

# Compact Moduli Spaces of Kähler–Einstein Fano Varieties

by

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## Abstract

We construct geometrically compactified moduli spaces of Kähler–Einstein Fano manifolds.

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## §1. Introduction

In this paper, we construct compactified moduli algebraic spaces of Fano manifolds which have Kähler–Einstein metrics or equivalently (thanks to [CDS], [Tia2], combined with [Ber], [Mab1], [Mab2]) are K-polystable, partially solving the (precise) conjecture in [OSS] formulated by C. Spotti, S. Sun and the present author. The K-stability was originally introduced by G. Tian [Tia1] and formulated in a purely algebraic way by S. Donaldson [Don0]. Brief explanations of the definition and the statement of the recent theorem on equivalence with existence of Kähler–Einstein metrics are given at the beginning of Section 2 and Subsection 3.2.

Roughly speaking, the main result of this paper is:

**Theorem 1.1** (Algebraic-geometric statement, over  $\mathbb{C}$ ). *For any positive integer  $n$ , there is a “canonical” algebraic compactification  $\bar{M}$  of the moduli space  $M$  of K-polystable smooth Fano manifolds of dimension  $n$ , whose boundary parametrises K-polystable (Kawamata-log-terminal  $\mathbb{Q}$ -Gorenstein smoothable)  $\mathbb{Q}$ -Fano varieties of the same dimension.*

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More precisely, the compactification  $\bar{M}$  is an algebraic space in the sense of Artin [Art]. For more details, see (Section 2 and) Theorem 2.3. We further expect the compactification to be a *projective scheme*, following the idea of Fujiki–Schumacher [FS]. See the precise formulation in [OSS, Subsections 3.4, 6.2] or our Section 2 (which follows [OSS]).

The corresponding complex differential-geometric (roughest) restatement of Theorem 1.1 is the following.

**Theorem 1.2** (Differential-geometric restatement). *The Gromov–Hausdorff compactification of the moduli space of Kähler–Einstein (smooth) Fano manifolds has a structure of compact Hausdorff Moishezon analytic space.*

This compactification statement extends that of the explicit 2-dimensional case in [OSS], which was previously proved in the case of complete intersection of two quadric 3-folds (i.e. degree 4 del Pezzo surfaces) in the old work of Mabuchi–Mukai [MM] much before the introduction of K-stability.

This contrasts with the “canonically polarised” case (i.e. of ample canonical class), an idea which (for dimensions higher than 1) goes back to Shepherd-Barron [SB]. This case was systematically studied by Kollár–Shepherd-Barron [KSB] for surfaces, extended by Alexeev to higher dimensions [Ale], and now being accomplished with technical details (a book by Professor Kollár [Kol] with all the details is expected to appear). In honor of the main contributors to the construction, that theory is often named *Kollár–Shepherd-Barron–Alexeev*, or briefly “KSBA”.

The novelty in our case is that all the varieties parametrised are *normal* (even Kawamata-log-terminal), hence irreducible, while KSBA degenerations are usually non-normal, as even the simplest case of stable curve [DM] can have up to  $3g - 3$  components.

However, those two moduli compactifications can be seen from a unified point of view, as examples of moduli of K-(semi)stable varieties, since the semi-log-canonical varieties of ample canonical class are also K-stable by [Od1] (“K-moduli”, cf. e.g. [Od0, Section 5], [Spo, Chapter 1]). Inspired by the breakthroughs [DS] and [Spo], in [OSS, Conjecture 6.2], a precise formulation of the K-moduli conjecture for Fano varieties case is worked out, and we will quickly review a part of this in the next section.

A key technical result may be of independent interest. Namely, we will establish the following deformation picture. The (easier) half of the following statement is proved in [OSS], and the rest essentially depends on [LWX1], [SSY], which in turn use the idea of *Donaldson’s continuity method* [CDS], [Tia2]. Our statement is as follows, but we again leave the detailed statement to Theorem 3.2.

**Theorem 1.3.** *If a Kähler–Einstein  $\mathbb{Q}$ -Fano variety  $X$  is  $\mathbb{Q}$ -Gorenstein smoothable, then in a local  $\mathbb{Q}$ -Gorenstein (Kuranishi) deformation space of  $X$ , which we denote by  $\text{Def}(X)$ , the existence of a Kähler–Einstein metric on the corresponding  $\mathbb{Q}$ -Fano variety is equivalent to the GIT polystability of the  $\text{Aut}(X)$ -action on  $\text{Def}(X)$ .*

As already mentioned, we have already proved in [OSS, Lemma 3.6] that the classical GIT polystability of points corresponds to Kähler–Einstein  $\mathbb{Q}$ -Fano varieties, which is the easier half of Theorem 1.3. This extends the picture of [Tia1], [Don1] for the commonly studied “Mukai–Umemura 3-fold” case, and the general result by Székelyhidi [Sze] which depends on the infinite-dimensional implicit function theorem. Our proof essentially depends on the recent developments for one-parameter deformations in [LWX1] and [SSY]. We expect that the  $\mathbb{Q}$ -Gorenstein smoothability condition is unnecessary but we do not know how to argue in that generality, using current technologies. See also the list of related questions for future research in the final section.

Actually many of the main technical ingredients of the proof are already present in previous papers [DS], [Spo], [Od2], [OSS] and recent [SSY], [LWX1], and this paper does not claim elaboration of essentially new ideas.

## §2. Precise formulation of K-moduli

In this section, we give a precise formulation of K-moduli. First, recall that the K-stability of a  $\mathbb{Q}$ -Fano variety  $X$  is, roughly speaking, defined as positivity of all the Donaldson–Futaki invariants (a variant of the GIT weight) associated to every one-parameter isotrivial degeneration of  $X$ . We will be more precise later in Subsection 3.2. The recent developments show the following.

**Theorem 2.1** ([CDS], [Tia2] for smooth  $X$ , [SSY] for singular  $X$ ). *For any  $\mathbb{Q}$ -Gorenstein smoothable klt  $\mathbb{Q}$ -Fano variety  $X$ , the existence of a Kähler–Einstein metric is equivalent to the K-polystability of  $X$ .*

For the definition of Kähler–Einstein metrics on singular klt (Kawamata-log-terminal)  $\mathbb{Q}$ -Fano varieties, we refer to [Ber] or [SSY] for instance.

