

Haagerup Approximation Property for Arbitrary von Neumann Algebras

by

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Abstract

We attempt presenting a notion of the Haagerup approximation property for an arbitrary von Neumann algebra by using its standard form. We also prove the expected heredity results for this property.

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§1. Introduction

In the remarkable paper [Ha2], U. Haagerup proves that the reduced C^* -algebra of the non-amenable free group F_d has Grothendieck's metric approximation property. He actually shows that there exists a sequence of normalized positive definite functions φ_n on F_d such that

- (a) $\varphi_n(s) \rightarrow 1$ for every $s \in F_d$;
- (b) φ_n vanishes at infinity for every n .

It is known that many classes of locally compact second countable groups possess such sequences, where pointwise convergence to 1 is replaced by uniform convergence on compact subsets, and it is called the *Haagerup approximation property*. See the book [C+] for more details.

In [Ch], M. Choda observes that a countable discrete group Γ has the Haagerup approximation property if and only if its group von Neumann algebra $L(\Gamma)$ admits a sequence of normal contractive completely positive maps Φ_n on $L(\Gamma)$ such that

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- (A) $\Phi_n \rightarrow \text{id}_{L(\Gamma)}$ in the point-ultraweak topology;
 (B) $\tau \circ \Phi_n \leq \tau$ and Φ_n extends to a compact operator T_n on $\ell^2(\Gamma)$ for every n , which is given by

$$T_n(x\xi_\tau) = \Phi_n(x)\xi_\tau \quad \text{for } x \in L(\Gamma),$$

where τ denotes the canonical tracial state on $L(\Gamma)$. After her work, many authors studied the Haagerup approximation property, including F. Boca [Bo], A. Connes and V. Jones [CJ], P. Jolissaint [Jo] and S. Popa [Po]. However, it is defined only for a finite von Neumann algebra. In the case of a non-finite von Neumann algebra, one runs into the problem of how to describe *vanishing at infinity* in (b) or *compactness* in (B) for a completely positive map.

After the systematic study of one-parameter families of convex cones in the Hilbert space on which a von Neumann algebra acts with a distinguished cyclic and separating vector by H. Araki [Ar], and the independent work by Connes [Co1], Haagerup [Ha1] proved that any von Neumann algebra is isomorphic to a von Neumann algebra M on a Hilbert space H such that there exists a conjugate-linear isometric involution J on H and a self-dual positive cone P in H with the following properties:

- (i) $JMJ = M'$;
- (ii) $J\xi = \xi$ for any $\xi \in P$;
- (iii) $aJaJP \subset P$ for any $a \in M$;
- (iv) $JcJ = c^*$ for any $c \in \mathcal{Z}(M) := M \cap M'$.

Such a quadruple (M, H, J, P) is called a *standard form* of the von Neumann algebra M .

Let \mathbb{M}_n denote the $n \times n$ complex matrices. Then $M \otimes \mathbb{M}_n$ operates in its standard form on $H \otimes \mathbb{M}_n$ with the self-dual positive cone $P^{(n)}$, where $P^{(1)} = P$. The partial order on $H \otimes \mathbb{M}_n$ induced by $P^{(n)}$ turns H into the matrix ordered Hilbert space in the sense of M. D. Choi and E. G. Effros [CE]. Thus we will say that an operator T on H is *completely positive* if $(T \otimes \text{id}_n)P^{(n)} \subset P^{(n)}$ for all $n \geq 1$. So an arbitrary von Neumann algebra M is defined to have the Haagerup approximation property if the identity of H can be approximated in the strong operator topology by contractive completely positive compact operators.

The Haagerup approximation property is also defined in other ways for a non-finite von Neumann algebra. One definition is the following: A σ -finite von Neumann algebra M with a faithful normal state φ is said to have the Haagerup approximation property for φ if there exists a net of unital completely positive φ -preserving normal maps Φ_n on M such that

- (A') $\Phi_n \rightarrow \text{id}_M$ in the point-ultraweak topology;
- (B') the following implementing operators T_n on H_φ are contractive and compact:

$$T_n(x\xi_\varphi) = \Phi_n(x)\xi_\varphi \quad \text{for } x \in M.$$

However we wonder whether this definition sufficiently captures the property of the corresponding compact operator T_n in (B) in the case where M is finite. More precisely, one of our main results is the following (Theorem 4.9):

Theorem A. *Let M be a σ -finite von Neumann algebra with a faithful normal state φ . Then M has the Haagerup approximation property if and only if there exists a net of normal contractive completely positive maps Φ_n on M such that*

- (A') $\Phi_n \rightarrow \text{id}_M$ in the point-ultraweak topology;
- (B'') the operators T_n on H_φ are contractive and compact, where

$$T_n(\Delta_\varphi^{1/4}x\xi_\varphi) = \Delta_\varphi^{1/4}\Phi_n(x)\xi_\varphi \quad \text{for } x \in M.$$

In [To], A. M. Torpe gives a characterization of semidiscrete von Neumann algebras in terms of matrix ordered Hilbert spaces: a von Neumann algebra M is semidiscrete if and only if the identity of the Hilbert space H with respect to its standard form can be approximated in the strong operator topology by completely positive contractions of finite rank. A similar characterization of semidiscrete von Neumann algebras is also given by M. Junge, Z.-J. Ruan and Q. Xu [JRX] in terms of non-commutative L^p -spaces. In particular, the non-commutative L^2 -spaces become standard forms, and hence their result is a generalization of Torpe's characterization of semidiscrete von Neumann algebras. Therefore it immediately follows that injectivity implies the Haagerup approximation property in our sense.

The Haagerup approximation property has various stabilities. Among them, we will prove the following result (Theorem 5.9):

Theorem B. *Let $N \subset M$ be an inclusion of von Neumann algebras. Suppose that there exists a norm one projection from M onto N . If M has the Haagerup approximation property, then so does N .*

In [CS], M. Caspers and A. Skalski independently introduce the notion of the Haagerup approximation property. Our formulation actually coincides with theirs because in either case, the Haagerup approximation property is preserved under taking crossed products by \mathbb{R} -actions. (See Remark 5.8.)

This paper is organized as follows: In Section 2, the basic notions are reviewed and we introduce the Haagerup approximation property for a von Neumann algebra. In Section 3, we study some permanence properties such as reduced von

Neumann algebras, tensor products, the commutant and the direct sums. In Section 4, we consider the case where M is a σ -finite von Neumann algebra with a faithful normal state φ ; we present the proof of Theorem A; and we discuss the free product of von Neumann algebras and examples. In Section 5, we study the crossed product of a von Neumann algebra by a locally compact group. We show that a von Neumann algebra has the Haagerup approximation property if and only if so does its core von Neumann algebra. Finally, the proof of Theorem B is presented.

§2. Definition

We first fix the notation and recall basic facts. Let M be a von Neumann algebra. We denote by M_{sa} and M^+ the sets of all self-adjoint elements and all positive elements in M , respectively. We also denote by M_* and M_*^+ the spaces of all normal linear functionals and all positive normal linear functionals on M , respectively.

Let us recall the definition of a standard form of a von Neumann algebra as given by Haagerup [Hal].

Definition 2.1. Let M be a von Neumann algebra, H a Hilbert space on which M acts, J a conjugate-linear isometry on H with $J^2 = 1_H$, and $P \subset H$ a closed convex cone which is self-dual, i.e.,

$$P = \{\xi \in H \mid \langle \xi, \eta \rangle \geq 0 \text{ for } \eta \in P\}.$$

Then the quadruple (M, H, J, P) is called a *standard form* if

- (i) $JMJ = M'$;
- (ii) $J\xi = \xi$ for any $\xi \in P$;
- (iii) $xJxJP \subset P$ for any $x \in M$;
- (iv) $JcJ = c^*$ for any $c \in \mathcal{Z}(M) := M \cap M'$.

Remark 2.2. Recently, Ando and Haagerup [AH, Lemma 3.19] proved that condition (iv) in the above definition can actually be dropped.

By the work of Araki [Ar], every functional $\varphi \in M_*^+$ is represented as $\varphi = \omega_{\xi_\varphi}$ by a unique vector $\xi_\varphi \in P$, where

$$\omega_{\xi_\varphi}(x) = \langle x\xi_\varphi, \xi_\varphi \rangle \quad \text{for } x \in M.$$

Moreover the *Araki–Powers–Størmer inequality* holds:

$$\|\xi_\varphi - \xi_\psi\|^2 \leq \|\varphi - \psi\| \leq \|\xi_\varphi - \xi_\psi\| \|\xi_\varphi + \xi_\psi\| \quad \text{for } \varphi, \psi \in M_*.$$

A vector $\xi \in H$ is said to be *self-adjoint* if $J\xi = \xi$. We denote by H_{sa} the set of all self-adjoint vectors in H . For $\xi, \eta \in H_{\text{sa}}$, we will write $\xi \geq \eta$ if $\xi - \eta \in P$. Note that for $\xi \in H_{\text{sa}}$ there exist unique vectors $\xi_+, \xi_- \in P$ such that $\xi = \xi_+ - \xi_-$ and $\langle \xi_+, \xi_- \rangle = 0$.

We next show that a faithful normal semifinite (f.n.s.) weight gives a standard form. We refer to the book of Takesaki [Ta2] for details.

Let φ be an f.n.s. weight on a von Neumann algebra M and let

$$n_\varphi := \{x \in M \mid \varphi(x^*x) < \infty\}.$$

Then H_φ is the completion of n_φ with respect to the norm

$$\|x\|_\varphi^2 := \varphi(x^*x) \quad \text{for } x \in n_\varphi.$$

We write the canonical injection $\Lambda_\varphi: n_\varphi \rightarrow H_\varphi$. Then

$$\mathcal{A}_\varphi := \Lambda_\varphi(n_\varphi \cap n_\varphi^*)$$

is an achieved left Hilbert algebra with the multiplication

$$\Lambda_\varphi(x) \cdot \Lambda_\varphi(y) := \Lambda_\varphi(xy) \quad \text{for } x \in n_\varphi \cap n_\varphi^*$$

and the involution

$$\Lambda_\varphi(x)^\sharp := \Lambda_\varphi(x^*) \quad \text{for } x \in n_\varphi \cap n_\varphi^*.$$

Let π_φ be the corresponding representation of M on H_φ . We always identify M with $\pi_\varphi(M)$.

Let S_φ be the closure of the conjugate-linear operator $\xi \mapsto \xi^\sharp$ on H_φ ; it has the polar decomposition

$$S_\varphi = J_\varphi \Delta_\varphi^{1/2},$$

where J_φ is the modular conjugation and Δ_φ is the modular operator. Then we have a self-dual positive cone

$$P_\varphi := \overline{\{\xi(J_\varphi\xi) \mid \xi \in \mathcal{A}_\varphi\}} \subset H_\varphi.$$

Note that P_φ is given by the closure of the set of $\Lambda_\varphi(x\sigma_{i/2}^\varphi(x)^*)$, where $x \in \mathcal{A}_\varphi$ is entire with respect to the modular automorphism group $\sigma_t^\varphi := \text{Ad } \Delta_\varphi^{it}|_M$.

Therefore the quadruple $(M, H_\varphi, J_\varphi, P_\varphi)$ is a standard form. A standard form is, in fact, unique up to a spatial isomorphism, and so it is independent of the choice of an f.n.s. weight φ .

Theorem 2.3 ([Ha1, Theorem 2.3]). *Let (M_1, H_1, J_1, P_1) and (M_2, H_2, J_2, P_2) be two standard forms and let $\pi: M_1 \rightarrow M_2$ be an isomorphism. Then there exists a unique unitary $u: H_1 \rightarrow H_2$ such that*

- (1) $\pi(x) = uxu^*$ for any $x \in M_1$;
- (2) $J_2 = uJ_1u^*$;
- (3) $P_2 = uP_1$.

Let us consider the $n \times n$ matrix algebra \mathbb{M}_n with the normalized trace tr_n . If we define the inner product on \mathbb{M}_n by

$$\langle x, y \rangle := \text{tr}_n(y^*x) \quad \text{for } x, y \in \mathbb{M}_n,$$

then \mathbb{M}_n can also be regarded as a Hilbert space. Moreover \mathbb{M}_n is an achieved left Hilbert algebra such that the modular operator is the identity operator on \mathbb{M}_n and the modular conjugation is the canonical involution $J_{\text{tr}_n} : x \mapsto x^*$. Hence the quadruple $(\mathbb{M}_n, \mathbb{M}_n, J_{\text{tr}_n}, \mathbb{M}_n^+)$ is a standard form.

Let (M, H, J, P) be a standard form and consider the von Neumann algebra $\mathbb{M}_n(M) := M \otimes \mathbb{M}_n$ on $\mathbb{M}_n(H) := H \otimes \mathbb{M}_n$. If we consider an f.n.s. weight $\varphi \otimes \text{tr}_n$ on $M \otimes \mathbb{M}_n$ for a fixed f.n.s. weight φ on M , then we can give a standard form of $\mathbb{M}_n(M)$ as mentioned before. However we give a standard form without using an f.n.s. weight. The following definition is given by Miura and Tomiyama [MT].

Definition 2.4 ([MT, Definition 2.1]). Let (M, H, J, P) be a standard form and $n \in \mathbb{N}$. A matrix $[\xi_{i,j}] \in \mathbb{M}_n(H)$ is said to be *positive* if

$$\sum_{i,j=1}^n x_i J x_j J \xi_{i,j} \in P \quad \text{for all } x_1, \dots, x_n \in M.$$

We denote by $P^{(n)}$ the set of all positive matrices $[\xi_{i,j}]$ in $\mathbb{M}_n(H)$.

Proposition 2.5 ([MT, Proposition 2.4], [SW1, Lemma 1.1]). Let (M, H, J, P) be a standard form and $n \in \mathbb{N}$. Then $(\mathbb{M}_n(M), \mathbb{M}_n(H), J^{(n)}, P^{(n)})$ is a standard form, where $J^{(n)} := J \otimes J_{\text{tr}_n}$.

Definition 2.6. Let (M_1, H_1, J_1, P_1) and (M_2, H_2, J_2, P_2) be two standard forms. We will say that a bounded linear (or conjugate-linear) operator $T : H_1 \rightarrow H_2$ is *n-positive* if

$$T^{(n)} P_1^{(n)} \subset P_2^{(n)},$$

where $T^{(n)} : \mathbb{M}_n(H_1) \rightarrow \mathbb{M}_n(H_2)$ is defined by $T^{(n)}([\xi_{i,j}]) := [T\xi_{i,j}]$.

