Publ. RIMS Kyoto Univ. **52** (2016), 63–81
DOI 10.4171/PRIMS/174

Basepoint-free Theorem of Reid–Fukuda Type for Quasi-log Schemes

by

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Abstract

We introduce various new operations for quasi-log structures. Then we prove a basepoint-free theorem of Reid–Fukuda type for quasi-log schemes as an application.

2010 Mathematics Subject Classification: Primary 14E30; Secondary 14C20.
Keywords: quasi-log schemes, basepoint-free theorem, minimal model program.

§1. Introduction

Let (X, Δ) be a log canonical pair and let $f : Y \to X$ be a resolution such that $K_Y + \Delta_Y = f^*(K_X + \Delta)$ and $\operatorname{Supp} \Delta_Y$ is a simple normal crossing divisor on Y. We set $S = \Delta_Y^{=1}$ and $\Delta_Y = S + B$. We consider the short exact sequence

$$0 \to \mathcal{O}_Y(-S + \lceil -B \rceil) \to \mathcal{O}_Y(\lceil -B \rceil) \to \mathcal{O}_S(\lceil -B \rceil) \to 0.$$

By the Kawamata–Viehweg vanishing theorem, we have $R^1 f_* \mathcal{O}_Y(-S+\lceil -B \rceil) = 0$. Therefore, we obtain

$$0 \to \mathcal{J}(X, \Delta) \to \mathcal{O}_X \to f_*\mathcal{O}_S(\lceil -B \rceil) \to 0$$

where $\mathcal{J}(X, \Delta) = f_*\mathcal{O}_Y(-S + \lceil -B \rceil)$ is the multiplier ideal sheaf of (X, Δ). Let $\operatorname{Nklt}(X, \Delta)$ be the non-klt locus of (X, Δ) with the reduced scheme structure. Then $f_*\mathcal{O}_S(\lceil -B \rceil) \simeq \mathcal{O}_{\operatorname{Nklt}(X,\Delta)}$. This data

$$f : (S, B|_S) \to \operatorname{Nklt}(X, \Delta)$$

is a typical example of *quasi-log schemes*. In general, $\operatorname{Nklt}(X, \Delta)$ is reducible and is not equidimensional. Note that the data

$$f : (Y, \Delta_Y) \to X$$

Communicated by S. Mukai. Received January 18, 2014. Revised July 20, 2015.

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also defines a natural quasi-log structure on X which is compatible with the original log canonical structure of (X, Δ). By the framework of quasi-log schemes, we can treat log canonical pairs and their non-klt loci on an equal footing.

The following theorem is the main theorem of this paper. It was stated in [A] without proof (see [A, Theorem 7.2] and Remark 1.4 below). For some related results, see [S, 10.4], [Fk1], [Fk2], [Fk3], [F1], [F7, 5. Basepoint-free theorem of Reid–Fukuda type], and [F10, Theorem 1.16]. Note that the comment by Professor Miles Reid in [S, §10] is the origin of this type of basepoint-free theorems.

Theorem 1.1 (Basepoint-free theorem of Reid–Fukuda type for quasi-log schemes). *Let $[X, \omega]$ be a quasi-log scheme, let $\pi : X \to S$ be a projective morphism between schemes, and let L be a π-nef Cartier divisor on X such that $qL - \omega$ is nef and log big over S with respect to $[X, \omega]$ for some positive real number q. Assume that $\mathcal{O}_{X_{-\infty}}(mL)$ is π-generated for every $m \gg 0$. Then $\mathcal{O}_X(mL)$ is π-generated for every $m \gg 0$.*

In [F2, Theorem 4.1], the author proved Theorem 1.1 with the extra assumption that $X_{-\infty} = \emptyset$. Note that the assumption $X_{-\infty} = \emptyset$ is harmless for applications to semi-log canonical pairs in [F10, Theorem 1.16]. We also note that Ambro's original statement (see [A, Theorem 7.2]) only requires that π is *proper*. Unfortunately, our proof needs the assumption that π is *projective* because we use Kodaira's lemma for big $\mathbb{R}$-divisors on (not necessarily normal) irreducible varieties (cf. [F10, Lemma A.10]). Therefore, Theorem 1.1 is slightly weaker than the original statement (see [A, Theorem 7.2]).

Remark 1.2. Precisely speaking, it is sufficient to assume that π is proper and that every qlc stratum C of $[X, \omega]$ is projective over S in Theorem 1.1. This is obvious from the proof of Theorem 1.1.

Remark 1.3. In Theorem 1.1, if $qL - \omega$ is ample, then it is well-known that $\mathcal{O}_X(mL)$ is π-generated for every $m \gg 0$ (see [A, Theorem 5.1], [F2, Theorem 3.66], and [F12, Theorem 6.5.1]). For the proof, see [F2, Theorem 3.66] (see also [F12, Section 6.5]).

We give a remark on [A, Theorem 7.2].

Remark 1.4. Although Ambro wrote that the proof of [A, Theorem 7.2] is parallel to [A, Theorem 5.1], this does not seem to be true as stated, as there are some technical problems in the inductive step of the proof. Steps 1, 2, and 4 in the proof of [A, Theorem 5.1] work without any modifications. In Step 3, $q'L - \omega'$ is π-nef but $q'L - \omega' = qL - \omega$ is not always nef and log big over S with respect to $[X, \omega']$, where $\omega' = \omega + cD$ and $q' = q + cm$. So, we cannot directly apply the argument in Step 1 in the proof of [A, Theorem 5.1] to this new quasi-log pair $[X, \omega']$.

As a special case of Theorem 1.1, we have:

Theorem 1.5 (Basepoint-free theorem of Reid–Fukuda type for log canonical pairs). *Let (X, B) be a log canonical pair. Let L be a π-nef Cartier divisor on X where $\pi : X \to S$ is a projective morphism between schemes. Assume that $qL - (K_X + B)$ is nef and log big over S with respect to (X, B) for some positive real number q. Then $\mathcal{O}_X(mL)$ is π-generated for every $m \gg 0$.*

Theorem 1.5 is nothing but [F2, Theorem 4.4] (see [F12, Corollary 6.9.4]). We believe that Theorem 1.5 holds under the weaker assumption that π is only *proper*. Note that we do not know the proof of Theorem 1.5 without using the theory of quasi-log schemes. The usual basepoint-free theorem for log canonical pairs, that is, Theorem 1.5 with the extra assumption that $qL - (K_X + B)$ is ample over S, can be proved without using quasi-log structures (see [F5, Theorem 13.1]). The proof in [F5] is much simpler than the arguments in this paper.

Remark 1.6. In Theorem 1.5, if every log canonical center C of (X, B) is projective over S, then we can prove Theorem 1.5 under the weaker assumption that $\pi : X \to S$ is only *proper*. This is because we can apply Theorem 1.5 to the non-klt locus $\mathrm{Nklt}(X, B)$ of (X, B). So, we may assume that $\mathcal{O}_X(mL)$ is π-generated on a non-empty Zariski open subset containing $\mathrm{Nklt}(X, B)$. In this case, we can prove Theorem 1.5 by applying the usual X-method to L on (X, B). We note that C is projective over S when $\dim C \leq 1$.

