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The Denominators of Normalized R-matrices of Types $A_{2n-1}^{(2)}$, $A_{2n}^{(2)}$, $B_n^{(1)}$ and $D_{n+1}^{(2)}$

by

Se-jin Oh

Abstract

The denominators of normalized R-matrices provide important information on finite-dimensional integrable representations over quantum affine algebras, and finite-dimensional graded representations over quiver Hecke algebras by the generalized quantum affine Schur–Weyl duality functors. We compute the denominators of all normalized R-matrices between fundamental representations of types $A_{2n-1}^{(2)}$ $(n \geq 3)$, $A_{2n}^{(2)}$ $(n \geq 2)$, $B_n^{(1)}$ $(n \geq 3)$ and $D_{n+1}^{(2)}$ $(n \geq 2)$. Thus we can conclude that the normalized R-matrices of types $A_{2n-1}^{(2)}$, $A_{2n}^{(2)}$, $B_n^{(1)}$ and $D_3^{(2)}$ have only simple poles, and those of type $D_{n+1}^{(2)}$ $(n \geq 3)$ have double poles under certain conditions.

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§1. Introduction

Let $\mathfrak{g}$ be an affine Kac–Moody algebra and $U_q'(\mathfrak{g})$ be the quantum affine algebra corresponding to $\mathfrak{g}$. The finite-dimensional integrable representations over $U_q'(\mathfrak{g})$ have been investigated by many authors during the past twenty years from different perspectives (see [1, 3, 4, 10, 12, 24, 27]). Among these aspects, we focus on the theory of *R-matrices* which has deep relationships with q-analysis, operator algebras, conformal field theories, statistical mechanical models, etc.

The purpose of this paper is to compute the denominators of *normalized R-matrices* between the fundamental representations $V(\varpi_{i_k})$'s over $U_q'(\mathfrak{g})$. Knowing the denominators is quite crucial to the study of finite-dimensional integrable representations by the following theorem:

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S. Oh: School of Mathematics, Korea Institute for Advanced Study, Seoul 130-722, Korea;
e-mail: sejin092@gmail.com

Theorem ([1, 24]). *Let M be a finite-dimensional irreducible integrable $U_q'(\mathfrak{g})$-module M. Then there exists a finite sequence*

$$((i_1, a_1), \ldots, (i_l, a_l)) \quad \text{in } (\{1, \ldots, n\} \times \mathbf{k}^\times)^l$$

such that

- $d_{i_k, i_{k'}}(a_{k'}/a_k) \neq 0$ *for* $1 \le k < k' \le l$ *and*
- M *is isomorphic to the head of* $\bigotimes_{i=1}^{l} V(\varpi_{i_k})_{a_k}$.

Moreover, such a sequence $((i_1, a_1), \ldots, (i_l, a_l))$ *is unique up to permutation. Here* $\mathbf{k} = \overline{\mathbb{C}(q)} \subset \bigcup_{m>0} \mathbb{C}((q^{1/m}))$ *and* $d_{i_k, i_{k'}}(z) \in \mathbf{k}[z]$ *denotes the denominator of the normalized R-matrix*

$$R^{\mathrm{norm}}_{i_k, i_{k'}}(z) \colon V(\varpi_{i_k}) \otimes V(\varpi_{i_{k'}})_z \to \mathbf{k}(z) \underset{\mathbf{k}[z^{\pm 1}]}{\otimes} (V(\varpi_{i_{k'}})_z \otimes V(\varpi_{i_k}))$$

satisfying

$$d_{i_k, i_{k'}}(z) R^{\mathrm{norm}}_{i_k, i_{k'}}(z) (V(\varpi_{i_k}) \otimes V(\varpi_{i_{k'}})_z) \subset V(\varpi_{i_{k'}})_z \otimes V(\varpi_{i_k}).$$

Thus the study of denominators is one of the first steps to study the category $\mathscr{C}_\mathfrak{g}$ consisting of finite-dimensional integrable representations over $U_q'(\mathfrak{g})$.

On the other hand, Kang, Kashiwara and Kim [18, 19] recently constructed the *quantum affine Schur–Weyl duality functor* $\mathcal{F}$ by considering the zeros of the denominators of normalized R-matrices. The way of constructing $\mathcal{F}$ can be described as follows: Let $\{V_s\}_{s \in \mathcal{S}}$ be a family of fundamental representations over $U_q'(\mathfrak{g})$. For an index set J and two maps $X : J \to \mathbf{k}^\times$, $s : J \to \mathcal{S}$, we can define a quiver $Q^J = (Q^J_0, Q^J_1)$ associated with (J, X, s) as (vertices) $Q^J_0 = J$, (arrows) for $i, j \in J$, we put $\mathtt{d}_{ij}$ arrows from i to j, where $\mathtt{d}_{ij}$ is the order of the zero of $d_{V_{s(i)}, V_{s(j)}}(z)$ at $X(j)/X(i)$.

Then we obtain a symmetric Cartan matrix $\mathsf{A}^J = (a^J_{ij})_{i,j \in J}$ associated with (J, X, s) by

$$a^J_{ij} = 2 \quad \text{if } i = j, \qquad a^J_{ij} = -\mathtt{d}_{ij} - \mathtt{d}_{ji} \quad \text{if } i \neq j.$$

Let R^J be the quiver Hecke algebra associated with the symmetric Cartan matrix A^J and the parameters [25, 26, 30]

$$\mathcal{Q}_{i,j}(u, v) = (u - v)^{\mathtt{d}_{ij}} (v - u)^{\mathtt{d}_{ji}} \quad \text{if } i \neq j, \qquad \mathcal{Q}_{i,i}(u, v) = 0 \quad \text{for all } i \in J.$$

Theorem ([18]). *There exists a functor* $\mathcal{F} : \mathrm{Rep}(R^J) \to \mathscr{C}_\mathfrak{g}$, *where* $\mathrm{Rep}(R^J)$ *denotes the category of finite-dimensional representations over* R^J, *which enjoys the following properties:*

(a) *$\mathcal{F}$ is a tensor functor; that is, there exist $U_q'(\mathfrak{g})$-module isomorphisms*

$$\mathcal{F}(R^J(0)) \simeq \mathbf{k} \quad \textit{and} \quad \mathcal{F}(M_1 \circ M_2) \simeq \mathcal{F}(M_1) \otimes \mathcal{F}(M_2)$$

for any $M_1, M_2 \in \mathrm{Rep}(R^J)$.

(b) *If the Cartan matrix A^J is of type A_n $(n \geq 1)$, D_n $(n \geq 4)$, E_6, E_7 or E_8, then the functor $\mathcal{F}$ is exact.*

Thus the generalized quantum affine Schur–Weyl duality functor provides the way of investigating the category $\mathscr{C}_\mathfrak{g}$ via the category $\mathrm{Rep}(R^J)$ and the other way around (see [20]).

Note that A^J depends on the choice of (J, X, s) and the denominators. Hence one may expect various exact functors defined on $\mathrm{Rep}(R^J)$ for a fixed algebra R^J. In the forthcoming papers by the author and his collaborators [21, 22], such situations will be considered, and the denominator formulas given in this paper will play an important role.

The denominators of all normalized R-matrices $R^{\mathrm{norm}}_{k,l}(z)$ for $A_n^{(1)}$, $C_n^{(1)}$ and $D_n^{(1)}$ were studied in [1, 6, 19], and the denominators of the normalized R-matrix $R^{\mathrm{norm}}_{1,1}(z)$ (resp. $R^{\mathrm{norm}}_{n,n}(z)$) between vector representations (resp. spin representations) for all classical affine types are given in [23, 28]. On the other hand, the explicit forms of the normalized R-matrix $R^{\mathrm{norm}}_{1,1}(z)$ for all classical affine types were studied in [7, 14, 15, 16]. With these results, we will compute the denominators $d_{k,l}(z)$ of all normalized R-matrices $R^{\mathrm{norm}}_{k,l}(z)$ by employing the frameworks given in [1, Appendix C] and [19, Appendix A].

Our main results are

$$d_{k,l}(z) = \prod_{s=1}^{\min(k,l)} (z^t - (-q^t)^{|k-l|+2s})(z^t - (p^*)^t(-q^t)^{2s-k-l})$$

if $V(\varpi_k)$ and $V(\varpi_l)$ are not spin representations, and

$$d_{k,n}(z) = \prod_{s=1}^{k} (z - (-1)^{n+k} q_s^{2n-2k-1+4s}) \qquad \text{if } \mathfrak{g} = B_n^{(1)} \ (n \geq 3) \text{ and } k < n,$$

$$d_{k,n}(z) = \prod_{s=1}^{k} (z^2 + (-q^2)^{n-k+2s}) \qquad \text{if } \mathfrak{g} = D_{n+1}^{(2)} \ (n \geq 2) \text{ and } k < n.$$

Here,

$$t = \begin{cases} 2 & \text{if } \mathfrak{g} = D_{n+1}^{(2)} \ (n \geq 2), \\ 1 & \text{otherwise}, \end{cases} \qquad q_s^2 = q \quad \text{and} \quad p^* := (-1)^{\langle \rho^\vee, \delta \rangle} q^{(\rho, \delta)}$$

for the *null root* δ (see (2.3)). Hence we conclude that

(a) $R^{\rm norm}_{k,l}(z)$ of $A^{(2)}_{2n-1}$ $(n \ge 3)$, $A^{(2)}_{2n}$ $(n \ge 2)$, $B^{(1)}_n$ $(n \ge 3)$ or $D^{(2)}_3$ has only simple poles,

(b) $R^{\rm norm}_{k,l}(z)$ of $D^{(2)}_{n+1}$ $(n \ge 3)$ has a double pole at $z = (-q^2)^{s/2}$ if

$$2 \le k, l \le n-1, \ k+l > n, \ 2n+2-k-l \le s \le k+l \text{ and } s \equiv k+l \bmod 2,$$

(c) $R^{\rm norm}_{k,l}(z)$ has a pole at $\pm(-q^t)^{\ell/t}$ only if $k \in \mathbb{Z}$ and $2 \le \ell \le (\rho, \delta)$ (see [9]).

This paper is organized as follows. In the first section, we briefly recall the notion of quantum affine algebras and R-matrices. In the next section, we give the $U_q'(\mathfrak{g})$-module structure of the vector representations and spin representations over $U_q'(\mathfrak{g})$. In the third section, we study morphisms from $V(\varpi_i)_a \otimes V(\varpi_j)_b$ to $V(\varpi_k)_c$, called the *Dorey rule*. Then we prove the existence of certain surjective homomorphisms which can be understood as a $D^{(2)}_{n+1}$-analogue of [19, Lemma A.3.2] and a *generalized Dorey rule* in the context of [20]. In the last section, we propose a general framework for computing the denominators, which originates from [19, Appendix A]. Then we compute $d_{1,n}(z)$ for $\mathfrak{g} = D^{(2)}_{n+1}$ $(n \ge 2)$ and the unknown denominators $d_{k,l}(z)$ of normalized R-matrices for $\mathfrak{g} = A^{(2)}_{2n-1}$ $(n \ge 3)$, $A^{(2)}_{2n}$ $(n \ge 2)$, $B^{(1)}_n$ $(n \ge 3)$ and $D^{(2)}_{n+1}$ $(n \ge 2)$, by using the results in the previous sections. In the appendix, we provide a table of $d_{k,l}(z)$ for all classical affine types for the reader's convenience.

§2. Quantum affine algebras and R-matrices

In this section, we briefly recall the background on quantum affine algebras, their finite-dimensional integral representations and R-matrices. We refer to [1, 18, 24] for precise statements and definitions.

§2.1. Quantum affine algebras and their representations

Let $I = \{0, 1, \dots, n\}$ be a set of indices and set $I_0 := I \setminus \{0\}$. An *affine Cartan datum* is a quadruple $(\mathsf{A}, \mathsf{P}, \Pi, \Pi^\vee)$ consisting of

(a) a matrix A of corank 1, called the *affine Cartan matrix* satisfying

$$\text{(i) } a_{ii} = 2 \ (i \in I), \quad \text{(ii) } a_{ij} \in \mathbb{Z}_{\le 0}, \quad \text{(iii) } a_{ij} = 0 \text{ if } a_{ji} = 0$$

with $\mathsf{D} = \mathrm{diag}(\mathsf{d}_i \in \mathbb{Z}_{>0} \mid i \in I)$ making DA symmetric,

(b) a free abelian group P of rank $n+2$, called the *weight lattice*,

(c) $\Pi = \{\alpha_i \mid i \in I\} \subset \mathsf{P}$, called the set of *simple roots*,

(d) $\Pi^\vee = \{h_i \mid i \in I\} \subset \mathsf{P}^\vee := \mathrm{Hom}(\mathsf{P}, \mathbb{Z})$, called the set of *simple coroots*,

which satisfy:

(1) $\langle h_i, \alpha_j \rangle = a_{ij}$ for all $i, j \in I$,

(2) Π and $\Pi^\vee$ are linearly independent sets,

(3) for each $i \in I$, there exists $\Lambda_i \in \mathsf{P}$ such that $\langle h_i, \Lambda_j \rangle = \delta_{ij}$ for all $j \in I$.

We set $\mathsf{Q} = \bigoplus_{i\in I} \mathbb{Z}\alpha_i$, $\mathsf{Q}_+ = \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$, $\mathsf{Q}^\vee = \bigoplus_{i\in I} \mathbb{Z}h_i$ and $\mathsf{Q}^\vee_+ = \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}h_i$. We choose the *imaginary root* $\delta = \sum_{i\in I} \mathsf{a}_i\alpha_i \in \mathsf{Q}_+$ and the *center* $c = \sum_{i\in I} \mathsf{c}_i h_i \in \mathsf{Q}^\vee_+$ such that $\{\lambda \in \mathsf{Q} \mid \langle h_i, \lambda\rangle = 0 \text{ for every } i \in I\} = \mathbb{Z}\delta$ and $\{h \in \mathsf{Q}^\vee \mid \langle h, \alpha_i\rangle = 0 \text{ for every } i \in I\} = \mathbb{Z}c$ (see [17, Chapter 4]).

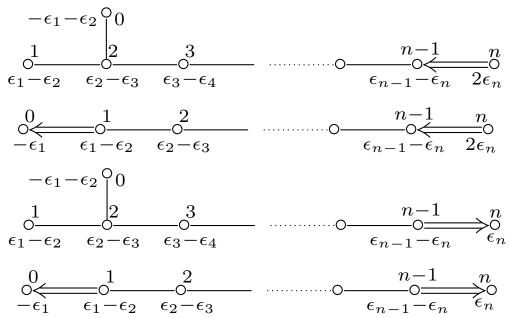
Remark 2.1. In this paper, we mainly deal with the affine types $A^{(2)}_{2n-1}$ $(n \geq 3)$, $A^{(2)}_{2n}$ $(n \geq 2)$, $B^{(1)}_n$ $(n \geq 3)$ and $D^{(2)}_{n+1}$ $(n \geq 2)$ with the following enumerations on their affine Dynkin diagrams (cf. [17, Chapter 4]):

$A^{(2)}_{2n-1}$ $(n \geq 3)$:

$A^{(2)}_{2n}$ $(n \geq 2)$:

$B^{(1)}_n$ $(n \geq 3)$:

$D^{(1)}_{n+1}$ $(n \geq 2)$:



Set $\mathfrak{h} = \mathbb{Q} \otimes_{\mathbb{Z}} \mathsf{P}^\vee$. Then there exists a symmetric bilinear form $(\ ,\ )$ on $\mathfrak{h}^*$ satisfying

$$\langle h_i, \lambda\rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)} \quad \text{for any } i \in I \text{ and } \lambda \in \mathfrak{h}^*.$$

We normalize the bilinear form by $\langle c, \lambda\rangle = (\delta, \lambda)$ for any $\lambda \in \mathfrak{h}^*$.

Denote by $\mathfrak{g}$ the affine Kac–Moody Lie algebra associated with $(\mathsf{A}, \mathsf{P}, \Pi, \Pi^\vee)$ and by W the Weyl group of $\mathfrak{g}$, generated by $(\mathsf{s}_i)_{i\in I}$. We define $\mathfrak{g}_0$ to be the subalgebra of $\mathfrak{g}$ generated by the Chevalley generators e_i, f_i, and h_i for $i \in I_0$. Then $\mathfrak{g}_0$ is a finite-dimensional simple Lie algebra.