Moreover we will say that T is *completely positive (c.p.)* if T is n -positive for any $n \in \mathbb{N}$.

We are now ready to give our definition of the Haagerup approximation property for a von Neumann algebra.

Definition 2.7. A W^* -algebra M has the *Haagerup approximation property* (HAP) if there exists a standard form (M, H, J, P) and a net of contractive completely positive (c.c.p.) compact operators T_n on H such that $T_n \rightarrow 1_H$ in the strong topology.

From this definition, it is clear that if a von Neumann algebra M_2 is isomorphic to M_1 which has the HAP, then so does M_2 . Moreover the property does not depend on the choice of a standard form. Indeed, let (M_1, H_1, J_1, P_1) and (M_2, H_2, J_2, P_2) be two standard forms of von Neumann algebras, and $\pi: M_1 \rightarrow M_2$ be an isomorphism. By Theorem 2.3, there is a unitary $u: H_1 \rightarrow H_2$ such that $\pi(x) = uxu^*$ for $x \in M_1$, $J_2 = uJ_1u^*$, and $P_2 = uP_1$. Let T_n^1 be a net of c.c.p. compact operators on H_1 as in the previous definition. Then one can easily check that $T_n^2 := uT_n^1u^*$ gives a desired net of c.c.p. compact operators on H_2 .

Remark 2.8. A notion of the HAP can also be defined for a matrix ordered Hilbert space in the sense of Choi and Effros [CE]. However, in this paper we only consider the case of a standard form of a von Neumann algebra.

In [To], Torpe gives a characterization of semidiscrete von Neumann algebras in terms of standard forms. In [JRX], Junge, Ruan and Xu also give a similar characterization of semidiscrete von Neumann algebras in terms of non-commutative L^p -spaces for $1 \leq p < \infty$. In particular, in the case where $p = 2$, the non-commutative L^2 -space gives a standard form. Hence their result is a generalization of Torpe’s characterization. As a corollary, injectivity implies the HAP.

Theorem 2.9 ([To, Theorem 2.1], [JRX, Theorem 3.2]). *Let (M, H, J, P) be a standard form. Then the following are equivalent:*

- (1) M is semidiscrete.
- (2) There exists a net of c.c.p. finite rank operators T_n on H such that $T_n \rightarrow 1_H$ in the strong topology.

Corollary 2.10. *If a von Neumann algebra M is injective, then M has the HAP.*

Remark 2.11. Unfortunately, Torpe’s paper [To] is unpublished. However, the implication (1) \Rightarrow (2) is proved by L. M. Schmitt [Sc] with Torpe’s techniques. We also comment on her proof of the other implication in Remark 4.10.

§3. Permanence properties

In this section, we study various permanence properties of the Haagerup approximation property.

§3.1. Reduction

We first recall the following result in [Ha1].

Lemma 3.1 ([Ha1, Corollary 2.5, Lemma 2.6]). *Let (M, H, J, P) be a standard form of a von Neumann algebra and q a projection of the form $q = pJpJ$, where $p \in M$ is a projection. Then*

- (1) *the induction $pap \mapsto qxq$ is an isomorphism from pMp onto qMq ;*
- (2) *the quadruple (qMq, qH, qJq, qP) is a standard form.*

Let (M, H, J, P) be a standard form and $p \in M$ be a projection with $q := pJpJ$. We write $M_q := qMq$, $H_q := qH$, $J_q := qJq$ and $P_q := qP$, respectively. On the one hand, we have a standard form

$$(\mathbb{M}_n(M_q), \mathbb{M}_n(H_q), J_q^{(n)}, P_q^{(n)}).$$

Notice that $(\mathbb{M}_n(M), \mathbb{M}_n(H), J^{(n)}, P^{(n)})$ is a standard form. Set $p^{(n)} := p \otimes 1_n \in \mathbb{M}_n(M)$ and $q^{(n)} := p^{(n)}J^{(n)}p^{(n)}J^{(n)}$. Then

$$q^{(n)} := p^{(n)}J^{(n)}p^{(n)}J^{(n)} = q \otimes 1_n.$$

On the other hand, by Lemma 3.1, we have a standard form

$$(q^{(n)}\mathbb{M}_n(M)q^{(n)}, q^{(n)}\mathbb{M}_n(H), q^{(n)}J^{(n)}q^{(n)}, q^{(n)}P^{(n)}).$$

Note that $\mathbb{M}_n(M_q) = q^{(n)}\mathbb{M}_n(M)q^{(n)}$, $\mathbb{M}_n(H_q) = q^{(n)}\mathbb{M}_n(H)$ and $J_q^{(n)} = q^{(n)}J^{(n)}q^{(n)}$. Moreover two standard forms, in fact, coincide.

Lemma 3.2. *In the above setting, $P_q^{(n)} = q^{(n)}P^{(n)}$.*

Proof. Let $[\xi_{i,j}] \in P^{(n)}$. For any $x_1, \dots, x_n \in M$, we have

$$\sum_{i,j=1}^n (qx_iq)(qJq)(qx_jq)(qJq)(q\xi_{i,j}) = q \sum_{i,j=1}^n (px_ip)J(px_jp)J\xi_{i,j} \in qP.$$

Hence $[q\xi_{i,j}] \in P_q^{(n)}$. Therefore $q^{(n)}P^{(n)} \subset P_q^{(n)}$.

To show the other inclusion, let $\xi \in P_q^{(n)}$. Then $\omega_\xi \in \mathbb{M}_n(M_q)_*$. Since $q^{(n)}P^{(n)}$ is a self-dual cone of a standard form of $\mathbb{M}_n(M_q)$, there exists $\eta \in q^{(n)}P^{(n)}$ such that $\omega_\xi = \omega_\eta$ in $\mathbb{M}_n(M_q)_*$. By the discussion above, we also have $\eta \in P_q^{(n)}$. By the uniqueness of ξ , we have $\xi = \eta \in q^{(n)}P^{(n)}$. Therefore $P_q^{(n)} = q^{(n)}P^{(n)}$. \square

Lemma 3.3. *For $x \in M$, $xJxJ$ is a c.p. operator.*

Proof. For $[\xi_{i,j}] \in P^{(n)}$, we have

$$[xJxJ\xi_{i,j}] = (x \otimes 1_n)(J \otimes J_{\text{tr}_n})(x \otimes 1_n)(J \otimes J_{\text{tr}_n})[\xi_{i,j}] \in P^{(n)}. \quad \square$$

Theorem 3.4. *Let (M, H, J, P) be a standard form and $p \in M$ a projection with $q := pJpJ$. If M has the HAP, then so does qMq . In particular, pMp also has the HAP.*

Proof. Since M has the HAP, there exists a net of c.c.p. compact operators T_n on H such that $T_n \rightarrow 1_H$ in the strong topology. Then $S_n := qT_nq$ gives a desired net for qMq by Lemma 3.3. By Lemma 3.1, pMp is isomorphic to qMq . Hence pMp also has the HAP. □

Proposition 3.5. *Let (M, H, J, P) be a standard form and (p_n) an increasing net of projections of M such that $p_n \rightarrow 1_H$ in the strong operator topology. If p_nMp_n has the HAP for all n , then so does M .*

Proof. Let $q_n := p_nJp_nJ$. By Lemma 3.1, q_nMq_n has the HAP for all n . Let F be a finite subset of H and $\varepsilon > 0$. Since $q_n \rightarrow 1$ in the strong topology, there exists n_F such that

$$\|q_{n_F}\xi - \xi\| < \varepsilon/2 \quad \text{for } \xi \in F.$$

Since $q_{n_F}Mq_{n_F}$ has the HAP, there exists a c.c.p. compact operator T on $q_{n_F}H$ such that

$$\|T(q_{n_F}\xi) - q_{n_F}\xi\| < \varepsilon/2 \quad \text{for } \xi \in F.$$

Now we define a c.c.p. compact operator $S := Tq_{n_F}$ on H . Since

$$\|S\xi - \xi\| \leq \|T(q_{n_F}\xi) - q_{n_F}\xi\| + \|q_{n_F}\xi - \xi\| < \varepsilon \quad \text{for } \xi \in F,$$

it follows that M has the HAP. □

§3.2. Norm one projection

Secondly, we consider an inclusion of von Neumann algebras, $N \subset M$ and study when N inherits the HAP from M . One answer will be presented in Theorem 5.9, which states that it is so when there exists a norm one projection from M onto N . In the following, let us prove this assuming normality.

Theorem 3.6. *Let $N \subset M$ be an inclusion of von Neumann algebras. Suppose that there exists a normal conditional expectation from M onto N . If M has the HAP, then so does N .*

Proof. Let $\mathcal{E}: M \rightarrow N$ be a normal conditional expectation. Take an increasing net of σ -finite projections p_n in N such that $p_n \rightarrow 1$ in the strong topology. Then we have a normal conditional expectation $\mathcal{E}_n: p_nMp_n \rightarrow p_nNp_n$, which is given by $\mathcal{E}_n(p_nxp_n) := \mathcal{E}(p_nxp_n) = p_n\mathcal{E}(x)p_n$ for $x \in M$. By Theorem 3.4, p_nMp_n has the HAP. Thanks to Proposition 3.5, N has the HAP if each p_nNp_n does. Hence we may and do assume that N is σ -finite.

Suppose that \mathcal{E} is faithful. Let ψ be a faithful normal state on N . Then $\varphi := \psi \circ \mathcal{E} \in M_*^+$ is also faithful. The projection E from H_φ onto $K_\varphi := \overline{N\xi_\varphi}$ is given by $E(x\xi_\varphi) = \mathcal{E}(x)\xi_\varphi$ for $x \in M$.

Thanks to [Ta2, IX, §4, Theorem 4.2], the modular operator Δ_φ and E commute. Thus it turns out that $E: H_\varphi \rightarrow K_\varphi$ is a c.p. operator, where we regard K_φ as the GNS Hilbert space of N with respect to ψ . Moreover the inclusion operator $V: K_\varphi \rightarrow H_\varphi$ is also a c.p. operator. Let T_n be a net of c.c.p. compact operators for M such that $T_n \rightarrow 1_H$ in the strong topology. Then ET_nV gives a net of c.c.p. compact operators such that $ET_nV \rightarrow 1_K$ in the strong topology, that is, N has the HAP.

If \mathcal{E} is not faithful, there exists a projection $e \in M \cap N'$ such that the central support of e in N' is the identity and $\{x \in M \mid \mathcal{E}(x^*x) = 0\} = M(1 - e)$. Moreover we obtain a faithful normal conditional expectation $\mathcal{E}': eMe \rightarrow Ne$, given by $\mathcal{E}'(x) = \mathcal{E}(x)e$ for $x \in eMe$. By Theorem 3.4, eMe has the HAP, and so does Ne by our discussion above. Let $x \in N$. Then $\mathcal{E}(xe) = x$, which implies that \mathcal{E} is an isomorphism from Ne onto N , and N has the HAP. \square

§3.3. Tensor product and commutant

Next we show the following theorem on tensor products.

Theorem 3.7. *Let M_1 and M_2 be von Neumann algebras. Then M_1 and M_2 have the HAP if and only if so does $M_1 \otimes M_2$.*

To prove this, we recall several results from [MT, SW1, SW2]. Let (M_1, H_1, J_1, P_1) and (M_2, H_2, J_2, P_2) be two standard forms of von Neumann algebras. For $\zeta \in H_1 \otimes H_2$, we define a bounded conjugate-linear map $r(\zeta): H_1 \rightarrow H_2$ by

$$r(\zeta)(\xi) := (\xi^* \otimes 1)\zeta \quad \text{for } \xi \in H_1.$$

Definition 3.8 ([MT, Definition 2.7]). For $n \in \mathbb{N}$, the set of all elements $\zeta \in H_1 \otimes H_2$ such that $r(\zeta)$ is a c.p. map from H_1 to H_2 is denoted by $P_1 \widehat{\otimes} P_2$.

Theorem 3.9 ([MT, Theorem 2.8], [SW2, Theorem 1]). *The cone $P_1 \widehat{\otimes} P_2$ contains $P_1 \otimes P_2$ and is the self-dual cone in $H_1 \otimes H_2$ such that $(M_1 \overline{\otimes} M_2, H_1 \otimes H_2, J_1 \otimes J_2, P_1 \widehat{\otimes} P_2)$ is a standard form.*

Corollary 3.10 ([MT, Corollary 2.9]). *The cone $P_1 \widehat{\otimes} P_2$ coincides with the closure of*

$$\left\{ \sum_{i,j=1}^n \xi_{i,j} \otimes \eta_{i,j} \mid n \in \mathbb{N}, [\xi_{i,j}] \in P_1^{(n)}, [\eta_{i,j}] \in P_2^{(n)} \right\}.$$

Under the identification $\mathbb{M}_n(M_1 \overline{\otimes} M_2) = M_1 \overline{\otimes} \mathbb{M}_n(M_2)$ and $\mathbb{M}_n(H_1 \otimes H_2) = H_1 \otimes \mathbb{M}_n(H_2)$, the self-dual positive cone $P_1 \widehat{\otimes} P_2^{(n)}$ gives a standard form of $\mathbb{M}_n(M_1 \overline{\otimes} M_2)$ by [SW1, Corollary 2.3].

Lemma 3.11. *If T_1 and T_2 are c.p. operators on H_1 and H_2 , respectively, then $T_1 \otimes T_2$ is a c.p. operator on $H_1 \otimes H_2$.*

Proof. Since T_1 and T_2 are c.p. operators, it suffices to show that $T_1 \otimes T_2$ is positive. Let $\zeta \in P_1 \widehat{\otimes} P_2$. By Corollary 3.10, we may assume that

$$\zeta = \sum_{i,j=1}^n \xi_{i,j} \otimes \eta_{i,j},$$

where $n \in \mathbb{N}$, $[\xi_{i,j}] \in P_1^{(n)}$, $[\eta_{i,j}] \in P_2^{(n)}$. Then

$$(T_1 \otimes T_2)\zeta = \sum_{i,j=1}^n T_1 \xi_{i,j} \otimes T_2 \eta_{i,j},$$

which belongs to $P_1 \widehat{\otimes} P_2$ by Corollary 3.10. □

Proof of Theorem 3.7. We show the “only if” part. Since M_i has the HAP, there exists a net of c.c.p. compact operators T_n^i on H_i such that $T_n^i \rightarrow 1_{H_i}$ in the strong topology for $i = 1, 2$. Then by Lemma 3.11, $T_n := T_n^1 \otimes T_n^2$ gives the desired net of c.c.p. compact operators on $H_1 \otimes H_2$. The “if” part follows from Theorem 3.6 with slice maps determined by states. □

The proof of the following theorem is inspired by [HT, Theorem 2.8].

