Uniform Resolvent Estimates for Magnetic Schrödinger Operators in a 2D Exterior Domain and their Applications to Related Evolution Equations

by

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Abstract

We consider the magnetic Schrödinger operator in an exterior domain $\Omega \subset \mathbb{R}^2$ with star-shaped boundary with respect to the origin. We prove uniform resolvent estimates under suitable decay and smallness conditions on the magnetic field and external potential. The results are then used to obtain smoothing properties for the corresponding evolution equations.

§1. Introduction and results

Let Ω be an exterior domain in $\mathbb{R}^2$ with a star-shaped complement with respect to the origin and smooth boundary $\partial\Omega$. In this paper we consider in Ω the magnetic Schrödinger equation

$$
(1.1) \qquad -\sum_{j=1}^{2}\{\partial_j + ib_j(x)\}^2 u + c(x)u - \kappa^2 u = f(x), \quad x \in \Omega,
$$

with Dirichlet boundary condition

$$
(1.2) \qquad u(x, \kappa) = 0, \quad x \in \partial\Omega.
$$

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Here $\partial_j = \partial/\partial x_j$ $(j = 1, 2)$, $i = \sqrt{-1}$, $\kappa \in \Pi_\pm = \{\kappa \in \mathbb{C}; \pm \operatorname{Re} \kappa > 0, \ \operatorname{Im} \kappa > 0\}$, $b_j(x)$ are real valued C^1-functions of $x \in \overline{\Omega} = \Omega \cup \partial\Omega$, $c(x)$ is a real valued continuous function of $x \in \overline{\Omega}$ and $f \in L^2(\Omega)$. Moreover, $b(x) = (b_1(x), b_2(x))$ represents a magnetic potential. Thus the magnetic field is defined by its rotation, $\nabla \times b(x) = \partial_1 b_2(x) - \partial_2 b_1(x)$.

We set $\nabla = (\partial_1, \partial_2)$, $\nabla_b = \nabla + ib(x)$, $\Delta_b = \nabla_b \cdot \nabla_b$, $r = |x|$, $\tilde{x} = x/r$ and $\partial_r = \tilde{x} \cdot \nabla$. The inner product and norm of $L^2(\Omega)$ are defined by

$$(f, g) = \int f(x)\overline{g(x)}\, dx \quad \text{and} \quad \|f\| = \sqrt{(f, f)}.$$

Here $\int dx$ denotes integration over Ω. Moreover, for $t > 0$ we set

$$\Omega_t = \{x \in \Omega; |x| < t\} \quad \text{and} \quad S_t = \{x \in \Omega; |x| = t\}.$$

Now we define an operator L acting in $L^2(\Omega)$ as follows:

(1.3) $\qquad \begin{cases} Lu = -\Delta_b u + c(x)u \quad \text{for } u \in \mathcal{D}(L), \\ \mathcal{D}(L) = \{u \in L^2(\Omega) \cap H^2_{\mathrm{loc}}(\overline{\Omega}); (-\Delta_b + c)u \in L^2, \ u|_{\partial\Omega} = 0\}. \end{cases}$

Here $H^j(\Omega)$ $(j = 1, 2, \dots)$ are the usual Sobolev spaces on Ω and $H^2_{\mathrm{loc}}(\overline{\Omega})$ is the space of functions that are H^2 on each compact set of $\overline{\Omega}$.

As is well known, if $c \in L^2(\Omega)$, then L is a lower semibounded selfadjoint operator in $L^2(\Omega)$ (see e.g. Mochizuki [13]). Moreover, if

(1.4) $\qquad\qquad \max\{|\nabla \times b(x)|, |c(x)|\} = o(r^{-1}) \quad \text{as } r \to \infty$

then the essential spectrum of L fills the whole nonnegative half-line $[0, \infty)$ (see Aliev–Eyvazov [1]).

The selfadjointness of L shows that its resolvent $R(\kappa^2) = (L - \kappa^2)^{-1}$ is defined for each $\kappa \in \Pi_\pm$, and $u = R(\kappa^2)f$ gives a unique L^2-solution of the exterior problem (1.1), (1.2).

In the following we exclude the case $\Omega = \mathbb{R}^2$ and assume that

(A0) $\qquad\qquad$ there exists $r_0 > 0$ such that $\min\{|x|; x \in \partial\Omega\} > r_0$.

Now the main purpose of this paper is to show the following theorem for the solution $u = R(\kappa^2)f$.

Theorem 1. *Assume that* (A0) *holds and*

(A1) $\qquad\qquad \{|\nabla \times b(x)|^2 + |c(x)|^2\}^{1/2} \le \dfrac{\epsilon_0}{r^2(1 + \log r/r_0)^2} \quad in \ \Omega,$

where $0 < \epsilon_0 < 1/(4\sqrt{21})$ and we mean $\log r/r_0 = \log(r/r_0)$.

(i) *For each $\kappa \in \Pi_\pm$, we have*

$$(1.5) \qquad \int (\operatorname{Im} \kappa r + 1) \frac{|u|^2}{r^2(1 + \log r/r_0)^2} \, dx \leq C_1 \int r^2(1 + \log r/r_0)^2 |f|^2 \, dx$$

with $C_1 = (2\sqrt{21})^2/(1 - (4\sqrt{21}\,\epsilon_0)^2)$. Moreover, if we define the vector function

$$(1.6) \qquad \theta(x, \kappa) = \nabla_b u + \tilde{x} \left(\frac{1}{2r} - i\kappa \right) u,$$

then for each $\kappa \in \Pi_\pm$,

$$(1.7) \qquad \int (\operatorname{Im} \kappa r + 1) \frac{|\theta|^2}{(4 + \log r/r_0)^2} \, dx \leq C_2 \int r^2(1 + \log r/r_0)^2 |f|^2 \, dx$$

where $C_2 = C_2(\epsilon_0) \leq 2 + 2(1 + \epsilon_0^2)C_1$.

(ii) *Let $\mu(r)$ be a smooth, positive L^1-function of $r \geq r_0$ satisfying also*

$$(1.8) \qquad 2r\mu'(r) \leq \mu(r) \quad and \quad \mu(r) \leq \frac{C_3}{(4 + \log r/r_0)^2}$$

for some $C_3 > 0$. Then for each $\kappa \in \Pi_\pm$,

$$(1.9) \quad \int \mu(r)\{|\nabla_b u|^2 + |\kappa u|^2\} \, dx \leq C_4 \int \{r^2(1 + \log r/r_0)^2 + \mu(r)^{-1}\}|f(x)|^2 \, dx$$

with $C_4 = \max\{2C_3C_2, 4\|\mu\|_{L^1}^2\}$, where $\|\mu\|_{L^1} = \int_{r_0}^\infty \mu(t) \, dt$.

