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Spectral Properties of the Linearized Semigroup of the Compressible Navier–Stokes Equation on a Periodic Layer

by

Yoshiyuki KAGEI and Naoki MAKIO

Abstract

The linearized problem for the compressible Navier–Stokes equation around a given constant state is considered in a periodic layer of $\mathbb{R}^n$ with $n \geq 2$, and spectral properties of the linearized semigroup are investigated. It is shown that the linearized operator generates a C_0-semigroup in L^2 over the periodic layer and the time-asymptotic leading part of the semigroup is given by a C_0-semigroup generated by an $n-1$-dimensional elliptic operator with constant coefficients that are determined by solutions of a Stokes system over the basic period domain.

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§1. Introduction

This paper is concerned with the initial boundary value problem for the following compressible Navier–Stokes equation in a periodic layer Ω:

$$(1.1) \qquad \begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \rho(\partial_t v + v \cdot \nabla v) - \mu \Delta v - (\mu + \mu')\nabla \operatorname{div} v + \nabla(P(\rho)) = 0, \\ v|_{\partial\Omega} = 0, \\ (\rho, v)|_{t=0} = (\rho_0, v_0). \end{cases}$$

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Y. Kagei: Faculty of Mathematics, Kyushu University, Nishi-ku, Motooka 744, Fukuoka 819-0395, Japan;
e-mail: kagei@math.kyushu-u.ac.jp
N. Makio: Simplex Inc., Toranomon Hills Mori Tower, 1-23-1 Toranomon, Minato-ku, Tokyo 105-6319, Japan

Here $\rho = \rho(x,t)$ and $v = {}^\top(v^1(x,t),\dots,v^n(x,t))$ denote the unknown density and velocity, respectively, at time t and position x; Ω is a periodic layer defined by

$$\Omega := \left\{x = (x', x_n);\ x' \in \mathbb{R}^{n-1},\ \omega_1(x') < x_n < \omega_2(x')\right\},$$

where ω_1 and ω_2 are nonconstant and smooth functions of x' satisfying the periodicity conditions $\omega_j\left(x' + \frac{2\pi}{\alpha_k}\boldsymbol{e}'_k\right) = \omega_j(x')$ $(j = 1,2;\ k = 1,\dots,n-1)$ with constants $\alpha_k > 0$ and $\boldsymbol{e}'_k := {}^\top(0,\dots,\overset{k}{1},\dots,0) \in \mathbb{R}^{n-1}$; μ and μ' are the viscosity coefficients that are constants satisfying

$$\mu > 0, \qquad \frac{2}{n}\mu + \mu' \geq 0;$$

P is the pressure for which we assume that P is a smooth function of ρ that satisfies

$$P'(\rho_*) > 0$$

for a given positive constant ρ_*. Here and in what follows, ${}^\top\cdot$ stands for transposition.

We are interested in the large time behavior of solutions to (1.1) around the constant equilibrium $u_s = {}^\top(\rho_*, 0)$. To establish a detailed asymptotic description of large time behavior, we study the spectral properties of the linearized semigroup for (1.1) around u_s as a first step of our analysis.

The system of equations for the perturbation is written as

$$\begin{cases} \partial_t \phi + \gamma \operatorname{div} w = f^0, \\ \partial_t w - \nu\Delta w - \tilde{\nu}\nabla \operatorname{div} w + \gamma\nabla\phi = f, \\ w|_{\partial\Omega} = 0, \\ u|_{t=0} = u_0 = (\phi_0, w_0). \end{cases} \tag{1.2}$$

Here $u = {}^\top(\phi, w)$ with $\phi := \frac{1}{\rho_*}(\rho - \rho_*)$ and $w := \frac{1}{\gamma}v$ denotes the (scaled) perturbation from $u_s := {}^\top(\rho_*, 0)$; ν, $\tilde{\nu}$ and γ are parameters given by

$$\nu := \frac{\mu}{\rho_*}, \qquad \tilde{\nu} := \frac{\mu + \mu'}{\rho_*}, \qquad \gamma := \sqrt{P'(\rho_*)};$$

and f^0 and f denote the nonlinearities

$$\begin{aligned} f^0 &:= -\gamma \operatorname{div}(\phi w), \\ f &:= -\frac{\phi}{1+\phi}\left\{\nu\Delta w + \tilde{\nu}\nabla \operatorname{div} w\right\} - \gamma w\cdot\nabla w \\ &\quad - \left\{\frac{1}{\gamma\rho_*(1+\phi)}\nabla(P(\rho_*(1+\phi))) - \frac{P'(\rho_*)}{\gamma}\nabla\phi\right\}. \end{aligned}$$

Large time behavior of solutions to the compressible Navier–Stokes equations has been extensively studied since the pioneering works by Matsumura–Nishida [16, 17, 18]. See, e.g., [5, 10, 11, 13, 14, 15, 20] and references therein. In [7, 8, 9], the stability of u_s was studied when the underlying domain is an n-dimensional infinite layer

$$\mathbb{R}^{n-1} \times (0,1) = \{x = (x', x_n);\ x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1},\ 0 < x_n < 1\}.$$

It was proved that u_s is stable under sufficiently small initial perturbations and the L^2 norm of the perturbation decays in the order of $t^{-(n-1)/4}$ as $t \to \infty$. Furthermore, it was shown that the perturbation behaves like a solution of an $n-1$-dimensional heat equation.

In this paper we extend the results on the asymptotic behavior of the linearized semigroup for (1.2) obtained in [7, 8] to the case of the periodic layer Ω. We will prove that the linearized semigroup behaves as $t \to \infty$ like a semigroup generated by an $n-1$-dimensional elliptic operator with constant coefficients. More precisely, we consider the linear problem

$$\partial_t u + Lu = 0, \qquad u|_{t=0} = u_0, \tag{1.3}$$

where $u = {}^\top(\phi, w)$ is the unknown; $u_0 = {}^\top(\phi_0, w_0)$ is a given initial datum; and L is the operator of the form

$$L := \begin{pmatrix} 0 & \gamma \operatorname{div} \\ \gamma\nabla & -\nu\Delta - \tilde{\nu}\nabla \operatorname{div} \end{pmatrix}.$$

It is shown that $-L$ generates a contraction C_0-semigroup e^{-tL} on $L^2(\Omega)$ and e^{-tL} is decomposed as

$$e^{-tL} = e^{-tL}\Pi + e^{-tL}(I - \Pi).$$

Here I is the identity operator and Π is a bounded projection on $L^2(\Omega)$; moreover,

$$\|e^{-tL}\Pi u_0\|_{L^2(\Omega)} \le C(1+t)^{-(n-1)/4}\|u_0\|_{L^1(\Omega)},$$
$$\|e^{-tL}(I-\Pi)u_0\|_{L^2(\Omega)} \le Ce^{-\beta t}\|u_0\|_{L^2(\Omega)}$$

and

$$\|e^{-tL}\Pi u_0 - [e^{-tH}\sigma_0]u^{(0)}\|_{L^2(\Omega)} \le Ct^{-(n-1)/4-1/2}\|u_0\|_{L^1(\Omega)}, \tag{1.4}$$

where β is a positive constant; and e^{-tH} is the C_0-semigroup in $L^2(\mathbb{R}^{n-1})$ generated by the operator $-H$ where

$$H\sigma := -\frac{\gamma^2}{\nu}\sum_{i,j=1}^{n-1} a_{ij}\partial_{x_i}\partial_{x_j}\sigma.$$

Here (a_{ij}) is a positive definite symmetric matrix with constant components; and σ_0 and $u^{(0)}$ are given by

$$\sigma_0 := \frac{|Q|}{|\Omega_{\mathrm{per}}|} \int_{\omega_1(x')}^{\omega_2(x')} \phi_0(x', x_n)\, dx_n, \qquad u^{(0)} := {}^{\top}(1, 0),$$

where Ω_{per} is the basic period domain given by

$$\Omega_{\mathrm{per}} := \{x = (x', x_n);\, x' \in Q,\, \omega_1(x') < x_n < \omega_2(x')\}$$

with the basic period cell $Q = \prod_{j=1}^{n-1}[-\pi/\alpha_j, \pi/\alpha_j)$. Here and in what follows, for a bounded domain D, $|D|$ denotes the volume of D. We note that the matrix (a_{ij}) is given by

$$a_{ij} := \frac{1}{|\Omega_{\mathrm{per}}|}(\nabla w^{(i)}, \nabla w^{(j)})_{L^2(\Omega_{\mathrm{per}})},$$

where $w^{(k)} = w^{(k)}(x', x_n)$ $(k = 1, \dots, n-1)$ are functions Q-periodic in x' satisfying the following Stokes system:

$$\begin{cases} \operatorname{div} w^{(k)} = 0, \\ -\Delta w^{(k)} + \nabla \phi^{(k)} = \boldsymbol{e}_k, \\ w^{(k)}|_{x_n = \omega_1(x'), \omega_2(x')} = 0 \end{cases}$$

for some $\phi^{(k)} = \phi^{(k)}(x', x_n)$ Q-periodic in x', where $\boldsymbol{e}_k := {}^{\top}(0, \dots, \overset{k}{1}, \dots, 0) \in \mathbb{R}^n$. Here and in what follows, we say that a function $f(x')$ is *Q-periodic* if $f(x' + \frac{2\pi}{\alpha_j}\boldsymbol{e}'_j) = f(x')$ for all $x' \in \mathbb{R}^{n-1}$ and $j = 1, \dots, n-1$, where $\boldsymbol{e}'_j = {}^{\top}(0, \dots, \underset{j}{1}, \dots, 0) \in \mathbb{R}^{n-1}$.

We will prove our results as follows. In the case of infinite layers analyzed in [7, 8, 9], the spectral properties of the linearized semigroup were investigated by using the Fourier transform in $x' \in \mathbb{R}^{n-1}$. In the case of the periodic layer Ω, the Fourier transform does not work well any longer; instead, we employ the Bloch wave decomposition which transforms the linearized problem (1.3) on Ω to the problem $\partial_t u + L_{\eta'} u = 0$ on Ω_{per} under Q-periodic boundary conditions in x'. Here $L_{\eta'}$ is the linear operator obtained by replacing the partial derivatives ∂_{x_j} $(j = 1, \dots, n-1)$ in L by $\partial_{x_j} + i\eta_j$ with parameter $\eta' = (\eta_1, \dots, \eta_{n-1}) \in Q^*$, where Q^* is the dual cell defined by $Q^* := \prod_{j=1}^{n-1}[-\alpha_j/2, \alpha_j/2)$.

When $|\eta'| \ll 1$, the operator $L_{\eta'}$ can be regarded as a perturbation of L_0; and analytic perturbation theory is applied to show that

$$\rho(-L_{\eta'}) \supset \{\operatorname{Re} \lambda > -\beta_0\} \setminus \{\lambda_{\eta'}\} \quad \text{for some } \beta_0 > 0,$$
$$\sigma(-L_{\eta'}) \cap \{|\lambda| < \beta_0/2\} = \{\lambda_{\eta'}\},$$

where

$$\lambda_{\eta'} = -\frac{\gamma^2}{\nu} \sum_{i,j=1}^{n-1} a_{ij}\eta_i\eta_j + O(|\eta'|^3)$$

as $\eta' \to 0$. It then follows that this part of e^{-tL} behaves as in (1.4). As for the remaining part of η', we establish some estimates for a modified Stokes system (see Section 4.3); and based on the established estimates we prove by an energy method that if $|\eta'| \geq r_0$ ($\eta' \in Q^*$), then

$$\rho(-L_{\eta'}) \supset \{\mathrm{Re}\,\lambda \geq -\beta_1\} \quad \text{for some } \beta_1 > 0,$$

and hence this part of e^{-tL} decays exponentially. We note that we consider the linearized operator L as an operator on L^2 as in [6], in contrast to [7, 8] where the underlying space is $H^1 \times L^2$. The L^2 setting will be useful for the stability analysis of stationary flows with nonzero velocity fields.

This paper is organized as follows. In Section 2 we introduce some notation, function spaces and state some properties of the Bloch wave decomposition. In Section 3 we state the main result of this paper. The proof of the main result is given in Sections 4–5. In Section 6 we give an outline of the proof of a lemma used in Section 4.3.

§2. Preliminaries

In this section we introduce the notation, function spaces and operators which will be used in this paper.

For a domain D and $1 \leq p \leq \infty$, the Lebesgue space over D is denoted by $L^p(D)$ and its norm is denoted by $\|\cdot\|_{L^p(D)}$. The symbol $W^{l,p}(D)$ stands for the lth order L^p Sobolev space and its norm is denoted by $\|\cdot\|_{W^{l,p}(D)}$. When $p = 2$, we denote $W^{l,2}(D)$ by $H^l(D)$ and its norm is denoted by $\|\cdot\|_{H^l(D)}$. We denote by $C_0^l(D)$ the set of all C^l functions whose support is compact in D. The completion of $C_0^l(D)$ in $W^{l,p}(D)$ is denoted by $W_0^{l,p}(D)$. In particular, we write $H_0^l(D)$ for $W_0^{l,2}(D)$.

We simply denote by $L^p(D)$ the set of all vector fields $W = {}^\top(w^1, \dots, w^n)$ on D whose components w^j ($j = 1, \dots, n$) belong to $L^p(D)$ and the norm is also denoted by $\|\cdot\|_{L^p(D)}$ if no confusion can occur. Similarly, the symbols $W^{l,p}(D)$ and $H^l(D)$ are also used for vector fields.

For $u = {}^\top(\phi, w)$ with $\phi \in W^{k,p}(D)$ and $w = {}^\top(w^1, \dots, w^n) \in W^{l,q}(D)$, we define the norm $\|u\|_{W^{k,p}(D) \times W^{l,q}(D)}$ by

$$\|u\|_{W^{k,p}(D) \times W^{l,q}(D)} := \|\phi\|_{W^{k,p}(D)} + \|w\|_{W^{l,q}(D)}.$$

We define the sets Q, Q^*, Ω_{per}, $\Sigma_{j,\pm}$ ($j = 1, \dots, n-1$) and Σ_n as follows:

$$Q := \prod_{i=1}^{n-1} [-\pi/\alpha_i, \pi/\alpha_i), \qquad Q^* := \prod_{i=1}^{n-1} [-\alpha_i/2, \alpha_i/2),$$

$$\begin{aligned}
\Omega_{\mathrm{per}} &:= \{x=(x',x_n);\, x'\in Q,\, \omega_1(x')<x_n<\omega_2(x')\},\\
\Sigma_{j,\pm} &:= \{x\in\overline{\Omega_{\mathrm{per}}};\, x_j=\pm\pi/\alpha_j\},\\
\Sigma_n &:= \{x\in\partial\Omega;\, x'\in Q,\, x_n=\omega_j(x'),\, j=1,2\}.
\end{aligned}$$

In the case $D=\Omega_{\mathrm{per}}$, we simply write $L^p(\Omega_{\mathrm{per}})$ as L^p, and likewise, $W^{k,p}(\Omega_{\mathrm{per}})$, $H^l(\Omega_{\mathrm{per}})$ as $W^{k,p}$, H^l, respectively. Similarly, the norms are also abbreviated to $\|\cdot\|_{H^l}$, $\|\cdot\|_{W^{k,p}}$, and, in particular, we write $\|\cdot\|_p$ for $\|\cdot\|_{L^p(\Omega_{\mathrm{per}})}$.

The inner product of $L^2(D)$ is defined by

$$(f,g)_{L^2(D)} := \int_D f(x)\overline{g(x)}\,dx, \qquad f,g\in L^2(D).$$

When $D=\Omega_{\mathrm{per}}$, we abbreviate it to (f,g). The dual space of $H_0^1(D)$ is denoted by $H^{-1}(D)$, and the pairing between $H^{-1}(D)$ and $H_0^1(D)$ is written as $[\cdot,\cdot]$. For $f\in L^2(\Omega_{\mathrm{per}})$, its mean value over Ω_{per} is denoted by $[\![f]\!]$, i.e.,

$$[\![f]\!] := (f,1) = \frac{1}{|\Omega_{\mathrm{per}}|}\int_{\Omega_{\mathrm{per}}} f(x)\,dx.$$

We often write $x\in\Omega$ as $x={}^\top(x',x_n)$, $x'={}^\top(x_1,\dots,x_{n-1})\in\mathbb{R}^{n-1}$. The partial derivatives of a function u are denoted by ∂_{x_j}, $\partial_{x_j}\partial_{x_k}$, and so on.