Theorem 3.12. *If M has the HAP, then M' has the HAP.*

Proof. Since a representation of a von Neumann algebra consists of an amplification, an induction and a spatial isomorphism, it suffices to prove the statement for $N = M \otimes 1_K$ or $Q = Mp'$ for a projection $p' \in M'$, where K denotes a Hilbert space. Taking the commutants of these, we obtain $N' = M' \overline{\otimes} \mathbb{B}(K)$ or $Q' = p'M'p'$. They have the HAP by Theorems 3.4 and 3.7. □

Corollary 3.13. *Let M be a von Neumann algebra and $p \in M$ be a projection with central support 1 in M . The von Neumann algebra M has the HAP if and only if pMp has the HAP. In particular, a factor M has the HAP if and only if a corner of M has the HAP.*

Proof. The “only if” part is nothing but Theorem 3.4. Conversely, suppose that pMp has the HAP. Then by Theorem 3.12, $(pMp)' = M'p$ has the HAP. Since

the central support of p in M' equals 1, the induction $M' \ni x \mapsto xp \in M'p$ is an isomorphism. Thus M' has the HAP, and so does M again by Theorem 3.12. \square

§3.4. Direct sum

Finally, we consider the direct sum of von Neumann algebras.

Theorem 3.14. *Let $(M_i)_{i \in I}$ be a family of von Neumann algebras. Then $\bigoplus_{i \in I} M_i$ has the HAP if and only if M_i has the HAP for all $i \in I$.*

Proof. We write $M := \bigoplus_{i \in I} M_i$. If M has the HAP, then M_i has the HAP by Theorem 3.4.

Conversely, let (M_i, H_i, J_i, P_i) be a standard form for $i \in I$. We denote

$$H := \bigoplus_{i \in I} H_i, \quad J := \bigoplus_{i \in I} J_i, \quad P := \bigoplus_{i \in I} P_i.$$

Then (M, H, J, P) is a standard form. Let F be a subset of I , and T_i be a c.c.p. compact operator on H_i for $i \in I$. Then we define a c.c.p. compact operator T_F on H by

$$T_F := \left(\bigoplus_{i \in F} T_i \right) p_F J p_F J,$$

where p_F is the projection of M onto $\bigoplus_{i \in F} M_i$.

Let $\varepsilon > 0$ and $\xi^1, \dots, \xi^m \in H$. We denote $\xi^k = \bigoplus_{i \in I} \xi_i^k$ with $\xi_i^k \in H_i$ for $1 \leq k \leq m$. Since $\|\xi^k\|^2 = \sum_{i \in I} \|\xi_i^k\|^2 < \infty$, there is a finite subset $F \subset I$ such that

$$\sum_{i \notin F} \|\xi_i^k\|^2 < \frac{\varepsilon}{2} \quad \text{for } 1 \leq k \leq m.$$

For each $i \in F$, since M_i has the HAP, there exists a c.c.p. compact operator T_i on H_i such that

$$\|T_i \xi_i^k - \xi_i^k\|^2 < \frac{\varepsilon}{2|F|} \quad \text{for } 1 \leq k \leq m.$$

Then

$$\|T_F \xi^k - \xi^k\|^2 = \sum_{i \in F} \|T_i \xi_i^k - \xi_i^k\|^2 + \sum_{i \notin F} \|\xi_i^k\|^2 < \varepsilon. \quad \square$$

Corollary 3.15. *Let π be a normal $*$ -homomorphism from M into N . Then M has the HAP if and only if $\pi(M)$ and $\ker \pi$ have the HAP.*

Proof. Take a central projection $z \in M$ such that $\ker \pi = Mz$ and $M(1 - z)$ is isomorphic to $\pi(M)$. Since $M = Mz \oplus M(1 - z)$, the corollary follows from Theorem 3.14. \square

§4. σ -finite von Neumann algebras

Let M be a σ -finite von Neumann algebra with a faithful state $\varphi \in M_*^+$. We denote by $(\pi_\varphi, H_\varphi, \xi_\varphi)$ the GNS construction of (M, φ) . We always identify M with $\pi_\varphi(M)$. We also denote by Δ_φ and J_φ the modular operator and the modular conjugation, respectively. Denote by P_φ the norm closure of the cone $\Delta_\varphi^{1/4} M^+ \xi_\varphi$ in H_φ . Then $(M, H_\varphi, J_\varphi, P_\varphi)$ is a standard form.

§4.1. Construction of completely positive maps

Let (M, H, J, P) be a standard form and $\xi_0 \in P$ be a cyclic and separating vector. Then we denote by Δ_{ξ_0} the associated modular operator. Note that the associated modular conjugation equals J by [Ha1, Lemma 2.9].

Lemma 4.1 (cf. [Col, Theorem 2.7], [AHW, Lemma 4.8]). *Let (M, H, J, P) be a standard form of a σ -finite von Neumann algebra M . Let $\xi_0 \in P$ be a cyclic and separating vector. Then the map $\Theta_{\xi_0} : M \rightarrow H$ defined by*

$$\Theta_{\xi_0}(x) := \Delta_{\xi_0}^{1/4} x \xi_0 \quad \text{for } x \in M$$

induces an order isomorphism between $\{x \in M_{\text{sa}} \mid -c1 \leq x \leq c1\}$ and $K_{\xi_0} := \{\xi \in H_{\text{sa}} \mid -c\xi_0 \leq \xi \leq c\xi_0\}$ for each $c > 0$. Moreover Θ_{ξ_0} is $\sigma(M, M_)$ - $\sigma(H, H)$ continuous.*

Proof. The first part of the lemma is proved in [AHW, Lemma 4.8]. We need to show that Θ_{ξ_0} is $\sigma(M, M_*)$ - $\sigma(H, H)$ continuous. Since

$$\Delta_{\xi_0}^{1/4} x \xi_0 = (\Delta_{\xi_0}^{1/4} + \Delta_{\xi_0}^{-1/4})^{-1} (x \xi_0 + J_{\xi_0} x^* \xi_0)$$

and $(\Delta_{\xi_0}^{1/4} + \Delta_{\xi_0}^{-1/4})^{-1}$ is bounded, the conclusion follows. □

Lemma 4.2. *Let (M, H, J, P) be a standard form and $\xi \in P$. Then*

- (1) $f_\xi : H \rightarrow \mathbb{C}, \zeta \mapsto \langle \zeta, \xi \rangle$, is a c.p. operator;
- (2) $g_\xi : \mathbb{C} \rightarrow H, z \mapsto z\xi$, is a c.p. operator.

Proof. (1) For $[\xi_{i,j}] \in P^{(n)}$, we have

$$f_\xi^{(n)}([\xi_{i,j}]) = [f_\xi(\xi_{i,j})] = [\langle \xi_{i,j}, \xi \rangle].$$

This is a positive matrix. Indeed, if $z_1, \dots, z_n \in \mathbb{C}$, then

$$\sum_{i,j=1}^n \langle \xi_{i,j}, \xi \rangle z_i \bar{z}_j = \left\langle \sum_{i,j=1}^n z_i J z_j J \xi_{i,j}, \xi \right\rangle \geq 0.$$

(2) Let $[z_{i,j}] \in \mathbb{M}_n^+$. Take $[w_{i,j}] \in \mathbb{M}_n$ so that $z_{i,j} = \sum_{k=1}^n w_{i,k} \overline{w_{j,k}}$. Then $g_\xi^{(n)}([z_{i,j}]) = [z_{i,j}\xi]$ belongs to $P^{(n)}$. Indeed, for $x_1, \dots, x_n \in M$, putting $y_k := \sum_{i=1}^n x_i w_{i,k}$, we have

$$\sum_{i,j=1}^n x_i J x_j J z_{i,j} \xi = \sum_{k=1}^n y_k J y_k J \xi \in P. \quad \square$$

Lemma 4.3. *Suppose that there exists a net of c.p. operators S_n on H_φ such that $S_n \rightarrow 1_{H_\varphi}$ in the strong topology. Then there exists a net of c.p. operators S'_n on H_φ satisfying the following:*

- (1) $S'_n \rightarrow 1_{H_\varphi}$ in the strong topology;
- (2) $S'_n - S_n$ has rank one for all n ;
- (3) $\|S'_n - S_n\| \rightarrow 0$;
- (4) $S'_n \xi_\varphi$ is cyclic and separating for all n .

In particular, if M has the HAP, then there exists a net of c.c.p. compact operators T_n on H_φ such that $T_n \rightarrow 1_{H_\varphi}$ in the strong topology and $T_n \xi_\varphi$ is cyclic and separating for all n .

Proof. Let (S_n) be a net of c.p. operators on H_φ such that $S_n \rightarrow 1_{H_\varphi}$ in the strong topology. Set $\eta_n := S_n \xi_\varphi \in P_\varphi$. Then we define $\xi_n := \eta_n + (\eta_n - \xi_\varphi)_- \in P_\varphi$. Since

$$\xi_n - \xi_\varphi = \eta_n + (\eta_n - \xi_\varphi)_- - \xi_\varphi = (\eta_n - \xi_\varphi)_+ \in P_\varphi,$$

we have $\xi_n \geq \xi_\varphi$. For any $\eta \in P_\varphi$, if $\langle \xi_n, \eta \rangle = 0$, then $\langle \xi_\varphi, \eta \rangle = 0$, and thus $\eta = 0$. By [Co1, Lemma 4.3], ξ_n is cyclic and separating.

Now we define a bounded operator S'_n on H_φ by

$$S'_n \xi := S_n \xi + \langle \xi, \xi_\varphi \rangle (\xi_n - \eta_n) \quad \text{for } \xi \in H_\varphi.$$

By Lemma 4.2, S'_n is a c.p. operator. Note that $S'_n - S_n$ has rank one and

$$S'_n \xi_\varphi = S_n \xi_\varphi + \langle \xi_\varphi, \xi_\varphi \rangle (\xi_n - \eta_n) = \xi_n.$$

Since

$$\|\xi_n - \eta_n\| = \|(\eta_n - \xi_\varphi)_-\| \leq \|\eta_n - \xi_\varphi\| = \|S_n \xi_\varphi - \xi_\varphi\| \rightarrow 0,$$

we have $\|S'_n \xi - \xi\| \rightarrow 0$ for any $\xi \in H_\varphi$, and $\|S'_n - S_n\| \rightarrow 0$.

If M has the HAP, then we may assume that the above operators S_n are compact with $\|S_n\| \leq 1$. Let $\xi \in H_\varphi$ with $\|\xi\| = 1$. Since $\|S_n\| \leq 1$, we obtain

$$0 \leq 1 - \|S_n\| \leq \|\xi\| - \|S_n \xi\| \leq \|\xi - S_n \xi\| \rightarrow 0.$$

Hence $\|S_n\| \rightarrow 1$, and thus $\|S'_n\| \rightarrow 1$. Then $T_n := \|S'_n\|^{-1} S'_n$ is a c.c.p. compact operator such that $T_n \rightarrow \text{id}_{H_\varphi}$ in the strong topology, and $T_n \xi_\varphi$ is cyclic and separating. □

Lemma 4.4 (cf. [Ar, Theorem 10]). *Let (M, H, J, P) be a standard form and $\xi_0 \in P$ be a cyclic and separating vector. If (ξ_n) is a net of cyclic and separating vectors in P such that $\xi_n \rightarrow \xi_0$, then $f(\Delta_{\xi_n}) \rightarrow f(\Delta_{\xi_0})$ in the strong topology for any $f \in C_0[0, \infty)$. In particular $(\Delta_{\xi_n}^{1/4} + \Delta_{\xi_n}^{-1/4})^{-1} \rightarrow (\Delta_{\xi_0}^{1/4} + \Delta_{\xi_0}^{-1/4})^{-1}$ in the strong topology.*

Lemma 4.5 (cf. [Wo, Theorem 1.1]). *Let (M, H, J, P) be a standard form and $\xi_0 \in P$ be a cyclic and separating vector. Let $C > 0$ and s be a positive sesquilinear form on $M \times M$ such that $s(x, y) \geq 0$ and $s(x, 1) \leq C\omega_{\xi_0}(x)$ for $x, y \in M^+$. Then*

$$s(x, x) \leq C\|\Delta_{\xi_0}^{1/4}x\xi_0\|^2 \quad \text{for } x \in M.$$

Lemma 4.6. *Let (M, H, J, P) be a standard form and $\eta_0 \in P$ be a cyclic and separating vector. Then for $x, y \in M^+$, one has*

$$0 \leq \langle \Delta_{\eta_0}^{1/4}x\eta_0, \Delta_{\eta_0}^{1/4}y\eta_0 \rangle \leq \|y\|\langle x\eta_0, \eta_0 \rangle.$$

Proof. Set $y' := JyJ \in M'$. Then

$$\begin{aligned} \langle \Delta_{\eta_0}^{1/4}x\eta_0, \Delta_{\eta_0}^{1/4}b\eta \rangle &= \langle \Delta_{\eta_0}^{1/2}x\eta_0, y\eta_0 \rangle = \langle Jy'\eta_0, J\Delta_{\eta_0}^{1/2}x\eta \rangle \\ &= \langle JyJ\eta_0, x\eta_0 \rangle = \langle xy'\eta_0, \eta_0 \rangle. \end{aligned}$$

Since xy' is positive and $xy' = x^{1/2}y'x^{1/2} \leq \|y'\|x = \|y\|x$, we are done. □

By applying the above lemmas, we can produce a c.p. operator from a c.p. map.