Now we explain our precise statement of the K-moduli existence, partially recalling [OSS]. The details on the local deformation picture will only be given in the final section (Theorem 3.2).

For partial self-containedness and convenience of the readers, we recall the notion of *KE moduli stack*, introduced for algebraically oriented people. We also note that in [OSS], the notion of *KE analytic moduli spaces* (for analytic oriented

people) was introduced as well. For the general theory of algebraic stacks, we refer to textbooks such as [LM].

For those who are not familiar with the stack language, we note that an algebraic stack (appearing here) is more or less an algebraic scheme (such as Hilbert scheme, Chow variety) attached with “glueing data” which identifies points on the scheme which “parametrise the same objects”. Artin stack is the most general category of algebraic stack, allowing “non-discrete automorphism groups” of the parametrised objects, while Deligne–Mumford stack is, roughly speaking, for those objects with only discrete automorphism groups. The point of introducing the stack language here is, more or less, to make the statement most precise with the information on flat families of Fano varieties (in connection with Kähler–Einstein metrics).

**Definition 2.2** ([OSS, Definition 3.13]). A moduli algebraic (Artin) stack  $\bar{\mathcal{M}}$  of a  $\mathbb{Q}$ -Gorenstein family of  $\mathbb{Q}$ -Fano varieties is called a *KE moduli stack* if:

- (i) There is a categorical moduli algebraic space  $\bar{M}$ .
- (ii) There is an étale covering  $\{[U_i/G_i]_i\}$  of  $\bar{\mathcal{M}}$  where  $U_i$  is an affine algebraic scheme and  $G_i$  is some reductive algebraic group, on which there is some  $G_i$ -equivariant  $\mathbb{Q}$ -Gorenstein flat family of  $\mathbb{Q}$ -Fano varieties.
- (iii) Closed  $G_i$ -orbits in  $U_i$  parametrise  $\mathbb{Q}$ -Gorenstein smoothable Kähler–Einstein  $\mathbb{Q}$ -Fano varieties via the families of (ii), and via the canonical map  $\varphi_i: U_i \rightarrow \bar{M}$ , each such orbit maps to a closed point of  $\bar{M}$  and every closed point of  $\bar{M}$  can be obtained in this manner for some  $i$ .

We call the coarse algebraic space  $\bar{M}$  of (i) a *KE moduli space*. If it is an algebraic variety, we also call it a *KE moduli variety*.

Recall that  $\bar{M}$  being the *coarse moduli algebraic space* of the Artin stack  $\bar{\mathcal{M}}$  means that there is a morphism  $\bar{\mathcal{M}} \rightarrow \bar{M}$  and it is universal among the morphisms from  $\bar{\mathcal{M}}$  to algebraic spaces. In our case, thanks to conditions (ii) and (iii),  $\bar{M}$  is also set-theoretically “nice”, i.e. bijectively corresponds to Kähler–Einstein  $\mathbb{Q}$ -Fano varieties.

For the definition of more differential-geometric “KE analytic moduli space”, we refer to [OSS, Definitions 3.14, 3.15] since we do not use this notion in this paper and moreover it naturally follows from our construction that  $\bar{M}$  satisfies the defining conditions of this notion.

In this paper, we prove Conjecture 6.2 of [OSS] in the  $\mathbb{Q}$ -Gorenstein smoothable case, i.e. we compactify the moduli of smooth Fano manifolds.

**Theorem 2.3** (Refined statement of existence of K-moduli). *Fix the dimension of  $\mathbb{Q}$ -Fano varieties under study to be  $n$ . There is a KE moduli stack  $\bar{\mathcal{M}}^{\text{GH}}$ , in the sense of [OSS]. In particular,  $\bar{\mathcal{M}}^{\text{GH}}$  has a coarse moduli algebraic space  $\bar{M}$  as a proper separated algebraic space, and  $\bar{\mathcal{M}}^{\text{GH}}$  is good in the sense of Alper [Alp].*

*Then from the Gromov–Hausdorff compactification  $M^{\text{GH}}$  (in the sense of [DS], [OSS]), which is a priori just a compact Hausdorff metric space, there is a homeomorphism*

$$\Phi: \bar{M}^{\text{GH}} \rightarrow \bar{M}$$

*such that  $[X]$  and  $\Phi([X])$  parametrise isomorphic  $\mathbb{Q}$ -Fano varieties for any  $[X] \in \bar{M}^{\text{GH}}$ .*

We remark that the above “Gromov–Hausdorff” is in the refined sense, that is, with respect to complex (algebraic) structures as defined and explained in [DS], [SSY] etc.

### §3. Proof of the main theorems

#### §3.1. Affine étale slice in the Hilbert scheme

We begin the proof of our Main Theorem 2.3, which will be completed at the end of Subsection 3.3. In the current subsection, we construct an affine slice around  $[X]$  inside an appropriate Hilbert scheme, where  $X$  is the  $\mathbb{Q}$ -Fano variety under study. In the next subsection, using that slice, we formulate and prove the local deformation picture of Kähler–Einstein metrics.

We fix the dimension  $n$  of the Fano varieties under study, and consider a finite disjoint union of components of the Hilbert scheme, which we denote by  $Hilb$ , which includes all smooth Kähler–Einstein Fano manifolds of dimension  $n$  and their Gromov–Hausdorff limits. Such finite type  $Hilb$  exists thanks to the recent breakthrough result by Donaldson–Sun [DS] and the “classical” boundedness result by Kollár–Miyaoaka–Mori [KMM]. In [DS], it is even proved that we can assume that both the Kähler–Einstein Fano manifolds and their Gromov–Hausdorff limits are all  $m$ -pluri-anticanonically embedded inside  $\mathbb{P}^N$  with some uniform exponent  $m$  and  $N = h^0(-mK_X) - 1$ . We work in this setting so that our construction a priori depends on  $m$  but we expect it does not (see Remark 3.5).