We will work in spaces of functions Q-periodic in x', and so we introduce the function spaces $L^2_{\mathrm{per}}(\Omega_{\mathrm{per}})$, $C^\infty_{\mathrm{per}}(\overline{\Omega_{\mathrm{per}}})$, $C^\infty_{0,\mathrm{per}}(\Omega_{\mathrm{per}})$, $H^l_{\mathrm{per}}(\Omega_{\mathrm{per}})$, $H^l_{0,\mathrm{per}}(\Omega_{\mathrm{per}})$ defined by

$$\begin{aligned}
L^2_{\mathrm{per}}(\Omega_{\mathrm{per}}) &:= \big\{u|_{\Omega_{\mathrm{per}}};\, u\in L^2_{\mathrm{loc}}(\overline{\Omega}),\, u\big(x'+\tfrac{2\pi}{\alpha_j}\boldsymbol{e}'_j,x_n\big)=u(x',x_n),\\
&\qquad\qquad (x',x_n)\in\Omega,\, 1\le j\le n-1\big\},\\
C^\infty_{\mathrm{per}}(\overline{\Omega_{\mathrm{per}}}) &:= \big\{u|_{\overline{\Omega_{\mathrm{per}}}};\, u\in C^\infty(\overline{\Omega}),\, u\big(x'+\tfrac{2\pi}{\alpha_j}\boldsymbol{e}'_j,x_n\big)=u(x',x_n),\\
&\qquad\qquad (x',x_n)\in\Omega,\, 1\le j\le n-1\big\},\\
C^\infty_{0,\mathrm{per}}(\Omega_{\mathrm{per}}) &:= \{u\in C^\infty_{\mathrm{per}}(\overline{\Omega_{\mathrm{per}}});\, u=0 \text{ in a neighborhood of } \partial\Omega\},\\
H^l_{\mathrm{per}}(\Omega_{\mathrm{per}}) &:= \text{the closure of } C^\infty_{\mathrm{per}}(\overline{\Omega_{\mathrm{per}}}) \text{ in } H^l(\Omega_{\mathrm{per}}),\\
H^l_{0,\mathrm{per}}(\Omega_{\mathrm{per}}) &:= \text{the closure of } C^\infty_{0,\mathrm{per}}(\Omega_{\mathrm{per}}) \text{ in } H^l(\Omega_{\mathrm{per}}).
\end{aligned}$$

Observe that $L^2_{\mathrm{per}}(\Omega_{\mathrm{per}})$ can be identified with $L^2(\Omega_{\mathrm{per}})$, and that

$$\begin{aligned}
H^l_{\mathrm{per}}(\Omega_{\mathrm{per}}) &= \{u\in H^l(\Omega_{\mathrm{per}});\, \partial^\beta_{x'}u|_{\Sigma_{j,-}}=\partial^\beta_{x'}u|_{\Sigma_{j,+}},\, 1\le j\le n-1,\, |\beta|\le l-1\},\\
H^1_{0,\mathrm{per}}(\Omega_{\mathrm{per}}) &= \{u\in H^1_{\mathrm{per}}(\Omega_{\mathrm{per}});\, u|_{\Sigma_n}=0\}.
\end{aligned}$$

We also set

$$\begin{aligned}
L^2_{*,\mathrm{per}}(\Omega_{\mathrm{per}}) &:= \{f\in L^2_{\mathrm{per}}(\Omega_{\mathrm{per}});\, [\![f]\!]=0\},\\
H^l_{*,\mathrm{per}}(\Omega_{\mathrm{per}}) &:= H^l_{\mathrm{per}}(\Omega_{\mathrm{per}})\cap L^2_{*,\mathrm{per}}(\Omega_{\mathrm{per}}).
\end{aligned}$$

For $\eta' \in \mathbb{R}^{n-1}$ we denote

$$\tilde{\eta}' = {}^{\top}(\eta', 0) \in \mathbb{R}^n,$$

and $\nabla_{\eta'}$ and $\Delta_{\eta'}$ are defined by

$$\nabla_{\eta'} := \nabla + i\tilde{\eta}' \quad \text{and} \quad \Delta_{\eta'} := \nabla_{\eta'} \cdot \nabla_{\eta'},$$

respectively.

We next introduce some operators. We denote by $\mathbb{P}_0$ and $\tilde{\mathbb{P}}$ the following $(n+1) \times (n+1)$ diagonal matrices:

$$\mathbb{P}_0 := \mathrm{diag}(1, 0, \ldots, 0), \qquad \tilde{\mathbb{P}} = \mathrm{diag}(0, 1, \ldots, 1).$$

Note that $\mathbb{P}_0 u = {}^{\top}(\phi, 0)$ and $\tilde{\mathbb{P}} u = {}^{\top}(0, w)$ for $u = {}^{\top}(\phi, w)$ with $w = {}^{\top}(w_1, \ldots, w_n)$.

We denote the kernel and range of an operator A by $\mathrm{Ker}\, A$ and $R(A)$, respectively.

For a function $f = f(x')$ ($x' \in \mathbb{R}^{n-1}$), we denote its Fourier transform by $\hat{f}$ or $\mathcal{F}[f]$:

$$\hat{f}(\xi') = \mathcal{F}[f](\xi') = \int_{\mathbb{R}^{n-1}} f(x') e^{-i\xi' \cdot x'}\, dx' \qquad (\xi' \in \mathbb{R}^{n-1}).$$

The inverse Fourier transform $\mathcal{F}^{-1}$ is defined by

$$\mathcal{F}^{-1}[f](x') = (2\pi)^{-(n-1)} \int_{\mathbb{R}^{n-1}} f(\xi') e^{i\xi' \cdot x'} d\xi' \qquad (x' \in \mathbb{R}^{n-1}).$$

We next introduce the Bloch wave decomposition. Let $\mathcal{S}(\mathbb{R}^{n-1})$ denote the Schwartz space on $\mathbb{R}^{n-1}$.

Definition 2.1. We define the operator T by setting, for $\varphi \in \mathcal{S}(\mathbb{R}^{n-1})$, $x' \in \mathbb{R}^{n-1}$, and $\eta' \in \mathbb{R}^{n-1}$,

$$\begin{aligned}
(2.1) \qquad & (T\varphi)(x', \eta') \\
& := \frac{1}{(2\pi)^{(n-1)/2} |Q|^{1/2}} \sum_{(k_1, \ldots, k_{n-1}) \in \mathbb{Z}^{n-1}} \hat{\varphi}\Big(\eta' + \sum_{j=1}^{n-1} k_j \alpha_j \boldsymbol{e}'_j\Big)\, e^{i \sum_{j=1}^{n-1} k_j \alpha_j x_j} \\
& = \frac{1}{|Q^*|^{1/2}} \sum_{(l_1, \ldots, l_{n-1}) \in \mathbb{Z}^{n-1}} \varphi\left(x' + \sum_{j=1}^{n-1} l_j \frac{2\pi}{\alpha_j} \boldsymbol{e}'_j\right) e^{-i\eta' \cdot (x' + \sum_{j=1}^{n-1} l_j \frac{2\pi}{\alpha_j} \boldsymbol{e}'_j)}.
\end{aligned}$$

We also define the operator U as follows. For a function $\varphi(x', \eta') \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ such that $\varphi(x', \eta')$ is Q-periodic in x' and $\varphi(x', \eta') e^{i\eta' \cdot x'}$ is Q^*-periodic in η', we define, for $x' \in \mathbb{R}^{n-1}$,

$$(2.2) \qquad (U\varphi)(x') := \frac{1}{|Q^*|^{1/2}} \int_{Q^*} \varphi(x', \eta') e^{i\eta' \cdot x'} d\eta'.$$

Note that $\varphi(x, \eta' + \alpha_j \boldsymbol{e}'_j) = \varphi(x', \eta') e^{-i\alpha_j \boldsymbol{e}'_j \cdot x'}$ ($j = 1, \ldots, n-1$).

The operators T and U have the following properties. See, e.g., [21, 22] for the details.

Proposition 2.2. (i) *$(T\varphi)(x', \eta')$ is Q-periodic in x' and $(T\varphi)(x', \eta')e^{i\eta'\cdot x'}$ is Q^*-periodic in η'.*

(ii) *T can be uniquely extended to an isometric operator from $L^2(\mathbb{R}^{n-1})$ to $L^2(Q^*; L^2(Q))$.*

(iii) *U is the inverse operator of T.*

(iv) *Let ψ be Q-periodic in x'. Then $T(\psi\varphi) = \psi T(\varphi)$.*

(v) *$T(\partial_{x_j}\varphi) = (\partial_{x_j} + i\eta_j)T\varphi$ $(j = 1, \dots, n-1)$ and T defines an isomorphism from $H^l(\mathbb{R}^{n-1})$ to $L^2(Q^*; H^l_{\mathrm{per}}(Q))$. (Here $H^l_{\mathrm{per}}(Q)$ denotes the space of Q-periodic functions belonging to $H^l(Q)$, as in the case of $H^l_{\mathrm{per}}(\Omega_{\mathrm{per}})$.)*

We next consider T as an operator acting on functions in $H^l(\Omega)$. Let $y = \Phi(x)$ be the transformation

$$y' = x', \qquad y_n = \frac{1}{\omega_2(x') - \omega_1(x')}(x_n - \omega_1(x')).$$

Then Φ is a diffeomorphism from Ω to $\mathbb{R}^{n-1} \times (0,1)$ and Φ transforms Q-periodic functions on Ω to those on $\mathbb{R}^{n-1} \times (0,1)$. We denote the inverse transform of Φ by Ψ and we define the operators Φ^* and Ψ^* by $[\Phi^* u](x) = u(\Phi(x))$ and $[\Psi^* u](y) = u(\Psi(y))$. Then Φ^* is an isomorphism from $H^l(\Omega)$ to $H^l(\mathbb{R}^{n-1} \times (0,1))$, and likewise from $H^l_{\mathrm{per}}(\Omega_{\mathrm{per}})$ to $H^l_{\mathrm{per}}(Q \times (0,1))$, where $H^l_{\mathrm{per}}(Q \times (0,1))$ denotes the space of Q-periodic functions belonging to $H^l(Q \times (0,1))$.

It is not difficult to see that Proposition 2.2 holds with $H^l(\mathbb{R}^{n-1})$ replaced by $H^l(\mathbb{R}^{n-1} \times (0,1))$, and likewise with $H^l_{\mathrm{per}}(Q)$ replaced by $H^l_{\mathrm{per}}(Q \times (0,1))$. It then follows that $\Phi^* T \Psi^*$ is an isomorphism from $H^l(\Omega)$ to $L^2(Q^*; H^l_{\mathrm{per}}(\Omega_{\mathrm{per}}))$. Using the second expression of T in Definition 2.1 and the periodicity of ω_j $(j = 1, 2)$, one can see that $\Phi^* T \Psi^* u = Tu$ for functions u on Ω. Therefore, we will write $\Phi^* T \Psi^* u$ as Tu if no confusion can occur.

§3. Main results

In this section we state the main results of this paper.

Let us consider the linear problem

$$\partial_t u + Lu = 0, \qquad u = {}^\top(\phi, w). \tag{3.1}$$

Here L is the operator on $L^2(\Omega)$ given by

$$L := \begin{pmatrix} 0 & \gamma \operatorname{div} \\ \gamma \nabla & -\nu\Delta - \tilde{\nu}\nabla \operatorname{div} \end{pmatrix} \tag{3.2}$$

with domain

$$D(L) = \left\{u = {}^{\top}(\phi, w) \in L^2(\Omega);\ w \in H_0^1(\Omega),\ Lu \in L^2(\Omega)\right\}. \tag{3.3}$$

Our main issue is to investigate the spectral properties of the semigroup generated by $-L$.

Theorem 3.1. *The operator $-L$ generates a contraction C_0-semigroup e^{-tL} on $L^2(\Omega)$, and*

$$\|e^{-tL}u_0\|_{L^2(\Omega)} \le \|u_0\|_{L^2(\Omega)} \qquad (u_0 \in L^2(\Omega)).$$

The semigroup e^{-tL} has the following properties.

Theorem 3.2. *There is a bounded projection $\Pi : L^2(\Omega) \to L^2(\Omega)$ with $\Pi L \subset L\Pi$ and $\Pi e^{-tL} = e^{-tL}\Pi$, and the following estimates hold uniformly for $t > 0$ and $u_0 \in L^1(\Omega) \cap L^2(\Omega)$:*

(i) $\|e^{-tL}\Pi u_0\|_{L^2(\Omega)} \le C(1+t)^{-(n-1)/4}\|u_0\|_{L^1(\Omega)}$,

(ii) $\|e^{-tL}(I-\Pi)u_0\|_{L^2(\Omega)} \le Ce^{-\beta t}\|u_0\|_{L^2(\Omega)}$,

(iii) $\|e^{-tL}\Pi u_0 - [e^{-tH}\sigma_0]u^{(0)}\|_{L^2(\Omega)} \le Ct^{-(n-1)/4-1/2}\|u_0\|_{L^1(\Omega)}$.

Here β is a positive constant; e^{-tH} is the C_0-semigroup in $L^2(\mathbb{R}^{n-1})$ generated by the operator $-H$ defined by

$$H\sigma := -\frac{\gamma^2}{\nu}\sum_{i,j=1}^{n-1} a_{ij}\partial_{x_i}\partial_{x_j}\sigma \qquad (\sigma \in D(H))$$

with domain $D(H) = H^2(\mathbb{R}^{n-1})$; and σ_0 and $u^{(0)}$ are given as follows:

$$\sigma_0 := \frac{|Q|}{|\Omega_{\mathrm{per}}|}\int_{\omega_1(x')}^{\omega_2(x')} \phi_0(x', x_n)\,dx_n, \qquad u^{(0)} := {}^{\top}(1,0).$$

Here the matrix (a_{ij}) satisfies

$$\sum_{i,j=1}^{n-1} a_{ij}\xi_i\xi_j \ge \kappa_0|\xi'|^2 \qquad (\xi' = (\xi_1,\ldots,\xi_{n-1}) \in \mathbb{R}^{n-1})$$

with a constant $\kappa_0 > 0$ independent of ξ'. Furthermore, $a_{ij} = (\nabla w^{(i)}, \nabla w^{(j)})_{L^2(\Omega_{\mathrm{per}})}$ with ${}^{\top}(\phi^{(k)}, w^{(k)})$ $(k = 1,\ldots,n-1)$ satisfying the following Stokes system in Ω_{per}:

$$\begin{cases} \operatorname{div} w^{(k)} = 0, \\ -\Delta w^{(k)} + \nabla\phi^{(k)} = \boldsymbol{e}_k, \\ w^{(k)}|_{\Sigma_{j,+}} = w^{(k)}|_{\Sigma_{j,-}}, \quad \phi^{(k)}|_{\Sigma_{j,+}} = \phi^{(k)}|_{\Sigma_{j,-}}, \quad w^{(k)}|_{\Sigma_n} = 0, \\ [\![\phi^{(k)}]\!] = 0 \end{cases} \tag{3.4}$$

for some $\phi^{(k)}$.

The proof of Theorem 3.2 will be given in Sections 4 and 5. To prove Theorem 3.2, we will consider the resolvent problem $\lambda u + L_{\eta'}u = f$ on $L^2_{\mathrm{per}}(\Omega_{\mathrm{per}})$ with parameter $\eta' \in Q^*$. In the case of $|\eta'| \le r_0$ for some small $r_0 > 0$, we regard $L_{\eta'}$ as a perturbation of L_0 and apply analytic perturbation theory to study the spectrum of $-L_{\eta'}$. For $\eta' \in Q^*$ with $|\eta'| \ge r_0$, we establish estimates for a modified Stokes system and apply an energy method. Based on the analysis of $-L_{\eta'}$, we give a proof of Theorem 3.2.

§4. Spectral properties of $L_{\eta'}$

In this section we investigate the spectral properties of $L_{\eta'}$.

§4.1. Formulation

Let us consider the resolvent problem for (3.1),

$$(\lambda + L)u = f, \qquad u \in D(L). \tag{4.1}$$

Here $\lambda \in \mathbb{C}$ is a resolvent parameter.

Applying Ψ^* to (4.1), we have

$$(\lambda + \Psi^* L)\Psi^* u = \Psi^* f \quad \text{in } \mathbb{R}^{n-1} \times (0,1). \tag{4.2}$$

Here $\Psi^* L$ is the differential operator of the form

$$\Psi^* L = \begin{pmatrix} 0 & \sum_{j=1}^n l_{12}^j(y', y_n)\partial_{y_j} \\ \sum_{j=1}^n l_{21}^j(y', y_n)\partial_{y_j} & \sum_{j,k=1}^n l_{22}^{j,k}(y', y_n)\partial_{y_j}\partial_{y_k} + \sum_{j=1}^n l_{22}^j(y', y_n)\partial_{y_j} \end{pmatrix}$$

with some l_{pq}^j and $l_{pq}^{j,k}$ $(p, q = 1, 2)$ Q-periodic in y'. We next apply T to (4.2). It then follows from Proposition 2.2(i), (iv) and (v) that (4.2) is transformed into the following problem on $Q \times (0,1)$:

$$(\lambda + \Psi^* L_{\eta'})T\Psi^* u = T\Psi^* f \qquad (\eta' \in Q^*) \tag{4.3}$$

with Q-periodic boundary condition in y'. Applying Φ^* to (4.3) we arrive at

$$(\lambda + L_{\eta'})Tu = Tf \qquad \text{on } \Omega_{\mathrm{per}} \tag{4.4}$$

with the dual parameter $\eta' \in Q^*$, where $L_{\eta'}$ is the operator on $L^2_{\mathrm{per}}(\Omega_{\mathrm{per}})$ of the form

$$L_{\eta'} := \begin{pmatrix} 0 & \gamma\,{}^\top\nabla_{\eta'} \\ \gamma\nabla_{\eta'} & -\nu\Delta_{\eta'} - \tilde{\nu}\nabla_{\eta'}{}^\top\nabla_{\eta'} \end{pmatrix}$$

with domain

$$D(L_{\eta'}) = \{u = {}^\top(\phi, w) \in L^2_{\mathrm{per}}(\Omega_{\mathrm{per}});\ L_{\eta'}u \in L^2_{\mathrm{per}}(\Omega_{\mathrm{per}}),\ w \in H^1_{0,\mathrm{per}}(\Omega_{\mathrm{per}})\}.$$

It is not difficult to see that $D(L_{\eta'}) = D(L_0)$ for all $\eta' \in Q^*$ and that $L_{\eta'}$ is a closed operator on $L^2_{\rm per}(\Omega_{\rm per})$.