The reader can find a different proof of Theorem 1.5 in [Fk3] when (X, B) is a log canonical *surface*, where Fukuda used the log minimal model program with scaling for divisorial log terminal surfaces.

More generally, we have:

Theorem 1.7. *Let X be a normal variety, let B be an effective $\mathbb{R}$-divisor on X such that $K_X + B$ is $\mathbb{R}$-Cartier, and let $\pi : X \to S$ be a projective morphism between schemes. Let L be a π-nef Cartier divisor on X such that $qL - (K_X + B)$ is nef and log big over S with respect to (X, B) for some positive real number q. Assume that $\mathcal{O}_{\mathrm{Nlc}(X,B)}(mL)$ is π-generated for every $m \gg 0$. Note that $\mathrm{Nlc}(X, B)$ denotes the non-lc locus of (X, B) and is defined by the non-lc ideal sheaf $\mathcal{J}_{\mathrm{NLC}}(X, B)$ of (X, B). Then $\mathcal{O}_X(mL)$ is π-generated for every $m \gg 0$.*

For the details of $\mathcal{J}_{\mathrm{NLC}}(X, B)$, see [F3] and [F5, §7. Non-lc ideal sheaves]. Theorem 1.7 is new and is a generalization of [F5, Theorem 13.1] and [F6, Theorem 9.1].

In this paper, we use the following convention.

Notation 1.8. The expression '... for every $m \gg 0$' means that 'there exists a positive integer m_0 such that ... for every $m \geq m_0$.'

We summarize the contents of this paper. In Section 2, we recall some basic definitions. In Section 3, we recall the basic definitions and properties of quasi-log schemes. Then we introduce various new operations for quasi-log structures (see Lemmas 3.12, 3.14, 3.15, and so on). Section 4 is devoted to the proof of the main theorem: Theorem 1.1.

We will work over $\mathbb{C}$, the complex number field, throughout this paper. For the standard notation of the log minimal model program, see, for example, [F5] and [F12]. For the basic definitions and properties of the theory of quasi-log schemes, see [F11] (see also [F12]). Note that [F12] is a completely revised and expanded version of the author's unpublished manuscript [F2]. For a gentle introduction to the theory of quasi-log schemes (varieties), see [F4]. In this paper, a *scheme* means a separated scheme of finite type over $\operatorname{Spec}\mathbb{C}$. A *variety* means a reduced scheme.

§2. Preliminaries

In this section, we recall some basic definitions.

2.1 (Operations for $\mathbb{R}$-divisors). Let D be an $\mathbb{R}$-divisor on an equidimensional variety X, that is, D is a finite formal $\mathbb{R}$-linear combination

$$D = \sum_i d_i D_i$$

of irreducible reduced subschemes D_i of codimension one. We define the *round-up* $\lceil D \rceil = \sum_i \lceil d_i \rceil D_i$ (resp. *round-down* $\lfloor D \rfloor = \sum_i \lfloor d_i \rfloor D_i$), where for every real number x, $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) is the integer defined by $x \leq \lceil x \rceil < x+1$ (resp. $x-1 < \lfloor x \rfloor \leq x$). The *fractional part* $\{D\}$ of D is $D - \lfloor D \rfloor$. We write

$$D^{<1} = \sum_{d_i<1} d_i D_i, \qquad D^{>1} = \sum_{d_i>1} d_i D_i, \qquad D^{=1} = \sum_{d_i=1} D_i.$$

We call D a *boundary* (resp. *subboundary*) $\mathbb{R}$-divisor if $0 \leq d_i \leq 1$ (resp. $d_i \leq 1$) for every i.

2.2 (Singularities of pairs). Let X be a normal variety and let Δ be an $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $f : Y \to X$ be a resolution such that $\operatorname{Exc}(f) \cup f_*^{-1}\Delta$, where $\operatorname{Exc}(f)$ is the exceptional locus of f and $f_*^{-1}\Delta$ is the strict transform of Δ on Y, has a simple normal crossing support. We can write

$$K_Y = f^*(K_X + \Delta) + \sum_i a_i E_i. \tag{2.1}$$

We say that (X, Δ) is *sub log canonical* (*sub lc*, for short) if $a_i \geq -1$ for every i. We usually write $a_i = a(E_i, X, \Delta)$ and call it the *discrepancy coefficient* of E_i with respect to (X, Δ). If (X, Δ) is sub log canonical and Δ is effective, then (X, Δ) is called *log canonical* (*lc*, for short).

It is well-known that there is the largest Zariski open subset U of X such that $(U, \Delta|_U)$ is sub log canonical. If there exist a resolution $f : Y \to X$ and a divisor E on Y such that $a(E, X, \Delta) = -1$ and $f(E) \cap U \neq \emptyset$, then $f(E)$ is called a *log canonical center* (an *lc center*, for short) with respect to (X, Δ). A closed subset C of X is called a *log canonical stratum* (an *lc stratum*, for short) of (X, Δ) if C is either a log canonical center of (X, Δ) or an irreducible component of X.

From now on, we assume that Δ is effective. In the above formula (2.1), we denote $\Delta_Y = -\sum_i a_i E_i$. Then

$$\mathcal{J}(X, \Delta) = f_* \mathcal{O}_Y(-\lfloor \Delta_Y \rfloor)$$

is a well-defined ideal sheaf on X and is known as the *multiplier ideal sheaf* associated to the pair (X, Δ). The closed subscheme $\mathrm{Nklt}(X, \Delta)$ defined by $\mathcal{J}(X, \Delta)$ is called the *non-klt locus* of (X, Δ). We set

$$\mathcal{J}_{\mathrm{NLC}}(X, \Delta) = f_* \mathcal{O}_Y(-\lfloor \Delta_Y \rfloor + \Delta_Y^{=1})$$

and call it the *non-lc ideal sheaf* associated to the pair (X, Δ). The closed subscheme $\mathrm{Nlc}(X, \Delta)$ is defined by $\mathcal{J}_{\mathrm{NLC}}(X, \Delta)$ and is called the *non-lc locus* of (X, Δ).

The notion of *nef and log big divisors* was first introduced in [S, 10.4] by Miles Reid. For the details of big $\mathbb{R}$-divisors on non-normal irreducible varieties, see [F10, Appendix A].

2.3 (Nef and log big divisors). Let X be a normal variety, let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier, and let $\pi : X \to S$ be a proper morphism between schemes. Let L be a Cartier divisor on X. We say that L is *nef and log big* over S with respect to (X, Δ) if L is nef over S and $L|_C$ is big over S for every lc stratum C of (X, Δ).

We close this section with:

Notation 2.4. A pair $[X, \omega]$ consists of a scheme X and an $\mathbb{R}$-Cartier divisor (or an $\mathbb{R}$-line bundle) on X.

§3. On quasi-log structures

In this section, we recall some definitions and basic properties of quasi-log schemes and prove some useful lemmas. We prove various new lemmas to make the theory of quasi-log schemes more flexible and more useful. For a quick introduction to the theory of quasi-log schemes (varieties), see [F4].

For the reader's convenience let us quickly recall the definitions of *globally embedded simple normal crossing pairs* and *quasi-log schemes*. For the details, see, for example, [F11, Section 3] and [F12, Chapters 5 and 6].