Proposition 4.7. *Let (M, H, J, P) be a standard form of a σ -finite von Neumann algebra M with cyclic and separating vectors $\xi_0, \eta_0 \in P$. Let Φ be a c.p. map on M such that $\omega_{\eta_0} \circ \Phi \leq C\omega_{\xi_0}$ for some $C > 0$. Then there exists a c.p. operator T on H with $\|T\| \leq (C\|\Phi\|)^{1/2}$ such that*

$$T(\Delta_{\xi_0}^{1/4}x\xi_0) = \Delta_{\eta_0}^{1/4}\Phi(x)\eta_0 \quad \text{for } x \in M.$$

Proof. We define a positive sesquilinear s_Φ on $M \times M$ by

$$s_\Phi(x, y) := \langle \Delta_{\eta_0}^{1/4}\Phi(x)\eta_0, \Delta_{\eta_0}^{1/4}\Phi(y)\eta_0 \rangle \quad \text{for } x, y \in M.$$

Note that the corresponding modular operators Δ_{ξ_0} and Δ_{η_0} may not coincide. However, by [Ha1, Lemma 2.9], we have $P = P_{\xi_0} = P_{\eta_0}$ and $J = J_{\xi_0} = J_{\eta_0}$ because $\xi_0, \eta_0 \in P$. Then one can easily check that

$$s_\Phi(x, y) \geq 0 \quad \text{for } x, y \in M^+.$$

Moreover for $x \in M^+$, by Lemma 4.6, we have

$$s_\Phi(x, 1) = \langle \Delta_{\eta_0}^{1/4} \Phi(x) \eta_0, \Delta_{\eta_0}^{1/4} \Phi(1) \eta_0 \rangle \leq \|\Phi(1)\| \langle \Phi(x) \eta_0, \eta_0 \rangle \leq C \|\Phi\| \omega_{\xi_0}(x).$$

By Lemma 4.5, we obtain

$$s_\Phi(x, x) = \|\Delta_{\eta_0}^{1/4} \Phi(x) \eta_0\|^2 \leq C \|\Phi\| \|\Delta_{\xi_0}^{1/4} x \xi_0\|^2 \quad \text{for } x \in M.$$

Hence there exists a bounded operator T on H with $\|T\| \leq (C \|\Phi\|)^{1/2}$ and with

$$T(\Delta_{\xi_0}^{1/4} x \xi_0) = \Delta_{\eta_0}^{1/4} \Phi(x) \eta_0 \quad \text{for } x \in M.$$

Finally we show that T is a c.p. operator. Let $(e_{i,j})$ be a system of matrix units for \mathbb{M}_n . For $[x_{i,j}] \in \mathbb{M}_n(M)^+$, we have

$$\begin{aligned} (T \otimes \text{id}_n)(\Delta_{\xi_0}^{1/4} \otimes \text{id}_n) \left(\sum_{i,j=1}^n x_{i,j} \otimes e_{i,j} \right) (\xi_0 \otimes 1_n) &= \sum_{i,j=1}^n T(\Delta_{\xi_0}^{1/4} x_{i,j} \xi_0) \otimes e_{i,j} \\ &= \sum_{i,j=1}^n \Delta_{\eta_0}^{1/4} \Phi(x_{i,j}) \eta_0 \otimes e_{i,j}. \end{aligned}$$

Since Φ is a c.p. map, $[\Phi(x_{i,j})] \in \mathbb{M}_n(M)^+$. Hence T is a c.p. operator. □

In Lemma 4.8 and Theorem 4.9 below, we deal with possibly non-contractive c.p. operators. So, we use the symbol S for a *not necessarily contractive* c.p. operator. Similarly, we employ the symbol Ψ for a *not necessarily contractive* c.p. map.

Lemma 4.8. *Let M be a σ -finite von Neumann algebra with a faithful state $\varphi \in M_*^+$. Suppose that there exists a net of compact c.p. operators S_n on H_φ such that $S_n \rightarrow 1_{H_\varphi}$ in the strong topology and $\sup_n \|S_n\| < \infty$. Then there exists a net of normal c.c.p. maps $\tilde{\Phi}_m$ on M and compact c.p. operators \tilde{S}_m on H_φ (over a new directed set) such that*

- $\tilde{\Phi}_m \rightarrow \text{id}_M$ in the point-ultraweak topology;
- $\sup_m \|\tilde{S}_m\| < \infty$;
- $\tilde{S}_m(\Delta_\varphi^{1/4} x \xi_\varphi) = \Delta_\varphi^{1/4} \tilde{\Phi}_m(x) \xi_\varphi$ for $x \in M$.

Proof. Let S_n be as stated above. By Lemma 4.3, we may and do assume that $\xi_n := S_n \xi_\varphi$ is cyclic and separating by taking sufficiently large n so that $\|S_n\|$ is uniformly bounded. Let Θ_{ξ_φ} and Θ_{ξ_n} be the maps given in Lemma 4.1. Let $x \in M_{\text{sa}}$. Take $c > 0$ so that $-c1 \leq x \leq c1$. Then $-c\xi_\varphi \leq \Delta_\varphi^{1/4} x \xi_\varphi \leq c\xi_\varphi$. Applying S_n to this inequality, we obtain $-c\xi_n \leq S_n \Delta_\varphi^{1/4} x \xi_\varphi \leq c\xi_n$, because S_n is

positive. In view of Lemma 4.1, the operator $\Theta_{\xi_n}^{-1}(S_n \Delta_\varphi^{1/4} x \xi_\varphi)$ in M is well-defined. Hence we can define a linear map $\Phi_n : M \rightarrow M$ by

$$\Phi_n = \Theta_{\xi_n}^{-1} \circ S_n \circ \Theta_{\xi_\varphi}.$$

In other words,

$$S_n(\Delta_\varphi^{1/4} x \xi_\varphi) = \Delta_{\xi_n}^{1/4} \Phi_n(x) \xi_n \quad \text{for } x \in M.$$

It is easy to check that Φ_n is a normal unital completely positive (u.c.p.) map.

Step 1. We will show that $\Phi_n \rightarrow \text{id}_M$ in the point-ultraweak topology.

Since normal functionals of the form $\omega_{y' \xi_\varphi}$ with $y' \in M'$ span a dense subspace in M_* , it suffices to show that

$$(4.1) \quad \langle \Phi_n(x) \xi_\varphi, y' \xi_\varphi \rangle \rightarrow \langle x \xi_\varphi, y' \xi_\varphi \rangle \quad \text{for } x \in M, y' \in M'_{\text{sa}}.$$

To prove this, we first claim that

$$(4.2) \quad \|\Delta_{\xi_n}^{1/4} \Phi_n(x) \xi_n - \Delta_\varphi^{1/4} x \xi_\varphi\| \rightarrow 0.$$

Indeed, since $S_n \rightarrow 1_{H_\varphi}$ in the strong topology, we have

$$\|S_n(\Delta_\varphi^{1/4} x \xi_\varphi) - \Delta_\varphi^{1/4} x \xi_\varphi\| \rightarrow 0.$$

Hence our claim (4.2) follows.

Secondly we claim that

$$(4.3) \quad \|\Delta_{\xi_n}^{-1/4} y' \xi_n - \Delta_\varphi^{-1/4} y' \xi_\varphi\| \rightarrow 0.$$

Indeed, if we set $y := Jy'J \in M_{\text{sa}}$, then (4.3) is equivalent to

$$\|\Delta_{\xi_n}^{1/4} y \xi_n - \Delta_\varphi^{1/4} y \xi_\varphi\| \rightarrow 0.$$

Since

$$\Delta_\varphi^{1/4} y \xi_\varphi = (J + 1)(\Delta_\varphi^{1/4} + \Delta_\varphi^{-1/4})^{-1} y \xi_\varphi$$

and

$$\Delta_{\xi_n}^{1/4} y \xi_n = (J + 1)(\Delta_{\xi_n}^{1/4} + \Delta_{\xi_n}^{-1/4})^{-1} y \xi_n,$$

our claim (4.3) is also equivalent to

$$\|(\Delta_{\xi_n}^{1/4} + \Delta_{\xi_n}^{-1/4})^{-1} y \xi_n - (\Delta_\varphi^{1/4} + \Delta_\varphi^{-1/4})^{-1} y \xi_\varphi\| \rightarrow 0.$$

However, this easily follows from Lemma 4.4 and $\|\xi_n - \xi_\varphi\| \rightarrow 0$.

Thus to prove (4.1), it suffices to show that

$$\langle \Phi_n(x) \xi_n, y' \xi_n \rangle \rightarrow \langle x \xi_\varphi, y' \xi_\varphi \rangle,$$

because $\|\xi_n - \xi_\varphi\| \rightarrow 0$. By (4.3), there is a constant $C_{y'} > 0$ and n_0 such that

$$\|\Delta_{\xi_n}^{-1/4} y' \xi_n\| \leq C_{y'} \quad \text{for all } n \geq n_0.$$

By using (4.2) and (4.3), we have

$$\begin{aligned} & |\langle \Phi_n(x) \xi_n, y' \xi_n \rangle - \langle x \xi_\varphi, y' \xi_\varphi \rangle| \\ &= |\langle \Delta_{\xi_n}^{1/4} \Phi_n(x) \xi_n, \Delta_{\xi_n}^{-1/4} y' \xi_n \rangle - \langle \Delta_\varphi^{1/4} x \xi_\varphi, \Delta_\varphi^{-1/4} y' \xi_\varphi \rangle| \\ &\leq |\langle \Delta_{\xi_n}^{1/4} \Phi_n(x) \xi_n - \Delta_\varphi^{1/4} x \xi_\varphi, \Delta_{\xi_n}^{-1/4} y' \xi_n \rangle| + |\langle \Delta_\varphi^{1/4} x \xi_\varphi, \Delta_{\xi_n}^{-1/4} y' \xi_n - \Delta_\varphi^{-1/4} y' \xi_\varphi \rangle| \\ &\leq C_{y'} \|\Delta_{\xi_n}^{1/4} \Phi_n(x) \xi_n - \Delta_\varphi^{1/4} x \xi_\varphi\| + \|\Delta_\varphi^{1/4} x \xi_\varphi\| \|\Delta_{\xi_n}^{-1/4} y' \xi_n - \Delta_\varphi^{-1/4} y' \xi_\varphi\| \rightarrow 0. \end{aligned}$$

This is our claim (4.1), that is, $\Phi_n \rightarrow \text{id}_M$ in the point-ultraweak topology.

Step 2. We will make a small perturbation of Φ_n .

Set $\varphi_n := \omega_{\xi_n} \in M_*^+$. Since $\|\xi_n - \xi_\varphi\| \rightarrow 0$, we have $\|\varphi_n - \varphi\| \rightarrow 0$ by the Araki–Powers–Størmer inequality. If we set $\psi_n := \varphi + (\varphi - \varphi_n)_-$, then $\varphi_n \leq \psi_n$. Thanks to Sakai’s Radon–Nikodym theorem [Sa, Theorem 1.24.3], there exists $h_n \in M$ with $0 \leq h_n \leq 1$ such that $\varphi_n(x) = \psi_n(h_n x h_n)$ for $x \in M$. We define a c.p. map $\Psi_n: M \rightarrow M$ by

$$\Psi_n(x) := h_n x h_n + (\varphi - \varphi_n)_-(h_n x h_n) 1 \quad \text{for } x \in M.$$

Note that $\|\psi_n - \varphi\| = \|(\varphi - \varphi_n)_-\| \leq \|\varphi - \varphi_n\| \rightarrow 0$. Since

$$\begin{aligned} \varphi(1 - h_n^2) &\leq \psi_n(1 - h_n^2) = \psi_n(1) - \varphi_n(1) = \|\psi_n - \varphi_n\| \\ &\leq \|\psi_n - \varphi\| + \|\varphi_n - \varphi\| \rightarrow 0, \end{aligned}$$

we have $(1 - h_n^2)^{1/2} \rightarrow 0$ in the strong topology. Moreover since

$$\|(1 - h_n) \xi\|^2 = \langle (1 - h_n)^2 \xi, \xi \rangle \leq \langle (1 - h_n^2) \xi, \xi \rangle = \|(1 - h_n^2)^{1/2} \xi\| \quad \text{for } \xi \in H_\varphi,$$

we have $h_n \rightarrow 1$ in the strong topology. Consequently, for $x \in M$, we have $h_n x h_n \rightarrow x$ in the strong topology. Therefore $\Psi_n \rightarrow \text{id}_M$ in the point-ultraweak topology. Since

$$\Psi_n(1) = h_n^2 + (\varphi - \varphi_n)_-(h_n^2) 1 \leq 1 + \|\varphi - \varphi_n\| =: C_n \rightarrow 1,$$

a c.p. map $\Phi'_n := \Psi_n / C_n$ is contractive and such that $\Phi'_n \rightarrow \text{id}_M$ in the point-ultraweak topology.

Moreover for $x \in M^+$ we have

$$\varphi \circ \Phi'_n(x) = \frac{1}{C_n} \varphi(\Psi_n(x)) = \frac{1}{C_n} \psi_n(h_n x h_n) = \frac{1}{C_n} \varphi_n(x) \leq \varphi_n(x).$$

By Proposition 4.7, there exists a c.c.p. operator T'_n on H_φ with

$$T'_n(\Delta_\varphi^{1/4}x\xi_n) = \Delta_\varphi^{1/4}\Phi'_n(x)\xi_\varphi \quad \text{for } x \in M.$$

Since $\Phi'_n \rightarrow \text{id}_M$ in the point-ultraweak topology, $T'_n \rightarrow 1_{H_\varphi}$ in the weak topology.

Now we define a normal c.c.p. map $\tilde{\Phi}_n := \Phi'_n \circ \Phi_n$ on M and a c.p. compact operator $\tilde{S}_n := T'_n S_n$ on H_φ , which satisfies $\sup_n \|\tilde{S}_n\| < \infty$. Then we have

$$\tilde{S}_n(\Delta_\varphi^{1/4}x\xi_\varphi) = \Delta_\varphi^{1/4}\tilde{\Phi}_n(x)\xi_\varphi \quad \text{for } x \in M.$$

We first claim that $\tilde{S}_n \rightarrow 1_{H_\varphi}$ in the weak topology. Indeed, for $\xi, \eta \in H_\varphi$,

$$\begin{aligned} |\langle \tilde{S}_n \xi, \eta \rangle - \langle \xi, \eta \rangle| &= |\langle T'_n S_n \xi, \eta \rangle - \langle \xi, \eta \rangle| \leq |\langle T'_n S_n \xi - T'_n \xi, \eta \rangle| + |\langle T'_n \xi - \xi, \eta \rangle| \\ &\leq \|S_n \xi - \xi\| \|\eta\| + |\langle T'_n \xi - \xi, \eta \rangle| \rightarrow 0. \end{aligned}$$

Next we claim that $\tilde{\Phi}_n \rightarrow \text{id}_M$ in the point-ultraweak topology. It suffices to show that

$$\langle \tilde{\Phi}_n(x)\xi_\varphi, y'\xi_\varphi \rangle \rightarrow \langle x\xi_\varphi, y'\xi_\varphi \rangle \quad \text{for } x \in M, y' \in M'.$$

Indeed,

$$\begin{aligned} \langle \tilde{\Phi}_n(x)\xi_\varphi, y'\xi_\varphi \rangle &= \langle \Delta_\varphi^{1/4}\tilde{\Phi}_n(x)\xi_\varphi, \Delta_\varphi^{-1/4}y'\xi_\varphi \rangle = \langle T_n(\Delta_\varphi^{1/4}x\xi_\varphi), \Delta_\varphi^{-1/4}y'\xi_\varphi \rangle \\ &\rightarrow \langle \Delta_\varphi^{1/4}x\xi_\varphi, \Delta_\varphi^{-1/4}y'\xi_\varphi \rangle = \langle x\xi_\varphi, y'\xi_\varphi \rangle. \end{aligned}$$

Finally, by taking suitable convex combinations, we can obtain c.p. compact operators \tilde{S}_m on H_φ and normal c.c.p. maps $\tilde{\Phi}_m$ on M so that $\tilde{S}_m \rightarrow 1_{H_\varphi}$ in the strong topology, $\sup_m \|\tilde{S}_m\| < \infty$, $\tilde{\Phi}_m \rightarrow \text{id}_M$ in the point-ultraweak topology and

$$\tilde{S}_m(\Delta_\varphi^{1/4}x\xi_\varphi) = \Delta_\varphi^{1/4}\tilde{\Phi}_m(x)\xi_\varphi \quad \text{for } x \in M. \quad \square$$

Now we are ready to prove the main theorem of this section.