Let γ be the smallest positive integer such that

$$\gamma(\alpha_i, \alpha_i)/2 \in \mathbb{Z} \quad \text{for any } i \in I.$$

Let q be an indeterminate. For $m, n \in \mathbb{Z}_{\geq 0}$ and $i \in I$, we define $q_i = q^{(\alpha_i,\alpha_i)/2}$ and

$$[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \qquad [n]_i! = \prod_{k=1}^{n} [k]_i, \qquad \begin{bmatrix} m \\ n \end{bmatrix}_i = \frac{[m]_i!}{[m-n]_i![n]_i!}.$$

Definition 2.2. The *quantum affine algebra* $U_q(\mathfrak{g})$ associated with $(\mathsf{A}, \mathsf{P}, \Pi, \Pi^\vee)$ is the associative algebra over $\mathbb{Q}(q^{1/\gamma})$ with 1 generated by e_i, f_i $(i \in I)$ and q^h $(h \in \gamma^{-1}\mathsf{P}^\vee)$ satisfying the following relations:

(1) $q^0 = 1, q^h q^{h'} = q^{h+h'}$ for $h, h' \in \gamma^{-1}\mathsf{P}^\vee$,

(2) $q^h e_i q^{-h} = q^{\langle h,\alpha_i\rangle} e_i,\ q^h f_i q^{-h} = q^{-\langle h,\alpha_i\rangle} f_i$ for $h \in \gamma^{-1}\mathsf{P}^\vee, i \in I$,

(3) $e_i f_j - f_j e_i = \delta_{ij} \dfrac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$, where $K_i = q_i^{h_i}$,

(4) $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k e_i^{(1-a_{ij}-k)} e_j e_i^{(k)} = \sum_{k=0}^{1-a_{ij}} (-1)^k f_i^{(1-a_{ij}-k)} f_j f_i^{(k)} = 0$ for $i \neq j$,

where $e_i^{(k)} = e_i^k/[k]_i!$ and $f_i^{(k)} = f_i^k/[k]_i!$.

Let $U_q^+(\mathfrak{g})$ (resp. $U_q^-(\mathfrak{g})$) be the subalgebra of $U_q(\mathfrak{g})$ generated by e_i's (resp. f_i's) and $U_q'(\mathfrak{g})$ be the subalgebra of $U_q(\mathfrak{g})$ generated by $e_i, f_i, K_i^{\pm 1}$ $(i \in I)$. We call $U_q'(\mathfrak{g})$ *also* the quantum affine algebra. Throughout this paper, we mainly deal with $U_q'(\mathfrak{g})$. When we deal with $U_q'(\mathfrak{g})$-modules, we regard the base field as $\mathbf{k}$, the algebraic closure of $\mathbb{C}(q)$ in $\bigcup_{m>0} \mathbb{C}((q^{1/m}))$.

For $U_q'(\mathfrak{g})$-modules M and N, $M \otimes N$ becomes a $U_q'(\mathfrak{g})$-module via the comultiplication Δ of $U_q'(\mathfrak{g})$:

$$\Delta(q^h) = q^h \otimes q^h, \quad \Delta(e_i) = e_i \otimes K_i^{-1} + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + K_i \otimes f_i. \tag{2.1}$$

Set $\mathsf{P}_{\mathrm{cl}} := \mathsf{P}/\mathbb{Z}\delta$ and let $\mathrm{cl}\colon \mathsf{P} \to \mathsf{P}_{\mathrm{cl}}$ be the canonical projection. We say that a $U_q'(\mathfrak{g})$-module M is *integrable* provided that

(a) M decomposes into P_{cl}-weight spaces, that is,

$$M = \bigoplus_{\mu \in \mathsf{P}_{\mathrm{cl}}} M_\mu, \quad \text{where} \quad M_\mu := \{v \in M \mid K_i v = q^{\langle h_i,\mu\rangle} v\},$$

(b) e_i and f_i $(i \in I)$ act on M nilpotently.

We denote by $\mathscr{C}_\mathfrak{g}$ the category of finite-dimensional integrable $U_q'(\mathfrak{g})$-modules. Note that $\mathscr{C}_\mathfrak{g}$ is a tensor category with the comultiplication (2.1).

Note that an irreducible module M in $\mathscr{C}_\mathfrak{g}$ contains a unique (up to a constant multiple) vector u_M of weight $\lambda \in \mathsf{P}_{\mathrm{cl}}$ with the following properties:

- $\langle c, \lambda\rangle = 0$ and $\langle h_i, \lambda\rangle \geq 0$ for all $i \in I_0$,
- $\mathrm{wt}(M) := \{\mu \in \mathsf{P}_{\mathrm{cl}} \mid M_\mu \neq 0\} \subset \lambda - \sum_{i \in I_0} \mathbb{Z}_{\geq 0} \mathrm{cl}(\alpha_i)$.

Such a weight λ is also unique. We call the vector u_M the *dominant extremal vector* of M and the weight λ the *dominant extremal weight* of M.

For an integrable $U_q'(\mathfrak{g})$-module M, the *affinization* M_{aff} of M is the P-graded $U_q'(\mathfrak{g})$-module

$$M_{\mathrm{aff}} = \bigoplus_{\lambda\in\mathsf{P}} (M_{\mathrm{aff}})_\lambda \quad \text{with} \quad (M_{\mathrm{aff}})_\lambda = M_{\mathrm{cl}(\lambda)}.$$

Here the actions e_i and f_i are defined so that they commute with the canonical projection $\mathrm{cl}\colon M_{\mathrm{aff}} \to M$.

We denote by $z_M\colon M_{\mathrm{aff}} \to M_{\mathrm{aff}}$ the $U_q'(\mathfrak{g})$-module automorphism of weight δ defined by $(M_{\mathrm{aff}})_\lambda \xrightarrow{\sim} (M_{\mathrm{aff}})_{\lambda+\delta}$. For $x \in \mathbf{k}^\times$, we define

$$M_x := M_{\mathrm{aff}}/(z_M - x)M_{\mathrm{aff}}.$$

Note that, for $M \in \mathscr{C}_\mathfrak{g}$ and $x \in \mathbf{k}^\times$, M_x is also contained in $\mathscr{C}_\mathfrak{g}$.

We embed P_{cl} into P by $\iota\colon \mathsf{P}_{\mathrm{cl}} \to \mathsf{P}$ which is given by $\iota(\mathrm{cl}(\Lambda_i)) = \Lambda_i$. For $u \in M_\lambda$ ($\lambda \in \mathsf{P}_{\mathrm{cl}}$) and an indeterminate z, let us denote by $u_z \in (M_{\mathrm{aff}})_{\iota(\lambda)}$ the element such that $\mathrm{cl}(u_z) = u$. With this notation, we have

$$e_i(u_z) = z^{\delta_{i,0}}(e_i u)_z, \quad f_i(u_z) = z^{-\delta_{i,0}}(f_i u)_z, \quad K_i(u_z) = (K_i u)_z.$$

Then we have $M_{\mathrm{aff}} \simeq M_z := \mathbf{k}[z, z^{-1}] \otimes M$ and hence the canonical action of z on M_z determines an automorphism z_M on M_{aff}. Thus u_z is the element $1 \otimes u \in \mathbf{k}[z, z^{-1}] \otimes M$ for $u \in M$. We sometimes use M_{z_M} instead of M_z to emphasize that z_M is an indeterminate depending on the module M.

Definition 2.3 ([1, §1.3]). For $i \in I_0$, the *ith fundamental module* is a unique finite-dimensional integrable $U_q'(\mathfrak{g})$-module $V(\varpi_i)$ satisfying the following properties:

(1) The weights of $V(\varpi_i)$ are contained in the convex hull of $\mathsf{W}_0\mathrm{cl}(\varpi_i)$.

(2) $V(\varpi_i)_{\mathrm{cl}(\varpi_i)} = \mathbb{C}(q)u_{\varpi_i}$, where u_{ϖ_i} is the *dominant extremal vector* of $V(\varpi_i)$.

(3) To any $\mu \in \mathsf{W}_0\mathrm{cl}(\varpi_i)$, we can associate a non-zero vector u_μ, called an *extremal vector of weight* μ, such that

$$\mathcal{S}_i \cdot u_\mu := u_{\mathsf{s}_i\mu} = \begin{cases} f_i^{(\langle h_i,\mu\rangle)} u_\mu & \text{if } \langle h_i,\mu\rangle \ge 0, \\ e_i^{(-\langle h_i,\mu\rangle)} u_\mu & \text{if } \langle h_i,\mu\rangle \le 0, \end{cases} \quad \text{for any } i \in I. \tag{2.2}$$

(4) v_{ϖ_i} generates $V(\varpi_i)$ as a $U_q'(\mathfrak{g})$-module.

For a $U_q'(\mathfrak{g})$-module M, we call ${}^*M$ the *right dual* and M^* the *left dual* of M if there exist $U_q'(\mathfrak{g})$-homomorphisms

$$M^* \otimes M \xrightarrow{\mathrm{tr}} \mathbf{k}, \quad \mathbf{k} \to M \otimes M^* \quad \text{and} \quad M \otimes {}^*M \xrightarrow{\mathrm{tr}} \mathbf{k}, \quad \mathbf{k} \to M^* \otimes M.$$

Note that $V(\varpi_i)_x$ is contained in $\mathscr{C}_\mathfrak{g}$ and has the right dual and left dual as follows:

$$V(\varpi_i)_x^* := V(\varpi_{i^*})_{x(p)^{-1}}, \quad {}^*V(\varpi_i)_x := V(\varpi_{i^*})_{xp} \quad \text{where} \tag{2.3}$$

- i^* is the image of i under the involution of I_0 determined by the action of w_0,
- $p := (-1)^{\langle \rho^\vee, \delta \rangle} q^{(\rho,\delta)}$,
- ρ is defined by $\langle h_i, \rho \rangle = 1$ and $\rho^\vee$ is defined by $\langle \rho^\vee, \alpha_i \rangle = 1$ for all $i \in I$.

An integrable $U_q'(\mathfrak{g})$-module M is called *good* if it has certain properties; we refer to [24] for the precise definition, which we do not need here. We just note that the fundamental modules are good $U_q'(\mathfrak{g})$-modules and any good $U_q'(\mathfrak{g})$-module is irreducible.

§2.2. Normalized and universal R-matrices

In this subsection, we recall the notions of R-matrices following [24, §8] and [20, §2.2]. Let us take a basis $\{P_\nu\}$ of $U_q^+(\mathfrak{g})$ and a basis $\{Q_\nu\}$ of $U_q^-(\mathfrak{g})$ which are dual to each other with respect to a suitable coupling on $U_q^+(\mathfrak{g}) \times U_q^-(\mathfrak{g})$. Then for $U_q'(\mathfrak{g})$-modules M and N define

$$R^{\mathrm{univ}}_{M,N}(u \otimes v) = q^{(\mathrm{wt}(u),\mathrm{wt}(v))} \sum_\nu P_\nu v \otimes Q_\nu u,$$

so that $R^{\mathrm{univ}}_{M,N}$ gives a $U_q'(\mathfrak{g})$-linear homomorphism from $M \otimes N$ to $N \otimes M$ under the assumption that the infinite sum is meaningful. We call $R^{\mathrm{univ}}_{M,N}$ the *universal R-matrix*.

For M and N in $\mathscr{C}_{\mathfrak{g}}$, $R^{\mathrm{univ}}_{M_{z_M},N_{z_N}}$ converges in the (z_N/z_M)-adic topology. The existence of the universal R-matrix for $M, N \in \mathscr{C}_{\mathfrak{g}}$ is proved in [8]. Hence we have a morphism of $\mathbf{k}[[z_N/z_M]] \otimes_{\mathbf{k}[z_N/z_M]} \mathbf{k}[z_M^{\pm 1}, z_N^{\pm 1}] \otimes U_q'(\mathfrak{g})$-modules

$$\begin{aligned} R^{\mathrm{univ}}_{M_{z_M},N_{z_N}} : \mathbf{k}[[z_N/z_M]] \underset{\mathbf{k}[z_N/z_M]}{\otimes} (M_{z_M} \otimes N_{z_N}) \\ \to \mathbf{k}[[z_N/z_M]] \underset{\mathbf{k}[z_N/z_M]}{\otimes} (N_{z_N} \otimes M_{z_M}). \end{aligned}$$

Note that the universal R-matrix has the following property:

$$R^{\mathrm{univ}}_{M,(N\otimes N')_{z_{N\otimes N'}}} = (\mathrm{id}_{N_{z_N}} \otimes R^{\mathrm{univ}}_{M,N'_{z_{N'}}}) \circ (R^{\mathrm{univ}}_{M,N_{z_N}} \otimes \mathrm{id}_{N'_{z_{N'}}}), \tag{2.4}$$

where $(N \otimes N')_{z_{N\otimes N'}} \simeq N_{z_N} \otimes N'_{z_{N'}}$.

We say that $R^{\mathrm{univ}}_{M_{z_M},N_{z_N}}$ is *rationally renormalizable* if there exist $a \in \mathbf{k}(z_N/z_M)$ and a $\mathbf{k}(z_N, z_M) \otimes U_q'(\mathfrak{g})$-homomorphism

$$\begin{aligned} R : \mathbf{k}(z_M, z_N) \underset{\mathbf{k}[z_M^{\pm 1}, z_N^{\pm 1}]}{\otimes} (M_{z_M} \otimes N_{z_N}) \\ \to \mathbf{k}(z_M, z_N) \underset{\mathbf{k}[z_M^{\pm 1}, z_N^{\pm 1}]}{\otimes} (N_{z_N} \otimes M_{z_M}) \end{aligned} \tag{2.5}$$

such that $R^{\mathrm{univ}}_{M_{z_M},N_{z_N}} = aR$.

For good modules M and N in $\mathscr{C}_{\mathfrak{g}}$, it is known that $R^{\mathrm{univ}}_{M_{z_M},N_{z_N}}$ is rationally renormalizable [1, Corollary 2.5]. More precisely, there exists $a(z_N/z_M) \in \mathbf{k}[z_M^{\pm 1}, z_N^{\pm 1}]^\times$ such that:

$$(2.6)\quad \begin{cases} \text{(i) } R^{\mathrm{univ}}_{M_{z_M},N_{z_N}}((u_M)_{z_M} \otimes (u_N)_{z_N}) = a(z_N/z_M)((u_N)_{z_N} \otimes (u_M)_{z_M}), \\ \quad\text{where } u_M \text{ and } u_N \text{ denote the dominant extremal vectors of } M \text{ and } N\text{, respectively,} \\ \text{(ii) } R^{\mathrm{norm}}_{M,N} := a(z_N/z_M)^{-1} R^{\mathrm{univ}}_{M_{z_M},N_{z_N}} \text{ is a unique } \mathbf{k}(z_N,z_M) \otimes U_q'(\mathfrak{g})\text{-homomorphism in (2.5) satisfying} \\ \qquad R^{\mathrm{norm}}_{M,N}((u_M)_{z_M} \otimes (u_N)_{z_N}) = (u_N)_{z_N} \otimes (u_M)_{z_M}. \end{cases}$$

We call $R^{\mathrm{norm}}_{M,N}$ the *normalized R-matrix.*

Let $d_{M,N}(u) \in \mathbf{k}[u]$ be a monic polynomial of *smallest degree* such that the image of $M_{z_M} \otimes N_{z_N}$ under the homomorphism $d_{M,N}(z_N/z_M)R^{\mathrm{norm}}_{M,N}$ is contained in $N_{z_N} \otimes M_{z_M}$. We call $d_{M,N}(u)$ the *denominator* of $R^{\mathrm{norm}}_{M,N}$. Thus we have

$$d_{M,N}(z_N/z_M)R^{\mathrm{norm}}_{M,N} \colon M_{z_M} \otimes N_{z_N} \to N_{z_N} \otimes M_{z_M}.$$

Lemma 2.4 ([1, Lemma C.15]). *Let M, N, O and P be irreducible modules in $\mathscr{C}_{\mathfrak{g}}$. If there exists a surjective $U_q'(\mathfrak{g})$-homomorphism $M \otimes N \twoheadrightarrow O$, then*

$$\frac{d_{P,M}(z)d_{P,N}(z)a_{P,O}(z)}{d_{P,O}(z)a_{P,M}(z)a_{P,N}(z)}, \frac{d_{M,P}(z)d_{N,P}(z)a_{O,P}(z)}{d_{O,P}(z)a_{M,P}(z)a_{N,P}(z)} \in \mathbf{k}[z^{\pm 1}].$$

§3. Vector and spin representations

In this section, we record the $U_q'(\mathfrak{g})$-module structure of

- $V(\varpi_1)$, called the *vector representation*,
- $V(\varpi_n)$, called the *spin representation*, for $\mathfrak{g} = B_n^{(1)}$ or $D_{n+1}^{(2)}$.

