

Analytic Singular Support Properties for Integral Operators in Hyperfunctions

by

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Abstract

In this paper, we continue our study of integral operators in [12] and give a result on the analytic singular support of the kernel hyperfunction similar to the main result in [12]. A new feature with respect to [12] is that we use classes of ultradifferentiable functions associated with a weight function and a topological description of related spaces of quasi-analytic functionals.

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§1. Introduction

In this paper, we continue the study of integral operators begun in [12] and give a result on the analytic singular support of the kernel hyperfunction. The main theorem in [12] concerned support properties of integral operators, whereas our present result will concern similar singular support properties.

We recall here some notation and explain the setting. Consider open sets $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$, and a hyperfunction \mathcal{K} defined on $V \times U$ which satisfies the condition

$$(1.1) \quad \{(x, y, 0, \eta) \in V \times U \times \mathbb{R}^n \times \mathbb{R}^m; \eta \neq 0\} \cap \text{WF}_A \mathcal{K} = \emptyset,$$

where $\text{WF}_A \mathcal{K}$ denotes the analytic wave front set of \mathcal{K} . We can then associate with \mathcal{K} a linear operator $T : \mathcal{A}'(U) \rightarrow \mathcal{B}(V)$ by

$$(1.2) \quad (Tu)(x) = \int_U \mathcal{K}(x, y)u(y) dy \quad \text{for } u \in \mathcal{A}'(U).$$

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Here $\mathcal{A}'(U)$ denotes the space of real-analytic functionals on U , $\mathcal{B}(V)$ the space of hyperfunctions on V , and the meaning of the integral in (1.2) is the one given by microlocal analysis to such expressions (see Sato–Kawai–Kashiwara [16]). Note that we shall identify $\mathcal{A}'(U)$ with the space $\mathcal{B}_c(U)$ of hyperfunctions on U with compact support. In this setting, T is said to be the integral operator associated with \mathcal{K} , and \mathcal{K} is said to be the kernel of T .

Our main result is the following

Theorem 1.1. *Let $T : \mathcal{A}'(U) \rightarrow \mathcal{B}(V)$ be the integral operator associated with some kernel $\mathcal{K} \in \mathcal{B}(V \times U)$ satisfying (1.1). Assume that $\text{singsupp}_A Tu \subset V$ is compact for every $u \in \mathcal{A}'(U)$. Then the projection $(V \times U) \cap \text{singsupp}_A \mathcal{K} \rightarrow U$ is proper.*

Here and later on, $\text{singsupp}_A u$ denotes the analytic singular support of u . We mention that [12, Theorem 1.1] had a statement which was quite similar to that of Theorem 1.1, in that it stated that if for an operator as in (1.2), $\text{supp } Tu$ is compact for every $u \in \mathcal{A}'(U)$, then the projection $(V \times U) \cap \text{supp } \mathcal{K} \rightarrow U$ must be proper.

Classical examples of operators which are as in Theorem 1.1 include the operators associated with parametrices for analytic pseudodifferential operators and analytic Fourier integral operators.

The argument in [12] was based on a preliminary decomposition of the kernel \mathcal{K} in terms of C^∞ functions. By studying non-analyticity of differentiable functions in a quantitative way we could avoid cohomological characterizations of the supports of the hyperfunctions in the image of T . A similar technique is again needed in this paper, but here we shall use related decompositions of \mathcal{K} with respect to an adapted class of possibly quasianalytic ultradifferentiable functions and we must then know that any real-analytic functional will be in one of the spaces of quasianalytic functionals associated with some class. The difference between the approaches in [12] and the present paper comes from the fact that to characterize the support of some hyperfunction is easier than to characterize its singular support. See Remark 6.5 for more details on what we mean by this statement. As for the classes of ultradifferentiable functions and quasianalytic functionals, we will use those introduced and studied by Braun–Meise–Taylor [5] and Heinrich–Meise [6].

The plan of the paper is as follows. In Section 2, we recall the notion of ultradifferentiable classes with respect to weight functions and quasianalytic functionals, from [5] and [6]. These classes will be called BMT-classes for short. We also cite a topological description of the spaces of quasianalytic functionals via the Fourier–Laplace transform, and an existence result for weight functions which may be large at infinity (see Theorem 2.7, Lemma 2.8 and the citations there).

In Section 3, we study twisted Radon transforms applied to ultradifferentiable functions. There we shall see that an important feature of BMT-classes is that they are stable under twisted Radon decompositions (Lemma 3.2). This fact will then lead to boundary value representations for ultradifferentiable kernels (Theorem 3.4). When our kernel functions \mathcal{K} in (1.2) are ultradifferentiable and when they operate on quasianalytic functionals, then the integrals there can also be given a meaning by duality. We have thus two ways of defining our integral operators, an initial one in terms of integration of hyperfunctions as in [16] and a second one by duality. Based on the result in Section 3, we establish in Section 4 that the two interpretations of the integral (1.2) are compatible (Corollary 4.1).

In Section 5, after showing in Lemma 5.2 that the inverse Fourier transforms of functions satisfying suitable decay estimates are ultradifferentiable, we prove a division theorem for a hyperfunction kernel. In it, a given kernel is written, up to a real-analytic remainder, in the form $P(D_x)\mathcal{K}'$, where $P(D_x)$ is a possibly infinite order elliptic partial differential operator in the x variable and \mathcal{K}' is an ultradifferentiable quotient (Theorem 5.1). Due to this, we can then give, in the final Section 6, the proof of our main result, Theorem 1.1, also using the fact that the space of real-analytic functionals is exhausted by the classes of quasianalytic functionals which appear in BMT theory.

We conclude the introduction by recalling [12, Theorems 1.2 and 1.3].

Theorem 1.2. *Let $\mathcal{K} \in \mathcal{B}(V \times U)$ be a hyperfunction satisfying (1.1) and let T be the associated operator defined in (1.2). If $Tu = 0$ for every $u \in \mathcal{A}'(U)$, then \mathcal{K} must vanish on $V \times U$.*

Theorem 1.3. *Let $\mathcal{K} \in \mathcal{B}(V \times U)$ be a hyperfunction satisfying (1.1). Assume that the operator $T : \mathcal{A}'(U) \rightarrow \mathcal{B}(V)$ defined in (1.2) actually maps $\mathcal{A}'(U)$ into $\mathcal{A}(V)$. Then \mathcal{K} is real-analytic on $V \times U$.*

The two theorems are essentially consequences of results of Kaneko [9], Bastin–Laubin [2, 3] and Kataoka–Oshima (see [8, Theorem 4.4.7'] for the latter). For the reason, see [12, Remark 1.7 and §2].

§2. Ultradifferentiability with respect to a weight function

We now recall part of the terminology and of the results on the notion of ultradifferentiability with respect to weight functions, and on quasianalytic functionals associated with weight functions, as introduced by Braun–Meise–Taylor [5] and Heinrich–Meise [6]. For the relation between this and the notions of ultradiffer-

entiability given by sequences, we refer to Bonet–Meise–Melikhov [4]. Also refer to [13] for the relation to so-called inhomogeneous Gevrey classes of [15].

First we give the notion of a Young conjugate (or Legendre transform) for increasing functions defined on $[0, \infty[$.

Definition 2.1 ([4, Definition 2.3]). Let $\varphi : [0, \infty[\rightarrow [0, \infty[$ be an increasing function satisfying $\varphi(0) = 0$ and $x = o(\varphi(x))$ as $x \rightarrow \infty$. Then the *Young conjugate* φ^* of φ is defined as

$$(2.1) \quad \varphi^* : [0, \infty[\rightarrow [0, \infty[, \quad \varphi^*(y) := \sup_{x \geq 0} (xy - \varphi(x)).$$

We also use the notation

$$(2.2) \quad \varphi_{[\lambda]}^*(y) := \lambda \varphi^*(y/\lambda),$$

which is the Young conjugate of $\lambda\varphi(\cdot)$, for $\lambda > 0$. Therefore, the family $\{\lambda\varphi\}_{\lambda>0}$ corresponds to the family $\{\varphi_{[\lambda]}^*\}_{\lambda>0}$.

Definition 2.2 ([4, Definition 2.1(a)]). Let $\omega : [0, \infty[\rightarrow [0, \infty[$ be a continuous increasing function which satisfies $\omega|_{[0,1]} \equiv 0$. It is called a *weight function* if it has the following properties:

- (α) $\omega(2t) = O(\omega(t))$ as $t \rightarrow \infty$,
- (β) $\log(1+t) = o(\omega(t))$ as $t \rightarrow \infty$,
- (γ) $\omega(t) = O(t)$ as $t \rightarrow \infty$,
- (δ) $\varphi : [0, \infty[\rightarrow [0, \infty[$ defined by $\varphi(x) := \omega(e^x)$ is convex.

A weight function $\omega(t)$ is called *quasianalytic* if $\int_0^\infty \frac{\omega(t)}{1+t^2} dt = \infty$. While we shall mainly work with quasianalytic cases, we shall not use properties specific to quasianalyticity.

For a weight function ω , we always consider the function φ defined in Definition 2.2(δ) and its Young conjugate φ^* . Note that for a weight function ω , the φ given in Definition 2.2(δ) always satisfies the hypothesis of Definition 2.1. In fact, $x = o(\varphi(x))$ as $x \rightarrow \infty$ in view of Definition 2.2(β).

Example 2.3. The function $\omega_A(t) := \max\{0, t - 1\}$ is a weight function corresponding to $\varphi_A(x) = e^x - 1$ with Young conjugate $\varphi_A^*(y) = y \log y - y + 1$.

