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Mixed Frobenius Structure and Local Quantum Cohomology

by

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Abstract

In a previous paper, the authors introduced the notion of mixed Frobenius structure (MFS) as a generalization of the structure of a Frobenius manifold. Roughly speaking, the MFS is defined by replacing a metric of the Frobenius manifold with a filtration on the tangent bundle equipped with metrics on its graded quotients. The purpose of the current paper is to construct a MFS on the cohomology of a smooth projective variety whose multiplication is the nonequivariant limit of the quantum product twisted by a concave vector bundle. We show that such a MFS is naturally obtained as the nonequivariant limit of the Frobenius structure in the equivariant setting.

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§1. Introduction

We continue our study of mixed Frobenius structure and local quantum cohomology initiated in [8].

§1.1. A mixed Frobenius algebra

Let K be a field. A finite-dimensional associative commutative K-algebra A equipped with a nondegenerate bilinear form g (called a *metric*) is called a *Frobenius algebra* if g is invariant under the product, i.e., $g(xy, z) = g(x, yz)$ for any $x, y, z \in A$.

In [8], the following generalization of the Frobenius algebra was introduced. Let A be a K-algebra as above. By definition, a *Frobenius filtration* $(I_\bullet, g_\bullet)$ on A consists of an exhaustive increasing filtration $I_\bullet$ by ideals and A-invariant met-

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rics $g_\bullet$ on its graded quotients (Definition 2.1). We call an algebra with a Frobenius filtration a *mixed Frobenius algebra*. If the filtration is trivial, this is nothing but the notion of Frobenius algebra. We show that any algebra over an algebraically closed field admits a Frobenius filtration (Theorem 2.3). This is in contrast to the fact that not all algebras admit invariant metrics.

One of the main results of this paper is to show that a mixed Frobenius K-algebra appears in the limit as $\lambda \to 0$ of a "Frobenius algebra over $K[\lambda]$" (§3). The precise statement is as follows. Let H_K^λ be a free $K[\lambda]$-module of finite rank equipped with a symmetric $K[\lambda]$-bilinear form $g^\lambda : H_K^\lambda \times H_K^\lambda \to K[\lambda, \lambda^{-1}]$. If g^λ is unimodular over $K[\lambda, \lambda^{-1}]$, then it defines on the K-vector space $H_K := H_K^\lambda / \lambda H_K^\lambda$ an exhaustive increasing filtration by subspaces and metrics on its graded quotients (Lemma 3.4). We call such a pair a *nondegenerate filtration*. Moreover, if H_K^λ is equipped with a $K[\lambda]$-algebra structure with respect to which g^λ is invariant, then the nondegenerate filtration is a Frobenius filtration on H_K with respect to the induced multiplication (Theorem 3.5). This construction is a generalization of the nilpotent construction in [8, §3.1] (cf. §3.2).

§1.2. A mixed Frobenius structure

A *Saito structure* (*without a metric*)[1] on a complex manifold M [11, §VII.1] is a triple consisting of a torsion-free flat affine connection ∇, a symmetric Higgs field $\Phi : T_M \to \operatorname{End} T_M$ and a vector field E called an Euler vector field satisfying certain compatibility conditions (see Definition 4.1). A symmetric Higgs field gives rise to a fiberwise commutative associative multiplication $\circ$ on T_M. If a Saito structure (∇, Φ, E) on M is further equipped with a $\circ$-invariant metric g on T_M compatible with the other data, then the Saito structure (∇, Φ, E) with the metric g is equivalent to a Frobenius manifold structure on M [2].

Now we introduce the notion of mixed Frobenius structure which generalizes the Frobenius manifold structure. The idea is to replace a $\circ$-invariant metric g with a Frobenius filtration $(I_\bullet, g_\bullet)$. Namely, we define a *mixed Frobenius structure* (MFS) on a manifold M to be a Saito structure (∇, Φ, E) on M together with a nondegenerate filtration $(I_\bullet, g_\bullet)$ on the tangent bundle T_M subject to various compatibility conditions (Definition 4.5). In particular, it is required that $(T_M, \circ, I_\bullet, g_\bullet)$ is a mixed Frobenius algebra. We arrived at this notion through our study of local mirror symmetry [1]. For details about the motivation, we refer to [8, §1.3]. Notice that we slightly modify the definition of MFS from [8] (cf. Remark 4.7).

[1] In this article, we call a Saito structure without a metric a *Saito structure* for short.

For the application to local quantum cohomology, it is necessary to consider a formal and logarithmic version of MFS. Let K be a subfield of $\mathbb{C}$. Let $R = K[[t_1, \ldots, t_n, q_1, \ldots, q_m]]$ and $M = \operatorname{Spf} R$ be the formal completion of $K^{n+m} = \operatorname{Spec} K[t, q]$ at the origin. We consider the logarithmic structure on M defined by the divisor $\{q_1 \cdots q_m = 0\}$ on K^{n+m}. We denote by $M^\dagger$ the resulting logarithmic formal scheme. We define a formal (and logarithmic) MFS on $M^\dagger$ in §5. As in the case of mixed Frobenius algebra, we show that a formal MFS on $M^\dagger$ is obtained in the limit as $\lambda \to 0$ of a "formal Frobenius structure over $K[\lambda]$" on $M^\dagger$ (Proposition 5.5).

§1.3. MFS from local quantum cohomology

Let X be a smooth complex projective variety and let $H_\mathbb{C} := H^{\mathrm{even}}(X, \mathbb{C})$. We choose a nef basis $\{\phi_1, \ldots, \phi_p\}$ of $H^2(X, \mathbb{Z})$ and extend it to a homogeneous basis $\{\phi_0 = 1, \phi_1, \ldots, \phi_p, \phi_{p+1}, \ldots, \phi_s\}$ of $H_\mathbb{C}$. Let $t_0, \ldots, t_s$ be the coordinates on $H_\mathbb{C}$ associated to the basis. We set $R = K[[t, q]]$ where $t = (t_0, t_{p+1}, \ldots, t_s)$ and $q = (q_1, \ldots, q_p)$ with $q_i = e^{t_i}$. Let $M^\dagger$ be the logarithmic formal scheme defined as in the previous subsection.

Fix a concave holomorphic vector bundle $\mathcal{V}$ on X (e.g., $\mathcal{V}$ is the dual of an ample line bundle). We construct a formal MFS on $M^\dagger$ from $\mathcal{V}$ as follows. Let us introduce the fiberwise S^1-action on $\mathcal{V}$ by scalar multiplication. Then, following Givental [3], we consider the S^1-equivariant Gromov–Witten invariants of X and the intersection pairing on X, both twisted by the inverse of the S^1-equivariant Euler class of $\mathcal{V}$. Using them, one can define the twisted quantum cup product $*_\mathcal{V}$ on $H_\mathbb{C}^\lambda := H_\mathbb{C} \otimes_\mathbb{C} \mathbb{C}[\lambda]$ where $\mathbb{C}[\lambda] = H^*_{S^1}(pt, \mathbb{C})$ is identified with the S^1-equivariant cohomology of a point. Identifying the logarithmic tangent sheaf $\mathcal{T}^\lambda_{M^\dagger} := \mathcal{T}_{M^\dagger} \otimes_\mathbb{C} \mathbb{C}[\lambda]$ of $M^\dagger$ over $\mathbb{C}[\lambda]$ with $\mathcal{O}_M \otimes_\mathbb{C} H^\lambda_\mathbb{C}$, we obtain a formal Frobenius structure over $\mathbb{C}[\lambda]$ on $M^\dagger$. Then, as an application of the results in §5, we obtain a formal MFS on $M^\dagger$ in the nonequivariant limit (i.e. the limit as $\lambda \to 0$) (Theorem 6.4).

As mentioned earlier, our motivation to study MFS comes from local mirror symmetry [1]. Relationships to [1] and to our previous work [7] are explained in §6.4.

§1.4. Conventions

(1) Let K be a field. A K-algebra means a finite-dimensional commutative associative K-algebra with a unit.
(2) Given a commutative ring R, an R-algebra structure on a free R-module means an associative commutative R-bilinear multiplication which admits a unit.