Definition 3.1 (Globally embedded simple normal crossing pairs). Let Y be a simple normal crossing divisor on a smooth variety M and let D be an $\mathbb{R}$-divisor on M such that $\operatorname{Supp}(D+Y)$ is a simple normal crossing divisor on M, and D and Y have no common irreducible components. We write $B_Y = D|_Y$ and consider the pair (Y, B_Y). We call (Y, B_Y) a *globally embedded simple normal crossing pair* and M the *ambient space* of (Y, B_Y). A *stratum* of (Y, B_Y) is the ν-image of a log canonical stratum of (Y^ν, Θ) where $\nu : Y^\nu \to Y$ is the normalization and $K_{Y^\nu} + \Theta = \nu^*(K_Y + B_Y)$.

The following lemma is obvious but important.

Lemma 3.2. *Let Y be a smooth irreducible variety and let B_Y be an $\mathbb{R}$-divisor on Y such that $\operatorname{Supp} B_Y$ is a simple normal crossing divisor. Then (Y, B_Y) is a globally embedded simple normal crossing pair.*

Proof. We set $M = Y \times \mathbb{C}$, $D = B_Y \times \mathbb{C}$, and $Y = Y \times \{0\}$. Then D and Y are divisors on M such that $D|_Y = B_Y$. This means that (Y, B_Y) is a globally embedded simple normal crossing pair. □

In this paper, we adopt the following definition of quasi-log schemes. Although it looks slightly different from Ambro's original definition, it is equivalent to [A, Definition 4.1].

Definition 3.3 (Quasi-log schemes). A *quasi-log scheme* is a scheme X endowed with an $\mathbb{R}$-Cartier divisor (or an $\mathbb{R}$-line bundle) ω on X, a proper closed subscheme $X_{-\infty} \subset X$, and a finite collection $\{C\}$ of reduced and irreducible subschemes of X such that there is a proper morphism $f : (Y, B_Y) \to X$ from a globally embedded simple normal crossing pair satisfying the following properties:

(1) $f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$.

(2) The natural map $\mathcal{O}_X \to f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil)$ induces an isomorphism
$$\mathcal{I}_{X_{-\infty}} \xrightarrow{\simeq} f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil - \lfloor B_Y^{>1}\rfloor),$$
where $\mathcal{I}_{X_{-\infty}}$ is the defining ideal sheaf of $X_{-\infty}$.

(3) The collection $\{C\}$ of subvarieties coincides with the image of (Y, B_Y)-strata that are not included in $X_{-\infty}$.

We simply write $[X, \omega]$ to denote the above data

$$\bigl(X, \omega, f : (Y, B_Y) \to X\bigr)$$

if there is no risk of confusion. Note that a quasi-log scheme X is the union of $\{C\}$ and $X_{-\infty}$. We also note that ω is called the *quasi-log canonical class* of $[X, \omega]$, which is defined up to $\mathbb{R}$-linear equivalence. We sometimes simply say that $[X, \omega]$ is a *quasi-log pair*. The subvarieties C are called the qlc strata of $[X, \omega]$, $X_{-\infty}$ is called the non-qlc locus of $[X, \omega]$, and $f : (Y, B_Y) \to X$ is called a *quasi-log resolution* of $[X, \omega]$. We sometimes use $\mathrm{Nqlc}(X, \omega)$ to denote $X_{-\infty}$.

For the details of the various equivalent definitions of quasi-log schemes, see [F11, Sections 3, 4, and 8].

Remark 3.4. A qlc stratum of $[X, \omega]$ was originally called a qlc center of $[X, \omega]$ in the literature. We change the terminology (see Definition 3.5 below).

Our definition of qlc centers is different from Ambro's original one in [A].

Definition 3.5 (Qlc centers). A closed subvariety C of X is called a *qlc center* of $[X, \omega]$ if C is a qlc stratum of $[X, \omega]$ which is not an irreducible component of X.

Definition 3.6 (Qlc pairs). Let $[X, \omega]$ be a quasi-log scheme. Assume that $X_{-\infty} = \emptyset$. Then we sometimes simply say that $[X, \omega]$ is a *qlc pair* or $[X, \omega]$ is a quasi-log scheme with only *quasi-log canonical singularities*.

We need the notion of nef and log big divisors on quasi-log schemes for Theorem 1.1.

Definition 3.7 (Nef and log big divisors for quasi-log schemes). Let L be an $\mathbb{R}$-Cartier divisor (or $\mathbb{R}$-line bundle) on a quasi-log pair $[X, \omega]$ and let $\pi : X \to S$ be a proper morphism between schemes. Then L is *nef and log big over S with respect to* $[X, \omega]$ if L is π-nef and $L|_C$ is π-big for every qlc stratum C of $[X, \omega]$.

The following theorem is a key result for the theory of quasi-log schemes. It follows from the Kollár-type torsion-free and vanishing theorem for simple normal crossing varieties. For the details, see [F2, Chapter 2], [F8], [F9], and [F12].

Theorem 3.8 (see [A, Theorems 4.4 and 7.3], [F2, Theorem 3.39], and [F12, Theorem 6.3.4]). *Let $[X, \omega]$ be a quasi-log scheme and let X' be the union of $X_{-\infty}$ with a* (*possibly empty*) *union of some qlc strata of* $[X, \omega]$. *Then we have the following properties.*

(i) *Assume that $X' \neq X_{-\infty}$. Then X' is a quasi-log scheme with $\omega' = \omega|_{X'}$ and $X'_{-\infty} = X_{-\infty}$. Moreover, the qlc strata of $[X', \omega']$ are exactly the qlc strata of $[X, \omega]$ that are included in X'.*

(ii) *Assume that $\pi : X \to S$ is a proper morphism between schemes. Let L be a Cartier divisor on X such that $L - \omega$ is nef and log big over S with respect to $[X, \omega]$. Then $R^i\pi_*(\mathcal{I}_{X'} \otimes \mathcal{O}_X(L)) = 0$ for every $i > 0$, where $\mathcal{I}_{X'}$ is the defining ideal sheaf of X' on X.*

We give a proof of Theorem 3.8 for the reader's convenience because the theory of quasi-log schemes is not popular yet.

Proof. By taking some blow-ups of the ambient space M of (Y, B_Y), we may assume that the union of all strata of (Y, B_Y) mapped to X', which is denoted by Y', is a union of irreducible components of Y (see [F11, Proposition 4.1]). We write $K_{Y'} + B_{Y'} = (K_Y + B_Y)|_{Y'}$ and $Y'' = Y - Y'$. We will prove that $f : (Y', B_{Y'}) \to X'$ gives the desired quasi-log structure on $[X', \omega']$. By construction, we have $f^*\omega' \sim_{\mathbb{R}} K_{Y'} + B_{Y'}$ on Y'. We set $A = \lceil -(B_{Y'}^{<1}) \rceil$ and $N = \lfloor B_{Y'}^{>1} \rfloor$. We consider the short exact sequence

$$0 \to \mathcal{O}_{Y''}(-Y') \to \mathcal{O}_Y \to \mathcal{O}_{Y'} \to 0.$$

By applying $\otimes \mathcal{O}_Y(A - N)$, we have

$$0 \to \mathcal{O}_{Y''}(A - N - Y') \to \mathcal{O}_Y(A - N) \to \mathcal{O}_{Y'}(A - N) \to 0.$$