Theorem 4.9. *Let M be a σ -finite von Neumann algebra with a faithful state $\varphi \in M_*^+$. Then the following statements are equivalent:*

- (1) M has the HAP.
- (2) There exists a net of compact c.p. operators S_n on H_φ such that $S_n \rightarrow 1_{H_\varphi}$ in the strong topology and $\sup_n \|S_n\| < \infty$.
- (3) There exists a net of normal c.c.p. maps Φ_n on M such that
 - (i) $\varphi \circ \Phi_n \leq \varphi$ for all n ;
 - (ii) $\Phi_n \rightarrow \text{id}_M$ in the point-ultraweak topology;
 - (iii) the associated c.c.p. operators T_n on H_φ are compact and $T_n \rightarrow 1_{H_\varphi}$ in the strong topology, where

$$T_n(\Delta_\varphi^{1/4}x\xi_\varphi) = \Delta_\varphi^{1/4}\Phi_n(x)\xi_\varphi \quad \text{for } x \in M.$$

Proof. The implications (1) \Rightarrow (2) and (3) \Rightarrow (1) are trivial.

(2) \Rightarrow (3). Let us take $\tilde{\Phi}_m$ and \tilde{S}_n as in the previous lemma. We will modify the normal c.c.p. maps $\tilde{\Phi}_m$ so that $\varphi \circ \tilde{\Phi}_m \leq \varphi$.

We define $\chi_m: M_* \rightarrow M_*$ by $\chi_m(\omega) := \omega \circ \tilde{\Phi}_m$ for $\omega \in M_*$. By a convexity argument, we may assume that $\|\chi_m(\omega) - \omega\| \rightarrow 0$ for $\omega \in M_*$. Set $\varphi_m := \chi_m(\varphi)$. Note that $\|\varphi_m - \varphi\| \rightarrow 0$. Since $\tilde{\Phi}_m(1) \rightarrow 1$ in the ultraweak topology, we may also assume that $\varphi_m(1) \neq 0$. Since

$$\psi_m := \varphi_m + (\varphi_m - \varphi)_- \geq \varphi,$$

by Sakai's Radon–Nikodym theorem there is $h_m \in M$ with $0 \leq h_m \leq 1$ such that $\varphi(x) = \psi_m(h_m x h_m)$ for $x \in M$. Then we define a normal c.p. map Ψ_m on M by

$$\Psi_m(x) := h_m x h_m + \frac{1}{\varphi_m(1)} (\varphi_m - \varphi)_-(h_m x h_m) 1 \quad \text{for } x \in M.$$

Note that

$$\begin{aligned} \varphi_m \circ \Psi_m(x) &= \varphi_m(h_m x h_m) + \frac{1}{\varphi_m(1)} (\varphi_m - \varphi)_-(h_m x h_m) \varphi_m(1) \\ &= \psi_m(h_m x h_m) = \varphi(x). \end{aligned}$$

Since

$$\begin{aligned} \varphi(1 - h_m^2) &\leq \psi_m(1 - h_m^2) = \psi_m(1) - \varphi(1) = \|\psi_m - \varphi\| \\ &\leq \|\varphi_m - \varphi\| + \|(\varphi_m - \varphi)_-\| \rightarrow 0, \end{aligned}$$

we have $h_m \rightarrow 1$ in the strong topology. Hence $h_m x h_m \rightarrow x$ in the strong topology for $x \in M$. Moreover, since

$$\|(\varphi_m - \varphi)_-(h_m x h_m)\| \leq \|\varphi_m - \varphi\| \|x\| \rightarrow 0 \quad \text{for } x \in M,$$

we have $\Psi_m \rightarrow \text{id}_M$ in the point-ultraweak topology.

Note that

$$\begin{aligned} \Psi_m(1) &= h_m^2 + \frac{1}{\varphi_m(1)} (\varphi_m - \varphi)_-(h_m^2) \\ &\leq 1 + \frac{1}{\varphi_m(1)} \psi_m(h_m^2) = 1 + \frac{1}{\varphi_m(1)} =: C_m \rightarrow 1, \end{aligned}$$

and for $x \in M^+$,

$$\begin{aligned} \varphi \circ \Psi_m(x) &= \varphi(h_m x h_m) + \frac{1}{\varphi_m(1)} (\varphi_m - \varphi)_-(h_m x h_m) \\ &\leq C_m \psi_m(h_m x h_m) = C_m \varphi(x). \end{aligned}$$

By Proposition 4.7, we obtain a c.p. operator S_m on H_φ with $\|S_m\| \leq C_m$ such that

$$S_m(\Delta_\varphi^{1/4}x\xi_\varphi) = \Delta_\varphi^{1/4}\Psi_m(x)\xi_\varphi \quad \text{for } x \in M.$$

We may and do assume that $\sup_m \|S_m\| \leq \sup_m C_m < \infty$. Notice that $S_m \rightarrow 1_{H_\varphi}$ in the weak topology, because $\Psi_m \rightarrow \text{id}_M$ in the point-ultraweak topology.

Finally we define a normal c.p. map $\Psi'_m := \tilde{\Phi}_m \circ \Psi_m$ on M and a c.p. compact operator $S'_m := \tilde{S}_m S_m$ on H_φ . Then $\varphi \circ \Psi'_m = \varphi$ and

$$S'_m(\Delta_\varphi^{1/4}x\xi_\varphi) = \Delta_\varphi^{1/4}\Psi'_m(x)\xi_\varphi \quad \text{for } x \in M.$$

Moreover for $\omega \in M_*$, we have

$$\begin{aligned} |\langle \Psi'_m(x) - x, \omega \rangle| &\leq |\langle \Psi_m(x), \chi_m(\omega) - \omega \rangle| + |\langle \Psi_m(x) - x, \omega \rangle| \\ &\leq C_m \|x\| \|\chi_m(\omega) - \omega\| + |\langle \Psi_m(x) - x, \omega \rangle| \rightarrow 0. \end{aligned}$$

Therefore $\Psi'_m \rightarrow \text{id}_M$ in the point-ultraweak topology, and thus $S'_m \rightarrow 1_{H_\varphi}$ in the weak topology, because $\sup_m \|S'_m\| < \infty$.

Note that

$$\Psi'_m(1) = \Phi_m(h_m^2) + \frac{1}{\varphi_m(1)}(\varphi_m - \varphi)_-(h_m^2)\Phi_m(1) \leq 1 + \frac{\|\varphi_m - \varphi\|}{\varphi_m(1)} =: C'_m \rightarrow 1.$$

We define a normal c.c.p. map Φ_m on M by $\Phi_m := \Psi'_m/C'_m$.

Note that $\varphi \circ \Phi_m \leq \varphi$ and $\Phi_m \rightarrow \text{id}_M$ in the point-ultraweak topology. By Proposition 4.7, we have a c.c.p. operator T_m on H_φ given by

$$T_m(\Delta_\varphi^{1/4}x\xi_\varphi) = \Delta_\varphi^{1/4}\Phi_m(x)\xi_\varphi \quad \text{for } x \in M.$$

Then $T_m = S'_m/C'_m$ is compact and $T_m \rightarrow 1_{H_\varphi}$ in the weak topology. By the convexity argument, we may and do assume that $\Phi_n \rightarrow \text{id}_M$ in the point-ultraweak topology, $T_n \rightarrow 1_{H_\varphi}$ in the strong topology and, moreover, $\varphi \circ \Phi_n \leq \varphi$ and

$$T_n(\Delta_\varphi^{1/4}x\xi_\varphi) = \Delta_\varphi^{1/4}\Phi_n(x)\xi_\varphi \quad \text{for } x \in M. \quad \square$$

Remark 4.10. The proof of Theorem 4.9 is essentially based on the one of [To]. The proof above can also be applied to show Theorem 2.9. Also note that we have proved the existence of c.c.p. maps Φ_n such that $\varphi \circ \Phi_n = \lambda_n \varphi$ for some $0 < \lambda_n \leq 1$. In particular, Φ_n is faithful.

§4.2. Commutativity of c.c.p. operators with modular groups

In this subsection, we study the Haagerup approximation property such that the approximating c.c.p. compact operators commute with a modular group.

Definition 4.11. Let M be a von Neumann algebra with a f.n.s. weight φ . We will say that M has the φ -Haagerup approximation property (φ -HAP) if the c.c.p. compact operators T_n of Definition 2.7 moreover commute with Δ_φ^{it} for all $t \in \mathbb{R}$.

In this case, we can take unital φ -preserving Φ_n 's as shown below.

Theorem 4.12. *Let M be a σ -finite von Neumann algebra with a faithful state $\varphi \in M_*^+$. If M has the φ -HAP, then there exist a net of c.c.p. compact operators T_n on H_φ with $T_n \rightarrow 1_{H_\varphi}$ in the strong topology, and a net of normal u.c.p. maps Φ_n on M with $\Phi_n \rightarrow \text{id}_M$ in the point-ultraweak topology such that*

- (1) $\varphi \circ \Phi_n = \varphi$ for all n ;
- (2) $T_n(\Delta_\varphi^{1/4}x\xi_\varphi) = \Delta_\varphi^{1/4}\Phi_n(x)\xi_\varphi$ for $x \in M$ for all n .

Proof. Suppose that M has the φ -HAP. Recall the proof of Theorem 4.9. We let the starting T_n commute with Δ_φ^{it} for all $t \in \mathbb{R}$. Then it is not so difficult to check that the final Φ_n commutes with σ_t^φ for all $t \in \mathbb{R}$. So, we have T_n and Φ_n as stated in Theorem 4.9 and they commute with the modular group. Thus we have c.c.p. compact operators T_n on H_φ and normal c.c.p. maps Φ_n on M such that

- $T_n(\Delta_\varphi^{1/4}x\xi_\varphi) = \Delta_\varphi^{1/4}\Phi_n(x)\xi_\varphi$ for all $x \in M$;
- $\sigma_t^\varphi \circ \Phi_n = \Phi_n \circ \sigma_t^\varphi$ for all $t \in \mathbb{R}$.

We will make a small perturbation of Φ_n so that its perturbation is unital. Set $\varphi_n := \varphi \circ \Phi_n$. Then $\varphi_n \circ \sigma_t^\varphi = \varphi_n$ for $t \in \mathbb{R}$. By [PT, Theorem 5.12], there exists $h_n \in M_\varphi$ with $0 \leq h_n \leq 1$ such that $\varphi_n(x) = \varphi(h_nx)$ for $x \in M$, where M_φ denotes the centralizer of φ ,

$$M_\varphi := \{x \in M \mid x\varphi = \varphi x\} = \{x \in M \mid \sigma_t^\varphi(x) = x \text{ for } t \in \mathbb{R}\}.$$

Note that $\varphi_n(1) = \varphi(h_n)$. We may assume that $h_n \neq 1$. We set

$$x_n := \frac{1}{\varphi(1-h_n)}(1-\Phi_n(1)) \quad \text{and} \quad y_n := 1-h_n.$$

Next we define a normal c.p. map Φ_n on M by

$$\Phi_n(x) := \Phi_n(x) + \varphi(y_nx)x_n \quad \text{for } x \in M.$$

Then $\varphi \circ \Phi_n = \varphi$. By Proposition 4.7, we obtain a c.p. operator S_n on M_φ with

$$S_n(\Delta_\varphi^{1/4}x\xi_\varphi) = \Delta_\varphi^{1/4}\Phi_n(x)\xi_\varphi \quad \text{for } x \in M.$$

Note that S_n is compact, because

$$\begin{aligned} S_n(\Delta_\varphi^{1/4}x\xi_\varphi) &= \Delta_\varphi^{1/4}\Phi_n(x)\xi_\varphi = \Delta_\varphi^{1/4}\Phi_n(x)\xi_\varphi + \varphi(y_nx)\Delta_\varphi^{1/4}x_n\xi_\varphi \\ &= T_n(\Delta_\varphi^{1/4}x\xi_\varphi) + \varphi(y_nx)\Delta_\varphi^{1/4}x_n\xi_\varphi, \end{aligned}$$

Moreover

$$\Phi_n(1) = \Phi_n(1) + \varphi(y_n)x_n = \Phi_n(1) + \varphi(1 - h_n) \frac{1}{\varphi(1 - h_n)} (1 - \Phi_n(1)) = 1.$$

Finally since $y_n \in M_\varphi$, we have

$$0 \leq \Psi_n(x) - \Phi_n(x) = \varphi(y_n x)x_n \leq \|x\|\varphi(y_n)x_n = \|x\|(1 - \Phi_n(1)) \quad \text{for } x \in M^+$$

Therefore $\Psi_n \rightarrow \text{id}_M$ in the point-ultraweak topology. □

Theorem 4.13. *Let (M_1, φ_1) and (M_2, φ_2) be two σ -finite von Neumann algebras with faithful normal states. If M_i has the φ_i -HAP, $i = 1, 2$, then the free product $(M_1, \varphi_1) \star (M_2, \varphi_2)$ has the $\varphi_1 \star \varphi_2$ -HAP.*