Consider two weight functions ω_1 and ω_2 , and let $\varphi_{j, [\lambda]}^*$ for $\lambda > 0$ and $j = 1, 2$ denote the functions given by (2.1) and (2.2) for $\varphi_j(x) := \omega_j(e^x)$ ($j = 1, 2$) respectively. Then

$$\forall t \geq 0, \omega_1(t) \leq \omega_2(t) \Rightarrow \forall y \geq 0, \varphi_{1, [\lambda]}^*(y) \geq \varphi_{2, [\lambda]}^*(y).$$

Proposition 2.4. *Let ω be a weight function. Define φ , φ^* and $\varphi_{[\lambda]}^*$ by Definition 2.2(δ), (2.1) and (2.2). Then:*

- (a) φ^* is a convex increasing function mapping $[0, \infty[$ into $[0, \infty[$, satisfying $\varphi^*(0) = 0$ and $\lim_{y \rightarrow \infty} \varphi^*(y)/y = \infty$.
- (b) $(\varphi^*)^* = \varphi$; more generally, $(\varphi_{[\lambda]}^*)^* = \lambda\varphi$.
- (c) $\varphi_{[\lambda]}^*(y)$ is decreasing in λ , that is, $\varphi_{[\lambda]}^*(y) \geq \varphi_{[\mu]}^*(y)$ for $0 < \lambda \leq \mu$, $y \geq 0$.
- (d) $\varphi_{[\lambda]}^*(p) + \varphi_{[\lambda]}^*(q) \leq \varphi_{[\lambda]}^*(p+q)$ for any $\lambda > 0$ and $p, q \geq 0$.
- (e) $\varphi_{[\lambda+\mu]}^*(p+q) \leq \varphi_{[\lambda]}^*(p) + \varphi_{[\mu]}^*(q)$ for any $\lambda, \mu > 0$ and $p, q \geq 0$.
- (f) $\lambda\varphi(x) - x \leq \sup_{p \in \mathbb{N}_0} (xp - \varphi_{[\lambda]}^*(p)) \leq \lambda\varphi(x)$ for any $\lambda > 0$ and $x \geq 0$.
- (g) For any $k, l_1, l_2 \geq 0$ and $\lambda > 0$, there exist $\mu > 0$ and $C \geq 0$ such that

$$(2.3) \quad \varphi_{[\lambda]}^*(y + l_1) + ky \leq \varphi_{[\mu]}^*(y + l_2) + C, \quad y \geq 0.$$

On the other hand, for any $k, l_1, l_2 \geq 0$ and $\lambda > 0$, there exist $\mu > 0$ and $C \geq 0$ such that

$$(2.4) \quad \varphi_{[\lambda]}^*(y + l_1) - ky \geq \varphi_{[\mu]}^*(y + l_2) - C, \quad y \geq 0.$$

(See [6, Lemma 2.6].)

- (h) If ω is sublinear, i.e., $\omega(t) = o(t)$ as $t \rightarrow \infty$, then

$$\varphi_{[\lambda]}^*(y)/y - \log y \rightarrow \infty \quad \text{as } y \rightarrow \infty,$$

for any $\lambda > 0$. In particular, $\inf_{y>0} (\varphi_{[\lambda]}^*(y) - y \log y - ay) > -\infty$ for any sublinear ω , $\lambda > 0$ and $a \in \mathbb{R}$.

Here in (f) and later on, \mathbb{N}_0 denotes $\{0, 1, 2, \dots\}$, while $\mathbb{N} = \{1, 2, \dots\}$.

Proof. (a) and (b). That φ^* is increasing and $\varphi^*(0) = 0$ follows directly from the definition. The convexity of φ^* follows from the fact that φ^* is the supremum of the convex functions $y \mapsto xy - \varphi(x)$. By definition, and from the convexity of φ ,

$$\varphi^{**}(x) = \sup_{y \geq 0} (xy - \varphi^*(y)) = \sup_{y \geq 0} \inf_{p \geq 0} ((x-p)y + \varphi(p)) = \varphi(x).$$

This also implies $\varphi^*(y)/y \geq x - \varphi(x)/y$ for any $x, y \geq 0$, therefore we have $\lim_{y \rightarrow \infty} \varphi^*(y)/y = \infty$, or $y = o(\varphi^*(y))$ as $y \rightarrow \infty$.

(c), (d) and (e) follow from the convexity of φ^* and $\varphi^*(0) = 0$. In fact, they follow from the intermediate inequalities

$$\begin{aligned} \varphi^*(y/\mu) &\leq (1 - \lambda/\mu)\varphi^*(0) + (\lambda/\mu)\varphi^*(y/\lambda) \quad \text{for } 0 < \lambda \leq \mu, \\ \varphi_{[\lambda]}^*(p) &\leq \frac{q\varphi_{[\lambda]}^*(0) + p\varphi_{[\lambda]}^*(p+q)}{p+q}, \quad \varphi_{[\lambda]}^*(q) \leq \frac{p\varphi_{[\lambda]}^*(0) + q\varphi_{[\lambda]}^*(p+q)}{q+p}, \end{aligned}$$

$$\varphi^* \left(\frac{\lambda(p/\lambda) + \mu(q/\mu)}{\mu + \lambda} \right) \leq \frac{\lambda\varphi^*(p/\lambda) + \mu\varphi^*(q/\mu)}{\mu + \lambda},$$

which are consequences of the convexity of φ^* and $\varphi_{[\lambda]}^*$.

(f) for the general $\lambda > 0$ follows from the special case $\lambda = 1$. The right inequality follows trivially from (b) so let us prove the left one. Denoting by $[y]$ the integer part of y , we obtain $xy - \varphi^*(y) \leq x([y] + 1) - \varphi^*([y])$ since $[y] \leq y < [y] + 1$ and φ^* is increasing. By taking the supremum in $y \geq 0$ and using (b), we have

$$\varphi(x) \leq \sup_{p \in \mathbb{N}_0} (x(p+1) - \varphi^*(p)) = \sup_{p \in \mathbb{N}_0} (xp - \varphi^*(p)) + x.$$

(g) was proved in [6, Lemma 2.6], but let us prove it also here for the convenience of the reader. Note that from (e) we have, for any $\lambda > \mu > 0$ and any $l_1, l_2 \geq 0$,

$$(2.5) \quad \sup_{y \geq 0} (\varphi_{[\lambda]}^*(y + l_1) - \varphi_{[\mu]}^*(y + l_2)) < \infty.$$

In fact, the left hand side is less than $\varphi_{[\lambda-\mu]}^*(l_1 - l_2)$ when $l_1 \geq l_2$, and zero otherwise. Note also that from Definition 2.2(α), there exists a constant $A > 1$ such that $\varphi(x+1) \leq A(\varphi(x)+1)$ for any $x \geq 0$, which implies $\varphi(x+k) \leq A^k(\varphi(x)+k)$ for $k \in \mathbb{N}_0$ and $x \geq 0$. Thus

$$\begin{aligned} \varphi_{[\lambda]}^*(y + l_1) + ky &= \sup_{x \geq 0} (x(y + l_1) - \lambda\varphi(x)) + ky \\ &= \sup_{x \geq 0} ((x+k)(y + l_1) - kl_1 - \lambda(\varphi(x) + k) + k\lambda) \\ &\leq k(\lambda - l_1) + \sup_{x \geq 0} ((x+k)(y + l_1) - \lambda A^{-k}\varphi(x + k)) \\ &\leq k(\lambda - l_1) + \varphi_{[\lambda/A^k]}^*(y + l_1). \end{aligned}$$

If we choose μ with $0 < \mu < \lambda/A^k$, then (2.3) follows from (2.5). Similarly,

$$\begin{aligned} \varphi_{[\lambda]}^*(y + l_1) - ky &= \sup_{x \geq 0} (x(y + l_1) - \lambda\varphi(x)) - ky \\ &\geq \sup_{x \geq k} ((x-k)(y + l_1) + kl_1 - \lambda\varphi(x)) \\ &\geq kl_1 + \sup_{x \geq k} ((x-k)(y + l_1) - \lambda A^k\varphi(x-k) - \lambda k A^k) \\ &= k(l_1 - kA^k) + \varphi_{[\lambda A^k]}^*(y + l_1). \end{aligned}$$

If we choose μ with $\mu > \lambda A^k$, then (2.4) follows from (2.5).

The second statement of (h) follows from the first one, which we will prove. The inequality $\varphi(x) \leq Ae^x + B$ for $x \geq 0$ and positive constants A and B implies

$$\varphi^*(y) \geq \sup_{x \geq 0} (xy - Ae^x - B) \geq y \log(y/A) - y - B \quad \text{for } y \geq A,$$

and therefore $\liminf_{y \rightarrow \infty} (\varphi^*(y)/y - \log y) \geq -1 - \log A$. Since $\varphi(x) = o(e^x)$ from the sublinearity of ω , we can take an arbitrary $A > 0$, which concludes the proof for $\lambda = 1$. In view of

$$\varphi_{[\lambda]}^*(y)/y - \log y = \varphi^*(y/\lambda)/(y/\lambda) - \log(y/\lambda) - \log \lambda,$$

the general case follows from the case $\lambda = 1$. □

Definition 2.5. Let ω be a weight function and $V \subset \mathbb{R}^n$ an open set. Then we define, for $\lambda > 0$ and $K \Subset V$, a quasi-norm $p_{\omega, \lambda, K}$ on $C^\infty(V)$ by

$$p_{\omega, \lambda, K}(f) := \sup_{x \in K, \alpha \in \mathbb{N}_0^n} |\partial^\alpha f(x)| \exp(-\varphi_{[\lambda]}^*(|\alpha|)),$$

and also define the spaces $\mathcal{E}_{(\omega)}(V)$ and $\mathcal{E}_{\{\omega\}}(V)$ of ω -ultradifferentiable functions of Beurling type, respectively of Roumieu type, on V by

$$\begin{aligned} \mathcal{E}_{(\omega)}(V) &:= \{f \in C^\infty(V); \forall K \Subset V, \forall m \in \mathbb{N}, p_{\omega, m, K}(f) < \infty\}, \\ \mathcal{E}_{\{\omega\}}(V) &:= \{f \in C^\infty(V); \forall K \Subset V, \exists m \in \mathbb{N}, p_{\omega, 1/m, K}(f) < \infty\}. \end{aligned}$$