By applying f_*, we obtain

$$\begin{aligned}0 \to f_*\mathcal{O}_{Y''}(A - N - Y') \to f_*\mathcal{O}_Y(A - N) \to f_*\mathcal{O}_{Y'}(A - N) \\ \to R^1f_*\mathcal{O}_{Y''}(A - N - Y') \to \cdots .\end{aligned}$$

By [F8, Theorem 1.1] and [F2, Theorem 2.39] (see also [F12, Theorem 5.6.3]), no associated prime of $R^1f_*\mathcal{O}_{Y''}(A - N - Y')$ is contained in $X' = f(Y')$. We note that

$$\begin{aligned}(A - N - Y')|_{Y''} - (K_{Y''} + \{B_{Y''}\} + B_{Y''}^{=1} - Y'|_{Y''}) = -(K_{Y''} + B_{Y''}) \\ \sim_{\mathbb{R}} -(f^*\omega)|_{Y''},\end{aligned}$$

where $K_{Y''} + B_{Y''} = (K_Y + B_Y)|_{Y''}$. Therefore, the connecting homomorphism $\delta : f_*\mathcal{O}_{Y'}(A - N) \to R^1f_*\mathcal{O}_{Y''}(A - N - Y')$ is zero. Thus we obtain the short exact sequence

$$0 \to f_*\mathcal{O}_{Y''}(A - N - Y') \to \mathcal{I}_{X_{-\infty}} \to f_*\mathcal{O}_{Y'}(A - N) \to 0.$$

We set $\mathcal{I}_{X'} = f_*\mathcal{O}_{Y''}(A - N - Y')$. Then $\mathcal{I}_{X'}$ defines a scheme structure on X'. We define $\mathcal{I}_{X'_{-\infty}} = \mathcal{I}_{X_{-\infty}}/\mathcal{I}_{X'}$. Then $\mathcal{I}_{X'_{-\infty}} \simeq f_*\mathcal{O}_{Y'}(A - N)$ by the above exact sequence. From the big commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & f_*\mathcal{O}_{Y''}(A-N-Y') & \longrightarrow & f_*\mathcal{O}_Y(A-N) & \longrightarrow & f_*\mathcal{O}_{Y'}(A-N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & f_*\mathcal{O}_{Y''}(A-Y') & \longrightarrow & f_*\mathcal{O}_Y(A) & \longrightarrow & f_*\mathcal{O}_{Y'}(A) & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{I}_{X'} & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{X'} & \longrightarrow & 0
\end{array}$$

we can see that $\mathcal{O}_{X'} \to f_*\mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1})\rceil)$ induces an isomorphism

$$\mathcal{I}_{X'_{-\infty}} \xrightarrow{\simeq} f_*\mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1})\rceil - \lfloor B_{Y'}^{>1}\rfloor).$$

Therefore, $[X', \omega']$ is a quasi-log pair such that $X'_{-\infty} = X_{-\infty}$. By construction, the property of qlc strata is obvious. So, we obtain the desired quasi-log structure of $[X', \omega']$ in (i).

Let $f : (Y, B_Y) \to X$ be a quasi-log resolution as in the proof of (i). If $X' = X_{-\infty}$ in that proof, then we can easily see that

$$f_*\mathcal{O}_{Y''}(A - N - Y') \simeq f_*\mathcal{O}_{Y''}(A - N) \simeq \mathcal{I}_{X_{-\infty}}.$$

Therefore, we always have $\mathcal{I}_{X'} \simeq f_*\mathcal{O}_{Y''}(A - N - Y')$. Note that

$$f^*(L - \omega) \sim_{\mathbb{R}} f^*L - (K_{Y''} + B_{Y''})$$

on Y'', where $K_{Y''} + B_{Y''} = (K_Y + B_Y)|_{Y''}$. Moreover

$$f^*L - (K_{Y''} + B_{Y''}) = (f^*L + A - N - Y')|_{Y''} - (K_{Y''} + \{B_{Y''}\} + B_{Y''}^{=1} - Y'|_{Y''})$$

and no stratum of $(Y'', \{B_{Y''}\} + B_{Y''}^{=1} - Y'|_{Y''})$ is mapped to $X_{-\infty}$. Then, by [F2, Theorem 3.38] (see also [F12, Theorem 5.7.3]),

$$R^i\pi_*(f_*\mathcal{O}_{Y''}(f^*L + A - N - Y')) = R^i\pi_*(\mathcal{I}_{X'} \otimes \mathcal{O}_X(L)) = 0$$

for every $i > 0$. Thus, we obtain the desired vanishing theorem in (ii). □

We usually call Theorem 3.8(i) *adjunction* for quasi-log schemes.

For the reader's convenience let us recall the following well-known lemma (see [A, Proposition 4.7], [F2, Proposition 3.44], and [F12, Lemma 6.3.5]).

Lemma 3.9. *Let $[X, \omega]$ be a quasi-log scheme with $X_{-\infty} = \emptyset$. Assume that every qlc stratum of $[X, \omega]$ is an irreducible component of X. Then X is normal.*

The following proof is different from Ambro's original one (see [A, Proposition 4.7]).

Proof. Let $f : (Y, B_Y) \to X$ be a quasi-log resolution. Since $X_{-\infty} = \emptyset$, we have $f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil) \simeq \mathcal{O}_X$. This implies that $f_*\mathcal{O}_Y \simeq \mathcal{O}_X$. Therefore, by assumption, every connected component of X is an irreducible component of X because f has connected fibers. Let $\nu : X^\nu \to X$ be the normalization. Without loss of generality, we may assume that X is irreducible and every stratum of (Y, B_Y) is mapped onto X. Thus the indeterminacy locus of $\nu^{-1} \circ f : Y \dashrightarrow X^\nu$ contains no strata of (Y, B_Y). By modifying (Y, B_Y) suitably by [F11, Proposition 4.1], we may assume that $f : Y \to X$ factors through X^ν:

$$\begin{array}{ccc} Y & & \\ {\scriptstyle \overline{f}}\downarrow & \searrow{\scriptstyle f} & \\ X^\nu & \xrightarrow{\nu} & X \end{array}$$

Note that the composition

$$\mathcal{O}_X \to \nu_*\mathcal{O}_{X^\nu} \to \nu_*\overline{f}_*\mathcal{O}_Y = f_*\mathcal{O}_Y \simeq \mathcal{O}_X$$

is an isomorphism. This implies that $\mathcal{O}_X \simeq \nu_*\mathcal{O}_{X^\nu}$. Therefore, X is normal. $\square$

We now introduce $\mathrm{Nqklt}(X, \omega)$, which is a generalization of the notion of non-klt loci (see 2.2).

Notation 3.10. Let $[X, \omega]$ be a quasi-log scheme. The union of $X_{-\infty}$ with all qlc centers of $[X, \omega]$ is denoted by $\mathrm{Nqklt}(X, \omega)$. The scheme structure of $\mathrm{Nqklt}(X, \omega)$ is defined in Theorem 3.8. If $\mathrm{Nqklt}(X, \omega) \neq X_{-\infty}$, then

$$[\mathrm{Nqklt}(X, \omega), \omega|_{\mathrm{Nqklt}(X,\omega)}]$$

is a quasi-log scheme by Theorem 3.8. Note that $\mathrm{Nqklt}(X, \omega)$ is denoted by $\mathrm{LCS}(X)$ and is called the *LCS locus* of a quasi-log scheme $[X, \omega]$ in [A, Definition 4.6].

