Publ. RIMS Kyoto Univ. **52** (2016), 1–41
DOI 10.4171/PRIMS/172

Nahm's Equations, Quiver Varieties and Parabolic Sheaves

by

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Abstract

We construct an isomorphism between moduli spaces of solutions of Nahm's equations over the circle and framed moduli spaces of locally free parabolic sheaves over $\mathbb{P}^1 \times \mathbb{P}^1$ through chainsaw quiver varieties.

2010 Mathematics Subject Classification: Primary 70S15; Secondary 14D21.
Keywords: Nahm's equations, handsaw quiver variety, chainsaw quiver variety, parabolic sheaf, instanton, caloron, monopole.

§0. Introduction

For anti-self-dual (ASD) connections over $\mathbb{R}^4$, the following relations are well studied:

$$
(0.1)\qquad
\begin{array}{ccc}
\begin{array}{c}\text{hyper-Kähler quotients}\\ \{(A,B,i,j)\} /\!\!/\!\!/ U(n)\end{array} & \overset{\text{(ii)}}{\longleftrightarrow} & \begin{array}{c}\text{symplectic + GIT quotients}\\ \{(A,B,i,j)\} /\!\!/ GL(n)\end{array}\\
\text{(i)}\updownarrow & & \updownarrow\text{(iv)}\\
\begin{array}{c}\text{framed moduli spaces of}\\ \text{ASD connections over } \mathbb{R}^4\end{array} & \overset{\text{(iii)}}{\longleftrightarrow} & \begin{array}{c}\text{framed moduli spaces of}\\ \text{locally free sheaves on } \mathbb{P}^2\end{array}
\end{array}
$$

Here the arrow $\longleftrightarrow$ means that there exists an isomorphism under some conditions. The relation (i) was obtained by Atiyah–Drinfeld–Hitchin–Manin, and is called the ADHM construction [ADHM]. Today, this relation is regarded as an example of ADHMN constructions. The relation (ii) follows from the general theory relating a hyper-Kähler quotient on the one hand, and a holomorphic symplectic quotient and a geometric invariant theory quotient on the other. This theory was studied by

Communicated by H. Nakajima. Received April 13, 2015. Revised July 2, 2015.

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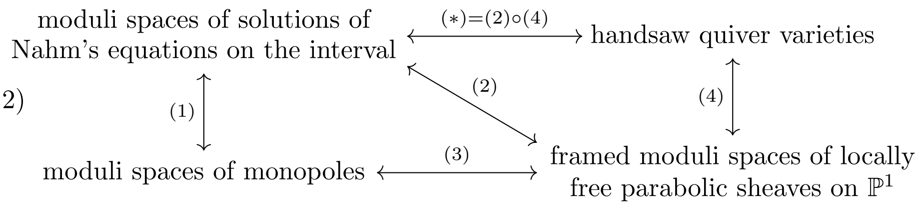
Kempf–Ness, Kirwan, and others [KeN, MFK]. The relation (iv) was constructed by Barth and Hulek [Bar, Hul], and the relation (iii) by Donaldson [D1] through (i), (ii) and (iv). The relations (i), (ii) and (iv) also appear in Nakajima's lecture notes [Nak3].

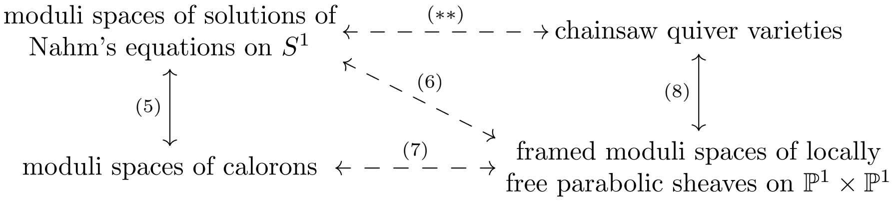
These relations among four spaces were also studied in other situations. For example, for ASD connections over ALE spaces, the relations (i) and (iv) were obtained by Kronheimer–Nakajima [KrN], and the relation (ii) also follows from the general theory. In this case, both sides of (ii) are known as quiver varieties and are well studied [Nak2]. Moreover, Bando [Ban] gave a direct analytic proof for (iii), and (iv) was extended to torsion free sheaves by Nakajima [Nak4].

In this paper, we focus on monopoles ($\mathbb{R}$-invariant ASD connections over $\mathbb{R}^4$) and calorons (ASD connections over $\mathbb{R}^3 \times S^1$). As analogues of the relations (0.1), the following relations are studied:

(0.2)



(0.3)



In fact, the relations (0.2) are obtained as special cases of (0.3). The arrow $\leftarrow - - \rightarrow$ means that the relation was only conjectured, for example in [CH1, CH2].

First we review the relations (0.2). For $SU(2)$-monopoles, the relation (1) was obtained by Nahm, Hitchin and Nakajima [Nah, Hi, Nak1]. This relation is the first example of ADHMN constructions. The relation (2) was given by Donaldson [D2], and (4) by Strømme [S]. For $SU(N)$-monopoles, the relation (1) was shown by Hurtubise–Murray [HurM], (2) by Hurtubise [Hur], and (4) by Finkelberg–Rybnikov and Nakajima [FR, Nak5]. Finally, the relation (3) was studied by Jarvis [J1, J2].

Now we consider the relation $(*)$. This relation is obtained as the composite of (2) and (4) as mentioned in (0.2), but we did not find its direct description anywhere. Hence we first construct $(*)$ explicitly. On the other hand, we can also regard (2) as the composite of $(*)$ and (4).

On the basis of these results, we next review the relations (0.3). The relation (5) was studied by Nye, Nye-Singer and Charbonneau–Hurtubise [Ny, NyS, CH1], as an example of ADHMN constructions. Furthermore Charbonneau–Hurtubise constructed (6) for rank 2 parabolic sheaves and suggested (7) [CH2]. This suggestion is supported by (3). The relation (8) was shown for torsion free parabolic sheaves by Finkelberg–Rybnikov and Braverman–Finkelberg [FR, BF].

The goal of this paper is to construct the relation (6) for general rank locally free parabolic sheaves.

Theorem 0.4. *The moduli space of solutions of Nahm's equations over the circle with rank* $(m_0, m_1, \ldots, m_{N-1})$ *is isomorphic to the framed moduli space of locally free parabolic sheaves over* $\mathbb{P}^1 \times \mathbb{P}^1$ *of rank* N *and degree* $(m_0, m_1, \ldots, m_{N-1})$.

We think that this approach can be generalized to the case of bow varieties describing instantons on ALF spaces by Cherkis [C].

We will prove Theorem 0.4 by composing the relations $(**)$ and (8). Here, we give $(**)$ in the same way as $(*)$, and we construct (8) as monads.

We remark that $(*)$ and $(**)$ can be regarded as analogues of (ii), but do not directly follow from the general theory. This is because the moduli spaces of solutions of Nahm's equations are considered as hyper-Kähler quotients, but in infinite-dimensional settings. Thus it becomes important to examine the stability conditions for the corresponding handsaw or chainsaw quiver varieties.

For (8), Finkelberg–Rybnikov and Braverman–Finkelberg constructed torsion free parabolic sheaves by using Biswas' method of constructing parabolic sheaves over orbifolds [Bis]. In more detail, they gave one sheaf (as opposed to a parabolic sheaf) on a variety Y as a monad; then Biswas' method leads to the corresponding parabolic sheaf on Y/Γ. On the other hand, we try to construct all sheaves of a parabolic sheaf as monads, as in [Nak5]. This approach also appears in [CH2]. Thus we give the relation (8) for locally free parabolic sheaves as $N+1$ monads and check that it can also be obtained for torsion free parabolic sheaves in the same way. This becomes another proof of Braverman–Finkelberg–Rybnikov's theorem.

This paper is organized as follows. In §1, we recall Nahm's equations according to [Hur, Bie2]. In §2, we describe the relation $(*)$ explicitly. In §3, we give the relation $(**)$; this is the first part of the proof of the main theorem. In §4, we construct the relation (8); this is the second part of the proof of the main theorem. In §5, we study a relation between (0.2) and (0.3), especially the relations (4) and (8).

§1. Preliminaries

§1.1. Nahm's equations

In this subsection we define $\mathcal{M}_{\mathrm{N}}(\mathcal{I};n,m)$ as the moduli space of solutions of Nahm's equations over the interval $\mathcal{I}$. Here n and m are nonnegative integers such that $m \le n$. The proofs of the theorems stated in this subsection are given in [Hur, Bie2, D2]. In particular, $\mathcal{M}_{\mathrm{N}}(\mathcal{I};n,m)$ is denoted by $F_n(n-m;|\mathcal{I}|)$ in [Bie2].

Set $\mathcal{I} = \{0 \le s \le 1\}$ and define $\sigma_k \in \mathfrak{su}(2)$ as

$$\sigma_0 = 0, \quad \sigma_1 = \frac{1}{2}\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad \sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_3 = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}.$$

Let ρ be the unitary representation of $\mathfrak{su}(2)$ which is the direct sum of the $(n-m)$-dimensional trivial representation and the standard m-dimensional irreducible representation, that is, $\rho = 0_{n-m} \oplus \mathrm{irr}_m \colon \mathfrak{su}(2) \to \mathfrak{u}(n)$. We define

$$\begin{aligned}
\mathcal{H} &:= \{(T_0,T_1,T_2,T_3) \in \Gamma(\mathcal{I},\mathfrak{u}(n)) \otimes \mathbb{H} \mid T_k - \rho(\sigma_k)/s \in L^2_1(\mathcal{I},\mathfrak{u}(n))\},\\
\mathcal{G}_{00} &:= \{u \in L^2_2(\mathcal{I},U(n)) \mid u(0) = u(1) = \mathrm{id}\},\\
\mathcal{G}_{0*} &:= \{u \in L^2_2(\mathcal{I},U(n)) \mid u(0) = \mathrm{id}\},\\
\mathcal{G}_{**} &:= \left\{u \in L^2_2(\mathcal{I},U(n)) \;\middle|\; u(0) \in \begin{pmatrix} U(n-m) & 0 \\ 0 & \mathrm{id}_m \end{pmatrix}\right\}.
\end{aligned}$$

We can regard $\mathcal{H}$ as having an infinite-dimensional hyper-Kähler structure induced by the quaternions $\mathbb{H}$. Further, $\mathcal{G}$ acts on $\mathcal{H}$ as follows:

$$u \cdot (T_0,T_1,T_2,T_3) = \left(uT_0u^{-1} - \frac{du}{ds}u^{-1}, uT_1u^{-1}, uT_2u^{-1}, uT_3u^{-1}\right).$$

Definition 1.1. We consider the following ordinary differential equations (*Nahm's equations*)

$$\begin{cases}
\frac{d}{ds}T_1 + [T_0,T_1] + [T_2,T_3] = 0,\\
\frac{d}{ds}T_2 + [T_0,T_2] + [T_3,T_1] = 0,\\
\frac{d}{ds}T_3 + [T_0,T_3] + [T_1,T_2] = 0.
\end{cases}$$

We denote the left hand sides by μ_I, μ_J and μ_K respectively.

These equations are preserved by the $\mathcal{G}_{00}$-action, and $\mu = (\mu_I,\mu_J,\mu_K)$ is regarded as the hyper-Kähler moment map of this action. Therefore, we define

$$\mathcal{M}_{\mathrm{N}}(\mathcal{I};n,m) := \mathcal{H} /\!\!/\!\!/_0\, \mathcal{G}_{00} = \mu_I^{-1}(0) \cap \mu_J^{-1}(0) \cap \mu_K^{-1}(0)/\mathcal{G}_{00}.$$

Here "N" means Nahm. A representative (T_0,T_1,T_2,T_3) of $\mathcal{M}_{\mathrm{N}}(\mathcal{I};n,m)$ has a single pole at the left end of $\mathcal{I}$ and is regular at the right end of $\mathcal{I}$. We write $\overline{\mathcal{M}}_{\mathrm{N}}(\mathcal{I};n,m)$ for the moduli space of solutions of Nahm's equations with a pole at the right end of $\mathcal{I}$ and regular at the left end of $\mathcal{I}$. $\mathcal{M}_{\mathrm{N}}(\mathcal{I};n,m)$ and $\overline{\mathcal{M}}_{\mathrm{N}}(\mathcal{I};n,m)$ correspond to each other under the transformation $s \mapsto -s$. One can change the length of the interval, but the differential structure of $\mathcal{M}_{\mathrm{N}}(\mathcal{I};n,m)$ does not depend on the length of $\mathcal{I}$.

Furthermore $U(n-m)$ and $U(n)$ act on $\mathcal{M}_{\mathrm{N}}(\mathcal{I};n,m)$ through the isomorphism $U(n-m)\times U(n) \cong \mathcal{G}_{**}/\mathcal{G}_{00}$, and the actions preserve the hyper-Kähler structure. The moment map of this action is given by

$$\mu_k = \begin{cases} \pi \circ T_k(0) & \text{for the } U(n-m)\text{-action,} \\ -T_k(1) & \text{for the } U(n)\text{-action,} \end{cases} \tag{1.2}$$

where $\pi\colon \mathfrak{u}(n) \to \mathfrak{u}(n-m)$ is the natural projection.

We pick one complex structure of the hyper-Kähler structure and we set

$$\alpha := \frac{1}{2}(T_0 + \sqrt{-1}\,T_1), \qquad \beta := \frac{1}{2}(T_2 + \sqrt{-1}\,T_3),$$
$$a := \frac{\sqrt{-1}}{2}\rho(\sigma_1), \qquad b := \frac{1}{2}\rho(\sigma_2 + \sqrt{-1}\,\sigma_3),$$

so $\alpha - a/s, \beta - b/s \in L^2_1(\mathcal{I}, \mathfrak{gl}(n))$. By using these, Nahm's equations can be written as

$$\mu_{\mathbb{C}} := \mu_J + \sqrt{-1}\,\mu_K = \frac{d}{ds}\beta + 2[\alpha,\beta] = 0,$$
$$\mu_{\mathbb{R}} := \sqrt{-1}\,\mu_I = \frac{d}{ds}(\alpha+\alpha^*) + 2([\alpha,\alpha^*] + [\beta,\beta^*]) = 0.$$

Let $\mathcal{G}^{\mathbb{C}}_{00}$ be the complexification of $\mathcal{G}_{00}$. It acts on $\mathcal{H}$ as follows:

$$g\cdot(\alpha,\beta) = \left(g\alpha g^{-1} - \frac{1}{2}\frac{dg}{ds}g^{-1}, g\beta g^{-1}\right).$$

This action preserves only the equation $\mu_{\mathbb{C}} = 0$.

Theorem 1.3 ([D2], [Hur, Theorem 2.18]). *For any $p \in \mu_{\mathbb{C}}^{-1}(0)$, $\mathcal{G}^{\mathbb{C}}_{00}\cdot p$ meets $\mu_{\mathbb{R}} = 0$, and $\mathcal{G}^{\mathbb{C}}_{00}\cdot p \cap \mu_{\mathbb{R}}^{-1}(0)$ consists of exactly one $\mathcal{G}_{00}$-orbit. In particular, $\mathcal{M}_{\mathrm{N}}(\mathcal{I};n,m)$ is isomorphic to $\mu_{\mathbb{C}}^{-1}(0)/\mathcal{G}^{\mathbb{C}}_{00}$. Furthermore, by using the $\mathcal{G}^{\mathbb{C}}_{0*}$-action, $\mathcal{M}_{\mathrm{N}}(\mathcal{I};n,m)$ has the normal form given below.*

Now, we describe $\mu_{\mathbb{C}}^{-1}(0)/\mathcal{G}^{\mathbb{C}}_{00}$ as the normal form. By the assumption on ρ, we can assume a and b are given by $a = \left(\begin{smallmatrix} 0_{n-m} & 0 \\ 0 & a_m \end{smallmatrix}\right)$ and $b = \left(\begin{smallmatrix} 0_{n-m} & 0 \\ 0 & b_m \end{smallmatrix}\right)$, where

$$a_m = \begin{pmatrix} -(m-1)/4 & & & 0 \\ & -(m-3)/4 & & \\ & & \ddots & \\ 0 & & & (m-1)/4 \end{pmatrix}, \quad b_m = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & 1 & 0 \end{pmatrix}.$$

We can take $u \in \mathcal{G}^{\mathbb{C}}_{0*}$ $(u(0) = \mathrm{id})$ which satisfies

$$\frac{du}{ds} = -2\Big(u\alpha - \frac{au}{s}\Big).$$

Then α and β are described by means of the following form (called Hurtubise's normal form [Hur, Proposition 1.15]):

$$u \cdot (\alpha, \beta) = \left(\frac{a}{s}, \begin{pmatrix} h & 0 & s^{(m-1)/2} g \\ s^{(m-1)/2} f & 0 & s^{m-1} e_0 \\ 0 & s^{-1}\,\mathrm{id}_{m-1} & e(s) \end{pmatrix}\right), \qquad e(s) = \begin{pmatrix} s^{m-2} e_1 \\ \vdots \\ s e_{m-2} \\ e_{m-1} \end{pmatrix},$$

where $h \in M(n-m, n-m; \mathbb{C})$, $g \in M(n-m, 1; \mathbb{C})$, $f \in M(1, n-m; \mathbb{C})$ and $e_k \in \mathbb{C}$. It is easy to check the above $u \cdot (\alpha, \beta)$ satisfies $\mu_{\mathbb{C}} = 0$. Set

$$X(n,m) = \left\{ \begin{pmatrix} h & 0 & g \\ f & 0 & e_0 \\ 0 & \mathrm{id}_{m-1} & e \end{pmatrix} \in \mathrm{End}(\mathbb{C}^n) \;\middle|\; \begin{matrix} h \in \mathrm{End}(\mathbb{C}^{n-m}),\ g \in \mathbb{C}^{n-m} \\ f \in (\mathbb{C}^{n-m})^*,\ e \in \mathbb{C}^{m-1},\ e_0 \in \mathbb{C} \end{matrix} \right\}.$$

Clearly, $X(n,m)$ is an $\{(n-m)^2 + 2n - m\}$-dimensional vector space. Therefore, we have the following isomorphism:

$$\text{(1.4)} \qquad \mu_{\mathbb{C}}^{-1}(0)/\mathcal{G}^{\mathbb{C}}_{00} \to GL(n) \times X(n,m), \qquad [(\alpha(s), \beta(s))] \mapsto (u(1), (u\cdot\beta)(1)).$$

Here we have identified $\mathcal{G}^{\mathbb{C}}_{0*}/\mathcal{G}^{\mathbb{C}}_{00}$ with $GL(n)$.

Remark 1.5. We can see that $X(n,m)$ above is the transversal slice for the nilpotent orbit of b. See also [Bie1, Theorem 1].

$\mu_{\mathbb{C}}^{-1}(0)/\mathcal{G}^{\mathbb{C}}_{00}$ and $GL(n) \times X(n,m)$ have a holomorphic symplectic structure induced by the hyper-Kähler structure of $\mathcal{M}_{\mathrm{N}}(\mathcal{I}; n, m)$. $GL(n-m)$ and $GL(n)$ act on $GL(n) \times X(n,m)$ through the above isomorphism and $GL(n-m) \times GL(n) \cong \mathcal{G}^{\mathbb{C}}_{**}/\mathcal{G}^{\mathbb{C}}_{00}$, preserving the holomorphic symplectic structure. These actions are explicitly described as

$$\text{(1.6)} \qquad (g_1, g_2) \cdot (u, \eta) = \left(\begin{pmatrix} g_1^{-1} & 0 \\ 0 & \mathrm{id}_m \end{pmatrix} u g_2^{-1}, \begin{pmatrix} g_1 & 0 \\ 0 & \mathrm{id}_m \end{pmatrix} \eta \begin{pmatrix} g_1^{-1} & 0 \\ 0 & \mathrm{id}_m \end{pmatrix} \right)$$

for $(g_1, g_2) \in GL(n-m) \times GL(n)$ and $(u, \eta) \in GL(n) \times X(n,m)$. These actions have moment maps

$$\text{(1.7)} \qquad \mu = \begin{cases} \pi(\eta) = h, & \text{for the } GL(n-m)\text{-action}, \\ -u^{-1} \begin{pmatrix} h & 0 & g \\ f & 0 & e_0 \\ 0 & \mathrm{id} & e \end{pmatrix} u & \text{for the } GL(n)\text{-action}. \end{cases}$$

These moment maps are induced by (1.2), in fact, we have

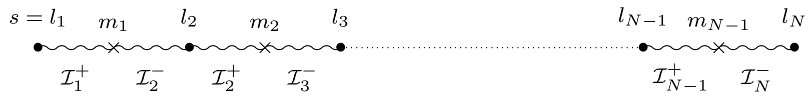
$$\pi(\beta(0)) = \pi(u(0)^{-1}\eta(0)u(0)) = h, \qquad \beta(1) = u(1)^{-1}\eta(1)u(1) = u^{-1}\eta u.$$

§2. Nahm's equations over the interval and a handsaw quiver

In this section, we first review the moduli space of solutions of Nahm's equations on the interval with rank $\vec{m}$. Then we recall a handsaw quiver variety, and describe a relation between these two spaces.