These spaces are endowed with the natural locally convex topologies given by

$$(2.6) \quad \mathcal{E}_{(\omega)}(V) = \varprojlim_{K \Subset V} \varinjlim_{m \in \mathbb{N}} \{f \in C^\infty(V); p_{\omega, m, K}(f) < \infty\},$$

$$(2.7) \quad \mathcal{E}_{\{\omega\}}(V) = \varprojlim_{K \Subset V} \varinjlim_{m \in \mathbb{N}} \{f \in C^\infty(V); p_{\omega, 1/m, K}(f) < \infty\},$$

where \varinjlim and \varprojlim denote the inductive and projective limits respectively. We denote by $\mathcal{E}'_{(\omega)}(V)$ and $\mathcal{E}'_{\{\omega\}}(V)$ the respective strong dual spaces, and their elements are called (ω) -ultradistributions (respectively $\{\omega\}$ -ultradistributions) with compact support in V . If ω is quasianalytic, they are also called quasianalytic functionals of (ω) -type (respectively of $\{\omega\}$ -type) on V .

By definition, we have the following continuous inclusion maps for weight functions ω and $\tilde{\omega}$:

$$\begin{aligned} \mathcal{E}_{(\omega)}(V) &\subset \mathcal{E}_{\{\omega\}}(V), \\ \mathcal{E}_{(\tilde{\omega})}(V) &\subset \mathcal{E}_{(\omega)}(V), \quad \mathcal{E}_{\{\tilde{\omega}\}}(V) \subset \mathcal{E}_{\{\omega\}}(V) \quad \text{if } \omega = O(\tilde{\omega}) \text{ as } t \rightarrow \infty, \\ \mathcal{E}_{\{\tilde{\omega}\}}(V) &\subset \mathcal{E}_{(\omega)}(V) \quad \text{if } \omega = o(\tilde{\omega}) \text{ as } t \rightarrow \infty. \end{aligned}$$

For example, if we take ω_A given in Example 2.3, then $\mathcal{E}_{\{\omega_A\}}(V) \simeq \mathcal{A}(V)$ as locally convex spaces, and therefore $\mathcal{A}(V)$ is continuously included in $\mathcal{E}_{(\omega)}(V)$ and in $\mathcal{E}_{\{\omega\}}(V)$ for any sublinear weight function ω . If moreover V is connected, then $\mathcal{E}_{(\omega_A)}(V) \simeq \mathcal{O}(\mathbb{C}^n)$. From now on, we restrict ourselves to considering $\mathcal{E}_{(\omega)}(V)$ and $\mathcal{E}'_{(\omega)}(V)$ only when ω is sublinear.

Note that $p_{\omega,\lambda,K}(f)$ increases in λ (see Proposition 2.4(c)). Therefore we may take any unbounded subset in \mathbb{N} as a cofinal set in the projective and inductive limits in m in (2.6) and in (2.7) respectively. Also note that the inequality in Proposition 2.4(e) with $\lambda = \mu$ shows that

$$p_{\omega,\lambda,K}(\partial^\beta f) \leq \exp(\varphi_{[\lambda]}^*(|\beta|)) \cdot p_{\omega,2\lambda,K}(f),$$

which proves the stability of $\mathcal{E}_{(\omega)}(V)$ and $\mathcal{E}_{\{\omega\}}(V)$ under derivation by ∂^β .

Let $*$ denote (ω) or $\{\omega\}$. The spaces $\mathcal{E}'_*(V)$ are canonically embedded in $\mathcal{A}'(V)$ if $*$ = (ω) and $\omega(t) = o(t)$, or if $*$ = $\{\omega\}$, since $\mathcal{A}(V)$ is dense in $\mathcal{E}'_*(V)$ (see [6, Proposition 3.2]).

Definition 2.6 ([6, Definition 3.4]). Let ω be a weight function and $V \subset \mathbb{R}^n$ an open convex set. For a compact set $K \Subset V$, we denote the *supporting function* of K by

$$H_K(\xi) := \sup_{x \in K} \langle x, \xi \rangle,$$

and define Banach spaces $A(K, \lambda)$ for $\lambda > 0$ by

$$A(K, \lambda) := \left\{ f \in \mathcal{O}(\mathbb{C}^n); \|f\|_{K,\lambda} := \sup_{\zeta \in \mathbb{C}^n} |f(\zeta)| \exp(-H_K(\operatorname{Im} \zeta) - \lambda \omega(|\zeta|)) < \infty \right\}.$$

We also define the locally convex spaces

$$A_{(\omega)}(\mathbb{C}^n, V) := \varinjlim_{K \Subset V} \varprojlim_{k \in \mathbb{N}} A(K, k), \quad A_{\{\omega\}}(\mathbb{C}^n, V) := \varinjlim_{K \Subset V} \varprojlim_{k \in \mathbb{N}} A(K, 1/k).$$

When V is convex and ω is sublinear, the Fourier–Laplace transform

$$\mathcal{F} : \mathcal{A}'(V) \ni u \mapsto \hat{u} \in \mathcal{O}(\mathbb{C}^n), \quad \hat{u}(\zeta) := \langle u, e^{-i\langle \cdot, \zeta \rangle} \rangle,$$

induces the following topological isomorphisms:

Theorem 2.7 ([6, Theorems 3.6 and 3.7]). *Let ω be a sublinear weight function and $V \subset \mathbb{R}^n$ an open convex set. Then the linear maps*

$$\mathcal{F} : \mathcal{E}'_{(\omega)}(V) \rightarrow A_{(\omega)}(\mathbb{C}^n, V), \quad \mathcal{F} : \mathcal{E}'_{\{\omega\}}(V) \rightarrow A_{\{\omega\}}(\mathbb{C}^n, V)$$

are isomorphisms between the corresponding locally convex spaces.

The following result on existence of weight functions is a special case of [5, Lemma 1.7 and Remark 1.8(1)].

Lemma 2.8. *For any sublinear function $\ell(t)$, there exists a sublinear weight function $\omega(t)$ such that $\ell(t) = o(\omega(t))$ as $t \rightarrow \infty$.*

These results imply that every real-analytic functional belongs to some class of quasianalytic functionals associated with some sublinear weight function.

§3. Twisted Radon transforms of ultradifferentiable functions

In this section, we will study properties of a given ultradifferentiable integral kernel using a twisted Radon transform. The main result in this section is Theorem 3.4.

Twisted Radon transforms are integral operators associated with twisted Radon kernels. Here we recall such a twisted Radon kernel in one of the forms in which such kernels appear in the literature. For notational simplicity we will recall the kernel in n variables, although when we work with it we will apply it in \mathbb{R}^{n+m} . We use in fact a Radon kernel of Bony type:

$$W(z, \xi) = \frac{(n-1)!}{(-2\pi i)^n} \cdot \frac{1 + i\langle z, \xi \rangle}{\Phi(z, \xi)^n}, \quad \Phi(z, \xi) = \langle z, \xi \rangle + i\langle z, z \rangle,$$

regarded as a function defined on the set

$$\{(z, \xi) \in \mathbb{C}^n \times S^{n-1}; \operatorname{Im} \Phi(z, \xi) > 0\} = \{|\operatorname{Im} z - \xi/2|^2 < 1/4 + |\operatorname{Re} z|^2\},$$

and its integral with respect to $\xi \in S^{n-1} \cap \Delta$, with Δ a proper cone in \mathbb{R}^n :

$$W(z, \Delta) := \int_{S^{n-1} \cap \Delta} W(z, \xi) d\sigma(\xi),$$

where $\sigma(\xi)$ denotes the standard surface element of S^{n-1} .

We recall some known results concerning $W(z, \xi)$. The following statements are valid not only for kernels of Bony type, but also for other twisted Radon kernels.

For a hyperfunction $u \in \mathcal{B}(\mathbb{R}^n)$ with compact support, and for a decomposition $\dot{\mathbb{R}}^n = \bigcup_{j=1}^J \Delta_j$ of $\dot{\mathbb{R}}^n = \mathbb{R}^n \setminus \{0\}$ into closed proper cones $\Delta_j \subset \dot{\mathbb{R}}^n$ with disjoint interiors, the integrals

$$(3.1) \quad F_j(z) := \int_{\mathbb{R}^n} W(z-x, \Delta_j) u(x) dx$$

for $j = 1, \dots, J$ define holomorphic functions on infinitesimal wedges of the form $\mathbb{R}^n + i\Delta_j^\perp 0$, and they give a family of defining functions of u , that is,

$$u(x) = \sum_{j=1}^J F_j(x + i\Delta_j^\perp 0).$$

Here and later on we use the following notation: for a cone $\Delta \subset \mathbb{R}^n$, Δ^\perp denotes the dual cone $\operatorname{Int}\{s \in \mathbb{R}^n; \forall \xi \in \Delta, \langle s, \xi \rangle > 0\}$ of Δ ; for cones $G' \subset G$ in \mathbb{R}^n , G' is said to be a *strict subcone* of G if the closure of $G' \cap S^{n-1}$ is included in the interior of G ; and for an open set $V \subset \mathbb{R}^n$ and an open cone $G \subset \mathbb{R}^n$, an open set $U \subset \mathbb{C}^n$ is called an *infinitesimal wedge* of the form $V + iG0$ if $U \subset V + iG$ and if for any

$K \Subset V$ and any strict subcone G' of G , there exists $d > 0$ with $U \supset K + iG'[d]$, where $G'[d] := \{s \in G'; |s| < d\}$.