Theorem 3.11 is also a key result for the theory of quasi-log schemes.

Theorem 3.11 (see [A, Proposition 4.8], [F5, Theorem 3.45], and [F12, Theorem 6.3.7]). *Assume that $[X, \omega]$ is a quasi-log scheme with $X_{-\infty} = \emptyset$. Then we have the following properties.*

(i) *The intersection of two qlc strata is a union of qlc strata.*

(ii) *For any closed point $x \in X$, the set of all qlc strata passing through x has a unique minimal element C_x. Moreover, C_x is normal at x.*

Proof. Let C_1 and C_2 be two qlc strata of $[X, \omega]$. We fix $P \in C_1 \cap C_2$. It is enough to find a qlc stratum C such that $P \in C \subset C_1 \cap C_2$. The union $X' = C_1 \cup C_2$ with $\omega' = \omega|_{X'}$ is a qlc pair having two irreducible components. Hence, it is not normal at P. By Lemma 3.9, $P \in \mathrm{Nqklt}(X', \omega')$. Therefore, there exists a qlc stratum C such that $P \in C \subset X'$. We may assume that $C \subset C_1$ with $\dim C < \dim C_1$. If $C \subset C_2$, then we are done. Otherwise, we repeat the argument with $C_1 = C$ and reach the conclusion in a finite number of steps. This finishes the proof of (i). The uniqueness of the minimal qlc stratum follows from (i), and the normality of the minimal stratum follows from Lemma 3.9. Thus, we have (ii). □

The following lemma is useful for some applications. It enables removing redundant components of Y from the quasi-log resolution $f : (Y, B_Y) \to X$.

Lemma 3.12. *Let $(X, \omega, f : (Y, B_Y) \to X)$ be a quasi-log scheme as in Definition 3.3. Then we can construct a new quasi-log resolution $f' : (Y', B_{Y'}) \to X$ such that*

- (i) *$f' : (Y', B_{Y'}) \to X$ gives the same quasi-log structure as one given by $f : (Y, B_Y) \to X$, and*
- (ii) *every irreducible component of Y' is mapped by f' to $\overline{X \setminus X_{-\infty}}$, the closure of $X \setminus X_{-\infty}$ in X.*

Proof. Let M be the ambient space of (Y, B_Y). By taking some blow-ups of M, we may assume that the union of all strata of (Y, B_Y) that are not mapped to $\overline{X \setminus X_{-\infty}}$, which is denoted by Y'', is a union of some irreducible components of Y (see [F11, Proposition 4.1]). We define $Y' = Y - Y''$ and $K_{Y''} + B_{Y''} = (K_Y + B_Y)|_{Y''}$. We may further assume that the union of all strata of (Y, B_Y) mapped to $\overline{X \setminus X_{-\infty}} \cap X_{-\infty}$ is a union of some irreducible components of Y by [F11, Proposition 4.1]. We consider the short exact sequence

$$0 \to \mathcal{O}_{Y''}(-Y') \to \mathcal{O}_Y \to \mathcal{O}_{Y'} \to 0.$$

We set $A = \lceil -(B_Y^{<1}) \rceil$ and $N = \lfloor B_Y^{>1} \rfloor$. By applying $\otimes \mathcal{O}_Y(A - N)$, we have

$$0 \to \mathcal{O}_{Y''}(A - N - Y') \to \mathcal{O}_Y(A - N) \to \mathcal{O}_{Y'}(A - N) \to 0.$$

Applying f_* yields

$$\begin{aligned} 0 \to f_*\mathcal{O}_{Y''}(A - N - Y') \to f_*\mathcal{O}_Y(A - N) \to f_*\mathcal{O}_{Y'}(A - N) \\ \to R^1 f_*\mathcal{O}_{Y''}(A - N - Y') \to \cdots . \end{aligned}$$

By [F8, Theorem 1.1] and [F2, Theorem 2.39] (see also [F12, Theorem 5.6.3]), no associated prime of $R^1 f_*\mathcal{O}_{Y''}(A-N-Y')$ is contained in $f(Y')\cap X_{-\infty}$. Note that

$$\begin{aligned}(A-N-Y')|_{Y''}-(K_{Y''}+\{B_{Y''}\}+B_{Y''}^{=1}-Y'|_{Y''}) &= -(K_{Y''}+B_{Y''})\\ &\sim_{\mathbb{R}} -(f^*\omega)|_{Y''}.\end{aligned}$$

Therefore, the connecting homomorphism

$$\delta : f_*\mathcal{O}_{Y'}(A-N)\to R^1 f_*\mathcal{O}_{Y''}(A-N-Y')$$

is zero. This implies that

$$0\to f_*\mathcal{O}_{Y''}(A-N-Y')\to \mathcal{I}_{X_{-\infty}}\to f_*\mathcal{O}_{Y'}(A-N)\to 0$$

is exact. The ideal sheaf $\mathcal{J} = f_*\mathcal{O}_{Y''}(A-N-Y')$ is zero when restricted to $X_{-\infty}$ because $\mathcal{J}\subset \mathcal{I}_{X_{-\infty}}$. On the other hand, $\mathcal{J}$ is zero on $X\setminus X_{-\infty}$ because $f(Y'')\subset X_{-\infty}$. Therefore, we obtain $\mathcal{J}=0$. Thus we have $\mathcal{I}_{X_{-\infty}} = f_*\mathcal{O}_{Y'}(A-N)$. So $f'=f|_{Y'} : (Y', B_{Y'})\to X$, where $K_{Y'}+B_{Y'} = (K_Y+B_Y)|_{Y'}$, gives the same quasi-log structure as the one given by $f:(Y,B_Y)\to X$, and has property (ii). □

The next lemma is obvious. We will sometimes use it implicitly in the theory of quasi-log schemes.

Lemma 3.13. *Let $[X,\omega]$ be a quasi-log scheme. Assume that $X = V\cup X_{-\infty}$ and $V\cap X_{-\infty}=\emptyset$. Then $[V,\omega|_V]$ is a quasi-log scheme with only quasi-log canonical singularities.*

By using Lemma 3.12, we obtain Lemma 3.14 below. Roughly speaking, it enables removing the irreducible components of X contained in $X_{-\infty}$ from the quasi-log pair $[X,\omega]$.