§2.1. Nahm's equations over the interval

We select $\vec{m} = (m_1, \ldots, m_{N-1}) \in \mathbb{Z}_{\geq 0}^{\oplus N-1}$ and $l_1 < \cdots < l_N$. Then define $\mathcal{I}_k^- = \{(l_{k-1} + l_k)/2 \leq s \leq l_k\}$ and $\mathcal{I}_k^+ = \{l_k \leq s \leq (l_k + l_{k+1})/2\}$.



We construct the moduli space of solutions of Nahm's equations over $\mathcal{I}_k^- \cup \mathcal{I}_k^+$ with rank (m_{k-1}, m_k) as a 0-parameter hyper-Kähler quotient:

$$\mathcal{M}_{\mathrm{N}}(\mathcal{I}_k^- \cup \mathcal{I}_k^+; m_{k-1}, m_k)$$
$$:= \begin{cases} \overline{\mathcal{M}}_{\mathrm{N}}(\mathcal{I}_k^-; m_{k-1}, m_{k-1} - m_k) \times \mathcal{M}_{\mathrm{N}}(\mathcal{I}_k^+; m_k, 0) /\!\!/\!\!/_0\, U(m_k) \\ \qquad\qquad \text{when } m_{k-1} > m_k, \\ \overline{\mathcal{M}}_{\mathrm{N}}(\mathcal{I}_k^-; m_k, 0) \times \mathcal{M}_{\mathrm{N}}(\mathcal{I}_k^+; m_k, 0) \times \mathbb{C}^{m_k} \times (\mathbb{C}^{m_k})^* /\!\!/\!\!/_0\, U(m_k) \\ \qquad\qquad \text{when } m_{k-1} = m_k, \\ \overline{\mathcal{M}}_{\mathrm{N}}(\mathcal{I}_k^-; m_{k-1}, 0) \times \mathcal{M}_{\mathrm{N}}(\mathcal{I}_k^+; m_k, m_k - m_{k-1}) /\!\!/\!\!/_0\, U(m_{k-1}) \\ \qquad\qquad \text{when } m_{k-1} < m_k, \end{cases}$$

where $U(m_k)$ acts on $\mathbb{C}^{m_k} \times (\mathbb{C}^{m_k})^*$ as $(v, w) \mapsto (gv, wg^{-1})$.

By using these moduli spaces, we define the moduli space of solutions of Nahm's equations over the interval (l_1, l_N) with rank $\vec{m}$ as follows:

$$\mathcal{M}_{\mathrm{N}}((l_1, l_N); \vec{m}, N) := \prod_{k=1}^{N} \mathcal{M}_{\mathrm{N}}(\mathcal{I}_k^- \cup \mathcal{I}_k^+; m_{k-1}, m_k) /\!\!/\!\!/_0 \prod_{k=1}^{N-1} U(m_k),$$

where we regard $m_0 = m_N = 0$ and $\mathcal{I}_1^- = \mathcal{I}_N^+ = \emptyset$. In [Bie2], the spaces $\mathcal{M}_{\mathrm{N}}(\mathcal{I}_k^- \cup \mathcal{I}_k^+; m_{k-1}, m_k)$ and $\mathcal{M}_{\mathrm{N}}((l_1, l_N); \vec{m}, N)$ are denoted by $F_{m_{k-1}, m_k}(|\mathcal{I}_k^-|, |\mathcal{I}_k^+|)$ and $F_\sigma(\mu)$ respectively.

Remark 2.1. $\mathcal{M}_{\mathrm{N}}(\mathcal{I}_k^- \cup \mathcal{I}_k^+; m_{k-1}, m_k)$ and $\mathcal{M}_{\mathrm{N}}((l_1, l_N); \vec{m}, N)$ are defined as 0-parameter hyper-Kähler quotients, but indeed these moduli spaces do not depend on their parameters (see also [T, Remarks 2.5 and 4.5]).

The relations (1) and (2) of (0.2) in the Introduction are written as follows.

Theorem 2.2 ([HurM]). *The moduli space of $SU(N)$-monopoles of charge $\vec{m}$ with maximal symmetry breaking is isomorphic to $\mathcal{M}_{\mathrm{N}}((l_1, l_N); \vec{m}, N)$.*

Theorem 2.3 ([Hur]). *The moduli space of degree $\vec{m}$ based rational maps from $\mathbb{P}^1$ to the rank N full flag variety is isomorphic to $\mathcal{M}_{\mathrm{N}}((l_1, l_N); \vec{m}, N)$.*

Now we study $\mathcal{M}_{\mathrm{N}}(\mathcal{I}_k^- \cup \mathcal{I}_k^+; m_{k-1}, m_k)$ and $\mathcal{M}_{\mathrm{N}}((l_1, l_N); \vec{m}, N)$ in more detail.

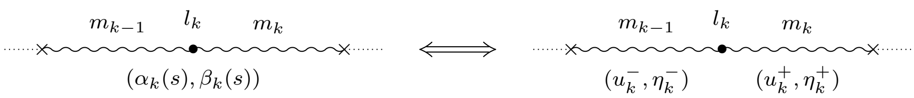
Proposition 2.4 ([Hur, Theorem 2.22]). *These spaces can be constructed as holomorphic symplectic quotients:*

$$\mathcal{M}_{\mathrm{N}}(\mathcal{I}_k^- \cup \mathcal{I}_k^+; m_{k-1}, m_k) \cong \begin{cases} \overline{\mathcal{M}}_{\mathrm{N}}(\mathcal{I}_k^-; m_{k-1}, m_{k-1} - m_k) \times \mathcal{M}_{\mathrm{N}}(\mathcal{I}_k^+; m_k, 0) /\!/ GL(m_k) & \text{when } m_{k-1} > m_k, \\ \overline{\mathcal{M}}_{\mathrm{N}}(\mathcal{I}_k^-; m_k, 0) \times \mathcal{M}_{\mathrm{N}}(\mathcal{I}_k^+; m_k, 0) \times \mathbb{C}^{m_k} \times (\mathbb{C}^{m_k})^* /\!/ GL(m_k) & \text{when } m_{k-1} = m_k, \\ \overline{\mathcal{M}}_{\mathrm{N}}(\mathcal{I}_k^-; m_{k-1}, 0) \times \mathcal{M}_{\mathrm{N}}(\mathcal{I}_k^+; m_k, m_k - m_{k-1}) /\!/ GL(m_{k-1}) & \text{when } m_{k-1} < m_k, \end{cases}$$

$$\mathcal{M}_{\mathrm{N}}((l_1, l_N); \vec{m}, N) \cong \prod_{k=1}^{N} \mathcal{M}_{\mathrm{N}}(\mathcal{I}_k^- \cup \mathcal{I}_k^+; m_{k-1}, m_k) \Big/\!\!\Big/ \prod_{k=1}^{N-1} GL(m_k).$$

This proposition can also be obtained by using the argument in [T, proof of Proposition 4.7]. We can write a representative of a point of $\mathcal{M}_{\mathrm{N}}((l_1, l_N); \vec{m}, N)$ as $(\alpha(s), \beta(s))$, which is defined on the whole (l_1, l_N). Let $(\alpha_k(s), \beta_k(s))$ be the restriction of $(\alpha(s), \beta(s))$ onto $\mathcal{I}_k^- \cup \mathcal{I}_k^+$. Then $(\alpha_k(s), \beta_k(s))$ becomes a representative of a point of $\mathcal{M}_{\mathrm{N}}(\mathcal{I}_k^- \cup \mathcal{I}_k^+; m_{k-1}, m_k)$.

We give an explicit description of $\mathcal{M}_{\mathrm{N}}(\mathcal{I}_k^- \cup \mathcal{I}_k^+; m_{k-1}, m_k)$ by using Proposition 2.4 and (1.4).



When $m_{k-1} > m_k$, we have

$$\begin{aligned} &\mathcal{M}_{\mathrm{N}}(\mathcal{I}_k^- \cup \mathcal{I}_k^+; m_{k-1}, m_k) \\ &\quad \cong GL(m_{k-1}) \times X(m_{k-1}, m_{k-1} - m_k) \times GL(m_k) \times X(m_k, 0) /\!/ GL(m_k) \\ &\quad \cong \left\{ \left(u_k^-, \begin{pmatrix} h_k^- & 0 & g_k^- \\ f_k^- & 0 & e_{0,k}^- \\ 0 & \mathrm{id} & e_k^- \end{pmatrix}, \eta_k^+, u_k^+ \right) \;\middle|\; -h_k^- + \eta_k^+ = 0 \right\} \Big/\!\!\Big/ GL(m_k) \\ &\quad = \left\{ \left(u_k^-, \begin{pmatrix} h_k^- & 0 & g_k^- \\ f_k^- & 0 & e_{0,k}^- \\ 0 & \mathrm{id} & e_k^- \end{pmatrix} \right) \right\} \cong GL(m_{k-1}) \times X(m_{k-1}, m_{k-1} - m_k). \end{aligned}$$

In the same way, we get

$$
\begin{aligned}
&\mathcal{M}_{\mathrm{N}}(\mathcal{I}_k^- \cup \mathcal{I}_k^+; m_{k-1}, m_k) \\
&\qquad \cong \begin{cases} GL(m_{k-1}) \times X(m_{k-1}, 0) \times \mathbb{C}^{m_k} \times (\mathbb{C}^{m_k})^* & \text{when } m_{k-1} = m_k, \\ GL(m_k) \times X(m_k, m_k - m_{k-1}) & \text{when } m_{k-1} < m_k. \end{cases}
\end{aligned}
$$

These statements mean that a representative $(\alpha_k(s), \beta_k(s))$ of a point of $\mathcal{M}_{\mathrm{N}}(\mathcal{I}_k^- \cup \mathcal{I}_k^+; m_{k-1}, m_k)$ can be described as

$$
\text{when } m_{k-1} > m_k, \quad \beta_k(s) = \begin{cases} (u_k^-)^{-1} \begin{pmatrix} h_k^- & 0 & g_k^- \\ f_k^- & 0 & e_{0,k}^- \\ 0 & \mathrm{id} & e_k^- \end{pmatrix} u_k^- & \text{for } s = \frac{l_{k-1}+l_k}{2}, \\ h_k^- & \text{for } l_k < s \le \frac{l_k + l_{k+1}}{2}, \end{cases}
$$

$$
\text{when } m_{k-1} = m_k, \quad \beta_k(s) = \begin{cases} u_k^- \eta_k^- u_k^- & \text{for } s = \frac{l_{k-1}+l_k}{2}, \\ \eta_k^- - v_k w_k & \text{for } l_k < s \le \frac{l_k+l_{k+1}}{2}, \end{cases}
$$

$$
\text{when } m_{k-1} < m_k, \quad \beta_k(s) = \begin{cases} h_k^+ & \text{for } \frac{l_{k-1}+l_k}{2} \le s < l_k, \\ (u_k^+)^{-1} \begin{pmatrix} h_k^+ & 0 & g_k^+ \\ f_k^+ & 0 & e_{0,k}^+ \\ 0 & \mathrm{id} & e_k^+ \end{pmatrix} u_k^+ & \text{for } s = \frac{l_k+l_{k+1}}{2}. \end{cases}
$$

Corollary 2.5. *$\mathcal{M}_{\mathrm{N}}(\mathcal{I}_k^- \cup \mathcal{I}_k^+; m_{k-1}, m_k)$ and $\mathcal{M}_{\mathrm{N}}((l_1, l_N); \vec{m}, N)$ are smooth manifolds, and their dimensions are given by*

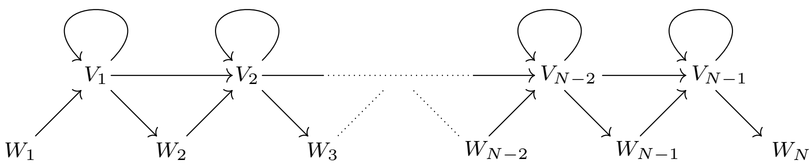
$$
\begin{aligned}
\dim_{\mathbb{C}} \mathcal{M}_{\mathrm{N}}(\mathcal{I}_k^- \cup \mathcal{I}_k^+; m_{k-1}, m_k) &= m_{k-1}^2 + m_{k-1} + m_k^2 + m_k, \\
\dim_{\mathbb{C}} \mathcal{M}_{\mathrm{N}}((l_1, l_N); \vec{m}, N) &= \sum_{k=1}^{N-1} 2m_k.
\end{aligned}
$$

Proof. We check that the stabilizer group of each point of both spaces is trivial. For $\mathcal{M}_{\mathrm{N}}(\mathcal{I}_k^- \cup \mathcal{I}_k^+; m_{k-1}, m_k)$, this was checked in the above argument. For $\mathcal{M}_{\mathrm{N}}((l_1, l_N); \vec{m}, N)$, it will be shown in Corollary 2.16. Then the dimensions are easily calculated. □

Remark 2.6. When we connect $\mathcal{M}_{\mathrm{N}}(\mathcal{I}_k^- \cup \mathcal{I}_k^+; m_{k-1}, m_k)$s linearly as in this subsection, their stabilizer groups are actually trivial. However, if we connect them circularly, their stabilizer groups may not be trivial. See also the comments following Theorem 3.2.

§2.2. Handsaw quiver variety

Choose $\vec{m} = (m_1, \ldots, m_{N-1}) \in \mathbb{Z}_{\ge 0}^{\oplus N-1}$. We consider the following diagram:



Here V_k is an m_k-dimensional vector space and W_k is a 1-dimensional vector space. Set

$$\begin{aligned}\mathbb{M}_{\mathrm{h}}(\vec{m},N) &:= \bigoplus_{k=1}^{N-1}\operatorname{End}V_k \oplus \bigoplus_{k=1}^{N-2}\operatorname{Hom}(V_k,V_{k+1}) \\ &\quad \oplus \bigoplus_{k=1}^{N}\operatorname{Hom}(W_k,V_k) \oplus \bigoplus_{k=1}^{N-1}\operatorname{Hom}(V_k,W_{k+1}), \\ \mathcal{G}_{\mathrm{h}}(\vec{m},N) &:= \prod_{k=1}^{N-1} GL(V_k).\end{aligned}$$

Here "h" means a handsaw quiver. $\mathcal{G}_{\mathrm{h}}$ acts on $\mathbb{M}_{\mathrm{h}}$ as

$$(g_k)\cdot(A_k,B_k,i_k,j_k) = (g_kA_kg_k^{-1}, g_{k+1}B_kg_k^{-1}, g_ki_k, j_kg_k^{-1}),$$

where $(A_k,B_k,i_k,j_k)\in\operatorname{End}V_k\oplus\operatorname{Hom}(V_k,V_{k+1})\oplus\operatorname{Hom}(W_k,V_k)\oplus\operatorname{Hom}(V_k,W_{k+1})$ and $g_k\in GL(V_k)$. We define maps $\mu_k\colon\mathbb{M}_{\mathrm{h}}\to\operatorname{Hom}(V_k,V_{k+1})$ for $k=1,\dots,N-1$ as

$$\mu_k(A,B,i,j) = A_{k+1}B_k - B_kA_k + i_{k+1}j_k.$$

Note that $\mu_k(g\cdot(A,B,i,j)) = g_{k+1}\mu_k(A,B,i,j)g_k^{-1}$. We write a set of subspaces $\{(S_1,\dots,S_{N-1})\mid S_k\subset V_k\}$ as $(S_1,\dots,S_{N-1})\subset(V_1,\dots,V_{N-1})$. We consider the following stability conditions:

(H-S1) for a set of subspaces $(S_1,\dots,S_{N-1})\subset(V_1,\dots,V_{N-1})$, if $A_k(S_k)\subset S_k$, $B_k(S_k)\subset S_{k+1}$, and $\operatorname{Ker}j_k\supset S_k$, then $S_k=0$.

(H-S2) for a set of subspaces $(T_1,\dots,T_{N-1})\subset(V_1,\dots,V_{N-1})$, if $A_k(T_k)\subset T_k$, $B_k(T_k)\subset T_{k+1}$, and $\operatorname{Im}i_k\subset T_k$, then $T_k=V_k$.

Then we define a *handsaw quiver variety* $\mathcal{M}_{\mathrm{h}}(\vec{m},N)$ as

$$\left\{(A,B,i,j)\in\bigcap_{k=1}^{N-1}\mu_k^{-1}(0)\;\middle|\;(A,B,i,j)\text{ satisfies (H-S1) and (H-S2)}\right\}\Big/\mathcal{G}_{\mathrm{h}}(\vec{m},N).$$

We recall the properties of a handsaw quiver variety. See also [Nak5], where $\mathcal{M}_{\mathrm{h}}(\vec{m},N)$ is denoted by $\mathfrak{L}_0^{\mathrm{reg}}(v,w)$ with $v=\vec{m}$ and $w=(1,\dots,1)$.

Theorem 2.7 ([FR, Nak5]). *The handsaw quiver variety $\mathcal{M}_{\mathrm{h}}(\vec{m},N)$ is isomorphic to the moduli space of degree $\vec{m}$ based rational maps from $\mathbb{P}^1$ to the rank N full flag variety.*

This theorem corresponds to the relation (4) of (0.2).

Remark 2.8. In fact, the moduli space of based rational maps from $\mathbb{P}^1$ to the full flag variety is an open subvariety of the framed moduli space of locally free parabolic sheaves on $\mathbb{P}^1$. The relation between these two spaces and handsaw quiver varieties is given in [Nak5, §3] and mentioned in §5 of this paper.

Proposition 2.9. *All $\mathcal{G}_{\mathrm{h}}$-orbits in $\mathcal{M}_{\mathrm{h}}(\vec{m}, N)$ are closed and their stabilizer groups are trivial.*

Proof. The stability conditions (H-S1) and (H-S2) coincide with King's stability condition [Kin]. Hence the assertion follows from geometric invariant theory. □

Lemma 2.10. *The differential of μ_k is surjective if $(A, B, i, j) \in \bigcap \mu_k^{-1}(0)$ satisfies* (H-S1) *or* (H-S2)*.*

Proof. The cokernel of the differential of μ_k is given by

$$\{\xi \in \mathrm{Hom}(V_{k+1}, V_k) \mid B_k\xi = 0,\ \xi A_{k+1} = A_k\xi,\ \xi B_k = 0,\ j_k\xi = 0,\ \xi i_{k+1} = 0\}.$$

$\mu_k = 0$ means that $\mathrm{Ker}\, B_k \cap \mathrm{Ker}\, j_k$ is A_k-invariant, thus is zero by (H-S1). Now ξ satisfies $\mathrm{Im}\,\xi \subset \mathrm{Ker}\, B_k \cap \mathrm{Ker}\, j_k = 0$, so $\xi = 0$.

In the same way, (H-S2) means that $\mathrm{Im}\, B_k \cup \mathrm{Im}\, i_{k+1} = V_{k+1}$, so $\xi = 0$ again from $\mathrm{Im}\, B_k \cup \mathrm{Im}\, i_{k+1} \subset \mathrm{Ker}\,\xi$. □

Thus $\mathcal{M}_{\mathrm{h}}(\vec{m}, N)$ is a smooth variety, and its dimension is given by

$$\dim_{\mathbb{C}} \mathcal{M}_{\mathrm{h}}(\vec{m}, N) = \sum_{k=1}^{N-1} 2m_k.$$

In the following two subsections, we prove the following theorem without using Theorems 2.3 and 2.7 (cf. $(*)$ in (0.2)):

Theorem 2.11. *$\mathcal{M}_{\mathrm{N}}((l_1, l_N); \vec{m}, N)$ and $\mathcal{M}_{\mathrm{h}}(\vec{m}, N)$ are isomorphic as varieties.*

In particular, we have

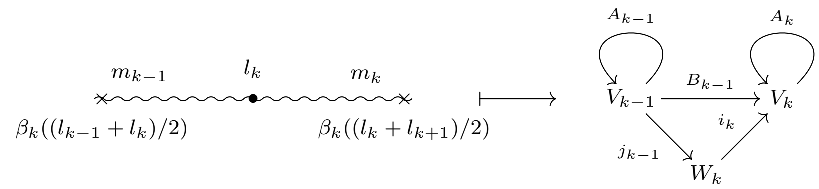
Corollary 2.12. *$\mathcal{M}_{\mathrm{h}}(\vec{m}, N)$ has a hyper-Kähler structure.*

Remark 2.13. The point of our proof is to characterize Hurtubise's normal form by stability conditions. When $N = 2$, Theorem 2.11 is proved by Donaldson [D2, Proposition (3.1)] in this sense.