If $\lambda \in \rho(-L_{\eta'})$, then, by (4.4), u can be written as

$$u = U(\lambda + L_{\eta'})^{-1} Tf.$$

Therefore, to investigate the resolvent of $-L$, we will consider the problem for $-L_{\eta'}$:

$$\lambda u + L_{\eta'} u = f, \qquad u \in D(L_0). \tag{4.5}$$

Before going further, we also introduce the adjoint operator of $L_{\eta'}$. We define

$$L^*_{\eta'} := \begin{pmatrix} 0 & -\gamma {}^\top\nabla_{\eta'} \\ -\gamma\nabla_{\eta'} & -\nu\Delta_{\eta'} - \tilde{\nu}\nabla_{\eta'} {}^\top\nabla_{\eta'} \end{pmatrix}$$

with domain

$$D(L^*_{\eta'}) = \{u = {}^\top(\phi, w) \in L^2_{\rm per}(\Omega_{\rm per});\ L^*_{\eta'} u \in L^2_{\rm per}(\Omega_{\rm per}),\ w \in H^1_{0,\rm per}(\Omega_{\rm per})\}.$$

One can see that $D(L^*_{\eta'}) = D(L^*_0)$ for all $\eta' \in Q^*$ and that $L^*_{\eta'}$ is the adjoint operator of $L_{\eta'}$.

§4.2. The case $|\eta'| \le r_0$

In this subsection we consider (4.5) with $|\eta'| \le r_0$ for some sufficiently small $r_0 > 0$. It is convenient to write

$$L_{\eta'} = L_0 + \sum_{j=1}^{n-1} \eta_j L^{(1)}_j + \sum_{j,k=1}^{n-1} \eta_j \eta_k L^{(2)}_{jk},$$

where

$$L_0 := \begin{pmatrix} 0 & \gamma\,\mathrm{div} \\ \gamma\nabla & -\nu\Delta - \tilde{\nu}\nabla\,\mathrm{div} \end{pmatrix},$$

$$L^{(1)}_j := i\begin{pmatrix} 0 & \gamma {}^\top\boldsymbol{e}_j \\ \gamma\boldsymbol{e}_j & -2\nu\partial_{x_j} - \tilde{\nu}\boldsymbol{e}_j\,\mathrm{div} - \tilde{\nu}\nabla({}^\top\boldsymbol{e}_j) \end{pmatrix}, \qquad L^{(2)}_{jk} := \begin{pmatrix} 0 & 0 \\ 0 & \nu\delta_{jk} I_n + \tilde{\nu}\boldsymbol{e}_j {}^\top\boldsymbol{e}_k \end{pmatrix}$$

with I_n being the $n \times n$ identity matrix. We also set

$$M_{\eta'} = \sum_{j=1}^{n-1} \eta_j L^{(1)}_j + \sum_{j,k=1}^{n-1} \eta_j \eta_k L^{(2)}_{jk},$$

namely,

$$M_{\eta'} := \begin{pmatrix} 0 & i\gamma {}^\top\tilde{\eta}' \\ i\gamma\tilde{\eta}' & \nu(|\eta'|^2 - 2i\tilde{\eta}' \cdot \nabla) - i\tilde{\nu}\tilde{\eta}' {}^\top(\nabla + i\tilde{\eta}') - i\tilde{\nu}\nabla {}^\top\tilde{\eta}' \end{pmatrix}.$$

Similarly, we write

$$L_{\eta'}^* = L_0^* + \sum_{j=1}^{n-1} \eta_j L_j^{(1)*} + \sum_{j,k=1}^{n-1} \eta_j \eta_k L_{jk}^{(2)*},$$

where

$$L_0^* := \begin{pmatrix} 0 & -\gamma \operatorname{div} \\ -\gamma \nabla & -\nu\Delta - \tilde{\nu}\nabla \operatorname{div} \end{pmatrix},$$
$$L_j^{(1)*} := i \begin{pmatrix} 0 & -\gamma {}^\top\boldsymbol{e}_j \\ -\gamma \boldsymbol{e}_j & -2\nu\partial_{x_j} - \tilde{\nu}\boldsymbol{e}_j \operatorname{div} - \tilde{\nu}\nabla({}^\top\boldsymbol{e}_j) \end{pmatrix},$$
$$L_{jk}^{(2)*} := \begin{pmatrix} 0 & 0 \\ 0 & \nu\delta_{jk} I_n + \tilde{\nu}\boldsymbol{e}_j {}^\top\boldsymbol{e}_k \end{pmatrix}.$$

We begin with the resolvent estimates for the case $\eta' = 0$ which imply the generation of a contraction semigroup e^{-tL_0}.

In what follows we write $X := L^2_{\mathrm{per}}(\Omega_{\mathrm{per}})$ for simplicity of notation.

Proposition 4.1. *We have* $\{\lambda; \operatorname{Re}\lambda > 0\} \subset \rho(-L_0)$*, and if* $\operatorname{Re}\lambda > 0$*, then*

$$\|(\lambda + L_0)^{-1} f\|_2 \le \frac{1}{\operatorname{Re}\lambda}\|f\|_2, \qquad \|\nabla\tilde{\mathbb{P}}(\lambda + L_0)^{-1} f\|_2 \le \frac{1}{(\nu \operatorname{Re}\lambda)^{1/2}}\|f\|_2.$$

The same conclusion holds for the adjoint operator L_0^**.*

Proof. Let $\operatorname{Re}\lambda > 0$. Since

$$\operatorname{Re}((\lambda + L_0)u, u) = \nu\|\nabla w\|_2^2 + \tilde{\nu}\|\operatorname{div} w\|_2^2 + \operatorname{Re}\lambda\|u\|_2^2, \tag{4.6}$$

we see that if $(\lambda + L_0)u = 0$, then $u = 0$, and so $\lambda + L_0$ is injective when $\operatorname{Re}\lambda > 0$. Observe also that if $\operatorname{Re}\lambda > 0$, then

$$\|u\|_2 \le \frac{1}{\operatorname{Re}\lambda}\|(\lambda + L_0)u\|_2, \tag{4.7}$$

$$\|\nabla w\|_2 \le \frac{1}{(\nu \operatorname{Re}\lambda)^{1/2}}\|(\lambda + L_0)u\|_2. \tag{4.8}$$

It follows from (4.7) that $R(\lambda + L_0)$ is a closed subspace of X. We note that these inequalities also hold with L_0 replaced by L_0^*. Let $v \in R(\lambda + L_0)^\perp$. Then, since $((\lambda + L_0)u, v) = 0$ for all $u \in D(L_0)$, we see that $v \in D(L_0^*)$ and $(\lambda + L_0^*)v = 0$. This, together with (4.7) with L_0 replaced by L_0^*, implies that $v = 0$. We thus conclude that $R(\lambda + L_0) = X$, that is, $\lambda + L_0$ is surjective. □

The following estimates show that $-L_{\eta'}$ also generates a contraction semigroup.

Proposition 4.2. *We have* $\{\lambda;\, \mathrm{Re}\,\lambda > 0\} \subset \rho(-L_{\eta'})$ *and the following estimates hold for* $\mathrm{Re}\,\lambda > 0$*:*

$$\|(\lambda + L_{\eta'})^{-1} f\|_2 \le \frac{1}{\mathrm{Re}\,\lambda}\|f\|_2,$$
$$\|\nabla\tilde{\mathbb{P}}(\lambda + L_{\eta'})^{-1} f\|_2 \le \frac{C}{\nu^{1/2}}\left(\frac{1}{(\mathrm{Re}\,\lambda)^{1/2}} + \frac{1}{\mathrm{Re}\,\lambda}\right)\|f\|_2.$$

The same conclusion holds for the adjoint operator $L^*_{\eta'}$*.*

Proof. We have

$$\mathrm{Re}\,((\lambda + L_{\eta'})u, u) = \mathrm{Re}\,\lambda\|u\|_2^2 + \nu\|\nabla_{\eta'} w\|_2^2 + \tilde{\nu}\|\nabla_{\eta'}\cdot w\|_2^2.$$

It then follows that if $\mathrm{Re}\,\lambda > 0$, then

$$\|u\|_2 \le \frac{1}{\mathrm{Re}\,\lambda}\|(\lambda + L_{\eta'})u\|_2.$$

We also have

$$\mathrm{Re}\,((\lambda + L_{\eta'})u, u) \ge \nu\|\nabla w\|_2^2 + (\mathrm{Re}\,\lambda - C)\|u\|_2^2$$

for a constant $C > 0$ uniformly for $\eta' \in Q^*$. Therefore, we deduce that

$$\nu\|\nabla w\|_2^2 \le \|(\lambda + L_{\eta'})u\|_2\|u\|_2 + C\|u\|_2^2 \le C\left(\frac{1}{\mathrm{Re}\,\lambda} + \frac{1}{(\mathrm{Re}\,\lambda)^2}\right)\|(\lambda + L_{\eta'})u\|_2^2,$$

which gives

$$\|\nabla w\|_2 \le \frac{C}{\nu^{1/2}}\left(\frac{1}{(\mathrm{Re}\,\lambda)^{1/2}} + \frac{1}{\mathrm{Re}\,\lambda}\right)\|(\lambda + L_{\eta'})u\|_2.$$

As in the proof of Proposition 4.1, one can now obtain the desired results. □

We next show that $\lambda = 0$ is a simple eigenvalue of $-L_0$.

Proposition 4.3. *There exists a constant* $\beta_0 > 0$ *such that* $\rho(-L_0) \supset \{\lambda \neq 0;\, \mathrm{Re}\,\lambda > -\beta_0\}$*. Furthermore,* $\lambda = 0$ *is a simple eigenvalue of* $-L_0$*, and for* $\lambda \neq 0$ *satisfying* $\mathrm{Re}\,\lambda > -\beta_0$*,*

$$(\lambda + L_0)^{-1} f = \frac{1}{\lambda}\Pi^{(0)} f + S_\lambda(I - \Pi^{(0)})f,$$

and the following estimates hold uniformly for λ *satisfying* $\mathrm{Re}\,\lambda > -\beta_0$*:*

$$\|S_\lambda(I-\Pi^{(0)})f\|_2 \le \frac{C}{\mathrm{Re}\,\lambda + \beta_0}\|f\|_2, \qquad \|\nabla\tilde{\mathbb{P}}S_\lambda(I-\Pi^{(0)})f\|_2 \le \frac{C}{(\mathrm{Re}\,\lambda + \beta_0)^{1/2}}\|f\|_2.$$

Here $\Pi^{(0)}$ *is the eigenprojection for the eigenvalue* $\lambda = 0$ *defined by*

$$\Pi^{(0)} u := (u, u^{(0)*})u^{(0)} = [\![\phi]\!]u^{(0)}, \qquad u = {}^\top(\phi, w),$$

where

$$u^{(0)} := {}^{\top}(1,0), \qquad u^{(0)*} := \frac{1}{|\Omega_{\rm per}|}{}^{\top}(1,0),$$

and S_λ *is the operator defined by*

$$S_\lambda := [(I-\Pi^{(0)})(\lambda+L_0)(I-\Pi^{(0)})]^{-1}.$$

The same conclusion holds with L_0, S_λ, $\Pi^{(0)}$ *replaced by* L_0^*, S_λ^*, $\Pi^{(0)*}$ *respectively where*

$$S_\lambda^* := [(I-\Pi^{(0)*})(\lambda+L_0^*)(I-\Pi^{(0)*})]^{-1}, \qquad \Pi^{(0)*}u := (u,u^{(0)})u^{(0)*}.$$

We give a proof of Proposition 4.3 for L_0 only since the case of L_0^* can be treated similarly. For the proof, we prepare the following two lemmas.

Lemma 4.4. *We have* $\operatorname{Ker} L_0 = \operatorname{span}\{u^{(0)}\}$ *and* $\Pi^{(0)}$ *is a bounded projection on* $L^2(\Omega_{\rm per})$ *with* $\Pi^{(0)}X = \operatorname{Ker} L_0$ *and* $\Pi^{(0)}L_0 \subset L_0\Pi^{(0)} = 0$.

Proof. Let $L_0u = 0$. It then follows from (4.6) that $\nabla w = 0$, and hence $\nabla\phi = 0$. This implies that $w = 0$ and $\phi = \text{const}$. This shows that $\operatorname{Ker} L_0 = \operatorname{span}\{u^{(0)}\}$. Clearly, $\Pi^{(0)}$ is a bounded projection onto $\operatorname{Ker} L_0$. For $u = {}^{\top}(\phi, w)$, we have $L_0\Pi^{(0)}u = {}^{\top}(0, \gamma\nabla[\![\phi]\!]) = 0$. On the other hand, for $u \in D(L_0)$, we have $\Pi^{(0)}L_0u = [\![\gamma \operatorname{div} w]\!]u^{(0)} = 0$. We thus conclude that $\Pi^{(0)}L_0 \subset L\Pi^{(0)} = 0$. □

Lemma 4.5. *We have* $\rho(-L_0|_{(I-\Pi^{(0)})X}) \supset \{\lambda; \operatorname{Re}\lambda > -\beta_0\}$ *with a positive constant* β_0*, and the estimates for* S_λ *in Proposition* 4.3 *hold true.*

Proof. We set $\mathcal{A} := -L_0|_{(I-\Pi^{(0)})X}$. Let us consider $\lambda u + \mathcal{A}u = f$. It is known that there exists a bounded linear operator $\mathcal{B} : L^2_*(\Omega_{\rm per}) \to H^1_{0,\rm per}(\Omega_{\rm per})$ such that for any $g \in L^2_{*,\rm per}(\Omega_{\rm per})$ we have $\operatorname{div}\mathcal{B}g = g$ and $\|\nabla\mathcal{B}g\|_2 \le c_0\|g\|_2$ for some constant $c_0 > 0$. See [1, 2, 4] for the details.

We follow the argument in [6]. We introduce a new inner product

$$(\!(u_1,u_2)\!) := (u_1,u_2) - \delta\{(w_1,\mathcal{B}\phi_2) + (\mathcal{B}\phi_1, w_2)\}$$

for $u_j = {}^{\top}(\phi_j, w_j)$ $(j = 1,2)$ with a constant $\delta > 0$ to be determined later. This pairing $(\!(u_1,u_2)\!)$ defines an inner product on $L^2_{*,\rm per}(\Omega_{\rm per}) \times L^2(\Omega_{\rm per})$ if $\delta > 0$ is sufficiently small. In fact, using the Poincaré inequality $\|w\|_2 \le c_1\|\nabla w\|_2$, we see that there exists a constant $C > 0$ such that

$$(\!(u,u)\!) = \|u\|_2^2 - \delta\{(w,\mathcal{B}\phi) + (\mathcal{B}\phi, w)\} \ge (1-\delta c_0c_1)\|u\|_2^2$$

and $(\!(u,u)\!) \le (1+\delta c_0c_1)\|u\|_2^2$. Therefore, $(\!(\cdot,\cdot)\!)$ is an inner product and the norm it defines is equivalent to the norm $\|\cdot\|_2$ if $\delta > 0$ is taken sufficiently small.

We denote $\mathcal{A}u = {}^\top(\mathcal{A}_1 u, \mathcal{A}_2 u)$. Note that $\int_{\Omega_{\rm per}} \mathcal{A}_1 u\, dx = 0$. We see that

$$\begin{aligned}
(\!(\mathcal{A}u, u)\!) &= (L_0 u, u) - \delta\{(\mathcal{A}_2 u, \mathcal{B}\phi) + (\mathcal{B}(\mathcal{A}_1 u), w)\} \\
&\geq \nu\|\nabla w\|_2^2 + \tilde{\nu}\|\mathrm{div}\, w\|_2^2 + \tfrac{1}{2}\delta\gamma\|\phi\|_2^2 - \delta\left\{\left(\frac{\nu^2 c_0^2}{\gamma} + \gamma c_1^2\right)\|\nabla w\|_2^2 + \frac{\tilde{\nu}^2}{\gamma}\|\mathrm{div}\, w\|_2^2\right\} \\
&\geq \tfrac{1}{2}\nu\|\nabla w\|_2^2 + \tfrac{1}{2}\tilde{\nu}\|\mathrm{div}\, w\|_2^2 + \tfrac{1}{2}\delta\gamma\|\phi\|_2^2
\end{aligned}$$

if $\delta > 0$ is taken suitably small. Therefore, we have

$$\begin{aligned}
(1 - \delta c_0 c_1)\,\mathrm{Re}\,\lambda\|u\|_2 + \tfrac{1}{2}\nu\|\nabla w\|_2^2 + \tfrac{1}{2}\tilde{\nu}\|\mathrm{div}\, w\|_2^2 + \tfrac{1}{2}\delta\gamma\|\phi\|_2^2 &\leq \mathrm{Re}\,(\!(f, u)\!) \\
&\leq C\|f\|_2\|u\|_2.
\end{aligned}$$

Setting $\beta_0 = \frac{1}{2(1-\delta c_0 c_1)}\min\{\delta\gamma, \frac{\nu}{2c_1^2}\}$ we find by the Poincaré inequality that

$$(\mathrm{Re}\,\lambda + \beta_0)\|u\|_2 \leq C\|f\|_2.$$

We thus conclude that if $\mathrm{Re}\,\lambda + \beta_0 > 0$, then

$$\|u\|_2 \leq \frac{C}{\mathrm{Re}\,\lambda + \beta_0}\|f\|_2 \quad \text{and} \quad \|\nabla w\|_2 \leq \frac{C}{(\mathrm{Re}\,\lambda + \beta_0)^{1/2}}\|f\|_2.$$

These estimates, together with Proposition 4.1, yield the desired results. □

We are now in a position to prove Proposition 4.3.

