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Microlocal Lefschetz Classes of Graph Trace Kernels

by

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Abstract

In this paper, we define the notion of graph trace kernels as a generalization of trace kernels. We associate a microlocal Lefschetz class with a graph trace kernel and prove that this class is functorial with respect to the composition of kernels. We apply graph trace kernels to the microlocal Lefschetz fixed point formula for constructible sheaves.

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§1. Introduction

In [KS14], Kashiwara and Schapira introduced the notion of trace kernels and the method to associate a microlocal Euler class with a trace kernel.

Let X be a C^∞-manifold and k be a field. We denote by ω_X the dualizing complex on X, that is, $\omega_X \simeq \mathrm{or}_X[d_X]$ where or_X is the orientation sheaf on X and d_X is the dimension of X. Denote by k_{Δ_X} and ω_{Δ_X} the direct image of k_X and ω_X respectively under the diagonal embedding $\delta\colon X \hookrightarrow X \times X$. Let $\pi\colon T^*X \to X$ be the cotangent bundle of X.

A *trace kernel* on X is a triplet (K, u, v) where K is an object of the derived category of sheaves $\mathsf{D}^{\mathrm{b}}(k_{X\times X})$ and u, v are morphisms

$$u\colon k_{\Delta_X} \to K, \qquad v\colon K \to \omega_{\Delta_X}. \tag{1.1}$$

One can naturally define the microlocal Euler class $\mu\mathrm{eu}(K, u, v)$ as an element of $H^0_\Lambda(T^*X; \mu hom(k_{\Delta_X}, \omega_{\Delta_X})) \simeq H^0_\Lambda(T^*X; \pi^{-1}\omega_X)$, where $\Lambda = \mathrm{SS}(K) \cap T^*_{\Delta_X}(X \times X)$. Kashiwara and Schapira proved the functoriality of the microlocal Euler classes:

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the microlocal Euler class of the composition $K_1 \circ K_2$ of two trace kernels is the composition of the microlocal Euler classes of K_1 and K_2 [KS14, Theorem 6.3].

On the other hand, microlocal Lefschetz classes of elliptic pairs (Guillermou [Gu96]) and Lefschetz cycles of constructible sheaves (Matsui–Takeuchi [MT10]) were introduced in order to prove the microlocal fixed point formula. For elliptic pairs, see Schapira–Schneiders [ScSn94]. For recent results on this subject, see also [IMT15] and [RTT13].

Let us recall the notion of Lefschetz cycles defined in [MT10]. Let X be a real analytic manifold and $\phi\colon X \to X$ be a morphism of manifolds. We denote by $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$ the bounded derived category of $\mathbb{R}$-constructible sheaves on X. Denote by ω_{Γ_ϕ} the direct image of ω_X under the graph map $\delta_\phi\colon X \hookrightarrow X \times X, x \mapsto (x, \phi(x))$. With a pair (F, Φ) of $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$ and $\Phi \in \mathrm{Hom}(\phi^{-1}F, F)$, one can associate a cohomology class $\mu\mathrm{Le}(F, \Phi, \phi) \in H^0_\Lambda(T^*X; \mu_{\Delta_X}(\omega_{\Gamma_\phi}))$, where $\Lambda := \mathrm{SS}(F) \cap T^*_{\Delta_X}(X \times X) \cap T^*_{\Gamma_\phi}(X \times X)$. This class is called the Lefschetz cycle or the *microlocal Lefschetz class* of the pair (F, Φ).

The microlocal Lefschetz classes of $\mathbb{R}$-constructible sheaves can be treated in the same way as the microlocal Euler classes of trace kernels. Define $\mathrm{D}_X F := \mathrm{R}\mathscr{H}om(F, \omega_X)$, the dual of F. The pair (F, Φ) gives natural morphisms

$$k_{\Delta_X} \to \mathrm{D}_X F \boxtimes F \to \omega_{\Gamma_\phi}. \tag{1.2}$$

The composition of the above morphisms defines a cohomology class in $H^0_\Lambda(T^*X; \mu hom(k_{\Delta_X}, \omega_{\Gamma_\phi})) \simeq H^0_\Lambda(T^*X; \mu_{\Delta_X}(\omega_{\Gamma_\phi}))$ and this class coincides with $\mu\mathrm{Le}(F, \Phi, \phi)$.

In this paper, we extend the notion of trace kernels so that we can treat fixed point formulas. Then we associate a microlocal Lefschetz class with such a kernel and prove the functoriality of the class.

For a C^∞-manifold X and a morphism of manifolds $\phi\colon X \to X$, a *ϕ-graph trace kernel* is a triplet (K, u, v) where $K \in \mathsf{D}^{\mathrm{b}}(k_{X\times X})$ and u, v are morphisms

$$u\colon k_{\Delta_X} \to K, \qquad v\colon K \to \omega_{\Gamma_\phi}. \tag{1.3}$$

One defines the microlocal Lefschetz class $\mu\mathrm{Le}(K, u, v, \phi)$ as an element of $H^0_\Lambda(T^*X; \mu hom(k_{\Delta_X}, \omega_{\Gamma_\phi}))$, where $\Lambda := \mathrm{SS}(K) \cap T^*_{\Delta_X}(X \times X) \cap T^*_{\Gamma_\phi}(X \times X)$. By (1.2), a pair (F, Φ) of $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(X)$ and $\Phi \in \mathrm{Hom}(\phi^{-1}F, F)$ defines a ϕ-graph trace kernel.