We define  $Hilb^{\text{KE}}$  to be the set which parametrises all  $m$ -pluri-anticanonically embedded Kähler–Einstein  $\mathbb{Q}$ -Fano varieties. Obviously  $Hilb^{\text{KE}}$  is an  $\text{SL}(N+1)$ -invariant (equivalently,  $\text{PGL}(N+1)$ -invariant) subset of  $Hilb$  but note that it does not have a scheme structure in general. In fact, as we will show in Subsection 3.3 without using the results of this subsection,  $Hilb^{\text{KE}}$  is a *constructible* subset in  $Hilb$ . So from now on, we replace  $Hilb$  by the Zariski closure of  $Hilb^{\text{KE}}$  so that

we can assume that  $\mathit{Hilb}^{\text{KE}}$  is dense inside  $\mathit{Hilb}$ . From now on, we work inside this replaced  $\mathit{Hilb}$ .

Take any point  $[X] \in \mathit{Hilb}^{\text{KE}}$ . From [CDS, III, Theorem 4], an extension of Matsushima's theorem [Mat], we know that the automorphism group  $\text{Aut}(X)$  is a reductive algebraic group. Note that  $\text{Aut}(X)$  is the isotropy (stabiliser) subgroup of the natural PGL-action on  $\mathit{Hilb}$ . Thus the isotropy subgroup of the SL-action on  $\mathit{Hilb}$ , which we denote by  $\tilde{\text{Aut}}(X)$ , is a central extension of  $\text{Aut}(X)$  by  $\mu_{N+1}$ , the finite group of  $(N+1)$ -th roots of unity isomorphic to  $\mathbb{Z}/(N+1)\mathbb{Z}$  which acts trivially on  $\mathit{Hilb}$ . The reason why we think also the SL-action and not only the PGL-action is sometimes needed is to make the action available at the level of the vector space  $H^0(X, -mK_X)$ , i.e. the cone over the projective space  $\mathbb{P}^N$ .

Also let us recall that  $\mathit{Hilb} \subset \mathbb{P}_*(V)$  with some SL-representation  $V$  from the construction of the Hilbert scheme by Grothendieck. (Here  $\mathbb{P}_*$  denotes covariant projectivisation unlike in Grothendieck's notation.) Noting that  $[X]$  corresponds to an  $\tilde{\text{Aut}}(X)$ -invariant one-dimensional vector space  $\mathbb{C}v \subset V$ , we can decompose  $\tilde{\text{Aut}}(X)$ 's linear representation as  $V = \mathbb{C}v \oplus V'$  where  $V'$  is also  $\tilde{\text{Aut}}(X)$ -invariant. (We are grateful to Jarod Alper for the clarification of this.) It is possible since we know  $\text{Aut}(X)$  is reductive. Then we can take an  $\text{Aut}(X)$ -invariant open subset  $U_{[X]}$  as  $\mathit{Hilb} \setminus \mathbb{P}_*(V')$ . It is also affine since  $\mathbb{P}_*(V')$  is an ample divisor of the original projective space  $\mathbb{P}_*(V)$ .

Note that this open neighborhood  $U_{[X]}$  of  $[X]$  is only  $\text{Aut}(X)$ -invariant (or equivalently  $\tilde{\text{Aut}}(X)$ -invariant), but *not* necessarily SL-invariant. However, the affineness of  $U_{[X]}$  enables us to apply the following techniques of taking étale slice mainly due to [Luna] (also known as Luna's "étale slice theorem", cf. [Dre, 5.3]). We include a short outline of the proof for the readers' convenience, partially because we slightly extend the original theorem of [Luna]; basically, the argument below is from the nice exposition [Dre] of Luna's theory [Luna]. First we can easily construct a closed immersion of  $U_{[X]}$  into an  $\text{Aut}(X)$ -acted smooth affine space  $\tilde{U}_{[X]}$  (cf. e.g. [Dre, Lemma 5.2]) with the same embedded dimension of  $[X] \in U_{[X]}$ . Then to prove an étale slice theorem of [Luna] (cf., e.g., [Dre, Lemma 5.1]), it is shown that there is an  $\text{Aut}(X)$ -equivariant affine regular map  $\varphi: \tilde{U}_{[X]} \rightarrow (T_{[X]}U_{[X]})$  which is étale at  $[X]$ . This again depends on the reductiveness of  $\text{Aut}(X)$ . We use this equivariant map as follows.

We decompose the  $\text{Aut}(X)$ -representation  $T_{[X]}U_{[X]}$  as  $T_{[X]}(\text{SL}(N+1)[X] \cap U_{[X]}) \oplus N$  with some  $\text{Aut}(X)$ -invariant vector subspace  $N$ . Then we define  $V_{[X]} := \varphi^{-1}N \cap U_{[X]} \subset U_{[X]}$ , which is an  $\text{Aut}(X)$ -invariant locally closed affine subset of  $\mathit{Hilb}$  including  $[X]$ . Then  $V_{[X]} \subset U_{[X]}$  is an étale slice in the sense of [Luna, Dre], in particular  $[V_{[X]}/\text{Aut}(X)] \rightarrow [U_{[X]}/\text{PGL}]$  is an étale morphism (between two quotient stacks). We omit more details and the rest of the proof of this known fact since it follows easily from the proof of [Dre, Theorem 5.3] or [AK, Subsection 2.2].

### §3.2. K-stability via the CM line bundle

Before proceeding, we briefly recall the fundamental relation between K-stability and the *CM line bundle* ([FS, PT]), which we regard as a definition of K-stability in this paper.

The CM line bundle, in our setting, is a certain SL-equivariant line bundle  $\lambda_{\text{CM}}$  on *Hilb* ([FS], [PT], [FR]). As the actual construction is a little complicated and we do not need it in this paper, we omit the details and refer to [FR].

In our setting, for a given positive integer parameter  $m$ , the  $K_{(m)}$ -stability of  $\mathbb{Q}$ -Fano varieties means the following (as in [Od2], just following [Don0]).