Lemma 3.14. *Let $[X,\omega]$ be a quasi-log scheme. Consider $X^\dagger = \overline{X\setminus X_{-\infty}}$ (the closure in X), with the reduced scheme structure. Then $[X^\dagger,\omega^\dagger]$, where $\omega^\dagger=\omega|_{X^\dagger}$, has a natural quasi-log structure induced by $[X,\omega]$. This means that*

(i) *C is a qlc stratum of $[X,\omega]$ if and only if C is a qlc stratum of $[X^\dagger,\omega^\dagger]$, and*
(ii) *$\mathcal{I}_{\mathrm{Nqlc}(X,\omega)} = \mathcal{I}_{\mathrm{Nqlc}(X^\dagger,\omega^\dagger)}$.*

Proof. Let $\mathcal{I}_{X^\dagger}$ be the defining ideal sheaf of $X^\dagger$ on X. Let $f' : (Y', B_{Y'})\to X$ be the quasi-log resolution constructed in the proof of Lemma 3.12. Note that

$$\mathcal{I}_{X_{-\infty}}\simeq f'_*\mathcal{O}_{Y'}(A-N)\simeq f'_*\mathcal{O}_{Y'}(-N)$$

and

$$f'(N) = X_{-\infty}\cap f'(Y') = X_{-\infty}\cap X^\dagger$$

set-theoretically, where $A = \lceil -(B_{Y'}^{<1}) \rceil$ and $N = \lfloor B_{Y'}^{>1} \rfloor$ (see [F11, Remark 3.8]). Therefore, we obtain

$$\mathcal{I}_{X^\dagger} \cap \mathcal{I}_{X_{-\infty}} = \mathcal{I}_{X^\dagger} \cap f'_*\mathcal{O}_{Y'}(-N') = \{0\}.$$

Thus we can construct the big commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{I}_{X_{-\infty}} & = & \mathcal{I}_{X^\dagger_{-\infty}} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_{X^\dagger} & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{X^\dagger} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_{X^\dagger} & \longrightarrow & \mathcal{O}_{X_{-\infty}} & \longrightarrow & \mathcal{O}_{X^\dagger_{-\infty}} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

By construction, f' factors through a map $f^\dagger : (Y', B_{Y'}) \to X^\dagger$, which gives the desired quasi-log structure on $[X^\dagger, \omega^\dagger]$. □

Lemma 3.15 below is needed in the proof of Theorem 1.1.

Lemma 3.15. *Let $[X, \omega]$ be a quasi-log scheme and let E be an effective $\mathbb{R}$-Cartier divisor on X, that is, $E = \sum_{i=1}^k e_i E_i$, where E_i is an effective Cartier divisor on X and $e_i \geq 0$ for every i. Set*

$$\widetilde{\omega} = \omega + \varepsilon E$$

with $0 < \varepsilon \ll 1$. Then $[X, \widetilde{\omega}]$ has a natural quasi-log structure with the following properties.

(i) *Let $\{C_i\}_{i \in I}$ be the set of qlc strata of $[X, \omega]$ contained in* $\operatorname{Supp} E$. *Write*

$$X^\star = \Big(\bigcup_{i \in I} C_i \Big) \cup \operatorname{Nqlc}(X, \omega)$$

as in Theorem 3.8. Then $\operatorname{Nqlc}(X, \widetilde{\omega})$ *coincides with $X^\star$ scheme-theoretically.*

(ii) *C is a qlc stratum of $[X, \widetilde{\omega}]$ if and only if C is a qlc stratum of $[X, \omega]$ with $C \not\subset$* $\operatorname{Supp} E$.

Proof. Let $f : (Y, B_Y) \to X$ be a quasi-log resolution as in Definition 3.3. By [F11, Proposition 4.1], the union of all strata of (Y, B_Y) mapped to $X^\star$, which is denoted by Y'', is a union of some irreducible components of Y. We write $Y' = Y - Y''$ and $K_{Y'} + B_{Y'} = (K_Y + B_Y)|_{Y'}$. By [F11, Proposition 4.1] we may further assume that $(Y', B_{Y'} + f^*E)$ is a globally embedded simple normal crossing pair. We consider $f : (Y', B_{Y'} + \varepsilon f^*E) \to X$ with $0 < \varepsilon \ll 1$. We set $A = \lceil -(B_Y^{<1}) \rceil$ and $N = \lfloor B_Y^{>1} \rfloor$. Then $X^\star$ is defined by the ideal sheaf $f_*\mathcal{O}_{Y'}(A - N - Y'')$ (see the proof of Theorem 3.8). Note that

$$\begin{aligned}(A - N - Y'')|_{Y'} &= -\lfloor B_{Y'} + \varepsilon f^*E \rfloor + (B_{Y'} + \varepsilon f^*E)^{=1} \\ &= \lceil -(B_{Y'} + \varepsilon f^*E)^{<1} \rceil - \lfloor (B_{Y'} + \varepsilon f^*E)^{>1} \rfloor.\end{aligned}$$

Therefore, if we define $\mathrm{Nqlc}(X, \widetilde{\omega})$ by the ideal sheaf

$$f_*\mathcal{O}_{Y'}(\lceil -(B_{Y'} + \varepsilon f^*E)^{<1} \rceil - \lfloor (B_{Y'} + \varepsilon f^*E)^{>1} \rfloor) = f_*\mathcal{O}_{Y'}(A - N - Y''),$$

then $f : (Y', B_{Y'} + \varepsilon f^*E) \to X$ gives the desired quasi-log structure on $[X, \widetilde{\omega}]$. □

The following lemma is a slight generalization of [F2, Lemma 3.71], which played a crucial role in the proof of the rationality theorem for quasi-log schemes (see [F2, Theorem 3.68] and [F12, Theorem 6.6.1]).

Lemma 3.16 (see [F2, Lemma 3.71] and [F12, Lemma 6.3.9]). *Let $[X, \omega]$ be a quasi-log scheme with $X_{-\infty} = \emptyset$ and let $x \in X$ be a closed point. Let $D_1, \dots, D_k$ be effective Cartier divisors on X such that $x \in \operatorname{Supp} D_i$ for every i. Let $f : (Y, B_Y) \to X$ be a quasi-log resolution. Assume that the normalization of $(Y, B_Y + \sum_{i=1}^k f^*D_i)$ is sub log canonical. This means that (Y^ν, Ξ) is sub log canonical, where $\nu : Y^\nu \to Y$ is the normalization and $K_{Y^\nu} + \Xi = \nu^*(K_Y + B_Y + \sum_{i=1}^k f^*D_i)$. Note that this requires that no irreducible component of Y is mapped into $\bigcup_{i=1}^k \operatorname{Supp} D_i$. Then $k \leq \dim_x X$. More precisely, $k \leq \dim_x C_x$, where C_x is the minimal qlc stratum of $[X, \omega]$ passing through x.*

Proof. We prove this lemma by induction on the dimension.

Step 1. By [F11, Proposition 4.1], we may assume that $(Y, B_Y + \sum_{i=1}^k f^*D_i)$ is a globally embedded simple normal crossing pair. Let i_0 be any positive integer with $1 \leq i_0 \leq k$. Note that $f_*\mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil) \simeq \mathcal{O}_X$. Therefore, for any irreducible component T of $\operatorname{Supp} D_{i_0}$, there is a stratum S of $(Y, B_Y + f^*D_{i_0})$ mapped onto T. Note that $f : (Y, B_Y + \sum_{i=1}^k f^*D_i) \to X$ gives a natural quasi-log structure on $[X, \omega + \sum_{i=1}^k D_i]$ with only quasi-log canonical singularities. We also note that $\operatorname{Supp} D_i$ and $\operatorname{Supp} D_j$ have no common irreducible components for $i \neq j$ by the condition $f_*\mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil) \simeq \mathcal{O}_X$.