As a P_{cl}-graded vector space, the vector representation can be expressed as follows [13, Chapter 11]:

$$V(\varpi_1) = \Big(\bigoplus_{j=1}^{n} \mathbf{k}v_j\Big) \oplus \Big(\bigoplus_{j=1}^{n} \mathbf{k}v_{\bar{j}}\Big) \oplus V$$

where

$\mathfrak{g}$	$A_{2n-1}^{(2)}$	$B_n^{(1)}$	$A_{2n}^{(2)}$	$D_{n+1}^{(2)}$
V	0	$\mathbf{k}v_0$	$\mathbf{k}v_\emptyset$	$\mathbf{k}v_0 \oplus \mathbf{k}v_\emptyset$

and

$$\mathrm{wt}(v_j) = \epsilon_j,\ \mathrm{wt}(v_{\bar{j}}) = -\epsilon_j \quad \text{for } j = 1, \dots, n \quad \text{and} \quad \mathrm{wt}(v_0) = \mathrm{wt}(v_\emptyset) = 0.$$

The actions of e_i, f_i and q^h are defined as follows:

$$q^h \cdot v_j = q^{\langle h, \mathrm{wt}(v_j)\rangle} v_j \quad \text{for } h \in \mathsf{P}_{\mathrm{cl}}^{\vee},$$

$$A_{2n-1}^{(2)}:\ e_i v_j = \begin{cases} v_i & \text{if } j = i+1,\ i \neq n, \\ v_{\overline{i+1}} & \text{if } j = \overline{i},\ i \neq n, \\ v_n & \text{if } j = \overline{n},\ i = n, \\ v_{\overline{2}} & \text{if } j = 1,\ i = 0, \\ v_{\overline{1}} & \text{if } j = 2,\ i = 0, \\ 0 & \text{otherwise,} \end{cases} \qquad f_i v_j = \begin{cases} v_{i+1} & \text{if } j = i,\ i \neq n, \\ v_{\overline{i}} & \text{if } j = \overline{i+1},\ i \neq n, \\ v_{\overline{n}} & \text{if } j = n,\ i = n, \\ v_1 & \text{if } j = \overline{2},\ i = 0, \\ v_2 & \text{if } j = \overline{1},\ i = 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$A_{2n}^{(2)}:\ e_i v_j = \begin{cases} v_i & \text{if } j = i+1,\ i \neq n, \\ v_{\overline{i+1}} & \text{if } j = \overline{i},\ i \neq n, \\ v_n & \text{if } j = \overline{n},\ i = n, \\ v_{\emptyset} & \text{if } j = 1,\ i = 0, \\ [2]_0 v_{\overline{1}} & \text{if } j = \emptyset,\ i = 0, \\ 0 & \text{otherwise,} \end{cases} \qquad f_i v_j = \begin{cases} v_{i+1} & \text{if } j = i,\ i \neq n, \\ v_{\overline{i}} & \text{if } j = \overline{i+1},\ i \neq n, \\ v_{\overline{n}} & \text{if } j = n,\ i = n, \\ v_{\emptyset} & \text{if } j = \overline{1},\ i = 0, \\ [2]_0 v_1 & \text{if } j = \emptyset,\ i = 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$B_n^{(1)}:\ e_i v_j = \begin{cases} v_i & \text{if } j = i+1,\ i \neq n, \\ v_{\overline{i+1}} & \text{if } j = \overline{i},\ i \neq n, \\ v_0 & \text{if } j = \overline{n},\ i = n, \\ [2]_n v_n & \text{if } j = 0,\ i = n, \\ v_{\overline{2}} & \text{if } j = 1,\ i = 0, \\ v_{\overline{1}} & \text{if } j = 2,\ i = 0, \\ 0 & \text{otherwise,} \end{cases} \qquad f_i v_j = \begin{cases} v_{i+1} & \text{if } j = i,\ i \neq n, \\ v_{\overline{i}} & \text{if } j = \overline{i+1},\ i \neq n, \\ v_0 & \text{if } j = n,\ i = n, \\ [2]_n v_{\overline{n}} & \text{if } j = 0,\ i = n, \\ v_1 & \text{if } j = \overline{2},\ i = 0, \\ v_2 & \text{if } j = \overline{1},\ i = 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$D_{n+1}^{(2)}:\ e_i v_j = \begin{cases} v_i & \text{if } j = i+1,\ i \neq n, \\ v_{\overline{i+1}} & \text{if } j = \overline{i},\ i \neq n, \\ v_0 & \text{if } j = \overline{n},\ i = n, \\ [2]_n v_n & \text{if } j = 0,\ i = n, \\ v_{\emptyset} & \text{if } j = 1,\ i = 0, \\ [2]_0 v_{\overline{1}} & \text{if } j = \emptyset,\ i = 0, \\ 0 & \text{otherwise,} \end{cases} \qquad f_i v_j = \begin{cases} v_{i+1} & \text{if } j = i,\ i \neq n, \\ v_{\overline{i}} & \text{if } j = \overline{i+1},\ i \neq n, \\ v_0 & \text{if } j = n,\ i = n, \\ [2]_n v_{\overline{n}} & \text{if } j = 0,\ i = n, \\ v_{\emptyset} & \text{if } j = \overline{1},\ i = 0, \\ [2]_0 v_1 & \text{if } j = \emptyset,\ i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $\mathfrak{g} = B_n^{(1)}$ or $\mathfrak{g} = D_{n+1}^{(2)}$, the spin representation $V(\varpi_n)$ is the $\mathbf{k}$-vector space with a basis

$$\mathsf{B}_{\mathrm{sp}} = \{(m_1, \ldots, m_n)\,;\, m_i = + \text{ or } -\}.$$

Its $U_q'(\mathfrak{g})$-module structure is given by defining the action of e_i, f_i and q^h as follows:

$$q^h v = q^{\langle h, \mathrm{wt}(v)\rangle} v \quad \text{for } h \in \mathsf{P}_{\mathrm{cl}}^\vee, \quad \text{where} \quad \mathrm{wt}(v) = \frac{1}{2}\sum_{k=1}^{n} m_k \epsilon_k,$$

$$B_n^{(1)}: \quad e_i v = \begin{cases} (m_1, \ldots, \overset{i}{+}, \overset{i+1}{-}, \ldots, m_n) & \text{if } i \neq n,\ m_i = -,\ m_{i+1} = +, \\ (m_1, \ldots, m_{n-1}, \overset{n}{+}) & \text{if } i = n,\ m_n = -, \\ (-, -, m_3, \ldots, m_n) & \text{if } i = 0,\ m_1 = m_2 = +, \\ 0 & \text{otherwise,} \end{cases}$$

$$f_i v = \begin{cases} (m_1, \ldots, \overset{i}{-}, \overset{i+1}{+}, \ldots, m_n) & \text{if } i \neq n,\ m_i = +,\ m_{i+1} = -, \\ (m_1, \ldots, m_{n-1}, \overset{n}{-}) & \text{if } i = n,\ m_n = +, \\ (+, +, m_3, \ldots, m_n) & \text{if } i = 0,\ m_1 = m_2 = -, \\ 0 & \text{otherwise,} \end{cases}$$

$$D_{n+1}^{(2)}: \quad e_i v = \begin{cases} (m_1, \ldots, \overset{i}{+}, \overset{i+1}{-}, \ldots, m_n) & \text{if } i \neq n,\ m_i = -,\ m_{i+1} = +, \\ (m_1, \ldots, m_{n-1}, \overset{n}{+}) & \text{if } i = n,\ m_n = -, \\ (-, m_2, \ldots, m_n) & \text{if } i = 0,\ m_1 = +, \\ 0 & \text{otherwise,} \end{cases}$$

$$f_i v = \begin{cases} (m_1, \ldots, \overset{i}{-}, \overset{i+1}{+}, \ldots, m_n) & \text{if } i \neq n,\ m_i = +,\ m_{i+1} = -, \\ (m_1, \ldots, m_{n-1}, \overset{n}{-}) & \text{if } i = n,\ m_n = +, \\ (+, m_2, \ldots, m_n) & \text{if } i = 0,\ m_1 = -, \\ 0 & \text{otherwise.} \end{cases}$$

§4. Surjective homomorphisms between integrable $U_q'(\mathfrak{g})$-modules

In this section, we first study the morphisms in

$$\mathrm{Hom}_{U_q'(\mathfrak{g})}(V(\varpi_i)_a \otimes V(\varpi_j)_b, V(\varpi_k)_c) \qquad \text{for } i, j, k \in I_0 \text{ and } a, b, c \in \mathbf{k}^\times.$$

These kinds of morphisms are known as *Dorey type morphisms* and have been investigated in [5] for the classical untwisted affine types $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$ and $D_n^{(1)}$. By the result of [20], $\dim(\mathrm{Hom}_{U_q'(\mathfrak{g})}(V(\varpi_i)_a \otimes V(\varpi_j)_b, V(\varpi_k)_c)) \leq 1$ and $V(\varpi_k)_c$ is the irreducible head of $V(\varpi_i)_a \otimes V(\varpi_j)_b$.

In the last part of this section, we study the surjective homomorphisms which can be understood as a $D_{n+1}^{(2)}$-analogue of the surjective homomorphisms given in [19, (A.17)]

$$\mathrm{Hom}_{U_q'(D_{n+1}^{(2)})}(V(\varpi_k)_a \otimes V(\varpi_l)_b, V(\varpi_n)_c \otimes V(\varpi_n)_d).$$

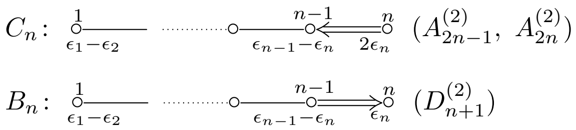
The existence of such morphisms can also be considered as a *generalized Dorey rule*, since $V(\varpi_n)_c \otimes V(\varpi_n)_d$ is simple and is the irreducible head of $V(\varpi_k)_a \otimes V(\varpi_l)_b$.

Hereafter, we will use the following convention frequently:

For a statement P, $\delta(P)$ is 1 if P is true and 0 if P is false.

By the result on $B_n^{(1)}$ in [5], *it suffices to consider when* $\mathfrak{g} = A_{2n-1}^{(2)}$ $(n \geq 3)$, $A_{2n}^{(2)}$ $(n \geq 2)$ *and* $D_{n+1}^{(2)}$ $(n \geq 2)$.

The finite Dynkin diagrams of $\mathfrak{g}_0$ associated with $\mathfrak{g}$ are given as follows:



We denote by $V_0(\varpi_i)$ for $i \in I_0$ the highest weight $U_q(\mathfrak{g}_0)$-module with highest weight ϖ_i.

Throughout this paper, we set

$$t = \begin{cases} 2 & \text{if } \mathfrak{g} = D_{n+1}^{(2)}, \\ 1 & \text{otherwise}, \end{cases} \quad \text{and} \quad \vartheta = \begin{cases} 1 & \text{if } \mathfrak{g} = B_n^{(1)} \text{ or } D_{n+1}^{(2)}, \\ 0 & \text{otherwise}. \end{cases} \tag{4.1}$$

§4.1. $i + j = k \leq n - \vartheta$

Recall that there exists an injective $U_q(\mathfrak{g}_0)$-module homomorphism

$$\Phi_{i,j} \colon V_0(\varpi_{i+j}) \rightarrowtail V_0(\varpi_i) \otimes V_0(\varpi_j) \quad \text{for } i + j \leq n - \vartheta$$

given by

$$u_\lambda \mapsto v_\lambda = \sum_{\lambda = \mu + \xi} C^\lambda_{\mu,\xi} u_\mu \otimes u_\xi \quad (C^\lambda_{\mu,\xi} \in \mathbf{k}) \tag{4.2}$$

where $\lambda \in \mathsf{W}_0 \cdot \varpi_{i+j}$ and μ (resp. ξ) runs over $\mathsf{W}_0 \cdot \varpi_i$ (resp. $\mathsf{W}_0 \cdot \varpi_j$).

For a positive integer $l \leq n - \vartheta$, we sometimes write $\lambda \in \mathrm{wt}(V_0(\varpi_l))$ as a sequence $(\lambda_1, \ldots, \lambda_n) \in \{1, 0, -1\}^n$ such that $\lambda = \sum_{k=1}^n \lambda_k \epsilon_k$. In (4.2), since $\Phi_{i,j}$ is a $U_q(\mathfrak{g}_0)$-homomorphism and $V(\varpi_{i+j})$ is generated by $u_{\varpi_{i+j}}$, we observe that

$$\lambda_k \geq 0 \ \Rightarrow \ \mu_k, \xi_k \geq 0 \quad \text{and} \quad \lambda_k \leq 0 \ \Rightarrow \ \mu_k, \xi_k \leq 0, \tag{4.3}$$

under the assumption that $C^\lambda_{\mu,\xi} \neq 0$. Since $\lambda_k, \mu_k, \xi_k \in \{1, 0, -1\}$ for all k, we conclude that

$$\mu_k \xi_k = 0 \quad \text{for all } 1 \le k \le n. \tag{4.4}$$

From (4.3), $C^{\lambda}_{\mu,\xi}$ must be the same as $C^{\mathsf{s}_k\lambda}_{\mathsf{s}_k\mu,\mathsf{s}_k\xi}$ whenever $\langle h_k, \lambda\rangle \neq 0$ for $k \in I_0$.

Proposition 4.1. *For $\lambda \in \mathsf{W}_0 \cdot \varpi_{i+j}$, $\mu \in \mathsf{W}_0 \cdot \varpi_i$ and $\xi \in \mathsf{W}_0 \cdot \varpi_j$ such that $\lambda = \mu + \xi$, set*

$$\begin{aligned} c^{\lambda}_{\mu,\xi} = {} & \#\{(a,b) \mid a<b,\ (\mu_a,\xi_a) = (0,1),\ \mu_b \neq 0\} \\ & + \#\{(a,b) \mid a<b,\ (\mu_a,\xi_a) = (-1,0),\ \xi_b \neq 0\}. \end{aligned} \tag{4.5}$$

Then the $C^{\lambda}_{\mu,\xi}$ in (4.2) is given by

$$C^{\lambda}_{\mu,\xi} = (-q_1)^{c^{\lambda}_{\mu,\xi}}.$$

Proof. First, we check that $c^{\mathsf{s}_k\lambda}_{\mathsf{s}_k\mu,\mathsf{s}_k\xi} = c^{\lambda}_{\mu,\xi}$ whenever $\langle h_k, \lambda\rangle \neq 0$ for $k \in I_0$. To do so, it suffices to consider $(a,b) = (k,k+1)$. Then one can easily check that

$$\begin{aligned} & \#\{(a,b) \mid a<b,\ (\mu_a,\xi_a) = (0,1),\ \mu_b \neq 0\} \\ & \qquad + \#\{(a,b) \mid a<b,\ (\mu_a,\xi_a) = (-1,0),\ \xi_b \neq 0\} \\ & = \#\{(a,b) \mid a<b,\ ((\mathsf{s}_k\mu)_a, (\mathsf{s}_k\xi)_a) = (0,1),\ (\mathsf{s}_k\mu)_b \neq 0\} \\ & \qquad + \#\{(a,b) \mid a<b,\ ((\mathsf{s}_k\mu)_a, (\mathsf{s}_k\mu)_a) = (-1,0),\ (\mathsf{s}_k\xi)_b \neq 0\}. \end{aligned}$$

Thus we can assume that $\lambda = \varpi_{i+j}$ since we only consider λ in $\mathsf{W}_0 \cdot \varpi_{i+j}$. If $k \ge i+j$, then $\langle h_k, \lambda\rangle = 0$ and hence $e_k v_\lambda = 0$. For $1 \le k < i+j$, we also have $e_k v_\lambda = 0$ since $\langle h_k, \lambda\rangle = 0$. On the other hand, by a direct computation with (4.2), the right hand side of the following equation must vanish:

$$e_k v_\lambda = \sum_{\substack{(\mu_k,\mu_{k+1})=(0,1) \\ (\xi_k,\xi_{k+1})=(1,0)}} C^{\lambda}_{\mu,\xi} q_1^{-1} v_{\mathsf{s}_k\mu} \otimes v_\xi + \sum_{\substack{(\mu_k,\mu_{k+1})=(1,0) \\ (\xi_k,\xi_{k+1})=(0,1)}} C^{\lambda}_{\mu,\xi} v_\mu \otimes v_{\mathsf{s}_k\xi}. \tag{4.6}$$

For $(\mu_k,\mu_{k+1}) = (0,1)$ and $(\xi_k,\xi_{k+1}) = (1,0)$, we can check that (4.5) yields

$$c^{\lambda}_{\mu,\xi} = c^{\lambda}_{\mathsf{s}_k\mu,\mathsf{s}_k\xi} + 1.$$

This implies that the right hand side of (4.6) vanishes when $C^{\lambda}_{\mu,\xi} = (-q_1)^{c^{\lambda}_{\mu,\xi}}$. Thus our assertion follows. □

Now we shall determine $x, y \in \mathbf{k}^{\times}$ such that there exists an injective $U_q'(\mathfrak{g})$-module homomorphism

$$V(\varpi_{i+j}) \rightarrowtail V(\varpi_i)_x \otimes V(\varpi_j)_y. \tag{4.7}$$

(4.8) The strategy in this subsection can be explained as follows: We have an injection of $U_q(\mathfrak{g}_0)$-modules,

$$V_0(\varpi_{i+j}) \rightarrowtail V(\varpi_i)_x \otimes V(\varpi_j)_y.$$

Using the characterization of $V(\varpi_{i+j})$ in Definition 2.3, it suffices to determine x and y satisfying the following equation which is induced by the desired $U_q'(\mathfrak{g})$-homomorphism in (4.7) and the action of s_0:

$$C^{\mathsf{s}_0\lambda}_{\mathsf{s}_0\mu,\mathsf{s}_0\xi} = f(x)g(y)C^{\lambda}_{\mu,\xi}, \quad \text{where}$$

(i) λ, μ and ξ are extremal weights and $\langle h_0, \lambda\rangle \neq 0$,
(ii) $f(x)$ and $g(y)$ arise from the action of $\mathcal{S}_0$ on $V(\varpi_i)_x$ and $V(\varpi_j)_y$, respectively.