**Definition 3.1.** As in the previous subsection, suppose that a (klt)  $\mathbb{Q}$ -Fano variety  $X$  is such that  $-mK_X$  is a very ample line bundle ( $m \in \mathbb{Z}_{>0}$ ). Then the  $\mathbb{Q}$ -Fano variety  $X$  (more precisely,  $(X, -K_X)$ ) is said to be  $K_{(m)}$ -stable if for any non-trivial one-parameter subgroup  $f: \mathbb{C}^* \rightarrow \text{SL}(H^0(-mK_X))$ , minus the weight of  $\lambda_{\text{CM}}|_{\lim_{t \rightarrow 0}(f(t)[X])}$  (called the *Donaldson–Futaki invariant* associated to  $f$ ) is positive. The one-parameter degeneration of  $X$  along  $\overline{f(\mathbb{C}^*) \cdot [X]} \subset \text{Hilb}$  is called a *test configuration* by [Don0].

Similarly,  $X$  is said to be  $K_{(m)}$ -semistable (resp.  $K_{(m)}$ -polystable) if all the Donaldson–Futaki invariants are non-negative (resp.  $X$  is semistable and the Donaldson–Futaki invariant of  $f$  is positive, or equivalently the orbit closure  $\overline{f(\mathbb{C}^*) \cdot [X]} \subset \text{Hilb}$  is contained in the SL-orbit of  $X$ ; such a degeneration is called a *product test configuration*).

$X$  is said to be  $K$ -stable (resp.  $K$ -semistable,  $K$ -polystable) if it is  $K_{(m)}$ -stable (resp.  $K_{(m)}$ -semistable,  $K_{(m)}$ -polystable) for all sufficiently divisible positive integer  $m$ .

### §3.3. Local GIT polystability

In this subsection, we apply [OSS, Lemma 3.6] to the  $\text{Aut}(X)$ -action on the affine étale slice  $V_{[X]}$  and see that

the points corresponding to some Kähler–Einstein  $\mathbb{Q}$ -Fano varieties are GIT polystable in  $V_{[X]}$  with respect to the  $\text{Aut}(X)$ -action;

we denote the polystable locus in  $V_{[X]}$  as  $V_{[X]}^{\text{ps}}$ . The following theorem shows that the converse to [OSS, Lemma 3.6] also holds in appropriate sense; this will be crucial for us later on.

**Theorem 3.2** (Local deformation picture of KE Fano varieties). *For small enough affine étale slice  $V_{[X]}$ , i.e. after shrinking  $V_{[X]}$  to an  $\text{Aut}(X)$ -invariant*

open affine neighborhood of  $[X]$  if necessary, we have

$$V_{[X]}^{\text{ps}} = V_{[X]} \cap \text{Hilb}^{\text{KE}}.$$

Recall that  $V_{[X]}^{\text{ps}}$  denotes the GIT polystable locus of the affine slice  $V_{[X]}$  in the Hilbert scheme, with respect to the  $\text{Aut}(X)$ -action.

This roughly says that, étale locally, the existence of Kähler–Einstein metrics on  $\mathbb{Q}$ -Fano varieties is equivalent to the classical GIT polystability, at least in the  $\mathbb{Q}$ -Gorenstein smoothable case (we expect this is so in the non-smoothable case as well). Note that the above statement is about the “local” deformation picture in the sense that we need to shrink  $V_{[X]}$  in general. Otherwise the statement is false and indeed the proof requires that shrinking.

This refines [Tia1, Section 7], [Don1, Subsection 5.3] which treated Mukai–Umemura (Fano) 3-folds, the  $\mathbb{Q}$ -Fano varieties case of [Sze] and of course [OSS, Lemma 3.6]. We expect that this will be a fundamental tool in the further study of Kähler–Einstein metrics on  $\mathbb{Q}$ -Fano varieties.

*Proof of Theorem 3.2.* The inclusion  $V_{[X]} \cap \text{Hilb}^{\text{KE}} \subset V_{[X]}^{\text{ps}}$  is exactly (a special case of) [OSS, Lemma 3.6] and here is the argument for the other inclusion, i.e.

$$V_{[X]}^{\text{ps}} \subset V_{[X]} \cap \text{Hilb}^{\text{KE}}.$$

We prove that this holds once we replace  $V_{[X]}$  with a small enough affine  $\text{Aut}(X)$ -invariant slice if necessary.

Note that the difference  $V_{[X]}^{\text{ps}} \setminus (V_{[X]} \cap \text{Hilb}^{\text{KE}})$  is constructible since both  $V_{[X]}^{\text{ps}}$  and  $V_{[X]} \cap \text{Hilb}^{\text{KE}}$  are constructible subsets. The constructibility of the polystable locus is a standard fact in geometric invariant theory. We now explain how to show the constructibility of  $V_{[X]} \cap \text{Hilb}^{\text{KE}} \subset V_{[X]}$ . Indeed, from [SSY, Theorem 1], we know the equivalence of K-polystability and existence of Kähler–Einstein metrics for  $\mathbb{Q}$ -Gorenstein smoothable Fano varieties in general. Moreover, combining [CDS, esp. II, Theorem 1, and III, Theorem 2], [SSY, 4.2.2] and the arguments of [Od2, esp. (2.4-8)], we know that it is also equivalent to the quantised “ $K_{(m)}$ -polystability” in the above sense of Subsection 3.2 for sufficiently divisible uniform  $m \gg 0$ , i.e. we can bound the exponent  $m$  for testing K-(poly)stability. For the readers’ convenience, we recall from [Od2, esp. (2.4-8)] that the main point of the uniform bound  $m$  was the uniform positive lower bounds of (small) *angles* of conical Kähler–Einstein metrics on all the  $\mathbb{Q}$ -Fano varieties parametrised in  $\text{Hilb}$ .

Then the proof of the constructibility of the  $K_{(m)}$ -polystable locus inside  $\text{Hilb}$  follows from the arguments in [Od2, esp. (2.10-12)] with the additional but simple concern about whether the test configurations are of product type or not.