§2. Mixed Frobenius algebra

§2.1. Frobenius filtration and mixed Frobenius algebra

Let K be a field. A nondegenerate symmetric bilinear form g on a K-vector space is called a *metric*. A pair $(I_\bullet, g_\bullet)$ consisting of an exhaustive increasing filtration $I_\bullet$ on a K-vector space by subspaces and a collection of metrics $g_\bullet$ on $I_\bullet/I_{\bullet-1}$ is called a *nondegenerate filtration* on the vector space.

Let A be a K-algebra. We say that a metric g on an A-module I is A-*invariant* if it satisfies the condition

$$g(a \cdot x, y) = g(x, a \cdot y) \qquad (a \in A,\ x, y \in I).$$

Definition 2.1. A *Frobenius filtration* on a K-algebra A is a nondegenerate filtration $(I_\bullet, g_\bullet)$ on A such that each filter $I_\bullet$ is an ideal of A and the metric $g_\bullet$ on $I_\bullet/I_{\bullet-1}$ is $A/I_{\bullet-1}$-invariant.

Definition 2.2. A *mixed Frobenius K-algebra* is a pair which consists of a K-algebra A and a Frobenius filtration $(I_\bullet, g_\bullet)$ on A.

§2.2. Existence of Frobenius filtrations

In this subsection, the field K is assumed to be algebraically closed.

Theorem 2.3. *Any finite-dimensional K-algebra A has a Frobenius filtration.*

Let $\mathfrak{N} = \sqrt{0}$ be the nilradical of A. Note that the finite-dimensionality of A implies that $\mathfrak{N}$ is a nilpotent ideal and that $\mathfrak{N}$ coincides with the Jacobson radical of A. It follows that $A/\mathfrak{N}$ is a semisimple algebra. Consider the decreasing sequence of ideals $A \supset \mathfrak{N} \supset \mathfrak{N}^2 \supset \cdots \supset \mathfrak{N}^{l-1} \supset \mathfrak{N}^l = 0$.

Lemma 2.4. *$\mathfrak{N}^i/\mathfrak{N}^{i+1}$ is a completely reducible $A/\mathfrak{N}^{i+1}$-module.*

Proof. Consider the exact sequence of A-modules

$$0 \to \mathfrak{N}/\mathfrak{N}^{i+1} \to A/\mathfrak{N}^{i+1} \to A/\mathfrak{N} \to 0.$$

It follows that $A/\mathfrak{N}^{i+1}$ acts on $\mathfrak{N}^i/\mathfrak{N}^{i+1}$ via $A/\mathfrak{N}$, since $\mathfrak{N}/\mathfrak{N}^{i+1}$ annihilates $\mathfrak{N}^i/\mathfrak{N}^{i+1}$. Then the semisimplicity of $A/\mathfrak{N}$ implies that $\mathfrak{N}^i/\mathfrak{N}^{i+1}$ is a completely reducible $A/\mathfrak{N}$-module, hence it is also a completely reducible $A/\mathfrak{N}^{i+1}$-module. □

Lemma 2.5. *Let B be a finite-dimensional K-algebra. Then for any simple B-module $S \neq 0$, we have $\dim_K S = 1$.*

Proof. Since S is a simple B-module, there is a maximal ideal $\mathfrak{m}$ of B such that $S \cong B/\mathfrak{m}$ as B-modules. The finite-dimensionality of B implies that the composition

$K \to B \to B/\mathfrak{m}$ is a field extension of finite degree. It then follows that $B/\mathfrak{m} \cong K$, since K is algebraically closed. □

Proof of Theorem 2.3. By the above two lemmas, $\mathfrak{N}^i/\mathfrak{N}^{i+1}$ is the direct sum of 1-dimensional simple $A/\mathfrak{N}^{i+1}$-modules. If we take a basis $x_{i,j}$ $(1 \le j \le \dim_K \mathfrak{N}^i/\mathfrak{N}^{i+1})$ of the simple modules and define a bilinear form $\langle\ ,\ \rangle_i$ on $\mathfrak{N}^i/\mathfrak{N}^{i+1}$ by

$$\langle x_{i,j}, x_{i,k}\rangle_i = \delta_{j,k},$$

then $\langle\ ,\ \rangle_i$ is an invariant metric. Thus the filtration $I_\bullet := \mathfrak{N}^{l-\bullet}$ with metrics $g_\bullet := \langle\ ,\ \rangle_{i-\bullet}$ is a Frobenius filtration on A. □

§3. Mixed Frobenius algebra from a localized $K[\lambda]$-metric

§3.1. Construction of a mixed Frobenius algebra

Let K be a field and let H_K be a K-vector space of dimension s. We set $H_K^\lambda := H_K \otimes_K K[\lambda]$ and identify H_K with the quotient module $H_K^\lambda/\lambda H_K^\lambda$. Let $\pi : H_K^\lambda \to H_K = H_K^\lambda/\lambda H_K^\lambda$ be the projection.

Definition 3.1. A *localized* $K[\lambda]$*-metric* on H_K^λ is a symmetric $K[\lambda]$-bilinear form $g^\lambda : H_K^\lambda \times H_K^\lambda \to K[\lambda, \lambda^{-1}]$ which is unimodular over $K[\lambda, \lambda^{-1}]$[2].

Now assume that a localized $K[\lambda]$-metric g^λ on H_K^λ is given. We will construct from g^λ a nondegenerate filtration on H_K.

Lemma 3.2. *There exist a pair of* $K[\lambda]$*-module bases* $\boldsymbol{x}_1, \dots, \boldsymbol{x}_s$ *and* $\boldsymbol{y}_1, \dots, \boldsymbol{y}_s$ *of* H_K^λ *and a set of integers* $\kappa_1 \ge \cdots \ge \kappa_s$ *satisfying*

$$g^\lambda(\boldsymbol{x}_i, \boldsymbol{y}_j) = \lambda^{-\kappa_i}\delta_{i,j}. \tag{3.1}$$

The integers κ_i *are uniquely determined by* g^λ *(but the bases are not).*

Proof. Let G be the matrix representation of g^λ with respect to a $K[\lambda]$-module basis of H_K^λ. Multiplying by λ^{k_0} with some $k_0 \in \mathbb{Z}$ if necessary, we assume that all entries of the matrix $\lambda^{k_0}G$ are polynomials. By the theorem of elementary divisors, $\lambda^{k_0}G$ can be transformed into a diagonal matrix by successive elementary transformations from the left and from the right. This means that there exist $K[\lambda]$-module bases $\{\boldsymbol{x}_i\}$, $\{\boldsymbol{y}_i\}$ of H_K^λ such that $\lambda^{k_0}g_\lambda(\boldsymbol{x}_i, \boldsymbol{y}_j) = \delta_{i,j}\, e_i$ where $e_1, \dots, e_s \in K[\lambda]$ are diagonal entries (i.e. the elementary divisors of $\lambda^{k_0}G$). The assumption of unimodularity over $K[\lambda, \lambda^{-1}]$ implies that e_i's are monomials. □

[2] This means that, given a $K[\lambda]$-basis of H_K^λ, the representation matrix of g^λ is unimodular over $K[\lambda, \lambda^{-1}]$.

Let us define a sequence of $K[\lambda]$-submodules by

$$I_k^\lambda = \{\boldsymbol{x} \in H_K^\lambda \mid \lambda^k g^\lambda(\boldsymbol{x}, \boldsymbol{y}) \in K[\lambda] \ (\forall \boldsymbol{y} \in H_K^\lambda)\} \quad (k \in \mathbb{Z}). \tag{3.2}$$

Concretely, I_k^λ is written as follows with the basis $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_s$ of Lemma 3.2:

$$I_k^\lambda = \bigoplus_{i:\, \kappa_i \le k} K[\lambda]\, \boldsymbol{x}_i \oplus \bigoplus_{i:\, \kappa_i > k} \lambda^{\kappa_i - k} K[\lambda]\, \boldsymbol{x}_i. \tag{3.3}$$

The same formula holds for the other basis $\{\boldsymbol{y}_i\}$.