Remark 1. (1) The function $\theta(x, \kappa)$ is used to define radiation conditions for the solution $u(x, \kappa)$ (see [13]), and (1.7) asserts their uniform estimates in $\kappa \in \Pi_\pm$.

(2) We can choose $\mu(r) = r^{-2}(1 + \log r/r_0)^{-2}$. In this case (1.8) holds with $C_3 = (4/r_0)^2$.

Remark 2. Theorem 1(i) implies that the operator L is nonnegative and 0 is not an eigenvalue of L. The above two assertions can also be proved directly. Note that

$$\liminf_{t \to \infty} \operatorname{Re} \int_{S_t} (\tilde{x} \cdot \nabla u)\overline{u} \, dS \leq 0$$

for $u \in \mathcal{D}(L)$ (see [15]). Then a direct proof follows from the identity

$$\operatorname{Re} \int_{\Omega_t} \{-\nabla_b \cdot \nabla_b u + c(x)u\}\overline{u} \, dx = \int_{\Omega_t} \{|\nabla_b u|^2 + c(x)|u|^2\} \, dx - \operatorname{Re} \int_{S_t} (\tilde{x} \cdot \nabla u)\overline{u} \, dS$$

combined with the modified Hardy inequality (Lemma 4) of §3 and the requirement $c(x) \geq -\epsilon_0 r^{-2}(1 + \log r/r_0)^{-2}$ with $\epsilon_0 < 1/4$.

As a corollary of Theorem 1, we are able to obtain space-time weighted estimates (smoothing properties, cf. Kato [8]) for the Schrödinger evolution equation

$$(1.10) \qquad i\frac{\partial u}{\partial t} - Lu = 0, \quad u(0) = f \in L^2(\Omega),$$

and for the relativistic Schrödinger evolution equation

$$(1.11) \qquad i\frac{\partial u}{\partial t} - \sqrt{L + m^2}\, u = 0, \quad u(0) = f \in L^2(\Omega),$$

with $m \geq 0$. Note that the smoothing effects for (1.11) give those for the Klein–Gordon equation ($m > 0$) or the wave equation ($m = 0$) in the energy space.

Theorem 2. *Assume that* (A0) *and* (A1) *hold.*

(i) *The solution operator* e^{-itL} *to* (1.10) *satisfies*

$$(1.12) \qquad \left| \int_0^{\pm\infty} \left\| r^{-1}(1 + \log r/r_0)^{-1} \int_0^t e^{-i(t-\tau)L} h(\tau)\, d\tau \right\|^2 dt \right|$$
$$\leq C_1 \left| \int_0^{\pm\infty} \| r(1 + \log r/r_0) h(t) \|^2\, dt \right|$$

for every $h(t)$ *satisfying* $r(1 + \log r/r_0)h(t) \in L^2(\mathbb{R} \times \Omega)$, *and*

$$(1.13) \qquad \left| \int_0^{\pm\infty} \| r^{-1}(1 + \log r/r_0)^{-1} e^{-itL} f \|^2\, dt \right| \leq 2\sqrt{C_1}\, \|f\|^2$$

for $f \in L^2(\Omega)$. *Here* $\|\cdot\|$ *means the norm of* $L^2(\Omega)$.

(ii) *The solution operator* $e^{-it\sqrt{L+m}}$ *to* (1.11) *satisfies*

$$(1.14) \qquad \left| \int_0^{\pm\infty} \| r^{-1}(1 + \log r/r_0)^{-1} e^{-it\sqrt{L+m^2}} f \|^2\, dt \right| \leq 8\sqrt{m^2 C_1 + 2C_4}\, \|f\|^2$$

for $f \in L^2(\Omega)$. *Here to determine* C_4 *we have chosen* $\mu(r) = r^{-2}(1 + \log r/r_0)^{-2}$.

For space dimension $n \geq 3$, Theorems 1 and 2 have been proved by Mochizuki [14], [15] with weight $r^{-1}(1 + \log r/r_0)^{-1}$ in (A1) replaced by r^{-1} for problems in both an exterior domain and the whole space $\mathbb{R}^n$. In case $n = 2$ some partial results have been obtained by Nakazawa [14], but without the resolvent estimate being given clearly in the duality form with weight $r^{-1}(1 + \log r/r_0)^{-1}$. To show (1.5) it is crucial to obtain a Hardy type inequality corresponding to the functions $\theta(x, \kappa)$ (see Proposition 3 and Lemma 3 below). In case $n \geq 3$ the inequality is not so far from the original Hardy inequality, and reads (see [15, Lemma 9])

$$\int \frac{(n-2)^2}{4r^2} |u|^2\, dx \leq \int \left\{ |\tilde{x} \cdot \theta|^2 + \frac{(n-1)(n-3)}{4r^2} |u|^2 \right\} dx.$$

In [15] the proof of Theorem 1(i) is done by combining this with another inequality for the solution $u = R(\kappa^2)f$ which corresponds to Lemma 2 below. In case $n = 2$ the above inequality does not work since the left side vanishes identically. Moreover, the definiteness of the integrand $|\tilde{x} \cdot \theta|^2 - \frac{1}{4r^2}|u|^2$ is not guaranteed. So, we need a delicate calculation to modify it. As a result, the function

$$\eta(r) = \frac{1}{2r(1 + \log r/r_0)}$$

is adopted to be a suitable weight satisfying the inequality

$$\int \eta^2 |u|^2 \, dx \le C \int \left(\operatorname{Im} \kappa r + \frac{1}{2} - \tilde{\epsilon} r^2 \eta^2 \right) \left\{ |\tilde{x} \cdot \theta|^2 - \frac{1}{4r^2}|u|^2 \right\} dx$$

with $C = C(\tilde{\epsilon}) > 0$ for small $\tilde{\epsilon} > 0$ (see Lemmas 2 and 3). The above weight is the same as the one for the Hardy inequality in an exterior domain of $\mathbb{R}^2$ (Remark 3 of §3). Note that the standard Hardy inequality does not in general hold in the whole space $\mathbb{R}^2$ (see e.g. Solomyak [19], Kobayashi–Misawa [10]).