Our main result is the functoriality of microlocal Lefschetz classes: the microlocal Lefschetz class of the composition $K_1 \circ K_2$ of two graph trace kernels is the composition of the microlocal Lefschetz classes of K_1 and K_2 (for a more precise statement, see Theorem 4.3). As an application, we prove the microlocal Lefschetz fixed point formula for constructible sheaves.

Finally, let us explain the difference between our construction and that of [KS14]. In the last section of [KS14], the authors have remarked that trace kernels can be adapted to the Lefschetz fixed point formula for constructible sheaves. However, their construction is not suitable to prove the functoriality of the cohomology classes. Therefore, we extend the notion of trace kernels itself and prove the functoriality by using the new framework.

§2. Preliminaries

§2.1. Review on sheaves

In this paper, all manifolds are assumed to be real manifolds of class C^∞. Throughout this paper, let k be a field of characteristic zero. We follow the notation of [KS90].

Let X be a manifold. We denote by $\pi_X\colon T^*X \to X$ its cotangent bundle. If there is no risk of confusion, we simply write π instead of π_X. For a submanifold M of X, we denote by $T^*_M X$ the conormal bundle to M. In particular, $T^*_X X$ denotes the zero-section of T^*X. We also denote by $a\colon T^*X \to T^*X$ the antipodal map defined by $(x;\xi) \mapsto (x;-\xi)$. A set $\Lambda \subset T^*X$ is said to be conic if it is invariant by the action of $\mathbb{R}^+$ on T^*X.

Let $f\colon X \to Y$ be a morphism of manifolds. With f one associates the maps

$$\begin{array}{ccccc} T^*X & \xleftarrow{\;f_d\;} & X \times_Y T^*Y & \xrightarrow{\;f_\pi\;} & T^*Y \\ & \searrow^{\pi_X} & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_Y} \\ & & X & \xrightarrow{\;f\;} & Y \end{array}$$

We denote by k_X the constant sheaf on X with stalk k and by $\mathsf{D}^{\mathrm{b}}(k_X)$ the bounded derived category of sheaves of k-vector spaces on X. One can define Grothendieck's six operations $\mathrm{R}f_*, f^{-1}, \mathrm{R}f_!, f^!, \overset{\mathrm{L}}{\otimes}, \mathrm{R}\mathscr{H}om$ as functors of derived categories of sheaves. Since the functor $\cdot \underset{k_X}{\otimes} \cdot$ is exact, we simply write $\otimes$ instead of $\overset{\mathrm{L}}{\otimes}$. One denotes by ω_X the dualizing complex on X. That is, if $\mathbf{a}_X\colon X \to \mathrm{pt}$ denotes the natural map, then $\omega_X := \mathbf{a}_X^! \, k_{\mathrm{pt}}$. One also denotes by $\omega_X^{\otimes -1} := \mathrm{R}\mathscr{H}om(\omega_X, k_X)$ the dual of ω_X. More generally, for a morphism $f\colon X \to Y$, we denote by $\omega_{X/Y} := f^! k_Y \simeq \omega_X \otimes f^{-1}\omega_Y^{\otimes -1}$ the relative dualizing complex. Note that $\omega_X \simeq \mathrm{or}_X[d_X]$, where or_X is the orientation sheaf on X and d_X is the dimension of X. Recall that there is a natural morphism of functors

$$\omega_{X/Y} \otimes f^{-1}(\cdot) \to f^!(\cdot). \tag{2.1}$$

We define the duality functor by

$$\mathrm{D}_X F := \mathrm{R}\mathscr{H}om(F, \omega_X). \tag{2.2}$$

For $F \in \mathsf{D}^{\mathrm{b}}(k_X)$, we denote by $\mathrm{Supp}(F)$ the support of F and by $\mathrm{SS}(F)$ its micro-support, a closed conic involutive subset of T^*X.