Lemma 3.3. *For $\boldsymbol{x}, \boldsymbol{y} \in I_k^\lambda$, $\operatorname{Res}_{\lambda=0} \lambda^{k-1} g^\lambda(\boldsymbol{x}, \boldsymbol{y})$ depends only on $\pi(\boldsymbol{x}), \pi(\boldsymbol{y}) \in H_K$. Moreover $\operatorname{Res}_{\lambda=0} \lambda^{k-1} g^\lambda(\boldsymbol{x}, \boldsymbol{y}) = 0$ if $\boldsymbol{x} \in I_{k-1}^\lambda$ or $\boldsymbol{y} \in I_{k-1}^\lambda$.*

Proof. Let us write $\boldsymbol{x}$, $\boldsymbol{y} \in I_k^\lambda$ as

$$\begin{aligned} \boldsymbol{x} &= \sum_{i:\, \kappa_i \le k} f_i(\lambda)\boldsymbol{x}_i + \sum_{i:\, \kappa_i > k} \lambda^{\kappa_i - k} f_i(\lambda)\boldsymbol{x}_i \quad (f_i(\lambda) \in K[\lambda]), \\ \boldsymbol{y} &= \sum_{i:\, \kappa_i \le k} h_i(\lambda)\boldsymbol{y}_i + \sum_{i:\, \kappa_i > k} \lambda^{\kappa_i - k} h_i(\lambda)\boldsymbol{y}_i \quad (h_i(\lambda) \in K[\lambda]). \end{aligned}$$

By (3.1), we obtain

$$\operatorname{Res}_{\lambda=0} \lambda^{k-1} g^\lambda(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i:\, \kappa_i = k} f_i(0) h_i(0). \tag{3.4}$$

The statement follows easily from this. □

Let

$$I_k := \pi(I_k^\lambda) = \bigoplus_{i:\, \kappa_i \le k} K\pi(\boldsymbol{x}_i) \quad (k \in \mathbb{Z}). \tag{3.5}$$

By Lemma 3.3, the following bilinear form g_k on I_k/I_{k-1} is well-defined:

$$g_k(\bar{x}, \bar{y}) = \operatorname{Res}_{\lambda=0} \lambda^{k-1} g^\lambda(\boldsymbol{x}, \boldsymbol{y}) \quad (x, y \in I_k), \tag{3.6}$$

where $x \mapsto \bar{x}$ denotes the projection $H_K \to H_K/I_{k-1}$ and $\boldsymbol{x}, \boldsymbol{y}$ are any lifts of x, y to I_k^λ.

Lemma 3.4. *$(I_\bullet, g_\bullet)$ is a nondegenerate filtration on H_K.*

Proof. The nondegeneracy of g_k follows from (3.4). □

Now assume that H_K^λ is equipped with an associative commutative $K[\lambda]$-algebra structure $*$ with unit. Let g^λ be a localized $K[\lambda]$-metric which is $*$-invariant, i.e.

$$g^\lambda(\boldsymbol{x} * \boldsymbol{y}, \boldsymbol{z}) = g^\lambda(\boldsymbol{x}, \boldsymbol{y} * \boldsymbol{z}) \quad (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in H^\lambda). \tag{3.7}$$

On H_K, we have the induced multiplication and the nondegenerate filtration $(I_\bullet, g_\bullet)$ defined in (3.5), (3.6).

Theorem 3.5. *$(H_K, I_\bullet, g_\bullet)$ is a mixed Frobenius algebra.*

Proof. The $*$-invariance (3.7) of g^λ implies that I_k^λ is an ideal. Therefore I_k is an ideal with respect to the induced multiplication $\circ$ on H_K. The $\circ$-invariance of g_k follows from the $*$-invariance of g^λ. □

§3.2. Nilpotent construction

Let (A, g) be a Frobenius K-algebra having nilpotent elements $n_1, \dots, n_r$ and let

$$\boldsymbol{n} = \lambda^r + n_1\lambda^{r-1} + \cdots + n_r \in A[\lambda].$$

As an example of Theorem 3.5, we consider the case $H_K^\lambda = A[\lambda]$ with the localized $K[\lambda]$-metric g^λ given by

$$g^\lambda(\boldsymbol{x}, \boldsymbol{y}) := g(\boldsymbol{x}\cdot\boldsymbol{y}, \boldsymbol{n}^{-1}) = \sum_{j\geq 0} \frac{1}{\lambda^{(j+1)r}} g(\boldsymbol{x}\cdot\boldsymbol{y}, (\lambda^r - \boldsymbol{n})^j) \qquad (\boldsymbol{x}, \boldsymbol{y} \in A[\lambda]).$$

Let us calculate the nondegenerate filtration $(I_\bullet, g_\bullet)$ defined by (3.5), (3.6). The ideals I_k^λ of $A[\lambda]$ defined in (3.2) are as follows.

Lemma 3.6. *We have*

$$I_k^\lambda = \begin{cases} \{\lambda^{-k}\boldsymbol{n}\cdot\boldsymbol{x} \mid \boldsymbol{x} \in A[\lambda]\} & (k \leq 0), \\ I_0^\lambda \oplus J_k^\lambda & (k > 0). \end{cases} \tag{3.8}$$

In the last line, the direct sum is that of A-modules and

$$J_k^\lambda = \{\boldsymbol{x} \in A^{<r}[\lambda] \mid \lambda^k\boldsymbol{x} \text{ is divisible by } \boldsymbol{n}\},$$

where $A^{<r}[\lambda] = \{\boldsymbol{x} \in A[\lambda] \mid \deg \boldsymbol{x} < r\}$. Here $\deg \boldsymbol{x}$ is the degree of $\boldsymbol{x}$ with respect to λ.

Proof. Since $\boldsymbol{n}$ is monic of degree r, any $\boldsymbol{x} \in A[\lambda]$ can be written uniquely as $\boldsymbol{x} = \boldsymbol{n}\cdot\boldsymbol{x}' + \boldsymbol{x}''$ with $\deg \boldsymbol{x}'' < r$.

First, we consider the case $k = 0$. It is easy to see that $\boldsymbol{x} \in I_0^\lambda$ if and only if $g^\lambda(\boldsymbol{x}'', y) = 0$ for any $y \in A$. If $\boldsymbol{x}'' = \sum_{i=1}^r a_i\lambda^{r-i}$ then we have

$$g^\lambda(\boldsymbol{x}'', y) = \sum_{1\leq i\leq r,\, j\geq 0} g(a_i(\lambda^r - \boldsymbol{n})^j, y)\lambda^{-(i+jr)} = \frac{g(a_1, y)}{\lambda} + o\left(\frac{1}{\lambda^2}\right).$$

From this equation, for $\boldsymbol{x}$ to be in I_0^λ, it is necessary to have $g(a_1, y) = 0$ for any $y \in A$, hence $a_1 = 0$. Then we have

$$g^\lambda(\boldsymbol{x}'', y) = \frac{g(a_2, y)}{\lambda^2} + o\left(\frac{1}{\lambda^3}\right),$$

hence $a_2 = 0$. Repeating this process, we obtain $\boldsymbol{x}'' = 0$.

Next, we consider the case $k < 0$. If $\boldsymbol{x} \in I_k^\lambda$ then $\boldsymbol{x} \in I_0^\lambda$. Therefore it can be written as $\boldsymbol{x} = \boldsymbol{n} \cdot \boldsymbol{x}'$. Since $\boldsymbol{x} \in I_k^\lambda$, we have $\lambda^k g^\lambda(\boldsymbol{x}, y) = \lambda^k g(\boldsymbol{x}', y) \in K[\lambda]$ for any $y \in A$. It then follows that the coefficients of $\boldsymbol{x}'$ up to degree $-k-1$ must be zero. Hence $\boldsymbol{x}'$ is divisible by λ^{-k}.

For $k > 0$, it is easy to see that $\boldsymbol{x} \in I_k^\lambda$ if and only if $\lambda^k \boldsymbol{x}''$ is divisible by $\boldsymbol{n}$. □

Let $N : A^{\oplus r} \to A^{\oplus r}$ be the homomorphism given by

$$N \begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix} = \begin{pmatrix} -n_1 & 1 & 0 & \cdots & 0 \\ -n_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -n_{r-1} & 0 & 0 & \cdots & 1 \\ -n_r & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix}.$$

The projections $A^{\oplus r} \to A$ to the first and the rth factors are denoted p_1, p_r.

Lemma 3.7. *We have*

$$I_k = \begin{cases} 0 & (k < 0), \\ \{x \cdot n_r \mid x \in A\} & (k = 0), \\ I_0 + J_k & (k > 0), \end{cases} \tag{3.9}$$

where $J_k = p_r(\operatorname{Ker} N^k)$.