§2.3. From solutions of Nahm's equations to handsaw quiver varieties

In this subsection, we give a correspondence from $\mathcal{M}_{\mathrm{N}}((l_1, l_N); \vec{m}, N)$ to $\mathcal{M}_{\mathrm{h}}(\vec{m}, N)$. First, we consider a piecewise correspondence from $\mathcal{M}_{\mathrm{N}}(\mathcal{I}_k^- \cup \mathcal{I}_k^+;$

$m_{k-1}, m_k)$ to $\{(A_{k-1}, B_{k-1}, A_k, i_k, j_{k-1})\} \subset \mathbb{M}_{\mathrm{h}}$. Then we check that $\prod GL(V_k)$-orbits in $\mathcal{M}_{\mathrm{N}}\ ((l_1, l_N); \vec{m}, N)$ correspond to $\mathcal{G}_{\mathrm{h}}$-orbits in $\mathcal{M}_{\mathrm{h}}(\vec{m}, N)$.



View $(\alpha(s), \beta(s))$ as $[(\alpha(s), \beta(s))] \in \mathcal{M}_{\mathrm{N}}((l_1, l_N); \vec{m}, N)$. We define $(A_{k-1}, A_k) := (\beta_k((l_{k-1} + l_k)/2), \beta_k((l_k + l_{k+1})/2))$. By using the description of $\mathcal{M}_{\mathrm{N}}(\mathcal{I}_k^- \cup \mathcal{I}_k^+; m_{k-1}, m_k)$ given in §2.1, we obtain the next proposition.

Proposition 2.14. *For each k, set*

$$
\begin{aligned}
&(A_{k-1}, B_{k-1}, A_k, i_k, j_{k-1}) \\
&:= \begin{cases}
\left((u_k^-)^{-1} \begin{pmatrix} h_k^- & 0 & g_k^- \\ f_k^- & 0 & e_{0,k}^- \\ 0 & \mathrm{id} & e_k^- \end{pmatrix} u_k^-, (\mathrm{id}\ \ 0\ \ 0) u_k^-, h_k^-, g_k^-, (0\ 0\ 1) u_k^- \right) & \\
\hfill \textit{when } m_{k-1} > m_k, \\
((u_k^-)^{-1} \eta_k^- u_k^-, u_k^-, \eta_k^- - v_k w_k, v_k, w_k u_k^-) \qquad \textit{when } m_{k-1} = m_k, \\
\left(h_k^+, (u_k^+)^{-1} \begin{pmatrix} \mathrm{id} \\ 0 \\ 0 \end{pmatrix}, (u_k^+)^{-1} \begin{pmatrix} h_k^+ & 0 & g_k^+ \\ f_k^+ & 0 & e_{0,k}^+ \\ 0 & \mathrm{id} & e_k^+ \end{pmatrix} u_k^+, (u_k^+)^{-1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, -f_k^+ \right) \\
\hfill \textit{when } m_{k-1} < m_k.
\end{cases}
\end{aligned}
$$

Then the $\prod GL(V_k)$-orbit of $(\alpha(s), \beta(s))$ corresponds to the $\mathcal{G}_{\mathrm{h}}$-orbit of (A, B, i, j). Moreover (A, B, i, j) satisfies $\mu = 0$ and the stability conditions (H-S1) *and* (H-S2)*.*

Proof. When $m_{k-1} > m_k$, we recall

$$
\mathcal{M}_{\mathrm{N}}(\mathcal{I}_k^- \cup \mathcal{I}_k^+; m_{k-1}, m_k) \cong \left\{ \left(u_k^-, \begin{pmatrix} h_k^- & 0 & g_k^- \\ f_k^- & 0 & e_{0,k}^- \\ 0 & \mathrm{id} & e_k^- \end{pmatrix} \right) \right\}.
$$

Then $(g_{k-1}, g_k) \in GL(V_{k-1}) \times GL(V_k)$ acts as follows (cf. (1.6)):

$$
(u_k^-, h_k^-, g_k^-, f_k^-, e_{0,k}^-, e_k^-) \mapsto \left(\begin{pmatrix} g_k^{-1} & 0 \\ 0 & \mathrm{id} \end{pmatrix} u_k^- g_{k-1}^{-1}, g_k h_k^- g_k^{-1}, g_k g_k^-, f_k^- g_k^{-1}, e_{0,k}^-, e_k^- \right).
$$

Hence

$$
\begin{aligned}
&(A_{k-1}, B_{k-1}, A_k, i_k, j_{k-1}) \\
&\qquad \mapsto (g_{k-1} A_{k-1} g_{k-1}^{-1}, g_k B_{k-1} g_{k-1}^{-1}, g_k A_k g_k^{-1}, g_k i_k, j_{k-1} g_{k-1}^{-1}).
\end{aligned}
$$

When $m_{k-1} \leq m_k$, we can check this in the same way. Thus we conclude that the $\prod GL(V_k)$-orbit of $(\alpha(s), \beta(s))$ corresponds to the $\mathcal{G}_{\mathrm{h}}$-orbit of (A, B, i, j).

We now check $\mu = 0$ and conditions (H-S1) and (H-S2). By using the $\mathcal{G}_{\mathrm{h}}$-action, when we only deal with terms involved in V_{k-1}, W_k and V_k, we can rewrite $(A_{k-1}, B_{k-1}, A_k, i_k, j_{k-1})$ as

$$(2.15)\qquad (A_{k-1}, B_{k-1}, A_k, i_k, j_{k-1}) = \begin{cases} \left(\begin{pmatrix} h_k & 0 & g_k \\ f_k & 0 & e_{0,k} \\ 0 & \mathrm{id} & e_k \end{pmatrix}, (\mathrm{id}\ 0\ 0), h_k, g_k, (0\ 0\ 1)\right) & \text{when } m_{k-1} > m_k, \\ (h_k, \mathrm{id}, h_k - v_k w_k, v_k, w_k) & \text{when } m_{k-1} = m_k, \\ \left(h_k, \begin{pmatrix} \mathrm{id} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} h_k & 0 & g_k \\ f_k & 0 & e_{0,k} \\ 0 & \mathrm{id} & e_k \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, -f_k\right) & \text{when } m_{k-1} < m_k. \end{cases}$$

Thus for each k, it is easy to check $\mu_{k-1}(A, B, i, j) = A_k B_{k-1} - B_{k-1} A_{k-1} + i_k j_{k-1} = 0$.

We now show that conditions (H-S1) and (H-S2) are satisfied. First we check (H-S1). Let $(S_1, \dots, S_{N-1})$ be a set of subspaces which is preserved by A and B, and included in $\operatorname{Ker} j$. We consider V_{N-1} and W_N, and use the description (2.15). Then we get

$$A_{N-1} = \begin{pmatrix} 0 & e_{0,N} \\ \mathrm{id}_{m_{N-1}-1} & e_N \end{pmatrix}, \qquad j_{N-1} = (0\ 1), \qquad B_{N-1} = A_N = i_N = 0,$$

and $S_{N-1} \subset \operatorname{Ker}(j_{N-1} A_{N-1}^{\lambda})$ for any $\lambda \in \mathbb{Z}_{\geq 0}$. But

$$j_{N-1} A_{N-1} = (0\ 0\ \cdots\ 0\ 0\ 1\ *), \qquad j_{N-1} A_{N-1}^2 = (0\ 0\ \cdots\ 0\ 1\ *\ *),$$
$$\dots, \qquad j_{N-1} A_{N-1}^{m_{N-1}-1} = (1\ *\ \cdots\ *\ *\ *\ *),$$

so $S_{N-1} = 0$.

Suppose that $S_k = 0$ and consider V_{k-1}, W_k and V_k. When $m_{k-1} \leq m_k$, we get $S_{k-1} = 0$ because B_{k-1} is injective. When $m_{k-1} > m_k$, we get

$$\operatorname{Ker} B_{k-1} \supset S_{k-1} = \left\{ \begin{pmatrix} 0 \\ * \\ * \end{pmatrix} \right\}, \qquad A_{k-1} = \begin{pmatrix} h_k & 0 & g_k \\ f_k & 0 & e_{0,k} \\ 0 & \mathrm{id} & e_k \end{pmatrix}, \qquad j_{k-1} = (0\ 0\ 1).$$

By the same argument as above, $S_{k-1} = 0$ follows from $S_{k-1} \subset \operatorname{Ker}(j_{k-1} A_{k-1}^{\lambda})$. Thus $S_k = 0$ for any k.

Next we check (H-S2). Let $(T_1, \dots, T_{N-1})$ be a set of subspaces which is preserved by A and B, and includes $\operatorname{Im} i$. For W_1 and V_1, we have

$$A_1 = \begin{pmatrix} 0 & e_{0,1} \\ \mathrm{id} & e_1 \end{pmatrix}, \qquad i_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad A_0 = B_0 = j_0 = 0,$$

and $\langle i_1, A_1 i_1, \dots, A_1^{m_1 - 1} i_1 \rangle = V_1$. This means $T_1 = V_1$.

Suppose that $T_{k-1} = V_{k-1}$. When $m_{k-1} \geq m_k$, we get $T_k = V_k$ because B_{k-1} is surjective. When $m_{k-1} < m_k$, we get

$$T_k \supset B_{k-1}(T_{k-1}) = \left\{ \begin{pmatrix} * \\ 0 \\ 0 \end{pmatrix} \right\}, \qquad A_k = \begin{pmatrix} h_k & 0 & g_k \\ f_k & 0 & e_{0,k} \\ 0 & \mathrm{id} & e_k \end{pmatrix}, \qquad i_k = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

So $T_k \supset \langle i_k, A_k i_k, \ldots, A_k^{m_k - m_{k-1} - 1} i_k \rangle = \left\{ \begin{pmatrix} 0 \\ * \\ * \end{pmatrix} \right\}$ and $T_k = V_k$. Thus we conclude $T_k = V_k$ for any k. □

Corollary 2.16. *$\mathcal{M}_{\mathrm{N}}((l_1, l_N); \vec{m}, N)$ is smooth.*

Proof. By the proof of Proposition 2.14, we can check that points of a $\prod GL(V_k)$-orbit in $\mathcal{M}_{\mathrm{N}}((l_1, l_N); \vec{m}, N)$ are bijectively mapped to points of the corresponding $\mathcal{G}_{\mathrm{h}}$-orbit in $\mathcal{M}_{\mathrm{h}}(\vec{m}, N)$. Thus Proposition 2.9 means that all stabilizer groups of $\prod GL(V_k)$-orbits in $\mathcal{M}_{\mathrm{N}}((l_1, l_N); \vec{m}, N)$ are trivial. □

In this way, we obtain a map from $\mathcal{M}_{\mathrm{N}}((l_1, l_N); \vec{m}, N)$ to $\mathcal{M}_{\mathrm{h}}(\vec{m}, N)$.

Remark 2.17. (i) For $[(\alpha, \beta)] \in \mathcal{M}_{\mathrm{N}}((l_1, l_N); \vec{m}, N)$, we can regard α as the connection for the endomorphism β, and the equation $\mu_{\mathbb{C}} = d\beta/ds + 2[\alpha, \beta] = 0$ means the parallel transportation of β by α. Thus, for $[(A, B, i, j)] \in \mathcal{M}_{\mathrm{h}}(\vec{m}, N)$, the equation $\mu_{k-1} = A_k B_{k-1} - B_{k-1} A_{k-1} + i_k j_{k-1} = 0$ can be interpreted as the parallel transportation from A_{k-1} to A_k by B_{k-1} because of Proposition 2.14.

(ii) The correspondence given in Proposition 2.14 links some conditions for Nahm's equations and the stability conditions (H-S1) and (H-S2). That is, (H-S1) forbids the existence of subsolutions of Nahm's equations, and this is related to the irreducibility of the representation of the residue at $s = l_k$ in the interval (l_1, l_N). In the same way, (H-S2) forbids the existence of quotient solutions of Nahm's equations.

§2.4. From handsaw quiver varieties to solutions to Nahm's equations

We prove existence of the inverse map. That is, for $[(A, B, i, j)] \in \mathcal{M}_{\mathrm{h}}(\vec{m}, N)$, it is enough to show that for each k, $(A_{k-1}, B_{k-1}, A_k, i_k, j_{k-1})$ has the description given in (2.15).

For a later purpose, we consider the following conditions:

(S1) for each k, if a subspace $S_k \subset V_k$ satisfies $A_k(S_k) \subset S_k$ and $S_k \subset \operatorname{Ker} B_k \cap \operatorname{Ker} j_k$, then $S_k = 0$.

(S2) for each k, if a subspace $T_k \subset V_k$ satisfies $A_k(T_k) \subset T_k$ and $T_k \supset \operatorname{Im} B_{k-1} \cup \operatorname{Im} i_k$, then $T_k = V_k$.

It is clear that for a point (A, B, i, j), these conditions are satisfied if (H-S1) and (H-S2) are. In this subsection, we construct the inverse map by using (S1) and (S2) instead of (H-S1) and (H-S2).

View (A, B, i, j) as $[(A, B, i, j)] \in \mathcal{M}_{\mathrm{h}}(\vec{m}, N)$.

Lemma 2.18. *B_k has full rank for any k.*

Proof. We write a pairing of a vector space V and its dual space V^* as $\langle -, - \rangle_V$. Suppose B_k is not full rank. Then we can take both $0 \neq x \in \operatorname{Ker} B_k$ and $0 \neq y \in \operatorname{Ker} {}^tB_k \subset V_{k+1}^*$. Then we have

$$0 = \langle (A_{k+1}B_k - B_kA_k + i_{k+1}j_k)x, y \rangle_{V_{k+1}} = \langle j_kx, {}^ti_{k+1}y \rangle_{W_{k+1}}.$$

This means either $\operatorname{Ker} B_k \subset \operatorname{Ker} j_k$ or $\operatorname{Ker} {}^tB_k \subset \operatorname{Ker} {}^ti_{k+1}$, because $\dim W_{k+1} = 1$.

In the former case, $B_kA_kx = 0$, so $\operatorname{Ker} B_k$ is A_k-invariant and contained in $\operatorname{Ker} j_k$. We can use (S1) for $S_k = \operatorname{Ker} B_k$, so $\operatorname{Ker} B_k = 0$. This contradicts $x \neq 0$.

In the latter case, $\operatorname{Ker} {}^tB_k$ is ${}^tA_{k+1}$-invariant and $\langle \operatorname{Im} i_{k+1}, \operatorname{Ker} {}^tB_k \rangle = 0$. Set $T_{k+1} = \{v \in V_{k+1} \mid \langle v, \operatorname{Ker} {}^tB_k \rangle = 0\}$. Then, since T_{k+1} satisfies the assumption of (S2), we have $T_{k+1} = V_{k+1}$ and $\operatorname{Ker} {}^tB_k = 0$. This contradicts $y \neq 0$.

Thus either $\operatorname{Ker} B_k = 0$ or $\operatorname{Coker} B_k = 0$. □

We show $(A_{k-1}, B_{k-1}, A_k, i_k, j_{k-1})$ is as described in (2.15) for a certain basis. We consider three cases depending on the relation between m_{k-1} and m_k.

(i) In the case of $m_{k-1} > m_k$, Lemma 2.18 means B_{k-1} is surjective and $\dim \operatorname{Ker} B_{k-1} = m_{k-1} - m_k$. We determine a basis of the subspace $\operatorname{Ker} B_{k-1} \subset V_{k-1}$. Recall that $A_{k-1}\colon \operatorname{Ker} B_{k-1} \cap \operatorname{Ker} j_{k-1} \to \operatorname{Ker} B_{k-1}$ from $\mu_{k-1} = 0$, and A_{k-1} is injective on $\operatorname{Ker} B_{k-1} \cap \operatorname{Ker} j_{k-1}$ because of (S1).

We define a filtration of the vector space $\operatorname{Ker} B_{k-1}$,

$$0 \subset \cdots \subset U_{k-1,2} \subset U_{k-1,1} \subset U_{k-1,0} = \operatorname{Ker} B_{k-1},$$

where $U_{k-1,\kappa} = \operatorname{Ker} B_{k-1} \cap \operatorname{Ker} j_{k-1} \cap \cdots \cap \operatorname{Ker}(j_{k-1}A_{k-1}^{\kappa-1})$ for $\kappa \geq 1$. The inclusions are naturally induced by $U_{k-1,\kappa+1} = U_{k-1,\kappa} \cap \operatorname{Ker}(j_{k-1}A_{k-1}^{\kappa}) \hookrightarrow U_{k-1,\kappa}$. Notice that A_{k-1} also induces injections $A_{k-1}\colon U_{k-1,\kappa+1} \hookrightarrow U_{k-1,\kappa}$, but in general, $A_{k-1}(U_{k-1,\kappa+1})$ does not coincide with $U_{k-1,\kappa+1}$ in $U_{k-1,\kappa}$. Here, we have $\dim U_{k-1,\kappa}/\, U_{k-1,\kappa+1} \leq 1$ from $U_{k-1,\kappa+1} = U_{k-1,\kappa} \cap \operatorname{Ker}(j_{k-1}A_{k-1}^{\kappa})$ and $\dim W_k = 1$. And if $\dim U_{k-1,\kappa}/U_{k-1,\kappa+1} = 0$ for some κ, then $A_{k-1}(U_{k-1,\kappa+1}) \subset U_{k-1,\kappa} = U_{k-1,\kappa+1}$. This means $U_{k-1,\kappa+1}$ satisfies the assumption of (S1), hence we have $U_{k-1,\kappa+1} = U_{k-1,\kappa} = 0$. Thus the above filtration is as follows:

$$0 = U_{k-1,m_{k-1}-m_k} \subsetneq U_{k-1,m_{k-1}-m_k-1} \subsetneq \cdots \subsetneq U_{k-1,1} \subsetneq U_{k-1,0} = \operatorname{Ker} B_{k-1},$$

where $\dim U_{k-1,\kappa} = m_{k-1} - m_k - \kappa$. Furthermore, $U_{k-1,\kappa}$ decomposes as $U_{k-1,\kappa} = A_{k-1}(U_{k-1,\kappa+1}) \oplus U_{k-1,m_{k-1}-m_k-1}$, because if there exists $v \in U_{k-1,\kappa+1}$ such that $A_{k-1}v \in U_{k-1,m_{k-1}-m_k-1}$, then $v \in U_{k-1,m_{k-1}-m_k}$ by definition, but $U_{k-1,m_{k-1}-m_k} = 0$. Thus we have the following lemma:

Lemma 2.19. *$\operatorname{Ker} B_{k-1}$ decomposes as*

$$\begin{aligned}\operatorname{Ker} B_{k-1} = U_{k-1,m_{k-1}-m_k-1} &\oplus A_{k-1}(U_{k-1,m_{k-1}-m_k-1}) \\ &\oplus \cdots \oplus A_{k-1}^{m_{k-1}-m_k-1}(U_{k-1,m_{k-1}-m_k-1}),\end{aligned}$$

and $j_{k-1}|_{\operatorname{Ker} B_{k-1}}$ is $(0,\dots,0,1)$.

Set $U' := \operatorname{Ker} j_{k-1} \cap \operatorname{Ker} j_{k-1}A_{k-1} \cap \cdots \cap \operatorname{Ker}(j_{k-1}\ A_{k-1}^{m_{k-1}-m_k-1}) \subset V_{k-1}$. Then $V_{k-1} = U' \oplus \operatorname{Ker} B_{k-1}$ from $\dim U' \geq m_{k-1} - (m_{k-1} - m_k) = m_k$ and $U' \cap \operatorname{Ker} B_{k-1} = U_{k-1,m_{k-1}-m_k} = 0$. Thus we have a decomposition

$$\begin{aligned}V_{k-1} = U' \oplus U_{k-1,m_{k-1}-m_k-1} &\oplus A_{k-1}(U_{k-1,m_{k-1}-m_k-1}) \\ &\oplus \cdots \oplus A_{k-1}^{m_{k-1}-m_k-1}(U_{k-1,m_{k-1}-m_k-1}).\end{aligned}$$

By identifying V_k as U', we can describe A_{k-1}, B_{k-1} and j_{k-1} as

$$(A_{k-1}, B_{k-1}, j_{k-1}) = \left(\begin{pmatrix} h_k & 0 & g_k \\ f_k & 0 & e_{0,k} \\ 0 & \mathrm{id} & e_k \end{pmatrix}, (\mathrm{id}\ \ 0\ 0), (0\ 0\ 1) \right),$$

and by $\mu_{k-1} = A_k B_{k-1} - B_{k-1}A_{k-1} + i_k j_{k-1} = 0$, we get $A_k = h_k$ and $i_k = g_k$.