Theorem 1(i) shows the L^2-smoothness of $r^{-1}(1 + \log r/r_0)^{-1}$. Apart from [14], [15], the corresponding results have been studied by Kato–Yajima [9] (see also Kuroda [11], Watanabe [20]), where the operator in question is the Laplace operator $L_0 = -\Delta$ in $\mathbb{R}^n$ ($n \ge 3$). Fourier transformation methods are used there, and the results are summarized for $n \ge 2$ as follows:

$$\|r^{-\alpha} L_0^\beta (L_0 - \kappa^2)^{-1} r^{-\alpha} f\| \le C |\kappa|^{2(\alpha+\beta-1)} \|f\|,$$

where $\alpha \in (1/2, n/2)$ and $\beta \in [-\alpha, 1 - \alpha]$ (Yafaev [21, Proposition 7.1.11]). Note that the right side is never bounded when $n = 2$ and $\beta = 0$. More explicitly, the asymptotic expansions of the resolvent $(-\Delta - \kappa^2)^{-1}$ as $\kappa \to 0$ have been well studied in suitable weighted L^2-spaces (see Murata [16], Yajima [23], Jensen–Nenciu [7]), and in case $n = 2$ a logarithmic singularity $\log \kappa$ appears in the leading term. Thus, the uniform estimates of Theorem 1(i) are not expected to hold for two-dimensional whole space problems.

We should mention here that when a suitable *non-trivial* magnetic field is introduced, one can expect some improvement of the above situation. In fact, the Hardy inequality in Remark 3 is known to hold without the unpleasant logarithmic factor for such a magnetic Laplacian $-\Delta_b$ in the whole $\mathbb{R}^2$ (Laptev–Weidl [12]). So, there is some possibility to develop a new low-energy estimate of $(-\Delta_b - \kappa^2)^{-1}$. In this paper, however, we do not enter into this problem. Our assumption (A1) requires only the smallness of the magnetic field, and it does not exclude the *zero* magnetic field $\nabla \times b(x) \equiv 0$.

For the exterior problem, Fourier transformation methods are not applicable, and in general it becomes difficult to know clearly the κ dependence of the resol-

vent kernel. It seems that, especially when $n = 2$, there are no results showing the resolvent expansions around zero and/or high energies. Resolvent estimates are mainly developed in connection with the study of the principle of limiting absorption. So, the results are restricted to κ in a compact domain in $\Pi_{\pm}\backslash\{0\}$ (e.g., [13], Reed–Simon [18]).

Results similar to Theorem 2 have been studied by many authors in connection with local smoothing properties (see, e.g., Kato–Yajima [9], Yajima [22], Ben-Artzi [2], Cuccagna–Schirmer [3], D'Ancona–Fanelli [4], Erdogan–Goldberg–Schlag [5] and Georgiev–Stefanov–Tarulli [6]). Note that these works are also restricted to the case of space dimension $n \geq 3$, and the vector potential $b(x)$ itself is required to be small and to decay sufficiently fast (the smallness is not required in [5]). On the other hand, no such requirement appears in our case, but the smallness of the magnetic field $\nabla \times b(x)$ is required. It seems difficult to remove it without any decay conditions on $b(x)$.

The rest of this paper is organized as follows. Theorems 1 and 2 are proved in §3 and §4, respectively. In §2 we prepare three propositions which are essential in proving Theorem 1.

§2. Preliminaries

The first two propositions are quadratic identities for a solution u of problem (1.1), (1.2).

We multiply (1.1) by $-\overline{i\kappa u}$ to obtain

$$(2.1) \qquad \nabla \cdot \{(\nabla_b u)\overline{i\kappa u}\} - \overline{i\kappa}\{|\nabla_b u|^2 + c(x)|u|^2 - \kappa^2|u|^2\} = -f\overline{i\kappa u},$$

and integrate the real part of this equation over Ω_t $(t > r_0)$. By the boundary condition (1.2),

$$\operatorname{Re}\int_{\Omega_t} \nabla \cdot \{(\nabla_b u)\overline{i\kappa u}\}\, dx = \operatorname{Re}\int_{S_t} (\tilde{x} \cdot \nabla_b u)\overline{i\kappa u}\, dS$$
$$= \frac{1}{2}\int_{S_t} \{-|\nabla_b u - i\kappa\tilde{x}u|^2 + |\nabla_b u|^2 + |\kappa u|^2\}\, dS,$$

and it follows that

$$(2.2) \qquad \frac{1}{2}\int_{S_t} \{-|\nabla_b u - i\kappa u|^2 + |\nabla_b u|^2 + |\kappa u|^2\}\, dS$$
$$+ \operatorname{Im}\kappa \int_{\Omega_t} (|\nabla_b u|^2 + c|u|^2 + |\kappa u|^2)\, dx = -\operatorname{Re}\int_{\Omega_t} f\overline{i\kappa u}\, dx.$$

Here

$$-|\nabla_b u - i\kappa\tilde{x}u|^2 = -\left|\nabla_b u + \tilde{x}\left(\frac{1}{2r} - i\kappa\right)u\right|^2 + \frac{1}{4r^2}|u|^2 + \operatorname{Im}\kappa\frac{1}{r}|u|^2 + \nabla\cdot\left\{\frac{1}{2r}\tilde{x}|u|^2\right\}.$$

The following proposition is a direct consequence of (2.2) multiplied by $\mu(t)$ and integrated over (r_0, ∞).

Proposition 1. *Let $u = R(\kappa^2)f$. Then*

$$\frac{1}{2}\int\left\{\left(\operatorname{Im}\kappa\frac{\mu}{r} + \frac{\mu}{4r^2} - \frac{\mu'}{2r}\right)|u|^2 + \mu(|\nabla_b u|^2 + |\kappa u|^2)\right\}dx$$

$$+ \operatorname{Im}\kappa\int_{r_0}^{\infty}\mu(t)\,dt\int_{\Omega_t}\left\{|\nabla_b u|^2 + c(x)|u|^2 + |\kappa u|^2\right\}dx$$

$$= \frac{1}{2}\int\mu|\theta|^2\,dx - \operatorname{Re}\int_{r_0}^{\infty}\mu(t)\,dt\int_{\Omega_t}f\overline{i\kappa u}\,dx,$$

where $\theta = \nabla_b u + \tilde{x}\left(\frac{1}{2r} - i\kappa\right)u$.