Proof. By Lemma 3.6, it is enough to show that $\pi(J_k^\lambda) = J_k$ for $k > 0$. Let $\rho : A^{<r}[\lambda] \to A^{\oplus r}$ be the isomorphism $\sum_{i=1}^r a_i \lambda^{r-i} \mapsto {}^t(a_1, \dots, a_r)$. Notice that $\rho^{-1} \circ N \circ \rho : A^{<r}[\lambda] \to A^{<r}[\lambda]$ maps $\boldsymbol{x}$ to the remainder of $\lambda \boldsymbol{x}$ divided by $\boldsymbol{n}$. By induction on k, we can show that

$$\lambda^k \boldsymbol{x} = \sum_{i=0}^{k-1} (p_1 \circ N^i \circ \rho)(\boldsymbol{x}) \lambda^{k-1-i} \cdot \boldsymbol{n} + (\rho^{-1} \circ N^k \circ \rho)(\boldsymbol{x}) \quad (\boldsymbol{x} \in A^{<r}[\lambda]). \tag{3.10}$$

Thus we obtain

$$J_k^\lambda = \{\boldsymbol{x} \in A^{<r}[\lambda] \mid \rho(\boldsymbol{x}) \in \operatorname{Ker} N^k\}.$$

From this $\pi(J_k^\lambda) = J_k$ follows. □

Lemma 3.8. *We have*

$$g_0(x \cdot n_r, y \cdot n_r) = g(x \cdot y, n_r),$$
$$g_k(\bar{x}, \bar{y}) = g(x, p_1(N^{k-1} \vec{y})) \quad (k > 0,\ x, y \in J_k),$$

where $\vec{y} \in \operatorname{Ker} N^k$ *is any lift of* y *satisfying* $p_r(\vec{y}) = y$.

Proof. The case $k = 0$ is clear. For $k > 0$, let $\boldsymbol{x}, \boldsymbol{y} \in J_k^\lambda$ be lifts of x, y. By (3.10), we have

$$g_k(\bar{x}, \bar{y}) = g^\lambda\left(\boldsymbol{x}, \frac{\lambda^k \boldsymbol{y}}{\boldsymbol{n}}\right)\Big|_{\lambda=0} = g(x, p_1(N^{k-1}\rho(\boldsymbol{y}))). \qquad \square$$

As a corollary of Theorem 3.5, we obtain

Proposition 3.9. *$(A, I_\bullet, g_\bullet)$ with $I_\bullet, g_\bullet$ given in Lemmas 3.7 and 3.8 is a mixed Frobenius algebra.*

When $r = 1$, $J_k = \{x \in A \mid n_1^k \cdot x = 0\}$ and $g_k(\bar{x}, \bar{y}) = g(x \cdot y, (-n_1)^{k-1})$ $(k > 0)$. This is the nilpotent construction in [8, §3] up to shifts of the filtration.

§4. Mixed Frobenius structure

In this section, the base field is $K = \mathbb{C}$, a manifold means a complex manifold and vector bundles are assumed to be holomorphic. For a manifold M, T_M denotes the tangent bundle, $\mathcal{T}_M$ its sheaf of local sections and we write $x \in \mathcal{T}_M$ to mean that x is a local section of T_M.

Although definitions here are for complex manifolds, they can be easily translated to C^∞-manifolds ($K = \mathbb{R}$).

§4.1. Saito structure

The following definition is due to Sabbah [11, Ch. VII].

Definition 4.1. Let M be a manifold. A *Saito structure* (*without a metric*) on M consists of

- a torsion-free flat connection ∇ on T_M,
- an associative and commutative $\mathcal{O}_M$-bilinear multiplication $\circ$ on $\mathcal{T}_M$ with a global unit section e, and
- a global vector field E on M (called the *Euler vector field*),

satisfying the following conditions.

(i) The multiplication C_x by $x \in \mathcal{T}_M$ regarded as a local section of $\operatorname{End} T_M$ satisfies

$$\nabla_x C_y - \nabla_y C_x = C_{[x,y]}, \tag{4.1}$$

and the unit vector field e is flat, i.e. $\nabla e = 0$.

(ii) The vector field E satisfies $\nabla(\nabla E) = 0$ and

$$[E, x \circ y] - [E, x] \circ y - x \circ [E, y] = x \circ y \qquad (x, y \in \mathcal{T}_M). \tag{4.2}$$

In this article, we call a Saito structure without a metric a *Saito structure* for short.

Remark 4.2. In [11], a Saito structure is defined in terms of the symmetric Higgs field instead of the multiplication. As explained in [11, Ch. 0.13], a symmetric Higgs field corresponds to a multiplication and Definition 4.1 is equivalent to that in [11].

Lemma 4.3. *Given a Saito structure $(\nabla, \circ, E)$ on a manifold M, there exists a local vector field $\mathcal{G} \in \mathcal{T}_M$ such that*

$$\nabla_x \nabla_y \mathcal{G} = x \circ y \tag{4.3}$$

for any flat vector fields $x, y \in \mathcal{T}_M$. Moreover $\nabla\nabla([E, \mathcal{G}] - \mathcal{G}) = 0$.

We call $\mathcal{G}$ satisfying (4.3) a (local) *potential vector field*.

Proof. Let $\{t_\alpha\}_\alpha$ be a local coordinate system on M whose corresponding local frame fields $\{\partial_\alpha\}_\alpha$ are ∇-flat. Let us write $\partial_\alpha \circ \partial_\beta = \sum_\gamma C^\gamma_{\alpha\beta} \partial_\gamma$. The commutativity implies $C^\gamma_{\alpha\beta} = C^\gamma_{\beta\alpha}$. Equation (4.1) is equivalent to $\partial_\alpha C^\delta_{\beta\gamma} = \partial_\beta C^\delta_{\alpha\gamma}$. Therefore there exist $\mathcal{G}^\gamma \in \mathcal{O}_M$ such that $\partial_\alpha \partial_\beta \mathcal{G}^\gamma = C^\gamma_{\alpha\beta}$. Then $\mathcal{G} := \sum_\gamma \mathcal{G}^\gamma \partial_\gamma$ satisfies (4.3). The second statement follows from (4.2). $\square$

Remark 4.4. It is known that a Frobenius manifold structure defined by Dubrovin [2] is equivalent to a Saito structure with a metric [11, Ch. VII, Prop. 2.2]. When M is a Frobenius manifold, (4.1) is equivalent to the potentiality condition, and the gradient vector field of the potential function is a potential vector field.

§4.2. Mixed Frobenius structure

Definition 4.5. A *mixed Frobenius structure* (MFS) on a manifold M consists of a Saito structure $(\nabla, \circ, E)$ together with

- an increasing sequence of subbundles $I_\bullet$ of T_M and
- metrics (i.e. nondegenerate symmetric $\mathcal{O}_M$-bilinear forms) $g_\bullet$ on $\mathcal{I}_\bullet / \mathcal{I}_{\bullet-1}$

satisfying the following conditions.

(i) $(\circ, I_\bullet, g_\bullet)$ is a mixed Frobenius algebra structure on T_M, i.e. $\mathcal{I}_k$ are ideals of $\mathcal{T}_M$ and all g_k's are $\circ$-invariant.

(ii) The subbundles I_k $(k \in \mathbb{Z})$ are preserved by ∇ and the metrics are compatible with ∇, i.e.

$$z g_k(\overline{x}, \overline{y}) = g_k(\overline{\nabla_z x}, \overline{y}) + g_k(\overline{x}, \overline{\nabla_z y}) \quad (k \in \mathbb{Z},\ z \in \mathcal{T}_M,\ x, y \in \mathcal{I}_k). \tag{4.4}$$

Here $x \mapsto \overline{x}$ denotes the projection $\mathcal{I}_k \to \mathcal{I}_k / \mathcal{I}_{k-1}$.