For a closed submanifold M of X, one denotes by $\mu_M \colon \mathsf{D}^{\mathrm{b}}(k_X) \to \mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(k_{T^*_M X})$ Sato's microlocalization functor along M, where $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(k_{T^*_M X})$ is the full subcategory of $\mathsf{D}^{\mathrm{b}}(k_{T^*_M X})$ consisting of $\mathbb{R}^+$-conic objects. We shall use the functor μhom defined in [KS90]. For $F_1, F_2 \in \mathsf{D}^{\mathrm{b}}(k_X)$, one defines the bifunctor

$$\mu hom \colon \mathsf{D}^{\mathrm{b}}(k_X)^{\mathrm{op}} \times \mathsf{D}^{\mathrm{b}}(k_X) \to \mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(k_{T^*X}),$$
$$\mu hom(F_1, F_2) := \mu_\Delta \mathrm{R}\mathscr{H}om(q_2^{-1} F_1, q_1^! F_2),$$

where q_1 and q_2 are the first and second projections from $X \times X$ and Δ is the diagonal. Note that the support of $\mu hom(F_1, F_2)$ satisfies

$$\mathrm{Supp}(\mu hom(F_1, F_2)) \subset \mathrm{SS}(F_1) \cap \mathrm{SS}(F_2). \tag{2.3}$$

Furthermore, we have the isomorphism

$$\mathrm{R}\pi_* \mu hom(F_1, F_2) \simeq \mu hom(F_1, F_2)|_X \simeq \mathrm{R}\mathscr{H}om(F_1, F_2). \tag{2.4}$$

§2.2. Compositions of kernels

We follow the notation of [KS14]. The results in this subsection are the same as in Section 3 of [KS14]. For the convenience of the readers, we give proofs of these results here.

Notation 2.1. (i) For a manifold X, we denote by $\delta \colon X \hookrightarrow X \times X$ the diagonal embedding and by Δ_X the diagonal set of $X \times X$.

(ii) Let X_i $(i = 1, 2, 3)$ be manifolds. For short, we write $X_{ij} := X_i \times X_j$, $X_{123} := X_1 \times X_2 \times X_3$, $X_{11223} := X_1 \times X_1 \times X_2 \times X_2 \times X_3$, etc.

(iii) Let $\phi_i : X_i \to X_i$ $(i = 1, 2, 3)$ be morphisms of manifolds. We write $\phi_{ij} := \phi_i \times \phi_j : X_{ij} \to X_{ij}$.

(iv) For simplicity, we shall write k_i instead of k_{X_i} and ω_i instead of ω_{X_i}, etc. We also write k_{Δ_i} instead of $k_{\Delta_{X_i}}$.

(v) We denote by π_i or π_{ij}, etc. the projection $T^*X_i \to X_i$ or $T^*X_{ij} \to X_{ij}$, etc.

(vi) We use the same symbol q_i for the projections $X_{ij} \to X_i$ and $X_{123} \to X_i$. We also denote by q_{ij} the projection $X_{123} \to X_{ij}$, by p_i the projection $T^*X_{ij} \to T^*X_i$, and by p_{ij} the projection $T^*X_{123} \to T^*X_{ij}$.

(vii) We denote by p_{j^a} (resp. p_{ij^a}) the composition of p_j (resp. p_{ij}) and the antipodal map on T^*X_j.

(viii) We denote by δ_2 the diagonal embedding $X_{123} \to X_{1223}$.

Recall the operations of composition of kernels defined in [KS14].

Definition 2.2 ([KS14]). We define the operations of composition of kernels as follows:

$$\underset{2}{\circ}\colon \mathsf{D}^{\mathrm{b}}(k_{12}) \times \mathsf{D}^{\mathrm{b}}(k_{23}) \to \mathsf{D}^{\mathrm{b}}(k_{13}),$$
$$(K_{12}, K_{23}) \mapsto K_{12} \underset{2}{\circ} K_{23} := \mathrm{R}q_{13!}\,(q_{12}^{-1}K_{12} \otimes q_{23}^{-1}K_{23}) \simeq \mathrm{R}q_{13!}\delta_2^{-1}(K_{12} \boxtimes K_{23}),$$
$$\underset{2}{*}\colon \mathsf{D}^{\mathrm{b}}(k_{12}) \times \mathsf{D}^{\mathrm{b}}(k_{23}) \to \mathsf{D}^{\mathrm{b}}(k_{13}),$$
$$(K_{12}, K_{23}) \mapsto K_{12} \underset{2}{*} K_{23} := \mathrm{R}q_{13*}\,(\omega^{\otimes -1}_{X_{123}/X_{1223}} \otimes \delta_2^{!}(K_{12} \boxtimes K_{23})).$$

By (2.1), we have a natural morphism $\delta_2^{-1}(\cdot) \to \omega^{\otimes -1}_{X_{123}/X_{1223}} \otimes \delta_2^{!}(\cdot)$. Combining this with the morphism $\mathrm{R}q_{13!} \to \mathrm{R}q_{13*}$, we obtain a natural morphism

$$K_{12} \underset{2}{\circ} K_{23} \to K_{12} \underset{2}{*} K_{23}. \tag{2.5}$$

This is an isomorphism if $p_{12^a}^{-1}(\mathrm{SS}(K_{12})) \cap p_{23}^{-1}(\mathrm{SS}(K_{23}))$ is proper over T^*X_{13}.

We now define the composition of kernels on cotangent bundles.