For contradiction, suppose that for any small enough affine  $\text{Aut}(X)$ -invariant slice  $V_{[X]}$  of  $[X]$ , we have  $V_{[X]}^{\text{ps}} \neq V_{[X]} \cap \text{Hilb}^{\text{KE}}$ .

Then we have an irreducible locally closed subvariety  $W$  inside  $V_{[X]}^{\text{ps}} \setminus (V_{[X]} \cap \text{Hilb}^{\text{KE}})$  whose closure meets  $[X]$ , and we take a sequence  $P_i$  in  $W$  converging to  $[X]$ . Otherwise, we can shrink  $V_{[X]}$  to make it satisfy  $V_{[X]}^{\text{ps}} = V_{[X]} \cap \text{Hilb}^{\text{KE}}$ . Now we fix our slice  $V_{[X]}$ .

We take any  $\text{SL}$ -equivariant compactification of the algebraic group  $\text{SL}$  (such as in [DP], or apply [Sum]), denote it by  $\overline{\text{SL}}$  and consider the rational map  $\varphi: \overline{V_{[X]}} \times \overline{\text{SL}} \dashrightarrow \text{Hilb}$  induced by the  $\text{SL}$ -action. Here  $\overline{V_{[X]}}$  denotes the Zariski closure of  $V_{[X]}$  inside  $\text{Hilb}$ . Then we take an  $\text{SL}$ -equivariant resolution of indeterminacy of  $\varphi$ ,

$$\tilde{\varphi}: T \rightarrow \text{Hilb}.$$

So  $T$  is a certain  $\text{SL}$ -equivariant blow up of  $\overline{V_{[X]}} \times \overline{\text{SL}}$  along some ideal co-supported on  $\overline{V_{[X]}} \times (\overline{\text{SL}} \setminus \text{SL})$ . Via the morphism from  $T$  to  $\text{Hilb}$ , we can regard  $T$  as a parameter space of Fano varieties and their degenerations.

Then, for the sequence  $P_i \in W \subset \overline{V_{[X]}} \simeq \overline{V_{[X]}} \times \{e\} \subset T$  ( $i = 1, 2, \dots$ ) which converges to  $[X] \in V_{[X]}$ , we take sequences  $P_{i,j} \in V_{[X]} \simeq V_{[X]} \times \{e\} \subset T$  ( $j = 1, 2, \dots$ ) for each  $i$  ( $= 1, 2, \dots$ ) which parametrise smooth Kähler–Einstein Fano manifolds  $X_{i,j}$  and converge to  $P_i$  when  $j$  goes to infinity.

Thanks to [DS], we know that (by taking subsequences) the Gromov–Hausdorff limit of  $X_{i,j}$  with Kähler–Einstein metrics exists as another Kähler–Einstein  $\mathbb{Q}$ -Fano variety  $Y_i$ . Furthermore, from their construction as a limit inside the Hilbert scheme (cf. [DS, Theorem 1.2]), we know that there is a sequence of elements of  $\text{SL}$  which we denote by  $\phi_{i,j}$  such that  $\lim_{j \rightarrow \infty} \phi_{i,j}(P_{i,j})$  represents a point  $Q_i$  which parametrises the ( $m$ -th pluri-anticanonically embedded) Kähler–Einstein Fano variety  $Y_i$ , for each fixed  $i$ . By the standard diagonal argument, it also follows from [DS] that  $\lim_{i \rightarrow \infty}^{\text{GH}} Y_i$  exists (limit in the (refined) Gromov–Hausdorff sense as in [DS]) as yet another Kähler–Einstein  $\mathbb{Q}$ -Fano variety  $Y$  where the corresponding point will be denoted by  $Q \in \text{Hilb}$ . As the blow up morphism  $T \rightarrow \overline{V_{[X]}} \times \overline{\text{SL}}$  is (topologically) a proper morphism, we can take all these points in  $T$ .

Our general idea is to apply the (recently obtained) separatedness theorem to the two “degenerations” of  $X_{i,j}$  to  $[X]$  and  $[Y] = Q \in T$ , both of which parametrise Kähler–Einstein  $\mathbb{Q}$ -Fano varieties. To lend precision to this idea, we proceed to some more algebro-geometric arguments.

Let  $T^\circ$  be the (open dense) subset of  $T$  which is the preimage of  $\text{SL} \subset \overline{\text{SL}}$ . Also set  $\partial T := T \setminus T^\circ$ . Consider some general affine curve  $C \subset T$  which passes through  $Q$  and intersects  $\partial T \cup (V_{[X]}^{\text{ps}} \setminus (V_{[X]} \cap \text{Hilb}^{\text{KE}}))$  only at the point  $\{Q\}$ .

On the other hand, take the natural retraction  $r: T^\circ \rightarrow \overline{V}_{[X]}$  induced by  $\mathrm{SL} \rightarrow \{e\}$  where  $e \in \mathrm{SL}$  is the unit of  $\mathrm{SL}$ , and partially complete  $C'^0 := r(C \setminus \{Q\})$  naturally to  $C'$  with  $i: C \simeq C'$ . Note that from the construction,  $r$  also naturally extends to a morphism

$$\tilde{r}: T \rightarrow \mathrm{Hilb}$$

from the whole  $T$ . Then from our construction, the image  $i(Q)$  is nothing but the original  $[X] \in \mathrm{Hilb}$ . We can see this as follows. Since  $i$  should preserve the image of  $\tilde{r}$ , we have  $\tilde{r}(i(Q)) = \tilde{r}(Q)$  and  $\tilde{r}(Q) = \tilde{r}(\lim_{i \rightarrow \infty}(Q_i)) = \tilde{r}(\lim_{i \rightarrow \infty}(\lim_{j \rightarrow \infty}(P_{i,j})) = \lim_{i \rightarrow \infty}(\tilde{r}(P_i)) = [X]$ . The last equality follows from our construction of  $P_i$ . (Here all the limits are taken in the usual analytic topology.)

The crucial result we now need is the following. Although we do not have any contribution on it in this paper, we recall the result as we need a comment (on how to combine [LWX1], [SSY], [CDS]) on the proof to make things rigorous. I thank S. Sun for the mathematical clarification of this point.