(iii) The subbundles I_k ($k \in \mathbb{Z}$) are preserved by $[E, -]$ and there exists a collection $\{D_k \in K\}_{k \in \mathbb{Z}}$ of numbers (called *charges*) such that

$$
\begin{aligned}
\text{(4.5)} \qquad & E g_k(\overline{x}, \overline{y}) - g_k(\overline{[E, x]}, \overline{y}) - g_k(\overline{x}, \overline{[E, y]}) \\
& \qquad = (2 - D_k) g_k(\overline{x}, \overline{y}) \qquad (k \in \mathbb{Z},\ x, y \in \mathcal{I}_k).
\end{aligned}
$$

A MFS with the trivial filtration $I_\bullet$ (i.e. $0 \subset T_M$) is the same as a Saito structure with a metric [11] and also the same as a Frobenius manifold structure [2].

Lemma 4.6. *If* $(\nabla, \circ, E, I_\bullet, g_\bullet)$ *is a MFS on a manifold* M*, then each* $\mathcal{I}_k \subset \mathcal{T}_M$ ($k \in \mathbb{Z}$) *is involutive.*

Proof. This follows from the condition that I_k is preserved by the torsion free affine connection ∇. □

As a consequence of this lemma, there exists a flat local coordinate system $\{t_{k\alpha}\}_{k \in \mathbb{Z},\, 1 \le \alpha \le \dim I_k / I_{k-1}}$ such that $\{t_{k\alpha}\}_{k \le l,\, 1 \le \alpha \le \dim I_k / I_{k-1}}$ is a local coordinate system of leaves of I_l.

Remark 4.7. The definition of MFS in this article is different from that in our previous article [8, Definition 6.2] in a few points.

Firstly the charges D_k are allowed to take any values. The advantage is that any mixed Frobenius algebra has a MFS (see Proposition 4.8 below) whereas the condition $D_k = D_0 - k$ in the old definition is quite restrictive.

Secondly the compatibility conditions of the multiplication with the connection and the Euler vector field are strengthened as we adopt the Saito structure (compare [8, (6.2), (6.9)] with (4.1), (4.2)). The reason for this change is the existence of a local potential vector field (Lemma 4.3) and the flat meromorphic connection [11] on the Saito structure. We believe that they may play important roles in formulating local mirror symmetry as an equivalence of MFS's (cf. §6.4).

§4.3. An algebra with a Frobenius filtration has a MFS

Let $(A, I_\bullet, g_\bullet)$ be a mixed Frobenius algebra. We assume that $A = \bigoplus_{d \in \mathbb{Z}} A_d$ is a graded algebra satisfying $I_k = \bigoplus_d I_k \cap A_d$. Moreover we assume that there exist $\{D_k \in \mathbb{Z} \mid k \in \mathbb{Z},\, I_k / I_{k-1} \neq 0\}$ such that

$$
g_k(x, y) = 0 \quad \text{unless} \quad |x| + |y| = D_k.
$$

Here $|x|$ denotes the degree of $x \in A$. Notice that any mixed Frobenius algebra satisfies this assumption with $A = A_0$ and $D_0 = 0$.

Let $\{e_{k\alpha} \mid k \in \mathbb{Z}, 1 \le \alpha \le \dim I_k / I_{k-1}\}$ be a homogeneous basis of A such that $\{e_{k\alpha} \mid k \le l, 1 \le \alpha \le \dim I_l / I_{l-1}\}$ is a basis of I_l. Let $\{t_{k\alpha}\}$ be the associated coordinates of A.

Proposition 4.8. *The trivial connection d, the multiplication on A, the vector field*

$$E = \sum_{k,\alpha}(1 - |e_{ka}|)t_{ka}\partial_{ka},$$

and the nondegenerate filtration $(I_\bullet, g_\bullet)$ form a MFS on A of charges $\{D_k\}$.

§5. Formal mixed Frobenius structure

In this section we will define a formal (and logarithmic) version of MFS using [4] as reference.

In this section, the base field K may be any subfield of $\mathbb{C}$.

§5.1. Notation

Fix $n \in \mathbb{Z}_{>0}$ and $m \in \mathbb{Z}_{\geq 0}$. Set $R = K[[t_1, \dots, t_n, q_1, \dots, q_m]]$ and

$$P = \{\Phi(t,q) \in R \mid \exists d_1, \dots, d_m \in \mathbb{Z}_{\geq 0} \text{ such that } q_1^{-d_1} \cdots q_m^{-d_m} \Phi(t,q) \in R^\times\}, \tag{5.1}$$

which is a submonoid of R. Let $M = \operatorname{Spf} R$ be the formal completion of $K^{n+m} = \operatorname{Spec} K[t,q]$ at the origin and let P_M be the constant sheaf on M with a stalk P. Denote by $M^\dagger$ the formal scheme M equipped with the logarithmic structure associated to $P_M \hookrightarrow \mathcal{O}_M$.

Let $\mathcal{T}_{M^\dagger}$ be the sheaf of logarithmic vector fields on $M^\dagger$ which is freely generated by $\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n}$ and $q_1 \frac{\partial}{\partial q_1}, \dots, q_m \frac{\partial}{\partial q_m}$ over $\mathcal{O}_M$. Namely, if we let

$$H_K = \bigoplus_{\alpha=1}^{n} K \frac{\partial}{\partial t_\alpha} \oplus \bigoplus_{i=1}^{m} K q_i \frac{\partial}{\partial q_i}, \tag{5.2}$$

then $\mathcal{T}_{M^\dagger} = \mathcal{O}_M \otimes_K H_K$. Define a flat connection ∇ on $\mathcal{T}_{M^\dagger}$ by $\nabla = d \otimes 1_{H_K}$. The Lie bracket $[\ ,\ ]$ satisfies

$$[x,y] = \nabla_x y - \nabla_y x \quad (x, y \in \mathcal{T}_{M^\dagger}). \tag{5.3}$$

§5.2. Formal mixed Frobenius structure

We keep the notation of §5.1.

Definition 5.1. A *formal Saito structure* on $M^\dagger$ consists of

- an $\mathcal{O}_M$-bilinear multiplication $\circ$ on $\mathcal{T}_{M^\dagger}$ and
- an element $E \in \mathcal{T}_{M^\dagger}$

satisfying the following conditions.

(i) The multiplication $\circ$ is compatible with ∇ in the sense that

$$\nabla_x(y \circ z) = \nabla_y(x \circ z) \quad (x, y, z \in H_K), \tag{5.4}$$

and the unit element e satisfies $\nabla e = 0$, i.e. $e \in H_K$.

(ii) The element E satisfies[3] $\nabla_x \nabla_y E = 0$ for $x, y \in H_K$ and

$$[E, x \circ y] - x \circ [E, y] - [E, x] \circ y = x \circ y \quad (x, y \in \mathcal{T}_{M^\dagger}). \tag{5.5}$$

If $(\circ, E)$ is a formal Saito structure on $M^\dagger$, then as in Lemma 4.3, there exists $\mathcal{G} \in K[\log q_1, \ldots, \log q_m] \otimes_K \mathcal{T}_{M^\dagger}$ satisfying (4.3) for $x, y \in H_K$.

Definition 5.2. A *formal mixed Frobenius structure* on $M^\dagger$ consists of

- a formal Saito structure $(\circ, E)$ on $M^\dagger$ and
- a nondegenerate filtration $(I_\bullet, g_\bullet)$ on H_K

satisfying the following conditions:

(i) $\mathcal{I}_k = \mathcal{O}_M \otimes_K I_k$ is an ideal and g_k extended $\mathcal{O}_M$-bilinearly to $\mathcal{I}_k$ is $\circ$-invariant, i.e.

$$g_k(x \circ y, z) = g_k(y, x \circ z) \quad (x \in \mathcal{T}_{M^\dagger},\ y, z \in \mathcal{I}_k). \tag{5.6}$$

(ii) $\mathcal{I}_k$ is preserved by $[E, -]$, i.e. $[E, x] \in \mathcal{I}_k$ $(x \in \mathcal{I}_k)$ and there exists a collection $\{D_k \in K \mid k \in \mathbb{Z}, I_k/I_{k-1} \neq 0\}$ of numbers, called *charges*, satisfying

$$E g_k(\bar{x}, \bar{y}) - g_k(\overline{[E, x]}, \bar{y}) - g_k(\bar{x}, \overline{[E, y]}) = (2 - D_k) g_k(\bar{x}, \bar{y}) \quad (x, y \in \mathcal{I}_k). \tag{5.7}$$

Here $x \mapsto \bar{x}$ denotes the projection $\mathcal{I}_k \to \mathcal{I}_k/\mathcal{I}_{k-1}$.