Proof of Proposition 4.3. We define

$$X_0 := \Pi^{(0)} X \quad \text{and} \quad X_1 := (I - \Pi^{(0)})X.$$

By Lemma 4.4, we have $X = X_0 \oplus X_1$ and $\rho(-L_0|_{X_0}) = \{\lambda;\ \lambda \neq 0\}$. This, together with Lemma 4.5, shows that $\{\lambda \neq 0;\ \mathrm{Re}\,\lambda > -\beta_0\} \subset \rho(-L_0)$,

$$(\lambda + L_0)^{-1} f = \frac{1}{\lambda}\Pi^{(0)} f + S_\lambda (I - \Pi^{(0)}) f,$$

and S_λ satisfies the desired estimates. □

We next derive the resolvent estimates for $-L_{\eta'}$ with $|\eta'| \leq r_0$.

Theorem 4.6. *There exists a constant $r_0 > 0$ such that if $\eta' \in Q^*$ satisfies $|\eta'| \leq r_0$, then*

$$\Sigma_1 := \{\lambda; \mathrm{Re}\,\lambda \geq -3\beta_0/4\} \cap \{\lambda;\ |\lambda| \geq \beta_0/2\} \subset \rho(-L_{\eta'}),$$

and the following estimates hold uniformly for $\lambda \in \Sigma_1$:

$$\|(\lambda + L_{\eta'})^{-1} f\|_2 \leq \frac{C}{\mathrm{Re}\,\lambda + \beta_0}\|f\|_2, \qquad \|\nabla\tilde{\mathbb{P}}(\lambda + L_{\eta'})^{-1} f\|_2 \leq \frac{C}{(\mathrm{Re}\,\lambda + \beta_0)^{1/2}}\|f\|_2.$$

The same conclusion holds with $L_{\eta'}$ replaced by $L_{\eta'}^*$.

Proof. Let $\lambda \in \Sigma_1$. By Proposition 4.3, we see that

$$\|(\lambda + L_0)^{-1} f\|_2 + \|\nabla \tilde{\mathbb{P}}(\lambda + L_0)^{-1} f\|_2 \le C_1 \|f\|_2 \tag{4.9}$$

uniformly for $\lambda \in \Sigma_1$. Here C_1 is a constant depending only on β_0. It then follows that

$$\|L_j^{(1)} u\|_2 \le C\{\|w\|_2 + \|\nabla w\|_2 + \|\phi\|_2\} \le CC_1 \|(\lambda + L_0)u\|_2, \tag{4.10}$$

$$\|L_{jk}^{(2)} u\|_2 \le C\|w\|_2 \le CC_1 \|(\lambda + L_0)u\|_2 \tag{4.11}$$

uniformly for $\lambda \in \Sigma_1$ and $u \in D(L_0)$. We thus obtain

$$\|M_{\eta'}(\lambda + L_0)^{-1} f\|_2 \le CC_1 |\eta'| \, \|f\|_2 \qquad (\lambda \in \Sigma_1)$$

uniformly for $\lambda \in \Sigma_1$ and $f \in X$. Therefore, if $r_0 > 0$ is a constant satisfying $r_0 < \frac{1}{CC_1}$, then $\lambda \in \rho(-L_{\eta'})$ for $|\eta'| \le r_0$ and

$$(\lambda + L_{\eta'})^{-1} = (\lambda + L_0)^{-1} \sum_{N=0}^{\infty} (-1)^N (M_{\eta'}(\lambda + L_0)^{-1})^N,$$

$$\|(\lambda + L_{\eta'})^{-1} f\|_2 \le \frac{C}{\operatorname{Re}\lambda + \beta_0} \sum_{N=0}^{\infty} \|M_{\eta'}(\lambda + L_0)^{-1}\|^N \|f\|_2 \le \frac{C}{\operatorname{Re}\lambda + \beta_0} \|f\|_2.$$

Similarly,

$$\|\nabla \tilde{\mathbb{P}}(\lambda + L_{\eta'})^{-1} f\|_2 \le \frac{C}{(\operatorname{Re}\lambda + \beta_0)^{1/2}} \|f\|_2.$$

The case of $L^*_{\eta'}$ can be proved similarly. □

We now show that $\sigma(-L_{\eta'}) \cap \{\lambda;\, |\lambda| < \beta_0/2\}$ consists of a simple eigenvalue whose real part is negative and of order $O(|\eta'|^2)$ as $\eta' \to 0$.

Theorem 4.7. *There exists a constant $r_0 > 0$ such that if $|\eta'| \le r_0$, then $\sigma(-L_{\eta'}) \cap \{\lambda;\, |\lambda| < \beta_0/2\} = \{\lambda_{\eta'}\}$. Here $\lambda_{\eta'}$ is a simple eigenvalue that satisfies*

$$\lambda_{\eta'} = -\frac{\gamma^2}{\nu} \kappa(\eta') + O(|\eta'|^3) \qquad (\eta' \to 0),$$

where

$$\kappa(\eta') := \sum_{j,k=1}^{n-1} a_{jk} \eta_j \eta_k, \qquad a_{jk} := \frac{1}{|\Omega_{\mathrm{per}}|} (\nabla w_1^{(j)}, \nabla w_1^{(k)}).$$

Here $w_1^{(k)}$ $(k = 1, \ldots, n-1)$ satisfy the Stokes system (3.4) for some $\phi_1^{(k)}$, and $\kappa(\eta') \ge \kappa_0 |\eta'|^2$ with some constant $\kappa_0 > 0$. As a result,

$$\operatorname{Re}\lambda_{\eta'} \le -\frac{\kappa_0}{2} \frac{\gamma^2}{\nu} |\eta'|^2.$$

Remark 4.8. A similar result holds for $L^*_{\eta'}$ with simple eigenvalue $\lambda^*_{\eta'} = \overline{\lambda}_{\eta'}$.

Remark 4.9. Since $\lambda_{\eta'} \to 0$ as $\eta' \to 0$, we see that for any $\beta \in (0, \beta_0/2)$, there exists a constant $r = r(\beta) > 0$ such that if $|\eta'| \le r(\beta)$, then $|\lambda_{\eta'}| < \beta$ and $\{\lambda; \operatorname{Re}\lambda \ge -3\beta_0/4\} \cap \{\lambda; |\lambda| \ge \beta\} \subset \rho(-L_{\eta'})$.

Proof of Theorem 4.7. In view of Proposition 4.3, (4.10) and (4.11), we can apply analytic perturbation theory to see that $\sigma(-L_{\eta'}) \cap \{\lambda; |\lambda| < \beta_0/2\}$ consists of a simple eigenvalue, say $\lambda_{\eta'}$, for sufficiently small η', and that $\lambda_{\eta'}$ is expanded as

$$\lambda_{\eta'} = \sum_{j=0}^{n-1} \lambda_j^{(1)} \eta_j + \sum_{j,k=0}^{n-1} \lambda_{jk}^{(2)} \eta_j \eta_k + O(|\eta'|^3)$$

with

$$\lambda_j^{(1)} := -(L_j^{(1)} u^{(0)}, u^{(0)*}),$$
$$\lambda_{jk}^{(2)} := -\tfrac{1}{2}((L_{jk}^{(2)} + L_{kj}^{(2)})u^{(0)}, u^{(0)*}) + \tfrac{1}{2}((L_j^{(1)} S L_k^{(1)} + L_k^{(1)} S L_j^{(1)})u^{(0)}, u^{(0)*}).$$

Here $S = S_\lambda|_{\lambda=0}$. See, e.g., [12, Chap. VII], [21, Chap. XII].

Let us compute $\lambda_j^{(1)}$. Since $(u, u^{(0)*}) = [\![\phi]\!]$ for $u = {}^\top(\phi, w)$ and $L_j^{(1)} u^{(0)} = {}^\top(0, i\gamma \boldsymbol{e}_j)$, we have $\lambda_j^{(1)} = 0$.

As for $\lambda_{jk}^{(2)}$, since $L_{jk}^{(2)} u^{(0)} = 0$, we have $[\![L_{jk}^{(2)} u^{(0)}]\!] = 0$. Furthermore,

$$\tfrac{1}{2}((L_j^{(1)} S L_k^{(1)} + L_j^{(1)} S L_k^{(1)})u^{(0)}, u^{(0)*}) = (L_j^{(1)} S L_k^{(1)} u^{(0)}, u^{(0)*}) = [\![L_j^{(1)} S L_k^{(1)} u^{(0)}]\!].$$

We compute $[\![L_j^{(1)} S L_k^{(1)} u^{(0)}]\!]$. Set $u_1 = {}^\top(\phi_1, w_1) := S L_k^{(1)} u_0$. Then u_1 is a solution of

$$L_0 u_1 = (I - \Pi^{(0)}) L_k^{(1)} u^{(0)} = L_k^{(1)} u^{(0)}, \qquad [\![\phi_1]\!] = 0,$$

that is,

$$\begin{cases} \gamma \operatorname{div} w_1 = 0, \\ -\nu \Delta w_1 + \gamma \nabla \phi_1 = i\gamma \boldsymbol{e}_k, \\ w_1|_{\Sigma_{j,+}} = w_1|_{\Sigma_{j,-}}, \quad \phi_1|_{\Sigma_{j,+}} = \phi_1|_{\Sigma_{j,-}}, \quad w_1|_{\Sigma_n} = 0, \quad [\![\phi_1]\!] = 0. \end{cases}$$

Lemma 4.5 implies that for each $k = 1, \dots, n-1$, there exists a unique solution $\tilde{u}_1^{(k)} = {}^\top(\tilde{\phi}_1^{(k)}, \tilde{w}_1^{(k)})$ of this system. Let $u_1^{(k)} = {}^\top(\phi_1^{(k)}, w_1^{(k)})$ be the unique solution of (3.4). Then $\tilde{\phi}_1^{(k)} = i\phi_1^{(k)}$ and $\tilde{w}_1^{(k)} = \frac{i\gamma}{\nu} w_1^{(k)}$, and hence

$$L_j^{(1)} S L_k^{(1)} u^{(0)} = i \begin{pmatrix} 0 & \gamma {}^\top\boldsymbol{e}_j \\ \gamma \boldsymbol{e}_j & -2\nu\partial_{x_j} - \tilde{\nu}\boldsymbol{e}_j \operatorname{div} - \tilde{\nu}\nabla({}^\top\boldsymbol{e}_j) \end{pmatrix} \begin{pmatrix} \tilde{\phi}_1^{(k)} \\ \tilde{w}_1^{(k)} \end{pmatrix} = -\frac{\gamma^2}{\nu} \begin{pmatrix} \boldsymbol{e}_j \cdot w_1^{(k)} \\ * \end{pmatrix}.$$

It then follows that

$$\begin{aligned}\lambda_{jk}^{(2)} &= [\![L_j^{(1)} S L_k^{(1)} u^{(0)}]\!] = -\frac{\gamma^2}{\nu}[\![\boldsymbol{e}_j \cdot w_1^{(k)}]\!] \\ &= -\frac{\gamma^2}{\nu}[\![(-\Delta w_1^{(j)} + \nabla \phi_1^{(j)}) \cdot w_1^{(k)}]\!] = -\frac{\gamma^2}{\nu}\frac{1}{|\Omega_{\mathrm{per}}|}(\nabla w_1^{(j)}, \nabla w_1^{(k)}).\end{aligned}$$

Let us show that the matrix $((\nabla w_1^{(j)}, \nabla w_1^{(k)}))_{j,k=1}^{n-1}$ is positive definite. We first observe that $w_1^{(1)}, \dots, w_1^{(n-1)}$ are linearly independent. In fact, suppose that $w_1 = \sum_{j=1}^{n-1} c_j w_1^{(j)} = 0$. Then $\phi_1 = \sum_{j=1}^{n-1} c_j \phi_1^{(j)}$ satisfies $\nabla \phi_1 = \sum_{j=1}^{n-1} c_j \boldsymbol{e}_j$. Therefore, ϕ_1 can be written as $\phi_1 = c + \sum_{j=1}^{n-1} c_j x_j$ with some constant c. Since ϕ_1 is Q-periodic in $x' = (x_1, \dots, x_{n-1})$ and $[\![\phi_1]\!] = 0$, we see that $c = c_1 = \cdots = c_{n-1} = 0$. We thus conclude that $w_1^{(1)}, \dots, w_1^{(n-1)}$ are linearly independent.

Set $V := \mathrm{span}\{w_1^{(1)}, \dots, w_1^{(n-1)}\}$ and take an orthonormal basis $\{f_1, \dots, f_{n-1}\}$ of V as a subspace of $H^1_{0,\mathrm{per}}(\Omega_{\mathrm{per}})$ with respect to the inner product $(w, v)_{H^1_{0,\mathrm{per}}} := (\nabla w, \nabla v)$. Then $w_1^{(m)}$ can be written as $w_1^{(m)} = \sum_{k=1}^{n-1} b_{mk} f_k$ for $m = 1, \dots, n-1$, and thus $(w_1^{(1)}, \dots, w_1^{(n-1)}) = (f_1, \dots, f_{n-1})B$, where $B = (\boldsymbol{b}_1, \dots, \boldsymbol{b}_{n-1})$ with $\boldsymbol{b}_m = {}^\top(b_{m1}, \dots, b_{m,n-1})$. It then follows that $\boldsymbol{b}_1, \dots, \boldsymbol{b}_{n-1}$ are linearly independent. We have $(\nabla w_1^{(m)}, \nabla w_1^{(l)}) = (w_1^{(m)}, w_1^{(l)})_{H^1_{0,\mathrm{per}}} = (BB^*)_{ml}$. Since BB^* is positive definite, so is the matrix $((\nabla w_1^{(m)}, \nabla w_1^{(l)}))_{j,k=1}^{n-1}$. It then follows that there is a constant $\kappa_0 > 0$ such that

$$\sum_{j,k=1}^{n-1} \lambda_{jk}^{(2)} \eta_j \eta_k = -\sum_{j,k=1}^{n-1} \frac{\gamma^2}{\nu}\frac{1}{|\Omega_{\mathrm{per}}|}(\nabla w_1^{(j)}, \nabla w_1^{(k)})\eta_j\eta_k = -\frac{\gamma^2}{\nu}|B^*\eta'|^2 \le -\kappa_0 \frac{\gamma^2}{\nu}|\eta'|^2$$

for all $\eta' \in \mathbb{R}^{n-1}$. Therefore, there exists $r_0 > 0$ such that if $|\eta'| \le r_0$, then

$$\mathrm{Re}\,\lambda_{\eta'} \le -\frac{\kappa_0}{2}\frac{\gamma^2}{\nu}|\eta'|^2. \qquad \square$$

Let $\Pi_{\eta'}$ be the eigenprojection for the eigenvalue $\lambda_{\eta'}$. Since $\Pi_{\eta'} e_{\eta'}^{-tL} = e^{\lambda_{\eta'} t}\Pi_{\eta'}$, we have the following estimate.

Theorem 4.10. *If $|\eta'| \le r_0$, then*

$$\| e^{-tL_{\eta'}} u_0 - e^{\lambda_{\eta'} t}\Pi_{\eta'} u_0 \|_2 \le C e^{-\frac{\beta_0}{2}t}\|u_0\|_2.$$

Theorem 4.10 follows from Theorems 4.6 and 4.7. See, e.g., [3, Chap. V, Theorem 1.11], [24].

We close this subsection with estimates for the eigenprojections $\Pi_{\eta'}$ and $\Pi_{\eta'}^*$ for the eigenvalues $\lambda_{\eta'}$ and $\lambda_{\eta'}^*$ $(= \overline{\lambda}_{\eta'})$ of $-L_{\eta'}$ and $-L_{\eta'}^*$, respectively.

Theorem 4.11. *For any nonnegative integer k, there exists a constant $r_k > 0$ such that the following estimates hold uniformly for $|\eta'| \le r_k$:*

(i) $\|\Pi_{\eta'} u\|_{H^k} \le C\|u\|_1$.

(ii) $\|(\Pi_{\eta'} - \Pi^{(0)})u\|_{H^k} \le C|\eta'|\,\|u\|_1$.

The same conclusion holds with $\Pi_{\eta'}$ replaced by $\Pi^*_{\eta'}$.

Proof. By Theorem 4.6 we have

$$\Pi_{\eta'} = \frac{1}{2\pi i}\int_{|\lambda|=\beta_0/2} (\lambda + L_{\eta'})^{-1}\, d\lambda, \qquad \Pi^*_{\eta'} = \frac{1}{2\pi i}\int_{|\lambda|=\beta_0/2} (\lambda + L^*_{\eta'})^{-1}\, d\lambda.$$

Furthermore, $u_{\eta'} = \Pi_{\eta'} u^{(0)}$ and $u^*_{\eta'} = \Pi^*_{\eta'} u^{(0)^*}$ are eigenfunctions of $-L_{\eta'}$ and $-L^*_{\eta'}$ for the eigenvalues $\lambda_{\eta'}$ and $\lambda^*_{\eta'} = \overline{\lambda_{\eta'}}$, respectively; and

$$\Pi_{\eta'} u = \frac{(u, u^*_{\eta'})}{(u_{\eta'}, u^*_{\eta'})} u_{\eta'}.$$

Note that $u_{\eta'}|_{\eta'=0} = \Pi^{(0)} u^{(0)} = u^{(0)}$ and $u^*_{\eta'}|_{\eta'=0} = \Pi^{(0)*} u^{(0)*} = u^{(0)*}$.