Proof. The proof is essentially given in [Bo, Proposition 3.9]. We will give a sketch. Assume that for $i = 1, 2$, there exists a net of normal u.c.p. maps Φ_n^i on M such that $\varphi_i \circ \Phi_n^i = \varphi_i$ and $\Phi_n^i \rightarrow \text{id}_{M_i}$ in the point-ultraweak topology. The corresponding c.c.p. compact operators T_n^i on H_{φ_i} are defined by

$$T_n^i(\Delta_{\varphi_i}^{1/4} x \xi_{\varphi_i}) = \Delta_{\varphi_i}^{1/4} \Phi_n^i(x) \xi_{\varphi_i} \quad \text{for } x \in M_i.$$

Set $(M, \varphi) := (M_1, \varphi_1) \star (M_2, \varphi_2)$. Then we obtain normal u.c.p. maps $\Phi_n := \Phi_n^1 \star \Phi_n^2$ such that $\varphi \circ \Phi_n = \varphi$ and Φ_n commutes with σ^φ . We write $H_{\varphi_i}^\circ := \ker \varphi_i$ for $i = 1, 2$. Since $T_n^i = 1 \oplus (T_n^i)^\circ$ on $H_{\varphi_i} = \mathbb{C}\xi_{\varphi_i} \oplus H_{\varphi_i}^\circ$, we can define $T_n := T_n^1 \star T_n^2$ on $(H, \xi) := (H_{\varphi_1}, \xi_{\varphi_1}) \star (H_{\varphi_2}, \xi_{\varphi_2})$ by

$$T_n \xi = \xi, \\ T_n(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) = (T_n^{i_1})^\circ \xi_{i_1} \otimes \cdots \otimes (T_n^{i_n})^\circ \xi_{i_n} \quad \text{for } i_1 \neq \cdots \neq i_n.$$

Then each T_n is the c.c.p. compact operator corresponding to Φ_n , and $T_n \rightarrow 1_H$ in the strong topology. □

Remark 4.14. Let M be a σ -finite von Neumann algebra with a faithful state $\varphi \in M_*^+$. Suppose that M has the HAP for φ in the sense of [D+, Definition 6.3], i.e., there exist a net of φ -preserving normal u.c.p. maps Φ_n on M with $\Phi_n \rightarrow \text{id}_M$ in the point-ultraweak topology, and a net of compact contractions T_n on H_φ with $T_n \rightarrow 1_{H_\varphi}$ in the strong topology such that

$$T_n(x\xi_\varphi) = \Phi_n(x)\xi_\varphi \quad \text{for } x \in M.$$

If the above normal u.c.p. normal maps Φ_n satisfy

$$\sigma_t^\varphi \circ \Phi_n = \Phi_n \circ \sigma_t^\varphi \quad \text{for all } t \in \mathbb{R},$$

then M has the φ -HAP in our sense. Indeed, for $x \in M^+$, as in [Ta2, VIII, §2, Lemma 2.3], we set

$$x_\gamma := \sqrt{\frac{\gamma}{\pi}} \int_{\mathbb{R}} \exp(-\gamma t^2) \sigma_t^\varphi(x) dt.$$

Then x_γ is entire for $\gamma > 0$. Hence

$$\begin{aligned} T_n(\Delta_\varphi^{1/4} x_\gamma \xi_\varphi) &= T_n(\sigma_{-i/4}^\varphi(x_\gamma) \xi_\varphi) = \Phi_n(\sigma_{-i/4}^\varphi(x_\gamma)) \xi_\varphi \\ &= \sigma_{-i/4}^\varphi(\Phi_n(x_\gamma)) \xi_\varphi = \Delta_\varphi^{1/4} \Phi_n(x_\gamma) \xi_\varphi. \end{aligned}$$

Since $x_\gamma \rightarrow x$ in the ultraweak topology as $\gamma \rightarrow \infty$, and

$$\Delta_\varphi^{1/4}(x_\gamma \xi_\varphi - x \xi_\varphi) = (J_\varphi + 1)(\Delta_\varphi^{1/4} + \Delta_\varphi^{-1/4})^{-1}(x_\gamma \xi_\varphi - x \xi_\varphi),$$

we have

$$T_n(\Delta_\varphi^{1/4} x \xi_\varphi) = \Delta_\varphi^{1/4} \Phi_n(x) \xi_\varphi \quad \text{for } x \in M.$$

Therefore the above compact contraction T_n is, in fact, a c.p. operator on H_φ , and thus M has the φ -HAP.

The following result states that the combination of the HAP and the existence of an almost periodic state φ implies φ -HAP.

Theorem 4.15. *Let M be a σ -finite von Neumann algebra with the HAP. If there exists a faithful almost periodic state $\varphi \in M_*^+$, then M has the φ -HAP.*

Proof. Thanks to [Co2], there exist a compact group G , an action $\sigma: G \rightarrow \text{Aut}(M)$ and a continuous group homomorphism $\rho: \mathbb{R} \rightarrow G$ such that $\sigma_t^\varphi = \sigma_{\rho(t)}$ for $t \in \mathbb{R}$ and ρ has dense range. Let $U: G \rightarrow \mathbb{B}(H_\varphi)$ be the associated unitary representation which implements σ . Note that $U_g P = P$ and $J_\varphi U_g = U_g J_\varphi$. Hence, U_g is a c.p. unitary operator.

Let (T_n) be a net of c.c.p. compact operators such that $T_n \rightarrow 1_{H_\varphi}$ in the strong topology. We set

$$\tilde{T}_n := \int_G U_g T_n U_g^* dg.$$

Then \tilde{T}_n belongs to $\mathbb{K}(H_\varphi)$ because the compactness of T implies the norm continuity of the map $G \ni g \mapsto U_g T_n U_g^* \in \mathbb{K}(H_\varphi)$. It is clear that \tilde{T}_n is contractive and commuting with $\Delta_\varphi^{it} = U_{\rho(t)}$ for all $t \in \mathbb{R}$. We will show the complete positivity

of \tilde{T}_n . Let $[\xi_{i,j}] \in P^{(m)}$, $m \in \mathbb{N}$. Take $x_1, \dots, x_m \in M$. Then we have

$$\begin{aligned} \sum_{i,j=1}^m x_i J_\varphi x_j J_\varphi \tilde{T}_n \xi_{i,j} &= \int_G dg \sum_{i,j=1}^m x_i J_\varphi x_j J_\varphi U_g T_n U_g^* \xi_{i,j} \\ &= \int_G dg U_g \sum_{i,j=1}^m \sigma_{g^{-1}}(x_i) J_\varphi \sigma_{g^{-1}}(x_j) J_\varphi T_n U_g^* \xi_{i,j}. \end{aligned}$$

Since $T_n U_g^*$ is a c.p. operator, $\sigma_{g^{-1}}(x_i) J_\varphi \sigma_{g^{-1}}(x_j) J_\varphi T_n U_g^* \xi_{i,j} \in P$ for each $g \in G$, and the integral above belongs to P . Hence \tilde{T}_n is a c.p. operator.

We will check that $\tilde{T}_n \rightarrow 1_{H_\varphi}$ in the strong topology. Let $\xi \in H_\varphi$. Then the set $K := \{U_g^* \xi \mid g \in G\}$ is norm compact, and $T_n \rightarrow 1_{H_\varphi}$ uniformly on K in the strong topology. Thus we are done. \square

Corollary 4.16. *Let M be a σ -finite von Neumann algebra with the HAP. If there exists a faithful almost periodic state $\varphi \in M_*^+$, then there exists a net of normal u.c.p. maps Φ_n on M such that*

- (1) $\varphi \circ \Phi_n = \varphi$ for all n ;
- (2) $\Phi_n \circ \sigma_t^\varphi = \sigma_t^\varphi \circ \Phi_n$ for all $t \in \mathbb{R}$;
- (3) $\Phi_n \rightarrow \text{id}_M$ in the point-ultraweak topology;
- (4) the associated operator T_n on H_φ is compact, where

$$T_n(x\xi_\varphi) = \Phi_n(x)\xi_\varphi \quad \text{for } x \in M.$$

Example 4.17. The following examples have the HAP for φ in the sense of [D+, Definition 6.3]. All known examples so far have the φ -HAP.

- the free Araki–Woods factors [HR];
- the free quantum groups [DCFY];
- the duals of quantum permutation groups [Br1];
- the duals of Wang’s quantum automorphism groups [Br2];
- the duals of quantum reflection groups [Le].

Remark 4.18. The Haar state h on a compact quantum group G is almost periodic. Thus if $L^\infty(G)$, the function algebra on G , has the HAP, then $L^\infty(G)$ has the h -HAP.

§5. Crossed products

Let G be a locally compact group and α an action of G on a von Neumann algebra M . Our main result in this section is the following.

Theorem 5.1. *If $M \rtimes_{\alpha} G$ has the HAP, then so does M .*

To prove this, we may and do assume that M is properly infinite by studying the tensor product $\mathbb{B}(\ell_2) \otimes M$ and the action $\text{id} \otimes \alpha$. Let β be the bidual action of α on $M \otimes \mathbb{B}(L^2(G))$. Then β has an invariant weight and β is cocycle conjugate to $\alpha \otimes \text{id}$. Thus we may and do assume that there exists a weight φ on M such that $\varphi \circ \alpha_t = \varphi$ for all $t \in G$.

Let $N := M \rtimes_{\alpha} G$ be the von Neumann algebra generated by the copy of M , $\pi_{\alpha}(M)$, and the copy of G , $\lambda^{\alpha}(G)$, as defined below:

$$(\pi_{\alpha}(x)\xi)(s) = \alpha_{s^{-1}}(x)\xi(s), \quad (\lambda^{\alpha}(t)\xi)(s) = \xi(t^{-1}s)$$

for $x \in M$, $s, t \in G$ and $\xi \in H_{\varphi} \otimes L^2(G)$.

Let $\hat{\varphi}$ be the dual weight of φ . Then for all $x \in n_{\varphi}$ and $f \in C_c(G)$, we obtain

$$\hat{\varphi}((\lambda^{\alpha}(f)\pi_{\alpha}(x))^* \lambda^{\alpha}(f)\pi_{\alpha}(x)) = \varphi(x^*x) \int_G |f(t)|^2 dt.$$

Hence $a := \lambda^{\alpha}(f)\pi_{\alpha}(x) \in n_{\hat{\varphi}}$ and $\|\Lambda_{\hat{\varphi}}(a)\| = \|\Lambda_{\varphi}(x)\|_{\varphi} \|f\|_2$. Actually, it is known that there exists a surjective isometry from $H_{\hat{\varphi}}$ onto $H_{\varphi} \otimes L^2(G)$ which maps $\Lambda_{\hat{\varphi}}(a)$ to $\Lambda_{\varphi}(x) \otimes f$. Thus we will regard $H_{\hat{\varphi}} = H_{\varphi} \otimes L^2(G)$ and

$$\Lambda_{\hat{\varphi}}(\lambda^{\alpha}(f)\pi_{\alpha}(x)) = \Lambda_{\varphi}(x) \otimes f \quad \text{for } x \in n_{\varphi}, f \in L^2(G).$$

Note that φ is α -invariant, and $\lambda^{\alpha}(t)$ is fixed by $\sigma^{\hat{\varphi}}$, that is, $\mathbb{C} \otimes L(G) = \{\lambda^{\alpha}(t) \mid t \in G\}''$ is contained in the centralizer $N_{\hat{\varphi}}$. The following formulae are frequently used:

$$\sigma_t^{\hat{\varphi}}(\pi_{\alpha}(x)) = \pi_{\alpha}(\sigma_t^{\varphi}(x)), \quad \sigma_t^{\hat{\varphi}}(\lambda^{\alpha}(f)) = \lambda^{\alpha}(f)$$

for all $t \in \mathbb{R}$, $x \in M$ and $f \in L^1(G)$.

Denote by Δ_G the modular function of G . In the following, dt denotes a left invariant Haar measure on G . Then $L^1(G)$ is a Banach $*$ -algebra equipped with the convolution product and the involution defined as follows:

$$(f * g)(t) := \int_G f(s)g(s^{-1}t) ds, \quad f^*(t) := \Delta_G(t^{-1})\overline{f(t^{-1})}$$

for $f, g \in L^1(G)$ and $t \in G$. We further recall the following useful formulae:

$$d(st) = dt, \quad d(ts) = \Delta_G(s)dt, \quad d(t^{-1}) = \Delta_G(t^{-1})dt.$$

For $g \in C_c(G)$, let us introduce a map $R_g: H_{\varphi} \rightarrow H_{\hat{\varphi}}$ satisfying

$$R_g\Lambda_{\varphi}(x) := \Lambda_{\hat{\varphi}}(\lambda^{\alpha}(g)\pi_{\alpha}(x)\lambda^{\alpha}(g)^*) \quad \text{for } x \in n_{\varphi}.$$

This map is bounded since

$$\Lambda_{\hat{\varphi}}(\lambda^\alpha(g)\pi_\alpha(x)\lambda^\alpha(g)^*) = J_{\hat{\varphi}}\lambda^\alpha(g)J_{\hat{\varphi}}\Lambda_{\hat{\varphi}}(\lambda^\alpha(g)\pi_\alpha(x)) = J_{\hat{\varphi}}\lambda^\alpha(g)J_{\hat{\varphi}}(\Lambda_\varphi(x) \otimes g),$$

and $\|R_g\| \leq \|g\|_1\|g\|_2$. We will improve this estimate as follows.

Lemma 5.2. *Let $g \in C_c(G)$. Then*

- (1) R_g is a c.p. operator;
- (2) $\|R_g\| \leq \|\Delta_G^{-1/2} \cdot (g^* * g)\|_2$.

Proof. (1) Let $x \in m_\varphi$ be an entire element with respect to σ^φ . Then $xJ_\varphi\Lambda_\varphi(x) = \Lambda_\varphi(x\sigma_{i/2}^\varphi(x)^*)$, and

$$\begin{aligned} R_gxJ_\varphi\Lambda_\varphi(x) &= R_g\Lambda_\varphi(x\sigma_{i/2}^\varphi(x)^*) = \Lambda_{\hat{\varphi}}(\lambda^\alpha(g)\pi_\alpha(x\sigma_{i/2}^\varphi(x)^*)\lambda^\alpha(g)^*) \\ &= \Lambda_{\hat{\varphi}}(\lambda^\alpha(g)\pi_\alpha(x) \cdot \sigma_{i/2}^{\hat{\varphi}}(\lambda^\alpha(g)\pi_\alpha(x))^*) \\ &= \lambda^\alpha(g)\pi_\alpha(x)J_{\hat{\varphi}}\Lambda_{\hat{\varphi}}(\lambda^\alpha(g)\pi_\alpha(x)), \end{aligned}$$

which belongs to $P_{\hat{\varphi}}$. Thus $R_gP_\varphi \subset P_{\hat{\varphi}}$.