Definition 2.3 ([KS14]). For kernels on cotangent bundles, we define the composition of kernels as follows:

$$\overset{a}{\underset{2}{\circ}}\colon \mathsf{D}^{\mathrm{b}}(k_{T^*X_{12}}) \times \mathsf{D}^{\mathrm{b}}(k_{T^*X_{23}}) \to \mathsf{D}^{\mathrm{b}}(k_{T^*X_{13}}),$$
$$(K_{12}, K_{23}) \mapsto K_{12} \overset{a}{\underset{2}{\circ}} K_{23} := \mathrm{R}p_{13!}(p_{12^a}^{-1}K_{12} \otimes p_{23}^{-1}K_{23}).$$

We also define the corresponding operations for subsets of cotangent bundles. Let $A \subset T^*X_{12}$ and $B \subset T^*X_{23}$. We set

$$A \overset{a}{\underset{2}{\times}} B := p_{12^a}^{-1}(A) \cap p_{23}^{-1}(B), \qquad A \overset{a}{\underset{2}{\circ}} B := p_{13}(A \overset{a}{\underset{2}{\times}} B).$$

In order to define a composition morphism, we need the following lemma. Let X, Y, S be manifolds. Let $q_X : X \to S$ and $q_Y : Y \to S$ be morphisms. Assume that

$$X \times_S Y \text{ is a submanifold of } X \times Y. \tag{2.6}$$

Let j be an embedding $X \times_S Y \hookrightarrow X \times Y$. Noticing that $(X \times_S Y) \times_{(X \times Y)} T^*(X \times Y) \simeq T^*X \times_S T^*Y$, we have the following morphisms:

$$T^*(X \times_S Y) \xleftarrow{j_d} T^*X \times_S T^*Y \xrightarrow{j_\pi} T^*X \times T^*Y. \tag{2.7}$$

Lemma 2.4. (cf. [KS90, Proposition 4.4.8]) *For $F_1, G_1 \in \mathsf{D}^{\mathrm{b}}(k_X)$ and $F_2, G_2 \in \mathsf{D}^{\mathrm{b}}(k_Y)$, there is a canonical morphism*

$$
\begin{aligned}
(2.8)\qquad & \mathrm{R}j_{d!}(\mu hom(G_1, F_1) \underset{S}{\boxtimes} \mu hom(G_2, F_2)) \\
&\qquad \to \mu hom(j^!(G_1 \boxtimes G_2) \otimes \omega^{\otimes -1}_{X \times_S Y/X \times Y}, F_1 \underset{S}{\boxtimes} F_2).
\end{aligned}
$$

Proof. First, we construct the morphism when $S = \mathrm{pt}$. By using the morphism $\mu_M(F) \boxtimes \mu_N(G) \to \mu_{M \times N}(F \boxtimes G)$ [KS90, Proposition 4.3.6], we obtain a chain of morphisms

$$
\begin{aligned}
& \mu_{\Delta_X} \mathrm{R}\mathscr{H}om({q_X}_2^{-1} G_1, {q_X}_1^! F_1) \boxtimes \mu_{\Delta_Y} \mathrm{R}\mathscr{H}om({q_Y}_2^{-1} G_2, {q_Y}_1^! F_2) \\
&\quad \to \mu_{\Delta_X \times \Delta_Y} (\mathrm{R}\mathscr{H}om({q_X}_2^{-1} G_1, {q_X}_1^! F_1) \boxtimes \mathrm{R}\mathscr{H}om({q_Y}_2^{-1} G_2, {q_Y}_1^! F_2)) \\
&\quad \to \mu_{\Delta_X \times \Delta_Y} (\mathrm{R}\mathscr{H}om({q_X}_2^{-1} G_1 \boxtimes {q_Y}_2^{-1} G_2, {q_X}_1^! F_1 \boxtimes {q_Y}_1^! F_2) \\
&\quad \simeq \mu_{\Delta_{X \times Y}} \mathrm{R}\mathscr{H}om(q_2^{-1}(G_1 \boxtimes G_2), q_1^!(F_1 \boxtimes F_2)).
\end{aligned}
$$

Next, we treat the general case. Using the morphism

$$
(2.9)\qquad \mathrm{R}j_{d!} j_\pi^{-1} \mu hom(G, F) \to \mu hom(j^! G \otimes \omega^{\otimes -1}_{X \times_S Y/X \times Y}, j^{-1} F)
$$

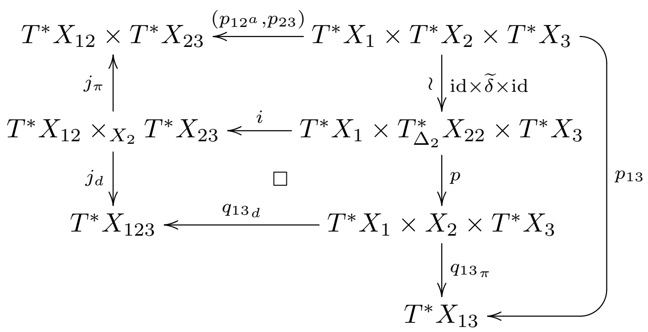
from [KS90, Proposition 4.4.7], we obtain a chain of morphisms

$$
\begin{aligned}
& \mathrm{R}j_{d!} j_\pi^{-1} (\mu hom(G_1, F_1) \boxtimes \mu hom(G_2, F_2)) \to \mathrm{R}j_{d!} j_\pi^{-1} \mu hom(G_1 \boxtimes G_2, F_1 \boxtimes F_2) \\
&\qquad \to \mu hom(j^!(G_1 \boxtimes G_2) \otimes \omega^{\otimes -1}_{X \times_S Y/X \times Y}, j^{-1}(F_1 \boxtimes F_2)). \qquad \square
\end{aligned}
$$

