\varphi(\xi)\, d^n\xi$$

as before. Now

$$|u_1(x,t)| \leq \int_0^t \int_{\mathbb{Q}_p^n} |\Lambda(x,t,\xi,\tau)|\, |f(\xi,\tau)|\, d^n\xi\, d\tau$$
$$\leq \|f(x,t)\|_{\mathfrak{M}_\lambda} \left\{ \int_0^t \int_{\mathbb{Q}_p^n} |\Lambda(x,t,\xi,\tau)|\, d^n\xi\, d\tau + \int_0^t \int_{\mathbb{Q}_p^n} |\Lambda(x,t,\xi,\tau)|\, \|\xi\|_p^\lambda\, d^n\xi\ d\tau \right\},$$

and by (5.12)–(5.13), (5.3) and Proposition 4.4,

$$|u_1(x,t)| \leq C_0 \|f(\cdot,t)\|_{\mathfrak{M}_\lambda} \left\{ \int_0^t (t-\tau)^{1+\frac{n-\alpha}{\alpha-n}}\, d\tau + \int_0^t (t-\tau)^{2-\frac{\lambda}{\alpha-n}+\frac{n-\alpha}{\alpha-n}}\, d\tau \right.$$
$$+ \sum_{k=1}^{N+1} \int_0^t (t-\tau)^{1+\frac{n-\alpha_k}{\alpha-n}}\, d\tau + (1+\|x\|_p^\lambda) \int_0^t (t-\tau)^{1+\frac{n-\alpha}{\alpha-n}}\, d\tau$$
$$\left. + (1+\|x\|_p^\lambda) \int_0^t (t-\tau)^{2-\frac{\lambda}{\alpha-n}+\frac{n-\alpha}{\alpha-n}}\, d\tau + (1+\|x\|_p^\lambda) \sum_{k=1}^{N+1} \int_0^t (t-\tau)^{1+\frac{n-\alpha_k}{\alpha-n}}\, d\tau \right\}$$
$$\leq \|f(\cdot,t)\|_{\mathfrak{M}_\lambda} \{C_1(T) + C_2(T)(1+\|x\|_p^\lambda)\}.$$

Hence,

$$\left| \frac{u_1(x,t)}{1+\|x\|_p^\lambda} \right| \leq \|f(\cdot,t)\|_{\mathfrak{M}_\lambda} \left\{ \frac{C_1(T)}{1+\|x\|_p^\lambda} + C_2(T) \right\}.$$

Similarly, one shows that

$$\left| \frac{u_2(x,t)}{1+\|x\|_p^\lambda} \right| \leq \|\varphi(\cdot)\|_{\mathfrak{M}_\lambda} \left\{ \frac{C_1'(T)}{1+\|x\|_p^\lambda} + C_2'(T) \right\},$$

therefore $\|u(\cdot,t)\|_{\mathfrak{M}_\lambda} \leq C(\|\varphi(\cdot)\|_{\mathfrak{M}_\lambda} + \|f(\cdot,t)\|_{\mathfrak{M}_\lambda})$. □

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