The great merit of twisted Radon integrals is that if we choose Δ_j suitably, then microanalyticity (microlocal regularity) of u can be read off from the properties of the F_j in a rather explicit form:

Theorem 3.1. *Let $u \in \mathcal{B}(\mathbb{R}^n)$ be a hyperfunction with compact support. Assume that $(V_0 \times \Delta_j) \cap \text{WF}_A u = \emptyset$ for some open set $V_0 \subset \mathbb{R}^n$. Then $F_j(z)$ given by (3.1) extends holomorphically to a neighborhood of V_0 .*

This theorem can be proved starting from the definition of integration of hyperfunctions and the definition of the analytic wave front set. For the theory of the twisted Radon transform, we refer to Kataoka [10], Kaneko [8] and Aoki–Kataoka–Yamazaki [1].

It is also known that the differentiability and Gevrey regularity of u are reflected in the corresponding properties of the F_j . The following lemma asserts that the same is true even when the regularity is measured in terms of the ultradifferentiable classes of Section 2. To do so, we consider $u = f\chi$ where f is a (possibly) quasianalytic function and χ is a cut-off function.

Lemma 3.2. *Let ω be a sublinear weight function as in Definition 2.2. Let $V_0 \Subset V \subset \mathbb{R}^n$ be open sets, $f \in \mathcal{E}_{(\omega)}(V)$, and let $\chi \in C_0^\infty(\mathbb{R}^n)$ be a differentiable function satisfying $\chi \equiv 1$ on a neighborhood of \bar{V}_0 and $\text{supp } \chi \subset V$. Let $\Delta_j \subset \mathbb{R}^n$ be closed proper cones ($j = 1, \dots, J$) with disjoint interiors satisfying $\mathbb{R}^n = \bigcup_{j=1}^J \Delta_j$, and define*

$$(3.2) \quad F_j(z) = \int_{\mathbb{R}^n} W(z-x, \Delta_j) f(x) \chi(x) dx$$

for $j = 1, \dots, J$. Then each $F_j(z)$ is a holomorphic function in an infinitesimal wedge of the form $\mathbb{R}^n + i\Delta_j^\perp 0$. More precisely, for any strict subcone $G_j \subset \Delta_j^\perp$, there exists a positive constant d such that $F_j(z) \in \mathcal{O}(\mathbb{R}^n + iG_j[d])$. For such choices of G_j and d , all the derivatives $F_j^{(\alpha)}(z) := \partial^\alpha F_j(z)$ ($\alpha \in \mathbb{N}_0^n$) extend continuously to $\mathbb{R}^n + i(G_j[d] \cup \{0\})$. Therefore we may denote by the same symbol $F_j^{(\alpha)}(z)$ the extended function on $\mathbb{R}^n + i(G_j[d] \cup \{0\})$, and by $F_j^{(\alpha)}(x)$ its restriction on \mathbb{R}^n . Furthermore if we restrict the domain of definition from \mathbb{R}^n to V_0 , then $F_j^{(\alpha)}$ belongs to $\mathcal{E}_{(\omega)}(V_0)$, and for any $\alpha \in \mathbb{N}_0^n$, the map

$$(3.3) \quad G_j[d] \cup \{0\} \ni s \mapsto F_j^{(\alpha)}(\cdot + is) \in \mathcal{E}_{(\omega)}(V_0)$$

is continuous.

Proof. The statements on the estimates of the domains of holomorphy of F_j and the extendibility of $F_j^{(\alpha)}$ are consequences of the fact that the F_j are the convolutions of $\chi(x)f(x) \in C_0^\infty(\mathbb{R}^n)$ with the twisted Radon kernel associated with Δ_j . The remaining statements follow from the estimate

$$(3.4) \quad \exists d > 0, \forall \lambda > 0, \exists C_\lambda > 0, \forall j \in \{1, \dots, J\}, \forall \alpha \in \mathbb{N}_0^n, \\ \sup_{x \in \bar{V}_0, s \in G_j[d]} |\partial^\alpha F_j(x + is)| \leq C_\lambda \exp(\varphi_{[\lambda]}^*(|\alpha|)).$$

In fact, assume for the moment that we have proved (3.4). Then, for any $x \in \bar{V}_0$, $s \in G_j[d]$, $\lambda > 0$, $1 \leq j \leq J$ and $\alpha, \beta \in \mathbb{N}_0^n$, from Proposition 2.4(e) we have

$$|\partial^{\alpha+\beta} F_j(x + is)| \leq C_{2\lambda} \exp(\varphi_{[2\lambda]}^*(|\alpha + \beta|)) \leq C_{2\lambda} \exp(\varphi_{[\lambda]}^*(|\alpha|)) \cdot \exp(\varphi_{[\lambda]}^*(|\beta|)),$$

which implies

$$p_{\omega, \lambda, \bar{V}_0}(F^{(\alpha)}(\cdot + is)) \leq C_{2\lambda} \exp(\varphi_{[\lambda]}^*(|\alpha|)).$$

Moreover, for s_1 and s_2 in $G_j[d]$,

$$|F_j^{(\alpha+\beta)}(x + is_1) - F_j^{(\alpha+\beta)}(x + is_2)| \leq |s_1 - s_2| \sum_{k=1}^n \sup_{s \in [s_1, s_2]} |\partial_{z_k} F_j^{(\alpha+\beta)}(x + is)| \\ \leq 2dn C_{2\lambda} \exp(\varphi_{[2\lambda]}^*(|\alpha + \beta| + 1)),$$

where $[s_1, s_2]$ denotes the line segment with endpoints s_1 and s_2 , and we have used $|s_1 - s_2| \leq 2d$. Therefore

$$p_{\omega, \lambda, \bar{V}_0}(F^{(\alpha)}(\cdot + is_1) - F^{(\alpha)}(\cdot + is_2)) \leq 2dn C_{2\lambda} \exp(\varphi_{[\lambda]}^*(|\alpha| + 1)).$$

In this way, we obtain the boundedness and the uniform continuity of the map (3.3) on $G_j[d]$.

Now we prove the estimate (3.4). To do so, we start with the following claim.

Claim 3.3. *Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ and let k, l be integers with $1 \leq k \leq n$ and $1 \leq l \leq \alpha_k$. With the notation*

$$\alpha_{<k} = (\alpha_1, \dots, \alpha_{k-1}, 0, \dots, 0), \quad \alpha_{>k} = (0, \dots, 0, \alpha_{k+1}, \dots, \alpha_n),$$

we have

$$(3.5) \quad \partial^\alpha F_j(z) \\ = \sum_{k=1}^n \sum_{1 \leq l \leq \alpha_k} \int_{\mathbb{R}^n} (\partial_{z_k}^{\alpha_k - l} \partial_z^{\alpha_{>k}} W(z - x, \Delta_j)) (\partial_{x_k}^{l-1} \partial_x^{\alpha_{<k}} f(x)) (\partial_{x_k} \chi(x)) dx \\ + \int_{\mathbb{R}^n} W(z - x, \Delta_j) (\partial_x^\alpha f(x)) \chi(x) dx.$$

Proof. This claim follows easily from the following formula of integration by parts. Let X_q ($q = 1, \dots, p$) be vector fields with adjoint operators tX_q . For $f, g \in C^p(\mathbb{R}^n)$ and $\chi \in C_0^1(\mathbb{R}^n)$, we have (e.g., by induction)

$$\int_{\mathbb{R}^n} ({}^tX_1 \cdots {}^tX_p g) f \chi \, dx = \sum_{q=1}^p \int_{\mathbb{R}^n} ({}^tX_{q+1} \cdots {}^tX_p g) (X_{q-1} \cdots X_1 f) (X_q \chi) \, dx + \int_{\mathbb{R}^n} g ((X_p \cdots X_1) f) \chi \, dx. \quad \square$$

(Notice that $\int ({}^tXg)(f\chi) \, dx = \int g((Xf)\chi + fX\chi) \, dx$. The main point in the procedure is that once one of the fields X_j falls on χ , we stop integrating by parts. This idea was used by [6] for the estimates of derivatives of convolutions with Gaussian kernels of ultradifferentiable functions multiplied by a cut-off function. Refer to the proof of [6, Proposition 3.2].)