In view of (4.9)–(4.11), we see that $(\lambda + L_{\eta'})^{-1}$ can be expanded as

$$(\lambda + L_{\eta'})^{-1} = (\lambda + L_0)^{-1} - (\lambda + L_0)^{-1} \sum_{j=1}^{n-1} \eta_j L_j^{(1)} (\lambda + L_0)^{-1} + R_{\eta'}(\lambda)$$

and

$$\|R_{\eta'}(\lambda) f\|_2 \le C|\eta'|^2 \|f\|_2, \qquad \|\nabla \tilde{\mathbb{P}} R_{\eta'}(\lambda) f\|_2 \le C|\eta'|^2 \|f\|_2$$

uniformly for $|\eta'| \le r_0$ and $|\lambda| = \beta_0/2$. We write

$$\begin{aligned} u_{\eta'} &= u^{(0)} + \frac{1}{2\pi i}\int_{|\lambda|=\beta_0/2} \Big(-\sum_{j=1}^{n-1} \eta_j (\lambda + L_0)^{-1} L_j^{(1)} (\lambda + L_0)^{-1} u^{(0)} \Big)\, d\lambda \\ &\quad + \frac{1}{2\pi i}\int_{|\lambda|=\beta_0/2} R_{\eta'}(\lambda) u^{(0)}\, d\lambda \\ &=: u^{(0)} + \sum_{j=1}^{n-1} \eta_j u_j^{(1)} + u^{(2)}. \end{aligned}$$

Using (4.9)–(4.11), we have

$$\|u_j^{(1)}\|_2 + \|\nabla \tilde{\mathbb{P}} u_j^{(1)}\|_2 \le C, \qquad \|u^{(2)}\|_2 + \|\nabla \tilde{\mathbb{P}} u^{(2)}\|_2 \le C|\eta'|^2.$$

Similarly,

$$u^*_{\eta'} = u^{(0)*} + \sum_{j=1}^{n-1} \eta_j u_j^{(1)*} + u^{(2)*},$$

with estimates

$$\|u_j^{(1)*}\|_2 + \|\nabla\tilde{\mathbb{P}}u_j^{(1)*}\|_2 \le C, \qquad \|u^{(2)*}\|_2 + \|\nabla\tilde{\mathbb{P}}u^{(2)*}\|_2 \le C|\eta'|^2.$$

It then follows that

$$(u_{\eta'}, u_{\eta'}^*) = (u^{(0)}, u^{(0)*}) + (u_{\eta'} - u^{(0)}, u_{\eta'}^*) + (u^{(0)}, u_{\eta'}^* - u^{(0)*}) \ge 1 - C|\eta'| \ge 1/2$$

for $|\eta'| \le r_0$ with $r_0 > 0$ replaced by a smaller one if necessary.

If we had the estimates $\|u_{\eta'}^*\|_\infty \le C$ and $\|\partial_x^\alpha u_{\eta'}\|_2 \le C$, then it would follow that $\|\partial_x^\alpha \Pi_{\eta'} u\|_2 \le C\|u\|_1\|u_{\eta'}^*\|_\infty\|\partial_x^\alpha u_{\eta'}\|_2 \le C\|u\|_1$. So we will deduce the estimates for $u_{\eta'}$ and $u_{\eta'}^*$, in other words, for $(\lambda + L_{\eta'})^{-1}u^{(0)}$ and $(\lambda + L_{\eta'}^*)^{-1}u^{(0)*}$.

In the remainder of the proof we only consider $(\lambda + L_{\eta'})^{-1}u^{(0)}$ since $(\lambda + L_{\eta'}^*)^{-1}u^{(0)*}$ can be estimated similarly. We also observe that the integral path of $u_{\eta'} = \frac{1}{2\pi i}\int_{|\lambda|=\beta_0/2}(\lambda + L_{\eta'})^{-1}u^{(0)}\,d\lambda$ can be deformed into $\{|\lambda| = \beta\} \subset \rho(-L_{\eta'})$.

We claim the following

Proposition 4.12. *Let m be a nonnegative integer. Then there exist constants $r_m > 0$ and $\beta_m > 0$ such that if $|\eta'| \le r_m$ and $\beta_m/2 \le |\lambda| \le \beta_m$, then we have $(\lambda + L_{\eta'})^{-1}u^{(0)} \in H^{m+1}_{\mathrm{per}}(\Omega_{\mathrm{per}}) \times (H^{m+2}_{\mathrm{per}} \cap H^1_{0,\mathrm{per}})(\Omega_{\mathrm{per}})$, and*

$$\|(\lambda + L_{\eta'})^{-1}u^{(0)}\|_{H^{m+1}\times H^{m+2}} \le C$$

uniformly for $|\eta'| \le r_m$ and $\beta_m/2 \le |\lambda| \le \beta_m$.

To prove Proposition 4.12, we employ the following lemma.

Lemma 4.13. *Let m be a nonnegative integer. Then there exists $\tilde{\beta}_m > 0$ such that if $|\lambda| \le \tilde{\beta}_m$, then $S_\lambda f \in H^{m+1}_{*,\mathrm{per}}(\Omega_{\mathrm{per}}) \times (H^{m+2}_{\mathrm{per}} \cap H^1_{0,\mathrm{per}})(\Omega_{\mathrm{per}})$ for any $f \in H^{m+1}_{*,\mathrm{per}}(\Omega_{\mathrm{per}}) \times H^m_{\mathrm{per}}(\Omega_{\mathrm{per}})$, and*

$$\|S_\lambda f\|_{H^{m+1}\times H^{m+2}} \le C\|f\|_{H^{m+1}\times H^m}$$

uniformly for λ with $|\lambda| \le \tilde{\beta}_m$.

The proof of Lemma 4.13 will be given later.

Proof of Proposition 4.12. We argue by induction on m. We denote $u := (\lambda + L_{\eta'})^{-1}u^{(0)}$. By Theorems 4.6 and 4.7, we have $\|u\|_{L^2\times H^1} \le C$ uniformly for $|\eta'| \le r_0$ and $\beta_0/4 \le |\lambda| \le \beta_0/2$ with r_0 replaced by a smaller one if necessary.

We write $(\lambda + L_0)u = u^{(0)} - M_{\eta'}u$ and decompose $u = {}^\top(\phi, w)$ as $u = [\![\phi]\!]u^{(0)} + u_1$, where $\Pi^{(0)}u = [\![\phi]\!]u^{(0)}$ and $u_1 = (I - \Pi^{(0)})u$. Then

$$[\![\phi]\!] = \frac{1}{\lambda}\{1 - i\gamma[\![\eta' \cdot w']\!]\}, \qquad (\lambda + L_0)u_1 = -(M_{\eta'}u_1 + [\![\phi]\!]M_{\eta'}u^{(0)} - \Pi^{(0)}M_{\eta'}u).$$

It then follows that

$$|[\![\phi]\!]| = \frac{1}{|\lambda|}\{1 + \gamma r_0\|w\|_2\} \le C \tag{4.12}$$

uniformly for $|\eta'| \le r_0$ and $\beta_1/2 \le |\lambda| \le \beta_1$ with $\beta_1 > 0$ to be determined later. On the other hand, we have

$$\|M_{\eta'}u_1 + [\![\phi]\!]M_{\eta'}u^{(0)} - \Pi^{(0)}M_{\eta'}u\|_{H^1\times L^2} \le C\|u\|_{L^2\times H^1} \le C$$

uniformly for $|\eta'| \le r_0$ and $\beta_0/4 \le |\lambda| \le \beta_0/2$. It then follows from Remark 4.9 and Lemma 4.13 that, with a suitable choice of $r_1, \beta_1 > 0$, the estimate $\|u_1\|_{H^1\times H^2} \le C$ holds uniformly for $|\eta'| \le r_1$ and $\beta_1/2 \le |\lambda| \le \beta_1$. This, together with (4.12), proves Proposition 4.12 for $m = 0$.

Assume that the proposition holds for $m = k$. We will show that it holds for $m = k + 1$. By the inductive assumption, we have

$$\|M_{\eta'}u_1 + [\![\phi]\!]M_{\eta'}u^{(0)} - \Pi^{(0)}M_{\eta'}u\|_{H^{k+2}\times H^{k+1}} \le C\|u\|_{H^{k+1}\times H^{k+2}} \le C$$

uniformly for $|\eta'| \le r_k$ and $\beta_k/2 \le |\lambda| \le \beta_k$. It then follows from Remark 4.9 and Lemma 4.13 that $\|u_1\|_{H^{k+2}\times H^{k+3}} \le C$ uniformly for $|\eta'| \le r_{k+1}$ and $\beta_{k+1}/2 \le |\lambda| \le \beta_{k+1}$. Combining this with (4.12), we conclude that the proposition holds for $m = k + 1$. □

We now continue the proof of Theorem 4.11. Let m be a nonnegative integer. By Proposition 4.12, we see that

$$(\lambda + L_{\eta'})^{-1}u^{(0)} \in H^{m+1}_{\rm per}(\Omega_{\rm per}) \times (H^{m+2}_{\rm per} \cap H^1_{0,\rm per})(\Omega_{\rm per}),$$
$$\|(\lambda + L_{\eta'})^{-1}u^{(0)}\|_{H^{m+1}\times H^{m+2}} \le C$$

uniformly for $|\eta'| \le r_m$ and $|\lambda| = \beta_m$. Deforming the integral path into $\{|\lambda| = \beta_m\}$, we deduce that $u_{\eta'} \in H^{m+1}_{\rm per}(\Omega_{\rm per}) \times H^{m+2}_{\rm per}(\Omega_{\rm per})$ and

$$\|u_{\eta'}\|_{H^{m+1}\times H^{m+2}} = \left\|\frac{1}{2\pi i}\int_{|\lambda|=\beta_m/2}(\lambda + L_{\eta'})^{-1}u^{(0)}\,d\lambda\right\|_{H^{m+1}\times H^{m+2}} \le C. \tag{4.13}$$

Taking $m = k-1$, we have $\|\partial_x^\alpha u_{\eta'}\|_2 \le C$ for $|\alpha| \le k$ and $|\eta'| \le r_k$. Similarly we can obtain (4.13) with $u_{\eta'}$ replaced by $u^*_{\eta'}$, and hence $\|u^*_{\eta'}\|_\infty \le C\|u^*_{\eta'}\|_{H^{[n/2]+1}} \le C$. It then follows that

$$\|\partial_x^\alpha \Pi_{\eta'}u\|_2 \le C\|u^*_{\eta'}\|_\infty\|\partial_x^\alpha u_{\eta'}\|_2\|u\|_1 \le C\|u\|_1.$$

This proves (i).

Let us next consider (ii). We write $\Pi_{\eta'}u - \Pi^{(0)}u$ as

$$\begin{aligned}\Pi_{\eta'}u - \Pi^{(0)}u &= \left(\frac{1}{(u_{\eta'}, u^*_{\eta'})} - 1\right)(u, u^{(0)*})u^{(0)} \\ &\quad + \frac{1}{(u_{\eta'}, u^*_{\eta'})}\{(u, u^*_{\eta'})u_{\eta'} - (u, u^{(0)*})u^{(0)}\} \\ &=: I_1 + I_2.\end{aligned}$$

As for I_1, we have

$$\begin{aligned}|(u_{\eta'}, u^*_{\eta'}) - 1| &= |(u_{\eta'} - u^{(0)}, u^*_{\eta'}) + (u^{(0)}, u^*_{\eta'} - u^{(0)*})| \\ &\le C\{\|u_{\eta'} - u^{(0)}\|_2 + \|u^*_{\eta'} - u^{(0)*}\|_2\}.\end{aligned}$$

Since

$$u_{\eta'} - u^{(0)} = \frac{1}{2\pi i}\int_{|\lambda|=\beta_m}(\lambda + L_0)^{-1}\sum_{N=1}^{\infty}(-1)^N[M_{\eta'}(\lambda + L_0)^{-1}]^N u^{(0)}\,d\lambda,$$

we have $\|u_{\eta'} - u^{(0)}\|_2 \le C|\eta'|$, and likewise $\|u^*_{\eta'} - u^{(0)*}\|_2 \le C|\eta'|$. We thus obtain

$$\|\partial_x^\alpha I_1\|_2 \le C|\eta'|\,|(u, u^{(0)*})\partial_x^\alpha u^{(0)}| \le C|\eta'|\,\|u\|_1\|u^{(0)*}\|_\infty\|\partial_x^\alpha u^{(0)}\|_\infty \le C|\eta'|\,\|u\|_1.$$

As for I_2, we have

$$\begin{aligned}&\|\partial_x^\alpha\{(u, u^*_{\eta'})u_{\eta'} - (u, u^{(0)*})u^{(0)}\}\|_2 \\ &\quad = \|(u, u^*_{\eta'} - u^{(0)*})\partial_x^\alpha u_{\eta'} + (u, u^{(0)*})\partial_x^\alpha(u_{\eta'} - u^{(0)})\|_2 \\ &\quad \le \|u\|_1\|u^*_{\eta'} - u^{(0)*}\|_\infty\|\partial_x^\alpha u_{\eta'}\|_2 + \|u\|_1\|u^{(0)*}\|_\infty\|\partial_x^\alpha(u_{\eta'} - u^{(0)})\|_2 \\ &\quad \le C\|u\|_1\{\|u^*_{\eta'} - u^{(0)*}\|_{H^{[n/2]+1}}\|u_{\eta'}\|_{H^k} + \|u_{\eta'} - u^{(0)}\|_{H^k}\}.\end{aligned}$$

Since $\|M_{\eta'}u\|_{H^k\times H^{k-1}} \le C|\eta'|\,\|u\|_{H^{k-1}\times H^k}$, with the aid of Lemma 4.13 we see that

$$\|(\lambda + L_0)^{-1}[M_{\eta'}(\lambda + L_0)^{-1}]^N u^{(0)}\|_{H^k\times H^k} \le (C|\eta'|)^N$$

uniformly for $|\eta'| \le r_k$ and $|\lambda| = \beta_k$. Taking $r_k > 0$ smaller if necessary, we obtain $\|u_{\eta'} - u^{(0)}\|_{H^k} \le C|\eta'|$. Similarly, we can obtain $\|u^*_{\eta'} - u^{(0)*}\|_{H^{[n/2]+1}} \le C|\eta'|$. It then follows that $\|\partial_x^\alpha I_2\|_2 \le C|\eta'|\,\|u\|_1$ for $|\alpha| \le k$. We thus conclude that

$$\|\partial_x^\alpha(\Pi_{\eta'} - \Pi^{(0)})u\|_2 \le C|\eta'|\,\|u\|_1$$

for $|\alpha| \le k$. □

In the remainder of this subsection we prove Lemma 4.13.

Proof of Lemma 4.13. We set $\check{u} := S_\lambda f$. Then

$$L_0\check{u} = f - \lambda\check{u},$$

which is regarded as an inhomogeneous Stokes system. This can be solved for $\check{u}$ if $|\lambda|$ is suitably small. In fact, let $f \in H^{m+1}_{*,\mathrm{per}}(\Omega_{\mathrm{per}}) \times H^m_{\mathrm{per}}(\Omega_{\mathrm{per}})$. Then, for each $\check{v} \in H^{m+1}_{*,\mathrm{per}}(\Omega_{\mathrm{per}}) \times (H^{m+2}_{\mathrm{per}} \cap H^1_{0,\mathrm{per}})(\Omega_{\mathrm{per}})$, there exists a unique $\check{u} \in H^{m+1}_{*,\mathrm{per}}(\Omega_{\mathrm{per}}) \times (H^{m+2}_{\mathrm{per}} \cap H^1_{0,\mathrm{per}})(\Omega_{\mathrm{per}})$ such that $L_0 \check{u} = f - \lambda \check{v}$ and

$$\|\check{u}\|_{H^{m+1}\times H^{m+2}} \le C|\lambda|\,\|\check{v}\|_{H^{m+1}\times H^{m+2}} + C\|f\|_{H^{m+1}\times H^m}.$$

See, e.g., [23, Chap. III, Theorem 1.5.3]. This estimate shows that the map $\check{v} \mapsto \check{u}$ is a contraction on $H^{m+1}_{*,\mathrm{per}}(\Omega_{\mathrm{per}}) \times (H^{m+2}_{\mathrm{per}} \cap H^1_{0,\mathrm{per}})(\Omega_{\mathrm{per}})$ when $|\lambda| \le \tilde{\beta}_m$ with suitably small $\tilde{\beta}_m$. □

§4.3. The case $|\eta'| \ge r_0$

In this subsection we investigate the spectrum of $-L_{\eta'}$ for $\eta' \in Q^*$ with $|\eta'| \ge r_0$. We have already shown in Proposition 4.2 that $-L_{\eta'}$ generates a contraction semigroup $e^{-tL_{\eta'}}$. We will show that $e^{-tL_{\eta'}}$ has an exponential decay estimate, uniformly for $\eta' \in Q^*$ with $|\eta'| \ge r_0$.