The action s_0 for $\mathfrak{g} = A^{(2)}_{2n-1}$ $(n \geq 3)$, $\mathfrak{g} = A^{(2)}_{2n}$ $(n \geq 2)$ and $\mathfrak{g} = D^{(2)}_{n+1}$ $(n \geq 2)$ can be summarized as follows:

$$\mathsf{s}_0(\varepsilon_1,\varepsilon_2,\varepsilon_3,\dots,\varepsilon_n) = \begin{cases} (-\varepsilon_2,-\varepsilon_1,\varepsilon_3,\dots,\varepsilon_n) & \text{if } \mathfrak{g} = A^{(2)}_{2n-1}, \\ (-\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n) & \text{if } \mathfrak{g} = A^{(2)}_{2n} \text{ or } D^{(2)}_{n+1}, \end{cases} \tag{4.9}$$

where $\varepsilon_k \in \{-1,0,1\}$ for $1 \leq k \leq n$.

Proposition 4.2. *Let $\mathfrak{g} = A^{(2)}_{2n-1}$ $(n \geq 3)$. Then the x, y in (4.7) are given as follows:*

$$x = (-q)^j \quad \textit{and} \quad y = (-q)^{-i}.$$

Proof. By (4.8)(i) and (4.9), it suffices to consider $\lambda \in \mathsf{W}_0 \cdot \varpi_{i+j}$ such that $\lambda_1, \lambda_2 \geq 0$. Thus it is enough to consider $\mu_1,\mu_2,\xi_1,\xi_2 \geq 0$ by (4.3). Then

$$\mathcal{S}_0 \cdot v_\lambda = v_{\mathsf{s}_0\lambda} = \sum C^{\lambda}_{\mu,\xi} x^{\delta(\mu_1=1)+\delta(\mu_2=1)} y^{\delta(\xi_1=1)+\delta(\xi_2=1)}\ v_{\mathsf{s}_0\mu} \otimes v_{\mathsf{s}_0\xi}.$$

Thus

$$C^{\mathsf{s}_0\lambda}_{\mathsf{s}_0\mu,\mathsf{s}_0\xi} = x^{\delta(\mu_1=1)+\delta(\mu_2=1)} y^{\delta(\xi_1=1)+\delta(\xi_2=1)} C^{\lambda}_{\mu,\xi}.$$

On the other hand, by (4.5),

$$\begin{aligned} c^{\mathsf{s}_0\lambda}_{\mathsf{s}_0\mu,\mathsf{s}_0\xi} - c^{\lambda}_{\mu,\xi} &= +\delta(\mu_1=1)\times\#\{b>1 \mid \xi_b \neq 0\} + \delta(\mu_2=1)\times\#\{b>2 \mid \xi_b\neq 0\} \\ &\quad - \delta(\xi_1=1)\times\#\{b>1 \mid \mu_b\neq 0\} - \delta(\xi_2=1)\times\#\{b>2 \mid \mu_b \neq 0\} \\ &= \delta(\mu_1=1)\times j + \delta(\mu_2=1)\times(j-\delta(\xi_2=1)) \\ &\quad - \delta(\xi_1=1)\times i - \delta(\xi_2=1)\times(i-\delta(\mu_2=1)). \end{aligned}$$

By (4.4), $\mu_i\xi_i = 0$ $(i = 1, 2)$ and hence we conclude that

$$c^{\mathsf{s}_0\lambda}_{\mathsf{s}_0\mu,\mathsf{s}_0\xi} - c^{\lambda}_{\mu,\xi} = -(\delta(\xi_1 = 1) + \delta(\xi_2 = 1)) \times i + (\delta(\mu_1 = 1) + \delta(\mu_2 = 1)) \times j.$$

Thus $x = (-q)^j$ and $y = (-q)^{-i}$ as desired. □

Proposition 4.3. *Let $\mathfrak{g} = A^{(2)}_{2n}$ $(n \geq 2)$. Then the x, y in (4.7) are given by*

$$x = (-q)^j \quad \text{and} \quad y = (-q)^{-i}.$$

Proof. For the same reason as in the preceding proposition, it suffices to consider $\lambda \in \mathsf{W}_0 \cdot \varpi_{i+j}$ such that $\langle h_0, \lambda\rangle < 0$ and hence $\lambda_1 = 1$. Then

$$\mathcal{S}_0 \cdot v_\lambda = v_{\mathsf{s}_0\lambda} = e^{(2)}_0 v_\lambda = C^{\lambda}_{\mu,\xi} x^{\delta(\mu_1=1)} y^{\delta(\xi_1=1)}\ v_{\mathsf{s}_0\mu} \otimes v_{\mathsf{s}_0\xi}.$$

Thus

$$C^{\mathsf{s}_0\lambda}_{\mathsf{s}_0\mu,\mathsf{s}_0\xi} = x^{\delta(\mu_1=1)} y^{\delta(\xi_1=1)} C^{\lambda}_{\mu,\xi}.$$

On the other hand, by (4.5),

$$C^{\mathsf{s}_0\lambda}_{\mathsf{s}_0\mu,\mathsf{s}_0\xi} = (-q)^{\delta(\mu_1=1)\times\#\{b>1|\xi_b\neq 0\}}(-q)^{-\delta(\xi_1=1)\times\#\{b>1|\mu_b\neq 0\}} C^{\lambda}_{\mu,\xi}.$$

Thus we conclude that $x = (-q)^j$ and $y = (-q)^{-i}$. □

Proposition 4.4. *Let $\mathfrak{g} = D^{(2)}_{n+1}$ $(n \geq 2)$. Then the x, y in (4.7) are given by*

$$x = (-q^2)^{j/2} \quad \text{and} \quad y = (-q^2)^{-i/2}.$$

Proof. It suffices to consider $\lambda \in \mathsf{W}_0 \cdot \varpi_{i+j}$ such that $\langle h_0, \lambda\rangle < 0$, and hence $\lambda_1 = 1$. Note that $q_1 = q^2$. Then

$$\mathcal{S}_0 \cdot v_\lambda = v_{\mathsf{s}_0\lambda} = e^{(2)}_0 v_\lambda = \sum C^{\lambda}_{\mu,\xi} x^{2\delta(\mu_1=1)} y^{2\delta(\xi_1=1)}\ v_{\mathsf{s}_0\mu} \otimes v_{\mathsf{s}_0\xi}.$$

Here $\mathsf{s}_0(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) = (-\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$. Thus

$$C^{\mathsf{s}_0\lambda}_{\mathsf{s}_0\mu,\mathsf{s}_0\xi} = x^{2\delta(\mu_1=1)} y^{2\delta(\xi_1=1)} C^{\lambda}_{\mu,\xi}.$$

On the other hand, by (4.5),

$$C^{\mathsf{s}_0\lambda}_{\mathsf{s}_0\mu,\mathsf{s}_0\xi} = (-q^2)^{\delta(\mu_1=1)\times\#\{b>1|\xi_b\neq 0\}}(-q^2)^{-\delta(\xi_1=1)\times\#\{b>1|\mu_b\neq 0\}} C^{\lambda}_{\mu,\xi}.$$

Thus we conclude that $x^2 = (-q^2)^j$ and $y^2 = (-q^2)^{-i}$ as desired. □

Theorem 4.5. *For $i + j = k \leq n - \vartheta$, there exists a surjective $U'_q(\mathfrak{g})$-module homomorphism*

$$p_{i,j} \colon V(\varpi_i)_{(-q^t)^{-j/t}} \otimes V(\varpi_j)_{(-q^t)^{i/t}} \twoheadrightarrow V(\varpi_k). \tag{4.10}$$

By taking duals, there exists an injective $U'_q(\mathfrak{g})$-module homomorphism

$$\iota_{i,j} \colon V(\varpi_k) \rightarrowtail V(\varpi_i)_{(-q^t)^{j/t}} \otimes V(\varpi_j)_{(-q^t)^{-i/t}}. \tag{4.11}$$

Proof. This immediately follows from the previous propositions. □

§4.2. $i = j = n,\ k < n$ for $\mathfrak{g} = D^{(2)}_{n+1}$

In this subsection, we fix $\mathfrak{g}$ as $D^{(2)}_{n+1}$. Recall that there exists an injective $U_q(B_n)$-module homomorphism $V_0(\varpi_k) \rightarrowtail V_0(\varpi_n) \otimes V_0(\varpi_n)$ given by

$$u_\lambda \mapsto v_\lambda = \sum_{\lambda=\mu+\xi} C^\lambda_{\mu,\xi} u_\mu \otimes u_\xi \tag{4.12}$$

where $\lambda \in \mathsf{W}_0 \cdot \varpi_i$ and $\mu, \xi \in \mathsf{W}_0 \cdot \varpi_n$.

We sometimes write $\mu \in \mathrm{wt}(V_0(\varpi_n))$ as a sequence $(\mu_1, \ldots, \mu_n) \in \{+,-\}^n$ such that

$$\mu = \sum_{k=1}^{n} \frac{\mu_k}{2} \varepsilon_k.$$

Proposition 4.6. *Set*

$$\begin{aligned} {}_1c^\lambda_{\mu,\xi} &= \#\{(a,b) \mid a < b,\ (\mu_a,\xi_a) = (-,+),\ (\mu_b,\xi_b) = (+,-)\},\\ {}_2c^\lambda_{\mu,\xi} &= \#\{a \mid (\mu_a,\xi_a) = (-,+)\},\\ \varphi(c) &= (-q)^c(-q^2)^{c(c-1)/2}. \end{aligned} \tag{4.13}$$

Then $C^\lambda_{\mu,\xi}$ in (4.12) is given by

$$C^\lambda_{\mu,\xi} = (-q^2)^{{}_1c^\lambda_{\mu,\xi}} \varphi({}_2c^\lambda_{\mu,\xi}).$$

Proof. As in Proposition 4.1, one can check that $C^{\mathsf{s}_k\lambda}_{\mathsf{s}_k\mu,\mathsf{s}_k\xi} = C^\lambda_{\mu,\xi}$ whenever $\langle h_k, \lambda\rangle \neq 0$ for $k \in I_0$. Thus we can assume that $\lambda = \varpi_i$ since we consider λ in $\mathsf{W}_0 \cdot \varpi_i$. If $1 \le k \le i$, then $e_k u_\lambda = 0$ since $\langle e_k, \lambda\rangle = 0$. Similarly, $\langle e_k, \lambda\rangle = 0$ for $k > i$. On the other hand, by a direct computation with (4.12), the right hand side of the following equation must vanish $(i < k)$:

$$e_k v_\lambda = \begin{cases} \displaystyle\sum_{\substack{(\mu_k,\mu_{k+1})=(-,+)\\(\xi_k,\xi_{k+1})=(+,-)}} C^\lambda_{\mu,\xi}(q^2)^{-1} u_{\mathsf{s}_k\mu} \otimes u_\xi \\ \displaystyle\qquad + \sum_{\substack{(\mu_k,\mu_{k+1})=(+,-)\\(\xi_k,\xi_{k+1})=(-,+)}} C^\lambda_{\mu,\xi} u_\mu \otimes u_{\mathsf{s}_k\xi} & \text{if } i < k < n,\\ \displaystyle\sum_{(\mu_n,\xi_n)=(-,+)} C^\lambda_{\mu,\xi} q^{-1} u_{\mathsf{s}_n\mu} \otimes u_\xi \\ \displaystyle\qquad + \sum_{(\mu_n,\xi_n)=(+,-)} u_\mu \otimes u_{\mathsf{s}_n\xi} & \text{if } k = n. \end{cases}$$

Equivalently,

$$C^\lambda_{\mu,\xi} = \begin{cases} -q^2 C^\lambda_{\mathsf{s}_k\mu,\mathsf{s}_k\xi} & \text{if } i < k < n \text{ and } (\mu_k,\xi_k) = (-,+),\\ (-q)^{-1} C^\lambda_{\mathsf{s}_n\mu,\mathsf{s}_n\xi} & \text{if } k = n \text{ and } \mu_n = +. \end{cases}$$

On the other hand, for $i < k < n$ and $(\mu_k, \xi_k) = (-,+)$, one can check that

$$
{}_1c^{\lambda}_{\mu,\xi} = \begin{cases} {}_1c^{\lambda}_{\mathsf{s}_k\mu,\mathsf{s}_k\xi} - 1 & \text{if } i < k < n \text{ and } (\mu_k,\xi_k) = (-,+), \\ {}_1c^{\lambda}_{\mathsf{s}_k\mu,\mathsf{s}_k\xi} + {}_2c^{\lambda}_{\mu,\xi} & \text{if } k = n \text{ and } \mu_n = +, \end{cases}
$$
$$
{}_2c^{\lambda}_{\mu,\xi} = \begin{cases} {}_2c^{\lambda}_{\mathsf{s}_k\mu,\mathsf{s}_k\xi} & \text{if } i < k < n \text{ and } (\mu_k,\xi_k) = (-,+), \\ {}_2c^{\lambda}_{\mathsf{s}_k\mu,\mathsf{s}_k\xi} - 1 & \text{if } k = n \text{ and } \mu_n = +, \end{cases}
$$

by using the formulas in (4.13). Thus our assertion follows. □

Theorem 4.7. *For $k \le n-1$, there exists a surjective $U_q'(D^{(2)}_{n+1})$-module homomorphism*

$$
(4.14) \qquad p_{n,k} \colon V(\varpi_n)_{\pm\sqrt{-1}(-q^2)^{-(n-k)/2}} \otimes V(\varpi_n)_{\mp\sqrt{-1}(-q^2)^{(n-k)/2}} \twoheadrightarrow V(\varpi_k).
$$

By taking duals, there exists an injective $U_q'(D^{(2)}_{n+1})$-module homomorphism

$$
(4.15) \qquad \iota_{n,k} \colon V(\varpi_k) \rightarrowtail V(\varpi_n)_{\pm\sqrt{-1}(-q^2)^{(n-k)/2}} \otimes V(\varpi_n)_{\mp\sqrt{-1}(-q^2)^{-(n-k)/2}}.
$$

Proof. We apply the same strategy of §4.1, i.e., we determine the x and y in (4.7). As in Proposition 4.2, we first consider $\lambda \in \mathsf{W}_0 \cdot \varpi_k$ with $\lambda_1 = 1$ and hence $\mu_1 = \xi_1 = +$. In this case,

$$
\mathcal{S}_0 \cdot v_\lambda = v_{\mathsf{s}_0\lambda} = e_0^{(2)} v_\lambda = \sum C^{\lambda}_{\mu,\xi} xy u_{\mathsf{s}_0\mu} \otimes u_{\mathsf{s}_0\xi}.
$$

On the other hand,

$$
{}_1c^{\lambda}_{\mu,\xi} = {}_1c^{\mathsf{s}_0\lambda}_{\mathsf{s}_0\mu,\mathsf{s}_0\xi}, \qquad {}_2c^{\lambda}_{\mu,\xi} = {}_2c^{\mathsf{s}_0\lambda}_{\mathsf{s}_0\mu,\mathsf{s}_0\xi}.
$$

Thus we conclude that $xy = 1$.