We continue the proof of Lemma 3.2, denoting by $F_{j,1,\alpha}(z)$ and by $F_{j,2,\alpha}(z)$ the first and the second terms on the right hand side of (3.5). Since $f \in \mathcal{E}_\omega(V)$, there exists for any $\mu > 0$ a constant $A_\mu > 0$ such that

$$(3.6) \quad |\partial_x^\gamma f(x)| \leq A_\mu \exp(\varphi_{[\mu]}^*(|\gamma|)) \quad \text{for } x \in \text{supp } \chi \text{ and } \gamma \in \mathbb{N}_0^n.$$

Note that the sum $\sum_{k=1}^n \sum_{1 \leq l \leq \alpha_k}$ of $F_{j,1,\alpha}$ consists of $|\alpha|$ terms, and it can be written in the form

$$F_{j,1,\alpha}(z) = \sum_{q=1}^p \int_{K_1} W_{\alpha,q}(z-x, \Delta_j) f_{\alpha,q}(x) \chi_{\alpha,q}(x) \, dx \quad (p = |\alpha|),$$

where each $W_{\alpha,q}$ (resp. $f_{\alpha,q}$, $\chi_{\alpha,q}$) is one of the derivatives of W (resp. f , χ) of order $p-q$ (resp. $q-1$, 1), and $K_1 := \bigcup_{k=1}^n \text{supp } \partial_{x_k} \chi$ is a compact set satisfying $\text{dist}(K_1, \bar{V}_0) > 0$. Therefore, we have

$$(3.7) \quad \|\chi_{\alpha,q}\|_{L^1(K_1)} \leq c_0 := \sup_{1 \leq k \leq n} \|\partial_{x_k} \chi\|_{L^1(K_1)} < \infty.$$

Since $W(z, \xi)$ is holomorphic in a neighborhood of $(\mathbb{R}^n \setminus \{0\}) \times S^{n-1}$, and since $\text{dist}(K_1, \bar{V}_0) > 0$, there exist positive constants d_1 , a_0 and a_1 such that

$$|\partial_z^\beta W(z-x, \xi)| \leq a_0 a_1^{|\beta|} |\beta|!$$

for $x \in K_1$, $\text{Re } z \in \bar{V}_0$, $|\text{Im } z| \leq d_1$, $\xi \in S^{n-1}$, $\beta \in \mathbb{N}_0^n$. Moreover, since ω is sublinear, we can use Proposition 2.4(h) and take, for any $\mu > 0$, a constant B_μ such that

$$(3.8) \quad \sup_{x \in K_1, \text{Re } z \in \bar{V}_0, |\text{Im } z| \leq d_1} |\partial_z^\beta W(z-x, \Delta_j)| \leq B_\mu \exp(\varphi_{[\mu]}^*(|\beta|)), \quad \forall j \in \{1, \dots, J\}, \forall \beta \in \mathbb{N}_0^n.$$

From (3.6)–(3.8), the first term $F_{j,1,\alpha}(z)$ on the right hand side of (3.5) is estimated, for any $\mu > 0$, as

$$(3.9) \quad |F_{j,1,\alpha}(z)| \leq \sum_{q=1}^p B_\mu \exp(\varphi_{[\mu]}^*(p-q)) \cdot A_\mu \exp(\varphi_{[\mu]}^*(q-1)) \cdot c_0 \\ \leq A_\mu B_\mu c_0 \cdot \exp(\varphi_{[\mu]}^*(p-1) + \log p) \quad \text{for } \operatorname{Re} z \in \bar{V}_0, |\operatorname{Im} z| < d_1.$$

Here we have used Proposition 2.4(d) to write $\varphi_{[\mu]}^*(p-q) + \varphi_{[\mu]}^*(q-1) \leq \varphi_{[\mu]}^*(p-1)$, and we have again denoted $p = |\alpha|$. Now for any given $\lambda > 0$, we can apply (2.4) in Proposition 2.4(g), and get a suitable μ and c with $\varphi_{[\lambda]}^*(p) - p \geq \varphi_{[\mu]}^*(p) - c$. We can use (3.9) for this μ to get $C_{1,\lambda}$ such that

$$(3.10) \quad |F_{j,1,\alpha}(z)| \leq C_{1,\lambda} \exp(\varphi_{[\lambda]}^*(|\alpha|)) \quad \text{for } \operatorname{Re} z \in \bar{V}_0, |\operatorname{Im} z| < d_1,$$

and for any $\lambda > 0, 1 \leq j \leq J, \alpha \in \mathbb{N}_0^n$.

As for the second term on the right hand side of (3.5), we write

$$F_{j,2,\alpha}(z) = \int_K dx \int_{S^{n-1} \cap \Delta_j} W(z-x, \xi) (\partial_x^\alpha f(x)) \chi(x) d\sigma(\xi) \quad (K := \operatorname{supp} \chi \subset \mathbb{R}^n),$$

and use the formula

$$W(z, \xi) = -\frac{1+i\langle z, \xi \rangle}{(2\pi i)^n} \left(\frac{1}{1+2i\langle z, \xi \rangle} \langle \xi, \partial_z \rangle \right)^n \log \Phi(z, \xi),$$

which follows from $\langle \xi, \partial_z \rangle \Phi(z, \xi) = 1 + 2i\langle z, \xi \rangle$. Integrating by parts, and using $\langle \xi, \partial_z \rangle (1 + i\langle z, \xi \rangle) = i$ and $\langle \xi, \partial_z \rangle (1 + 2i\langle z, \xi \rangle) = 2i$, we get

$$F_{j,2,\alpha}(z) = \int_K dx \int_{S^{n-1} \cap \Delta_j} \log \Phi(z-x, \xi) \sum_{\substack{k \in \{0,1\}, l, m \in \mathbb{N}_0, \\ k+l+m=n}} b_{k,l,m} \\ \times \frac{(1+i\langle z-x, \xi \rangle)^{1-k}}{(1+2i\langle z-x, \xi \rangle)^{n+l}} \langle \xi, \partial_x \rangle^m \{(\partial_x^\alpha f(x)) \chi(x)\} d\sigma(\xi) \\ = \int_K dx \int_{S^{n-1} \cap \Delta_j} \log \Phi(z-x, \xi) \sum_{\substack{k \in \{0,1\}, l, m \in \mathbb{N}_0, \\ k+l+m=n}} b_{k,l,m} \sum_{\substack{\beta, \gamma \in \mathbb{N}_0^n, \\ |\beta|+|\gamma|=m}} \frac{m!}{\beta! \gamma!} \xi^{\beta+\gamma} \\ \times \frac{(1+i\langle z-x, \xi \rangle)^{1-k}}{(1+2i\langle z-x, \xi \rangle)^{n+l}} (\partial_x^{\alpha+\beta} f(x)) (\partial_x^\gamma \chi(x)) d\sigma(\xi) \\ = \int_K dx \int_{S^{n-1} \cap \Delta_j} \log \Phi(z-x, \xi) \\ \times \sum_{|\beta|+|\gamma| \leq n} b_{\beta,\gamma}(z-x, \xi) (\partial_x^{\alpha+\beta} f(x)) (\partial_x^\gamma \chi(x)) d\sigma(\xi),$$

where $b_{k,l,m}$ are some constants, and $b_{\beta,\gamma}(z, \xi)$ are analytic functions on $\{(z, \xi) \in \mathbb{C}^n \times S^{n-1}; |\operatorname{Im} z| < 1/2\}$. In fact, $(1 + 2i\langle z, \xi \rangle)^{2n} b_{\beta,\gamma}(z, \xi)$ are polynomials in $\langle z, \xi \rangle$ and ξ , and $1 + 2i\langle z, \xi \rangle$ does not vanish if $|\operatorname{Im} z| < 1/2$ and $|\xi| = 1$. (The exact form of these constants is not needed in the argument, since there are only finitely many of them.)

Therefore, using (3.6), $F_{j,2,\alpha}(z)$ is estimated as

$$(3.11) \quad |F_{j,2,\alpha}(z)| \leq A_{2\lambda} \exp(\varphi_{[2\lambda]}^*(|\alpha| + n)) \|\chi\|_{C^n(K)} \\ \times \sum_{|\beta|+|\gamma| \leq n} \sup_{x \in K, \xi \in S^{n-1}} |b_{\beta,\gamma}(z - x, \xi)| \cdot \int_K \sup_{\xi \in S^{n-1} \cap \Delta_j} |\log \Phi(z - x, \xi)| dx \\ \times \int_{S^{n-1} \cap \Delta_j} d\sigma(\xi) \quad \text{for } |\operatorname{Im} z| < 1/2,$$

and for any $\lambda > 0, 1 \leq j \leq J, \alpha \in \mathbb{N}_0^n$.

By Proposition 2.4(e), we have

$$(3.12) \quad A_{2\lambda} \exp(\varphi_{[2\lambda]}^*(|\alpha| + n)) \leq A_{2\lambda} \exp(\varphi_{[\lambda]}^*(n)) \cdot \exp(\varphi_{[\lambda]}^*(|\alpha|)).$$

Since G_j is a strict subcone of Δ_j^\perp , we have $\inf_{\xi \in S^{n-1} \cap \Delta_j, s \in G_j} \langle s/|s|, \xi \rangle > 0$, and therefore we can take a constant d_2 with

$$0 < d_2 < \min \left\{ 1/2, \min_{1 \leq j \leq J} \inf_{\xi \in S^{n-1} \cap \Delta_j, s \in G_j} \langle s/|s|, \xi \rangle \right\}.$$

Since $d_2 < 1/2$, we have

$$(3.13) \quad c_1 := \sup_{z \in \bar{V}_0 + iG_j[d_2]} \sum_{|\beta|+|\gamma| \leq n} \sup_{x \in K, \xi \in S^{n-1}} |b_{\beta,\gamma}(z - x, \xi)| < \infty.$$

On the other hand, since $0 < d_2 < \inf_{\xi \in S^{n-1} \cap \Delta_j, s \in G_j} \langle s/|s|, \xi \rangle$ for any j , we have $\langle s, \xi \rangle - |s|^2 \geq 0$ for $s \in G_j[d_2]$. Using also $\operatorname{Im} \Phi(z - x, \xi) = \langle \operatorname{Im} z, \xi \rangle - |\operatorname{Im} z|^2 + |\operatorname{Re} z - x|^2$, we get the estimates

$$\operatorname{Im} \log \Phi(z - x, \xi) \in [0, \pi], \\ \operatorname{Re} \log \Phi(z - x, \xi) \geq \log \operatorname{Im} \Phi(z - x, \xi) \geq 2 \log |\operatorname{Re} z - x|,$$

for $z \in \bar{V}_0 + iG_j[d_2]$, and therefore

$$(3.14) \quad c_2 := \sup_{z \in \bar{V}_0 + iG_j[d_2]} \int_K \sup_{\xi \in S^{n-1} \cap \Delta_j} |\log \Phi(z - x, \xi)| dx < \infty.$$

Finally, from (3.11)–(3.14) we get

$$(3.15) \quad |F_{j,2,\alpha}(z)| \leq C_{2,\lambda} \exp(\varphi_{[\lambda]}^*(|\alpha|)) \quad \text{for } z \in \bar{V}_0 + iG_j[d_2],$$

and for any $\lambda > 0$, $1 \leq j \leq J$, $\alpha \in \mathbb{N}_0^n$, where $C_{2,\lambda} := c_1 c_2 |S^{n-1}| \cdot \|\chi\|_{C^n(K)} \cdot A_{2\lambda} \exp(\varphi_{[\lambda]}^*(n))$.