Proposition 2.5 ([KS14]). *For $G_1, F_1 \in \mathsf{D}^{\mathrm{b}}(k_{12})$ and $G_2, F_2 \in \mathsf{D}^{\mathrm{b}}(k_{23})$, there is a composition morphism*

$$
(2.10)\qquad \mu hom(G_1, F_1) \overset{a}{\underset{2}{\circ}} \mu hom(G_2, F_2) \to \mu hom(G_1 \underset{2}{*} G_2, F_1 \underset{2}{\circ} F_2).
$$

Proof. We shall apply Lemma 2.4 for $X_{12} \to X_2$ and $X_{23} \to X_2$. In this case, $X_{12} \times_{X_2} X_{23} \simeq X_{123}$ and j is the diagonal embedding $X_{123} \hookrightarrow X_{1223}$. Consider the following commutative diagram:



where $\widetilde{\delta}$ is defined by $(x_2; \xi_2) \mapsto (x_2, x_2; -\xi_2, \xi_2)$.

By Lemma 2.4, we get a morphism

$$\mathrm{R}j_{d!}j_{\pi}^{-1}(\mu hom(G_1,F_1)\boxtimes\mu hom(G_2,F_2)) \to \mu hom(j^!(G_1\boxtimes G_2)\otimes\omega^{\otimes -1}_{X_{123}/X_{1223}}, j^{-1}(F_1\boxtimes F_2)). \tag{2.11}$$

Set $G := j^!(G_1\boxtimes G_2)\otimes\omega^{\otimes -1}_{X_{123}/X_{1223}} \in \mathsf{D}^{\mathrm{b}}(k_{123})$ and $F := j^{-1}(F_1\boxtimes F_2)\in\mathsf{D}^{\mathrm{b}}(k_{123})$. Combining (2.11) with the morphism

$$\mathrm{R}q_{13_{\pi}!}q_{13_d}^{-1}\mu hom(G,F)\to\mu hom(\mathrm{R}q_{13*}G,\mathrm{R}q_{13!}F) \tag{2.12}$$

from [KS90, Proposition 4.4.7], we get a morphism

$$\begin{aligned}\mathrm{R}q_{13_{\pi}!}q_{13_d}^{-1}\mathrm{R}j_{d!}j_{\pi}^{-1}(\mu hom(G_1,F_1)\boxtimes\mu hom(G_2,F_2)) &\to \mu hom(\mathrm{R}q_{13*}G,\mathrm{R}q_{13!}F)\\ &\simeq \mu hom(G_1\underset{2}{*}G_2, F_1\underset{2}{\circ}F_2).\end{aligned}$$

By the above commutative diagram, we have

$$\begin{aligned}\mathrm{R}q_{13_{\pi}!}q_{13_d}^{-1}\mathrm{R}j_{d!}j_{\pi}^{-1} &\simeq \mathrm{R}q_{13_{\pi}!}\mathrm{R}p_!i^{-1}j_{\pi}^{-1}\simeq\mathrm{R}q_{13_{\pi}!}\mathrm{R}p_!(\mathrm{id}\times\widetilde{\delta}\times\mathrm{id})_!(p_{12^a},p_{23})^{-1}\\ &\simeq \mathrm{R}p_{13!}(p_{12^a},p_{23})^{-1}.\end{aligned}$$

Thus, the result follows from the isomorphisms

$$\begin{aligned}&\mathrm{R}q_{13_{\pi}!}q_{13_d}^{-1}\mathrm{R}j_{d!}j_{\pi}^{-1}(\mu hom(G_1,F_1)\boxtimes\mu hom(G_2,F_2))\\ &\quad\simeq \mathrm{R}p_{13!}(p_{12^a},p_{23})^{-1}(\mu hom(G_1,F_1)\boxtimes\mu hom(G_2,F_2))\\ &\quad\simeq \mathrm{R}p_{13!}(p_{12^a}^{-1}\mu hom(G_1,F_1)\otimes p_{23}^{-1}\mu hom(G_2,F_2)).\end{aligned}$$ □

§3. Definition of graph trace kernels

§3.1. Microlocal homology associated with morphisms

Let X be a manifold and $\phi\colon X\to X$ be a morphism of manifolds. We shall identify X with the diagonal Δ_X of $X\times X$ and write Δ instead of Δ_X if there is no risk of confusion. We shall also identify T^*X with $T^*_{\Delta}(X\times X)$ by means of the map

$$\delta^a_{T^*X}\colon T^*X\to T^*(X\times X),\qquad (x;\xi)\mapsto(x,x;\xi,-\xi). \tag{3.1}$$