Consider $\lambda \in \mathsf{W}_0 \cdot \varpi_i$ with $\langle h_0, \lambda \rangle = 0$. Equivalently $\lambda_1 = 0$ and hence $-\mu_1 = \xi_1$. In this case,

$$
0 = e_0 v_\lambda = \sum_{(\mu_1,\xi_1)=(+,-)} C^{\lambda}_{\mu,\xi} q^{-1} x u_{\mathsf{s}_0\mu} \otimes u_\xi + \sum_{(\mu_1,\xi_1)=(-,+)} C^{\lambda}_{\mu,\xi} y u_\mu \otimes u_{\mathsf{s}_0\xi}.
$$

Thus, for $\mu_1 = +$, we have

$$
C^{\lambda}_{\mathsf{s}_0\mu,\mathsf{s}_0\xi} = C^{\lambda}_{\mu,\xi}(-q)^{-1} \times \frac{x}{y} = C^{\lambda}_{\mu,\xi}(-q)^{-1} \times x^2.
$$

On the other hand,

$$
{}_1c^{\lambda}_{\mathsf{s}_0\mu,\mathsf{s}_0\xi} = {}_1c^{\lambda}_{\mu,\xi} + \#\{b > 1 \mid (\mu_b,\xi_b) = (+,-)\} \ \text{ and } \ {}_2c^{\lambda}_{\mathsf{s}_0\mu,\mathsf{s}_0\xi} = {}_2c^{\lambda}_{\mu,\xi} + 1.
$$

Thus $x^2 = -(-q^2)^{n-k}$, which yields our assertion. □

§4.3. $j = 1$ and $i = k = n$ for $\mathfrak{g} = A_{2n}^{(2)}$

In this subsection, we show that there exists a surjective $U_q'(A_{2n}^{(2)})$-homomorphism

$$V(\varpi_n)_{(-q)^{-1}} \otimes V(\varpi_1)_{(-q)^n} \twoheadrightarrow V(\varpi_n). \tag{4.16}$$

Indeed, we do not use (4.16) in this paper; we present it for the sake of forthcoming works.

Similarly to the previous subsections, we shall determine the relations among a, b and c such that

$$V(\varpi_n)_a \rightarrowtail V(\varpi_1)_b \otimes V(\varpi_n)_c. \tag{4.17}$$

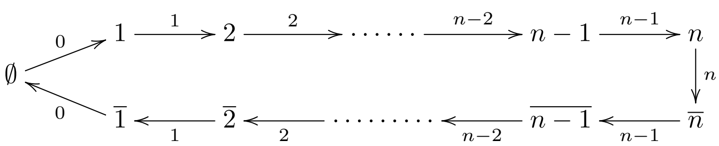
Recall that (see [29, Table 1])

$$V(\varpi_n) \simeq \bigoplus_{j=0}^{n} V_0(\varpi_j) \quad \text{as } U_q(C_n)\text{-modules.} \tag{4.18}$$

Here $V_0(\varpi_0)$ is the trivial $U_q(C_n)$-module $\mathbf{k}$. Thus

$$V_0(\varpi_n)^{\oplus 2} \rightarrowtail V(\varpi_1) \otimes V(\varpi_n) \quad \text{as } U_q(C_n)\text{-modules.}$$

The crystal graph of $V(\varpi_1)$ is given by (see [13, Example 11.1.4])



We denote by u the dominant integral weight vector of $V(\varpi_n)$ with weight $\varpi_n = \sum_{i \in I_0} \varepsilon_i$.

For $i_1, \ldots, i_k, j_1, \ldots, j_l \in I_0$, we let $\mathsf{u}[\overline{i_1}, \ldots, \overline{i_l}, \widehat{j_1}, \ldots, \widehat{j_l}]$ be the vector in $V_0(\varpi_{n-l})$, a $U_q(C_n)$-submodule of $V(\varpi_i)$, with weight

$$\mathrm{wt}(\mathsf{u}[\overline{i_1}, \ldots, \overline{i_k}, \widehat{j_1}, \ldots, \widehat{j_l}]) = \mathrm{wt}(\mathsf{u}) - \sum_{s=1}^{k} 2\varepsilon_{i_s} - \sum_{t=1}^{l} \varepsilon_{j_t},$$

if such a weight vector exists in $V_0(\varpi_{n-l})$. Note that, by (4.18), the vector is unique if it exists.

The map (4.17), if it exists, sends u to

$$\mathsf{v} = v_\emptyset \otimes \mathsf{u} + (-qb^{-1}c)\Big(\sum_{k=1}^{n} (-q)^{k-1} v_k \otimes \mathsf{u}[\widehat{k}]\Big).$$

Here v should be a unique (up to a constant) vector in $V(\varpi_1)_b \otimes V(\varpi_n)_c$ in the sense that $e_i\mathsf{v} = 0$ for $i \in I_0$ and $f_0\mathsf{v} = 0$, since u is a unique vector in $V(\varpi_n)_a$ with these properties.

In $V(\varpi_n)_a$, we have

$$\mathcal{S}_0 \cdot \mathsf{u} = a\mathcal{S}_w \cdot \mathsf{u} \quad \text{where} \quad \mathcal{S}_w = \mathcal{S}_1 \cdots \mathcal{S}_n \text{ for } w = \mathsf{s}_1 \cdots \mathsf{s}_n \in \mathsf{W}_0. \tag{4.19}$$

On the other hand,

$$\mathcal{S}_0 \cdot \mathsf{v} = e_0^{(2)}\mathsf{v} = c\ v_{\overline{1}} \otimes \mathsf{u}[\widehat{1}] - qc\ v_{\emptyset} \otimes \mathsf{u}[\overline{1}] - qb^{-1}cd \sum_{k \neq 1} (-q)^{k-1} v_k \otimes \mathsf{u}[\overline{1}, \widehat{k}],$$

$$\begin{aligned}\mathcal{S}_w \cdot \mathsf{v} &= f_1^{(2)} \cdots f_{n-1}^{(2)} f_n \mathsf{v} \\ &= v_{\emptyset} \otimes \mathsf{u}[\overline{1}] + (-qb^{-1}c)\Big(\sum_{k \neq n} (-q)^{k-1} v_{k+1} \otimes \mathsf{u}[\overline{1}, \widehat{k+1}]\Big) \\ &\quad + (-qb^{-1}c)(-q)^{n-1} v_{\overline{1}} \otimes \mathsf{u}[\widehat{1}],\end{aligned}$$

where d is an element in $\mathbf{k}^{\times}$ such that

$$e_0^{(2)}\mathsf{u}[\widehat{k}] = d \times \mathsf{u}[\overline{1}, \widehat{k}] \quad \text{for } k \neq 1 \text{ in } V(\varpi_n)_c. \tag{4.20}$$

By (4.19), we conclude that

$$a = -qc, \qquad b = a(-q)^n, \qquad d = c. \tag{4.21}$$

Now, it suffices to show that $d = c = 1$.

Proposition 4.8. *For $1 \neq k \in I_0$, the coefficient d in (4.20) is 1, i.e.,*

$$e_0^{(2)}\mathsf{u}[\widehat{k}] = \mathsf{u}[\overline{1}, \widehat{k}] \quad \textit{in } V(\varpi_n)_c.$$

Proof. By Definition 2.2(3), we have

$$f_1 e_0 \mathsf{u}[\widehat{2}] = e_0 f_1 \mathsf{u}[\widehat{2}] = e_0 \mathsf{u}[\widehat{1}] = [2]_0 \mathsf{u}[\overline{1}].$$

Thus

$$e_1 e_0 \mathsf{u}[\widehat{2}] = [2]_0 e_1^{(2)} \mathsf{u}[\overline{1}] = [2]_0 \mathsf{u}[\overline{2}].$$

From the actions e_i $(i \in I)$ on $V(\varpi_n)_c$, we have

$$e_0 e_1 e_0^{(2)} \mathsf{u}[\widehat{2}] = c e_0 e_1 \mathsf{u}[\overline{1}, \widehat{2}] = c e_0 \mathsf{u}[\overline{2}, \widehat{1}] = c[2]_0 \mathsf{u}[\overline{1}, \overline{2}]. \tag{4.22}$$

Since all vectors in $V(\varpi_n)$ are annihilated by the action $e_0^{(3)}$, the quantum Serre relation in Definition 2.2(4) implies that

$$\begin{aligned}e_0 e_1 e_0^{(2)} \mathsf{u}[\widehat{2}] &= (e_1 e_0^{(3)} + e_0^{(2)} e_1 e_0 - e_0^{(3)} e_1)\mathsf{u}[\widehat{2}] \\ &= e_0^{(2)} e_1 e_0 \mathsf{u}[\widehat{2}] = [2]_0 e_0^{(2)} \mathsf{u}[\overline{2}] = [2]_0 u[\overline{1}, \overline{2}].\end{aligned} \tag{4.23}$$

From (4.21), (4.22) and (4.23), we conclude that $d = c = 1$ as desired. □

Now, we have the following theorem.

Theorem 4.9. *There exists a surjective $U_q'(A_{2n}^{(2)})$-module homomorphism*

$$p_{1,n}\colon V(\varpi_n)_{(-q)^{-1}} \otimes V(\varpi_1)_{(-q)^n} \twoheadrightarrow V(\varpi_n). \tag{4.24}$$

By taking duals, there exists an injective $U_q'(A_{2n}^{(2)})$-module homomorphism

$$\iota_{1,n}\colon V(\varpi_n) \rightarrowtail V(\varpi_1)_{(-q)^n} \otimes V(\varpi_n)_{(-q)^{-1}}. \tag{4.25}$$

§4.4. $D_{n+1}^{(2)}$-analogue of the surjective homomorphisms given in [19, (A.17)]

This subsection is devoted to proving the following lemma:

Lemma 4.10. *Let $\eta, \eta' \in \{\sqrt{-1}, -\sqrt{-1}\}$ and $1 \le k, l \le n-1$ such that $k+l=n$. Then there exists a surjective $U_q'(D_{n+1}^{(2)})$-module homomorphism*

$$V(\varpi_k)_{\eta(-q^2)^{-l/2}} \otimes V(\varpi_l)_{\eta'(-q^2)^{k/2}} \twoheadrightarrow V(\varpi_n)_{-1} \otimes V(\varpi_n).$$

Proof. Note that $\eta/\eta' = \pm 1$. By Theorem 4.7, there are injective $U_q'(D_{n+1}^{(2)})$-homomorphisms

$$\psi_1\colon V(\varpi_k)_{\eta(-q^2)^{-l/2}} \rightarrowtail V(\varpi_n)_{-1} \otimes V(\varpi_n)_{(-q^2)^{(-n+k-l)/2}},$$

$$\psi_2\colon V(\varpi_l)_{\eta'(-q^2)^{k/2}} \rightarrowtail V(\varpi_n)_{-(-q^2)^{(n+k-l)/2}} \otimes V(\varpi_n),$$

by taking duals. Then we obtain $\varphi = (\mathrm{id}_{V(\varpi_n)_{-1}} \otimes \mathrm{tr} \otimes \mathrm{id}_{V(\varpi_n)}) \circ (\psi_1 \otimes \psi_2)$,

$$\varphi\colon V(\varpi_k)_{\eta(-q^2)^{-l/2}} \otimes V(\varpi_l)_{\eta'(-q^2)^{k/2}} \to V(\varpi_n)_{-1} \otimes V(\varpi_n),$$

since $V(\varpi_n)_{(-q^2)^{(-n+k-l)/2}}$ and $V(\varpi_n)_{-(-q^2)^{(n+k-l)/2}}$ are dual to each other.

Applying the argument of [19, Lemma A.3.2], we have

$$\varphi(v \otimes w) \equiv \mathrm{tr}(u_{-\varpi_n} \otimes u_{\varpi_n}) v_1 \otimes w_1 \mod \bigoplus_{\lambda \neq -\varpi_k+\varpi_n} (V(\varpi_n)_{-1})_\lambda \otimes V(\varpi_n)_{-\varpi_k+\varpi_l-\lambda},$$

where

- v is the $U_q(B_n)$-lowest weight vector of $V(\varpi_k)_{\eta(-q^2)^{-l/2}}$ of weight $-\varpi_k$,
- w is the $U_q(B_n)$-highest weight vector of $V(\varpi_l)_{\eta'(-q^2)^{k/2}}$ of weight ϖ_l,
- v_1 is a non-zero vector of $V(\varpi_n)_{-1}$ of weight $-\varpi_k + \varpi_n$,
- w_1 is a non-zero vector of $V(\varpi_n)$ of weight $\varpi_l - \varpi_n$.

Thus φ is non-zero. As $V(\varpi_n)_{-1} \otimes V(\varpi_n)$ is irreducible, our assertion follows. □

§5. The computation of denominators between fundamental representations

For simplicity, we write $R^{\mathrm{norm}}_{k,l}$ for $R^{\mathrm{norm}}_{V(\varpi_k),V(\varpi_l)}$, $d_{k,l}$ for $d_{V(\varpi_k),V(\varpi_l)}$, and $a_{k,l}$ for $a_{V(\varpi_k),V(\varpi_l)}$ of §2.2.

By the result of [1, Appendix A] and [2], the denominator $d_{k,l}(z)$ and the element $a_{k,l}(z) \in \mathbf{k}(z)$ are symmetric with respect to the indices k and l:

$$d_{k,l}(z) = d_{l,k}(z) \quad \text{and} \quad a_{k,l}(z) = a_{l,k}(z). \tag{5.1}$$

§5.1. General framework

In this subsection, we propose the strategy for computing $d_{k,l}(z)$, which originates from [1, Appendix C] and [19, Appendix A].

Note that we have a surjective homomorphism

$$p_{l-1,1} \colon V(\varpi_{l-1})_{(-q^t)^{-1/t}} \otimes V(\varpi_1)_{(-q^t)^{l-1/t}} \twoheadrightarrow V(\varpi_l) \quad \text{if } l \le n-\vartheta, \tag{5.2}$$

by the previous section.

Assumption 5.1. (A) We know $a_{k,l'}(z)$ for $k \in I_0$ and $l' \le l-1$.
(B) We know $d_{1,1}(z)$ for all $\mathfrak{g}$, and $d_{1,n}(z)$ for $\mathfrak{g} = B_n^{(1)}$ or $\mathfrak{g} = D_{n+1}^{(2)}$.