**Theorem 3.3** ([LWX1, Thm. 1.1 of v1]+[SSY, Thm. 1.1], [CDS]). *Let  $\mathcal{X}$  and  $\mathcal{Y}$  be two  $\mathbb{Q}$ -Gorenstein flat deformations of Kähler–Einstein  $\mathbb{Q}$ -Fano varieties over a smooth curve  $C \ni 0$ . Suppose that  $\mathcal{X}_t \cong \mathcal{Y}_t$  for  $t \neq 0$  and further that these are all smooth (i.e. generically smooth). If  $\mathcal{X}_0$  and  $\mathcal{Y}_0$  are both  $K$ -polystable, then they are isomorphic  $\mathbb{Q}$ -Fano varieties.*

This follows from the combination of [LWX1, v1] and [SSY, Theorem 1.1]. Note that for *separatedness*, [SSY, Corollary 1.2] has to assume that  $\mathcal{X}_0$  and  $\mathcal{Y}_0$  have discrete automorphism groups, while [LWX1, Remark 6.11] needs to assume that  $\mathcal{X}_0$  and  $\mathcal{Y}_0$  have reductive automorphism groups. But from [SSY, Theorem 1.1] we know both  $\mathcal{X}_0$  and  $\mathcal{Y}_0$  admit KE metrics, so satisfy the *reductivity* assumption of [LWX1, v1] by [CDS, III, Theorem 4]. (The author had once attempted to prove this *separatedness* with Professor Richard Thomas, but the arguments had a technical gap.)

We apply the theorem above to the two families of  $\mathbb{Q}$ -Fano varieties corresponding to  $C \subset T$  and  $C' \subset T$ . Then we can show that  $Q$  is in the  $\mathrm{SL}$ -orbit of  $[X] \in \mathrm{Hilb}$ , hence in  $T^\circ$  in particular. Recall that  $Q$  was defined as the limit of  $Q_i$ . Hence for  $i \gg 0$ ,  $Q_i$  is also in  $T^\circ$ . Then by [OSS, Lemma 3.6],  $i(Q_i) \in V_{[X]}$ , which is well-defined, is GIT polystable with respect to the action of  $\mathrm{Aut}(X)$ .

Then we get a contradiction from the general geometric invariant theory [Mum] since  $i(Q_i)$  and  $P_i$  are both GIT polystable, while being the limits of sequences which parametrise the same polystable point. This completes the proof.  $\square$

**Proposition 3.4.** *Let  $X$  be an arbitrary  $\mathbb{Q}$ -Gorenstein smoothable Kähler–Einstein  $\mathbb{Q}$ -Fano variety and denote the corresponding point in the Hilbert scheme as  $[X]$  which represents an  $m$ -pluri-anticanonically embedding of  $X$ . Then there is a small enough affine  $\text{Aut}(X)$ -invariant slice  $V_{[X]}$  of the natural  $\text{PGL}$ -action on  $\text{Hilb}$  such that an open neighborhood (in the analytic topology) of  $[\bar{X}]$  in the GIT (categorical) quotient  $V_{[X]}/\text{Aut}(X)$  naturally maps homeomorphically to  $\bar{M}^{\text{GH}}$  (which eventually becomes an étale algebraic morphism with the algebraic structure on the latter).*

*Analytically speaking, this is equivalent to saying that there is an open subset  $W$  of  $[X]$  in  $V_{[X]}$  and an analytically open neighborhood  $N$  of  $[X] \in \bar{M}^{\text{GH}}$  such that there is a natural homeomorphism*

$$N \rightarrow (W \cap V_{[X]}^{\text{ps}})/\text{Aut}(X),$$

*preserving the  $\mathbb{Q}$ -Fano varieties being parametrised.*

*Proof.* The continuity from  $N$  to  $(W \cap V_{[X]}^{\text{ps}})/\text{SL}$  follows from Donaldson–Sun [DS, (proof of) Theorem 1.2]. The quotient space  $\text{Hilb}^{\text{KE}}/\text{SL}$  satisfies the Hausdorff axiom due to the separatedness Theorem 3.3 proved by [LWX1]+[SSY], while  $\bar{M}^{\text{GH}}$  is compact by the Gromov compactness theorem. It is a general theorem that a continuous bijection from a compact topological space (now  $\bar{M}^{\text{GH}}$ ) to a Hausdorff space (now  $\text{Hilb}^{\text{KE}}/\text{SL}$ ) is automatically a homeomorphism.  $\square$

Summarising the above discussions, we conclude the proof of our main theorem 2.3, the moduli construction, as follows.

*Proof of Theorem 2.3.* For each  $[X_i] \in \text{Hilb}^{\text{KE}}$ , i.e.  $X_i$  is a smooth Kähler–Einstein Fano  $n$ -dimensional manifold or a Gromov–Hausdorff limit of such manifolds (hence a  $\mathbb{Q}$ -Fano variety with Kähler–Einstein metric by [DS]), consider  $V_{[X_i]}$  constructed in Subsection 3.1. We replace  $V_{[X_i]}$  by its open  $\text{Aut}(X_i)$ -invariant open neighborhood, if necessary, to make it satisfy the requirement in Theorem 3.2.

Note that for each  $X_i$ ,  $\text{PGL} \cdot V_{[X_i]}$  is a Zariski open subset in  $\text{Hilb}$ . This follows from the fact that since we constructed  $V_{[X_i]} \subset U_{[X_i]}$  as an étale slice,  $\text{PGL} \times_{\text{Aut}(X_i)} V_{[X_i]} \rightarrow \text{Hilb}$  is an étale morphism, so in particular an open morphism. Thus by quasi-compactness of  $\text{Hilb}$ , we only need finitely many sets  $\text{PGL} \cdot V_{[X_i]}$  to cover  $\text{Hilb}^{\text{KE}}$ .