Remark 5.3 (on the convergent case). Let $(\circ, E)$ (resp.$(\circ, E, I_\bullet, g_\bullet)$) be a formal Saito structure (resp. a formal MFS) on $M^\dagger$ and let $C_{\alpha\beta}^\gamma \in \mathcal{O}_M$ $(1 \leq \alpha, \beta, \gamma \leq n+m)$ denote the structure constants of $\circ$ with respect to the basis $(x_1, \ldots, x_{n+m}) = (\partial_{t_1}, \ldots, \partial_{t_n}, q_1\partial_{q_1}, \ldots, q_m\partial_{q_m})$. If there exists an open neighborhood U' of $0 \in K^{n+m} = \operatorname{Spec} K[t, q]$ where all $C_{\alpha\beta}^\gamma$ converge, then $(\nabla, \circ, E)$ (resp. $(\nabla, \circ, E, I_\bullet, g_\bullet)$) is a Saito structure (resp. a MFS) on $U = U' \cap \{q_1 \cdots q_m \neq 0\}$ with local flat coordinates $t_1, \ldots, t_n, \log q_1, \ldots, \log q_m$. In the case when the filtration $I_\bullet$ is trivial, then $(U, \circ, E, e, g_\bullet)$ is a Frobenius manifold with logarithmic poles along the divisor $\{q_1 \cdots q_m = 0\}$ (see [10] for the definition).

[3] If we write $E = \sum_{\alpha=1}^n E_\alpha \partial_{t_\alpha} + \sum_{i=1}^m E_i q_i \partial_{q_i}$, the condition $\nabla\nabla E = 0$ implies that E_α $(1 \leq \alpha \leq n)$ and E_i $(1 \leq i \leq m)$ are linear polynomials in t independent of q.

§5.3. Localized formal Frobenius structure over $K[\lambda]$

We still keep the notation of §5.1 and use superscripts λ for objects tensored with $K[\lambda]$: $\mathcal{O}_M^\lambda := \mathcal{O}_M \otimes_K K[\lambda]$, $H_K^\lambda = H_K \otimes_K K[\lambda]$, and $\mathcal{T}_{M^\dagger}^\lambda := \mathcal{T}_{M^\dagger} \otimes_K K[\lambda] = \mathcal{O}_M \otimes_K H_K^\lambda$. We have a flat connection ∇ on $\mathcal{T}_{M^\dagger}^\lambda$ defined by the $K[\lambda]$-linear extension of that introduced in §5.1.

Definition 5.4. A *localized formal Frobenius structure over* $K[\lambda]$ on $M^\dagger$ consists of

- an $\mathcal{O}_M^\lambda$-bilinear multiplication $*$ on $\mathcal{T}_{M^\dagger}^\lambda$,
- an element $\boldsymbol{E} \in \mathcal{T}_{M^\dagger}^\lambda$, and
- a localized $K[\lambda]$-metric g^λ on H_K^λ,

satisfying the following conditions.

(i) The multiplication $*$ is compatible with ∇ in the sense that

$$\nabla_{\boldsymbol{x}}(\boldsymbol{y} * \boldsymbol{z}) = \nabla_{\boldsymbol{y}}(\boldsymbol{x} * \boldsymbol{z}) \quad (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in H_K^\lambda), \tag{5.8}$$

and the unit $\boldsymbol{e}$ satisfies $\nabla \boldsymbol{e} = 0$, i.e. $\boldsymbol{e} \in H_K^\lambda$.

(ii) The element $\boldsymbol{E}$ satisfies $\nabla_{\boldsymbol{x}} \nabla_{\boldsymbol{y}} \boldsymbol{E} = 0$ for $\boldsymbol{x}, \boldsymbol{y} \in H_K^\lambda$ and

$$[\boldsymbol{E}^\lambda, \boldsymbol{x} * \boldsymbol{y}] - \boldsymbol{x} * [\boldsymbol{E}^\lambda, \boldsymbol{y}] - [\boldsymbol{E}^\lambda, \boldsymbol{x}] * \boldsymbol{y} = \boldsymbol{x} * \boldsymbol{y} \quad (\boldsymbol{x}, \boldsymbol{y} \in \mathcal{T}_{M^\dagger}^\lambda), \tag{5.9}$$

where $\boldsymbol{E}^\lambda := \boldsymbol{E} + \lambda \frac{\partial}{\partial \lambda}$.

(iii) g^λ, extended $\mathcal{O}_M^\lambda$-bilinearly to $\mathcal{T}_{M^\dagger}^\lambda$, is $*$-invariant.

(iv) There exists $D \in K$ (called a *charge*) satisfying

$$\begin{aligned} \boldsymbol{E}^\lambda g^\lambda(\boldsymbol{x}, \boldsymbol{y}) - g^\lambda([\boldsymbol{E}^\lambda, \boldsymbol{x}], \boldsymbol{y}) - g^\lambda(\boldsymbol{x}, [\boldsymbol{E}^\lambda, \boldsymbol{y}]) \\ = (2 - D) g^\lambda(\boldsymbol{x}, \boldsymbol{y}) \quad (\boldsymbol{x}, \boldsymbol{y} \in \mathcal{T}_{M^\dagger}^\lambda). \end{aligned} \tag{5.10}$$

Proposition 5.5. *Let* $(*, \boldsymbol{E}, g^\lambda)$ *be a localized formal Frobenius structure over* $K[\lambda]$ *of charge* D *on* $M^\dagger$. *Let* $\circ$ *be the multiplication on* $\mathcal{T}_{M^\dagger}$ *induced by* $\pi : \mathcal{T}_{M^\dagger}^\lambda \to \mathcal{T}_{M^\dagger} = \mathcal{T}_{M^\dagger}^\lambda / \lambda \mathcal{T}_{M^\dagger}^\lambda$, $E = \pi(\boldsymbol{E})$, *and let* $(I_\bullet, g_\bullet)$ *be the nondegenerate filtration on* H_K *induced from the localized* $K[\lambda]$*-metric* g^λ *(see Lemma 3.4). Then* $(\circ, E, I_\bullet, g_\bullet)$ *is a formal MFS on* $M^\dagger$ *of charges* $\{D_k = D - k\}$.

Proof. First, the conditions (i) and (ii) of Definition 5.1 follow from (i) and (ii) in Definition 5.4 respectively.

The $*$-invariance of g^λ implies that $\mathcal{O}_M \otimes I_k^\lambda =: \mathcal{I}_k^\lambda \subset \mathcal{T}_{M^\dagger}^\lambda$ is an ideal with respect to $*$, which in turn implies that $\mathcal{I}_k$ is an ideal with respect to $\circ$. It also implies the $\circ$-invariance of the metrics $g_\bullet$.

Equation (5.10) implies that the Lie bracket $[\boldsymbol{E}^\lambda, -]$ preserves $\mathcal{I}_k^\lambda$. From this it follows that $[E, -]$ preserves $\mathcal{I}_k$. Equation (5.10) also implies (5.7) as follows. For $x, y \in \mathcal{I}_k$, we have

$$\begin{aligned} Eg_k(\bar{x}, \bar{y}) &= \mathrm{Res}_{\lambda=0}\, \lambda^{k-1}(k + \boldsymbol{E}^\lambda) g^\lambda(\boldsymbol{x}, \boldsymbol{y}) \\ &\overset{(5.10)}{=} k g_k(\bar{x}, \bar{y}) \\ &\quad + \mathrm{Res}_{\lambda=0}\, \lambda^{k-1}\{g^\lambda([\boldsymbol{E}^\lambda, \boldsymbol{x}], \boldsymbol{y}) + g^\lambda(\boldsymbol{x}, [\boldsymbol{E}^\lambda, \boldsymbol{y}]) + (2-D) g^\lambda(\boldsymbol{x}, \boldsymbol{y})\} \\ &= (2 - D + k) g_k(\bar{x}, \bar{y}) + g_k([E, x], y) + g_k(x, [E, y]), \end{aligned}$$

where $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{I}_k^\lambda$ are lifts of x, y. □

§6. Local quantum cohomology

In this section, K denotes either $\mathbb{R}$ or $\mathbb{C}$.