By defining $d := \min\{d_1, d_2\}$, the estimates (3.10) and (3.15) give (3.4), which concludes the proof of Lemma 3.2. \square

Theorem 3.4. *Let $V \times U \subset \mathbb{R}^n \times \mathbb{R}^m$ be an open set and consider $\mathcal{K} \in \mathcal{E}_{(\omega)}(V \times U)$ satisfying (1.1). Then, for any $V_0 \times U_0 \Subset V \times U$, \mathcal{K} admits, on $V_0 \times U_0$, a boundary value representation*

$$\mathcal{K}(x, y) = \sum_{j=0}^J b_{\hat{G}_j}(F_j(z, w)),$$

where $\hat{G}_j \subset \mathbb{R}^{n+m}$ ($j = 0, 1, \dots, J$) are open convex cones which include $G_j \times \{0\}$ for some open convex cones $G_j \subset \mathbb{R}^n$, the F_j belong to $\mathcal{O}(V_0 \times U_0 + i\hat{G}_j[d]) \cap C(V_0 \times U_0 + i(\hat{G}_j[d] \cup \{0\}))$ with some $d > 0$, such that for each fixed $z \in V_0 + i(G_j[d] \cup \{0\})$, the map $U_0 \ni y \mapsto F_j(z, y)$ is an ultradifferentiable function in $\mathcal{E}_{(\omega)}(U_0)$ even in the case $z \in V_0$ (which means in particular that z is real), and that for any $\alpha \in \mathbb{N}_0^n$ and any j , the correspondence

$$(3.16) \quad V_0 + i(G_j[d] \cup \{0\}) \ni z \mapsto \partial_x^\alpha F_j(z, \cdot) \in \mathcal{E}_{(\omega)}(U_0)$$

is continuous. In particular,

$$\mathcal{K}(x, y) = \sum_{j=0}^J F_j(x, y).$$

Proof. We take a cut-off function $\chi \in C_0^\infty(V \times U)$ with $\chi \equiv 1$ in a neighborhood of $\overline{V_0 \times U_0}$. We can take a decomposition $\mathbb{R}^{n+m} = \bigcup_{j=1}^J \Delta_j$ into proper closed convex cones $\Delta_j \subset \mathbb{R}^{n+m}$ with disjoint interiors and a number $J_0 < J$ such that

$$(3.17) \quad \Delta_j^\perp \cap (\mathbb{R}^n \times \{0\}) \neq \emptyset \quad (j \leq J_0),$$

$$(3.18) \quad (V_0 \times U_0 \times \Delta_j) \cap \text{WF}_A \mathcal{K} = \emptyset \quad (j > J_0).$$

In fact, since \mathcal{K} satisfies (1.1) and since $V_0 \times U_0 \Subset V \times U$, there exists a constant $\varepsilon > 0$ satisfying

$$\text{WF}_A \mathcal{K} \cap \{(x, y; \xi, \eta); (x, y) \in \overline{V_0 \times U_0}, |\xi| \leq 2\varepsilon|\eta|\} = \emptyset.$$

Then we may construct a covering $\mathbb{R}^{n+m} = \bigcup_j \Delta_j$ consisting of proper closed convex cones Δ_j with disjoint interiors, satisfying the additional conditions

$$\bigcup_{j=1}^{J_0} \Delta_j \subset \{|\xi| > \varepsilon|\eta|\}, \quad \bigcup_{j=J_0+1}^J \Delta_j \subset \{|\xi| < 2\varepsilon|\eta|\}.$$

Then for $j \leq J_0$, $\Delta_j \cap (\{0\} \times \mathbb{R}^m) = \emptyset$ implies (3.17), and (3.18) is trivially satisfied.

Now we apply Lemma 3.2 to the data $V_0 \times U_0 \Subset V \times U \subset \mathbb{R}^{n+m}$, $\mathcal{K} \in \mathcal{E}_{(\omega)}(V \times U)$, $\chi \in C_0^\infty(\mathbb{R}^{n+m})$ and $\Delta_j \subset \mathbb{R}^{n+m}$, and get holomorphic functions $F_j(z, w)$ on infinitesimal wedges of the form $\mathbb{R}^n + i\Delta_j^\perp 0$.

Let us first consider $j = 1, \dots, J_0$. For these j , we can take cones $G_j \subset \mathbb{R}^n$ and strict subcones $\hat{G}_j \subset \mathbb{R}^{n+m}$ of Δ_j^\perp such that $G_j \times \{0\} \subset \hat{G}_j$, since the Δ_j satisfy (3.17). Moreover, when $z = x + is \in V_0 + i(G_j[d] \cup \{0\})$ is fixed, the function $U_0 \ni y \mapsto \partial_x^\alpha F_j(z, y)$ is a restriction of the function $V_0 \times U_0 \ni (x, y) \mapsto \partial_x^\alpha F_j((x, y) + i(s, 0))$ with respect to the x variable. Therefore, it belongs to $\mathcal{E}_{(\omega)}(U_0)$. In addition, the continuity of

$$G_j[d] \cup \{0\} \ni s \mapsto \partial_x^\alpha F_j(\cdot + is, \cdot) \in \mathcal{E}_{(\omega)}(V_0 \times U_0)$$

(coming from the continuity of (3.3)) implies the locally uniform continuity of

$$G_j[d] \cup \{0\} \ni s \mapsto \partial_x^\alpha F_j(x + is, \cdot) \in \mathcal{E}_{(\omega)}(U_0)$$

in $x \in V_0$, which also implies the continuity of (3.16).

Next we consider $j = J_0 + 1, \dots, J$. In view of (3.18), we can apply Theorem 3.1 to $V_0 \times U_0 \Subset V \times U \subset \mathbb{R}^{n+m}$, $\chi(x, y)\mathcal{K}(x, y)$ and $\Delta_j \subset \mathbb{R}^{n+m}$. Since F_j is real-analytic in a neighborhood of $\overline{V_0 \times U_0}$, we can ignore Δ_j and take $\hat{G}_j = \mathbb{R}^{n+m}$ and $G_j = \mathbb{R}^n$. Then the map

$$V_0 + i\{s; |s| < d\} \ni z \mapsto \partial_x^\alpha F_j(z, \cdot) \in \mathcal{A}(U_0)$$

is well-defined and the usual Cauchy estimate proves its continuity. The continuity of the canonical embedding $\mathcal{A}(U_0) \hookrightarrow \mathcal{E}_{(\omega)}(U_0)$ gives the desired estimate for these j .

In this way we have proved the continuity of (3.16) for both cases, which concludes the proof of Theorem 3.4. \square

§4. Compatibility questions related to the definition of T

We discuss here the integral operator $u \mapsto \int \mathcal{K}(x, y)u(y) dy$ of (1.2) in the special case when $\mathcal{K} \in \mathcal{E}_{(\omega)}(V \times U)$ (still satisfying (1.1)) and $u \in \mathcal{E}'_{(\omega)}(U)$. We recall that initially in (1.2) the integration $\int \mathcal{K}(x, y)u(y) dy$ is the one along fibers in the theory of hyperfunctions, but we have already rewritten it in [12, (4.3)] as an explicit formula in terms of holomorphic defining functions for the kernel. It will be convenient to take for these defining functions the ones given by Theorem 3.4. We can then in fact give a new interpretation for our integrals using the duality between ultradifferentiable functions and the corresponding functionals.

In this section, we study the compatibility between the two interpretations so obtained for (1.2) in our special case. We refer to [12, Remark 6.2(ii)] for a similar compatibility question in the case of differentiable kernels and Radon measures. As a corollary to the results in Section 3 we can in fact prove

Corollary 4.1. *Let ω be a weight function and let $V \times U \subset \mathbb{R}^n \times \mathbb{R}^m$ be an open set. Then for any $\mathcal{K} \in \mathcal{E}_{(\omega)}(V \times U)$ satisfying (1.1) and any $u \in \mathcal{E}'_{(\omega)}(U)$, we have*

$$(4.1) \quad \int_U \mathcal{K}(x, y)u(y) dy = \langle u, \mathcal{K}(x, \cdot) \rangle.$$

Here the left hand side is the integration of the hyperfunction $\mathcal{K}u$ with u regarded as a hyperfunction on U with compact support, and the right hand side is a continuous function on V whose value for each fixed x is computed by the duality bilinear form $\mathcal{E}'_{(\omega)}(U) \times \mathcal{E}_{(\omega)}(U) \rightarrow \mathbb{C}$, and is regarded as a hyperfunction on V .

Note that $\mathcal{K}u$ as a product of hyperfunctions is well-defined because of the wave front set estimate (1.1), and also that the restriction $y \mapsto \mathcal{K}(x, y)$ at each fixed x belongs to $\mathcal{E}_{(\omega)}(U)$ in a trivial way.