Consider the action $\alpha \otimes \text{id}_n$ on $M \otimes \mathbb{M}_n$ for $n \geq 1$. Let $\tilde{R}_g: H_\psi \rightarrow H_{\hat{\psi}}$ be the map as defined above, where $\psi := \varphi \otimes \text{tr}_n$. We have proved that \tilde{R}_g is positive. By the natural identification $H_\psi = H_\varphi \otimes \mathbb{M}_n$ and $\hat{\psi} = \hat{\varphi} \otimes \text{tr}_n$, the map $\tilde{R}_g = R_g \otimes \text{id}_n$ is positive. Hence R_g is n -positive for all n .

(2) Let $x \in n_\varphi$. Then

$$\begin{aligned} \pi_\alpha(x)\lambda^\alpha(g)^* &= \pi_\alpha(x)\lambda^\alpha(g^*) = \pi_\alpha(x) \int_G g^*(t)\lambda^\alpha(t) dt \\ &= \int_G g^*(t)\lambda^\alpha(t)\pi_\alpha(\alpha_{t^{-1}}(x)) dt. \end{aligned}$$

Since $\lambda^\alpha(g)\lambda^\alpha(t) = \Delta_G(t^{-1})\lambda^\alpha(g_{t^{-1}})$, where $g_{t^{-1}}(s) := g(st^{-1})$, we have

$$\lambda^\alpha(g)\pi_\alpha(x)\lambda^\alpha(g)^* = \int_G \Delta_G(t^{-1})g^*(t)\lambda^\alpha(g_{t^{-1}})\pi_\alpha(\alpha_{t^{-1}}(x)) dt.$$

Then

$$\begin{aligned} R_g\Lambda_\varphi(x) &= \int_G \Delta_G(t^{-1})g^*(t)\Lambda_\varphi(\alpha_{t^{-1}}(x)) \otimes g_{t^{-1}} dt \\ &= \int_G g^*(t^{-1})\Lambda_\varphi(\alpha_t(x)) \otimes g_t dt. \end{aligned}$$

Hence for $y \in n_\varphi$, we obtain

$$\begin{aligned} \langle R_g \Lambda_\varphi(x), R_g \Lambda_\varphi(y) \rangle &= \int_{G \times G} g^*(t^{-1}) \overline{g^*(s^{-1})} \langle \Lambda_\varphi(\alpha_t(x)) \otimes g_t, \Lambda_\varphi(\alpha_s(y)) \otimes g_s \rangle ds dt \\ &= \int_{G \times G} g^*(t^{-1}) \overline{g^*(s^{-1})} \varphi(y^* \alpha_{s^{-1}t}(x)) \langle g_{s^{-1}t}, g \rangle ds dt \\ &= \int_{G \times G} g^*(t^{-1} s^{-1}) \overline{g^*(s^{-1})} \varphi(y^* \alpha_t(x)) \langle g_t, g \rangle ds dt. \end{aligned}$$

Since

$$\begin{aligned} \int_G g^*(t^{-1} s^{-1}) \overline{g^*(s^{-1})} ds &= \int_G g^*(t^{-1} s) \overline{g^*(s)} \Delta_G(s^{-1}) ds \\ &= \int_G \Delta_G(t^{-1}) \cdot \Delta_G(t^{-1} s)^{-1} g^*(t^{-1} s) \overline{g^*(s)} ds \\ &= \int_G \Delta_G(t^{-1}) \cdot \overline{(g^*)^*(s^{-1} t) g^*(s)} ds = \Delta_G(t^{-1}) \overline{(g^* * g)(t)}, \end{aligned}$$

and $\langle g_t, g \rangle = (g^* * g)(t)$, we have

$$\langle R_g \Lambda_\varphi(x), R_g \Lambda_\varphi(y) \rangle = \int_G \Delta_G(t^{-1}) |g^* * g(t)|^2 \varphi(y^* \alpha_t(x)) dt.$$

This implies that

$$(5.1) \quad R_g^* R_g \Lambda_\varphi(x) = \int_G \Delta_G(t^{-1}) |g^* * g(t)|^2 \Lambda_\varphi(\alpha_t(x)) dt,$$

and so

$$\|R_g^* R_g\| \leq \int_G \Delta_G(t^{-1}) |g^* * g(t)|^2 dt = \|\Delta_G^{-1/2} \cdot (g^* * g)\|_2^2. \quad \square$$

Remark 5.3. If there exists a non-zero $x \in n_\varphi \cap M^\alpha$, then the equality (5.1) implies $\|R_g\| = \|\Delta_G^{-1/2} \cdot (g^* * g)\|_2$.

Now let \mathcal{U} be the collection of all compact neighborhoods of the neutral element $e \in G$. We equip \mathcal{U} with the structure of a directed set with $U \leq V$ if and only if $V \subset U$ for $U, V \in \mathcal{U}$.

For each $U \in \mathcal{U}$, take a non-zero $g_U \in C_c(G)$ such that $\text{supp } g_U \subset U$. Now let

$$k_U(t) := \|\Delta_G^{-1/2} \cdot (g_U^* * g_U)\|_2^{-2} \Delta_G(t^{-1}) |(g_U^* * g_U)(t)|^2 \quad \text{for } t \in G.$$

Note that $g_U^* * g_U$ is non-zero since so is g_U .

The following lemma is a direct consequence of the definition.

Lemma 5.4. *The function k_U has the following properties:*

- $k_U(t) \geq 0$ for all $t \in G$;
- $\text{supp } k_U \subset U^{-1}U$;
- $\int_G k_U(t) dt = 1$.

In particular, for any continuous function f on G ,

$$\lim_U \int_G k_U(t)f(t) dt = f(e).$$

Lemma 5.5. *Let R_{g_U} be as before. Then*

- (1) $S_U := \|\Delta_G^{-1/2} \cdot (g_U^* * g_U)\|_2^{-1} R_{g_U}$ is a c.c.p. operator from H_φ into $H_{\hat{\varphi}}$;
- (2) $S_U^* S_U \rightarrow 1_{H_\varphi}$ in the strong topology of $\mathbb{B}(H_\varphi)$.

Proof. (1) It is clear from Lemma 5.2 that S_U is a c.c.p. operator.

(2) Let $x \in n_\varphi$. By (5.1), we have

$$\|S_U^* S_U \Lambda_\varphi(x) - \Lambda_\varphi(x)\| \leq \int_G k_U(t) \|\Lambda_\varphi(\alpha_t(x)) - \Lambda_\varphi(x)\| dt.$$

Applying Lemma 5.4 to $f(t) := \|\Lambda_\varphi(\alpha_t(x)) - \Lambda_\varphi(x)\|$, we are done. □

Proof of Theorem 5.1. Let \mathcal{F} be the collection of all finite sets contained in n_φ . It is trivial that $\{\Lambda_\varphi(x) \mid x \in F\}_{F \in \mathcal{F}}$ forms a net of finite sets in H_φ whose union over $F \in \mathcal{F}$ is dense in H_φ .

Let $F \in \mathcal{F}$ be a non-empty set. Employing Lemma 5.5, we can take $U_F \in \mathcal{U}$ so that

$$(5.2) \quad \|S_{U_F}^* S_{U_F} \Lambda_\varphi(x) - \Lambda_\varphi(x)\| < 1/|F| \quad \text{for } x \in F.$$

Next, let T_γ be a net of c.c.p. compact operators on $H_{\hat{\varphi}}$ such that $T_\gamma \rightarrow 1$ in the strong topology of $\mathbb{B}(H_{\hat{\varphi}})$. Then we can find γ_F such that

$$(5.3) \quad \|T_{\gamma_F} S_{U_F} \Lambda_\varphi(x) - S_{U_F} \Lambda_\varphi(x)\| < 1/|F| \quad \text{for } x \in F.$$

Now set $\tilde{T}_F := S_{U_F}^* T_{\gamma_F} S_{U_F}$. Then \tilde{T}_F is a c.c.p. compact operator on H_φ , and by (5.2) and (5.3), we have

$$\|\tilde{T}_F \Lambda_\varphi(x) - \Lambda_\varphi(x)\| < 2/|F| \quad \text{for all } x, y \in F, F \in \mathcal{F}.$$

This implies that $\tilde{T}_F \rightarrow 1_{H_\varphi}$ in the strong topology. □

Corollary 5.6. *Let G be a locally compact abelian group and α an action on a von Neumann algebra. Then M has the HAP if and only if so does $M \rtimes_\alpha G$.*

Proof. The “if” part is nothing but Theorem 5.1. To prove the “only if” part, suppose that M has the HAP. Then so does $M \otimes \mathbb{B}(L^2(G))$ by Corollary and Theorem 3.7. The Takesaki duality states that $M \otimes \mathbb{B}(L^2(G))$ is isomorphic to $(M \rtimes_{\alpha} G) \rtimes_{\hat{\alpha}} \hat{G}$. Hence $M \rtimes_{\alpha} G$ has the HAP by Theorem 5.1. \square

It is well-known that the crossed product $M \rtimes_{\sigma^{\varphi}} \mathbb{R}$ does not depend on the choice of the f.n.s. weight φ . So, we denote it by \widetilde{M} and call it the *core* of M . The reader is referred to [FT], [Ta2] for the cores.

Corollary 5.7. *Let M be a von Neumann algebra and \widetilde{M} the core. Then M has the HAP if and only if so does \widetilde{M} .*

Remark 5.8. M. Caspers and A. Skalski independently introduced the notion of the Haagerup approximation property for arbitrary von Neumann algebras in their setting. One may wonder whether the two definitions differ or not. Actually, these formulations are equivalent as shown below though we give an indirect proof using cores. In either way, a von Neumann algebra has the HAP if and only if so does its core. (See [CS, Corollary 5.10, Theorem 6.6].) Thus we may and do assume that M is finite or of type II_{∞} . If M is of type II_{∞} , then M has a finite projection e with central support 1. Considering the corner eMe , we may and do assume that M is finite. (See [CS, Lemma 4.1, Proposition 5.9].) Then it is fairly trivial that our definition coincides with [CS, Definition 3.1] for a faithful normal tracial state by Theorem 4.9.

As an application of Corollary 5.7, we will prove the following result, which generalizes Theorem 3.6.

Theorem 5.9. *Let $N \subset M$ be an inclusion of von Neumann algebras. Suppose that there exists a norm one projection from M onto N . If M has the HAP, then so does N .*

To prove this, we may assume that N and M are properly infinite by considering $N \otimes \mathbb{B}(\ell^2) \subset M \otimes \mathbb{B}(\ell^2)$ if necessary. Let \widetilde{M} be the core of M , which has the HAP by Corollary 5.7. Note that there exists a norm one projection from \widetilde{M} onto M , obtained by averaging the dual action on \widetilde{M} . Thus we may assume that M is semifinite. Let $N = Q \rtimes_{\theta} \mathbb{R}$ be a continuous decomposition of N for some \mathbb{R} -action θ on a semifinite von Neumann algebra Q . By Corollary 5.6, it suffices to prove that Q has the HAP.

Therefore we may assume that N and M are semifinite. Let $p \in N$ be a finite projection with central support 1 in N . By Corollary 3.13, our task is reduced to proving that pNp has the HAP. So, we may assume that N is finite and also σ -finite by the usual reduction argument with Proposition 3.5.

In the following discussion, τ_N and τ_M denote a faithful normal tracial state on N and a f.n.s. tracial weight on M , respectively. Thanks to [Ha3, Theorem 5.1], there exists a unique f.n.s. operator valued weight T from M onto N such that $\tau_M = \tau_N \circ T$.

Recall the following lemma [A-D, Lemma 3.7] originally due to Connes (see [Co2, p. 102]).

Lemma 5.10. *Let N and M be as in Theorem 5.9. Then for any $\delta > 0$ and a finite subset $F \subset N$, there exists a normal state φ on M such that*

$$(5.4) \quad \|\varphi|_N - \tau_N\|_{N_*} < \delta,$$

$$(5.5) \quad \|a\varphi - \varphi a\|_{M_*} < \delta \quad \text{for all } a \in F.$$

In the following, we will use the notations

$$|x|_{\tau_M} := \tau_M(|x|), \quad \|x\|_{\tau_M} := \tau_M(x^*x)^{1/2} \quad \text{for } x \in M,$$

as well as the notations $|\cdot|_{\tau_N}$ and $\|\cdot\|_{\tau_N}$. An important fact is that they satisfy the triangle inequality by the tracial property.

Lemma 5.11. *Let N and M be as in Theorem 5.9. Then for any $\varepsilon > 0$ and a finite subset $F \subset N$, there exists $b \in n_{\tau_M} \cap M^+$ and a projection $e \in N$ such that*

- $\tau_M(b^2) \leq 1$;
- $(1 - \varepsilon)e \leq T(b^2) \leq (1 + \varepsilon)e$;
- $\tau_N(1 - e) < \varepsilon$;
- $\|\Lambda_{\tau_M}(ab) - \Lambda_{\tau_M}(ba)\| < \varepsilon$ for all $a \in F$.