Next we set $v = e^{-i\kappa r}r^{1/2}u$, $g = e^{-i\kappa r}r^{1/2}f$ and rewrite (1.1) as follows:

$$(2.3) \qquad -\nabla_b\cdot\nabla_b v + \left(-2i\kappa + \frac{1}{r}\right)\tilde{x}\cdot\nabla_b v + \left(c - \frac{1}{4r^2}\right)v = g.$$

Let $\phi = \phi(r) = e^{-2\operatorname{Im}\kappa r}r^{-1}\varphi(r)$, where $\varphi(r)$ is a smooth, positive function of $r > 0$. We multiply (2.3) by $\phi\overline{\tilde{x}\cdot\nabla_b v}$. Note that

$$-\nabla_b\cdot\nabla_b v\phi\overline{\tilde{x}\cdot\nabla_b v} = -\nabla\cdot\{\nabla_b v\phi\overline{\tilde{x}\cdot\nabla_b v}\} + (\nabla\phi\cdot\nabla_b v)\overline{\tilde{x}\cdot\nabla_b v}$$

$$+ \phi\nabla_b v\cdot\overline{\nabla_b(\tilde{x}\cdot\nabla_b v)}.$$

In the last term of the right side we use the identity

$$(\partial_j + ib_j)\{\tilde{x}_k(\partial_k + ib_k)v\} = \tilde{x}_k(\partial_k + ib_k)\{(\partial_j + ib_j)v\}$$

$$+ \frac{\delta_{jk} - \tilde{x}_j\tilde{x}_k}{r}(\partial_k + ib_k)v + i\tilde{x}_k(\partial_j b_k - \partial_k b_j)v.$$

Then it follows that

$$\operatorname{Re}\{\phi\nabla_b v\cdot\overline{\nabla_b(\tilde{x}\cdot\nabla_b v)}\} = \frac{1}{2}\nabla\cdot\{\phi\tilde{x}|\nabla_b v|^2\} - \left(\frac{\phi'}{2} + \frac{\phi}{2r}\right)|\nabla_b v|^2$$

$$+ \frac{\phi}{r}(|\nabla_b v|^2 - |\tilde{x}\cdot\nabla_b v|^2) - \phi\operatorname{Re}\{(\tilde{x}\times\nabla_b v)\overline{(\nabla\times ib)v}\},$$

and we finally obtain

$$(2.4) \qquad - \operatorname{Re} \nabla \cdot \{(\phi \nabla_b v)\overline{\tilde{x} \cdot \nabla_b v}\} + \phi' |\tilde{x} \cdot \nabla_b v|^2 + \frac{1}{2} \nabla \cdot \{\phi \tilde{x} |\nabla_b v|^2\}$$

$$- \left(\frac{\phi'}{2} + \frac{\phi}{2r}\right) |\nabla_b v|^2 + \frac{\phi}{r} (|\nabla_b v|^2 - |\tilde{x} \cdot \nabla_b v|^2)$$

$$- \phi \operatorname{Re}\{(\tilde{x} \times \nabla_b v)\overline{(\nabla \times ib)v}\} + \phi\left(2 \operatorname{Im} \kappa + \frac{1}{r}\right) |\tilde{x} \cdot \nabla_b v|^2$$

$$+ \operatorname{Re} \phi\left\{\left(c - \frac{1}{4r^2}\right) v \overline{\tilde{x} \cdot \nabla_b v}\right\} = \operatorname{Re} \phi\{g \overline{\tilde{x} \cdot \nabla_b v}\}.$$

Note here $\nabla_b v = e^{-i\kappa r} r^{1/2} \theta$, and

$$\phi'(r) = \phi(r)\left(-2 \operatorname{Im} \kappa - \frac{1}{r} + \frac{\varphi'}{\varphi}\right),$$

$$- \operatorname{Re}\left\{\frac{\phi}{4r^2} v \overline{\tilde{x} \cdot \nabla_b v}\right\} = -\frac{1}{2} \nabla \cdot \left\{\frac{\varphi}{4r^2} \tilde{x} |u|^2\right\} + \varphi\left(- \operatorname{Im} \kappa - \frac{1}{r} + \frac{\varphi'}{2\varphi}\right) \frac{1}{4r^2} |u|^2.$$

Substituting these relations to (2.4) and integrating both sides over Ω_t, we obtain

Proposition 2. *Let $u = R(\kappa^2)f$. Then for t large,*

$$- \int_{S_t} \varphi\left\{|\tilde{x} \cdot \theta|^2 - \frac{1}{2}|\theta|^2 + \frac{1}{8r^2}|u|^2\right\} dS - \int_{\partial\Omega} \varphi\left\{(\nu \cdot \nabla u)(\tilde{x} \cdot \overline{\nabla u}) - \frac{1}{2}(\nu \cdot \tilde{x})|\nabla u|^2\right\} dS$$

$$+ \int_{\Omega_t} \varphi\left[-\left(\frac{1}{r} - \frac{\varphi'}{\varphi}\right)|\tilde{x} \cdot \theta|^2 + \left(\operatorname{Im} \kappa + \frac{1}{r} - \frac{\varphi'}{2\varphi}\right)\left\{|\theta|^2 - \frac{1}{4r^2}|u|^2\right\}\right.$$

$$\left. + \operatorname{Re}\{-(\nabla \times ib)u \overline{\tilde{x} \times \theta} + cu \overline{\tilde{x} \cdot \theta}\}\right] dx = \operatorname{Re} \int_{\Omega_t} \varphi f \overline{\tilde{x} \cdot \theta} \, dx,$$

where $\nu = \nu(x)$ is the outer unit normal to the boundary $\partial\Omega$.

Let $H_{0,b}^1 = H_{0,b}^1(\Omega)$ denote the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{H_b^1}^2 = \int \{|u(x)|^2 + |\nabla_b u(x)|^2\} \, dx.$$

Obviously $\mathcal{D}(L) \subset H_{0,b}^1$.