Proof. It suffices to prove the equality on each relatively compact open set $V_0 \Subset V$. We take a relatively compact open set $U_0 \Subset U$ which includes $\text{supp } u$. Applying Theorem 3.4 to \mathcal{K} and $V_0 \times U_0$, we get a representation $\mathcal{K}(x, y) = \sum_{j=0}^J b_{G_j}(F_j(z, w))$ on $V_0 \times U_0$ satisfying the additional properties stated in the theorem. In view of [12, (4.3)], this representation enables us to calculate the left hand side of (4.1) as

$$\int_U \mathcal{K}(x, y)u(y) dy = \sum_{j=0}^J b_{G_j}(f_j(z)),$$

where each f_j is a holomorphic function on $V_0 + iG_j[d]$ given by

$$(4.2) \quad f_j(z) := \int_{U_0} F_j(z, y)u(y) dy = \langle u, F_j(z, \cdot) \rangle \quad \text{for } z \in V_0 + iG_j[d].$$

On the other hand, it follows, again from Theorem 3.4, that the right hand side of (4.2) defines a continuous function on $V_0 + i(G_j[d] \cup \{0\})$, since $z \mapsto F_j(z, \cdot) \in \mathcal{E}_{(\omega)}(U_0)$ is continuous on that set. Therefore each $f_j(z)$ admits a boundary value in $\mathcal{E}_{(\omega)}(U_0)$ and we have

$$b_{G_j}(f_j(z)) = \langle u, F_j(x, \cdot) \rangle.$$

We conclude that

$$\begin{aligned} \int_U \mathcal{K}(x, y)u(y) dy &= \sum_{j=0}^J b_{G_j}(f_j(z)) = \sum_{j=0}^J \langle u, F_j(x, \cdot) \rangle \\ &= \left\langle u, \sum_{j=0}^J F_j(x, \cdot) \right\rangle = \langle u, \mathcal{K}(x, \cdot) \rangle \end{aligned}$$

as hyperfunctions on V_0 . □

§5. Division of the kernel

The following result will play the role of [12, Theorem 5.3], but is here adapted to an ultradifferentiable setting.

Theorem 5.1. *Let $\mathcal{K} \in \mathcal{B}(V \times U)$ be a hyperfunction satisfying the wave front set estimate (1.1), and ω a sublinear weight function. Then there exist an elliptic (possibly infinite order) partial differential operator $P(D_x)$ with constant coefficients in the variable x , a function $\mathcal{K}' \in \mathcal{E}_{(\omega)}(V \times U)$, which also satisfies the wave front set estimate*

$$(5.1) \quad \{(x, y; 0, \eta) \in V \times U \times \mathbb{R}^n \times \mathbb{R}^m; \eta \neq 0\} \cap \text{WF}_A \mathcal{K}' = \emptyset,$$

and a real-analytic function \mathcal{K}'' on $V \times U$ such that $\mathcal{K} = P(D_x)\mathcal{K}' + \mathcal{K}''$.

As a preparation for the proof of Theorem 5.1, we start with the following lemma.

Lemma 5.2. *Let ω be a weight function. For a measurable function $\mu(\xi)$ on \mathbb{R}^n satisfying*

$$\mu(\xi) \exp(\lambda\omega(|\xi|)) \in L^1(\mathbb{R}^n) \quad \text{for any } \lambda > 0,$$

the inverse Fourier transform $(\mathcal{F}^{-1}\mu)(x)$ belongs to $\mathcal{E}_{(\omega)}(\mathbb{R}^n)$.

Proof. We have, for any $\alpha \in \mathbb{N}_0^n$ and any $\lambda > 0$,

$$\begin{aligned} \|\xi^\alpha \mu(\xi)\|_{L^1(\mathbb{R}^n)} &\leq \| |\xi|^{|\alpha|} e^{-\lambda\omega(|\xi|)} \|_{L^\infty(\mathbb{R}^n)} \cdot \|\mu(\xi) e^{\lambda\omega(|\xi|)}\|_{L^1(\mathbb{R}^n)} \\ &\leq \exp(\varphi_{[\lambda]}^*(|\alpha|)) \cdot \|\mu(\xi) e^{\lambda\omega(|\xi|)}\|_{L^1(\mathbb{R}^n)}. \end{aligned}$$

Therefore $\mathcal{F}^{-1}\mu$ is differentiable, and satisfies

$$p_{\omega, \lambda, K}(\mathcal{F}^{-1}\mu) \leq (2\pi)^{-n} \|\mu(\xi) e^{\lambda\omega(|\xi|)}\|_{L^1(\mathbb{R}^n)}$$

for any $K \in \mathbb{R}^n$. □

Proof of Theorem 5.1. The proof is similar to that of [12, Theorem 5.3]. In order to explain the differences in the argument, we recall now the steps in the proof of that theorem and mention how they have to be adapted to our present needs.

Step 1. We first take compact subsets $K_1 \Subset K_2 \Subset \dots$ in $V \times U$ with $\bigcup_j K_j = V \times U$, hyperfunctions $\mathcal{K}_j \in \mathcal{B}(\mathbb{R}^{n+m})$ with compact support each of which coincides with \mathcal{K} in a neighborhood of K_j , positive constants c_j and sublinear functions ℓ_j such that

$$\begin{aligned} \text{WF}_A \mathcal{K}_j \cap \{(x, y; \xi, \eta); (x, y) \in K_j\} &\subset \{(x, y; \xi, \eta); |\eta| \leq c_j |\xi|\}, \\ |\mathcal{F}\mathcal{K}_j(\xi, \eta)| &\leq \exp \ell_j(\xi, \eta) \quad \text{for any } (\xi, \eta) \in \mathbb{R}^{n+m}. \end{aligned}$$

Next we take a function $\tilde{\omega}(t)$ satisfying $\omega(t) = o(\tilde{\omega}(t))$ and $\tilde{\omega}(t) = o(t)$ as $t \rightarrow \infty$, define sublinear functions $\tilde{\ell}_j$ by

$$(5.2) \quad \tilde{\ell}_j(\xi) := \sup_{|\eta| \leq 2c_j|\xi|} (\ell_j(\xi, \eta) + |(\xi, \eta)|^{1/2} + \tilde{\omega}(|\xi|)),$$

and take a sublinear function ℓ and constants c'_j such that $\tilde{\ell}_j(\xi) \leq \ell(\xi) + c'_j$ for any $\xi \in \mathbb{R}^n$ and j (see e.g. [12, Lemma 5.1]). The formula (5.2) replaces [12, (5.7)], where however no $\tilde{\omega}$ was present. We then get the estimate

$$|\mathcal{FK}_j(\xi, \eta)| \leq A_j \exp(\ell(\xi) - |(\xi, \eta)|^{1/2} - \tilde{\omega}(|\xi|)) \quad \text{if } |\eta| \leq 2c_j|\xi|,$$

with constants $A_j := e^{c'_j}$. Associated with this ℓ , we take an infra-exponential elliptic symbol $P(\zeta)$ ($\zeta \in \mathbb{C}^n$) and $c > 0$ satisfying

$$|P(\zeta)| \geq e^{\ell(\text{Re } \zeta)} \quad \text{if } |\text{Im } \zeta| < c|\text{Re } \zeta|$$

(see Kawai [11], Kaneko [7]).

Step 2. We define measurable functions $Q_j(\xi, \eta)$ by

$$Q_j(\xi, \eta) = \begin{cases} \mathcal{FK}_j(\xi, \eta)/P(\xi) & \text{if } |\eta| \leq 2c_j|\xi|, \\ 0 & \text{if } |\eta| > 2c_j|\xi|. \end{cases}$$

They satisfy

$$(5.3) \quad \begin{aligned} & \text{supp } Q_j \subset \{|\eta| \leq 2c_j|\xi|\}, \\ & |Q_j(\xi, \eta)| \leq A_j \exp(-|(\xi, \eta)|^{1/2} - \tilde{\omega}(|\xi|)) \quad \text{for } (\xi, \eta) \in \mathbb{R}^{n+m}. \end{aligned}$$

(This corresponds to [12, (5.9)].) The estimate (5.3) implies $\mathcal{F}^{-1}Q_j \in \mathcal{E}_{(\omega)}(\mathbb{R}^{n+m})$ in view of Lemma 5.2. Now we define the function \mathcal{K}'_j on $\text{Int } K_j$ as the restriction to $\text{Int } K_j$ of the iterated integral

$$\int_{|\eta| \leq \frac{3}{2}c_j|\xi|} d\sigma(\xi, \eta) \int_{\mathbb{R}^{n+m}} W(x - \tilde{x}, y - \tilde{y}, \xi, \eta)(\chi \cdot \mathcal{F}^{-1}Q_j)(\tilde{x}, \tilde{y}) d\tilde{x} d\tilde{y}$$

with some cut-off function $\chi \in C_0^\infty(\mathbb{R}^{n+m})$ satisfying $\chi \equiv 1$ on K_j , and it follows from Lemma 3.2 that $\mathcal{K}'_j \in \mathcal{E}_{(\omega)}(\text{Int } K_j)$. On the other hand, the estimates of analytic wave front sets as in [12, proof of Theorem 5.3, Step 2] show also in this case that

$$\text{WF}_A \mathcal{K}'_j \subset \{(x, y; \xi, \eta); |\eta| \leq c_j|\xi|\}, \quad \mathcal{K}_j - P(D_x)\mathcal{K}'_j \in \mathcal{A}(\text{Int } K_j).$$

Step 3. Again, it follows from the argument used in [12, proof of Theorem 5.3, Step 3] that $\mathcal{K}'_j - \mathcal{K}'_k \in \mathcal{A}(\text{Int } K_j)$ for any $j < k$, and therefore we can take

$\mathcal{L}_j \in \mathcal{A}(\text{Int } K_j)$ with $\mathcal{K}'_j - \mathcal{K}'_k = \mathcal{L}_j - \mathcal{L}_k$ for any $j < k$. Since each \mathcal{K}'_j belongs to $\mathcal{E}_{(\omega)}(\text{Int } K_j)$, so does $\mathcal{K}'_j - \mathcal{L}_j$, and the family $(\mathcal{K}'_j - \mathcal{L}_j)_j$ defines a global section $\mathcal{K}' \in \mathcal{E}_{(\omega)}(V \times U)$. We can also show, arguing as in [12, loc.cit.], that $\mathcal{K}'' := \mathcal{K} - P(D_x)\mathcal{K}'$ is real-analytic. The operator $P(D_x)$ and the functions \mathcal{K}' and \mathcal{K}'' have all the desired properties. \square

§6. Proof of Theorem 1.1

We first prove the following intermediate result.