With these assumptions and (2.4), consider the commutative diagram

(5.3)
$$\begin{array}{ccc}
V(\varpi_k) \otimes V(\varpi_{l-1})_{(-q^t)^{-1/t}z} \otimes V(\varpi_1)_{(-q^t)^{l-1/t}z} & \xrightarrow{\mathrm{id}_{V(\varpi_k)} \otimes p_{l-1,1}} & V(\varpi_k) \otimes V(\varpi_l)_z \\
\downarrow {\scriptstyle R^{\mathrm{univ}}_{k,l-1}((-q^t)^{-1/t}z) \otimes \mathrm{id}_{V(\varpi_1)_{(-q^t)^{l-1/t}z}}} & & \\
V(\varpi_{l-1})_{(-q^t)^{-1/t}z} \otimes V(\varpi_k) \otimes V(\varpi_1)_{(-q^t)^{l-1/t}z} & & \downarrow {\scriptstyle R^{\mathrm{univ}}_{k,l}(z)} \\
\downarrow {\scriptstyle \mathrm{id}_{V(\varpi_{l-1})_{(-q^t)^{-1/t}z}} \otimes R^{\mathrm{univ}}_{k,1}((-q^t)^{l-1/t}z)} & & \\
V(\varpi_{l-1})_{(-q^t)^{-1/t}z} \otimes V(\varpi_1)_{(-q^t)^{l-1/t}z} \otimes V(\varpi_k) & \xrightarrow{p_{l-1,1} \otimes \mathrm{id}_{V(\varpi_k)}} & V(\varpi_l)_z \otimes V(\varpi_k)
\end{array}$$

Then we have

(5.4)
$$\begin{array}{ccc}
v_{[1,\ldots,k]} \otimes v_{[1,\ldots,l-1]} \otimes v_l & \longmapsto & v_{[1,\ldots,k]} \otimes v_{[1,\ldots,l-1,l]} \\
\downarrow & & \\
a_{k,l-1}((-q^t)^{-1/t}z) v_{[1,\ldots,l-1]} \otimes v_{[1,\ldots,k]} \otimes v_l & & \downarrow \\
\downarrow & & \\
a_{k,l-1}((-q^t)^{-1/t}z) a_{k,1}((-q^t)^{l-1/t}z) v_{[1,\ldots,l-1]} \otimes w & \longmapsto & a_{k,l}(z) v_{[1,\ldots,l-1,l]} \otimes v_{[1,\ldots,k]}
\end{array}$$

where

- $v_{[1,\dots,a]}$ is the dominant extremal weight vector of $V(\varpi_a)$ for $a \in I_0$,
- $w = R^{\rm norm}_{k,1}((-q^t)^{l-1/t} z)(v_{[1,\dots,k]} \otimes v_l)$.

By considering the vector w, we can get an equation explaining the relationship between

$$a_{k,l-1}(-q^{-1}z)a_{k,1}((-q)^{l-1}z) \quad \text{and} \quad a_{k,l}(z).$$

By Assumption 5.1(A), we can compute $a_{k,l}(z)$ by induction.

After getting $a_{k,l}(z)$, we use the formulas in Lemma 2.4, by applying two surjective homomorphisms of §4,

$$p_{k-1,1}\colon V(\varpi_{k-1})_{(-q^t)^{-1/t}} \otimes V(\varpi_1)_{(-q^t)^{k-1/t}} \twoheadrightarrow V(\varpi_k), \tag{5.5}$$

$$p^*_{k-1,1}\colon V(\varpi_k)_{(-q^t)^{-1/t}} \otimes V(\varpi_1)_{(p^*)(-q^t)^{-k/t}} \twoheadrightarrow V(\varpi_{k-1}), \tag{5.6}$$

and setting $W = V(\varpi_l)$ or $V(\varpi_n)$, to get two elements in $\mathbf{k}[z^{\pm 1}]$ which are described in terms of $d_{k,l}(z)$'s and $a_{k,l}(z)$'s. Here (5.6) is a non-zero composition of $U'_q(\mathfrak{g})$-homomorphisms given as follows:

$$\begin{aligned} V(\varpi_k)_{(-q^t)^{-1/t}} \otimes V(\varpi_1)_{(p^*)(-q^t)^{-k/t}} & \\ &\hookrightarrow V(\varpi_{k-1}) \otimes V(\varpi_1)_{(-q^t)^{-k/t}} \otimes V(\varpi_1)_{(p^*)(-q^t)^{-k/t}} \\ &\twoheadrightarrow V(\varpi_{k-1}) \otimes \mathbf{k} \simeq V(\varpi_{k-1}). \end{aligned}$$

Note that (5.6) is surjective since it is non-zero and $V(\varpi_{k-1})$ is irreducible.

Since we know the forms of $a_{k,l}(z)$'s, two elements in $\mathbf{k}[z^{\pm 1}]$ can be described in terms of $d_{k,l}(z)$'s and polynomials in $\mathbf{k}[z]$ (up to a constant multiple in $\mathbf{k}[z^{\pm 1}]^\times$).

By the assumptions, we know $d_{1,1(z)}$, $d_{1,n}(z)$ and hence we can compute $d_{k,l}(z)$ and $d_{k,n}(z)$, by manipulating the two elements in $\mathbf{k}[z^{\pm 1}]$ and using induction.

The denominator $d_{1,1}(z)$ of $R^{\rm norm}_{1,1}(z)\colon V(\varpi_1) \otimes V(\varpi_1)_z \to V(\varpi_1)_z \otimes V(\varpi_1)$ is computed in [23] (see also [14] for $\mathfrak{g} = A_2^{(2)}$) to be

$$d_{1,1}(z) = (z^t - (q^2)^t)(z^t - (p^*)^t). \tag{5.7}$$

The denominator $d_{1,n}(z)$ of $R^{\rm norm}_{1,n}(z)\colon V(\varpi_1) \otimes V(\varpi_n)_z \to V(\varpi_n)_z \otimes V(\varpi_1)$ for $\mathfrak{g} = B_n^{(1)}$ is computed in [7] to be

$$d_{1,n}(z) = d_{n,1}(z) = z - (-1)^{n+1} q_s^{2n+1}. \tag{5.8}$$

Considering Assumption 5.1, the only missing part is the denominator $d_{1,n}(z)$ for $\mathfrak{g} = D_{n+1}^{(2)}$ $(n \geq 2)$.

§5.2. The denominator $d_{1,n}(z)$ for $\mathfrak{g} = D_{n+1}^{(2)}$ $(n \geq 2)$

To compute the denominator $d_{1,n}(z)$ for $\mathfrak{g} = D_{n+1}^{(2)}$ $(n \geq 2)$, we follow the notation and arguments given in [23, Section 4].

By the $U_q'(D_{n+1}^{(2)})$-module structure of $V(\varpi_1)$ and $V(\varpi_n)$ in §3, we have

$$V(\varpi_1) \simeq V_0(\varpi_1) \oplus V_0(0) \quad \text{and} \quad V(\varpi_n) \simeq V_0(\varpi_n) \quad \text{as } U_q(B_n)\text{-modules.}$$

Here $V_0(\varpi_n)$ (resp. $V_0(0)$) is the highest $U_q(B_n)$-module with highest weight ϖ_n (resp. 0). Thus

$$V(\varpi_n) \otimes V(\varpi_1) \simeq V_0(\lambda) \oplus V_0(\varpi_n)^{\oplus 2} \quad \text{as } U_q(B_n)\text{-modules,}$$

where $\lambda = (3/2, 1/2, \ldots, 1/2)$. Let

$$m_n^+ = (+, \ldots, +) \quad \text{and} \quad m^i = (+, \ldots, +, \overset{i}{-}, +, \ldots, +) \quad (1 \leq i \leq n)$$

be elements in $V(\varpi_n)$. Then by direct calculation we have:

Lemma 5.2. *Let u_λ, $u^1_{\varpi_n}$ and $u^2_{\varpi_n}$ be the $U_q(B_n)$-highest weight vectors with weights λ, ϖ_n and ϖ_n in $V(\varpi_n)_x \otimes V(\varpi_1)_y$ respectively. Then:*

(a) $u_\lambda = (m_n^+) \otimes v_1$,
(b) $u^1_{\varpi_n} = [2]_0^{-1}(m_n^+) \otimes v_\emptyset$,
(c) $u^2_{\varpi_n} = \sum_{k=1}^n (-1)^k q^{2k}(m^{n+1-k}) \otimes v_{n+1-k} + [2]_n^{-1}(m_n^+) \otimes v_0$.

Lemma 5.3. *Let $\tilde{u}_\lambda$, $\tilde{u}^1_{\varpi_n}$ and $\tilde{u}^2_{\varpi_n}$ be the $U_q(B_n)$-highest weight vectors with weights λ, ϖ_n and ϖ_n in $V(\varpi_1)_y \otimes V(\varpi_n)_x$, respectively. Then:*

(a) $\tilde{u}_\lambda = v_1 \otimes (m_n^+)$,
(b) $\tilde{u}^1_{\varpi_n} = [2]_0^{-1} v_\emptyset \otimes (m_n^+)$,
(c) $\tilde{u}^2_{\varpi_n} = \sum_{k=1}^n (-1)^{n+1-k} q^{-2(n+1-k)} v_k \otimes (m^k) + q^{-1}[2]_n^{-1} v_0 \otimes (m_n^+)$.

Hence $R^{\mathrm{norm}}_{1,n} \colon V(\varpi_1)_y \otimes V(\varpi_n)_x \to V(\varpi_n)_x \otimes V(\varpi_1)_y$ can be expressed by

$$R^{\mathrm{norm}}_{1,n}(\tilde{u}_\lambda) = u_\lambda \quad \text{and} \quad R^{\mathrm{norm}}_{1,n}(\tilde{u}^i_{\varpi_n}) = \sum_{j=1}^{2} a^{\varpi_n}_{ji} u^j_{\varpi_n}.$$

The following lemmas can be obtained by direct calculations.

Lemma 5.4. *For the highest weight vectors defined in Lemma 5.2, we have*

(a) $f_0(u^1_{\varpi_n}) = x^{-1}y^{-1}(q^{-1}x)u_\lambda$,
(b) $f_0(u^2_{\varpi_n}) = x^{-1}y^{-1}((-1)^n q^{2n} y)u_\lambda$,
(c) $e_1 \cdots e_{n-1} e_n^{(2)} e_{n-1} \cdots e_2 e_1 e_0(u^1_{\varpi_n}) = (y)u_\lambda$,
(d) $e_1 \cdots e_{n-1} e_n^{(2)} e_{n-1} \cdots e_2 e_1 e_0(u^2_{\varpi_n}) = (q^{-1}x)u_\lambda$,

in $V(\varpi_n)_x \otimes V(\varpi_1)_y$.

Lemma 5.5. *For the highest weight vectors defined in Lemma 5.3, we have*

(a) $f_0(\tilde{u}^1_{\varpi_n}) = x^{-1}y^{-1}(x)\tilde{u}_\lambda$,
(b) $f_0(\tilde{u}^2_{\varpi_n}) = x^{-1}y^{-1}((-1)^n q^{-2n-2}y)\tilde{u}_\lambda$,
(c) $e_1 \cdots e_{n-1}e_n^{(2)}e_{n-1} \cdots e_2e_1e_0(\tilde{u}^1_{\varpi_n}) = (q^{-1}y)\tilde{u}_\lambda$,
(d) $e_1 \cdots e_{n-1}e_n^{(2)}e_{n-1} \cdots e_2e_1e_0(\tilde{u}^2_{\varpi_n}) = (q^{-1}x)\tilde{u}_\lambda$,

in $V(\varpi_1)_y \otimes V(\varpi_n)_x$.

From these lemmas, we obtain

$$\begin{pmatrix} q^{-1}y^{-1} & (-1)^n q^{2n}x^{-1} \\ y & q^{-1}x \end{pmatrix} (a^{\varpi_n}_{ij}) = \begin{pmatrix} y^{-1} & (-1)^n q^{-2n-2}x^{-1} \\ q^{-1}y & q^{-1}x \end{pmatrix},$$

and hence

$$(a^{\varpi_n}_{ij}) = \frac{1}{z^2+(-q^2)^{n+1}} \begin{pmatrix} qz^2-(-1)^n q^{2n+1} & (-1)^n(q^{-2n-1}-q^{2n+1})z \\ (1-q^2)z & z^2-(-1)^n q^{-2n} \end{pmatrix},$$

where $z = xy^{-1}$.

Hence we conclude that

$$d_{1,n}(z) = d_{n,1}(z) = z^2 + (-q^2)^{n+1} \quad \text{for } \mathfrak{g} = D^{(2)}_{n+1}\ (n \geq 2). \tag{5.9}$$

§5.3. Denominators between fundamental representations

Write

$$d_{k,l}(z) = \prod_\nu (z - x_\nu).$$

For rational functions $f, g \in \mathbf{k}(z)$, we write $f \equiv g$ if there exists an element $a \in \mathbf{k}[z^{\pm 1}]^\times$ such that $f = ag$.

Lemma 5.6 ([1]). *For $k, l \in I_0$, we have*

$$\begin{aligned} & a_{k,l}(z)a_{k,l}((p^*)^{-1}z) \equiv \frac{d_{k,l}(z)}{d_{k,l}(p^*z^{-1})}, \\ & a_{k,l}(z) = q^{(\varpi_k,\varpi_l)} \prod_\nu \frac{(p^*x_\nu z; p^{*2})_\infty (p^*x_\nu^{-1}z; p^{*2})_\infty}{(x_\nu z; p^{*2})_\infty (p^{*2}x_\nu^{-1}z; p^{*2})_\infty}, \end{aligned} \tag{5.10}$$

where $(z;q)_\infty = \prod_{s=0}^\infty (1-q^s z)$.

Now we list the triples (δ, c, p^*) for each $\mathfrak{g}$:

$\mathfrak{g}$	δ	c	p^*
$A_{2n-1}^{(2)}$	$\alpha_0+\alpha_1+2(\alpha_2+\cdots+\alpha_{n-1})+\alpha_n$	$h_0+h_1+2(h_2+\cdots+h_n)$	$-(-q)^{2n}$
$A_{2n}^{(2)}$	$2(\alpha_0+\cdots+\alpha_{n-1})+\alpha_n$	$h_0+2(h_1+\cdots+h_n)$	$(-q)^{2n+1}$
$B_n^{(1)}$	$\alpha_0+\alpha_1+2(\alpha_2+\cdots+\alpha_n)$	$h_0+h_1+2(h_2+\cdots+h_{n-1})+h_n$	$-(-q)^{2n-1}$
$D_{n+1}^{(2)}$	$\alpha_0+\alpha_1+\cdots+\alpha_n$	$h_0+2(h_1+\cdots+h_{n-1})+h_n$	$-(-q^2)^n$

Table 1. (δ, c, p^*) for each affine type

By Lemma 5.6 and (5.7), we can compute $a_{1,1}(z)$ for all $\mathfrak{g}$:

$$a_{1,1}(z)=\begin{cases} q\dfrac{\langle 2n+2\rangle\langle 2n-2\rangle}{\langle 2n\rangle^2}\,\dfrac{[4n][0]}{[2][4n-2]} & \text{if } \mathfrak{g}=A_{2n-1}^{(2)}\ (n\ge 3),\\ q\dfrac{[2n+3][2n-1]}{[2n+1]^2}\,\dfrac{[4n+2][0]}{[2][4n]} & \text{if } \mathfrak{g}=A_{2n}^{(2)}\ (n\ge 2),\\ q\dfrac{[2n+1][2n-3]}{[2n-1]^2}\,\dfrac{[4n-2][0]}{[2][4n-4]} & \text{if } \mathfrak{g}=B_{n}^{(1)}\ (n\ge 3),\\ q\dfrac{\{n+1\}\{n-1\}}{\{n\}^2}\,\dfrac{\{2n\}\{0\}}{\{1\}\{2n-1\}} & \text{if } \mathfrak{g}=D_{n+1}^{(2)}\ (n\ge 2),\end{cases} \tag{5.11}$$

where, for $a\in\mathbb{Z}$ and $b\in\frac{1}{2}\mathbb{Z}$,

$$[a]=((-q)^a z;p^{*2})_\infty,\quad \langle a\rangle=(-(-q)^a z;p^{*2})_\infty,$$
$$\{b\}=((-q^2)^b z;p^{*2})_\infty\times(-(-q^2)^b z;p^{*2})_\infty.$$

Note that, for $a\in\mathbb{Z}$ and $b\in\frac{1}{2}\mathbb{Z}$, we have

$$\begin{aligned} &[a]/[a+4n]\equiv z-(-q)^{-a},\ \langle a\rangle/\langle a+4n\rangle\equiv z+(-q)^{-a} && \text{if } \mathfrak{g}=A_{2n-1}^{(2)}\ (n\ge 3),\\ &[a]/[a+4n+2]\equiv z-(-q)^{-a} && \text{if } \mathfrak{g}=A_{2n}^{(2)}\ (n\ge 2),\\ &[a]/[a+4n-2]\equiv z-(-q)^{-a} && \text{if } \mathfrak{g}=B_{n}^{(1)}\ (n\ge 3),\\ &\{b\}/\{b+2n\}\equiv z^2-(-q^2)^{-2b} && \text{if } \mathfrak{g}=D_{n+1}^{(2)}\ (n\ge 2).\end{aligned}$$

Following [16], [7, (3.12)] and [15, (3.7)], we recall that the image of $v_k\otimes v_l$ $(k\neq l\in I_0)$ under the normalized R-matrix

$$R_{1,1}^{\mathrm{norm}}(z)\colon V(\varpi_1)\otimes V(\varpi_1)_z\to V(\varpi_1)_z\otimes V(\varpi_1)$$

is given by

$$R_{1,1}^{\mathrm{norm}}(z)(v_k\otimes v_l)=\frac{(1-(q^2)^t)z^{t\times\delta(k\succ l)}}{z^t-(q^2)^t}(v_k\otimes v_l)+\frac{q^t(z^t-1)}{z^t-(q^2)^t}(v_l\otimes v_k). \tag{5.12}$$

Here $\succ$ is the linear order on the labeling set of the basis of $V(\varpi_1)$ (see [13, Section 8]).