Proposition 3. *Let $\eta = \eta(r)$ and $\xi = \xi(r)$ be smooth, positive functions of $r \geq 0$, and let t be large. Then for each $u \in H_{0,b}^1$,*

$$\int_{\Omega_t} \xi\left\{|\tilde{x} \cdot \theta|^2 - \frac{1}{4r^2}|u|^2\right\} dx = \int_{\Omega_t} \xi|\tilde{x} \cdot \nabla_b u - i\kappa u - \eta u|^2 \, dx$$

$$+ \int_{S_t} \xi\left(\frac{1}{2r} + \eta\right)|u|^2 \, dS - \int_{\Omega_t} \xi'\left(\frac{1}{2r} + \eta\right)|u|^2 \, dx$$

$$+ \int_{\Omega_t} \xi\left\{2 \operatorname{Im} \kappa\left(\frac{1}{2r} + \eta\right)|u|^2 - \left(\frac{1}{r}\eta + \eta' + \eta^2\right)|u|^2\right\} dx.$$

Proof. Note that

$$|\tilde{x} \cdot \theta|^2 = \left|\tilde{x} \cdot \nabla_b u + \frac{1}{2r}u - i\kappa u - \eta u + \eta u\right|^2$$

$$= |\tilde{x} \cdot \nabla_b u - i\kappa u - \eta u|^2 + \nabla \cdot \left\{\tilde{x}\left(\frac{1}{2r} + \eta\right)|u|^2\right\}$$

$$+ 2\operatorname{Im}\kappa\left(\frac{1}{2r} + \eta\right)|u|^2 + \frac{1}{4r^2}|u|^2 - \left(\frac{1}{r}\eta + \eta' + \eta^2\right)|u|^2.$$

Multiplying by $\xi(r)$ and integrating over Ω_t, we deduce the desired identity since $u|_{\partial\Omega} = 0$. $\qquad\square$

§3. Proof of Theorem 1

In this section we shall prove Theorem 1 by a series of lemmas for the solution $u = R(\kappa^2)f$ of (1.1), (1.2).

Lemma 1. *Let $\varphi = \varphi(r)$ be a positive increasing function of $r > r_0$ satisfying*

$$(3.1) \qquad\qquad \frac{\varphi'(r)}{\varphi(r)} \leq \frac{1}{r}.$$

Then

$$\int \varphi\left\{\left(\operatorname{Im}\kappa + \frac{\varphi'}{2\varphi}\right)|\theta|^2 - \left(\operatorname{Im}\kappa + \frac{1}{r} - \frac{\varphi'}{2\varphi}\right)\frac{1}{4r^2}|u|^2\right\}dx$$

$$\leq \int \varphi\{|f| + (|\nabla \times b|^2 + |c|^2)^{1/2}|u|\}|\theta|\,dx.$$

Proof. In Proposition 2 we let $t \to \infty$. Then since $\varphi(r) = O(r)$ as $r \to \infty$, we have

$$\liminf_{t\to\infty} \int_{S_t} \varphi\left\{|\tilde{x} \cdot \theta|^2 - \frac{1}{2}|\theta|^2 + \frac{1}{8r^2}|u|^2\right\}dS = 0.$$

On the other hand, since $\nabla u = (\nu \cdot \nabla u)\nu$ on $\partial\Omega$, the starshapedness of $\partial\Omega$ implies that

$$\int_{\partial\Omega} \varphi\left\{-(\nu \cdot \nabla u)(\tilde{x} \cdot \overline{\nabla u}) + \frac{1}{2}(\nu \cdot \tilde{x})|\nabla u|^2\right\}dS = -\frac{1}{2}\int_{\partial\Omega}\varphi(\nu \cdot \tilde{x})|\nu \cdot \nabla u|^2\,dS \geq 0.$$

Thus, the desired inequality follows if we note that

$$\left(\frac{1}{r} - \frac{\varphi'}{\varphi}\right)\{|\theta|^2 - |\tilde{x} \cdot \theta|^2\} \geq 0,$$

$$|-(\nabla \times ib)u\overline{\tilde{x} \times \theta} + cu\,\overline{\tilde{x} \cdot \theta}| \leq (|\nabla \times b|^2 + |c|^2)^{1/2}|u|\,|\theta|. \qquad\square$$

Lemma 2. *For any $\epsilon > 0$,*

$$\int \left(\text{Im}\,\kappa r + \frac{1}{2} - \frac{2\epsilon}{(1 + \log r/r_0)^2} \right) \left\{ |\theta|^2 - \frac{1}{4r^2}|u|^2 \right\} dx$$

$$\leq \frac{1}{4\epsilon} \int r^2 (1 + \log r/r_0)^2 |f|^2 \, dx + \left(\frac{\epsilon_0^2}{\epsilon} + 2\epsilon \right) \int \frac{|u|^2}{4r^2(1 + \log r/r_0)^2} \, dx.$$

Proof. We choose $\varphi = r$ in Lemma 1 and use the Schwarz inequality. Then noting (A1), we have

$$\int \left(\text{Im}\,\kappa r + \frac{1}{2} \right) \left\{ |\theta|^2 - \frac{1}{4r^2}|u|^2 \right\} dx$$

$$\leq \int \left\{ \frac{r^2(1 + \log r/r_0)^2 |f|^2}{4\epsilon} + \frac{\epsilon_0^2 |u|^2}{4\epsilon r^2(1 + \log r/r_0)^2} \right\} dx$$

$$+ \int \frac{2\epsilon}{(1 + \log r/r_0)^2} \left\{ |\theta|^2 - \frac{1}{4r^2}|u|^2 \right\} dx + \int \frac{2\epsilon |u|^2}{4r^2(1 + \log r/r_0)^2} \, dx,$$

which proves the lemma. □

Lemma 3. *Let $u \in H^1_{0,b}$. Then for each $0 < \epsilon < 1/38$,*

$$\int \left(\text{Im}\,\kappa r + \frac{1}{2} - 19\epsilon \right) \frac{|u|^2}{4r^2(1 + \log r/r_0)^2} \, dx$$

$$\leq \int \left(\text{Im}\,\kappa r + \frac{1}{2} - \frac{2\epsilon}{(1 + \log r/r_0)^2} \right) \left\{ |\tilde{x} \cdot \theta|^2 - \frac{1}{4r^2}|u|^2 \right\} dx.$$