Step 2. In this step, we assume that $\dim_x X = 1$. If x is a qlc stratum of $[X, \omega]$, then we have $k = 0$. Therefore, we may assume that x is not a qlc stratum of $[X, \omega]$.

By shrinking X around x, we may assume that every stratum of (Y, B_Y) is mapped onto X. Then X is irreducible and normal (see Lemma 3.9), and $f : Y \to X$ is flat. In this case, $f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil) \simeq \mathcal{O}_X$ implies $k \leq 1 = \dim_x X$.

Step 3. We assume that $\dim_x X \geq 2$. If x is a qlc stratum of $[X, \omega]$, then $k = 0$. So we may assume that x is not a qlc stratum of $[X, \omega]$. Let C be the minimal qlc stratum of $[X, \omega]$ passing through x. By shrinking X around x, we may assume that C is normal (see Theorem 3.11). By [F11, Proposition 4.1], we may assume that the union of all strata of (Y, B_Y) mapped to C, which is denoted by Y', is a union of some irreducible components of Y. Then $f : (Y', B_{Y'}) \to C$ gives a natural quasi-log structure induced by the original quasi-log structure $f : (Y, B_Y) \to X$ (see Theorem 3.8). Therefore, by induction on the dimension, we have $k \leq \dim_x C \leq \dim_x X$ when $\dim_x C < \dim_x X$. Thus we may assume that X is the unique qlc stratum of $[X, \omega]$. Note that $f : (Y, B_Y + f^*D_1) \to X$ gives a natural quasi-log structure on $[X, \omega + D_1]$ with only quasi-log canonical singularities. Let X' be the union of qlc strata of $[X, \omega + D_1]$ contained in $\operatorname{Supp} D_1$. Then $[X', (\omega + D_1)|_{X'}]$ is a qlc pair with $\dim_x X' < \dim_x X$ (see Step 1). Note that $[X', (\omega + D_1)|_{X'}]$ with $D_2|_{X'}, \ldots, D_k|_{X'}$ satisfies a condition similar to the original one for $[X, \omega]$ with $D_1, \ldots, D_k$ (see Step 1). Therefore, $k - 1 \leq \dim_x X' < \dim_x X$. This implies $k \leq \dim_x X$.

Anyway, we have obtained the desired inequality $k \leq \dim_x C_x$, where C_x is the minimal qlc stratum of $[X, \omega]$ passing through x. □

§4. Proof of Theorem 1.1

Step 1. If $\dim X \setminus X_{-\infty} = 0$, then the assertion of Theorem 1.1 obviously holds true. From now on, we assume that Theorem 1.1 holds for any quasi-log scheme Z with $\dim Z \setminus Z_{-\infty} < \dim X \setminus X_{-\infty}$.

Step 2. We take a qlc stratum C of $[X, \omega]$. We write $X' = C \cup X_{-\infty}$. Then X' has a natural quasi-log structure induced by $[X, \omega]$ (see Theorem 3.8). By the vanishing theorem (see Theorem 3.8), we have $R^1\pi_*(\mathcal{I}_{X'} \otimes \mathcal{O}_X(mL)) = 0$ for every $m \geq q$. Therefore, $\pi_*\mathcal{O}_X(mL) \to \pi_*\mathcal{O}_{X'}(mL)$ is surjective for every $m \geq q$. Thus, we may assume that $\overline{X \setminus X_{-\infty}}$ is irreducible for the proof of Theorem 1.1 by using the commutative diagram

$$\begin{array}{ccccc} \pi^*\pi_*\mathcal{O}_X(mL) & \longrightarrow & \pi^*\pi_*\mathcal{O}_{X'}(mL) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \mathcal{O}_X(mL) & \longrightarrow & \mathcal{O}_{X'}(mL) & \longrightarrow & 0 \end{array}$$

Step 3. Let $f:(Y,B_Y)\to X$ be a quasi-log resolution. By Lemma 3.12, we may assume that every irreducible component of Y is mapped to $\overline{X\setminus X_{-\infty}}$. We may further assume that S is affine.

Step 4. In this step, we assume that X is the disjoint union of $X_{-\infty}$ and a qlc stratum C of $[X,\omega]$. We further assume that C is the unique qlc stratum of $[X,\omega]$. In this case, we may assume that $X_{-\infty}=\emptyset$ by Lemma 3.12. By Lemma 3.9, X is normal. By Kodaira's lemma, we can write $qL-\omega\sim_{\mathbb{R}} A+E$ on X, where A is a π-ample $\mathbb{Q}$-divisor on X and E is an effective $\mathbb{R}$-Cartier divisor on X. We define $\widetilde{\omega}=\omega+\varepsilon E$ with $0<\varepsilon\ll 1$. Then $[X,\widetilde{\omega}]$ is a quasi-log scheme with $\mathrm{Nqlc}(X,\widetilde{\omega})=\emptyset$ (see Lemma 3.15). Note that

$$qL-\widetilde{\omega}\sim_{\mathbb{R}}(1-\varepsilon)(qL-\omega)+\varepsilon A$$

is π-ample. Therefore, by the basepoint-free theorem for quasi-log schemes (see [A, Theorem 5.1], [F2, Theorem 3.66], Remark 1.3, and [F12, Theorem 6.5.1]), we see that $\mathcal{O}_X(mL)$ is π-generated for every $m\gg 0$.

Step 5. From now on, by Step 4, we may assume that there is a qlc center C' of $[X,\omega]$ or assume that $C\cap X_{-\infty}\neq\emptyset$, where $X=C\cup X_{-\infty}$. We set

$$X'=\Big(\bigcup_{i\in I}C_i\Big)\cup X_{-\infty}$$

as in Theorem 3.8, where $\{C_i\}_{i\in I}$ is the set of all qlc centers of $[X,\omega]$, equivalently, $X'=\mathrm{Nqklt}(X,\omega)$. Then, by induction on the dimension or the assumption on $\mathcal{O}_{X_{-\infty}}(mL)$, $\mathcal{O}_{X'}(mL)$ is π-generated for every $m\gg 0$. By the same arguments as in Step 2, that is, the surjectivity of the restriction map $\pi_*\mathcal{O}_X(mL)\to\pi_*\mathcal{O}_{X'}(mL)$ for every $m\geq q$, $\mathcal{O}_X(mL)$ is π-generated in a neighborhood U of X' for every large and positive integer m. Note that $C\cap U\neq\emptyset$. In particular, for every prime number p and every large positive integer l, $\mathcal{O}_X(p^lL)$ is π-generated in the above neighborhood U of $X'=\mathrm{Nqklt}(X,\omega)$.

Step 6. In this step, we prove the following claim.

Claim. *If the relative base locus* $\mathrm{Bs}_\pi|p^lL|$ (*with the reduced scheme structure*) *is not empty, then there is a positive integer* a *such that* $\mathrm{Bs}_\pi|p^{al}L|$ *is strictly smaller than* $\mathrm{Bs}_\pi|p^lL|$.