(ii) In the case of $m_{k-1} = m_k$, Lemma 2.18 means B_{k-1} is an isomorphism, so we can write $B_{k-1} = \mathrm{id}\colon V_{k-1} \to V_k$. Then from $\mu_{k-1} = 0$, we have $A_k = A_{k-1} - i_k j_{k-1}$.

(iii) In the case of $m_{k-1} < m_k$, Lemma 2.18 means B_{k-1} is injective and $\dim \operatorname{Ker} {}^tB_{k-1} = m_k - m_{k-1}$. We can use Lemma 2.19 for $({}^tA_k, {}^tB_{k-1}, {}^ti_k)$ because of (S2), so we have

$$\begin{aligned}\operatorname{Ker} {}^tB_{k-1} = U^*_{k,m_k-m_{k-1}-1} &\oplus {}^tA_k(U^*_{k,m_k-m_{k-1}-1}) \\ &\oplus \cdots \oplus {}^tA_k^{m_k-m_{k-1}-1}(U^*_{k,m_k-m_{k-1}-1}),\end{aligned}$$

where $U^*_{k,m_k-m_{k-1}-1} = \operatorname{Ker} {}^tB_{k-1} \cap \operatorname{Ker} {}^ti_k \cap \operatorname{Ker} {}^ti_k {}^tA_k \cap \cdots \cap \operatorname{Ker} {}^ti_k {}^tA_k^{m_k-m_{k-1}-2} \subset V_k^*$.

Fix $0 \neq w \in U^*_{k,m_k-m_{k-1}-1}$ and take $v \in \operatorname{Im} i_k$ such that $\langle v, {}^tA_k^{m_k-m_{k-1}-1}w\rangle = 1$. Then we can check that $(\operatorname{Ker} {}^tB_{k-1})^* = \mathbb{C}v \oplus \mathbb{C}(A_k v) \oplus \cdots \oplus \mathbb{C}(A_k^{m_k-m_{k-1}}v)$. Thus we have a decomposition

$$V_k = \operatorname{Im} B_{k-1} \oplus \operatorname{Im} i_k \oplus \operatorname{Im} A_k i_k \oplus \cdots \oplus \operatorname{Im} A_k^{m_k-m_{k-1}} i_k.$$

By identifying V_{k-1} as $\operatorname{Im} B_{k-1}$, we can describe B_{k-1}, A_k and i_k as

$$(B_{k-1}, A_k, i_k) = \left(\begin{pmatrix} \mathrm{id} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} h_k & 0 & g_k \\ f_k & 0 & e_{0,k} \\ c_k & \mathrm{id} & e_k \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right),$$

and from $\mu_{k-1} = A_k B_{k-1} - B_{k-1} A_{k-1} + i_k j_{k-1} = 0$, we get $A_{k-1} = h_k$, $c_k = 0$ and $j_{k-1} = -f_k$.

Thus we have proved the following proposition:

Proposition 2.20. *Suppose* (A, B, i, j) *satisfies* $\mu = 0$ *and conditions* (S1) *and* (S2). *Then*

$$(A_{k-1}, B_{k-1}, A_k, i_k, j_k)$$
$$= \begin{cases} \left((u_k^-)^{-1} \begin{pmatrix} h_k^- & 0 & g_k^- \\ f_k^- & 0 & e_{0,k}^- \\ 0 & \mathrm{id} & e_k^- \end{pmatrix} u_k^-, (\mathrm{id}\ \ 0\ \ 0)u_k^-, h_k^-, g_k, (0\ \ 0\ \ 1)u_k^- \right) \\ \hfill \textit{when } m_{k-1} > m_k, \\ \left((u_k^-)^{-1} \eta_k^- u_k^-, u_k^-, \eta_k^- - v_k w_k, v_k, w_k u_k^- \right) \qquad \textit{when } m_{k-1} = m_k, \\ \left(h_k^+, (u_k^+)^{-1} \begin{pmatrix} \mathrm{id} \\ 0 \\ 0 \end{pmatrix}, (u_k^+)^{-1} \begin{pmatrix} h_k^+ & 0 & g_k^+ \\ f_k^+ & 0 & e_{0,k}^+ \\ 0 & \mathrm{id} & e_k^+ \end{pmatrix} u_k^+, (u_k^+)^{-1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, -f_k \right) \\ \hfill \textit{when } m_{k-1} < m_k, \end{cases}$$

for some $u_k^{\pm}, h_k^{\pm}, f_k^{\pm}, g_k^{\pm}, e_{0,k}^{\pm}, e_k^{\pm}, \eta_k^-, v_k$ *and* w_k.

Proposition 2.20 defines a map from $\mathcal{M}_{\mathrm{h}}(\vec{m}, N)$ to $\mathcal{M}_{\mathrm{N}}((l_1, l_N); \vec{m}, N)$. And this map is just the inverse of that given by Proposition 2.14, so we obtain Theorem 2.11.

The arguments in this section also imply

Corollary 2.21. *For* $(A, B, i, j) \in \bigcap \mu_k^{-1}(0) \subset \mathbb{M}_{\mathrm{h}}$, *conditions* (H-S1) *and* (H-S2) *are equivalent to* (S1) *and* (S2).

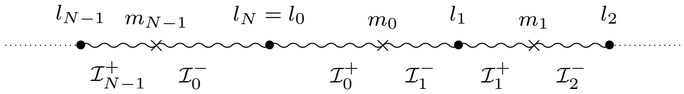
§3. Nahm's equations over the circle and a chainsaw quiver

In this section, we recall the moduli space of solutions of Nahm's equations over the circle and a chainsaw quiver variety from §2. Then we construct a relation between these two spaces.

§3.1. Nahm's equations over the circle

Select $\vec{m} = (m_0, m_1, \ldots, m_{N-1}) \in \mathbb{Z}_{\geq 0}^{\oplus N}$ and $l_0 < l_1 < \cdots < l_N$, and define $\mathcal{I}_k^- = \{(l_{k-1} + l_k)/2 \leq s \leq l_k\}$ and $\mathcal{I}_k^+ = \{l_k \leq s \leq (l_k + l_{k+1})/2\}$. By identifying

$s = l_0$ and $s = l_N$, we regard $[l_0, l_N]/\sim$ as S^1.



By using the moduli space $\mathcal{M}_{\mathrm{N}}(\mathcal{I}_k^- \cup \mathcal{I}_k^+, m_{k-1}, m_k)$, we define the moduli space of solutions of Nahm's equations over the circle $[l_0, l_N]/\sim$ with rank $\vec{m}$ as follows:

$$\mathcal{M}_{\mathrm{N}}(S^1; \vec{m}, N) := \prod_{k=0}^{N-1} \mathcal{M}_{\mathrm{N}}(\mathcal{I}_k^- \cup \mathcal{I}_k^+, m_{k-1}, m_k) /\!/\!/_0 \prod_{k=0}^{N-1} U(m_k).$$

Remark 3.1. When $m_k > 0$ for each k, $\mathcal{M}_{\mathrm{N}}(S^1; \vec{m}, N)$ may depend on the parameter of the hyper-Kähler quotient. In this paper, we only deal with 0-parameter cases as above.

The relation (5) of (0.3) in the Introduction can be written as the following theorem.

Theorem 3.2 ([CH1]). *The framed moduli space of calorons of charge $\vec{m}$ is isomorphic to the moduli space of solutions of Nahm's equations over the circle with rank $\vec{m}$.*

In the same way as Proposition 2.4, we have

$$\mathcal{M}_{\mathrm{N}}(S^1; \vec{m}, N) \cong \prod_{k=0}^{N-1} \mathcal{M}_{\mathrm{N}}(\mathcal{I}_k^- \cup \mathcal{I}_k^+, m_{k-1}, m_k) /\!/ \prod_{k=0}^{N-1} GL(m_k).$$

Here, on the right hand side, we only consider *closed* $\prod GL(m_k)$-orbits (see also [KeN, Nak3, T]). In general, $\mathcal{M}_{\mathrm{N}}(S^1; \vec{m}, N)$ is not smooth. For example, view $(\alpha(s), \beta(s))$ as $[(\alpha(s), \beta(s))] \in \mathcal{M}_{\mathrm{N}}(S^1; \vec{m}, N)$. Then we have

$$\left[\left(\begin{pmatrix} c_0 & 0 \\ 0 & \alpha(s) \end{pmatrix}, \begin{pmatrix} c_1 & 0 \\ 0 & \beta(s) \end{pmatrix}\right)\right] \in \mathcal{M}_{\mathrm{N}}(S^1; \vec{m} + \vec{1}, N) \tag{3.3}$$

for $c_k \in \mathbb{C}$ and $\vec{1} = (1, \ldots, 1)$. Obviously this representative has a stabilizer group $\{(\lambda \oplus \mathrm{id}_{m_0}, \ldots, \lambda \oplus \mathrm{id}_{m_{N-1}}) \mid \lambda \in GL(1)\} \subset \prod GL(m_k + 1)$. We define a regular subset $\mathcal{M}_{\mathrm{N}}^{\mathrm{reg}}(S^1; \vec{m}, N)$ as

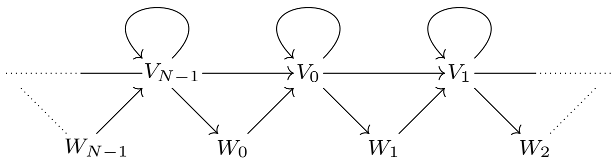
$$\left\{[(\alpha(s), \beta(s))] \in \mathcal{M}_{\mathrm{N}}(S^1; \vec{m}, N) \;\middle|\; \prod_{k=0}^{N-1} GL(m_k) \text{ acts on } (\alpha(s), \beta(s)) \text{ freely}\right\}.$$

If it is nonempty, its dimension is given by

$$\dim_{\mathbb{C}} \mathcal{M}_{\mathrm{N}}^{\mathrm{reg}}(S^1; \vec{m}, N) = \sum_{k=0}^{N-1} 2m_k.$$

§3.2. Chainsaw quiver variety

Take $\vec{m} = (m_0, \dots, m_{N-1}) \in \mathbb{Z}_{\geq 0}^{\oplus N}$. We consider the following diagram:



Here V_k is an m_k-dimensional vector space and W_k is a 1-dimensional vector space. Set

$$\begin{aligned}\mathbb{M}_{\mathrm{c}}(\vec{m}, N) &:= \bigoplus_{k=0}^{N-1} \operatorname{End} V_k \oplus \bigoplus_{k=0}^{N-2} \operatorname{Hom}(V_k, V_{k+1}) \\ &\quad \oplus \bigoplus_{k=0}^{N-1} \operatorname{Hom}(W_k, V_k) \oplus \bigoplus_{k=0}^{N-1} \operatorname{Hom}(V_k, W_{k+1}), \\ \mathcal{G}_{\mathrm{c}}(\vec{m}, N) &:= \prod_{k=0}^{N-1} GL(V_k).\end{aligned}$$

Here "c" means a chainsaw quiver. $\mathcal{G}_{\mathrm{c}}$ acts on $\mathbb{M}_{\mathrm{c}}$ as

$$(g_k) \cdot (A_k, B_k, i_k, j_k) = (g_k A_k g_k^{-1}, g_{k+1} B_k g_k^{-1}, g_k i_k, j_k g_k^{-1}).$$

We define maps $\mu_k \colon \mathbb{M}_{\mathrm{c}} \to \operatorname{Hom}(V_k, V_{k+1})$ for $k = 0, \dots, N-1$ as

$$\mu_k(A, B, i, j) = A_{k+1} B_k - B_k A_k + i_{k+1} j_k.$$

As before, these maps satisfy $\mu_k(g \cdot (A, B, i, j)) = g_{k+1} \mu_k(A, B, i, j) g_k^{-1}$. We consider the following stability conditions:

(C-S1) for a set of subspaces $(S_0, \dots, S_{N-1}) \subset (V_0, \dots, V_{N-1})$, if $A_k(S_k) \subset S_k$, $B_k(S_k) \subset S_{k+1}$, and $\operatorname{Ker} j_k \supset S_k$, then $S_k = 0$.

(C-S2) for a set of subspaces $(T_0, \dots, T_{N-1}) \subset (V_0, \dots, V_{N-1})$, $A_k(T_k) \subset T_k$, $B_k(T_k) \subset T_{k+1}$, and $\operatorname{Im} i_k \subset T_k$, then $T_k = V_k$.

Then we define a *chainsaw quiver variety* $\mathcal{M}_{\mathrm{c}}(\vec{m}, N)$ as

$$\left\{(A, B, i, j) \in \bigcap_{k=0}^{N-1} \mu_k^{-1}(0) \;\middle|\; (A, B, i, j) \text{ satisfies (C-S1) and (C-S2)}\right\} \Big/ \mathcal{G}_{\mathrm{c}}(\vec{m}, N).$$

By the same argument as for Proposition 2.9, we obtain the following proposition:

Proposition 3.4. *All $\mathcal{G}_{\mathrm{c}}$-orbits in $\mathcal{M}_{\mathrm{c}}(\vec{m}, N)$ are closed and their stabilizer groups are trivial.*

By this proposition and the same argument as for Lemma 2.10, $\mathcal{M}_{\mathrm{c}}(\vec{m},N)$ becomes a smooth variety, and its dimension is given by

$$\dim_{\mathbb{C}}\mathcal{M}_{\mathrm{c}}(\vec{m},N)=\sum_{k=0}^{N-1}2m_k.$$

In the next subsection we prove the following analogue of Theorem 2.11 (cf. $(**)$ in (0.3)):

Theorem 3.5. *$\mathcal{M}_{\mathrm{N}}^{\mathrm{reg}}(S^1;\vec{m},N)$ and $\mathcal{M}_{\mathrm{c}}(\vec{m},N)$ are isomorphic as varieties.*

Corollary 3.6. *$\mathcal{M}_{\mathrm{c}}(\vec{m},N)$ has a hyper-Kähler structure.*

§3.3. Solutions of Nahm's equations and chainsaw quiver varieties

In this subsection, we prove Theorem 3.5.

First we have a map from $\mathcal{M}_{\mathrm{c}}(\vec{m},N)$ to $\mathcal{M}_{\mathrm{N}}^{\mathrm{reg}}(S^1;\vec{m},N)$. The correspondence is given by Proposition 2.20 because conditions (S1) and (S2) follow from (C-S1) and (C-S2), and by Proposition 3.4, the image is included in the regular subset.

On the other hand, we define a map from $\mathcal{M}_{\mathrm{N}}^{\mathrm{reg}}(S^1;\vec{m},N)$ to $\mathcal{M}_{\mathrm{c}}(\vec{m},N)$ in the same way as in Proposition 2.14.

Proposition 3.7. *Suppose that $[(A,B,i,j)]\in\bigcap\mu_k^{-1}(0)/\mathcal{G}_{\mathrm{c}}$ is defined from $[(\alpha(s),\beta(s))]\in\mathcal{M}_{\mathrm{N}}^{\mathrm{reg}}(S^1;\vec{m},N)$ as in Proposition 2.14. Then (A,B,i,j) satisfies conditions* (C-S1) *and* (C-S2).

It is clear that this proposition leads to Theorem 3.5. In order to prove this, we need the following lemma.

Lemma 3.8. *Suppose that $(A_{k-1},B_{k-1},A_k,i_k,j_{k-1})$ is given by (2.15). Let S_{k-1} be an A_{k-1}-invariant subspace of V_{k-1} and T_k be an A_k-invariant subspace of V_k.*

(i) *If $S_{k-1}\subset\operatorname{Ker}j_{k-1}$, then $B_{k-1}(S_{k-1})$ is A_k-invariant and $\dim B_{k-1}(S_{k-1})=\dim S_{k-1}$.*
(ii) *If $T_k\supset\operatorname{Im}i_k$, then the inverse image $B_{k-1}^{-1}(T_k)$ is A_{k-1}-invariant and $\operatorname{codim}B_{k-1}^{-1}(T_k)=\operatorname{codim}T_k$.*

Proof. From $A_{k-1}(S_{k-1})\subset S_{k-1}$, $j_{k-1}(S_{k-1})$ and $\mu_{k-1}=0$, we have

$$A_kB_{k-1}(S_{k-1})=B_{k-1}A_{k-1}(S_{k-1})-i_kj_{k-1}(S_{k-1})\subset B_{k-1}(S_{k-1}),$$

and from $A_k(T_k)\subset T_k$, $\operatorname{Im}i_k\subset T_k$ and $\mu_{k-1}=0$, we get

$$B_{k-1}A_{k-1}(B_{k-1}^{-1}(T_k))=A_kB_{k-1}(B_{k-1}^{-1}(T_k))+i_kj_{k-1}(B_{k-1}^{-1}T_k)\subset T_k.$$

Hence both $B_{k-1}(S_{k-1})$ and $B_{k-1}^{-1}(T_k)$ are A-invariant.

When $m_{k-1} \le m_k$, $\dim B_{k-1}(S_{k-1}) = \dim S_{k-1}$ follows from injectivity of B_{k-1}. When $m_{k-1} > m_k$, from the proof of Proposition 2.14 we conclude that $S_{k-1} \cap \operatorname{Ker} B_{k-1} = 0$. Thus $\dim B_{k-1}(S_{k-1}) = \dim S_{k-1}$ for any m_{k-1} and m_k.

When $m_{k-1} \ge m_k$, $\operatorname{codim} B_{k-1}^{-1}(T_k) = \operatorname{codim} T_k$ follows from surjectivity of B_{k-1}. When $m_{k-1} < m_k$, from the proof of Proposition 2.14 we conclude that $T_k \supset (\operatorname{Ker} {}^tB_{k-1})^*$. Thus $\operatorname{codim} B_{k-1}^{-1}(T_k) = \operatorname{codim} T_k$ for any m_{k-1} and m_k. □

Proof of Proposition 3.7. Let $(S_0, \dots, S_{N-1})$ be a set of subspaces satisfying the assumption of (C-S1). Suppose that $\min_k \dim S_k > 0$. Then we can assume without loss of generality that $\dim S_k \ge \dim S_0 > 0$ for any k. Set $S_{0,0} = S_0$ and $S_{k+1,0} = B_k(S_{k,0}) \subset V_{k+1}$ for $k = 0, 1, \dots, N-2$. Then $S_{k,0} \subset S_k \subset \operatorname{Ker} j_k$, $A_k(S_{k,0}) \subset S_{k,0}$ and $\dim S_{k,0} = \dim S_0$ for any k by Lemma 3.8. We define a 1-parameter subgroup of $\prod GL(V_k)$ by

$$g(t) = (g_0(t), g_1(t), \dots, g_{N-1}(t)), \qquad g_k(t) = \begin{pmatrix} t\, \mathrm{id}_{S_{k,0}} & 0 \\ 0 & \mathrm{id}_{S_{k,0}^\perp} \end{pmatrix} \in GL(V_k).$$

Since the assumption $[(\alpha(s), \beta(s))] \in \mathcal{M}_{\mathrm{N}}^{\mathrm{reg}}(S^1; \vec{m}, N)$ means that the $\prod GL(V_k)$-orbit of $(\alpha(s), \beta(s))$ is closed, we have

$$\begin{aligned} \prod GL(V_k) \cdot (\alpha(s), \beta(s)) \ni \lim_{t \to 0} g(t) \cdot (\alpha(s), \beta(s)) \\ = \left(\begin{pmatrix} \alpha'(s) & 0 \\ 0 & \alpha''(s) \end{pmatrix}, \begin{pmatrix} \beta'(s) & 0 \\ 0 & \beta''(s) \end{pmatrix} \right). \end{aligned}$$

But this representative has stabilizer group $GL(1) \subset \prod GL(V_k)$ (see also (3.3) and the following discussion), contrary to the assumption $[(\alpha(s), \beta(s))] \in \mathcal{M}_{\mathrm{N}}^{\mathrm{reg}}(S^1; \vec{m}, N)$.

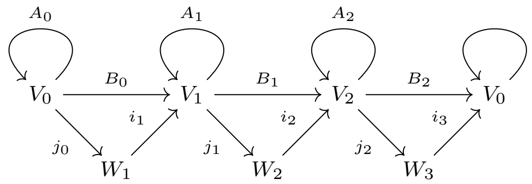
Thus $\min_k \dim S_k = 0$. In this case, $S_k = 0$ for any k from the proof of Proposition 2.14. Therefore we have (C-S1). For (C-S2), we can argue in the same way. □

Corollary 3.9. *For $(A, B, i, j) \in \bigcap \mu_k^{-1}(0) \subset \mathbb{M}_c$, conditions* (C-S1) *and* (C-S2) *are equivalent to* (S1), (S2) *and regularity.*

§4. Chainsaw quiver variety and parabolic sheaf

We consider the relation between a chainsaw quiver variety and the framed moduli space of parabolic sheaves over $\mathbb{P}^1 \times \mathbb{P}^1$. This was originally given by Finkelberg–Rybnikov and Braverman–Finkelberg [FR, BF]. We describe it more explicitly in this section. However it seems hard to understand the procedure for general N.