Proof. We choose

$$\eta(r) = \frac{1}{2r(1 + \log r/r_0)}, \quad \xi(r) = \text{Im}\,\kappa r + \frac{1}{2} - \frac{2\epsilon}{(1 + \log r/r_0)^2}$$

in Proposition 3. Obviously

$$\liminf_{t \to \infty} \int_{S_t} \xi \left(\frac{1}{2r} + \eta \right) |u|^2 \, dS = 0.$$

Moreover, by definition $\xi(r) > 0$ when $r > r_0$ and

$$\frac{1}{r}\eta + \eta' + \eta^2 = \frac{-1}{4r^2(1 + \log r/r_0)^2}.$$

Thus, the identity of Proposition 3 implies

$$\int \left(\operatorname{Im} \kappa r + \frac{1}{2} - \frac{2\epsilon}{(1 + \log r/r_0)^2} \right) \left\{ |\tilde{x} \cdot \theta|^2 - \frac{1}{4r^2} |u|^2 \right\} dx$$

$$\geq \int \left(\operatorname{Im} \kappa r + \frac{1}{2} - \frac{2\epsilon}{(1 + \log r/r_0)^2} \right)$$

$$\times \left\{ 2 \operatorname{Im} \kappa \left(\frac{1}{2r} + \eta \right) |u|^2 + \frac{1}{4r^2(1 + \log r/r_0)^2} |u|^2 \right\} dx$$

$$- \int \left(\operatorname{Im} \kappa + \frac{4\epsilon}{r(1 + \log r/r_0)^3} \right) \left(\frac{1}{2r} + \eta \right) |u|^2 \, dx$$

$$= \int \left(\operatorname{Im} \kappa r + \frac{1}{2} - \frac{2\epsilon}{(1 + \log r/r_0)^2} \right) \frac{|u|^2}{4r^2(1 + \log r/r_0)^2} \, dx$$

$$+ \int \left(2(\operatorname{Im} \kappa)^2 r - \frac{4 \operatorname{Im} \kappa \epsilon}{(1 + \log r/r_0)^2} - \frac{4\epsilon}{r(1 + \log r/r_0)^3} \right) \left(\frac{1}{2r} + \eta \right) |u|^2 \, dx.$$

Since

$$2(\operatorname{Im} \kappa)^2 r - \frac{4 \operatorname{Im} \kappa \epsilon}{(1 + \log r/r_0)^2} \geq \frac{-2\epsilon^2}{r(1 + \log r/r_0)^4}$$

and $0 < 1/2r + \eta \leq 1/r$, noting

$$\frac{1}{2} - 2\epsilon - 8\epsilon^2 - 16\epsilon \geq \frac{1}{2} - 19\epsilon,$$

we deduce the desired inequality. □

Proof of Theorem 1(i). Combining Lemmas 2 and 3 we have, for $0 < \epsilon < 1/38$,

$$(3.2) \qquad \int \left(\operatorname{Im} \kappa r + \frac{1}{2} - 19\epsilon \right) \frac{|u|^2}{4r^2(1 + \log r/r_0)^2} \, dx$$

$$\leq \frac{1}{4\epsilon} \int r^2(1 + \log r/r_0)^2 |f|^2 \, dx + \left(\frac{\epsilon_0^2}{\epsilon} + 2\epsilon \right) \int \frac{|u|^2}{4r^2(1 + \log r/r_0)^2} \, dx.$$

Here we require ϵ further to satisfy

$$(3.3) \qquad \frac{1}{2} - 19\epsilon - \frac{\epsilon_0^2}{\epsilon} - 2\epsilon > 0,$$

or equivalently

$$\epsilon_0^2 < \epsilon \left(\frac{1}{2} - 21\epsilon \right) = -21 \left(\epsilon - \frac{1}{84} \right)^2 + \frac{1}{336}.$$

This is possible if $\epsilon_0 < 1/(4\sqrt{21})$. In this case, we can choose $\epsilon = 1/84$ in (3.3). Then since

$$\frac{1}{4\epsilon} \left(\frac{1}{2} - 21\epsilon - \frac{\epsilon_0^2}{4\epsilon} \right)^{-1} = \frac{(2\sqrt{21})^2}{1 - (4\sqrt{21} \, \epsilon_0)^2},$$

the desired inequality (1.5) follows from (3.2).

To show (1.7) we return to the inequality of Lemma 1 choosing $\varphi = r/(4 + \log r/r_0)^2$. Concerning its left side, it follows from (1.5) that

$$(3.4) \quad \int \varphi \left(\operatorname{Im} \kappa + \frac{1}{r} - \frac{\varphi'}{2\varphi} \right) \frac{|u|^2}{4r^2} \, dx \le \int (\operatorname{Im} \kappa r + 1) \frac{|u|^2}{4r^2(1 + \log r/r_0)^2} \, dx$$

$$\le \frac{1}{4} C_1 \int r^2 (1 + \log r/r_0)^2 |f|^2 \, dx.$$

Moreover, since $1/r - \varphi'/\varphi \ge 0$ and

$$\varphi \left(\operatorname{Im} \kappa + \frac{\varphi'}{2\varphi} \right) = \varphi \left(\operatorname{Im} \kappa + \frac{1}{2r} - \frac{1}{r(4 + \log r/r_0)} \right) \ge \varphi \left(\operatorname{Im} \kappa + \frac{1}{4r} \right),$$

it follows that

$$(3.5) \quad \int \varphi \left(\operatorname{Im} \kappa + \frac{\varphi'}{2\varphi} \right) |\theta|^2 \, dx \ge \int \left(\operatorname{Im} \kappa r + \frac{1}{4} \right) \frac{|\theta|^2}{(4 + \log r/r_0)^2} \, dx.$$

On the other hand, by the Schwarz inequality the right side of the inequality of Lemma 1 is estimated as

$$(3.6) \quad \int \frac{r}{(4 + \log r/r_0)^2} \left\{ |f| + (|\nabla \times b|^2 + |c|^2)^{1/2} |u| \right\} |\theta| \, dx$$

$$\le \frac{1}{4\epsilon} \frac{1}{4^2} \int \left\{ r^2 |f|^2 + \epsilon_0^2 \frac{|u|^2}{r^2(1 + \log r/r_0)^4} \right\} dx + 2\epsilon \int \frac{|\theta|^2}{(4 + \log r/r_0)^2} \, dx.$$

Thus, applying (1.5) once more, from Lemma 1 and (3.4)–(3.6) we obtain

$$\int \left(\operatorname{Im} \kappa r + \frac{1}{4} - 2\epsilon \right) \frac{|\theta|^2}{(4 + \log r/r_0)^2} \, dx \le \frac{1}{4} C_1 \int r^2 (1 + \log r/r_0)^2 |f|^2 \, dx$$

$$+ \frac{1}{4^3 \epsilon} \left\{ \int r^2 |f|^2 \, dx + \epsilon_0^2 C_1 \int r^2 (1 + \log r/r_0)^2 |f|^2 \, dx \right\}$$

$$\le \frac{1}{4} \left\{ C_1 + \frac{1}{4^2 \epsilon} (1 + \epsilon_0^2 C_1) \right\} \int r^2 (1 + \log r/r_0)^2 |f|^2 \, dx,$$

which proves the inequality (1.7) if we choose $\epsilon = 1/16$. $\square$

To proceed to the proof of Theorem 1(ii), we require one more lemma.