We first introduce an inner product of $H^1_{0,\mathrm{per}}(\Omega_{\mathrm{per}})$ in terms of $\nabla_{\eta'}$.

Proposition 4.14. *Let $\eta' \in Q^*$. Then $(\nabla_{\eta'} w, \nabla_{\eta'} v)$ defines an inner product in $H^1_{0,\mathrm{per}}(\Omega_{\mathrm{per}})$. Furthermore, $\|\nabla_{\eta'} w\|_2$ is equivalent to $\|w\|_{H^1}$ for $w \in H^1_{0,\mathrm{per}}(\Omega_{\mathrm{per}})$, and the estimate*

$$C^{-1}\|w\|_{H^1} \le \|\nabla_{\eta'} w\|_2 \le C\|w\|_{H^1}$$

holds uniformly for $\eta' \in Q^*$ and $w \in H^1_{0,\mathrm{per}}(\Omega_{\mathrm{per}})$.

Proof. It suffices to show that $\|\nabla_{\eta'} w\|_2^2 = (\nabla_{\eta'} w, \nabla_{\eta'} w)$ is equivalent to $\|w\|^2_{H^1}$ for $w \in H^1_{0,\mathrm{per}}(\Omega_{\mathrm{per}})$. Let $w \in H^1_{0,\mathrm{per}}(\Omega_{\mathrm{per}})$. Then by the Poincaré inequality,

$$\|w\|_{H^1} \le C\|\nabla w\|_2 \le C'(\|\nabla'_{\eta'} w\|_2^2 + \|\partial_{x_n} w\|_2^2)^{1/2} = C'\|\nabla_{\eta'} w\|_2 \le C''\|w\|_{H^1}. \qquad \square$$

Before going further, we introduce some notation. We denote

$$\nabla'_{\eta'} := \nabla' + i\eta'.$$

Here ∇' denotes the gradient with respect to $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. We note that $\Delta_{\eta'} = \nabla'_{\eta'} \cdot \nabla'_{\eta'} + \partial^2_{x_n}$. We define

$$D_{\eta'}(w) := \nu\|\nabla_{\eta'} w\|_2^2 + \tilde{\nu}\|\nabla_{\eta'} \cdot w\|_2^2 = \nu\|\nabla'_{\eta'} w\|_2^2 + \nu\|\partial_{x_n} w\|_2^2 + \tilde{\nu}\|\nabla'_{\eta'} \cdot w' + \partial_{x_n} w^n\|_2^2.$$

In what follows we denote the projection $I - \Pi^{(0)}$ by Π_1:

$$\Pi_1 := I - \Pi^{(0)}.$$

Recall that $X_0 = \Pi^{(0)} X$ and $X_1 = \Pi_1 X$.

To study problem (4.5) for $\eta' \in Q^*$ with $|\eta'| \geq r_0$, we decompose u into its $\Pi^{(0)}$-part and Π_1-part in X, namely,

$$u = \sigma u^{(0)} + u_1, \tag{4.14}$$

where $\sigma = (u, u^{(0)*}) = [\![\phi]\!] \in \mathbb{C}$, $u^{(0)} = {}^\top(1,0)$ and $u_1 = {}^\top(\phi_1, w_1) \in X_1$. We note that

$$[\![\phi_1]\!] = 0. \tag{4.15}$$

It is easy to see that problem (4.5) reduces to

$$\begin{cases} \lambda\sigma + i\gamma[\![\tilde{\eta}' \cdot w_1]\!] = [\![f^0]\!], \\ \lambda u_1 + L_{\eta'} u_1 - \Pi^{(0)} M_{\eta'} u_1 + M_{\eta'}(\sigma u^{(0)}) = f_1, \end{cases} \tag{4.16}$$

where $\sigma \in \mathbb{C}$, $u_1 = {}^\top(\phi_1, w_1) \in D(L_{\eta'}) \cap X_1$ and $f_1 = \Pi_1 f := {}^\top(f_1^0, \tilde{f}_1) \in X_1$. We observe that

$$\Pi^{(0)} M_{\eta'} u_1 = i\gamma[\![\tilde{\eta}' \cdot w_1]\!] u^{(0)}, \qquad M_{\eta'}(\sigma u^{(0)}) = {}^\top(0, i\gamma\tilde{\eta}'\sigma).$$

We begin with the following

Proposition 4.15. *We have*

$$\operatorname{Re}\lambda(|\sigma|^2 + |u_1|^2) + D_{\eta'}(w_1) = \operatorname{Re}\{[\![f^0]\!]\overline{\sigma} + (f_1, u_1)\}. \tag{4.17}$$

Proof. Multiplying the first equation of (4.16) by $\overline{\sigma}$, we get

$$\lambda|\sigma|^2 + i\gamma[\![\tilde{\eta}' \cdot w_1]\!]\overline{\sigma} = [\![f^0]\!]\overline{\sigma}.$$

Taking the inner product of the second equation of (4.16) with u_1, we have

$$\lambda\|u_1\|_2^2 + (L_{\eta'}u_1, u_1) + (M_{\eta'}(\sigma u^{(0)}), u_1) - (\Pi^{(0)} M_{\eta'} u_1, u_1) = (f_1, u_1).$$

We add these two equations to obtain

$$\begin{aligned} \lambda(|\sigma|^2 + \|u_1\|_2^2) &+ (L_{\eta'}u_1, u_1) + i\gamma[\![\tilde{\eta}' \cdot w_1]\!]\overline{\sigma} \\ &+ (M_{\eta'}(\sigma u^{(0)}), u_1) - (\Pi^{(0)} M_{\eta'} u_1, u_1) = [\![f^{(0)}]\!]\overline{\sigma} + (f_1, u_1). \end{aligned}$$

Since $\operatorname{Re}(L_{\eta'}u_1, u_1) = D_{\eta'}(w_1)$ and

$$\operatorname{Re}\{i\gamma[\![\tilde{\eta}' \cdot w_1]\!]\overline{\sigma} + (M_{\eta'}(\sigma u^{(0)}), u_1)\} = \operatorname{Re}\{2i \operatorname{Im}(i\gamma[\![\tilde{\eta}' \cdot w_1]\!]\overline{\sigma})\} = 0,$$

and

$$(\Pi^{(0)} M_{\eta'} u_1, u_1) = (i\gamma[\![\tilde{\eta}' \cdot w_1]\!], \phi_1) = i\gamma[\![\tilde{\eta}' \cdot w_1]\!][\![\overline{\phi}_1]\!] = 0,$$

we obtain the desired conclusion. □

For later use, we next derive an estimate for λw_1.

Proposition 4.16. *We have*

$$\operatorname{Re}\overline{\lambda}\, D_{\eta'}(w_1) + |\lambda|^2\|w_1\|_2^2 \le C\{\|f_1\|_2^2 + \|w_1\|_2^2 + |[\![f^0]\!]|^2 + \|\nabla_{\eta'} w_1\|_2^2\}. \tag{4.18}$$

Proof. We write the second equation of (4.16) as

$$\lambda\phi_1 + \gamma \operatorname{div}_{\eta'} w_1 = f_1^0, \tag{4.19}$$

$$\lambda w_1 - \nu\Delta_{\eta'} w_1 - \tilde{\nu}\nabla_{\eta'}(\nabla_{\eta'}\cdot w_1) + \gamma\nabla_{\eta'}\phi_1 + i\gamma\sigma\tilde{\eta}' = \tilde{f}_1. \tag{4.20}$$

Here we define

$$\operatorname{div}_{\eta'} w := \nabla_{\eta'}\cdot w - i[\![\tilde{\eta}'\cdot w]\!].$$

We take the inner product of (4.20) with λw_1. Then the real part of the resulting equation yields

$$|\lambda|^2\|w_1\|_2^2 + \operatorname{Re}\overline{\lambda}\, D_{\eta'}(w_1) = \operatorname{Re}\{\gamma\overline{\lambda}(\phi_1, \nabla_{\eta'}\cdot w_1) - i\overline{\lambda}\gamma\sigma[\![\tilde{\eta}'\cdot\overline{w_1}]\!] + \overline{\lambda}(\tilde{f}_1, w_1)\}.$$

Equation (4.19) gives $\phi = \frac{1}{\lambda}f_1^0 - \frac{\gamma}{\lambda}\operatorname{div}_{\eta'} w_1$, and hence

$$\begin{aligned}&\operatorname{Re}|\lambda|^2\|w_1\|_2^2 + \operatorname{Re}\overline{\lambda}\, D_{\eta'}(w_1)\\ &= \operatorname{Re}\left\{\frac{\gamma\overline{\lambda}}{\lambda}(f_1^0, \nabla_{\eta'}\cdot w_1) - \frac{\overline{\lambda}}{\lambda}\gamma^2(\operatorname{div}_{\eta'} w_1, \nabla_{\eta'}\cdot w_1) - i\overline{\lambda}\gamma\sigma[\![\tilde{\eta}'\cdot\overline{w_1}]\!] + \overline{\lambda}(\tilde{f}_1, w_1)\right\}.\end{aligned} \tag{4.21}$$

By the first equation of (4.16), we have $\sigma = \frac{1}{\lambda}[\![f^0]\!] - \frac{i\gamma}{\lambda}[\![\tilde{\eta}', w_1]\!]$. Therefore,

$$\begin{aligned}\text{R.H.S. of (4.21)} &= \operatorname{Re}\left\{\frac{\gamma\overline{\lambda}}{\lambda}(f_1^0, \nabla_{\eta'}\cdot w_1) - \frac{\overline{\lambda}}{\lambda}\gamma^2(\operatorname{div}_{\eta'} w_1, \nabla_{\eta'}\cdot w_1) - i\gamma\frac{\overline{\lambda}}{\lambda}[\![f^0]\!][\![\tilde{\eta}'\cdot\overline{w_1}]\!]\right.\\ &\qquad\left. - \frac{\overline{\lambda}}{\lambda}\gamma^2|[\![\tilde{\eta}'\cdot w_1]\!]|^2 + \overline{\lambda}(\tilde{f}_1, w_1)\right\}\\ &\le \epsilon|\lambda|^2\|w_1\|_2^2 + C\{\|\nabla_{\eta'} w_1\|_2^2 + |[\![f^0]\!]|^2 + \|f_1^0\|_2^2 + (1/\epsilon)\|\tilde{f}_1\|_2^2\}\end{aligned}$$

for any $\epsilon > 0$, where C is a positive constant independent of ϵ. Taking ϵ suitably small, we see that

$$|\lambda|^2\|w_1\|_2^2 + \operatorname{Re}\overline{\lambda}\, D_{\eta'}(w_1) \le C_1\{\|f_1\|_2^2 + |[\![f^0]\!]|^2 + \|\nabla_{\eta'} w_1\|_2^2\}. \qquad \square$$

We next derive a coercive estimate for σ.

Proposition 4.17. *We have*

$$\begin{aligned}&\operatorname{Re}\lambda|\sigma|^2 + \frac{c_2\gamma^2}{2\nu}|\eta'|^2|\sigma|^2\\ &\qquad \le C\{(1 + 1/|\eta'|^2)|[\![f^0]\!]|^2 + \|\tilde{f}_1\|_2^2 + |\lambda|^2\|w_1\|_2^2 + D_{\eta'}(w_1)\},\end{aligned} \tag{4.22}$$

where c_2 is a positive constant independent of γ, ν and $\eta' \in Q^*$.

To prove Proposition 4.17, we prepare several lemmas.

Lemma 4.18. *Let $f^0 \in L^2_{*,\mathrm{per}}(\Omega_{\mathrm{per}})$ and let $\tilde{f} \in H^{-1}_{\mathrm{per}}(\Omega_{\mathrm{per}})$. Then there exists a unique ${}^\top(\phi, w) \in L^2_{*,\mathrm{per}}(\Omega_{\mathrm{per}}) \times H^1_{0,\mathrm{per}}(\Omega_{\mathrm{per}})$ satisfying*

$$
\begin{cases} \mathrm{div}_{\eta'} w = f^0, \\ -\Delta_{\eta'} w + \nabla_{\eta'} \phi = \tilde{f}, \\ \phi|_{\Sigma_{j,+}} = \phi|_{\Sigma_{j,-}}, \quad w|_{\Sigma_{j,+}} = w|_{\Sigma_{j,-}}, \quad w|_{\Sigma_n} = 0. \end{cases} \tag{4.23}
$$

Furthermore,

$$
\|\phi\|_2 + \|\nabla_{\eta'} w\|_2 \le C\{\|f^0\|_2 + \|\tilde{f}\|_{H^{-1}_{\mathrm{per}}(\Omega_{\mathrm{per}})}\}.
$$

Lemma 4.18 can be proved in a similar manner to [23, Chap. III, proof of Theorem 1.4.1]. An outline of the proof of Lemma 4.18 will be given in Section 6.

Setting $f^0 = 0$ and $\tilde{f} = \boldsymbol{e}_k$ in Lemma 4.18, we have the following

Lemma 4.19. *Let ${}^\top(\phi^{(1)}_{1,k,\eta'}, w^{(1)}_{1,k,\eta'}) \in L^2_{*,\mathrm{per}}(\Omega_{\mathrm{per}}) \times H^1_{0,\mathrm{per}}(\Omega_{\mathrm{per}})$ be the pair of functions satisfying*

$$
\begin{cases} \mathrm{div}_{\eta'} w^{(1)}_{1,k,\eta'} = 0, \\ -\Delta_{\eta'} w^{(1)}_{1,k,\eta'} + \nabla_{\eta'} \phi^{(1)}_{1,k,\eta'} = \boldsymbol{e}_k, \\ \phi_{1,k,\eta'}|_{\Sigma_{j,+}} = \phi_{1,k,\eta'}|_{\Sigma_{j,-}}, \quad w^{(1)}_{1,k,\eta'}|_{\Sigma_{j,+}} = w^{(1)}_{1,k,\eta'}|_{\Sigma_{j,-}}, \quad w_{1,k,\eta'}|_{\Sigma_n} = 0. \end{cases} \tag{4.24}
$$

Then there exists a constant $C > 0$ such that

$$
\|w^{(1)}_{1,k,\eta'}\|_{H^1} + \|\phi^{(1)}_{1,k,\eta'}\|_2 \le C \qquad (k = 1, \dots, n-1)
$$

uniformly for $\eta' \in Q^*$ with $|\eta'| \ge r_0$.

Lemma 4.20. *For each $k = 1, \dots, n-1$, let ${}^\top(\phi^{(1)}_{1,k,\eta'}, w^{(1)}_{1,k,\eta'}) \in L^2_{*,\mathrm{per}}(\Omega_{\mathrm{per}}) \times H^1_{0,\mathrm{per}}(\Omega_{\mathrm{per}})$ be the pair of functions satisfying (4.24). Then $w^{(1)}_{1,1,\eta'}, \dots, w^{(1)}_{1,n-1,\eta'}$ are linearly independent.*

Proof. Let $w := c_1 w^{(1)}_{1,1,\eta'} + \cdots + c_{n-1} w^{(1)}_{1,n-1,\eta'} = 0$. It then follows from (4.24) that $\nabla_{\eta'} \tilde{\phi} = \sum_{j=1}^{n-1} c_j \boldsymbol{e}_j$. Here $\tilde{\phi} := c_1 \phi^{(1)}_{1,1,\eta'} + \cdots + c_{n-1} \phi^{(1)}_{1,n-1,\eta'}$. Since $|\eta'| \ge r_0$, there exists j such that $\eta_j \ne 0$. For this η_j, since $(\partial_{x_j} + i\eta_j)\tilde{\phi} = c_j$, we have $\partial_{x_j}(e^{i\eta_j x_j}\tilde{\phi}) = c_j e^{i\eta_j x_j}$. This implies that there exists a function $a(\check{x}_j)$ ($\check{x}_j = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n-1})$) such that

$$
e^{i\eta_j x_j}\tilde{\phi} = a(\check{x}_j) + \frac{c_j}{i\eta_j} e^{i\eta_j x_j},
$$

that is,

$$
\tilde{\phi} = a(\check{x}_j) e^{-i\eta_j x_j} + \frac{c_j}{i\eta_j}.
$$

Since $\tilde{\phi}$ is Q-periodic in x', we see that $a(\check{x}_j)$ is Q-periodic and $a(\check{x}_j)e^{-i(\pi/\alpha_j)\eta_j} = a(\check{x}_j)e^{i(\pi/\alpha_j)\eta_j}$. Since $0 < |\eta_j| \le \alpha_j/2$, we have $a(\check{x}_j) = 0$, and hence, $\tilde{\phi} = c_j/(i\eta_j)$. But, since $[\![\tilde{\phi}]\!] = 0$, we have $c_j = 0$, and so $\tilde{\phi} = 0$. This implies that $\sum_{j=1}^{n-1} c_j \boldsymbol{e}_j = 0$. We thus conclude that $c_j = 0$ $(j = 1, \dots, n-1)$. □

We are now in a position to prove Proposition 4.17.