Thus, for simplicity, we only describe the $N = 3$ case, but we do not use this assumption essentially. Recall the $N = 3$ chainsaw quiver variety:



Here the left V_0 and the right V_0 are identified, and we renamed W_0 as W_3. Set $\vec{m} = (m_0, m_1, m_2) = (\dim V_0, \dim V_1, \dim V_2)$. Recall that the maps μ_k are given by

$$\begin{aligned} \mu_0(A, B, i, j) &= A_1 B_0 - B_0 A_0 + i_1 j_0, \\ \mu_1(A, B, i, j) &= A_2 B_1 - B_1 A_1 + i_2 j_1, \\ \mu_2(A, B, i, j) &= A_0 B_2 - B_2 A_2 + i_3 j_2. \end{aligned}$$

The stability conditions are

(C-S1) for a set of subspaces $(S_0, S_1, S_2) \subset (V_0, V_1, V_2)$, if $A_k(S_k) \subset S_k$, $B_k(S_k) \subset S_{k+1}$, and $\operatorname{Ker} j_k \supset S_k$, then $S_k = 0$.

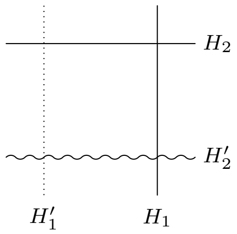
(C-S2) for a set of subspaces $(T_0, T_1, T_2) \subset (V_0, V_1, V_2)$, if $A_k(T_k) \subset T_k$, $B_k(T_k) \subset T_{k+1}$, and $\operatorname{Im} i_k \subset T_k$, then $T_k = V_k$.

This chainsaw quiver variety is written as $\mathcal{M}_{\mathrm{c}}(\vec{m}, 3)$.

§4.1. Parabolic sheaf

In this subsection, we define a parabolic sheaf over $\mathbb{P}^1 \times \mathbb{P}^1$. First we fix notation for $\mathbb{P}^1 \times \mathbb{P}^1$ and its divisors.

Let $[x_0 : x_1]$ be the homogeneous coordinate on $\mathbb{P}^1$. For $\mathbb{P}^1 \times \mathbb{P}^1 = \{([x_0 : x_1], [y_0 : y_1])\}$, we write divisors as follows:



$$\begin{aligned} H_1 &:= \{x_0 = 0\} = \{([0:1], [y_0 : y_1])\}, \\ H_1' &:= \{x_1 = 0\} = \{([1:0], [y_0 : y_1])\}, \\ H_2 &:= \{y_0 = 0\} = \{([x_0 : x_1], [0:1])\}, \\ H_2' &:= \{y_1 = 0\} = \{([x_0 : x_1], [1:0])\}. \end{aligned}$$

Clearly, H_k and H_k' are linearly equivalent. The intersection numbers are given by $H_1^2 = H_2^2 = 0$ and $H_1 \cdot H_2 = 1$. For a sheaf $\mathfrak{F}$ over $\mathbb{P}^1 \times \mathbb{P}^1$, we write $\mathfrak{F}(p, q)$ for $\mathfrak{F} \otimes \mathcal{O}(pH_1 + qH_2)$. Moreover, sometimes we regard a vector space V as a locally free sheaf $V \otimes \mathcal{O}$.

Definition 4.1. Let $n \in \mathbb{Z}_{>0}$ and $\vec{d} = (d_0, d_1, \dots, d_{n-1}) \in \mathbb{Z}_{\geq 0}^{\oplus n}$, and set $\mathcal{W} := \mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_n$, where $\mathcal{W}_k \cong \mathcal{O}_{\mathbb{P}^1}$ for each k. Let $\{\mathfrak{F}_\bullet\}$ be an infinite flag of locally free sheaves over $\mathbb{P}^1 \times \mathbb{P}^1$: $\cdots \subset \mathfrak{F}_{-1} \subset \mathfrak{F}_0 \subset \mathfrak{F}_1 \subset \cdots$.

$(\{\mathfrak{F}_\bullet\}, \Phi_1, \Phi_2)$ is called an (H_1, H_2)-*framed locally free parabolic sheaf* of rank n and degree $\vec{d}$ with respect to H_2' if:

1. $\mathfrak{F}_{k-n} = \mathfrak{F}_k(-H_2')$ for any $k \in \mathbb{Z}$;
2. $\mathrm{ch}_1(\mathfrak{F}_k) = k[H_2']$ for any $k \in \mathbb{Z}$;
3. $\mathrm{ch}_2(\mathfrak{F}_k) = -d_i[H_1] \wedge [H_2]$ for $i \equiv k \pmod n$;
4. $\mathfrak{F}_0/\mathfrak{F}_k$ and $\mathfrak{F}_k/\mathfrak{F}_{-n}$ are supported on H_2' for $-n \leq k \leq 0$;
5. $\Phi_1 : \mathfrak{F}_k|_{H_1} \to \mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_{n+k} \oplus \mathcal{W}_{n+k+1}(-1) \oplus \cdots \oplus \mathcal{W}_n(-1)$ is an isomorphism for $-n \leq k \leq 0$;
6. $\Phi_2 : \mathfrak{F}_k|_{H_2} \to \mathcal{W}$ is an isomorphism for $-n \leq k \leq 0$ and $\Phi_2|_{H_1} = \Phi_1|_{H_2}$ on $H_1 \cap H_2$,

where $[-]\colon H_2(\mathbb{P}^1 \times \mathbb{P}^1) \to H^2(\mathbb{P}^1 \times \mathbb{P}^1)$ is the Poincaré dual.

We define $\mathcal{P}(\vec{d}, n)$ as the moduli space of framed locally free parabolic sheaves of rank n and degree $\vec{d}$ defined in Definition 4.1. This moduli space is often called the framed moduli space of locally free parabolic sheaves.

Theorem 4.2. *The chainsaw quiver variety $\mathcal{M}_{\mathrm{c}}(\vec{m}, N)$ is isomorphic to the framed moduli space $\mathcal{P}(\vec{m}, N)$ of locally free parabolic sheaves.*

We prove this theorem in the following two subsections in a way different from [FR, BF]. In §4.2, we construct a map from a chainsaw quiver variety to a framed parabolic sheaf, and in §4.3 we give the inverse map.

Remark 4.3. In [FR, BF], the authors consider torsion free parabolic sheaves (as opposed to locally free). Their moduli space is denoted by $\mathcal{P}_{\underline{d}}$ with $\underline{d} = \vec{d}$.

Conditions (C-S1) and (C-S2) make the corresponding parabolic sheaves locally free by Proposition 4.5. See also Remark 4.6 and Theorem 4.25.

The behavior of parabolic sheaves on H_2' is studied in §4.4.

The isomorphism $\mathbb{P}^1 \times \mathbb{P}^1 \setminus H_1 \cup H_2 \cup H_2' \cong \mathbb{R}^3 \times S^1$ may support the suggestion that the relation (7) holds in (0.3).

§4.2. Chainsaw quiver to parabolic sheaf

In this section, we construct a framed locally free parabolic sheaf over $\mathbb{P}^1 \times \mathbb{P}^1$ from a point of a chainsaw variety.

4.2.1. Monad. We define the following diagram and maps by using the terms of $\mathbb{M}_{\mathrm{c}}(\vec{m}, 3)$:

(4.4)

$$\begin{array}{ccccc}
\mathfrak{F}_0\colon V_0(-1,0) & \xrightarrow{\quad\alpha_0\quad} & \begin{matrix} V_0(-1,1)\oplus V_0 \\ W_1\oplus W_2\oplus W_3 \end{matrix} & \xrightarrow{\quad\beta_0\quad} & V_0(0,1) \\
\Big\uparrow{\scriptstyle f_0=B_2} & & \Big\uparrow{\scriptstyle g_0=\begin{pmatrix} B_2 & 0 & 0 & 0 \\ 0 & B_2 & i_3 & 0 \\ 0 & 0 & 0 & \mathrm{id}_{123} \end{pmatrix}} & & \Big\uparrow{\scriptstyle h_0=\begin{pmatrix} B_2 & i_3 \end{pmatrix}} \\
\mathfrak{F}_{-1}\colon V_2(-1,0) & \xrightarrow{\quad\alpha_{-1}\quad} & \begin{matrix} V_2(-1,1)\oplus V_2 \\ W_3\oplus W_1\oplus W_2\oplus W_3 \end{matrix} & \xrightarrow{\quad\beta_{-1}\quad} & (V_2\oplus W_3)(0,1) \\
\Big\uparrow{\scriptstyle f_{-1}=B_1} & & \Big\uparrow{\scriptstyle g_{-1}=\begin{pmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_1 & i_2 & 0 \\ 0 & 0 & 0 & \mathrm{id}_{3123} \end{pmatrix}} & & \Big\uparrow{\scriptstyle h_{-1}=\begin{pmatrix} B_1 & i_2 & 0 \\ 0 & 0 & \mathrm{id}_3 \end{pmatrix}} \\
\mathfrak{F}_{-2}\colon V_1(-1,0) & \xrightarrow{\quad\alpha_{-2}\quad} & \begin{matrix} V_1(-1,1)\oplus V_1 \\ W_2\oplus W_3\oplus W_1\oplus W_2\oplus W_3 \end{matrix} & \xrightarrow{\quad\beta_{-2}\quad} & (V_1\oplus W_2\oplus W_3)(0,1) \\
\Big\uparrow{\scriptstyle f_{-2}=B_0} & & \Big\uparrow{\scriptstyle g_{-2}=\begin{pmatrix} B_0 & 0 & 0 & 0 \\ 0 & B_0 & i_1 & 0 \\ 0 & 0 & 0 & \mathrm{id}_{23123} \end{pmatrix}} & & \Big\uparrow{\scriptstyle h_{-2}=\begin{pmatrix} B_0 & i_1 & 0 \\ 0 & 0 & \mathrm{id}_{23} \end{pmatrix}} \\
\mathfrak{F}_{-3}\colon V_0(-1,0) & \xrightarrow{\quad\alpha_{-3}\quad} & \begin{matrix} V_0(-1,1)\oplus V_0 \\ W_1\oplus W_2\oplus W_3\oplus W_1\oplus W_2\oplus W_3 \end{matrix} & \xrightarrow{\quad\beta_{-3}\quad} & (V_0\oplus W_1\oplus W_2\oplus W_3)(0,1)
\end{array}$$

where id_{123} is the identity map for $W_1 \oplus W_2 \oplus W_3$. Here α_k and β_k are defined as

$$\alpha_0 = \begin{pmatrix} y_1 - y_0 B_{210} \\ x_1 - x_0 A_0 \\ x_0 j_0 \\ x_0 j_1 B_0 \\ x_0 j_2 B_{10} \end{pmatrix}, \quad \beta_0 = (x_1 - x_0 A_0 \quad -y_1 + y_0 B_{210} \quad y_0 B_{21} i_1 \quad y_0 B_2 i_2 \quad y_0 i_3),$$

$$\alpha_{-1} = \begin{pmatrix} y_1 - y_0 B_{102} \\ x_1 - x_0 A_2 \\ x_0 j_2 \\ x_0 j_0 B_2 \\ x_0 j_1 B_{02} \\ x_0 j_2 B_{102} \end{pmatrix}, \quad \beta_{-1} = \begin{pmatrix} x_1 - x_0 A_2 & -y_1 + y_0 B_{102} & y_0 B_{10} i_3 & y_0 B_1 i_1 & y_0 i_2 & 0 \\ x_0 j_2 & 0 & -y_1 & 0 & 0 & y_0 \end{pmatrix},$$

$$\alpha_{-2} = \begin{pmatrix} y_1 - y_0 B_{021} \\ x_1 - x_0 A_1 \\ x_0 j_1 \\ x_0 j_2 B_1 \\ x_0 j_0 B_{21} \\ x_0 j_1 B_{021} \\ x_0 j_2 B_{1021} \end{pmatrix}, \quad \beta_{-2} = \begin{pmatrix} x_1 - x_0 A_1 & -y_1 + y_0 B_{021} & y_0 B_{02} i_2 & y_0 B_0 i_3 & y_0 i_1 & 0 & 0 \\ x_0 j_1 & 0 & -y_1 & 0 & 0 & y_0 & 0 \\ x_0 j_2 B_1 & 0 & 0 & -y_1 & 0 & 0 & y_0 \end{pmatrix},$$

$$\alpha_{-3} = \begin{pmatrix} y_1 - y_0 B_{210} \\ x_1 - x_0 A_0 \\ x_0 j_0 \\ x_0 j_1 B_0 \\ x_0 j_2 B_{10} \\ x_0 j_0 B_{210} \\ x_0 j_1 B_{0210} \\ x_0 j_2 B_{10210} \end{pmatrix}, \quad \beta_{-3} = \begin{pmatrix} x_1 - x_0 A_0 & -y_1 + y_0 B_{210} & y_0 B_{21} i_1 & y_0 B_2 i_2 & y_0 i_3 & 0 & 0 & 0 \\ x_0 j_0 & 0 & -y_1 & 0 & 0 & y_0 & 0 & 0 \\ x_0 j_1 B_0 & 0 & 0 & -y_1 & 0 & 0 & y_0 & 0 \\ x_0 j_2 B_{10} & 0 & 0 & 0 & -y_1 & 0 & 0 & y_0 \end{pmatrix},$$

where $B_{210} = B_2 B_1 B_0 : V_0 \to V_0$. These maps are essentially given in [CH2]. When $(A, B, i, j) \in \bigcap \mu_k^{-1}(0)$, it follows that $\beta_k \alpha_k = 0$ and each square commutes (i.e. $\alpha_0 f_0 = g_0 \alpha_{-1}$ etc.).

Form now on, we assume $[(A, B, i, j)] \in \mathcal{M}_{\mathrm{c}}(\vec{m}, 3)$. We define $\mathfrak{F}_k$ as the middle cohomology of the $(k+1)$th row of the complex in (4.4), that is, $\mathfrak{F}_k =$

$\operatorname{Ker}\beta_k/\operatorname{Im}\alpha_k$. First we see that $\mathfrak{F}_k$ is a rank 3 locally free sheaf. This follows from the next proposition.

Proposition 4.5. *Let* $(A, B, i, j) \in \bigcap \mu_k^{-1}(0)$.

(i) α_k *is injective if and only if* (A, B, i, j) *satisfies condition* (C-S1).

(ii) β_k *is surjective if and only if* (A, B, i, j) *satisfies condition* (C-S2).

Proof. First we prove the "if" parts. When $x_0 = 0$ or $y_0 = 0$, both assertions are clear. So we prove these assertions on the fiber over $\{([1 : x_1], [1 : y_1])\}$.

(i) Suppose $\alpha_0(v_0) = 0$. We get $B_2B_1B_0v_0 = y_1v_0$, $A_0v_0 = x_1v_0$, $j_0v_0 = 0$, $j_1B_0v_0 = 0$ and $j_2B_1B_0v_0 = 0$. Set

$$S_0 = \mathbb{C}v_0, \quad S_1 = B_0(S_0), \quad S_2 = B_1(S_1).$$

Then $A_0(S_0) \subset S_0$. Further, $\mu_0(v_0) = 0$ and $\mu_1(v_0) = 0$ imply $A_1(S_1) \subset S_1$ and $A_2(S_2) \subset S_2$. Thus (S_0, S_1, S_2) satisfies the assumption of (C-S1), so we have $S_0 = 0$, $v_0 = 0$ and injectivity of α_0. For α_k $(-3 \le k \le -1)$, the same argument is valid.

(ii) Surjectivity of $\beta_0 \colon V_0 \oplus V_0 \oplus W_1 \oplus W_2 \oplus W_3 \to V_0$ is equivalent to injectivity of ${}^t\beta_0 \colon V_0^* \to V_0^* \oplus V_0^* \oplus W_1^* \oplus W_2^* \oplus W_3^*$. Here it is easy to see that "$(S_0', S_1', S_2') \subset (V_0^*, V_1^*, V_2^*)$ with ${}^tA_k(S_k') \subset S_k'$, ${}^tB_{k-1}(S_k') \subset S_{k-1}'$ and $\operatorname{Ker}{}^ti_k \supset S_k'$" is equivalent to "$({S_0'}^{\perp}, {S_1'}^{\perp}, {S_2'}^{\perp}) \subset (V_0, V_1, V_2)$ with $A_k({S_k'}^{\perp}) \subset {S_k'}^{\perp}$, $B_{k-1}\,({S_{k-1}'}^{\perp}) \subset {S_k'}^{\perp}$ and $\operatorname{Im} i_k \subset {S_k'}^{\perp}$", where $S^{\perp} = \{v \in V \mid \langle v, S\rangle = 0\}$ for $S \subset V^*$. Thus by the above argument, injectivity of ${}^t\beta_0$ follows from (C-S2).

For β_k $(-3 \le k \le -1)$, a little more argument is needed. Since β_{-1} is surjective onto V_2, for any $v_2 \in V_2$ there exists $(v_2', v_2'', w_3', w_1, w_2, 0) \in V_2 \oplus V_2 \oplus W_3 \oplus W_1 \oplus W_2 \oplus W_3$ mapped by β_{-1} to v_2. Then for any $w_3 \in W_3$, we have $\beta_{-1}(v_2', v_2'', w_3', w_1, w_2, w_3 - j_2v_2' + y_1w_3') = (v_2, w_3)$. Thus β_k is also surjective for $-3 \le k \le -1$.

Next we prove the "only if" parts. Suppose that (C-S1) is not satisfied. Then we have a set of subspaces (S_0, S_1, S_2) such that

$$A_k(S_k) \subset S_k, \quad B_k(S_k) \subset S_{k+1}, \quad S_k \subset \operatorname{Ker} j_k, \quad (S_0, S_1, S_2) \ne (0, 0, 0).$$

Assume $S_0 \ne 0$. Then $A_0B_2B_1B_0|_{S_0} - B_2B_1B_0A_0|_{S_0} = 0$, so there exists $0 \ne v_0 \in S_0$ such that

$$A_0v_0 = \lambda_1v_0, \quad B_2B_1B_0v_0 = \lambda_2v_0, \quad \text{for some } (\lambda_1, \lambda_2) \in \mathbb{C}^2.$$

Then $\alpha_0(v_0) = 0$ at $([1 : \lambda_1], [1 : \lambda_2])$, so α_0 is not injective. In the same way, $S_1 \ne 0$ and $S_2 \ne 0$ imply noninjectivity of α_1 and α_2 respectively.

In the same way we can check that surjectivity of β_k implies (C-S2). □

This proposition means that when (A, B, i, j) satisfies (C-S1), the image of α_k defines a subbundle, so $\mathfrak{F}_k$ becomes a locally free sheaf.

Remark 4.6. Even if we do not assume (C-S1), the above proof implies that α_k becomes injective as a sheaf homomorphism. In this case, $\mathfrak{F}_k$ becomes a torsion free sheaf (see also [Nak3, Chapter 2]).

Next, we consider the map $\tilde{g}_k \colon \mathfrak{F}_{k-1} \to \mathfrak{F}_k$ induced by (f_k, g_k, h_k).