Proof of Claim. Note that $\mathrm{Bs}_\pi|p^{al}L|\subseteq\mathrm{Bs}_\pi|p^lL|$ for every positive integer a. We consider $[X^\dagger,\omega^\dagger]$ as in Lemma 3.14. Since $(qL-\omega)|_{X^\dagger}$ is nef and big over S, we can write

$$qL|_{X^\dagger}-\omega^\dagger\sim_{\mathbb{R}}A+E$$

on $X^\dagger$ by Kodaira's lemma (cf. [F10, Lemma A.10]), where A is a π-ample $\mathbb{Q}$-divisor on $X^\dagger$ and E is an effective $\mathbb{R}$-Cartier divisor on $X^\dagger$. We note that $X^\dagger$ is projective over S and that $X^\dagger$ is not necessarily normal (see Example 4.1). By Lemma 3.15, we have a new quasi-log structure on $[X^\dagger, \widetilde{\omega}]$, where $\widetilde{\omega} = \omega^\dagger + \varepsilon E$ with $0 < \varepsilon \ll 1$, such that

$$\mathrm{Nqlc}(X^\dagger, \widetilde{\omega}) = \Big(\bigcup_{i \in I} C_i\Big) \cup \mathrm{Nqlc}(X^\dagger, \omega^\dagger), \tag{4.1}$$

where $\{C_i\}_{i\in I}$ is the set of qlc centers of $[X^\dagger, \omega^\dagger]$ contained in $\mathrm{Supp}\, E$.

We write $n = \dim X^\dagger$. Let $D_1, \dots, D_{n+1}$ be general members of $|p^l L|$. Let $f : (Y, B_Y) \to X^\dagger$ be a quasi-log resolution of $[X^\dagger, \widetilde{\omega}]$. We consider $f : (Y, B_Y + \sum_{i=1}^{n+1} f^* D_i) \to X^\dagger$. We define

$$c = \sup_{t \geq 0} \left\{ t \;\middle|\; \begin{array}{l} \text{the normalization of } (Y, B_Y + t\sum_{i=1}^{n+1} f^* D_i) \text{ is} \\ \text{sub log canonical over } X^\dagger \setminus \mathrm{Nqlc}(X^\dagger, \widetilde{\omega}) \end{array} \right\}.$$

Then $c < 1$ by Lemma 3.16. We have $c > 0$ by Step 5. Thus,

$$f : \Big(Y, B_Y + c\sum_{i=1}^{n+1} f^* D_i\Big) \to X^\dagger$$

gives a quasi-log structure on $[X^\dagger, \widetilde{\omega} + c\sum_{i=1}^{n+1} D_i]$. Note that $[X^\dagger, \widetilde{\omega} + c\sum_{i=1}^{n+1} D_i]$ has only quasi-log canonical singularities on $X^\dagger \setminus \mathrm{Nqlc}(X^\dagger, \widetilde{\omega})$. By construction, there is a qlc center C_0 of $[X^\dagger, \widetilde{\omega} + c\sum_{i=1}^{n+1} D_i]$ contained in $\mathrm{Bs}_\pi |p^l L|$. We set $\widetilde{\omega} + c\sum_{i=1}^{n+1} D_i = \overline{\omega}$. Then

$$C_0 \cap \mathrm{Nqlc}(X^\dagger, \overline{\omega}) = \emptyset$$

because

$$\mathrm{Bs}_\pi |p^l L| \cap \mathrm{Nqklt}(X, \omega) = \emptyset.$$

Note that $\mathrm{Nqlc}(X^\dagger, \overline{\omega}) = \mathrm{Nqlc}(X^\dagger, \widetilde{\omega})$ by construction. We also note that

$$(q + c(n+1)p^l) L|_{X^\dagger} - \overline{\omega} \sim_{\mathbb{R}} (1 - \varepsilon)(qL|_{X^\dagger} - \omega^\dagger) + \varepsilon A$$

is ample over S. Therefore,

$$\pi_* \mathcal{O}_{X^\dagger}(mL) \to \pi_* \mathcal{O}_{C_0}(mL) \oplus \pi_* \mathcal{O}_{\mathrm{Nqlc}(X^\dagger, \overline{\omega})}(mL) \tag{4.2}$$

is surjective for every $m \geq q + c(n+1)p^l$. Moreover, $\pi_* \mathcal{O}_{C_0}(mL)$ is π-generated for every $m \gg 0$ by the basepoint-free theorem for quasi-log schemes (see [A, Theorem 5.1], [F2, Theorem 3.66], Remark 1.3, and [F12, Theorem 6.5.1]). Note that $[C_0, \overline{\omega}|_{C_0}]$ is a quasi-log scheme with only quasi-log canonical singularities by Theorem 3.8 and Lemma 3.13. Therefore, we can construct a section s of $\mathcal{O}_{X^\dagger}(p^{al} L)$

for some positive integer a such that $s|_{C_0}$ is not zero and s is zero on $\mathrm{Nqlc}(X^\dagger, \overline{\omega})$ by (4.2). Thus s is zero on

$$\mathrm{Nqlc}(X^\dagger, \overline{\omega}) = \mathrm{Nqlc}(X^\dagger, \widetilde{\omega}) = \Big(\bigcup_{i\in I} C_i\Big) \cup \mathrm{Nqlc}(X^\dagger, \omega^\dagger)$$

by (4.1). In particular, s is zero on $\mathrm{Nqlc}(X^\dagger, \omega^\dagger)$. So, s can be seen as a section of $\mathcal{O}_X(p^{al}L)$ because $\mathcal{I}_{\mathrm{Nqlc}(X^\dagger,\omega^\dagger)} = \mathcal{I}_{\mathrm{Nqlc}(X,\omega)}$ by construction (see Lemma 3.14). Therefore, $\mathrm{Bs}_\pi |p^{al}L|$ is strictly smaller than $\mathrm{Bs}_\pi |p^l L|$. This completes the proof of Claim. □

Step 7. By Step 6 and noetherian induction, $\mathcal{O}_X(p^l L)$ and $\mathcal{O}_X(p'^{l'} L)$ are both π-generated for large l and l', where p and p' are distinct prime numbers. So, there exists a positive integer m_0 such that $\mathcal{O}_X(mL)$ is π-generated for every $m \geq m_0$.

Thus we obtain the desired basepoint-free theorem. □

Example 4.1. Let C be a nodal curve on a smooth surface. Then $[C, K_C]$ is a quasi-log scheme with only quasi-log canonical singularities. In this case, C is not normal.

Finally, we prove Theorems 1.5 and 1.7.

Proof of Theorems 1.5 and 1.7. Let $f : Y \to X$ be a resolution such that $K_Y + B_Y = f^*(K_X + B)$ and $\mathrm{Supp}\, B_Y$ is a simple normal crossing divisor on Y. By Lemma 3.2, (Y, B_Y) is a globally embedded simple normal crossing pair. Then $f : (Y, B_Y) \to X$ defines a quasi-log structure on $[X, K_X + B]$. We note that $\mathcal{J}_{\mathrm{NLC}}(X, B)$ coincides with the defining ideal sheaf of $\mathrm{Nqlc}(X, K_X + B)$ and that C is a qlc stratum of $[X, K_X + B]$ if and only if C is a log canonical stratum of (X, B). Therefore, Theorem 1.7 is a special case of Theorem 1.1. Moreover, Theorem 1.5 is a special case of Theorem 1.7. □

Acknowledgements

The author was partially supported by Grant-in-Aid for Young Scientists (A) 24684002 from JSPS. He thanks Kento Fujita for his comments. He also thanks the referee for useful comments.

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