We denote by $\delta_\phi=(\mathrm{id}_X,\phi)\colon X\hookrightarrow X\times X$ the graph map of ϕ and by $\Gamma_\phi=\delta_\phi(X)$ the graph of ϕ. Set $k_{\Gamma_\phi}:=(\delta_\phi)_*k_X$, $\omega_{\Gamma_\phi}:=(\delta_\phi)_*\omega_X$ and $\omega^{\otimes -1}_{\Gamma_\phi}:=(\delta_\phi)_*\omega_X^{\otimes -1}$. If $\phi=\mathrm{id}_X$, we shall write δ for δ_ϕ, k_Δ for k_{Γ_ϕ}, etc.

Definition 3.1. Let Λ be a closed conic subset of T^*X. We set

(i) $\mathscr{MH}_\Lambda(\phi):=\mathrm{R}\Gamma_\Lambda(\delta^a_{T^*X})^{-1}\mu hom(k_\Delta,\omega_{\Gamma_\phi})$,

(ii) $\mathbb{MH}_\Lambda(\phi):=\mathrm{R}\Gamma(T^*X;\mathscr{MH}_\Lambda(\phi))$,

(iii) $\mathbb{MH}^n_\Lambda(\phi):=H^n(\mathbb{MH}_\Lambda(\phi))$.

Let $\phi_i\colon X_i \to X_i$ $(i = 1, 2, 3)$ be morphisms of manifolds. We write Δ_i for $\Delta_{X_i} \subset X_{ii}$, etc.

Lemma 3.2. *We have natural morphisms:*

(i) $\omega_{\Gamma_{\phi_{12}}} \underset{22}{\circ} (k_{\Gamma_{\phi_2}} \boxtimes \omega_{\Gamma_{\phi_3}}) \to \omega_{\Gamma_{\phi_{13}}}$,

(ii) $k_{\Delta_{13}} \to k_{\Delta_{12}} \underset{22}{*} (\omega_{\Delta_2}^{\otimes -1} \boxtimes k_{\Delta_3})$.

Proof. We denote by δ_{22} the diagonal embedding $X_{112233} \hookrightarrow X_{11222233}$.

(i) We have morphisms

$$\begin{aligned}\omega_{\Gamma_{\phi_{12}}} \underset{22}{\circ} (k_{\Gamma_{\phi_2}} \boxtimes \omega_{\Gamma_{\phi_3}}) &= \mathrm{R}q_{1133!}\delta_{22}^{-1}(\omega_{\Gamma_{\phi_1}} \boxtimes \omega_{\Gamma_{\phi_2}} \boxtimes k_{\Gamma_{\phi_2}} \boxtimes \omega_{\Gamma_{\phi_3}}) \\ &\simeq \mathrm{R}q_{1133!}(\omega_{\Gamma_{\phi_1}} \boxtimes \omega_{\Gamma_{\phi_2}} \boxtimes \omega_{\Gamma_{\phi_3}}) \to \omega_{\Gamma_{\phi_{13}}}.\end{aligned}$$

(ii) The proof is the same as that of [KS14, Lemma 4.3]. □

Proposition 3.3. *We have a natural composition morphism*

$$\mu hom(k_{\Delta_{12}}, \omega_{\Gamma_{\phi_{12}}}) \underset{22}{\overset{a}{\circ}} \mu hom(k_{\Delta_{23}}, \omega_{\Gamma_{\phi_{23}}}) \to \mu hom(k_{\Delta_{13}}, \omega_{\Gamma_{\phi_{13}}}). \tag{3.2}$$

Proof. We have

$$\begin{aligned}\mu hom(k_{\Delta_{23}}, \omega_{\Gamma_{\phi_{23}}}) &\simeq \mu hom((\omega_2^{\otimes -1} \boxtimes k_{233}) \otimes k_{\Delta_{23}}, (\omega_2^{\otimes -1} \boxtimes k_{233}) \otimes \omega_{\Gamma_{\phi_{23}}}) \\ &\simeq \mu hom(\omega_{\Delta_2}^{\otimes -1} \boxtimes k_{\Delta_3}, k_{\Gamma_{\phi_2}} \boxtimes \omega_{\Gamma_{\phi_3}}).\end{aligned}$$