Proof of Proposition 4.17. We multiply the first equation of (4.16) by $\bar{\sigma}$ and take the real part of the resulting equation to obtain

$$\mathrm{Re}\,\lambda|\sigma|^2 + \mathrm{Re}\{i\gamma[\![\tilde{\eta}' \cdot w_1]\!]\overline{\sigma}\} = \mathrm{Re}\,\{[\![f^0]\!]\overline{\sigma}\}. \tag{4.25}$$

Let us estimate $\mathrm{Re}\{i\gamma[\![\tilde{\eta}' \cdot w_1]\!]\overline{\sigma}\}$ on the left-hand side of (4.25). To do so, we decompose w_1 in the following way. In (4.19) and (4.20) we decompose ϕ_1 and w_1 as follows:

$$\phi_1 = \phi_1^{(1)} + \phi_1^{(2)}, \qquad w_1 = w_1^{(1)} + w_1^{(2)},$$

where ${}^\top(\phi_1^{(1)}, w_1^{(1)})$ and ${}^\top(\phi_1^{(2)}, w_1^{(2)})$ satisfy the following systems:

$$\begin{cases} \mathrm{div}_{\eta'}\, w_1^{(1)} = 0, \\ -\Delta_{\eta'} w_1^{(1)} + \frac{\gamma}{\nu}\nabla_{\eta'}\phi_1^{(1)} = -\frac{i\gamma\sigma}{\nu}\tilde{\eta}', \\ \phi_1^{(1)}|_{\Sigma_{j,+}} = \phi_1^{(1)}|_{\Sigma_{j,-}}, \quad w_1^{(1)}|_{\Sigma_{j,+}} = w_1^{(1)}|_{\Sigma_{j,-}}, \quad w_1^{(1)}|_{\Sigma_n} = 0, \\ [\![\phi_1^{(1)}]\!] = 0 \end{cases} \tag{4.26}$$

and

$$\begin{cases} \gamma\,\mathrm{div}_{\eta'}\, w_1^{(2)} = f_1^0 - \lambda\phi_1, \\ -\nu\Delta_{\eta'} w_1^{(2)} + \gamma\nabla_{\eta'}\phi_1^{(2)} = \tilde{f}_1 - \lambda w_1 + \tilde{\nu}\nabla_{\eta'}(\nabla_{\eta'} \cdot w_1), \\ \phi_1^{(2)}|_{\Sigma_{j,+}} = \phi_1^{(2)}|_{\Sigma_{j,-}}, \quad w_1^{(2)}|_{\Sigma_{j,+}} = w_1^{(2)}|_{\Sigma_{j,-}}, \quad w_1^{(1)}|_{\Sigma_n} = 0, \\ [\![\phi_1^{(2)}]\!] = 0. \end{cases} \tag{4.27}$$

Let us estimate $i[\![\tilde{\eta}' \cdot w_1^{(1)}]\!]$. We see from (4.26) that ${}^\top(\phi_1^{(1)}, w_1^{(1)})$ can be written as

$$\begin{pmatrix} \phi_1^{(1)} \\ w_1^{(1)} \end{pmatrix} = -\frac{i\gamma}{\nu}\sigma \sum_{k=1}^{n-1} \eta_k \begin{pmatrix} \frac{\nu}{\gamma}\phi_{1,k,\eta'}^{(1)} \\ w_{1,k,\eta'}^{(1)} \end{pmatrix}.$$

Since $[\![\phi_{1,k,\eta'}^{(1)}]\!] = 0$, we see that $0 = i[\![\tilde{\eta}' \cdot w_{1,k,\eta'}^{(1)}]\!][\![\overline{\phi_{1,k,\eta'}^{(1)}}]\!] = (i[\![\tilde{\eta}' \cdot w_{1,k,\eta'}^{(1)}]\!], \phi_{1,k,\eta'}^{(1)})$, which implies that $(w_{1,k,\eta'}^{(1)}, \nabla_{\eta'}\phi_{1,k,\eta'}^{(1)}) = -(\mathrm{div}_{\eta'}\, w_{1,k,\eta'}^{(1)}, \phi_{1,k,\eta'}^{(1)}) = 0$. Taking this into account, we have

$$
\begin{aligned}
i[\![\tilde{\eta}'\cdot w_1^{(1)}]\!] &= \frac{\gamma}{\nu}\sigma\sum_{j,k=1}^{n-1}\eta_j\eta_k(w_{1,j,\eta'}^{(1)},\boldsymbol{e}_k)\\
&= \frac{\gamma}{\nu}\sigma\sum_{j,k=1}^{n-1}\eta_j\eta_k(w_{1,j,\eta'}^{(1)},-\Delta_{\eta'}w_{1,k,\eta'}^{(1)}+\nabla_{\eta'}\phi_{1,k,\eta'}^{(1)})\\
&= \frac{\gamma}{\nu}\sigma\sum_{j,k=1}^{n-1}\eta_j\eta_k(\nabla_{\eta'}w_{1,k,\eta'}^{(1)},\nabla_{\eta'}w_{1,j,\eta'}^{(1)}).
\end{aligned}
$$

Let $\{f_{1,\eta'},\dots,f_{n-1,\eta'}\}$ be an orthonormal basis of $\mathrm{span}\{w_{1,k,\eta'}^{(1)}\}_{k=1}^{n-1}$ in $H^1_{0,\mathrm{per}}(\Omega_{\mathrm{per}})$. Then $w_{1,k,\eta'}^{(1)}$ can be written as $w_{1,k,\eta'}^{(1)}=\sum_{m=1}^{n-1}b_{km,\eta'}f_{m,\eta'}$, and therefore

$$
(\nabla_{\eta'}w_{1,j,\eta'}^{(1)},\nabla_{\eta'}w_{1,k,\eta'}^{(1)})=\sum_{l,m}b_{jm,\eta'}\overline{b_{kl,\eta'}}(\nabla_{\eta'}f_{m,\eta'},\nabla_{\eta'}f_{l,\eta'})=(B_{\eta'}B_{\eta'}^*)_{jk}.
$$

Here $B_{\eta'}$ is the $(n-1)\times(n-1)$ matrix given by $B_{\eta'}:=(b_{jk,\eta'})_{j,k=1}^{n-1}$. Note that $B_{\eta'}$ is nonsingular since $\{w_{1,k,\eta'}^{(1)}\}_{k=1}^{n-1}$ are linearly independent by Lemma 4.20. Thus $B_{\eta'}B_{\eta'}^*$ is positive definite for each η', and

$$
i[\![\tilde{\eta}'\cdot w_1^{(1)}]\!]=\frac{\gamma}{\nu}\sigma|B_{\eta'}^*\eta'|^2, \tag{4.28}
$$

$$
|B_{\eta'}^*\eta'|^2\ge c_2|\eta'|^2 \tag{4.29}
$$

uniformly for $\eta'\in Q^*$ with $|\eta'|\ge r_0$. Here c_2 is the number given by

$$
c_2:=\inf_{\eta'\in Q^*,\,|\eta'|\ge r_0}c_{2,\eta'}
$$

with

$$
c_{2,\eta'}:=\min\{\lambda;\ \lambda \text{ is an eigenvalue of } B_{\eta'}B_{\eta'}^*\}>0.
$$

Let us show that $c_2>0$. To do so, we first show that, for each $j,k=1,\dots,n-1$, $b_{jk,\eta'}=(\nabla_{\eta'}w_{1,j,\eta'}^{(1)},\nabla_{\eta'}w_{1,k,\eta'}^{(1)})$ is continuous in η'. Once this is shown, then, by the continuity of the eigenvalue with respect to the components of a matrix, we will have $c_2>0$.

We define $u_{1,k,\eta'}^{(1)}:={}^\top(\phi_{1,k,\eta'}^{(1)},w_{1,k,\eta'}^{(1)})$. Then $u_{1,k,\eta'}^{(1)}$ is in $D(L_0)\cap X_1$ and $(L_0+\Pi_1M_{\eta'})u_{1,k,\eta'}^{(1)}=f_{1,k}$ with $f_{1,k}:={}^\top(0,\boldsymbol{e}_k)$. By Lemma 4.18, $L_0+\Pi_1M_{\eta'}$ has a bounded inverse $(L_0+\Pi_1M_{\eta'})^{-1}$ on X_1 and

$$
\|(L_0+\Pi_1M_{\eta'})^{-1}f\|_{L^2\times H^1}\le C\|f\|_2 \tag{4.30}
$$

uniformly for $\eta' \in Q^*$ and $f \in X_1$. On the other hand, we see from (4.10) and (4.11) that

$$\|\Pi_1(M_{\eta'+h'} - M_{\eta'})u\|_2 \le C|h'|\,\|(L_0 + \Pi_1 M_{\eta'})u\|_2$$

for $u \in D(L_0) \cap X_1$ and $h' \in \mathbb{R}^{n-1}$ with $|h'| \le 1$. This, together with (4.30), implies that for each fixed $f \in X_1$, $(L_0 + \Pi_1 M_{\eta'})^{-1} f$ is analytic in $\eta' \in Q^*$ in $L^2_{*,\mathrm{per}}(\Omega_{\mathrm{per}}) \times H^1_{0,\mathrm{per}}(\Omega_{\mathrm{per}})$. Since $u^{(1)}_{1,k,\eta'} = (L_0 + \Pi_1 M_{\eta'})^{-1} f_{1,k}$, we find that $w^{(1)}_{1,k,\eta'}$ is analytic in $\eta' \in Q^*$ in $H^1_{0,\mathrm{per}}(\Omega_{\mathrm{per}})$. Thus $b_{jk,\eta'} = (\nabla_{\eta'} w^{(1)}_{1,j,\eta'}, \nabla_{\eta'} w^{(1)}_{1,k,\eta'})$ is continuous in η', and hence the eigenvalues of $B_{\eta'} B^*_{\eta'}$ are continuous in η'. Since $c_{2,\eta'}$ is positive for each η' and is continuous in η', we deduce that

$$c_2 = \inf_{\eta' \in Q^*,\, |\eta'| \ge r_0} c_{2,\eta'} > 0.$$

By (4.28) and (4.29),

$$\mathrm{Re}\,\{i\gamma[\![\tilde{\eta}' \cdot w_1^{(1)}]\!]\bar{\sigma}\} = \mathrm{Re}\left\{\frac{\gamma^2}{\nu}|B^*_{\eta'}\eta'|^2\sigma\bar{\sigma}\right\} \ge c_2\frac{\gamma^2}{\nu}|\eta'|^2|\sigma|^2.$$

As for $\mathrm{Re}\{i\gamma[\![\tilde{\eta}' \cdot w_1^{(2)}]\!]\bar{\sigma}\}$, by Proposition 4.14, we have

$$\mathrm{Re}\,\{i\gamma[\![\tilde{\eta}' \cdot w_1^{(2)}]\!]\bar{\sigma}\} \le \epsilon\frac{\gamma^2}{\nu}|\eta'|^2|\sigma|^2 + \frac{C\nu}{\epsilon}\|\nabla_{\eta'} w_1^{(2)}\|_2^2$$

for all $\epsilon > 0$ with $C > 0$ independent of ϵ. On the other hand, using Lemma 4.18, we see from (4.27) that

$$\begin{aligned}\|\nabla_{\eta'} w_1^{(2)}\|_2 &\le C\{\|\mathrm{div}_{\eta'}\, w_1\|_2 + \|\tilde{f}_1 - \lambda w_1 + \tilde{\nu}\nabla_{\eta'}(\nabla_{\eta'} \cdot w_1)\|_{H^{-1}_{\mathrm{per}}(\Omega_{\mathrm{per}})}\} \\ &\le C\{D_{\eta'}(w_1) + \|\lambda w_1\|_2 + \|\tilde{f}_1\|_2\},\end{aligned}$$

and hence

$$\mathrm{Re}\{i\gamma[\![\tilde{\eta}' \cdot w_1]\!]\bar{\sigma}\} \le \epsilon\frac{\gamma^2}{\nu}|\sigma|^2 + \frac{C}{\epsilon}\{D_{\eta'}(w_1) + \|\lambda w_1\|_2 + \|\tilde{f}_1\|_2\}.$$

Taking $\epsilon = \frac{1}{4}c_2$, we arrive at

$$\mathrm{Re}\,\lambda|\sigma|^2 + \frac{3}{4}\,\frac{\gamma^2 c_2}{\nu}|\sigma|^2|\eta'|^2 \le C_3\{|[\![f^0]\!]|\,|\sigma| + \|\tilde{f}_1\|_2^2 + |\lambda|^2\|w_1\|_2^2 + D_{\eta'}(w_1)\},$$

which yields

$$\begin{aligned}&\mathrm{Re}\,\lambda|\sigma|^2 + \frac{c_0\gamma^2}{2\nu}|\eta'|^2|\sigma|^2 \\ &\qquad \le C_2\{(1 + 1/|\eta'|^2)|[\![f^0]\!]|^2 + \|\tilde{f}_1\|_2^2 + |\lambda|^2\|w_1\|_2^2 + D_{\eta'}(w_1)\}. \qquad \square\end{aligned}$$

We now establish the resolvent estimate for $-L_{\eta'}$ with $|\eta'| \ge r_0$.

Theorem 4.21. *Let $\eta' \in Q^*$ satisfy $|\eta'| \geq r_0$. Then there exists a constant $\beta_1 > 0$ such that $\{\lambda; \mathrm{Re}\,\lambda > -\beta_1\} \subset \rho(-L_{\eta'})$ and if $\mathrm{Re}\,\lambda > -\beta_1$, then*

$$\|(\lambda + L_{\eta'})^{-1} f\|_2 + \|\nabla \tilde{\mathbb{P}}(\lambda + L_{\eta'})^{-1} f\|_2 \leq \frac{C}{(\mathrm{Re}\,\lambda + \beta_1)^{1/2}} \|f\|_2.$$

The same conclusion holds with $L_{\eta'}$ replaced by $L^*_{\eta'}$.

Proof. Set $E[u] = (1 + b_2)|\sigma|^2 + \|u_1\|_2^2 + b_1 D_{\eta'}(w_1)$ with constants b_1, $b_2 > 0$ to be determined later. It suffices to show that

$$E[u] \leq \frac{C}{\mathrm{Re}\,\lambda + \beta_1} \{|[\![f^0]\!]|^2 + \|f_1^0\|_2^2 + \|\tilde{f}_1\|_2^2\}.$$

Consider (4.17) + (4.18) $\times\, b_1$. Then taking $b_1 > 0$ suitably small, we have

$$\begin{aligned} \mathrm{Re}\,\lambda(|\sigma|^2 + \|u_1\|_2^2 + b_1 D_{\eta'}(w_1)) + \tfrac{1}{4} D_{\eta'}(w_1) + b_1|\lambda|^2\|w_1\|_2^2 \\ \leq C\{|[\![f^0]\!]||\sigma| + |(f_1^0, \phi_1)| + \|f_1\|_2^2 + |[\![f^0]\!]|^2\}. \end{aligned} \tag{4.31}$$

We next consider (4.22) $\times\, b_2$ + (4.31). Then with a suitably small $b_2 > 0$, we have

$$\begin{aligned} \mathrm{Re}\,\lambda E[u] + \tfrac{1}{4} D_{\eta'}(w_1) + \frac{b_1}{2}|\lambda|^2\|w_1\|_2^2 + \frac{b_2}{4}\frac{c_0\gamma^2}{\nu}|\eta'|^2|\sigma|^2 \\ \leq C\{(1 + 1/|\eta'|^2)|[\![f^0]\!]|^2 + |(f_1^0, \phi_1)| + \|f_1\|_2^2\}. \end{aligned} \tag{4.32}$$

Since ${}^\top(\phi_1, w_1) \in L^2_{*,\mathrm{per}}(\Omega_{\mathrm{per}}) \times H^1_{0,\mathrm{per}}(\Omega_{\mathrm{per}})$ and

$$\begin{cases} -\Delta_{\eta'} w_1 + \nabla_{\eta'}(\frac{\gamma}{\nu}\phi_1) = \frac{1}{\nu}\tilde{f}_1 - \frac{1}{\nu}\{\lambda w_1 - \tilde{\nu}\nabla_{\eta'}(\nabla_{\eta'} \cdot w_1) + i\gamma\sigma\tilde{\eta}'\}, \\ \mathrm{div}_{\eta'}\, w_1 = \frac{1}{\gamma}\{f_1^0 - \lambda\phi_1\}, \end{cases}$$

we see from Lemma 4.18 that

$$\begin{aligned} \|\phi_1\|_2^2 &\leq C\frac{\nu^2}{\gamma^2}\bigg\{\|\mathrm{div}_{\eta'}\, w_1\|_2^2 + \frac{1}{\nu^2}\|\tilde{f}_1\|_2^2 + \frac{1}{\nu^2}|\lambda|^2\|w_1\|_2^2 \\ &\qquad + \frac{\tilde{\nu}^2}{\nu^2}\|\nabla_{\eta'}(\nabla_{\eta'} \cdot w_1)\|^2_{H^{-1}_{\mathrm{per}}(\Omega_{\mathrm{per}})} + \frac{\gamma^2}{\nu^2}|\eta'|^2|\sigma|^2\bigg\} \\ &\leq C\bigg\{\frac{(\nu + \tilde{\nu})^2}{\gamma^2}\|\nabla_{\eta'} w_1\|_2^2 + \frac{1}{\gamma^2}\|\tilde{f}_1\|_2^2 + \frac{1}{\gamma^2}|\lambda|^2\|w_1\|_2^2 + |\eta'|^2|\sigma|^2\bigg\}. \end{aligned} \tag{4.33}$$

We consider (4.33) $\times\, b_3$ + (4.32). Taking $b_3 > 0$ suitably small, we have

$$\begin{aligned} \mathrm{Re}\,\lambda E[u] + \tfrac{1}{8} D_{\eta'}(w_1) + \frac{b_1}{4}|\lambda|\,\|w_1\|_2^2 + \frac{b_3}{2}\|\phi_1\|_2^2 + \frac{b_2 c_2}{8}\frac{\gamma^2}{\nu}|\eta'|^2|\sigma|^2 \\ \leq C\{(1 + 1/|\eta'|^2)|[\![f^0]\!]|^2 + \|f_1^0\|_2^2 + \|\tilde{f}_1\|_2^2 + |(f_1^0, \phi_1)|\} \\ \leq \frac{b_3}{4}\|\phi_1\|_2^2 + C\{(1 + 1/|\eta'|^2)|[\![f^0]\!]|^2 + \|f_1^0\|_2^2 + \|\tilde{f}_1\|_2^2\}, \end{aligned}$$

and hence

$$\operatorname{Re}\lambda E[u] + \tfrac{1}{8}D_{\eta'}(w_1) + \frac{b_1}{4}|\lambda|\,\|w_1\|_2^2 + \frac{b_3}{4}\|\phi_1\|_2^2 + \frac{b_2c_2}{8}\frac{\gamma^2}{\nu}|\eta'|^2|\sigma|^2 \\ \leq C\{(1+1/|\eta'|^2)|[\![f^0]\!]|^2 + \|f_1^0\|_2^2 + \|\tilde{f}_1\|_2^2\}.$$

Using the Poincaré inequality, we have $\frac{1}{16}D_{\eta'}(w_1) + \frac{b_3}{4}\|\phi_1\|_2^2 + \frac{b_2c_2}{8}|\eta'|^2|\sigma|^2 \geq \beta_1 E[u]$ for some constant $\beta_1 = \beta_1(r_0) > 0$. We thus obtain

$$(\operatorname{Re}\lambda + \beta_1)E[u] + \tfrac{1}{16}D_{\eta'}(w_1) \leq C\{|[\![f^0]\!]|^2 + \|f_1^0\|_2^2 + \|\tilde{f}_1\|_2^2\}$$

for η' with $|\eta'| \geq r_0$. □

We have already shown in Proposition 4.2 that $-L_{\eta'}$ generates a contraction semigroup $e^{-tL_{\eta'}}$. Theorem 4.21 implies that $e^{-tL_{\eta'}}$ decays exponentially for $\eta' \in Q^*$ with $|\eta'| \geq r_0$.