Lemma 4. *For each $u \in H^1_{0,b}$ and $t > r_0$ we have*

$$\int_{\Omega_t} \frac{|u|^2}{4r^2(1 + \log r/r_0)^2} \, dx \le \int_{\Omega_t} |\tilde{x} \cdot \nabla_b u|^2 \, dx.$$

Proof. In the identity

$$|\tilde{x} \cdot \nabla_b u|^2 = \left| \tilde{x} \cdot \nabla_b u - \frac{u}{2r(1 + \log r/r_0)} \right|^2 - \frac{|u|^2}{4r^2(1 + \log r/r_0)^2}$$
$$+ 2 \operatorname{Re} \left\{ \tilde{x} \cdot \nabla_b u \frac{\overline{u}}{2r(1 + \log r/r_0)} \right\},$$

the last term on the right can be rewritten as

$$\nabla \cdot \left\{ \tilde{x} \frac{|u|^2}{2r(1 + \log r/r_0)} \right\} + \frac{|u|^2}{2r^2(1 + \log r/r_0)^2}.$$

Integrate both sides over Ω_t. Then the assertion follows from the identity

$$\int_{\Omega_t} |\tilde{x} \cdot \nabla_b u|^2 \, dx = \int_{\Omega_t} \left| \tilde{x} \cdot \nabla_b u - \frac{u}{2r(1 + \log r/r_0)} \right|^2 dx$$
$$+ \int_{S_t} \frac{|u|^2}{2r(1 + \log r/r_0)} \, dS + \int_{\Omega_t} \frac{|u|^2}{4r^2(1 + \log r/r_0)^2} \, dx. \qquad \square$$

Remark 3. The above lemma gives a modification of the standard Hardy inequality

$$\int \frac{|u|^2}{4r^2(1 + \log r/r_0)^2} \, dx \leq \int |\tilde{x} \cdot \nabla u|^2 \, dx$$

in an exterior domain.

Proof of Theorem 1(ii). We return to the identity of Proposition 1. By (A1),

$$c(x) \geq \frac{-1}{4r^2(1 + \log r/r_0)^2}.$$

Then from Lemma 4 we have

$$\operatorname{Im} \kappa \int_{r_0}^{\infty} \mu \, dr \int_{\Omega_r} \{ |\nabla_b u|^2 + c(x)|u|^2 + |\kappa u|^2 \} \, dx \geq 0,$$

and

$$\frac{1}{2} \int \left\{ \left(\operatorname{Im} \kappa \frac{\mu}{r} + \frac{\mu}{4r^2} - \frac{\mu'}{2r} \right) |u|^2 + \mu(|\nabla_b u|^2 + |\kappa u|^2) \right\} dx$$
$$\leq \frac{1}{2} \int \mu |\theta|^2 \, dx + \int_{r_0}^{\infty} \mu(t) \, dt \int_{\Omega_t} |f(x)| \, |i\kappa u| \, dx.$$

By condition (1.8) on $\mu(t)$, this inequality reduces to

$$\frac{1}{2} \int \mu \left(|\nabla_b u|^2 + |\kappa u|^2 \right) dx \leq \frac{1}{2} C_3 \int \frac{|\theta|^2}{(4 + \log r/r_0)^2} \, dx + \|\mu\|_{L^1} \int |f| \, |i\kappa u| \, dx.$$

Thus, the desired inequality (1.9) is concluded if we note (1.7) and

$$\|\mu\|_{L^1} \int |f| \, |i\kappa u| \, dx \le \|\mu\|_{L^1}^2 \int \mu^{-1}|f|^2 \, dx + \frac{1}{4} \int \mu|\kappa u|^2 \, dx. \qquad \square$$

§4. Proof of Theorem 2

The following proposition summarizes abstract results which allow us to employ the resolvent estimate for a selfadjoint operator to obtain a space-time weighted estimate for the associated evolution equation. For the proof of the proposition see, e.g., [14], [18].

Let Λ be a selfadjoint operator in a Hilbert space $\mathcal{H}$, and for $z \in \mathbb{C} \setminus \mathbb{R}$ let $\mathcal{R}(z)$ be the resolvent of Λ. Suppose that A is a densely defined, closed operator from $\mathcal{H}$ to another Hilbert space $\mathcal{H}_1$.

Proposition 4. *Assume that there exists $C > 0$ such that*

$$(4.1) \qquad \sup_{z \notin \mathbb{R}} \|A\mathcal{R}(z)A^*f\|_{\mathcal{H}_1} < \sqrt{C} \, \|f\|_{\mathcal{H}_1}$$

for $f \in \mathcal{D}(A^*)$. Then

$$(4.2) \qquad \left| \int_0^{\pm\infty} \left\| \int_0^t A e^{-i(t-\tau)\Lambda} A^* g(\tau) \, d\tau \right\|_{\mathcal{H}_1}^2 dt \right| \le C \left| \int_0^{\pm\infty} \|g(t)\|_{\mathcal{H}_1}^2 \, dt \right|,$$

$$(4.3) \qquad \sup_{t \in \mathbb{R}_\pm} \left\| \int_0^t e^{i\tau\Lambda} A^* g(\tau) \, d\tau \right\|_{\mathcal{H}}^2 \le 2\sqrt{C} \left| \int_0^{\pm\infty} \|g(t)\|_{\mathcal{H}_1}^2 \, dt \right|$$

for each $g \in L^2(\mathbb{R}; \mathcal{D}(A^*))$, and

$$(4.4) \qquad \left| \int_0^{\pm\infty} \|A e^{-it\Lambda} f\|_{\mathcal{H}_1}^2 \, dt \right| \le 2\sqrt{C} \, \|f\|_{\mathcal{H}}^2$$

for each $f \in \mathcal{H}$.