§6.1. Notation

Let X be a smooth complex projective variety. Let $\mathcal{V} \to X$ be a concave[4] vector bundle of rank r. Let $S^1 = \mathrm{U}(1)$ act on $\mathcal{V}$ by the scalar multiplication on the fiber. The generator of the S^1-equivariant cohomology of a point is denoted λ.

Let $H_K := H^{\mathrm{even}}(X, K)$. We fix a basis $\{\phi_1, \dots, \phi_p\}$ of $H^2(X, \mathbb{Z})$ satisfying the condition that $\int_C \phi_i \geq 0$ for any curve $C \subset X$.[5] We also fix a homogeneous basis $\{\phi_0 = 1, \phi_1, \dots, \phi_p, \phi_{p+1}, \dots, \phi_s\}$ of H_K.

Let $t_0, \dots, t_s$ be the coordinates on H_K associated to the basis. We set $R = K[[t, q]]$ where $t = (t_0, t_{p+1}, \dots, t_s)$ and $q = (q_1, \dots, q_p)$ with $q_i = e^{t_i}$. As in §5.1, we consider the formal scheme $M = \mathrm{Spf}\, R$ with a fixed logarithmic structure defined by the monoid (5.1) and denote it by $M^\dagger$. We identify H_K with the linear space of derivations on R defined in (5.2) by

$$\begin{cases} \phi_\alpha \mapsto \frac{\partial}{\partial t_\alpha} & (\alpha = 0,\ p+1, \dots, s), \\ \phi_i \mapsto q_i \frac{\partial}{\partial q_i} & (1 \leq i \leq p). \end{cases} \tag{6.1}$$

Hence $\mathcal{T}_{M^\dagger} = \mathcal{O}_M \otimes_K H_K$. The same notations $\mathcal{O}_M^\lambda$, H_K^λ and $\mathcal{T}_{M^\dagger}^\lambda$ as in §5.3 will be used.

---

[4] A vector bundle $\mathcal{V}$ is *concave* if $H^0(C, f^*\mathcal{V}) = 0$ for any genus zero stable map (f, C) to X of nonzero degree.

[5] The existence of such a basis follows from the fact that the Mori cone $\overline{NE}_{\mathbb{R}}(X)$ of a smooth projective variety X does not contain a straight line (see e.g. [6, Corollary 1.19]). If σ denotes the image of $\overline{NE}_{\mathbb{R}}(X)$ in $H_2(X, \mathbb{R})$, the dual cone $\sigma^\vee = \{x \in H^2(X, \mathbb{R}) \mid \langle x, y \rangle \geq 0,\ y \in \sigma\}$ is of maximal dimension. Therefore there exists an integral basis $\phi_1, \dots, \phi_p$ of $H^2(X, \mathbb{R})$ such that $\phi_i \in \sigma^\vee$.

We put the grading on the vector space H_K by setting $|\phi| = k$ if $\phi \in H^{2k}(X, K)$. We also put the gradings on the rings $\mathcal{O}_M$ and $\mathcal{O}_M^\lambda$ by $|t_\alpha| = 1 - |\phi_\alpha|$ $(\alpha = 0, p+1, \dots, s)$, $|\lambda| = 1$ and $|q_i| = \xi_i$, where ξ_i are defined by

$$c_1(X) + c_1(\mathcal{V}) = \sum_{i=1}^{p} \xi_i \phi_i. \tag{6.2}$$

Then we have the induced gradings on $\mathcal{T}_{M^\dagger}$ and $\mathcal{T}_{M^\dagger}^\lambda$.

Let

$$\boldsymbol{E} = \sum_{\alpha=0}^{s} (1 - |\phi_\alpha|) t_\alpha \frac{\partial}{\partial t_\alpha} + \sum_{i=1}^{p} \xi_i q_i \frac{\partial}{\partial q_i}, \qquad \boldsymbol{E}^\lambda = \boldsymbol{E} + \lambda \frac{\partial}{\partial \lambda}. \tag{6.3}$$

Then, for a homogeneous $f \in \mathcal{O}_M^\lambda$ and $\boldsymbol{x} \in \mathcal{T}_{M^\dagger}^\lambda$, we have

$$\boldsymbol{E}^\lambda f = |f| f, \qquad [\boldsymbol{E}^\lambda, \boldsymbol{x}] = (|\boldsymbol{x}| - 1)\boldsymbol{x}. \tag{6.4}$$

§6.2. Localized formal Frobenius structure over $K[\lambda]$

The following material can be found in [3]. Let g^λ be a localized $K[\lambda]$-metric on H_K^λ defined by

$$g^\lambda(\phi, \varphi) = \int_X \phi \cup \varphi \cup \frac{1}{e_{S^1}(\mathcal{V})} \tag{6.5}$$

where $e_{S^1}(\mathcal{V})$ is the S^1-equivariant Euler class of $\mathcal{V}$:

$$e_{S^1}(\mathcal{V}) = \lambda^r + c_1(\mathcal{V})\lambda^{r-1} + \cdots + c_r(\mathcal{V}).$$

Lemma 6.1. *g^λ and $\boldsymbol{E}$ (in (6.3)) satisfy (5.10) with $D = \dim_{\mathbb{C}} X + r$.*

Proof. By degree consideration, g^λ satisfies

$$g^\lambda(\phi_\alpha, \phi_\beta) = \eta_{\alpha\beta} \lambda^{|\phi_\alpha| + |\phi_\beta| - \dim_{\mathbb{C}} X - r} \qquad (\eta_{\alpha\beta} \in K). \tag{6.6}$$

This together with (6.4) implies the lemma. □

We define a multiplication on $\mathcal{T}_{M^\dagger}^\lambda$ as follows. For $x_1, \dots, x_m \in H_K$ and $d \in H_2(X, \mathbb{Z})$, let

$$\langle x_1, \dots, x_m \rangle_{\mathcal{V},d} = \int_{[\overline{M}_{0,m}(X,d)]^{\mathrm{vir}}} \prod_{i=1}^{m} \mathrm{ev}_i^* x_i \cup e_{S^1}(-R^\bullet \mu_* \mathrm{ev}_{m+1}^* \mathcal{V}) \in K[\lambda] \tag{6.7}$$

where $\overline{M}_{0,m}(X,d)$ is the moduli stack of genus zero stable maps to X of degree d with m marked points, $\mathrm{ev}_i : \overline{M}_{0,m}(X,d) \to X$ is the evaluation map at the

ith marked point, and $\mu : \overline{M}_{0,m+1}(X,d) \to \overline{M}_{0,m}(X,d)$ is the forgetful map. We define the multiplication $*_{\mathcal{V}}$ on $\mathcal{T}^{\lambda}_{M^\dagger}$ by

$$(6.8)\qquad g^{\lambda}(\boldsymbol{x} *_{\mathcal{V}} \boldsymbol{y}, \boldsymbol{z}) = \sum_{d}\sum_{m\geq 0}\frac{1}{m!}\langle \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \underbrace{\tau,\dots,\tau}_{m}\rangle_{\mathcal{V},d} \qquad (\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\in\mathcal{T}^{\lambda}_{M^\dagger})$$
$$= \sum_{d}\sum_{m\geq 0}\frac{1}{m!}\langle \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \underbrace{\tau_{\geq 4},\dots,\tau_{\geq 4}}_{m}\rangle_{\mathcal{V},d}\, q^d.$$

In the first line, $\tau = \sum_{\alpha=0}^{s} t_\alpha \phi_\alpha$, and in the second line, $\tau_{\geq 4} = \sum_{\alpha=p+1}^{s} t_\alpha\phi_\alpha$ and $q^d = e^{\int_d (t_1\phi_1+\cdots+t_p\phi_p)}$. In passing to the second line, the fundamental class axiom and the divisor axiom of Gromov–Witten theory (see, e.g., [9, III, §5]) are used.