We note that  $\varphi_i: [V_{[X_i]}/\text{Aut}(X_i)] \rightarrow [\text{Hilb}/\text{PGL}]$  is an étale morphism between two quotient stacks, since again the morphism  $\text{PGL} \times_{\text{Aut}(X_i)} V_{[X_i]} \rightarrow U_{[X_i]} \subset \text{Hilb}$  is strongly étale (in the sense of [Dre, Subsection 1.1]). Note that it is a priori *not* necessarily an open immersion (of algebraic stacks) because the slice  $V_{[X_i]}$  is just an étale slice. Glueing together  $[V_{[X_i]}/\text{Aut}(X_i)]$  via  $\varphi_i$ s, which is by definition

possible inside  $[Hilb/PGL]$ , we obtain  $[W/PGL]$  with  $W = \bigcup_i PGL \cdot V_{[X_i]} \subset Hilb$ , a moduli Artin stack which we denote as  $\bar{\mathcal{M}}$ . Furthermore, as the property [Dre, Subsection 1.1(ii)] of the étale slice  $V_{[X_i]}$  (cf. also [Dre, 5.3]) shows, the categorical quotients  $V_{[X_i]}/\text{Aut}(X_i)$  glue together to form a coarse moduli algebraic space  $\bar{M}$  of the Artin stack  $\bar{\mathcal{M}}$ .

The fact that it is a KE moduli stack in the sense of Definition 2.2 ([OSS]) now follows from Theorem 3.2. Indeed, condition (iii) of Definition 2.2 is exactly the statement of Theorem 3.2, and we have proved condition (i) above. The remaining (ii), which says that the flat family on  $V_{[X_i]}$  is  $\mathbb{Q}$ -Gorenstein flat (once we shrink  $V_{[X_i]}$  enough), can be easily checked as follows (see also [OSS, (2.4)] for essentially the same arguments). Actually in general if we have a point  $[X]$  in  $Hilb$  corresponding to some normal variety  $X$ , its deformation parametrised in a neighborhood in  $Hilb$  is automatically  $\mathbb{Q}$ -Gorenstein. We denote the locus of  $Hilb$  which parametrises normal varieties as  $Hilb_{\text{normal}} \subset Hilb$ ; as is well known, that is automatically an open subset. We denote its subset which parametrises singular (but normal) varieties as  $Hilb_{\text{normal.singular}}$ . Let us take a log resolution of singularities of the pair  $(Hilb_{\text{normal}}, Hilb_{\text{normal.singular}})$  after Hironaka as  $f: S \rightarrow Hilb$ , so that  $f^{-1}(Hilb_{\text{normal.singular}})$  is a (simple normal crossing) Cartier divisor  $\Sigma$  of  $S$ . Then we have a flat projective family  $\pi: (\mathcal{X}, \mathcal{O}_{\mathcal{X}}(1)) \rightarrow S$  and

$$(1) \quad \mathcal{O}_{\mathcal{X}}(1)|_{\mathcal{X} \setminus \pi^{-1}(\Sigma)} \sim_{(S \setminus \Sigma)} \mathcal{O}_{(\mathcal{X} \setminus \pi^{-1}(\Sigma))}(-mK_{\mathcal{X} \setminus \pi^{-1}(\Sigma)}).$$

This implies that there are Weil divisors  $\mathcal{D}, \mathcal{D}'$  of  $\mathcal{X}$  with  $\mathcal{O}_{\mathcal{X}}(\mathcal{D}) = \mathcal{O}_{\mathcal{X}}(1)$ ,  $\mathcal{O}_{\mathcal{X}}(\mathcal{D}') = \mathcal{O}_{\mathcal{X}}(-mK_{\mathcal{X}})$  (the latter is only a reflexive sheaf), and with  $\mathcal{D} - \mathcal{D}'$  supported on  $\pi^{-1}(\Sigma)$ . But any (a priori Weil) divisor supported on the central fiber is a pull back of a (Cartier) divisor of  $S$  supported on  $\Sigma$  since all the fibers of  $\pi$  are irreducible now. Hence, we get  $\mathcal{O}(1) \sim_C \mathcal{O}(-mK_{\mathcal{X}})$ .

Furthermore, the subset  $Hilb_{\text{klt}}$  of  $Hilb_{\text{normal.singular}}$  which parametrises (Kawamata-)log-terminal varieties is a Zariski open subset, which follows from the arguments of [AH, Appendix A] (even easier, since we only treat normal varieties). In particular,  $V_{[X_i]}$  only parametrises  $\mathbb{Q}$ -Fano varieties, since each variety parametrised in  $V_{[X_i]}$  has some isotrivial degeneration to a variety parametrised in  $V_{[X_i]}^{\text{PS}}$ , which is automatically a  $\mathbb{Q}$ -Fano variety. Summarising, we have proved assertion (ii) of Definition 2.2.

The topological space structure part is proved in Proposition 3.4. Indeed, Proposition 3.4 shows that the Gromov–Hausdorff compactification  $\bar{M}^{\text{GH}}$  is homeomorphic to the coarse moduli space  $\bar{M}$  constructed above. In particular,  $\bar{M}$  satisfies the Hausdorff second axiom (essentially follows from [CDS]+[LWX1](v1)+[SSY], cf. Theorem 3.3). This completes the proof of Theorem 2.3.  $\square$

**Remark 3.5** (added in revision, 20th March 2015). Our constructions of the moduli stacks  $\bar{\mathcal{M}}$  and their coarse moduli spaces  $\bar{M}$  a priori depend on the positive integer parameter  $m$  (recall that we consider the  $m$ -th pluri-anticanonical polarisation of the  $\mathbb{Q}$ -Fano varieties). However, we strongly believe that they actually do *not* depend on the sufficiently divisible  $m$ . Indeed, we can prove this under the following two hypotheses. To the best of the author’s knowledge (as of March, 2015) full proofs of the hypotheses below are not available yet, although the 2nd revision of [LWX1] has partial affirmative results (cf. Section 7 there) in this direction.

- (i) The K-semistability is an *open condition* for any  $\mathbb{Q}$ -Gorenstein flat projective family of  $\mathbb{Q}$ -Fano ( $\mathbb{Q}$ -Gorenstein smoothable) varieties.
- (ii) Any ( $\mathbb{Q}$ -Gorenstein smoothable) K-semistable  $\mathbb{Q}$ -Fano variety, say  $X$ , has a test configuration whose central fibre is a KE  $\mathbb{Q}$ -Fano variety  $Y$  (which is K-polystable by [Ber]).