Proposition 5.7. *For $1 \le k, l \le n - \vartheta$, we have*

$$a_{k,l}(z) \equiv \begin{cases} \dfrac{[|k-l|][4n-|k-l|]}{[k+l][4n-k-l]} \dfrac{\langle 2n+k+l\rangle\langle 2n-k-l\rangle}{\langle 2n+|k-l|\rangle\langle 2n-|k-l|\rangle} & \text{if } \mathfrak{g} = A^{(2)}_{2n-1}, \\ \dfrac{[|k-l|][4n+2-|k-l|]}{[k+l][4n+2-k-l]} \dfrac{[2n+1+k+l][2n+1-k-l]}{[2n+1+|k-l|][2n+1-|k-l|]} & \text{if } \mathfrak{g} = A^{(2)}_{2n}, \\ \dfrac{[|k-l|][2n+k+l-1][2n-k-l-1][2n-|k-l|-1]}{[k+l][2n+k-l-1][2n-k+l-1][2n-k-l-2]} & \text{if } \mathfrak{g} = B^{(1)}_{n}, \\ \dfrac{\{|\frac{k-l}{2}|\}\{2n-|\frac{k-l}{2}|\}\{n+\frac{k+l}{2}\}\{n-\frac{k+l}{2}\}}{\{\frac{k+l}{2}\}\{2n-\frac{k+l}{2}\}\{n+|\frac{k-l}{2}|\}\{n-|\frac{k-l}{2}|\}} & \text{if } \mathfrak{g} = D^{(2)}_{n+1}. \end{cases} \tag{5.13}$$

Proof. We give the proof only for $\mathfrak{g}$ of type $A^{(2)}_{2n-1}$. For the other $\mathfrak{g}$, one can apply the same argument. We first consider $k = 1$.

By (5.11), our assertion for $k = l = 1$ holds. Applying the commutative diagram (5.3) for $k = 1$, we have

$$a_{1,l-1}(-q^{-1}z)a_{1,1}((-q)^{l-1}z)v_{[1,\dots,l-1]} \otimes w \mapsto a_{1,l}(z)v_{[1,\dots,l-1,l]} \otimes v_1, \tag{5.14}$$

where

$$w = R^{\mathrm{norm}}_{1,1}((-q)^{l-1}z)(v_1 \otimes v_l) = \frac{q((-q)^{l-1}z-1)}{(-q)^{l-1}z-q^2} v_l \otimes v_1 + \frac{1-q^2}{(-q)^{l-1}z-q^2} v_1 \otimes v_l.$$

Since $v_{[1,\dots,l-1]} \otimes v_1$ vanishes under the map $p_{l-1,1}$, (5.14) indicates that

$$\begin{aligned} a_{1,l}(z) &= a_{1,l-1}(-q^{-1}z)a_{1,1}((-q)^{l-1}z)\frac{q((-q)^{l-1}z-1)}{(-q)^{l-1}z-q^2} \\ &\equiv a_{1,l-1}(-q^{-1}z)a_{1,1}((-q)^{l-1}z)\frac{[l-1]}{[4n+l-1]}\frac{[4n+l-3]}{[l-3]}. \end{aligned}$$

Hence our assertion for $k = 1$ follows by induction on l:

$$a_{1,l}(z) = a_{l,1}(z) \equiv \frac{[l-1][4n-l+1]}{[l+1][4n-l-1]} \frac{\langle 2n-l-1\rangle\langle 2n+l+1\rangle}{\langle 2n+l-1\rangle\langle 2n-l+1\rangle}. \tag{5.15}$$

By (5.1), we now assume $2 \le l \le k \le n$. By direct calculation, one can show that

$$\begin{aligned} f_{l-1}f_{l-2}\cdots f_1(v_{[1,\dots,k]} \otimes v_1) &= v_{[1,\dots,k]} \otimes v_l, \\ f_{l-1}f_{l-2}\cdots f_1(v_1 \otimes v_{[1,\dots,k]}) &= v_l \otimes v_{[1,\dots,k]}. \end{aligned}$$

Since $R^{\mathrm{norm}}_{k,1}$ is a $U'_q(\mathfrak{g})$-homomorphism and sends $v_{[1,\dots,k]} \otimes v_1$ to $v_1 \otimes v_{[1,\dots,k]}$, we have

$$R^{\mathrm{norm}}_{k,1}(z)(v_{[1,\dots,k]} \otimes v_l) = v_l \otimes v_{[1,\dots,k]}.$$

Thus, the image in (5.4),

$$a_{k,l-1}(-q^{-1}z)a_{k,1}((-q)^{l-1}z)v_{[1,\dots,l-1]}\otimes w \mapsto a_{k,l}(z)v_{[1,\dots,l-1,l]}\otimes v_{[1,\dots,k]}$$

for $w = R^{\mathrm{norm}}_{k,1}((-q)^{l-1}z)(v_{[1,\dots,k]}\otimes v_l) = v_l\otimes v_{[1,\dots,k]}$, implies that

$$a_{k,l}(z) = a_{k,l-1}(-q^{-1}z)\ a_{k,1}((-q)^{l-1}z) \quad (2\le l\le k\le n). \tag{5.16}$$

Hence one can obtain our assertion by induction on l. □

Theorem 5.8. *For $1\le k,l\le n-\vartheta$, we have*

$$d_{k,l}(z) = \prod_{s=1}^{\min(k,l)} (z^t-(-q^t)^{|k-l|+2s})(z^t-(p^*)^t(-q^t)^{2s-k-l}). \tag{5.17}$$

Proof. For $1\le k,l\le n-\vartheta$, set

$$D_{k,l}(z) = \prod_{s=1}^{\min(k,l)} (z^t-(-q^t)^{|k-l|+2s})(z^t-(p^*)^t(-q^2)^{2s-k-l}). \tag{5.18}$$

Observe that $D_{k,l}(z)$ behaves similarly to $d_{k,l}(z)$. Namely (cf. (5.1), (5.7) and (5.10)),

$$D_{1,1}(z) = d_{1,1}(z), \qquad D_{k,l}(z) = D_{l,k}(z), \tag{5.19}$$

$$\frac{D_{k,l}(z)}{D_{k,l}(p^*z^{-1})} \equiv a_{k,l}(z)a_{k,l}((p^*)^{-1}z) \equiv \frac{d_{k,l}(z)}{d_{k,l}(p^*z^{-1})}. \tag{5.20}$$

By calculations, one can check that

$$D_{k,l}(z) = D_{k,l-1}((-q^t)^{-1/t}z)D_{k,1}((-q^t)^{l-1/t}z) \quad \text{for } 2\le k\le n-\vartheta, \tag{5.21}$$

which is similar to (5.16).

Now we give a proof for $\mathfrak{g} = D^{(2)}_{n+1}$, since this case is most complicated. For the other $\mathfrak{g}$, the argument is similar.

We shall show that $D_{k,l}(z) = d_{k,l}(z)$ indeed. Our assertion for $k=l=1$ is presented in (5.9). Assume that $1\le k\le n-1$ and $2\le l\le n-1$.

From the surjective homomorphism in Theorem 4.5,

$$p_{l-1,1}\colon V(\varpi_{l-1})_{(-q^2)^{-1/2}}\otimes V(\varpi_1)_{(-q^2)^{(l-1)/2}} \twoheadrightarrow V(\varpi_l),$$

and the first formula in Lemma 2.4 with $P = V(\varpi_k)$, we have the following element in $\mathbf{k}[z^{\pm1}]$:

$$\frac{d_{k,l-1}((-q^2)^{-1/2}z)d_{k,1}((-q^2)^{(l-1)/2}z)}{d_{k,l}(z)}\ \frac{a_{k,l}(z)}{a_{k,l-1}((-q^2)^{-1/2}z)a_{k,1}((-q^2)^{(l-1)/2}z)}. \tag{5.22}$$

In particular, if $2 \le l \le k \le n-1$, then

$$\frac{d_{k,l-1}((-q^2)^{-1/2}z)d_{k,1}((-q^2)^{(l-1)/2}z)}{d_{k,l}(z)} \in \mathbf{k}[z^{\pm 1}], \tag{5.23}$$

since (cf. (5.16))

$$\frac{a_{k,l}(z)}{a_{k,l-1}((-q^2)^{-1/2}z)a_{k,1}((-q^2)^{(l-1)/2}z)} \in \mathbf{k}[z^{\pm 1}]^{\times}$$

by the computation using (5.13). Using (5.13) once again, for $k = 1 < l$, one can compute that

$$\frac{a_{1,l}(z)}{a_{1,l-1}((-q^2)^{-1/2}z)a_{1,1}((-q^2)^{(l-1)/2}z)} \equiv \frac{z^2-(-q^2)^{1-l}}{z^2-(-q^2)^{3-l}} \quad \text{for } 2 \le l \le n-1.$$

Set $k = 1$ and then replace l with k in (5.22). Then (5.22) becomes

$$\begin{aligned}&\frac{d_{1,k-1}((-q^2)^{-1/2}z)D_{1,1}((-q^2)^{(k-1)/2}z)}{d_{1,k}(z)}\frac{z^2-(-q^2)^{1-k}}{z^2-(-q^2)^{3-k}} \\ &\quad\equiv \frac{d_{1,k-1}((-q^2)^{-1/2}z)(z^2-(-q^2)^{2n-k+1})(z^2-(-q^2)^{1-k})}{d_{1,k}(z)} \in \mathbf{k}[z^{\pm 1}]\end{aligned} \tag{5.24}$$

for $2 \le k \le n-1$, since $D_{1,1}(z) = d_{1,1}(z)$.

On the other hand, from the surjective homomorphism

$$V(\varpi_k)_{(-q^2)^{-1/2}} \otimes V(\varpi_1)_{-(-q^2)^{(2n-k)/2}} \to V(\varpi_{k-1}),$$

and the second formula in Lemma 2.4 with $P = V(\varpi_l)$, we have the following element in $\mathbf{k}[z^{\pm 1}]$:

$$\frac{d_{1,l}(-(-q^2)^{(k-2n)/2}z)d_{k,l}((-q^2)^{1/2}z)}{d_{k-1,l}(z)}\,\frac{a_{k-1,l}(z)}{a_{k,l}((-q^2)^{1/2}z)a_{1,l}(-(-q^2)^{(k-2n)/2}z)}. \tag{5.25}$$

By the computations using (5.13), we have

$$\begin{aligned}&\frac{a_{k-1,l}(z)}{a_{k,l}((-q^2)^{1/2}z)a_{1,l}(-(-q^2)^{(k-2n)/2}z)} \\ &\quad\equiv \begin{cases} \dfrac{z^2-(-q^2)^{2n-k-l-1}}{z^2-(-q^2)^{2n-k-l+1}} & \text{if } 1 \le l < k \le n-1, \\[2ex] \dfrac{z^2-(-q^2)^{2n-k-l-1}}{z^2-(-q^2)^{2n-k-l+1}}\,\dfrac{z^2-(-q^2)^{-1}}{z^2-(-q^2)^{1}} & \text{if } 2 \le l = k \le n-1.\end{cases}\end{aligned}$$

Thus the element (5.25) in $\mathbf{k}[z^{\pm 1}]$ can be written as follows:

$$\frac{d_{1,l}(-(-q^2)^{(k-2n)/2}z)d_{k,l}((-q^2)^{1/2}z)}{d_{k-1,l}(z)}\,\frac{z^2-(-q^2)^{2n-k-l-1}}{z^2-(-q^2)^{2n-k-l+1}} \qquad \text{if } 1\le l<k\le n-1, \tag{5.26}$$

and

$$\frac{d_{1,l}(-(-q^2)^{(k-2n)/2}z)d_{k,l}((-q^2)^{1/2}z)}{d_{k-1,l}(z)}\,\frac{z^2-(-q^2)^{2n-k-l-1}}{z^2-(-q^2)^{2n-k-l+1}}\,\frac{z^2-(-q^2)^{-1}}{z^2-(-q^2)^{1}} \qquad \text{if } 2\le l=k\le n-1. \tag{5.27}$$

Setting $l=1$ in (5.26), we obtain

$$\frac{D_{1,1}(-(-q^2)^{(k-2n-1)/2}z)d_{k,1}(z)}{d_{k-1,1}((-q^2)^{-1/2}z)}\,\frac{z^2-(-q^2)^{2n-k-1}}{z^2-(-q^2)^{2n-k+1}}\in\mathbf{k}[z^{\pm 1}] \tag{5.28}$$

for $2\le k\le n-1$.

Now we claim that

$$d_{1,k}(z)=D_{1,k}(z)=(z^2-(-q^2)^{k+1})(z^2-(-q^2)^{2n-k+1}) \qquad \text{for } 2\le k\le n-1.$$

With (5.19), we can start induction on k. Thus (5.24) can be written in the form

$$\begin{aligned}&\frac{D_{1,k-1}((-q^2)^{-1/2}z)(z^2-(-q^2)^{2n+1-k})(z^2-(-q^2)^{1-k})}{d_{1,k}(z)}\\ &\equiv\frac{(z^2-(-q^2)^{k+1})(z^2-(-q^2)^{2n-k+3})(z^2-(-q^2)^{2n+1-k})(z^2-(-q^2)^{1-k})}{d_{1,k}(z)}\\ &\qquad\in\mathbf{k}[z^{\pm 1}] \quad \text{for } 2\le k\le n-1.\end{aligned} \tag{5.29}$$

Now we claim that

$$z=\pm(-q^2)^{(1-k)/2},\pm(-q^2)^{(2n-k+3)/2} \text{ are not zeros of } d_{1,k}(z). \tag{5.30}$$

If (5.30) is true, we have

$$\frac{D_{1,k}(z)}{d_{1,k}(z)}=\frac{(z^2-(-q^2)^{k+1})(z^2-(-q^2)^{2n+1-k})}{d_{1,k}(z)}\in\mathbf{k}[z^{\pm 1}] \tag{5.31}$$

for $2\le k\le n-1$.

Since $(1-k)/2\le 0$, we have $\pm(-q^2)^{(1-k)/2}\notin\mathbb{C}[[q]]$. Then [19, Theorem 2.2.1(i)] tells us that $\pm(-q^2)^{(1-k)/2}$ cannot be zeros of $d_{1,k}(z)$.

Now we shall show that $z = \pm(-q^2)^{(2n-k+3)/2}$ are not zeros of $d_{1k}(z)$. By (5.29), we know that $z = \pm(-q^2)^{(k-3)/2}$ are not zeros of $d_{1,k}(z)$. Since

$$\frac{D_{k,l}(z)}{d_{k,l}(z)} \equiv \frac{D_{k,l}(p^* z^{-1})}{d_{k,l}(p^* z^{-1})}$$

by (5.20), one can check that the fact that

$$z = \pm(-q^2)^{(k-3)/2} \text{ are not poles of } D_{1,k}(z)/d_{1,k}(z),$$

implies

$$z = \pm(-q^2)^{(2n-k+3)/2} \text{ are not poles of } D_{1,k}(-(-q^2)^n z^{-1})/d_{1,k}(-(-q^2)^n z^{-1})$$

Hence $\pm(-q^2)^{(2n-k+3)/2}$ cannot be zeros of $d_{1,k}(z)$ and hence (5.30) holds.

By induction on k in (5.28), we also obtain

$$\begin{aligned} &\frac{d_{1,k}(z)}{(z^2-(-q^2)^{k+1})(z^2-(-q^2)^{2n+1-k})} \in \mathbf{k}[z^{\pm 1}] && \text{if } k \neq n-1,\\ &\frac{d_{1,k}(z)}{z^2-(-q^2)^{2n+1-k}} \in \mathbf{k}[z^{\pm 1}] && \text{if } k = n-1. \end{aligned}$$

By Theorem 4.5 and Lemma 4.10, $d_{1,k}(z)$ has zeros at $\pm(-q^2)^{(k+1)/2}$ for $1 \le k \le n-1$. Thus

$$\frac{d_{1,k}(z)}{D_{1,k}(z)} = \frac{d_{1,k}(z)}{(z^2-(-q^2)^{k+1})(z^2-(-q^2)^{2n+1-k})} \in \mathbf{k}[z^{\pm 1}] \tag{5.32}$$

for $2 \le k \le n-1$.