Proof. We may and do assume that F consists of unitary operators. Let us take $1 \geq \delta > 0$ small enough so that $10\delta^{1/4} < \varepsilon^2$, $1 - \varepsilon < (1 - \delta^{1/4})^2$, and $(1 + \delta^{1/4})^2 < 1 + \varepsilon$. Applying Lemma 5.10 to δ and F , we obtain a state $\varphi \in M_*$ satisfying (5.4) and (5.5). Take the unique vector $\xi \in P_M$ such that $\varphi = \omega_\xi$, where P_M denotes the natural cone of M realized in the GNS Hilbert space H_{τ_M} . We may and do assume that $\xi = \Lambda_{\tau_M}(b)$ for some positive $b \in n_{\tau_M}$. Then we have

$$\varphi(x) = \tau_M(bxb) = \tau_M(b^2x) = \tau_N(T(b^2)x) \quad \text{for } x \in N.$$

In particular, $1 = \varphi(1) = \tau_N(T(b^2))$, and thus $h := T(b^2)$ is an operator in $L^1(N, \tau_N)_+$, where $L^1(N, \tau_N)_+$ denotes the positive operators in $L^1(N, \tau_N)$, the non-commutative L^1 -space with respect to the finite von Neumann algebra $\{N, \tau_N\}$. The L^1 -norm is denoted by $|\cdot|_{\tau_N}$. The L^2 -space of $\{N, \tau_N\}$ and the L^2 -norm are denoted by $L^2(N, \tau_N)$ and $\|\cdot\|_{\tau_N}$. For more details about the non-commutative L^p -space with respect to a faithful normal semifinite tracial weight,

the reader may refer to [Ta2, IX.2]. Then (5.4) implies

$$(5.6) \quad |h - 1|_{\tau_N} < \delta.$$

Applying the Araki–Powers–Størmer inequality to (5.5), we have

$$(5.7) \quad \|\Lambda_{\tau_M}(ubu^*) - \Lambda_{\tau_M}(b)\| \leq \|u\varphi u^* - \varphi\|^{1/2} < \delta^{1/2}.$$

Thus our task is to modify h so that we get a suitable estimate of the operator norm of h . Using the Araki–Powers–Størmer inequality, we have

$$(5.8) \quad \|h^{1/2} - 1\|_{\tau_N}^2 \leq |h - 1|_{\tau_N} < \delta \quad \text{by (5.6)}.$$

Let $h = \int_0^\infty \lambda de(\lambda)$ be the spectral decomposition. Set

$$\begin{aligned} \alpha_\delta &:= (1 - \delta^{1/4})^2, & \beta_\delta &:= (1 + \delta^{1/4})^2, \\ e_1 &:= e([0, \alpha_\delta]), & e_2 &:= e((\beta_\delta, \infty]). \end{aligned}$$

Then it follows from (5.8) that

$$\begin{aligned} \delta^{1/2} \tau_N(e_1) &\leq \int_{[0, \alpha_\delta]} |\lambda^{1/2} - 1|^2 d\tau(e(\lambda)) \leq \|h^{1/2} - 1\|_{\tau_N}^2 < \delta, \\ \delta^{1/2} \tau_N(e_2) &\leq \int_{(\beta_\delta, \infty]} |\lambda^{1/2} - 1|^2 d\tau(e(\lambda)) \leq \|h^{1/2} - 1\|_{\tau_N}^2 < \delta. \end{aligned}$$

Thus

$$(5.9) \quad |e_1|_{\tau_N} = \tau_N(e_1) < \delta^{1/2}, \quad |e_2|_{\tau_N} = \tau_N(e_2) < \delta^{1/2}.$$

Define $e := e([\alpha_\delta, \beta_\delta]) \in N$ and $b' := (eb^2e)^{1/2} \in M$. Then

$$\begin{aligned} \tau_N(1 - e) &= \tau_N(e_1) + \tau_N(e_2) < 2\delta^{1/2} < \varepsilon, \\ \tau_M(b'^2) &= \tau_M(eb^2e) = \varphi(e) \leq 1. \end{aligned}$$

Moreover,

$$T(b'^2) = T(eb^2e) = eT(b^2)e = ehe \leq \beta_\delta e \leq (1 + \varepsilon)e,$$

and similarly $(1 - \varepsilon)e \leq T(b'^2)$.

Next we have

$$\begin{aligned} |T(b'^2) - 1|_{\tau_N} &= |ehe - 1|_{\tau_N} \leq |e(h - 1)e|_{\tau_N} + |e - 1|_{\tau_N} \leq |h - 1|_{\tau_N} + |e - 1|_{\tau_N} \\ &< \delta + 2\delta^{1/2} < 3\delta^{1/2} \quad \text{by (5.6), (5.9)}. \end{aligned}$$

Let $(1 - e)b^2 = v|(1 - e)b^2|$ be the polar decomposition with a partial isometry v in M . Since

$$|(1 - e)h|_{\tau_N} \leq |(1 - e)(h - 1)|_{\tau_N} + |1 - e|_{\tau_N} < \delta + 2\delta^{1/2} < 3\delta^{1/2},$$

we have

$$\begin{aligned}
 (5.10) \quad |b^2(1-e)|_{\tau_M} &= |(1-e)b^2|_{\tau_M} = \tau_M(v^*(1-e)b^2) = \tau_M(bv^*(1-e)b) \\
 &\leq \tau_M(bv^*vb)^{1/2}\tau_M(b(1-e)b)^{1/2} \\
 &\leq \tau_M(b^2)^{1/2}\tau_M((1-e)b^2)^{1/2} = \tau_N((1-e)h)^{1/2} \\
 &= |(1-e)h|_{\tau_N}^{1/2} < \sqrt{3}\delta^{1/4}.
 \end{aligned}$$

Hence

$$(5.11) \quad |b^2 - eb^2e|_{\tau_M} \leq |(1-e)b^2|_{\tau_M} + |eb^2(1-e)|_{\tau_M} < 2\sqrt{3}\delta^{1/4}.$$

Then for $u \in F$, we have

$$\begin{aligned}
 \|\Lambda_{\tau_M}(ub') - \Lambda_{\tau_M}(b'u)\|^2 &= \|\Lambda_{\tau_M}(ub'u^*) - \Lambda_{\tau_M}(b')\|^2 \leq |ub'^2u^* - b'^2|_{\tau_M} \\
 &\leq |u(e-1)b^2eu^*|_{\tau_M} + |ub^2(e-1)u^*|_{\tau_M} \\
 &\quad + |ub^2u^* - b^2|_{\tau_M} + |b^2 - eb^2e|_{\tau_M} \\
 &\leq 4\sqrt{3}\delta^{1/4} + |ub^2u^* - b^2|_{\tau_M} \quad \text{by (5.10), (5.11)}.
 \end{aligned}$$

In the above, the second inequality follows from the Araki–Powers–Størmer inequality. Using again the Araki–Powers–Størmer inequality and (5.7), we obtain

$$\begin{aligned}
 \|\Lambda_{\tau_M}(ub') - \Lambda_{\tau_M}(b'u)\|^2 &\leq 4\sqrt{3}\delta^{1/4} + 2\|\Lambda_{\tau_M}(ubu^*) - \Lambda_{\tau_M}(b)\|_{\tau_M} \\
 &< 4\sqrt{3}\delta^{1/4} + 2\delta^{1/2} < 10\delta^{1/4} < \varepsilon^2.
 \end{aligned}$$

Therefore, b' does the job. □

Let b be as in Lemma 5.11. We define an operator $R_b: H_{\tau_N} \rightarrow H_{\tau_M}$ by

$$R_b(x\xi_{\tau_N}) := \Lambda_{\tau_M}(b^{1/2}xb^{1/2}) \quad \text{for } x \in N.$$

We now show that R_b is a well-defined bounded operator. Let $x, y \in N$. Then

$$\begin{aligned}
 \langle R_b(x\xi_{\tau_N}), R_b(y\xi_{\tau_N}) \rangle &= \langle \Lambda_{\tau_M}(b^{1/2}xb^{1/2}), \Lambda_{\tau_M}(b^{1/2}yb^{1/2}) \rangle = \tau_M(b^{1/2}y^*bx^{1/2}) \\
 &= \tau_M(by^*bx) = \langle \Lambda_{\tau_M}(bx), \Lambda_{\tau_M}(yb) \rangle.
 \end{aligned}$$

We have

$$\|\Lambda_{\tau_M}(bx)\|^2 = \tau_M(x^*b^2x) = \tau_N(x^*T(b^2)x) \leq (1 + \varepsilon)\tau_N(x^*x),$$

so $\|\Lambda_{\tau_M}(bx)\| \leq (1 + \varepsilon)^{1/2}\|x\xi_{\tau_N}\|$. Similarly, $\|\Lambda_{\tau_M}(yb)\| \leq (1 + \varepsilon)^{1/2}\|y\xi_{\tau_N}\|$. Hence

$$|\langle R_b(x\xi_{\tau_N}), R_b(y\xi_{\tau_N}) \rangle| \leq (1 + \varepsilon)\|x\xi_{\tau_N}\| \|y\xi_{\tau_N}\|.$$

This shows that $\|R_b\| \leq (1 + \varepsilon)^{1/2}$.

Lemma 5.12. *Let $\varepsilon > 0$ and $F \subset N$ be as before. Take b and e as in Lemma 5.11. Let R_b be the associated operator defined above. Then*

- (1) R_b is a c.p. operator from H_{τ_N} into H_{τ_M} ;
- (2) $|\langle R_b^* R_b(x\xi_{\tau_N}), y\xi_{\tau_N} \rangle - \langle x\xi_{\tau_N}, y\xi_{\tau_N} \rangle| < \varepsilon\|y\| + 2\varepsilon\|x\|\|y\|$ for all $x \in F$ and $y \in N$.

Proof. (1) This is trivial.

(2) Since $\|\Lambda_{\tau_M}(yb)\| = \tau_M(by^*yb)^{1/2} \leq \|y\|\tau_M(b^2)^{1/2} \leq \|y\|$, we have

$$\begin{aligned} |\langle R_b^* R_b(x\xi_{\tau_N}), y\xi_{\tau_N} \rangle - \langle \Lambda_{\tau_M}(xb), \Lambda_{\tau_M}(yb) \rangle| &= |\langle \Lambda_{\tau_M}(bx) - \Lambda_{\tau_M}(xb), \Lambda_{\tau_M}(yb) \rangle| \\ &\leq \|\Lambda_{\tau_M}(bx) - \Lambda_{\tau_M}(xb)\| \|\Lambda_{\tau_M}(yb)\| \\ &\leq \varepsilon\|y\|, \end{aligned}$$

and

$$\begin{aligned} &|\langle \Lambda_{\tau_M}(xb), \Lambda_{\tau_M}(yb) \rangle - \langle x\xi_{\tau_N}, y\xi_{\tau_N} \rangle| \\ &= |\tau_M(by^*xb) - \tau_N(y^*x)| = |\tau_M(y^*xb^2) - \tau_N(y^*x)| = |\tau_N(y^*x(T(b^2) - 1))| \\ &\leq |\tau_N(y^*x(T(b^2) - e))| + |\tau_N(y^*x(e - 1))| \\ &\leq \|x\|\|y\|\|T(b^2) - e\| + \|x\|\|y\|\tau_N(1 - e) \leq 2\varepsilon\|x\|\|y\|. \end{aligned}$$

Hence we are done. \square

Proof of Theorem 5.9. We have assumed that N is finite and M is semifinite. Let τ_N, τ_M and T be as before. The proof below is similar to that of Theorem 5.1.

Let \mathcal{F} be the collection of all finite subsets in the norm unit ball of N . Then \mathcal{F} is a directed set as before. Applying Lemmas 5.11 and 5.12 to ε and $F \in \mathcal{F}$, we obtain $b(\varepsilon, F) \in n_{\tau_M} \cap M_+$ such that

$$|\langle R_{b(\varepsilon, F)}^* R_{b(\varepsilon, F)}(x\xi_{\tau_N}), y\xi_{\tau_N} \rangle - \langle x\xi_{\tau_N}, y\xi_{\tau_N} \rangle| < 3\varepsilon \quad \text{for all } x, y \in F.$$

Since M has the HAP, there exists a c.c.p. compact operator $T_{(\varepsilon, F)}$ on H_{τ_M} such that

$$|\langle R_{b(\varepsilon, F)}^* T_{(\varepsilon, F)} R_{b(\varepsilon, F)}(x\xi_{\tau_N}), y\xi_{\tau_N} \rangle - \langle R_{b(\varepsilon, F)}^* R_{b(\varepsilon, F)}(x\xi_{\tau_N}), y\xi_{\tau_N} \rangle| < \varepsilon$$

for all $x, y \in F$. If we set $U_{(\varepsilon, F)} := R_{b(\varepsilon, F)}^* T_{(\varepsilon, F)} R_{b(\varepsilon, F)}$, then $U_{(\varepsilon, F)}$ is a c.p. compact operator on H_{τ_N} , because $T_{(\varepsilon, F)}$ is compact. It turns out that $U_{(\varepsilon, F)}$ converges to 1 weakly from the fact that $\|U_{(\varepsilon, F)}\| \leq \|R_{b(\varepsilon, F)}\|^2 \leq 1 + \varepsilon$ and

$$|\langle U_{(\varepsilon, F)}(x\xi_{\tau_N}), y\xi_{\tau_N} \rangle - \langle x\xi_{\tau_N}, y\xi_{\tau_N} \rangle| < 4\varepsilon \quad \text{for all } x, y \in F.$$

Then the net $(1 + \varepsilon)^{-1}U_{(\varepsilon, F)}$ does the job. \square

Let G be a locally compact quantum group in the sense of [KV]. Roughly speaking, G consists of a von Neumann algebra $L^\infty(G)$ and a coproduct $\Delta: L^\infty(G) \rightarrow L^\infty(G) \otimes L^\infty(G)$. Then G is said to be *amenable* if there exists a state m on $L^\infty(G)$, called an *invariant mean* on G , such that $(\text{id} \otimes m) \circ \Delta(x) = m(x)$.

Let α be an action of G on a von Neumann algebra M , that is, α is a unital faithful normal $*$ -homomorphism from M into $M \otimes L^\infty(G)$ such that $(\alpha \otimes \text{id}) \circ \alpha = (\text{id} \otimes \Delta) \circ \alpha$. If m is an invariant mean on G , then the map $(\text{id} \otimes m) \circ \alpha$ is a norm one projection from M onto $M^\alpha := \{x \in M \mid \alpha(x) = x \otimes 1\}$, the fixed point algebra. Thus the following result is an immediate consequence of Theorem 5.9.

Corollary 5.13. *Let G be an amenable locally compact quantum group. Let α be an action of G on a von Neumann algebra M . If M has the HAP, then the fixed point algebra M^α has the HAP.*

By a duality argument, we can generalize Theorem 5.1 as follows.

Corollary 5.14. *Let G be a locally compact quantum group whose dual quantum group is amenable. Let α be an action of G on a von Neumann algebra M . If $M \rtimes_\alpha G$ has the HAP, then so does M .*

Finally, we present a generalization of Corollary 5.6 that is obtained from the previous corollary and the fact that $M \rtimes_\alpha G$ equals the fixed point algebra of $M \otimes \mathbb{B}(L^2(G))$ under the G -action.

Corollary 5.15. *Let G be an amenable locally compact quantum group whose dual quantum group is also amenable. Let α be an action of G on a von Neumann algebra M . Then $M \rtimes_\alpha G$ has the HAP if and only if so does M .*

Remark 5.16. If we apply the same proof of Theorem 5.9 to the inclusion $N \subset M$ such that M is semidiscrete, then we can show that N is semidiscrete. In particular, this gives a proof of the fact that injectivity implies semidiscreteness. Indeed, let M be an injective von Neumann algebra which acts on a Hilbert space H . Then we have a norm one projection E from $\mathbb{B}(H)$ onto M . Since $\mathbb{B}(H)$ is semidiscrete, so is M .

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