Our proof of the desired  $m$ -independence of our moduli  $\bar{\mathcal{M}}$  and  $\bar{M}$ , under the above hypotheses, is simple, as follows. The hypotheses imply that  $W$  coincides exactly with the (open) locus of  *$\mathbb{Q}$ -Gorenstein smoothable K-semistable  $\mathbb{Q}$ -Fano varieties*, which we denote as  $Hilb^{sss}$ . We prove this as follows. Recall that each  $\mathbb{Q}$ -Fano variety corresponding to a point of  $W$  isotrivially degenerates to a KE  $\mathbb{Q}$ -Fano variety parametrised in  $Hilb^{KE}$  by our Theorem 3.2 and standard GIT. That fact, combined with (i), implies  $W \subset Hilb^{sss}$ . On the other hand, (ii) and [DS] (especially their uniform bound of “ $k$ ”) imply  $Hilb^{sss} \subset W$  straightforwardly. Thus our KE moduli stack  $\bar{\mathcal{M}}$ , which is isomorphic to the quotient stack  $[W/PGL]$  whose definition involved  $m$ , is exactly the moduli Artin stack of  $\mathbb{Q}$ -Gorenstein flat projective families of K-semistable  $\mathbb{Q}$ -Gorenstein smoothable  $\mathbb{Q}$ -Fano varieties of dimension  $n$ . It is this universality which automatically implies that the moduli stacks  $\bar{\mathcal{M}}$  do not depend on  $m$ . In particular, their coarse moduli spaces  $\bar{M}$  do not depend on  $m$  either.

We also make a brief remark about the relation with the 2nd version of [LWX1]: the moduli space constructed there is the semi-normalisation of the reduced subscheme of our moduli.

#### §4. Future work

We list some interesting problems on the K-moduli of Fano varieties, possibly with personal bias. Most of them (perhaps except Question 2) are natural and being shared among the community of this subject; we just write them down for the record.

**Question 1.** *How about concrete examples of  $\mathbb{Q}$ -Fano varieties?*

As far as we know, the only fully settled case appears in [MM], [OSS] which handle ( $\mathbb{Q}$ -Gorenstein smoothable) del Pezzo surfaces. We suppose that [OSS, Lemma 3.6] and our Theorem 3.2 will be one of the key tools for this direction. For example, the author is tempted to expect that many of the standard GIT moduli spaces of hypersurfaces, such as cubic 3-folds and 4-folds ([All], [Laza], [Yok1], [Yok2]), are examples of our  $K$ -moduli spaces (cf. [OSS, Theorem 3.4 and Subsection 4.2]). This last prediction is partially inspired by discussions with Julius Ross.

**Question 2.** *How to construct the Gromov–Hausdorff limit of Kähler–Einstein Fano manifolds (and the  $K$ -moduli) in a purely algebraic way?*

It is natural to expect that the (refined) GH limit, in the sense of [DS], [OSS] etc., is simply equivalent to the  $K$ -polystable limit and so, partially in view of [LX], [Od2, last section] (etc.), characterised by the minimality of the degree of (a family version of) the Donaldson–Futaki invariant. And we further expect that the construction will essentially need the idea and theory of the Minimal Model Program.

**Question 3.** *How about non-smoothable  $\mathbb{Q}$ -Fano varieties?*

This is a much more general case, morally about the moduli space all of whose members parametrise *singular* (log-terminal)  $\mathbb{Q}$ -Fano varieties. At this moment, our paper and [SSY], [LWX1] etc. all heavily depend on the ( $\mathbb{Q}$ -Gorenstein) smoothability of Fano varieties considered, in order to apply [CDS], [Tia2] which are for smooth Fano *manifolds*. But many algebraically oriented people agree that it is natural to expect completely the same picture for *general  $\mathbb{Q}$ -Fano varieties*.

**Question 4.** *What about the projectivity of our moduli space?*

The expectation is that the “descended”  $\mathbb{Q}$ -line bundle from the CM line bundle [FS], [PT], explained (with the proof of descending) at the end of [OSS], will be *ample* on the coarse compact moduli space  $\bar{M}$ , ensuring the projectivity. The expectation is based on the general *Weil–Petersson* metrics as in [FS]. Indeed, by [FS], any compact analytic subset of the coarse moduli space of *smooth* KE Fano manifolds with *discrete* automorphism groups (constructed in [Od2]) is projective. However, in the general case, there are two main technical difficulties: the presence of non-discrete automorphism groups (involving  $K$ -*semistable* varieties) and the log-terminal singularities.

(Added in revision, March, 2015: Two and a half months after the first manuscript of this paper appeared, [LWX2] announced a partial progress along this line, claiming the quasi-projectivity of the open locus  $M$  of  $\bar{M}$ .)

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This paper originally grew out from much more personal and incomplete notes sent to and shared with Cristiano Spotti, Song Sun, Chengjian Yao, from October 2014, that is, three months after the results of [SSY] were communicated to the author. The latter happened in July of 2014 during the visit of S. Sun to Kyoto and Tokyo, and also there were several seminar talks by the authors of [SSY], a few months before the appearance of [SSY]. The author is grateful for all clarifications of their results as well as for their helpful comments on the draft, and would like to say that they also made essential contributions to the present paper partially through [SSY] (and [OSS], [DS]). We also thank Jarod Alper for his kind communications about Subsection 3.1.

When the author started to expect “K-moduli” [Od0, Section 5], he could not handle even the Fano case and the partial proof obtained here just makes it clear that he is watching the beauty “on the shoulders of (modern) giants”, especially for the case of this paper, as no essentially new idea is brought in, but simply certain ideas and standard arguments are combined. I would like to take this opportunity to thank all the professors, colleagues and friends for their tutorials.

While finishing the first manuscript of this paper, the author learnt of the possibility of partial overlap of the paper with the revision (the 2nd version) of [LWX1]. The author wishes to clarify that we worked independently and both results appeared on the internet the same day.

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