Lemma 4.7. *$\tilde{g}_k$ is an isomorphism on every fiber over $\mathbb{P}^1 \times \mathbb{P}^1 \setminus H_2'$.*

Proof. We consider injectivity of $\tilde{g}_0$. Take $(v_2, v_2', w_3', w_1, w_2, w_3) \in \operatorname{Ker} \beta_{-1}$ and suppose that there exists $v_0 \in V_0$ such that $\alpha_0(v_0) = g_0(v_2, v_2', w_3', w_1, w_2, w_3)$. Then we get $\alpha_{-1}(v_2 + y_0 B_1 B_0 v_0) = y_1 \cdot (v_2,\, v_2', w_3', w_1, w_2, w_3)$ by the following calculation. $(v_2, v_2', w_3', w_1, w_2, w_3) \in \operatorname{Ker} \beta_{-1}$ means

$$\begin{cases} (x_1 - x_0 A_2)v_2 + (-y_1 + y_0 B_{102})v_2' + y_0 B_{10} i_3 w_3' + y_0 B_1 i_1 w_1 + y_0 i_2 w_2 = 0, \\ x_0 j_2 v_2 - y_1 w_3' + y_0 w_3 = 0, \end{cases} \tag{4.8}$$

and $\alpha_0(v_0) = g_0(v_2, v_2', w_3', w_1, w_2, w_3)$ means

$$\begin{pmatrix} y_1 v_0 - y_0 B_{210} v_0 \\ x_1 v_0 - x_0 A_0 v_0 \\ x_0 j_0 v_0 \\ x_0 j_1 B_0 v_0 \\ x_0 j_2 B_{10} v_0 \end{pmatrix} = \begin{pmatrix} B_2 v_2 \\ B_2 v_2' + i_3 w_3' \\ w_1 \\ w_2 \\ w_3 \end{pmatrix}. \tag{4.9}$$

From (4.8), (4.9), $\mu_0 = 0$ and $\mu_1 = 0$, we have

$$\begin{aligned} (x_1 - x_0 A_2) v_2 &= y_1 v_2' - y_0 B_{10}(B_2 v_2' + i_3 w_3') - y_0 B_1 i_1 w_1 - y_0 i_2 w_2 \\ &= y_1 v_2' - y_0 B_{10}(x_1 v_0 - x_0 A_0 v_0) - x_0 y_0 B_1 i_1 j_0 v_0 - x_0 y_0 i_2 j_1 B_0 v_0 \\ &= y_1 v_2' - x_1 y_0 B_{10} v_0 + x_0 y_0 A_2 B_{10} v_0. \end{aligned}$$

By using these equalities, we obtain

$$\begin{aligned} \alpha_{-1}(v_2 + y_0 B_{10} v_0) &= \begin{pmatrix} (y_1 - y_0 B_{102})(v_2 + y_0 B_{10} v_0) \\ (x_1 - x_0 A_2)(v_2 + y_0 B_{10} v_0) \\ x_0 j_2 (v_2 + y_0 B_{10} v_0) \\ x_0 j_0 B_2 (v_2 + y_0 B_{10} v_0) \\ x_0 j_1 B_{02} (v_2 + y_0 B_{10} v_0) \\ x_0 j_2 B_{102} (v_2 + y_0 B_{10} v_0) \end{pmatrix} \\ &= \begin{pmatrix} y_1 v_2 - y_0 B_{10}(y_1 v_0 - y_0 B_{210} v_0) + y_0 B_{10}(y_1 - y_0 B_{210}) v_0 \\ y_1 v_2' - x_1 y_0 B_{10} v_0 + x_0 y_0 A_2 B_{10} v_0 + y_0 (x_1 - x_0 A_2) B_{10} v_0 \\ y_1 w_3' - y_0 w_3 + y_0 x_0 j_2 B_{10} v_0 \\ x_0 j_0 (y_1 v_0 - y_0 B_{210} v_0) + x_0 y_0 j_0 B_{210} v_0 \\ x_0 j_1 B_0 (y_1 v_0 - y_0 B_{210} v_0) + x_0 y_0 j_1 B_{0210} v_0 \\ x_0 j_2 B_{10} (y_1 v_0 - y_0 B_{210} v_0) + x_0 y_0 j_2 B_{10210} v_0 \end{pmatrix} = \begin{pmatrix} y_1 v_2 \\ y_1 v_2' \\ y_1 w_3' \\ y_1 w_1 \\ y_1 w_2 \\ y_1 w_3 \end{pmatrix}. \end{aligned}$$

This means that when $y_1 \neq 0$, $\tilde{g}_0$ is injective and surjective because the dimensions of the fibers of $\mathfrak{F}_0$ and $\mathfrak{F}_{-1}$ are same. For $\tilde{g}_k$ $(k = -1, -2)$, the same argument is valid. □

This lemma means that $\tilde{g}_k$ is injective as a sheaf homomorphism. Thus we have a finite flag of sheaves $\mathfrak{F}_{-3} \subset \mathfrak{F}_{-2} \subset \mathfrak{F}_{-1} \subset \mathfrak{F}_0$.

Proposition 4.10. *$\mathfrak{F}_{-N}$ and $\mathfrak{F}_0(-H_2')$ are isomorphic.*

By using this proposition, we can inductively define infinitely many sheaves $\{\mathfrak{F}_k \mid k \in \mathbb{Z}\}$ by

$$\mathfrak{F}_k := \mathfrak{F}_{k-3}(H_2') \quad \text{for any } k \in \mathbb{Z}.$$

Thus, from the finite flag, we have the following infinite flag of sheaves:

$$\begin{array}{cccccccccc} \cdots & \mathfrak{F}_{-4} & & \mathfrak{F}_{-3} & \subset \mathfrak{F}_{-2} \subset & \mathfrak{F}_{-1} & \subset & \mathfrak{F}_0 & \mathfrak{F}_1 & \cdots \\ & \| & & \| & & & & \| & \| & \\ \cdots & \mathfrak{F}_{-1}(-H_2') & \subset & \mathfrak{F}_0(-H_2') & & & & \mathfrak{F}_{-3}(H_2') & \subset \mathfrak{F}_{-2}(H_2') & \cdots \end{array} \tag{4.11}$$

In order to prove Proposition 4.10, we need the next lemma:

Lemma 4.12. *$\tilde{g}_0 \cdots \tilde{g}_{-N+1}$ is the 0-map on H_2'.*

Proof. On $H_2' = \{y_1 = 0\}$, we have

$$\begin{aligned} \begin{pmatrix} 0 \\ \beta_{-3}(v_0, v_0', w_1', w_2', w_3', w_1, w_2, w_3) \end{pmatrix} &= \begin{pmatrix} -B_{210}v_0 + B_{210}v_0 \\ (x_1 - x_0A_0)v_0 + B_{210}v_0' + B_{21}i_1w_1' + B_2i_2w_2' + i_3w_3' \\ x_0j_0v_0 + w_1 \\ x_0j_1B_0v_0 + w_2 \\ x_0j_2B_{10}v_0 + w_3 \end{pmatrix} \\ &= \alpha_0(v_0) + g_0g_{-1}g_{-2}(v_0, v_0', w_1', w_2', w_3', w_1, w_2, w_3). \end{aligned}$$

Thus we obtain $\tilde{g}_0\tilde{g}_{-1}\tilde{g}_{-2}|_{H_2'}([(v_0, v_0', w_1', w_2', w_3', w_1, w_2, w_3)]) = 0 \in \mathfrak{F}_0$. □

Proof of Proposition 4.10. We consider the following two short exact sequences:

$$0 \to \mathfrak{F}_0(-H_2') \xrightarrow{y_1} \mathfrak{F}_0 \to Q \to 0,$$
$$0 \to \mathfrak{F}_{-3} \xrightarrow{\tilde{g}_0\tilde{g}_{-1}\tilde{g}_{-2}} \mathfrak{F}_0 \to Q' \to 0,$$

where Q and Q' are quotient sheaves. Obviously $Q \cong \mathfrak{F}_0|_{H_2'}$. From Lemma 4.7, Q' is supported on H_2', and from Lemma 4.12, $Q' \cong \mathfrak{F}_0|_{H_2'}$. Thus $Q \cong Q'$ and hence there exists an isomorphism between $\mathfrak{F}_{-3}$ and $\mathfrak{F}_0(-H_2')$. □

4.2.2. Proof of Theorem 4.2, from $\mathcal{M}_c(\vec{m}, N)$ to $\mathcal{P}(\vec{m}, N)$. We check that the infinite flag $\{\mathfrak{F}_\bullet\}$ of sheaves constructed by (4.4) and (4.11) satisfies the six conditions of Definition 4.1.

The first condition is satisfied by the definition of the infinite flag (4.11), and the fourth condition follows from Lemma 4.7. By calculation, we can check the second and third conditions; for example, we have

$$\begin{aligned}\mathrm{ch}(\mathfrak{F}_{-1}) &= \mathrm{ch}(V_2(-1,1) \oplus V_2 \oplus W_3 \oplus W_1 \oplus W_2 \oplus W_3) \\ &\quad - \mathrm{ch}(V_2(-1,0)) - \mathrm{ch}((V_2 \oplus W_3)(0,1)) \\ &= m_2 + 4 + m_2(1 - [H_1] + [H_2] - [H_1] \wedge [H_2]) \\ &\quad - m_2(1 - [H_1]) - (m_2+1)(1 + [H_2]) \\ &= 3 - [H_2] - m_2[H_1] \wedge [H_2].\end{aligned}$$

Here we have used the intersection numbers of H_1 and H_2. We check the fifth and sixth conditions:

Lemma 4.13. *For $-N \leq k \leq 0$, we have*

$$\begin{aligned}\mathfrak{F}_k|_{H_1} &\cong (W_1 \oplus \cdots \oplus W_{N+k}) \otimes \mathcal{O}_{\mathbb{P}^1} \oplus (W_{N+k+1} \oplus \cdots \oplus W_N) \otimes \mathcal{O}_{\mathbb{P}^1}(-1), \\ \mathfrak{F}_k|_{H_2} &\cong (W_1 \oplus \cdots \oplus W_N) \otimes \mathcal{O}_{\mathbb{P}^1}.\end{aligned}$$

Proof. For example, we check the assertions for $\mathfrak{F}_{-1}$. Restriction to H_2 means $y_0 = 0$, so we have

$$\alpha_{-1}|_{H_2} = \begin{pmatrix} \mathrm{id} \\ x_1 - x_0 A_2 \\ x_0 j_2 \\ x_0 j_0 B_2 \\ x_0 j_1 B_{02} \\ x_0 j_2 B_{102} \end{pmatrix}, \quad \beta_{-1}|_{H_2} = \begin{pmatrix} x_1 - x_0 A_2 & -\mathrm{id} & 0 & 0 & 0 & 0 \\ x_0 j_2 & 0 & -\mathrm{id} & 0 & 0 & 0 \end{pmatrix}.$$

This implies

$$\begin{aligned}\mathfrak{F}_{-1}|_{H_2} &= H^0\big(V_2 \otimes \mathcal{O}_{H_2}(-1) \xrightarrow{\alpha_{-1}|_{H_2}} V_2 \otimes \mathcal{O}_{H_2}(-1) \oplus V_2 \oplus W_3 \oplus W_1 \oplus W_2 \oplus W_3 \\ &\qquad \xrightarrow{\beta_{-1}|_{H_2}} V_2 \oplus W_3\big) \\ &\cong (W_1 \oplus W_2 \oplus W_3) \otimes \mathcal{O}_{\mathbb{P}^1}.\end{aligned}$$

And restriction to H_1 means $x_0 = 0$, so we have

$$\begin{aligned}\alpha_{-1}|_{H_1} &= \begin{pmatrix} y_1 - y_0 B_{102} \\ \mathrm{id} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \\ \beta_{-1}|_{H_1} &= \begin{pmatrix} \mathrm{id} & -y_1 + y_0 B_{102} & y_0 B_{10} i_3 & y_0 B_1 i_1 & y_0 i_2 & 0 \\ 0 & 0 & -y_1 & 0 & 0 & y_0 \end{pmatrix}.\end{aligned}$$

This implies

$$\begin{aligned}\mathfrak{F}_{-1}|_{H_1} &= H^0\big(V_3 \xrightarrow{\alpha_{-1}|_{H_1}} V_3 \otimes \mathcal{O}_{H_1}(1) \oplus V_3 \oplus W_3 \oplus W_1 \oplus W_2 \oplus W_3 \\ &\qquad\qquad \xrightarrow{\beta_{-1}|_{H_1}} (V_3 \oplus W_3) \otimes \mathcal{O}_{H_1}(1)\big) \\ &\cong \operatorname{Ker}\big((-y_1, 0, 0, y_0)\colon W_3 \oplus W_1 \oplus W_2 \oplus W_3 \to W_3 \otimes \mathcal{O}_{H_1}(1)\big) \\ &\cong (W_1 \oplus W_2) \otimes \mathcal{O}_{\mathbb{P}^1} \oplus W_3 \otimes \mathcal{O}_{\mathbb{P}^1}(-1),\end{aligned}$$

where we have used $\operatorname{Ker}\{(-y_1, y_0)\colon \mathcal{O}^{\oplus 2} \to \mathcal{O}(1)\} \cong \mathcal{O}(-1)$. □

It is clear that the above isomorphisms satisfy $\Phi_1(\mathfrak{F}_k|_{H_1})|_{H_2} = \Phi_2(\mathfrak{F}_k|_{H_2})|_{H_1}$.

Thus $\{\mathfrak{F}_\bullet\}$ satisfies the conditions of Definition 4.1 for $\mathcal{W}_k = W_k \otimes \mathcal{O}_{\mathbb{P}^1}$. This gives a map from $\mathcal{M}_{\mathrm{c}}(\vec{m}, N)$ to $\mathcal{P}(\vec{m}, N)$.

§4.3. Parabolic sheaf to chainsaw quiver

Next, conversely, we construct a point of a chainsaw quiver variety from a framed locally free parabolic sheaf. We use the idea of Nakajima [Nak3].

4.3.1. Resolution of the diagonal. Let $p_i\colon (\mathbb{P}^1 \times \mathbb{P}^1) \times (\mathbb{P}^1 \times \mathbb{P}^1) \to \mathbb{P}^1 \times \mathbb{P}^1$ be the projection to the ith factor, and $\Delta_{\mathbb{P}^1 \times \mathbb{P}^1}$ be the diagonal in $(\mathbb{P}^1 \times \mathbb{P}^1) \times (\mathbb{P}^1 \times \mathbb{P}^1)$, that is,

$$([x_0' : x_1'], [y_0' : y_1'], [x_0 : x_1], [y_0 : y_1]) \begin{array}{l} \xmapsto{p_1} ([x_0' : x_1'], [y_0' : y_1']), \\ \xmapsto{p_2} ([x_0 : x_1], [y_0 : y_1]), \end{array}$$

$$\Delta_{\mathbb{P}^1 \times \mathbb{P}^1} = \{([x_0 : x_1], [y_0 : y_1], [x_0 : x_1], [y_0 : y_1])\}.$$

Set $\mathcal{O}(p, q) \boxtimes \mathcal{O}(r, s) := p_1^*\mathcal{O}(p, q) \otimes p_2^*\mathcal{O}(r, s)$. We consider the complex

$$C^\bullet\colon \mathcal{O}(-1,-1) \boxtimes \mathcal{O}(-1,-1) \xrightarrow{d^{-1}} \begin{matrix} \mathcal{O}(-1,0) \boxtimes \mathcal{O}(-1,0) \\ \mathcal{O}(0,-1) \boxtimes \mathcal{O}(0,-1) \end{matrix} \xrightarrow{d^0} \mathcal{O}(0,0) \boxtimes \mathcal{O}(0,0)$$

with

$$d^{-1} = \begin{pmatrix} y_0' \boxtimes y_1 - y_1' \boxtimes y_0 \\ x_0' \boxtimes x_1 - x_1' \boxtimes x_0 \end{pmatrix}, \qquad d^0 = (x_0' \boxtimes x_1 - x_1' \boxtimes x_0 \quad -y_0' \boxtimes y_1 + y_1' \boxtimes y_0).$$

Proposition 4.14. *The complex*

$$0 \to C^{-1} \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} C^1 \xrightarrow{t} \mathcal{O}_{\Delta_{\mathbb{P}^1 \times \mathbb{P}^1}} \to 0$$

gives a resolution of $\mathcal{O}_{\Delta_{\mathbb{P}^1 \times \mathbb{P}^1}}$.

Proof. It is easy to check that d^{-1} and $(d^0)^\vee\colon {C^1}^\vee \to {C^0}^\vee$ are injective on $(\mathbb{P}^1\times\mathbb{P}^1)\times(\mathbb{P}^1\times\mathbb{P}^1)\setminus\Delta_{\mathbb{P}^1\times\mathbb{P}^1}$. This means that d^{-1} is injective as a sheaf homomorphism, and the cokernel of d^0 is isomorphic to $\mathcal{O}_{\Delta_{\mathbb{P}^1\times\mathbb{P}^1}}$. □

4.3.2. Vanishing theorem

Lemma 4.15. *For an $(H_1\cup H_2)$-framed flat sheaf $\mathcal{F}$, we have*

$$H^0(\mathcal{F}(p,q))=0 \quad \text{for } p\le -1 \text{ or } q\le -1,$$
$$H^2(\mathcal{F}(p,q))=0 \quad \text{for } p\ge -1 \text{ or } q\ge -1.$$

Proof. The short exact sequence

$$0\to\mathcal{O}(-1,0)\xrightarrow{x_0}\mathcal{O}(0,0)\to\mathcal{O}_{H_1}\to 0$$

induces the long exact sequence

$$\begin{aligned}0&\to H^0(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{F}(p-1,q))\to H^0(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{F}(p,q))\to H^0(H_1,\mathcal{F}|_{H_1}\otimes\mathcal{O}_{H_1}(q))\\ &\to H^1(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{F}(p-1,q))\to H^1(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{F}(p,q))\to H^1(H_1,\mathcal{F}|_{H_1}\otimes\mathcal{O}_{H_1}(q))\\ &\to H^2(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{F}(p-1,q))\to H^2(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{F}(p,q))\to 0.\end{aligned}$$

Then we have

$$H^0(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{F}(p-1,q))\cong H^0(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{F}(p,q)) \quad \text{for } q\le -1,$$
$$H^2(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{F}(p-1,q))\cong H^2(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{F}(p,q)) \quad \text{for } q\ge -1,$$

from the framing condition on H_1, that is, $\mathcal{F}|_{H_1}\otimes\mathcal{O}_{H_1}(q)\cong\mathcal{O}_{H_1}(q)^{\oplus\operatorname{rank}\mathcal{F}}$. From Serre's vanishing theorem, $H^0(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{F}(p,q))=H^2(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{F}(p,q))=0$ for sufficiently large p. Thus the assertion for q holds. In the same way, we can prove the assertion for p. □

Corollary 4.16. *Let $\{\mathfrak{F}_\bullet\}$ be a parabolic sheaf defined in Definition 4.1. For $k\le 0$, $p\ge -1$ and $q\le -1$, we have*

$$H^0(\mathbb{P}^1\times\mathbb{P}^1,\mathfrak{F}_k(p,q))=H^2(\mathbb{P}^1\times\mathbb{P}^1,\mathfrak{F}_k(p,q))=0.$$

Proof. If we prove the assertion for $-3\le k\le 0$, the assertion for $k<-3$ follows from $\mathfrak{F}_{k-3}=\mathfrak{F}_k(-H_2')$.

We recall that $\mathfrak{F}_{-3}(p,q)\cong\mathfrak{F}_0(p,q-1)$ and $\mathfrak{F}_0$ is an $(H_1\cup H_2)$-framed sheaf. Hence by Lemma 4.15, we get $H^0(\mathfrak{F}_{-3}(p,q))=H^0(\mathfrak{F}_0(p,q))=0$ from $q\le -1$ and $H^2(\mathfrak{F}_{-3}(p,q))=H^2(\mathfrak{F}_0(p,q))=0$ from $p\ge -1$. Then the inclusions $\mathfrak{F}_{-3}\hookrightarrow\mathfrak{F}_{-2}\hookrightarrow\mathfrak{F}_{-1}\hookrightarrow\mathfrak{F}_0$ induce the injective maps

$$0=H^0(\mathfrak{F}_{-3}(p,q))\xrightarrow{\text{inj.}}H^0(\mathfrak{F}_{-2}(p,q))\xrightarrow{\text{inj.}}H^0(\mathfrak{F}_{-1}(p,q))\xrightarrow{\text{inj.}}H^0(\mathfrak{F}_0(p,q))=0,$$

so $H^0(\mathfrak{F}_k(p,q)) = 0$ for $-3 \le k \le 0$. The fourth condition of Definition 4.1 means $H^2(\mathfrak{F}_k/\mathfrak{F}_{k-1}) = 0$ for $-3 \le k \le 0$, so we have

$$0 = H^2(\mathfrak{F}_{-3}(p,q)) \xrightarrow{\text{surj.}} H^2(\mathfrak{F}_{-2}(p,q)) \xrightarrow{\text{surj.}} H^2(\mathfrak{F}_{-1}(p,q)) \xrightarrow{\text{surj.}} H^2(\mathfrak{F}_0(p,q)) = 0,$$

and $H^2(\mathfrak{F}_k(p,q)) = 0$ for $-3 \le k \le 0$. □

We recall that the total Chern characters of the sheaves $\mathfrak{F}_k$ of a parabolic sheaf $\{\mathfrak{F}_\bullet\}$ are given by

$$\begin{aligned}
\mathrm{ch}(\mathfrak{F}_0) &= 3 - d_0[H_1]\wedge[H_2],\\
\mathrm{ch}(\mathfrak{F}_{-1}) &= 3 - [H_2] - d_2[H_1]\wedge[H_2],\\
\mathrm{ch}(\mathfrak{F}_{-2}) &= 3 - 2[H_2] - d_1[H_1]\wedge[H_2],\\
\mathrm{ch}(\mathfrak{F}_{-3}) &= 3 - 3[H_2] - d_0[H_1]\wedge[H_2] = \mathrm{ch}(\mathfrak{F}_0(-H_2')).
\end{aligned}$$

Since the Todd genus of $\mathbb{P}^1\times\mathbb{P}^1$ is given by $\mathrm{Td} = 1 + [H_1] + [H_2] + [H_1]\wedge[H_2]$, we can calculate the dimensions of the first cohomology of sheaves by Corollary 4.16 and the Riemann–Roch–Hirzebruch theorem (see Table 4.1).