Recall that $d_{1,k}(z) \in \mathbf{k}[z]$ is a monic polynomial of smallest degree. By considering (5.31) and (5.32) together, our assertion holds for $l = 1$:

$$d_{1,k}(z) = (z^2-(-q^2)^{k+1})(z^2-(-q^2)^{2n+1-k}) = D_{1,k}(z) \quad (2 \le k \le n-1).$$

Now we apply induction on $k+l$. Applying induction at (5.23) with (5.21), we have

$$\begin{aligned} \frac{d_{k,l-1}((-q^2)^{-1/2}z)d_{k,1}((-q^2)^{(l-1)/2}z)}{d_{k,l}(z)} &= \frac{D_{k,l-1}((-q^2)^{-1/2}z)D_{k,1}((-q^2)^{(l-1)/2}z)}{d_{k,l}(z)}\\ &= \frac{D_{k,l}(z)}{d_{k,l}(z)} \in \mathbf{k}[z^{\pm 1}]. \end{aligned}$$

Let $\phi_{k,l}(z) \in \mathbf{k}[z^{\pm 1}]$ satisfy $D_{k,l}(z) = d_{k,l}(z)\phi_{k,l}(z)$. We claim that $\phi_{k,l}(z) = 1$. Note that

$$\frac{D_{1,l}(-(-q^2)^{(k-2n)/2}z)D_{k,l}((-q^2)^{1/2}z)}{D_{k-1,l}(z)} \frac{z^2-(-q^2)^{2n-k-l-1}}{z^2-(-q^2)^{2n-k-l+1}}$$
$$= \begin{cases} (z^2-(-q^2)^{4n-k-l+1})(z^2-(-q^2)^{2n-k-l-1}) & \text{if } l<k, \\ (z^2-(-q^2)^{4n-k-l+1})(z^2-(-q^2)^{2n-k-l-1}) & \\ \quad \times (z^2-(-q^2)^{2n+1})(z^2-(-q^2)) & \text{if } l=k. \end{cases}$$

By (5.26), (5.27) and induction on $k+l$, the above elements can be written in the form

$$\frac{(z^2-(-q^2)^{4n-k-l+1})(z^2-(-q^2)^{2n-k-l-1})}{\phi_{k,l}((-q^2)^{1/2}z)} \in \mathbf{k}[z^{\pm 1}] \quad \text{if } l<k,$$
$$\frac{(z^2-(-q^2)^{4n-k-l+1})(z^2-(-q^2)^{2n-k-l-1})}{\phi_{k,l}((-q^2)^{1/2}z)} \times (z^2-(-q^2)^{2n+1})(z^2-(-q^2)^{-1}) \in \mathbf{k}[z^{\pm 1}] \quad \text{if } l=k.$$

Recall that $\phi_{k,l}((-q^2)^{1/2}z)$ divides $D_{k,l}((-q^2)^{1/2}z)$. Thus we conclude that

$$\phi_{k,l}(z) = 1 \quad \text{if } k+l<n, \tag{5.33}$$

$$\frac{z^2-(-q^2)^{2n-k-l}}{\phi_{k,l}(z)} \in \mathbf{k}[z^{\pm 1}] \quad \text{if } k+l \geq n, \tag{5.34}$$

since

- $(z^2-(-q^2)^{4n-k-l+1})(z^2-(-q^2)^{2n-k-l-1})$ is not a factor of $D_{k,l}((-q^2)^{1/2}z)$ for $k+l<n$,
- $(z^2-(-q^2)^{4n-k-l+1})(z^2-(-q^2)^{2n+1})(z^2-(-q^2)^{-1})$ is not a factor of $D_{k,l}((-q^2)^{1/2}z)$ for $k+l \geq n$.

Now our assertion holds if $z = \pm(-q^2)^{(2n-k-l)/2}$ are not zeros of $\phi_{k,l}(z)$ for $k+l \geq n$. From (5.20), one can see that

$$\phi_{k,l}(-(-q^2)^n z^{-1}) \equiv \phi_{k,l}(z). \tag{5.35}$$

Thus it suffices to prove that $z = \pm(-q^2)^{(k+l)/2}$ are not zeros of $\phi_{k,l}(z)$ for $k+l \geq n$.

(a) If $k+l>n$, then $n > 2n-k-l$ and hence $\phi_{k,l}(z) = 1$ by (5.33) and (5.35).

(b) Now assume $k+l=n$. Then Lemma 4.10 tells us that $d_{k,l}(z)$ has zeros at $z = \pm(-q^2)^{(k+l)/2}$. By the definition of $D_{k,l}(z)$, $\pm(-q^2)^{(k+l)/2}$ are zeros of multiplicity 1. Thus $\pm(-q^2)^{(k+l)/2}$ cannot be zeros of $\phi_{k,l}(z)$ when $k+l=n$. □

Now we shall compute $d_{k,n}(z)$ for $\mathfrak{g} = B_n^{(1)}$ and $\mathfrak{g} = D_{n+1}^{(2)}$. By Lemma 5.6, (5.8) and (5.9), we have

$$a_{1,n}(z) \equiv \begin{cases} \dfrac{[2n-3]_{(n+1)}[6n-1]_{(n+1)}}{[2n+1]_{(n+1)}[6n-5]_{(n+1)}} & \text{if } \mathfrak{g} = B_n^{(1)}, \\ \dfrac{\left\{\frac{3n+1}{2}\right\}'\left\{\frac{n-1}{2}\right\}'}{\left\{\frac{3n-1}{2}\right\}'\left\{\frac{n+1}{2}\right\}'} & \text{if } \mathfrak{g} = D_{n+1}^{(2)}, \end{cases} \tag{5.36}$$

where, for $a, k \in \mathbb{Z}$ and $b \in \frac{1}{2}\mathbb{Z}$,

$$[a]_{(k)} := ((-1)^k q_s^a z; p^{*2})_\infty, \qquad \{b\}' := (-\sqrt{-1}(-q^2)^b; p^{*2})_\infty(\sqrt{-1}(-q^2)^b; p^{*2})_\infty.$$

W prove the statement below only for $\mathfrak{g} = B_n^{(1)}$. For $\mathfrak{g} = D_{n+1}^{(2)}$, one can apply the same arguments.

Proposition 5.9. *For $1 \le l \le n-1$, we have*

$$a_{l,n}(z) \equiv \begin{cases} \dfrac{[2n-2l-1]_{(n+l)}[6n+2l-3]_{(n+l)}}{[2n+2l-1]_{(n+l)}[6n-2l-3]_{(n+l)}} & \textit{if } \mathfrak{g} = B_n^{(1)}, \\ \dfrac{\left\{\frac{3n+l}{2}\right\}'\left\{\frac{n-l}{2}\right\}'}{\left\{\frac{3n-l}{2}\right\}'\left\{\frac{n+l}{2}\right\}'} & \textit{if } \mathfrak{g} = D_{n+1}^{(2)}. \end{cases} \tag{5.37}$$

Proof. By (5.36), it suffices to consider $2 \le l \le n-1$. Applying the commutative diagram (5.3) with $k = n$, and (5.4), we have

$$a_{n,l-1}(-q^{-1}z)a_{n,1}((-q)^{l-1}z)v_{[1,\dots,l-1]} \otimes w \mapsto a_{n,l}(z)v_{[1,\dots,l-1,l]} \otimes m_n^+,$$

where $w = R_{n,1}^{\mathrm{norm}}((-q)^{l-1}z)(m_n^+ \otimes v_l)$ for the highest weight vector m_n^+ of $V(\varpi_n)$.

Since m_n^+ vanishes under the action f_i $(1 \le i \le l-1)$, as in the proof of Proposition 5.7 we have

$$w = R_{n,1}^{\mathrm{norm}}((-q)^{l-1}z)(m_n^+ \otimes v_l) = v_l \otimes m_n^+,$$

and hence

$$a_{n,l}(z) = a_{n,l-1}(-q^{-1}z)\ a_{n,1}((-q)^{l-1}z) \qquad \text{for } 2 \le l \le n-1. \tag{5.38}$$

By (5.36) and induction on l, our assertion follows. ◻

Theorem 5.10. *For $1 \le k \le n-1$, we have*

$$d_{k,n}(z) = \begin{cases} \displaystyle\prod_{s=1}^{k}(z - (-1)^{n+k} q_s^{2n-2k-1+4s}) & \textit{if } \mathfrak{g} = B_n^{(1)}, \\ \displaystyle\prod_{s=1}^{k}(z^2 + (-q^2)^{n-k+2s}) & \textit{if } \mathfrak{g} = D_{n+1}^{(2)}. \end{cases} \tag{5.39}$$

Proof. By (5.8), it suffices to consider $2 \le k \le n-1$. From the surjective homomorphism in Theorem 4.5,

$$V(\varpi_{k-1})_{(-q)^{-1}} \otimes V(\varpi_1)_{(-q)^{k-1}} \twoheadrightarrow V(\varpi_k),$$

the first formula in Lemma 2.4 with $W = V(\varpi_n)$ yields

$$\frac{d_{k-1,n}(-q^{-1}z)d_{1,n}((-q)^{k-1}z)}{d_{k,n}(z)} \frac{a_{k,n}(z)}{a_{k-1,n}(-q^{-1}z)a_{1,n}((-q)^{k-1}z)} \in \mathbf{k}[z^{\pm 1}].$$

By (5.38), this element can be written more simply as

$$\text{(5.40)} \qquad \frac{d_{k-1,n}(-q^{-1}z)d_{n,1}((-q)^{k-1}z)}{d_{k,n}(z)} \equiv \frac{d_{k-1,n}(-q^{-1}z)(z-(-1)^{n+k}q_s^{2n-2k+3})}{d_{k,n}(z)} \in \mathbf{k}[z^{\pm 1}].$$

On the other hand, for each $2 \le k \le n-1$, we have a surjective homomorphism

$$V(\varpi_k)_{-q^{-1}} \otimes V(\varpi_1)_{-(-q)^{2n-1-k}} \twoheadrightarrow V(\varpi_{k-1}).$$

Then the second formula in Lemma 2.4 with $P = V(\varpi_n)$ yields

$$\text{(5.41)} \quad \frac{d_{1,n}(-(-q)^{k+1-2n}z)d_{k,n}(-qz)}{d_{k-1,n}(z)} \frac{a_{k-1,n}(z)}{a_{1,n}(-(-q)^{k+1-2n}z)a_{k,n}(-qz)} \in \mathbf{k}[z^{\pm 1}].$$

Using (5.37), the second factor of (5.41) can be written as

$$\frac{a_{k-1,n}(z)}{a_{1,n}(-(-q)^{k+1-2n}z)a_{k,n}(-qz)} \equiv \frac{z-(-1)^{n+k+1}q_s^{2n-2k-3}}{z-(-1)^{n+k+1}q_s^{2n-2k+1}},$$

and hence (5.41) becomes

$$\frac{d_{k,n}(-qz)}{d_{k-1,n}(z)} \frac{(z-(-1)^{n+k+1}q_s^{6n-2k-1})(z-(-1)^{n+k+1}q_s^{2n-2k-3})}{(z-(-1)^{n+k+1}q_s^{2n-2k+1})} \in \mathbf{k}[z^{\pm 1}].$$

By the induction hypothesis,

$$z = (-1)^{n+k+1}q_s^{6n-2k-1} \quad \text{and} \quad (-1)^{n+k+1}q_s^{2n-2k-3}$$

are *not* zeros of $d_{k-1,n}(z)$. Hence

$$\text{(5.42)} \qquad \frac{d_{k,n}(-qz)}{d_{k-1,n}(z)(z-(-1)^{n+k+1}q_s^{2n-2k+1})} \in \mathbf{k}[z^{\pm 1}],$$

which is equivalent to

$$\frac{d_{k,n}(z)}{d_{k-1,n}(-q^{-1}z)(z-(-1)^{n+k}q_s^{2n-2k+3})} \in \mathbf{k}[z^{\pm 1}].$$

Combining (5.40) and (5.42) gives our assertion:

$$\begin{aligned} d_{k,n}(z) &\equiv d_{k-1,n}(-q^{-1}z)(z-(-1)^{n+k}q_s^{2n-2k+3}) \\ &= \prod_{s=1}^{k}(z-(-1)^{n+k}q_s^{2n-2k-1+4s}). \end{aligned}$$ □

Remark 5.11. In conclusion, we can observe that

for all $k, l \in I_0$, $R^{\mathrm{norm}}_{k,l}(z)$ has only simple poles unless $\mathfrak{g} = D^{(2)}_{n+1}$ $(n \geq 3)$.

For $\mathfrak{g} = D^{(2)}_{n+1}$ $(n \geq 3)$, $R^{\mathrm{norm}}_{k,l}(z)$ has a double pole at $z = \pm(-q^2)^{s/2}$ if

$$2 \leq k, l \leq n-1,\ k+l > n,\ 2n+2-k-l \leq s \leq k+l \text{ and } s \equiv k+l \bmod 2.$$

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Appendix: The table of denominators

Type	k,l	Denominators
$A^{(1)}_n$	$1 \leq k,l \leq n$	$d_{k,l}(z) = \prod_{s=1}^{\min(k,l,n+1-k,n+1-l)} (z-(-q)^{2s+\lvert k-l\rvert})$
$B^{(1)}_n$	$1 \leq k,l \leq n-1$	$d_{k,l}(z) = \prod_{s=1}^{\min(k,l)} (z-(-q)^{\lvert k-l\rvert+2s})(z+(-q)^{2n-k-l-1+2s})$
$q_s^2 = q$	$1 \leq k \leq n-1$	$d_{k,n}(z) = \prod_{s=1}^{k} (z-(-1)^{n+k}q_s^{2n-2k-1+4s})$
	$k=l=n$	$d_{n,n}(z) = \prod_{s=1}^{n} (z-(q_s)^{4s-2})$
$C^{(1)}_n$	$1 \leq k,l \leq n$	$d_{k,l}(z) = \prod_{s=1}^{\min(k,l,n-k,n-l)} (z-(-q_s)^{\lvert k-l\rvert+2s}) \prod_{i=1}^{\min(k,l)} (z-(-q_s)^{2n+2-k-l+2s})$

$D_n^{(1)}$	$1\le k,l\le n-2$	$d_{k,l}(z)=\prod_{s=1}^{\min(k,l)}(z-(-q)^{\lvert k-l\rvert+2s})(z-(-q)^{2n-2-k-l+2s})$
	$1\le k\le n-2$	$d_{k,n-1}(z)=d_{k,n}(z)=\prod_{s=1}^{k}(z-(-q)^{n-k-1+2s})$
	$\{k,l\}=\{n,n-1\}$	$d_{n,n-1}(z)=d_{n-1,n}(z)=\prod_{s=1}^{\lfloor n-1/2\rfloor}(z-(-q)^{4s})$
	$k=l\in\{n,n-1\}$	$d_{n,n}(z)=d_{n-1,n-1}(z)=\prod_{s=1}^{\lfloor n/2\rfloor}(z-(-q)^{4s-2})$
$A_{2n-1}^{(2)}$	$1\le k,l\le n$	$d_{k,l}(z)=\prod_{s=1}^{\min(k,l)}(z-(-q)^{\lvert k-l\rvert+2s})(z+(-q)^{2n-k-l+2s})$
$A_{2n}^{(2)}$	$1\le k,l\le n$	$d_{k,l}(z)=\prod_{s=1}^{\min(k,l)}(z-(-q)^{\lvert k-l\rvert+2s})(z-(-q)^{2n+1-k-l+2s})$
$D_{n+1}^{(2)}$	$1\le k,l\le n-1$	$d_{k,l}(z)=\prod_{s=1}^{\min(k,l)}(z^2-(-q^2)^{\lvert k-l\rvert+2s})(z^2-(-q^2)^{2n-k-l+2s})$
	$1\le k\le n-1$	$d_{k,n}(z)=\prod_{s=1}^{k}(z^2+(-q^2)^{n-k+2s})$
	$k=l=n$	$d_{n,n}(z)=\prod_{s=1}^{n}(z+(-q^2)^{s})$

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