Lemma 6.2. *$(\mathcal{T}^{\lambda}_{M^\dagger}, *_{\mathcal{V}})$ is a graded ring. Hence the multiplication $*_{\mathcal{V}}$ and $\boldsymbol{E}$ in (6.3) satisfy (5.9).*

Proof. The lemma follows from the degree axiom of Gromov–Witten theory. □

Proposition 6.3. *$(g^{\lambda}, *_{\mathcal{V}}, \boldsymbol{E})$ is a localized formal Frobenius structure over $K[\lambda]$ of charge $\dim_{\mathbb{C}} X + r$ on $M^\dagger$.*

Proof. By the definition of $*_{\mathcal{V}}$, it is clear that g^{λ} is $*_{\mathcal{V}}$-invariant and satisfies (5.8). □

§6.3. Formal mixed Frobenius structure from local quantum cohomology

Theorem 6.4. *The collection $(\circ_{\mathcal{V}}, E, I_\bullet, g_\bullet)$ of the following data determines a formal MFS of charges $\{\dim_{\mathbb{C}} X + r - k\}_{k\in\mathbb{Z}}$ on $M^\dagger$;*

- *the multiplication $\circ_{\mathcal{V}}$ on $\mathcal{T}_{M^\dagger}$ induced from the multiplication $*_{\mathcal{V}}$ on $\mathcal{T}^{\lambda}_{M^\dagger}$,*
- *the Euler vector field E which has the same expression as $\boldsymbol{E}$ in (6.3),*
- *a nondegenerate filtration $(I_\bullet, g_\bullet)$ on H_K constructed in Lemma 3.4.*

Proof. Applying Proposition 5.5 to the localized formal Frobenius structure over $K[\lambda]$ in Proposition 6.3, we obtain the result. □

Remark 6.5 (on convergence of the formal MFS). If $\mathcal{V} \to X$ is a negative line bundle, it can be shown that the structure constants of $\circ_{\mathcal{V}}$ are convergent if those of the quantum product of X are convergent, e.g. if X is a smooth projective toric variety [5]. The proof is completely the same as Iritani's [5] except that it is necessary to modify the proof of his Lemma 4.2. For a pair of such X and a negative line bundle $\mathcal{V}$, the formal MFS described in this subsection is actually a MFS on some open subset of H_K (see Remark 5.3).

Let us describe the MFS in Theorem 6.4 concretely. The multiplication $\circ_{\mathcal{V}}$ on $\mathcal{T}_{M^\dagger}$ is as follows. For $d \neq 0,\ x_1, \dots, x_m \in H_K$, let

$$\langle x_1, \dots, x_m \rangle_{\mathcal{V},d}^{\lambda=0} = \int_{[\overline{M}_{0,m}(X,d)]^{\mathrm{vir}}} \prod_{i=1}^{m} \mathrm{ev}_i^* x_i \cup e(R^1\mu_* \mathrm{ev}_{m+1}^* \mathcal{V}), \tag{6.9}$$

where e denotes the (nonequivariant) Euler class. Then a potential vector field $\mathcal{G}$ for $\circ_{\mathcal{V}}$ (cf. Lemma 4.3) is given by

$$\mathcal{G} = \sum_{\alpha=0}^{s} (\partial_\alpha \Phi_{\mathrm{cl}})\phi^\alpha + \sum_{\alpha=1}^{s} (\partial_\alpha \Phi_{\mathrm{qu}}) c_r(\mathcal{V}) \cup \phi^\alpha, \tag{6.10}$$

where $\partial_\alpha = \frac{\partial}{\partial t_\alpha}$ and

$$\Phi_{\mathrm{cl}} = \frac{1}{3!}\int_X \tau \cup \tau \cup \tau, \qquad \Phi_{\mathrm{qu}} = \sum_{d\neq 0}\sum_{m \geq 0} \frac{q^d}{m!} \langle \underbrace{\tau_{\geq 4}, \dots, \tau_{\geq 4}}_{m} \rangle_{\mathcal{V},d}^{\lambda=0},$$

and $\{\phi^\alpha\}$ is a basis of H_K dual to $\{\phi_\alpha\}$ with respect to the intersection form of X.

By the result of §3.2, the nondegenerate filtration $(I_\bullet, g_\bullet)$ on H_K is

$$\begin{aligned} I_k &= 0 \quad (k<0),\\ I_0 &= \{x \cup c_r(\mathcal{V}) \mid x \in H_K\},\\ I_k &= I_0 + J_k, \qquad J_k = p_r(\mathrm{Ker}\, N^k), \end{aligned} \tag{6.11}$$

where

$$N = \begin{pmatrix} -c_1(\mathcal{V}) & 1 & 0 & \cdots & 0\\ -c_2(\mathcal{V}) & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -c_{r-1}(\mathcal{V}) & 0 & 0 & \cdots & 1\\ -c_r(\mathcal{V}) & 0 & 0 & \cdots & 0 \end{pmatrix} : H_K^{\oplus r} \to H_K^{\oplus r}$$

and p_r is the projection to the rth factor. The metrics g_k on I_k/I_{k-1} are given by

$$\begin{aligned} g_0(c_r(\mathcal{V}) \cup x,\ c_r(\mathcal{V}) \cup y) &= \int_X c_r(\mathcal{V}) \cup x \cup y \qquad (x, y \in H_K),\\ g_k(\overline{x},\ \overline{y}) &= \int_X x \cup p_1(N^{k-1}\vec{y}) \qquad (k>0,\ x, y \in J_k), \end{aligned} \tag{6.12}$$

where $\vec{y} \in \mathrm{Ker}\, N^k$ is any lift of y.

Remark 6.6 (on the nilradical of $\circ_{\mathcal{V}}$). If $\int_C (c_1(X) + c_1(\mathcal{V})) \leq 0$ for any curve $C \subset X$, then $\phi_\alpha \circ_{\mathcal{V}} \phi_\beta \in \mathcal{O}_M \otimes_K H^{\geq |\phi_\alpha| + |\phi_\beta|}(X, K)$ by the degree axiom. Therefore for such $(X, \mathcal{V})$, the nilradical of $(\mathcal{T}_{M^\dagger}, \circ_{\mathcal{V}})$ is $\mathcal{O}_M \otimes_K H^{\geq 2}(X, K)$.

§6.4. Remarks on local mirror symmetry

Let X be a Fano toric surface and $\mathcal{V} = K_X$ the canonical bundle. Take $\phi_{p+1} = \phi^0$. Then

$$\mathcal{G} = \sum_{\alpha=0}^{p+1} (\partial_\alpha \Phi_{\mathrm{cl}})\phi^\alpha + \sum_{i=1}^{p} k_i(\partial_i \Phi_{\mathrm{qu}})\phi_{p+1},$$

where

$$\Phi_{\mathrm{qu}} = \sum_{d \neq 0} N_d\, q^d, \qquad N_d = \int_{[\bar{M}_{0,0}(X,d)]^{\mathrm{vir}}} e(R^1\mu_*\,\mathrm{ev}_{m+1}^*\, K_X),$$

and the k_i are defined by $\sum_{i=1}^{p} k_i\phi_i = c_1(K_X)$. The coefficient of ϕ_{p+1} in $\mathcal{G}$ above is nothing but the function $\mathcal{F}_{\mathrm{local}}$ in [1, §6.3].

Next, let us discuss the relationship with the mirror side of the story. Let Δ be the fan polytope of X. There is a certain family of curves $\mathcal{C} \to \mathcal{M}(\Delta)$ in $(\mathbb{C}^*)^2$ associated to Δ. It was shown that

$$H^*(X,\mathbb{C}) \cong H^2((\mathbb{C}^*)^2, C_z) \qquad (z \in \mathcal{M}(\Delta))$$

as $\mathbb{C}$-vector spaces and that the weight filtration of the mixed Hodge structure on $H^2((\mathbb{C}^*)^2, C_z)$ coincides with Frobenius filtration (up to shifts). Compare [7, §8] with (6.11) and [8, (8.8)].

Under the mirror map, $\mathcal{F}_{\mathrm{local}}$ corresponds to a double logarithmic period of $\omega_0(z) = \left[\left(\frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0\right)\right] \in H^2((\mathbb{C}^*)^2, C_z)$, and $\{g_0(\phi_i \circ_{K_X} \phi_j, c_1(K_X))\}_{1\le i,j\le p}$ is essentially equal to the Yukawa coupling defined in [7, §6].

It would be desirable to construct a MFS on $H^2((\mathbb{C}^*)^2, C_z)$ which is compatible with its variation of mixed Hodge structures and which agrees with the MFS on $H^*(X,\mathbb{C})$ under the mirror map.

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