Table 4.1. Dimensions of the first cohomology of sheaves

	$h^1(\mathfrak{F}_k(-1,-2))$	$h^1(\mathfrak{F}_k(-1,-1))$	$h^1(\mathfrak{F}_k(0,-1))$	$h^1(\mathfrak{F}_k(0,-2))$
$k=0$	d_0	d_0	d_0	d_0+3
$k=-1$	d_2	d_2	d_2+1	d_2+4
$k=-2$	d_1	d_1	d_1+2	d_1+5
$k=-3$	d_0	d_0	d_0+3	d_0+6

4.3.3. Inverse construction. For any coherent sheaf $\mathcal{F}$, we consider the double complex $R^\bullet p_{2*}(p_1^*\mathcal{F}\otimes C^\bullet)$. From [Nak3, p. 19], if we take the cohomology of $\{C^\bullet\}$ first, we have

$$E_2^{p,q} = R^q p_{2*}(H^p(p_1^*\mathcal{F}\otimes C^\bullet)) = \begin{cases}\mathcal{F} & \text{for } (p,q) = (0,0),\\ 0 & \text{for } (p,q)\neq(0,0).\end{cases}$$

On the other hand, if we take the direct image first, the E_1-term of the spectral sequence is given by

$$H^q(\mathcal{F}(-1,-1))(-1,-1) \to \begin{matrix} H^q(\mathcal{F}(-1,0))(-1,0) \\ H^q(\mathcal{F}(0,-1))(0,-1)\end{matrix} \to H^q(\mathcal{F}(0,0))(0,0) \tag{4.17}$$

for $q = 0,1,2$, and this spectral sequence must converge to $\mathcal{F}$ at $(p,q) = (0,0)$. Therefore, when we set $\mathcal{F} = \mathfrak{F}_k(0,-1)$ for a parabolic sheaf $\{\mathfrak{F}_\bullet\}$ and use Corollary

4.16, we can conclude that the middle cohomology of the complex

$$H^1(\mathfrak{F}_k(-1,-2))(-1,-1) \to \begin{matrix} H^1(\mathfrak{F}_k(-1,-1))(-1,0) \\ H^1(\mathfrak{F}_k(0,-2))(0,-1) \end{matrix} \to H^1(\mathfrak{F}_k(0,-1))(0,0)$$

coincides with $\mathfrak{F}_k(0,-1)$. Thus the flag $\mathfrak{F}_{-3} \subset \mathfrak{F}_{-2} \subset \mathfrak{F}_{-1} \subset \mathfrak{F}_0$ is described as

(4.18)

$$\begin{array}{ccccc}
H^1(\mathfrak{F}_0(-1,-2))(-1,0) & \xrightarrow{\alpha_0} & \begin{matrix} H^1(\mathfrak{F}_0(-1,-1))(-1,1) \\ H^1(\mathfrak{F}_0(0,-2)) \end{matrix} & \xrightarrow{\beta_0} & H^1(\mathfrak{F}_0(0,-1))(0,1) \\
\uparrow f_0 & & \uparrow g_0 & & \uparrow h_0 \\
H^1(\mathfrak{F}_{-1}(-1,-2))(-1,0) & \xrightarrow{\alpha_{-1}} & \begin{matrix} H^1(\mathfrak{F}_{-1}(-1,-1))(-1,1) \\ H^1(\mathfrak{F}_{-1}(0,-2)) \end{matrix} & \xrightarrow{\beta_{-1}} & H^1(\mathfrak{F}_{-1}(0,-1))(0,1) \\
\uparrow f_{-1} & & \uparrow g_{-1} & & \uparrow h_{-1} \\
H^1(\mathfrak{F}_{-2}(-1,-2))(-1,0) & \xrightarrow{\alpha_{-2}} & \begin{matrix} H^1(\mathfrak{F}_{-2}(-1,-1))(-1,1) \\ H^1(\mathfrak{F}_{-2}(0,-2)) \end{matrix} & \xrightarrow{\beta_{-2}} & H^1(\mathfrak{F}_{-2}(0,-1))(0,1) \\
\uparrow f_{-2} & & \uparrow g_{-2} & & \uparrow h_{-2} \\
H^1(\mathfrak{F}_{-3}(-1,-2))(-1,0) & \xrightarrow{\alpha_{-3}} & \begin{matrix} H^1(\mathfrak{F}_{-3}(-1,-1))(-1,1) \\ H^1(\mathfrak{F}_{-3}(0,-2)) \end{matrix} & \xrightarrow{\beta_{-3}} & H^1(\mathfrak{F}_{-3}(0,-1))(0,1)
\end{array}$$

where α_k and β_k are respectively induced by d^{-1} and d^0 of the complex $C^\bullet$, and f_k, g_k and h_k are induced by the inclusion $\mathfrak{F}_{k-1} \hookrightarrow \mathfrak{F}_k$. Here notice that convergence of (4.17) means that α_k is injective and β_k is surjective as sheaf homomorphisms. Furthermore, since $\mathfrak{F}_k$ is a locally free sheaf, α_k must be injective and β_k surjective as fiberwise homomorphisms.

Now we study the vector spaces $H^1(\mathfrak{F}_k(-1,-2))$, $H^1(\mathfrak{F}_k(-1,-1))$, $H^1(\mathfrak{F}_k(0,-2))$ and $H^1(\mathfrak{F}_k(0,-1))$ for $-3 \le k \le 0$. By using the fifth and sixth conditions of Definition 4.1, we have the following exact sequences:

(4.19)

$$0 \to H^1(\mathfrak{F}_k(-1,-2)) \xrightarrow{y'_0} H^1(\mathfrak{F}_k(-1,-1)) \to 0,$$

$$0 \to H^1(\mathfrak{F}_k(-1,-1)) \xrightarrow{x'_0} H^1(\mathfrak{F}_k(0,-1)) \to H^1(H_1, \mathfrak{F}_k|_{H_1} \otimes \mathcal{O}_{H_1}(-1)) \to 0,$$

$$0 \to H^1(\mathfrak{F}_k(-1,-2)) \xrightarrow{x'_0} H^1(\mathfrak{F}_k(0,-2)) \to H^1(H_1, \mathfrak{F}_k|_{H_1} \otimes \mathcal{O}_{H_1}(-2)) \to 0,$$

$$0 \to H^0(H_2, \mathfrak{F}_k|_{H_2}) \to H^1(\mathfrak{F}_k(0,-2)) \xrightarrow{y'_0} H^1(\mathfrak{F}_k(0,-1)) \to 0.$$

The relations of the first cohomology are summarized as

$$(4.20) \qquad \begin{array}{ccccc}
 & \overset{y'_0 \text{ isom.}}{\nearrow} & H^1(\mathfrak{F}_k(-1,-1)) & \overset{x'_0 \text{ inj.}}{\searrow} & \\
H^1(\mathfrak{F}_k(-1,-2)) & & & & H^1(\mathfrak{F}_k(0,-1)) \\
 & \underset{x'_0 \text{ inj.}}{\searrow} & H^1(\mathfrak{F}_k(0,-2)) & \underset{y'_0 \text{ surj.}}{\nearrow} &
\end{array}$$

Table 4.2. Relations between cohomology and W_k

	$H^1(H_1, \mathfrak{F}_k\|_{H_1} \otimes \mathcal{O}_{H_1}(-1))$	$H^1(H_1, \mathfrak{F}_k\|_{H_1} \otimes \mathcal{O}_{H_1}(-2))$	$H^0(H_2, \mathfrak{F}_k\|_{H_2})$
$k = 0$	0	W	W
$k = -1$	W_3	$W_3 \oplus W$	W
$k = -2$	$W_2 \oplus W_3$	$W_2 \oplus W_3 \oplus W$	W
$k = -3$	$W_1 \oplus W_2 \oplus W_3$	$W_1 \oplus W_2 \oplus W_3 \oplus W$	W

By using these data, we define V_k and W_k. We set $W_k := H^0(H_2, \mathcal{W}_k)$ and $W := W_1 \oplus W_2 \oplus W_3$. Then by the fifth and sixth conditions, we have Table 4.2, and because of (4.20), we can define V_k as

$$\begin{aligned} V_{k+3} := H^1(\mathfrak{F}_k(-1,-2)) &\cong H^1(\mathfrak{F}_k(-1,-1)) \\ &\cong \operatorname{Im} x_0' \subset H^1(\mathfrak{F}_k(0,-1)) \\ &\cong \operatorname{Im} x_0' \subset H^1(\mathfrak{F}_k(0,-2)). \end{aligned}$$

Here $V_3 = H^1(\mathfrak{F}_0(-1,-2)) \cong H^1(\mathfrak{F}_{-3}(-1,-1)) = V_0$. Thus from (4.19) and Table 4.2, we have Table 4.3. Note that the respective dimensions are given in Table 4.1.

Table 4.3. The first cohomology and vector spaces

	$H^1(\mathfrak{F}_k(-1,-2))$	$H^1(\mathfrak{F}_k(-1,-1))$	$H^1(\mathfrak{F}_k(0,-1))$	$H^1(\mathfrak{F}_k(0,-2))$
$k = 0$	V_0	V_0	V_0	$V_0 \oplus W$
$k = -1$	V_2	V_2	$V_2 \oplus W_3$	$V_2 \oplus W_3 \oplus W$
$k = -2$	V_1	V_1	$V_1 \oplus W_2 \oplus W_3$	$V_1 \oplus W_2 \oplus W_3 \oplus W$
$k = -3$	V_0	V_0	$V_0 \oplus W$	$V_0 \oplus W \oplus W$

4.3.4. A normal form. From now on, we define maps A_k, B_k, i_k and j_k, and describe α_k and β_k in terms of A, B, i and j. First, from the inclusion $\mathfrak{F}_{k-1} \hookrightarrow \mathfrak{F}_k$, we define B_0, B_1 and B_2 as follows:

$$\begin{aligned} &B_0 \colon H^1(\mathfrak{F}_{-3}(-1,-2)) \to H^1(\mathfrak{F}_{-2}(-1,-2)), \\ &B_1 \colon H^1(\mathfrak{F}_{-2}(-1,-2)) \to H^1(\mathfrak{F}_{-1}(-1,-2)), \\ &B_2 \colon H^1(\mathfrak{F}_{-1}(-1,-2)) \to H^1(\mathfrak{F}_0(-1,-2)) = H^1(\mathfrak{F}_{-3}(-1,-1)). \end{aligned}$$

The isomorphism $y_0' \colon H^1(\mathfrak{F}_k(-1,-2)) \to H^1(\mathfrak{F}_k(-1,-1))$ induces

$$\begin{aligned} &H^1(\mathfrak{F}_{-3}(-1,-1)) \xrightarrow{B_0} H^1(\mathfrak{F}_{-2}(-1,-1)) \\ &\qquad\qquad \xrightarrow{B_1} H^1(\mathfrak{F}_{-1}(-1,-1)) \xrightarrow{B_2} H^1(\mathfrak{F}_0(-1,-1)). \end{aligned}$$

On the other hand, the inclusion $\mathfrak{F}_{k-3}(-1,-1) \hookrightarrow \mathfrak{F}_k(-1,-1)$ coincides with $y_1'\colon \mathfrak{F}_k(-1,-2) \to \mathfrak{F}_k(-1,-1)$. Thus $y_0' \otimes y_1 - y_1' \otimes y_0\colon H^1(\mathfrak{F}_k(-1,-2))\,(-1,0) \to H^1(\mathfrak{F}_k(-1,-1))(-1,1)$ is described as

$$\begin{aligned}
y_1 - B_{102}y_0 &\colon H^1(\mathfrak{F}_{-1}(-1,-2))(-1,0) \to H^1(\mathfrak{F}_{-1}(-1,-1))(-1,1),\\
y_1 - B_{021}y_0 &\colon H^1(\mathfrak{F}_{-2}(-1,-2))(-1,0) \to H^1(\mathfrak{F}_{-2}(-1,-1))(-1,1),\\
y_1 - B_{210}y_0 &\colon H^1(\mathfrak{F}_{-3}(-1,-2))(-1,0) \to H^1(\mathfrak{F}_{-3}(-1,-1))(-1,1).
\end{aligned}$$

By composing the inclusion $V_{k+3} \hookrightarrow H^1(\mathfrak{F}_k(0,-2))$, the map $-y_0' \otimes y_1 + y_1' \otimes y_0\colon H^1(\mathfrak{F}_k(0,-2)) \to H^1(\mathfrak{F}_k(0,-1))(0,1)$, and the projection $H^1(\mathfrak{F}_k(0,-1)) \to V_{k+3}$, we have

$$\begin{aligned}
-y_1 + B_{102}y_0 &\colon V_2 \hookrightarrow H^1(\mathfrak{F}_{-1}(0,-2)) \to H^1(\mathfrak{F}_{-1}(0,-1))(0,1) \to V_2(-1,1),\\
-y_1 + B_{021}y_0 &\colon V_1 \hookrightarrow H^1(\mathfrak{F}_{-2}(0,-2)) \to H^1(\mathfrak{F}_{-2}(0,-1))(0,1) \to V_1(-1,1),\\
-y_1 + B_{210}y_0 &\colon V_0 \hookrightarrow H^1(\mathfrak{F}_{-3}(0,-2)) \to H^1(\mathfrak{F}_{-3}(0,-1))(0,1) \to V_0(-1,1).
\end{aligned}$$

Next, we define A_0, A_1 and A_2 as the composites of $x_1'\colon H^1(\mathfrak{F}_k(-1,\,-2)) \to H^1(\mathfrak{F}_k(0,-2))$ and the projection $H^1(\mathfrak{F}_k(0,-2)) \to V_{k+3}$:

$$\begin{aligned}
A_2 &\colon V_2 \cong H^1(\mathfrak{F}_{-1}(-1,-2)) \to H^1(\mathfrak{F}_{-1}(0,-2)) \to V_2,\\
A_1 &\colon V_1 \cong H^1(\mathfrak{F}_{-2}(-1,-2)) \to H^1(\mathfrak{F}_{-2}(0,-2)) \to V_1,\\
A_0 &\colon V_0 \cong H^1(\mathfrak{F}_{-3}(-1,-2)) \to H^1(\mathfrak{F}_{-3}(0,-2)) \to V_0.
\end{aligned}$$

We also have the commutative diagram

$$\begin{array}{ccccc}
H^1(\mathfrak{F}_{-1}(-1,-2)) & \xrightarrow{x_1'} & H^1(\mathfrak{F}_{-1}(0,-2)) & \longrightarrow & V_2\\
\| {\scriptstyle y_0'} & & \downarrow {\scriptstyle y_0'} & & \|\\
H^1(\mathfrak{F}_{-1}(-1,-1)) & \xrightarrow{x_1'} & H^1(\mathfrak{F}_{-1}(0,-1)) & \longrightarrow & V_2
\end{array}$$

so the composite of $x_1'\colon H^1(\mathfrak{F}_k(-1,-1)) \to H^1(\mathfrak{F}_k(0,-1))$ and the projection $H^1(\mathfrak{F}_k(0,-1)) \to V_{k+3}$ coincides with A_{k+3}:

$$\begin{aligned}
A_2 &\colon V_2 \cong H^1(\mathfrak{F}_{-1}(-1,-1)) \to H^1(\mathfrak{F}_{-1}(0,-1)) \to V_2,\\
A_1 &\colon V_1 \cong H^1(\mathfrak{F}_{-2}(-1,-1)) \to H^1(\mathfrak{F}_{-2}(0,-1)) \to V_1,\\
A_0 &\colon V_0 \cong H^1(\mathfrak{F}_{-3}(-1,-1)) \to H^1(\mathfrak{F}_{-3}(0,-1)) \to V_0.
\end{aligned}$$

Summarizing, the restriction of the diagram (4.18) to V_k is

$$\begin{array}{ccccc}
V_0(-1,0) & \xrightarrow{\begin{pmatrix} y_1-y_0B_{210} \\ x_1-x_0A_0 \end{pmatrix}} & V_0(-1,1)\oplus V_0 & \xrightarrow{(x_1-x_0A_0\ \ -y_1+y_0B_{210})} & V_0(0,1) \\
\uparrow{\scriptstyle B_2} & & \uparrow{\scriptstyle \begin{pmatrix} B_2 & 0 \\ 0 & B_2 \end{pmatrix}} & & \uparrow{\scriptstyle B_2} \\
V_2(-1,0) & \xrightarrow{\begin{pmatrix} y_1-y_0B_{102} \\ x_1-x_0A_2 \end{pmatrix}} & V_2(-1,1)\oplus V_2 & \xrightarrow{(x_1-x_0A_2\ \ -y_1+y_0B_{102})} & V_2(0,1) \\
\uparrow{\scriptstyle B_1} & & \uparrow{\scriptstyle \begin{pmatrix} B_1 & 0 \\ 0 & B_1 \end{pmatrix}} & & \uparrow{\scriptstyle B_1} \\
V_1(-1,0) & \xrightarrow{\begin{pmatrix} y_1-y_0B_{021} \\ x_1-x_0A_1 \end{pmatrix}} & V_1(-1,1)\oplus V_1 & \xrightarrow{(x_1-x_0A_1\ \ -y_1+y_0B_{021})} & V_1(0,1) \\
\uparrow{\scriptstyle B_0} & & \uparrow{\scriptstyle \begin{pmatrix} B_0 & 0 \\ 0 & B_0 \end{pmatrix}} & & \uparrow{\scriptstyle B_0} \\
V_0(-1,0) & \xrightarrow{\begin{pmatrix} y_1-y_0B_{210} \\ x_1-x_0A_0 \end{pmatrix}} & V_0(-1,1)\oplus V_0 & \xrightarrow{(x_1-x_0A_0\ \ -y_1+y_0B_{210})} & V_0(0,1)
\end{array}$$

Note that this diagram does not commute.

Third, we consider the composite of $x'_1\colon H^1(\mathfrak{F}_{-1}(-1,-2)) \to H^1(\mathfrak{F}_{-1}\,(0,-2))$ and the projection $H^1(\mathfrak{F}_{-1}(0,-2)) \to W_3 \oplus W$. Notice that $W_3 \oplus W_1 \oplus W_2 \oplus W_3$ was defined as $H^1(H_1, \mathfrak{F}_{-1}|_{H_1} \otimes \mathcal{O}(-2))$ and Serre duality means

$$H^1(H_1, \mathcal{W}_3 \otimes \mathcal{O}(-3)) \cong H^0(H_1, \mathcal{W}_3^\vee \otimes \mathcal{O}(1))^\vee \cong W_3(y'_0)^\vee \oplus W_3(y'_1)^\vee. \tag{4.21}$$

Hence

$$H^1(H_1, \mathfrak{F}_{-1}|_{H_1} \otimes \mathcal{O}(-2)) \cong W_3(y'_0)^\vee \oplus W_1(y'_1)^\vee \oplus W_2(y'_1)^\vee \oplus W_3(y'_1)^\vee. \tag{4.22}$$

Further the fifth condition of Definition 4.1 and (4.22) imply that the map $H^1(H_1, \mathfrak{F}_{-1}|_{H_1} \otimes \mathcal{O}(-2)) \to H^1(H_1, \mathfrak{F}_0|_{H_1} \otimes \mathcal{O}(-2))$ induced by $\mathfrak{F}_{-1} \hookrightarrow \mathfrak{F}_0$ is

$$(0\ \ \mathrm{id}_{123})\colon W_3 \oplus W_1 \oplus W_2 \oplus W_3 \to W_1 \oplus W_2 \oplus W_3. \tag{4.23}$$

Then we define j_0, j_1 and j_2 as the following composites:

$$\begin{aligned}
j_2\colon V_2 &\xrightarrow{x'_1} H^1(\mathfrak{F}_{-1}(0,-2)) \to H^1(H_1, \mathfrak{F}_{-1}|_{H_1} \otimes \mathcal{O}(-2)) \to W_3(y'_0)^\vee, \\
j_1\colon V_1 &\xrightarrow{x'_1} H^1(\mathfrak{F}_{-2}(0,-2)) \to H^1(H_1, \mathfrak{F}_{-2}|_{H_1} \otimes \mathcal{O}(-2)) \to W_2(y'_0)^\vee, \\
j_0\colon V_0 &\xrightarrow{x'_1} H^1(\mathfrak{F}_{-3}(0,-2)) \to H^1(H_1, \mathfrak{F}_{-3}|_{H_1} \otimes \mathcal{O}(-2)) \to W_1(y'_0)^\vee.
\end{aligned}$$

We have

$$\begin{aligned}
j_2B_{102}\colon V_2 &\to H^1(\mathfrak{F}_{-1}(0,-2)) \to H^1(H_1, \mathfrak{F}_{-1}|_{H_1} \otimes \mathcal{O}(-2)) \to W_3(y'_1)^\vee, \\
j_1B_{021}\colon V_1 &\to H^1(\mathfrak{F}_{-2}(0,-2)) \to H^1(H_1, \mathfrak{F}_{-2}|_{H_1} \otimes \mathcal{O}(-2)) \to W_2(y'_1)^\vee, \\
j_0B_{210}\colon V_0 &\to H^1(\mathfrak{F}_{-3}(0,-2)) \to H^1(H_1, \mathfrak{F}_{-3}|_{H_1} \otimes \mathcal{O}(-2)) \to W_1(y'_1)^\vee.
\end{aligned} \tag{4.24}$$

On the other hand, we consider the composite of $W_3 \hookrightarrow H^1(\mathfrak{F}_{-1}\,(0,-2))$, the map $H^1(\mathfrak{F}_{-1}(0,-2)) \to H^1(\mathfrak{F}_0(0,-2))$, and the projection $H^1(\mathfrak{F}_0(0,\,-2)) \to V_0$.