Applying Proposition 2.5 and Lemma 3.2, we get a chain of morphisms

$$\begin{aligned}&\mu hom(k_{\Delta_{12}}, \omega_{\Gamma_{\phi_{12}}}) \underset{22}{\overset{a}{\circ}} \mu hom(k_{\Delta_{23}}, \omega_{\Gamma_{\phi_{23}}}) \\ &\quad\simeq \mu hom(k_{\Delta_{12}}, \omega_{\Gamma_{\phi_{12}}}) \underset{22}{\overset{a}{\circ}} \mu hom(\omega_{\Delta_2}^{\otimes -1} \boxtimes k_{\Delta_3}, k_{\Gamma_{\phi_2}} \boxtimes \omega_{\Gamma_{\phi_3}}) \\ &\quad\to \mu hom(k_{\Delta_{12}} \underset{22}{*} (\omega_{\Delta_2}^{\otimes -1} \boxtimes k_{\Delta_3}), \omega_{\Gamma_{\phi_{12}}} \underset{22}{\circ} (k_{\Gamma_{\phi_2}} \boxtimes \omega_{\Gamma_{\phi_3}})) \\ &\quad\to \mu hom(k_{\Delta_{13}}, \omega_{\Gamma_{\phi_{13}}}).\end{aligned}$$

□

Corollary 3.4. *Let Λ_{ij} be a closed conic subset of T^*X_{ij} $(ij = 12, 23)$ and assume that*

$$\Lambda_{12} \underset{2}{\overset{a}{\times}} \Lambda_{23} \to T^*X_{13} \text{ is proper.} \tag{3.3}$$

Set $\Lambda_{13} := \Lambda_{12} \underset{2}{\overset{a}{\circ}} \Lambda_{23} \cap (\delta^a_{T^*X_{13}})^{-1}(T^*_{\Gamma_{\phi_{13}}} X_{1313})$. The composition of kernels induces a morphism

$$\underset{2}{\overset{a}{\circ}}\colon \mathbb{MH}_{\Lambda_{12}}(\phi_{12}) \otimes \mathbb{MH}_{\Lambda_{23}}(\phi_{23}) \to \mathbb{MH}_{\Lambda_{13}}(\phi_{13}). \tag{3.4}$$

In particular, a cohomology class $\lambda \in \mathbb{MH}^0_{\Lambda_{12}}(\phi_{12})$ *defines a morphism*

$$\lambda \overset{a}{\underset{2}{\circ}} \colon \mathbb{MH}_{\Lambda_{23}}(\phi_{23}) \to \mathbb{MH}_{\Lambda_{13}}(\phi_{13}). \tag{3.5}$$

Proof. Noticing that

$$\mathbb{MH}_{\Lambda_{12}}(\phi_{12}) \simeq \mathrm{R}\Gamma_{\delta^a_{T^*X_{12}}\Lambda_{12}}(T^*X_{1122}; \mu hom(k_{\Delta_{12}}, \omega_{\Gamma_{\phi_{12}}})), \tag{3.6}$$

$$\mathbb{MH}_{\Lambda_{23}}(\phi_{23}) \simeq \mathrm{R}\Gamma_{\delta^a_{T^*X_{23}}\Lambda_{23}}(T^*X_{2233}; \mu hom(k_{\Delta_{23}}, \omega_{\Gamma_{\phi_{23}}})), \tag{3.7}$$

we obtain a chain of morphisms

$$\begin{aligned}
&\mathbb{MH}_{\Lambda_{12}}(\phi_{12}) \otimes \mathbb{MH}_{\Lambda_{23}}(\phi_{23}) \\
&\quad\to \mathrm{R}\Gamma_{\delta^a_{T^*X_{12}}\Lambda_{12} \overset{a}{\underset{2}{\circ}} \delta^a_{T^*X_{23}}\Lambda_{23}}(T^*X_{1133}; \mu hom(k_{\Delta_{12}}, \omega_{\Gamma_{\phi_{12}}}) \overset{a}{\underset{22}{\circ}} \mu hom(k_{\Delta_{23}}, \omega_{\Gamma_{\phi_{23}}})) \\
&\quad\to \mathrm{R}\Gamma_{\delta^a_{T^*X_{13}}\Lambda_{13}}(T^*X_{1133}; \mu hom(k_{\Delta_{13}}, \omega_{\Gamma_{\phi_{13}}})) \simeq \mathbb{MH}_{\Lambda_{13}}(\phi_{13}).
\end{aligned}$$

Here, the first morphism comes from the assumption (3.3) and the second one is given by Proposition 3.3. □

§3.2. Microlocal Lefschetz classes of graph trace kernels

Let $\phi \colon X \to X$ be a morphism of manifolds.

Definition 3.5. A *ϕ-graph trace kernel* (K, u, v) is the data of $K \in \mathsf{D}^{\mathrm{b}}(k_{X\times X})$ together with morphisms

$$k_\Delta \xrightarrow{u} K \quad \text{and} \quad K \xrightarrow{v} \omega_{\Gamma_\phi}. \tag{3.8}$$

In particular, the original trace kernels defined in [KS14] are id_X-graph trace kernels. If there is no risk of confusion, we simply write K instead of (K, u, v).

For a ϕ-graph trace kernel K, we set

$$\begin{aligned}
\mathrm{SS}_{\Delta,\phi}(K) &:= \mathrm{SS}(K) \cap T^*_\Delta(X \times X) \cap T^*_{\Gamma_\phi}(X \times X) \\
&= (\delta^a_{T^*X})^{-1}(\mathrm{SS}(K) \cap T^*_{\Gamma_\phi}(X \times X)).
\end{aligned}$$

Definition 3.6. Let (K, u, v) be a ϕ-graph trace kernel.