Theorem 4.22. *The estimate*

$$\|e^{-tL_{\eta'}}u_0\|_2 \leq Ce^{-\frac{\beta_1}{2}t}\|u_0\|_2$$

holds uniformly for $\eta' \in Q^*$ *satisfying* $|\eta'| \geq r_0$.

This follows from Theorem 4.21 and [3, Chap. V, Theorem 1.11].

§5. Proof of Theorems 3.1 and 3.2

In this section we give proofs of Theorems 3.1 and 3.2.

Proof of Theorem 3.1. As in the proof of Proposition 4.1, one can show that $\{\lambda;\, \operatorname{Re}\lambda > 0\} \subset \rho(-L)$ and if $\operatorname{Re}\lambda > 0$, then

$$\text{(5.1)}\qquad \begin{aligned} &\|(\lambda + L)^{-1}f\|_{L^2(\Omega)} \leq \frac{1}{\operatorname{Re}\lambda}\|f\|_{L^2(\Omega)}, \\ &\|\nabla\tilde{\mathbb{P}}(\lambda + L)^{-1}f\|_{L^2(\Omega)} \leq \frac{1}{(\nu\operatorname{Re}\lambda)^{1/2}}\|f\|_{L^2(\Omega)}. \end{aligned}$$

Therefore, $-L$ generates a contraction semigroup e^{-tL} on $L^2(\Omega)$. □

Proof of Theorem 3.2. We set

$$\Pi := U\chi_0\Pi_{\eta'}T, \qquad \chi_0(\eta') := \begin{cases} 1, & |\eta'| \leq r_0, \\ 0, & |\eta'| \geq r_0. \end{cases}$$

It then follows from Proposition 2.2 that $\Pi^2 = \Pi$. Furthermore, by Theorem 4.7,

$$e^{-tL}\Pi u_0 = U\chi_0 e^{-tL_{\eta'}}\Pi_{\eta'}Tu_0 = U\chi_0 e^{\lambda_{\eta'}t}\Pi_{\eta'}Tu_0.$$

Since

$$\sup_{\eta' \in Q^*} \|Tu_0\|_1 \le C\|u_0\|_{L^1(\Omega)},$$

we see from Theorems 4.7 and 4.11 that

$$\begin{aligned} &\|e^{-tL}\Pi u_0\|^2_{L^2(\Omega)} \\ &\quad \le C\int_{\eta' \in Q^*} \|\chi_0 e^{-tL_{\eta'}}\Pi_{\eta'}Tu_0\|_2^2\, d\eta' \le C\int_{|\eta'|\le r_0} e^{-\frac{\kappa_0}{2}\frac{\gamma^2}{\nu}|\eta'|^2 t}\|\Pi_{\eta'}Tu_0\|_2^2\, d\eta' \\ &\quad \le C\int_{|\eta'|\le r_0} e^{-\frac{\kappa_0}{2}\frac{\gamma^2}{\nu}|\eta'|^2 t}\|Tu_0\|_1^2\, d\eta' \le Ct^{-(n-1)/2}\|u_0\|^2_{L^1(\Omega)}. \end{aligned}$$

On the other hand,

$$\|e^{-tL}\Pi u_0\|^2_{L^2(\Omega)} \le C\int_{|\eta'|\le r_0} d\eta'\, \|u_0\|^2_{L^1(\Omega)} \le C\|u_0\|^2_{L^1(\Omega)}.$$

We thus obtain

$$\|e^{-tL}\Pi u_0\|_{L^2(\Omega)} \le C(1+t)^{-(n-1)/4}\|u_0\|_{L^1(\Omega)}.$$

This proves (i) of Theorem 3.2.

As for the estimate for $e^{-tL}(I-\Pi)u_0$, we write it as

$$\begin{aligned} e^{-tL}(I-\Pi)u_0 &= U\chi_0 e^{-tL_{\eta'}}(I-\Pi_{\eta'})T + U(1-\chi_0)e^{-tL_{\eta'}}T \\ &= U\chi_0(e^{-tL_{\eta'}} - e^{\lambda_{\eta'}t}\Pi_{\eta'})T + U(1-\chi_0)e^{-tL_{\eta'}}T. \end{aligned}$$

It follows from Theorems 4.10 and 4.22 that

$$\|e^{-tL}(I-\Pi)u_0\|_{L^2(\Omega)} \le Ce^{-\beta t}\|u\|_{L^2(\Omega)},$$

where $\beta = \frac{1}{2}\min\{\beta_0, \beta_1\}$. This proves (ii) of Theorem 3.2.

Let us prove (iii) of Theorem 3.2. We write

$$e^{-tL}\Pi u_0 = U\chi_0 e^{\lambda\eta' t}\Pi^{(0)}Tu_0 + U\chi_0 e^{\lambda\eta' t}(\Pi_{\eta'} - \Pi^{(0)})Tu_0 =: J_1 + J_2.$$

For $\ell' = (\ell_1, \dots, \ell_{n-1}) \in \mathbb{Z}^{n-1}$, we denote $\Omega_{\mathrm{per},\ell'} := \big\{\big(x' + \sum_{j=1}^{n-1}\frac{2\pi\ell_j}{\alpha_j}\boldsymbol{e}'_j, x_n\big);\ (x', x_n) \in \Omega_{\mathrm{per}}\big\}$. By the definition of T, we have

$$\begin{aligned} \Pi^{(0)}Tu_0 &= \left[\int_{\Omega_{\mathrm{per}}} [\![(T\phi_0)(x',\cdot)]\!]\, dx\right]u^{(0)} \\ &= \left[\frac{1}{|\Omega_{\mathrm{per}}|\,|Q^*|^{1/2}}\sum_{\ell'\in\mathbb{Z}}\int_{\Omega_{\mathrm{per},\ell'}} \phi_0(x)e^{-i\eta'\cdot x'}\, dx\right]u^{(0)} \\ &= \left[\frac{1}{|\Omega_{\mathrm{per}}|\,|Q^*|^{1/2}}\int_{\Omega} \phi_0(x)e^{-i\eta'\cdot x'}\, dx\right]u^{(0)} = \frac{1}{(2\pi)^{(n-1)/2}|Q|^{1/2}}\hat{\sigma}_0(\eta')u^{(0)}, \end{aligned}$$

where

$$\sigma_0(x') = \frac{|Q|}{|\Omega_{\rm per}|}\int_{\omega_1(x')}^{\omega_2(x')} \phi_0(x', x_n)\, dx_n.$$

It then follows that

$$J_1 = \left[\frac{1}{(2\pi)^{n-1}}\int_{Q^*} \chi_0 e^{\lambda\eta' t}\widehat{\sigma_0}(\eta')e^{i\eta'\cdot x'}\, d\eta'\right]u^{(0)} = [e^{-tH}\sigma_0(x')]u^{(0)} + J_1^{(1)} + J_1^{(2)}.$$

Here

$$\begin{aligned} J_1^{(1)} &:= \mathcal{F}^{-1}[(\chi_0 - 1)e^{-\frac{\gamma^2}{\nu}\kappa(\eta')t}\widehat{\sigma_0}(\eta')]u^{(0)}, \\ J_1^{(2)} &:= \left[\frac{1}{(2\pi)^{n-1}}\int_{Q^*}\chi_0(e^{\lambda_{\eta'} t} - e^{-\frac{\gamma^2}{\nu}\kappa(\eta')t})\widehat{\sigma_0}(\eta')e^{i\eta'\cdot x'}\, d\eta'\right]u^{(0)}. \end{aligned}$$

By the Plancherel Theorem,

$$\begin{aligned} \|J_1^{(1)}\|_{L^2(\Omega)}^2 &\le \bar{d}\|\mathcal{F}^{-1}[(\chi_0 - 1)e^{-\frac{\gamma^2}{\nu}\kappa(\eta')t}\widehat{\sigma_0}]u^{(0)}\|_{L^2(\mathbb{R}^{n-1})}^2 \\ &= (2\pi)^{-(n-1)}\bar{d}\|(\chi_0 - 1)e^{-\frac{\gamma^2}{\nu}\kappa(\eta')t}\widehat{\sigma_0}\|_{L^2(\mathbb{R}^{n-1})}^2 \end{aligned}$$

with $\bar{d} := \max_{x'\in\mathbb{R}^{n-1}}\{\omega_2(x') - \omega_1(x')\} > 0$. Since $\mathrm{supp}(\chi_0 - 1) = \{|\eta'| \ge r_0\}$, we see that

$$\|(\chi_0 - 1)e^{-\frac{\gamma^2}{\nu}\kappa(\eta')t}\widehat{\sigma_0}\|_{L^2(\mathbb{R}^{n-1})}^2 \le Ct^{-(n-1)/2}e^{-\frac{\gamma^2}{\nu}r_0^2 t}\|\phi_0\|_{L^1(\Omega)}^2,$$

and hence

$$\|J_1^{(1)}\|_2 \le Ct^{-(n-1)/4}e^{-\frac{\gamma^2}{2\nu}r_0^2 t}\|\phi_0\|_{L^1(\Omega)}.$$

As for $J_1^{(2)}$, we have

$$e^{\lambda_{\eta'} t} - e^{-\kappa(\eta')t} = (\lambda_{\eta'} + \kappa(\eta'))t\int_0^1 e^{-\kappa(\eta')t + \theta(\lambda_{\eta'} + \kappa(\eta'))t}d\theta.$$

Since $\lambda_{\eta'} = -\frac{\gamma^2}{\nu}\kappa(\eta') + O(|\eta'|^3)$, we obtain

$$|e^{\lambda_{\eta'} t} - e^{-\frac{\gamma^2}{\nu}\kappa(\eta')t}| \le C|\eta'|^3 t e^{-\frac{\kappa_0}{2}\frac{\gamma^2}{\nu}|\eta'|^2 t} \le C|\eta'|e^{-\frac{\kappa_0}{4}\frac{\gamma^2}{\nu}|\eta'|^2 t},$$

and hence

$$\begin{aligned} \|J_1^{(2)}\|_{L^2(\Omega)}^2 &\le C\int_{|\eta'|\le r_0}|\eta'|^2 e^{-\frac{\kappa_0}{2}\frac{\gamma^2}{\nu}|\eta'|^2 t}\, d\eta' \left(\sup_{\eta'\in\mathbb{R}^{n-1}}|\widehat{\sigma_0}(\eta')|\right)^2 \\ &\le Ct^{-(n-1)/2-1}\|\phi_0\|_{L^1(\Omega)}^2. \end{aligned}$$

Concerning J_2, we see from Theorem 4.11 that

$$\begin{aligned} \|J_2\|_{L^2(\Omega)} &\le C\|\chi_0 e^{\lambda_{\eta'} t}(\Pi_{\eta'} - \Pi^{(0)})Tu_0\|_{L^2(Q^*;L^2(\Omega_{\rm per}))} \\ &\le C\big\|\chi_0|\eta'|e^{\lambda_{\eta'} t}\|Tu_0\|_1\big\|_{L^2(Q^*)} \le C(1+t)^{-(n-1)/4-1/2}\|u_0\|_{L^1(\Omega)}. \end{aligned}$$

We thus obtain the desired estimate. □

§6. Outline of proof of Lemma 4.18

In this section we outline the proof of Lemma 4.18. We only give several lemmas necessary for the proof since Lemma 4.18 can be proved by an argument similar to the proof of [23, Chap. III, Theorem 1.4.1], where the proof for the Stokes system (i.e., $\eta' = 0$) is given.

We begin with

Lemma 6.1. *For $u \in L^2(\Omega_{\mathrm{per}})$,*

$$\|u\|_2 \le C\{\|\nabla_{\eta'} u\|_{H^{-1}_{\mathrm{per}}(\Omega_{\mathrm{per}})} + \|u\|_{H^{-1}_{\mathrm{per}}(\Omega_{\mathrm{per}})}\}.$$

This can be proved in a similar manner to [19, Chap. 3, Lemma 7.1]. (Cf. [23, Chap. II, Lemma 1.1.3].)

Lemma 6.2. *For $u \in L^2_{*,\mathrm{per}}(\Omega_{\mathrm{per}})$,*

$$\|u\|_2 \le C_1\|\nabla_{\eta'} u\|_{H^{-1}_{\mathrm{per}}(\Omega_{\mathrm{per}})} \le C_1C_2\|u\|_2.$$

This follows from Lemma 6.1 as in [23, Chap. II, proof of Lemma 1.5.4].

Lemma 6.3. (i) *For every $g \in L^2_{*,\mathrm{per}}(\Omega_{\mathrm{per}})$, there exists $w \in H^1_{0,\mathrm{per}}(\Omega_{\mathrm{per}})$ such that*

$$\mathrm{div}_{\eta'}\, w = g, \qquad \|\nabla_{\eta'} w\|_2 \le C\|g\|_2.$$

(ii) *For every $f \in H^{-1}_{\mathrm{per}}(\Omega_{\mathrm{per}})$ satisfying*

$$[f, w] = 0 \quad \text{for all } w \in H^1_{0,\mathrm{per}}(\Omega_{\mathrm{per}}) \text{ with } \mathrm{div}_{\eta'}\, w = 0,$$

there exists a unique $p \in L^2_{*,\mathrm{per}}(\Omega_{\mathrm{per}})$ such that

$$\nabla_{\eta'} p = f, \qquad \|p\|_2 \le C\|f\|_{H^{-1}_{\mathrm{per}}(\Omega_{\mathrm{per}})}.$$

One can prove this by using Lemma 6.2 as in [23, Chap. II, proof of Lemma 2.1.1].

We define

$$H^1_{0,\sigma}(\Omega_{\mathrm{per}}) := \{w \in H^1_{0,\mathrm{per}}(\Omega_{\mathrm{per}});\ \mathrm{div}_{\eta'}\, w = 0\}.$$

Lemma 6.4. *For every $f \in H^{-1}_{0,\mathrm{per}}(\Omega_{\mathrm{per}})$, there exists a unique $w \in H^1_{0,\sigma}(\Omega_{\mathrm{per}})$ satisfying*

$$(\nabla_{\eta'} w, \nabla_{\eta'} v) = [f, v]$$

for all $v \in H^1_{0,\sigma}(\Omega_{\mathrm{per}})$, and

$$\|\nabla_{\eta'} w\|_2 \le C\|f\|_{H^{-1}_{\mathrm{per}}(\Omega_{\mathrm{per}})}.$$

Furthermore, there exists a unique $\phi \in L^2_{*,\mathrm{per}}(\Omega_{\mathrm{per}})$ *such that*

$$-\Delta_{\eta'} w - f = -\nabla_{\eta'}\phi$$

and

$$\|\phi\|_2 \leq C\|f\|_{H^{-1}_{\mathrm{per}}(\Omega_{\mathrm{per}})}.$$

This can be proved in a similar manner to [23, Chap. III, proof of Theorem 1.3.1].

Lemma 4.18 now follows from Lemmas 6.3 and 6.4 as in [23, Chap. III, proof of Theorem 1.4.1]. □

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