Then we define i_1, i_2 and i_3 as the following composites:

$$\begin{aligned}
i_3 &\colon W_3(y_0')^\vee \hookrightarrow H^1(\mathfrak{F}_{-1}(0,-2)) \to H^1(\mathfrak{F}_0(0,-2)) \to V_0,\\
i_2 &\colon W_2(y_0')^\vee \hookrightarrow H^1(\mathfrak{F}_{-2}(0,-2)) \to H^1(\mathfrak{F}_{-1}(0,-2)) \to V_2,\\
i_1 &\colon W_1(y_0')^\vee \hookrightarrow H^1(\mathfrak{F}_{-3}(0,-2)) \to H^1(\mathfrak{F}_{-2}(0,-2)) \to V_1.
\end{aligned}$$

Here we consider the composite of $V_0 \to H^1(\mathfrak{F}_0(0,-2)) \to H^1(H_1, \mathfrak{F}_0|_{H_1} \otimes \mathcal{O}(-2))$ $\to W_3$ as above. From (4.23) and (4.24), we have

$$\begin{array}{ccc}
V_0 = H^1(\mathfrak{F}_0(-1,-2)) & \xrightarrow{\hspace{6em}} & W_3(y_1')^\vee \\
{\scriptstyle B_2}\big\uparrow & & {\scriptstyle (0\ 1)}\big\uparrow \\
V_2 = H^1(\mathfrak{F}_{-1}(-1,-2)) & \xrightarrow{\ j_2 \oplus j_2 B_{102}\ } & W_3(y_0')^\vee \oplus W_3(y_1')^\vee
\end{array}$$

By using this diagram, we conclude that the composite of $V_0 \to H^1(\mathfrak{F}_0(0,-2)) \to$ $H^1(H_1, \mathfrak{F}_0|_{H_1} \otimes \mathcal{O}(-2)) = W_1 \oplus W_2 \oplus W_3$ is

$$\begin{pmatrix} j_0 \\ j_1 B_0 \\ j_2 B_{10} \end{pmatrix} : V_0 \to W_1 \oplus W_2 \oplus W_3.$$

Therefore, the left hand part of the diagram (4.18) is

$$\begin{array}{ccc}
V_0(-1,0) & \xrightarrow{\begin{pmatrix} y_1 - y_0 B_{210} \\ x_1 - x_0 A_0 \\ x_0 j_0 \\ x_0 j_1 B_0 \\ x_0 j_2 B_{10} \end{pmatrix}} & V_0(-1,1) \oplus V_0 \oplus W_1 \oplus W_2 \oplus W_3 \\
{\scriptstyle B_2}\big\uparrow & & \big\uparrow{\scriptstyle \begin{pmatrix} B_2 & 0 & 0 & 0 \\ 0 & B_2 & i_3 & 0 \\ 0 & 0 & 0 & \mathrm{id}_{123} \end{pmatrix}} \\
V_2(-1,0) & \xrightarrow{\begin{pmatrix} y_1 - y_0 B_{102} \\ x_1 - x_0 A_2 \\ x_0 j_2 \\ x_0 j_0 B_2 \\ x_0 j_1 B_{02} \\ x_0 j_2 B_{102} \end{pmatrix}} & V_2(-1,1) \oplus V_2 \oplus W_3 \oplus W_1 \oplus W_2 \oplus W_3 \\
{\scriptstyle B_1}\big\uparrow & & \big\uparrow{\scriptstyle \begin{pmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_1 & i_2 & 0 \\ 0 & 0 & 0 & \mathrm{id}_{3123} \end{pmatrix}} \\
V_1(-1,0) & \xrightarrow{\begin{pmatrix} y_1 - y_0 B_{021} \\ x_1 - x_0 A_1 \\ x_0 j_1 \\ x_0 j_2 B_1 \\ x_0 j_0 B_{21} \\ x_0 j_1 B_{021} \\ x_0 j_2 B_{1021} \end{pmatrix}} & V_1(-1,1) \oplus V_1 \oplus W_2 \oplus W_3 \oplus W_1 \oplus W_2 \oplus W_3 \\
{\scriptstyle B_0}\big\uparrow & & \big\uparrow{\scriptstyle \begin{pmatrix} B_0 & 0 & 0 & 0 \\ 0 & B_0 & i_1 & 0 \\ 0 & 0 & 0 & \mathrm{id}_{23123} \end{pmatrix}} \\
V_0(-1,0) & \xrightarrow{\begin{pmatrix} y_1 - y_0 B_{210} \\ x_1 - x_0 A_0 \\ x_0 j_0 \\ x_0 j_1 B_0 \\ x_0 j_2 B_{10} \\ x_0 j_0 B_{210} \\ x_0 j_1 B_{0210} \\ x_0 j_2 B_{10210} \end{pmatrix}} & V_0(-1,1) \oplus V_0 \oplus W_1 \oplus W_2 \oplus W_3 \oplus W_1 \oplus W_2 \oplus W_3
\end{array}$$

By commutativity of this diagram, we have

$$\mu_0 = A_1 B_0 - B_0 A_0 + i_1 j_0 = 0,$$
$$\mu_1 = A_2 B_1 - B_1 A_1 + i_2 j_1 = 0,$$
$$\mu_2 = A_0 B_2 - B_2 A_2 + i_3 j_2 = 0.$$

Lastly, we consider the composite of $W_3 \oplus W \hookrightarrow H^1(\mathfrak{F}_{-1}(0,-2))$, $\beta_{-1}\colon H^1(\mathfrak{F}_{-1}(0,-2)) \to H^1(\mathfrak{F}_{-1}(0,-1))(0,1)$ and the projection $H^1(\mathfrak{F}_{-1}(0,-1))(0,1) \to W_3(0,1)$. This map is induced by

$$-y_0' \otimes y_1 + y_1' \otimes y_0\colon H^1(H_1, \mathfrak{F}_{-1}|_{H_1} \otimes \mathcal{O}(-2)) \to H^1(H_1, \mathfrak{F}_{-1}|_{H_1} \otimes \mathcal{O}(-1))(0,1).$$

Thus, from (4.21), we have

$$\beta_{-1}|_{W_3 \oplus W} = (-y_1\ 0\ 0\ y_0)\colon W_3 \oplus W_1 \oplus W_2 \oplus W_3 \to W_3(0,1).$$

Summarizing the argument in §4.3.4, from $\beta_k \alpha_k = 0$ and commutativity of (4.18), we conclude that α_k, β_k, g_k, f_k and h_k are as described in (4.4).

4.3.5. Completion of the proof of Theorem 4.2. The data (A, B, i, j) defined in §4.3.4 are described as a point of $\mathbb{M}_c(\vec{m}, N)$. Moreover, the stability conditions (C-S1) and (C-S2) follow from injectivity of α_k and surjectivity of β_k by Proposition 4.5. Thus we obtain a map from $\mathcal{P}(\vec{m}, N)$ to $\mathcal{M}_c(\vec{m}, N)$. It is easy to see that the composite $\mathcal{P}(\vec{m}, N) \to \mathcal{M}_c(\vec{m}, N) \to \mathcal{P}(\vec{m}, N)$ is the identity. By the same argument as in [NakY, §5.4], the composite $\mathcal{M}_c(\vec{m}, N) \to \mathcal{P}(\vec{m}, N) \to \mathcal{M}_c(\vec{m}, N)$ is also the identity. Thus, the proof of Theorem 4.2 is complete.

As a corollary from the proof of Theorem 4.2 and Remark 4.6, we obtain Braverman–Finkelberg–Rybnikov's theorem.

Theorem 4.25 ([FR, BF]). *The variety $\{(A, B, i, j) \in \bigcap \mu_k^{-1}(0) \subset \mathbb{M}_c(\vec{m}, N) \mid (A, B, i, j)$ satisfies (C-S2)$\}/\mathcal{G}_c(\vec{m}, N)$ is isomorphic to the moduli space of (H_1, H_2)-framed torsion free parabolic sheaves over $\mathbb{P}^1 \times \mathbb{P}^1$ of rank N and degree $\vec{m}$ with respect to H_2'.*

Theorems 3.5 and 4.2 lead to Theorem 0.4:

Theorem 4.26. *$\mathcal{M}_{\mathrm{N}}^{\mathrm{reg}}(S^1; \vec{m}, N)$ is isomorphic to $\mathcal{P}(\vec{m}, N)$.*

§4.4. Behavior on the parabolic structure

In this subsection, we study the behavior of parabolic sheaves on H_2' by using our monad description. On $H_2' = \{y_1 = 0\}$, (4.4) descends as follows:

(4.27)
$$
\begin{array}{ccccc}
\mathfrak{F}_0|_{H_2'} : V_0(-1) & \xrightarrow{\begin{pmatrix} -B_{210} \\ x_1 - x_0 A_0 \\ x_0 j_0 \\ x_0 j_1 B_0 \\ x_0 j_2 B_{10} \end{pmatrix}} & \begin{matrix} V_0(-1) \oplus V_0 \\ W_1 \oplus W_2 \oplus W_3 \end{matrix} & \xrightarrow{(x_1 - x_0 A_0 \ B_{210} \ B_{21} i_1 \ B_2 i_2 \ i_3)} & V_0 \\
\uparrow B_2 & & \uparrow \begin{pmatrix} B_2 & 0 & 0 & 0 \\ 0 & B_2 & i_3 & 0 \\ 0 & 0 & 0 & \mathrm{id}_{12} \\ -x_0 j_2 & 0 & 0 & 0 \end{pmatrix} & & \uparrow B_2 \\
\mathfrak{F}_{-1}|_{H_2'} : V_2(-1) & \xrightarrow{\begin{pmatrix} -B_{102} \\ x_1 - x_0 A_2 \\ x_0 j_2 \\ x_0 j_0 B_2 \\ x_0 j_1 B_{02} \end{pmatrix}} & \begin{matrix} V_2(-1) \oplus V_2 \\ W_3 \oplus W_1 \oplus W_2 \end{matrix} & \xrightarrow{(x_1 - x_0 A_2 \ B_{102} \ B_{10} i_3 \ B_1 i_1 \ i_2)} & V_2 \\
\uparrow B_1 & & \uparrow \begin{pmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_1 & i_2 & 0 \\ 0 & 0 & 0 & \mathrm{id}_{31} \\ -x_0 j_1 & 0 & 0 & 0 \end{pmatrix} & & \uparrow B_1 \\
\mathfrak{F}_{-2}|_{H_2'} : V_1(-1) & \xrightarrow{\begin{pmatrix} -B_{021} \\ x_1 - x_0 A_1 \\ x_0 j_1 \\ x_0 j_2 B_1 \\ x_0 j_0 B_{21} \end{pmatrix}} & \begin{matrix} V_1(-1) \oplus V_1 \\ W_2 \oplus W_3 \oplus W_1 \end{matrix} & \xrightarrow{(x_1 - x_0 A_1 \ B_{021} \ B_{02} i_2 \ B_0 i_3 \ i_1)} & V_1 \\
\uparrow B_0 & & \uparrow \begin{pmatrix} B_0 & 0 & 0 & 0 \\ 0 & B_0 & i_1 & 0 \\ 0 & 0 & 0 & \mathrm{id}_{23} \\ -x_0 j_0 & 0 & 0 & 0 \end{pmatrix} & & \uparrow B_0 \\
\mathfrak{F}_{-3}|_{H_2'} : V_0(-1) & \xrightarrow{\begin{pmatrix} -B_{210} \\ x_1 - x_0 A_0 \\ x_0 j_0 \\ x_0 j_1 B_0 \\ x_0 j_2 B_{10} \end{pmatrix}} & \begin{matrix} V_0(-1) \oplus V_0 \\ W_1 \oplus W_2 \oplus W_3 \end{matrix} & \xrightarrow{(x_1 - x_0 A_0 \ B_{210} \ B_{21} i_1 \ B_2 i_2 \ i_3)} & V_0
\end{array}
$$

We have the following proposition.

Proposition 4.28. *For a framed locally free parabolic sheaf $\{\mathfrak{F}_\bullet\}$ as in Definition 4.1, the restriction of $\mathfrak{F}_k/\mathfrak{F}_{k-1}$ $(-n+1 \leq k \leq 0)$ to H_2' is locally free.*

Proof. For $\mathfrak{F}_0|_{H_2'}$ and $\mathfrak{F}_{-1}|_{H_2'}$ given in (4.27), we consider the following commutative diagram:

$$
\begin{array}{ccccc}
\mathfrak{F}_0|_{H_2'} : V_0(-1) & \xrightarrow{\begin{pmatrix} -B_{210} \\ x_1 - x_0 A_0 \\ x_0 j_0 \\ x_0 j_1 B_0 \\ x_0 j_2 B_{10} \end{pmatrix}} & \begin{matrix} V_0(-1) \oplus V_0 \\ W_1 \oplus W_2 \oplus W_3 \end{matrix} & \xrightarrow{(x_1 - x_0 A_0 \ B_{210} \ B_{21} i_1 \ B_2 i_2 \ i_3)} & V_0 \\
\uparrow B_2 & & \uparrow g_0 = \begin{pmatrix} B_2 & 0 & 0 & 0 \\ 0 & B_2 & i_3 & 0 \\ 0 & 0 & 0 & \mathrm{id}_{12} \\ -x_0 j_2 & 0 & 0 & 0 \end{pmatrix} & & \uparrow B_2 \\
\mathfrak{F}_{-1}|_{H_2'} : V_2(-1) & \xrightarrow{\begin{pmatrix} -B_{102} \\ x_1 - x_0 A_2 \\ x_0 j_2 \\ x_0 j_0 B_2 \\ x_0 j_1 B_{02} \end{pmatrix}} & \begin{matrix} V_2(-1) \oplus V_2 \\ W_3 \oplus W_1 \oplus W_2 \end{matrix} & \xrightarrow{(x_1 - x_0 A_2 \ B_{102} \ B_{10} i_3 \ B_1 i_1 \ i_2)} & V_2 \\
\uparrow \mathrm{id} & & \uparrow \phi_{-1} = \begin{pmatrix} B_{10} & 0 & 0 \\ 0 & \mathrm{id} & 0 \\ 0 & 0 & \mathrm{id} \\ -x_0 j_0 & 0 & 0 \\ -x_0 j_1 B_0 & 0 & 0 \end{pmatrix} & & \uparrow B_{10} \\
\mathfrak{K}_{-1} : V_2(-1) & \xrightarrow{a_{-1} = \begin{pmatrix} -B_2 \\ x_1 - x_0 A_2 \\ x_0 j_2 \end{pmatrix}} & V_0(-1) \oplus V_2 \oplus W_3 & \xrightarrow[b_{-1} = (x_1 - x_0 A_0 \ B_2 \ i_3)]{} & V_0
\end{array}
$$

$b_{-1}a_{-1} = 0$ follows from the equation, and injectivity of a_{-1} and surjectivity of b_{-1} follow from stability conditions (cf. Proposition 4.5). Thus $\mathfrak{K}_{-1} = \operatorname{Ker} b_{-1}/\operatorname{Im} a_{-1}$ becomes a locally free sheaf. A straightforward calculation shows $\operatorname{Im}\tilde{\phi}_{-1} = \operatorname{Ker}\tilde{g}_0$ on each fiber. Therefore when $\tilde{\phi}_{-1}$ is injective on each fiber, we conclude that $\operatorname{Im}\tilde{g}_0$ and $(\mathfrak{F}_0/\mathfrak{F}_{-1})|_{H_2'}$ are locally free.

To prove $\tilde{\phi}_{-1}$ is injective on each fiber take $(v_0, v_2, w_3) \in \operatorname{Ker} b_{-1}$ and suppose that there exists $v_2' \in V_2$ such that $\alpha_{-1}(v_2') = \phi_{-1}(v_0, v_2, w_3)$. Then we can show $a_{-1}(v_2') = (v_0, v_2, w_3)$ on each fiber, so $\tilde{\phi}_{-1}$ is injective. □

§5. From chainsaw quiver varieties to handsaw quiver varieties

When $m_0 = 0$, a chainsaw quiver variety $\mathcal{M}_{\mathrm{c}}(\vec{m}, N)$ becomes a handsaw quiver variety $\mathcal{M}_{\mathrm{h}}(\vec{m}', N)$, where $\vec{m}' = (m_1, \dots, m_{N-1})$. Then we can observe that Theorem 4.2 descends to Theorem 2.7.

We consider the $N = 3$ case, that is, $\vec{m} = (0, m_1, m_2)$. Then $A_0 = B_0 = j_0 = B_2 = i_3 = 0$. On H_2', the monad (4.4) splits as follows (see also (4.27)):

$$
\begin{array}{ccc}
0 & \xrightarrow{\;0\;} & 0 \\
{\scriptstyle 0}\uparrow & & \uparrow{\scriptstyle 0} \\
V_2(-1) & \xrightarrow{\begin{pmatrix} x_1 - x_0 A_2 \\ x_0 j_2 \end{pmatrix}} & V_2 \oplus W_3 \\
{\scriptstyle B_1}\uparrow & & \uparrow{\scriptstyle \begin{pmatrix} B_1 & i_2 & 0 \\ 0 & 0 & \mathrm{id}_3 \end{pmatrix}} \\
V_1(-1) & \xrightarrow{\begin{pmatrix} x_1 - x_0 A_1 \\ x_0 j_1 \\ x_0 j_2 B_1 \end{pmatrix}} & V_1 \oplus W_2 \oplus W_3 \\
{\scriptstyle 0}\uparrow & & \uparrow{\scriptstyle \begin{pmatrix} i_1 & 0 \\ 0 & \mathrm{id}_{23} \end{pmatrix}} \\
0 & \xrightarrow{\;0\;} & W_1 \oplus W_2 \oplus W_3
\end{array}
\qquad
\begin{array}{ccc}
W_1 \oplus W_2 \oplus W_3 & \xrightarrow{\;0\;} & 0 \\
{\scriptstyle \begin{pmatrix} 0 & \mathrm{id}_{12} \\ -x_0 j_2 & 0 \end{pmatrix}}\uparrow & & \uparrow{\scriptstyle 0} \\
V_2(-1) \oplus W_1 \oplus W_2 & \xrightarrow{(x_1 - x_0 A_2 \;\; B_1 i_1 \;\; i_2)} & V_2 \\
{\scriptstyle \begin{pmatrix} B_1 & 0 \\ 0 & \mathrm{id}_1 \\ -x_0 j_1 & 0 \end{pmatrix}}\uparrow & & \uparrow{\scriptstyle B_1} \\
V_1(-1) \oplus W_1 & \xrightarrow{(x_1 - x_0 A_1 \;\; i_1)} & V_1 \\
{\scriptstyle 0}\uparrow & & \uparrow{\scriptstyle 0} \\
0 & \xrightarrow{\;0\;} & 0
\end{array}
$$

We can see that the left hand part essentially appears in [S] and the right hand part is given in [Nak5]. From [Nak5, Lemma 3.2], Theorem 4.2 leads to Theorem 2.7, and Theorem 4.25 leads to the following theorem:

Theorem 5.1 ([FR, Nak5]). *The variety* $\{(A, B, i, j) \in \bigcap \mu_k^{-1}(0) \subset \mathbb{M}_{\mathrm{h}}(\vec{m}', N) \mid (A, B, i, j)$ *satisfies* (H-S2)$\}/\mathcal{G}_{\mathrm{h}}(\vec{m}', N)$ *is isomorphic to the moduli space of* ∞*-framed locally free parabolic sheaves over* $\mathbb{P}^1$ *of rank* N *and degree* $\vec{m}'$*.*

Acknowledgements

The author would like to express his thanks to his supervisor, Professor Hiraku Nakajima for his continuous encouragement and advice.

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