Proof of Theorem 2(i). Set $\Lambda = L$, $\mathcal{H} = \mathcal{H}_1 = L^2$ and $A = r^{-1}(1 + \log r/r_0)^{-1}$ (multiplication operator). Then $A^* = A$ and $\mathcal{R}(z) = R(z)$, and if we let $z = \kappa^2$, then it follows from Theorem 1(i) that

$$\begin{aligned} \|AR(z)A^*f\| &= \|r^{-1}(1 + \log r/r_0)^{-1} R(z) A^* f\| \\ &\le \sqrt{C_1} \, \|r(1 + \log r/r_0) A^* f\| = \sqrt{C_1} \, \|f\|. \end{aligned}$$

Thus, estimates (4.2) and (4.3) hold, and can be written as

$$\left| \int_0^{\pm\infty} \left\| r^{-1}(1 + \log r/r_0)^{-1} \int_0^t e^{-i(t-\tau)L} h(\tau) \, d\tau \right\|^2 dt \right|$$

$$\leq C_1 \left| \int_0^{\pm\infty} \| r(1 + \log r/r_0) h(t) \|^2 \, dt \right|,$$

$$\left| \int_0^{\pm\infty} \| r^{-1}(1 + \log r/r_0)^{-1} e^{-itL} f \|^2 \, dt \right| \leq 2\sqrt{C_1} \, \|f\|^2,$$

where $h(t) = r^{-1}(\log(1 + \log r/r_0)^{-1} g(t)$ and $f \in L^2(\Omega)$. By the limit procedure the first inequality is continuously extended to $h(t)$ satisfying $r(1 + \log r/r_0) h(t) \in L^2(\mathbb{R} \times \Omega)$, as desired. $\square$

To show Theorem 2(ii) we consider the Klein–Gordon equation

$$i\partial_t u = \Lambda u, \quad u(t) = \{w(t), \partial_t w(t)\}, \quad \Lambda = \begin{pmatrix} 0 & i \\ -i(L + m^2) & 0 \end{pmatrix}$$

in the energy space $\mathcal{H} = \dot{H}_b^1 \times L^2$, where $\dot{H}_b^1$ is the completion of $C_0^\infty(\Omega)$ in the norm

$$\|f_1\|_{\dot{H}_b^1}^2 = \frac{1}{2} \int \{|\nabla_b f_1|^2 + (c(x) + m^2)|f_1|^2\} \, dx.$$

Then Λ with domain

$$\mathcal{D}(\Lambda) = \{f_1 \in \dot{H}_b^1; \, \Delta_b f_1 \in L^2\} \times \{f_2 \in \dot{H}_b^1 \cap L^2\}$$

forms a selfadjoint operator in $\mathcal{H}$, and its resolvent is given by

$$\mathcal{R}(z) = (L + m^2 - z^2)^{-1} \begin{pmatrix} z & i \\ -i(L + m^2) & z \end{pmatrix}.$$

Let $A : \mathcal{H} \to \mathcal{H}_1 = L^2$ be defined by

$$Af = r^{-1}(1 + \log r/r_0)^{-1} \sqrt{L + m^2} \, f_1 \quad \text{for } f = \{f_1, f_2\} \in \mathcal{H}.$$

Then the adjoint operator A^* is given by

$$A^* g = \{2\sqrt{L + m^2}^{-1} r^{-1}(1 + \log r/r_0)^{-1} g, 0\} \quad \text{for } g \in L^2.$$

Proof of Theorem 2(ii). By definition

(4.5) $$AR(z)A^* g = 2r^{-1} \log(1 + \log r/r_0)^{-1}$$
$$\times z(L + m^2 - z^2)^{-1} r^{-1}(1 + \log r/r_0)^{-1} g$$

for $g \in \mathcal{D}(A^*)$. Then since

$$\int |r^{-1}(1 + \log r/r_0)^{-1} z (L + m^2 - z^2)^{-1} f|^2 \, dx$$

$$\leq m^2 \int r^{-2}(1 + \log r/r_0)^{-2} |(L + m^2 - z^2)^{-1} f|^2 \, dx$$

$$+ \int r^{-2}(1 + \log r/r_0)^{-2} |-m^2 + z^2| \, |(L + m^2 - z^2)^{-1} f|^2 \, dx,$$

using (1.5) and (1.9) with $\mu = r^{-2}(1 + \log r/r_0)^{-2}$ of Theorem 1, we obtain

$$\|A\mathcal{R}(z)A^* g\| \leq 2\sqrt{m^2 C_1 + 2C_4} \, \|g\|.$$

We return to Proposition 4 with this inequality. Let $w(t)$ be the first component of the solution $e^{-it\Lambda} f$. Then (4.4) shows that

$$\left| \int_0^{\pm\infty} \|A e^{-it\Lambda} f\|^2 \, dt \right| = \left| \int_0^{\pm\infty} \|r^{-1}(1 + \log r/r_0)^{-1} \sqrt{L + m^2} \, w(t)\|^2 \, dt \right|$$

$$\leq 4\sqrt{m^2 C_1 + 2C_4} \, \|f\|_{\mathcal{H}}^2.$$

Since

$$w(t) = \cos(t\sqrt{L + m^2}) f_1 + \sqrt{L + m^2}^{-1} \sin(t\sqrt{L + m^2}) f_2,$$

choosing $f = \{\sqrt{L + m^2}^{-1} g, 0\}$ and $f = \{0, g\}$ for $g \in L^2$, we obtain

$$\left| \int_0^{\pm\infty} \|r^{-1}(1 + \log r/r_0)^{-1} \cos(t\sqrt{L + m^2}) g\|^2 \, dt \right| \leq 2\sqrt{m^2 C_1 + 2C_4} \, \|g\|^2$$

and

$$\left| \int_0^{\pm\infty} \|r^{-1}(1 + \log r/r_0)^{-1} \sin(t\sqrt{L + m^2}) g\|^2 \, dt \right| \leq 2\sqrt{m^2 C_1 + 2C_4} \, \|g\|^2,$$

respectively. These inequalities imply assertion (ii).   □

Acknowledgements

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