Proposition 6.1. *Let T and \mathcal{K} be as in Theorem 1.1, $K \Subset U$ a compact convex set, and let $\ell : \mathbb{C}^n \rightarrow \mathbb{R}_+$ be a sublinear function. Then there exists $L \Subset V$ such that $u \in \mathcal{A}'_\ell(K)$ implies $\text{singsupp}_A Tu \subset L$. Here $\mathcal{A}'_\ell(K)$ denotes the Banach space*

$$(6.1) \quad \mathcal{A}'_\ell(K) := \left\{ u \in \mathcal{A}'(K); \sup_{\eta \in \mathbb{C}^n} |\hat{u}(\eta)| \exp(-H_K(\text{Im } \eta) - \ell(\eta)) < \infty \right\}.$$

We also quote the following theorem (see [12, Theorem 6.3]).

Theorem 6.2. *Let $T : X \rightarrow C^\infty(V)$ be a continuous linear map from some Fréchet space X to $C^\infty(V)$. Assume that for every $u \in X$, $\text{singsupp}_A Tu \subset V$ is compact. Then there exists a compact set $L \subset V$ such that $\text{singsupp}_A Tu \subset L$ for any $u \in X$.*

If T were a continuous linear map from $\mathcal{A}'_\ell(K)$ into $C^\infty(V)$ for any ℓ , then Proposition 6.1 would be an easy consequence of Theorem 6.2, but in the general case we cannot expect to have $T(\mathcal{A}'_\ell(K)) \subset C^\infty(V)$. For this reason we use at first the division result of Theorem 5.1 for the kernels.

Proof of Proposition 6.1. We can take a sublinear weight function $\omega(t)$ with $\ell(t) = o(\omega(t))$ as $t \rightarrow \infty$ using Lemma 2.8. By applying Theorem 5.1, we have a decomposition

$$(6.2) \quad \mathcal{K}(x, y) = P(D_x)\mathcal{K}'(x, y) + \mathcal{K}''(x, y),$$

where $P(D_x)$ is elliptic, $\mathcal{K}' \in \mathcal{E}_{(\omega)}(V \times U)$ satisfies the wave front set estimate (5.1), and $\mathcal{K}'' \in \mathcal{A}(V \times U)$. With the notation $(T'u)(x) = \int_U \mathcal{K}'(x, y)u(y) dy$, we have $\text{singsupp}_A Tu = \text{singsupp}_A T'u$ as pointed out in [12, Remark 5.4]. Therefore, it suffices to prove the present proposition for T' instead of T , which means that we may assume from the beginning that $\mathcal{K} \in \mathcal{E}_{(\omega)}(V \times U)$.

It can be seen from Definition 2.6, (6.1), and the estimate $\ell(t) = o(\omega(t))$, that $\mathcal{F}(\mathcal{A}'_\ell(K))$ is continuously embedded into $A_{(\omega)}(\mathbb{C}^n, U)$. Therefore Theorem 2.7 implies that the embedding $\mathcal{A}'_\ell(K) \hookrightarrow \mathcal{E}'_{(\omega)}(U)$ is also continuous.

Moreover, since $\partial_x^\alpha \mathcal{K} \in \mathcal{E}_{(\omega)}(V \times U)$ for any $\alpha \in \mathbb{N}_0^n$, Corollary 4.1 yields

$$\partial_x^\alpha(Tu)(x) = \int_U \partial_x^\alpha \mathcal{K}(x, y)u(y) dy = \langle u, \partial_x^\alpha \mathcal{K}(x, \cdot) \rangle \quad \text{for } u \in \mathcal{E}'_{(\omega)}(U).$$

Therefore T induces a continuous linear map from $\mathcal{E}'_{(\omega)}(U)$ to $C^\infty(V)$.

Combining these facts, we conclude that $T : \mathcal{A}'_\ell(K) \rightarrow C^\infty(V)$ is a continuous linear map. Since $\mathcal{A}'_\ell(K)$ is a Banach space and $\text{singsupp}_A Tu$ is compact for every $u \in \mathcal{A}'_\ell(K)$, we can now finish the proof by applying Theorem 6.2. \square

Theorem 6.3. *Let T and \mathcal{K} be as in Theorem 1.1. Then for any $K \Subset U$, there exists $L \Subset V$ such that $u \in \mathcal{A}'(K)$ implies $\text{singsupp}_A Tu \subset L$.*

Remark 6.4. The main difference between Proposition 6.1 and Theorem 6.3 is of course that, in the proposition, the compact set L in the conclusion might, in principle, depend on ℓ . By using $\bigcup_\ell \mathcal{A}'_\ell(K) = \mathcal{A}'(K)$, and by availing ourselves of the fact that the set of sublinear functions is very big, we shall now show that L can be chosen independently of ℓ .

Proof of Theorem 6.3. First consider the case when K is convex. Fix T and \mathcal{K} , and then take an increasing sequence $L_1 \subset L_2 \subset \dots$ of compact subsets of V with

$$(6.3) \quad \bigcup_j \text{Int } L_j = V.$$

Assume for contradiction that there is no $L \Subset V$ for which $u \in \mathcal{A}'(K)$ implies $\text{singsupp}_A Tu \subset L$. Then, for each j , we can find $u_j \in \mathcal{A}'(K)$ with $\text{singsupp}_A Tu_j \not\subset L_j$. Each u_j belongs to $\mathcal{A}'_{\ell_j}(K)$ with some sublinear ℓ_j , and we take a sublinear function ℓ and constants c'_j such that $\ell_j(\eta) \leq \ell(\eta) + c'_j$ for any $\eta \in \mathbb{R}^n$ and j (see again e.g. [12, Lemma 5.1]). The latter estimate yields $\mathcal{A}'_{\ell_j}(K) \subset \mathcal{A}'_\ell(K)$, therefore all the u_j belong to $\mathcal{A}'_\ell(K)$. Let now $L \subset V$ be a compact set associated with ℓ by Proposition 6.1. Then $\text{singsupp}_A Tu_j \subset L$ for any j . Moreover, by (6.3), there exists j^0 satisfying $L \subset \text{Int } L_{j^0}$ and in particular $\text{singsupp}_A Tu_j \subset L_{j^0}$ for any j . This contradicts the choice of the u_j .

Next we consider the general case, when K is not necessarily convex. We start with a finite covering $K \subset \bigcup_{j=1}^k K_j$ consisting of convex compact sets $K_j \subset U$, and take $L_j \Subset V$ for each j such that $u \in \mathcal{A}'(K_j)$ implies $\text{singsupp}_A Tu \subset L_j$. Then we can use the flabbiness of \mathcal{B} to obtain $\mathcal{A}'(K) \subset \sum_j \mathcal{A}'(K_j)$. Therefore $L := \bigcup_j L_j$ is a compact set with the properties required for K . \square

Proof of Theorem 1.1. We shall show that for any compact set $K \Subset U$, there exists a compact set $L \Subset V$ such that $\text{singsupp}_A \mathcal{K} \cap (V \times K) \subset L \times K$.

Once K is fixed, we consider a second compact set \tilde{K} with $K \Subset \tilde{K} \Subset U$ and associate with it a compact set $L \Subset V$ as in Theorem 6.3, that is, $u \in \mathcal{A}'(\tilde{K})$ must imply $\text{singsupp}_A Tu \subset L$. This means that the integral operator from $\mathcal{A}'(\text{Int } \tilde{K})$ to $\mathcal{B}(V \setminus L)$ associated with the kernel $\mathcal{K}|_{(V \setminus L) \times \text{Int } \tilde{K}}$ satisfies the assumptions of Theorem 1.3. Therefore, $\mathcal{K}|_{(V \setminus L) \times \text{Int } \tilde{K}}$ is real-analytic and $\text{singsupp}_A \mathcal{K} \cap (V \times K) \subset L \times K$. \square

Remark 6.5. We can now make some comments on why it seemed necessary in this paper to use results from the theory of ultradifferentiable functions and their duals, as opposed to the situation in [12]. Actually, in both papers, a key ingredient is Theorem 6.2 which is applied to an operator of type $T'u = \int_U \mathcal{K}'(x, y)u(y) dy$, where \mathcal{K}' is related to \mathcal{K} by a division with remainder of the form (6.2). There are however differences in the way this idea can be implemented. In [12] we could show that $T' \equiv 0$ in some region if $T'u = 0$ there for every Radon measure u , by applying the rather strong uniqueness theorem [12, Theorem 1.6] for integral operators of the form (1.2). In the present situation, there is no reasonable counterpart to [12, Theorem 1.6] to test real-analyticity by applying T' only to Radon measures, and we had to use Theorem 1.3 instead. It seemed impossible to find a decomposition of type (6.2) which works well simultaneously for all $u \in \mathcal{A}'(K)$, but as we have explained, it is possible to find a suitable decomposition which works well for all $u \in \mathcal{A}'_\ell(K)$ for a fixed sublinear function ℓ . In fact, a decomposition which is good enough for the argument is when \mathcal{K}' is in a BMT class for some weight function ω related to ℓ as above.

We mention incidentally that the reason why there is no full analogue to [12, Theorem 1.6] for real-analyticity is discussed in [14, Lemma 2.6 and Example 2.7].

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