(i) The morphism u defines an element $\widetilde{u}$ in $H^0_{\mathrm{SS}(K)\cap T^*_\Delta(X\times X)}(T^*X; \mu hom(k_\Delta, K))$. The *microlocal Lefschetz class* $\mu\mathrm{Le}(K, \phi) \in H^0_{\mathrm{SS}_{\Delta,\phi}(K)}(T^*X; \mu hom(k_\Delta, \omega_{\Gamma_\phi}))$ of K is the image of $\widetilde{u}$ under the morphism $\mu hom(k_\Delta, K) \to \mu hom(k_\Delta, \omega_{\Gamma_\phi})$ associated with v.

(ii) Let $\Lambda \subset T^*X$ be a closed conic subset containing $\mathrm{SS}_{\Delta,\phi}(K)$. We denote by $\mu\mathrm{Le}_\Lambda(K, \phi)$ the image of $\widetilde{u}$ in $H^0_\Lambda(T^*X; \mu hom(k_\Delta, \omega_{\Gamma_\phi}))$.

Therefore, we have

$$\mu\mathrm{Le}_\Lambda(K,\phi) \in \mathbb{MH}^0_\Lambda(\phi). \tag{3.9}$$

If there is no risk of confusion, we simply write $\mu\mathrm{Le}(K,\phi)$ instead of $\mu\mathrm{Le}_\Lambda(K,\phi)$.

We denote by $\mathrm{Le}(K,\phi)$ the restriction of $\mu\mathrm{Le}(K,\phi)$ to the zero-section X of T^*X and call it the *Lefschetz class* of K. Note that

$$\begin{aligned}\mu hom(k_\Delta, \omega_{\Gamma_\phi})|_{(X\times X)} &\simeq \mathrm{R}\mathscr{H}om(k_\Delta, \omega_{\Gamma_\phi}) \simeq (\delta_\phi)_*\mathrm{R}\mathscr{H}om(k_{(\delta_\phi)^{-1}(\Delta)}, \omega_X)\\ &\simeq (\delta_\phi)_*\mathrm{R}\Gamma_M(\omega_X),\end{aligned}$$

where $M := \{x \in X; \phi(x) = x\}$ is the fixed point set of ϕ. Since $\mathrm{R}\Gamma_M(\omega_X) \simeq \delta^{-1}(\delta_\phi)_*\mathrm{R}\Gamma_M(\omega_X)$, we have

$$\mathrm{Le}(K,\phi) \in H^0_M(X;\omega_X). \tag{3.10}$$

Graph trace kernels for constructible sheaves. Denote by $\mathsf{D}^{\mathrm{b}}_{\mathrm{cc}}(K_X)$ the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(k_X)$ consisting of cohomologically constructible sheaves (see [KS90, Section 3.4]).

Lemma 3.7. *Let $F \in \mathsf{D}^{\mathrm{b}}_{\mathrm{cc}}(k_X)$ and $\Phi\colon \phi^{-1}F \to F$ be a morphism in $\mathsf{D}^{\mathrm{b}}_{\mathrm{cc}}(k_X)$. There exist natural morphisms in $\mathsf{D}^{\mathrm{b}}_{\mathrm{cc}}(k_{X\times X})$,*

$$k_\Delta \to \mathrm{D}_X F \boxtimes F, \tag{3.11}$$

$$\mathrm{D}_X F \boxtimes F \to \omega_{\Gamma_\phi}. \tag{3.12}$$

In other words, a pair (F,Φ) of an object $F \in \mathsf{D}^{\mathrm{b}}_{\mathrm{cc}}(k_X)$ and a morphism $\Phi\colon \phi^{-1}F \to F$ defines naturally a ϕ-graph trace kernel.

Proof. (i) The morphism id_F induces a morphism

$$k_X \to \mathrm{R}\mathscr{H}om(F,F) \simeq \delta^!(\mathrm{D}_X F \boxtimes F). \tag{3.13}$$

Hence, (3.11) is obtained by adjunction.

(ii) Noticing that $\delta_\phi^{-1}(\mathrm{D}_X F \boxtimes F) \simeq \mathrm{D}_X F \otimes \phi^{-1}F$, we have a chain of morphisms

$$\delta_\phi^{-1}(\mathrm{D}_X F \boxtimes F) \xrightarrow{\Phi} \mathrm{D}_X F \otimes F \to \omega_X. \tag{3.14}$$

Therefore, (3.12) is obtained by adjunction. □

We denote by $\mathrm{TK}_\phi(F,\Phi)$ the ϕ-graph trace kernel associated with the pair (F,Φ) of $F \in \mathsf{D}^{\mathrm{b}}_{\mathrm{cc}}(k_X)$ and $\Phi\colon \phi^{-1}F \to F$. The graph trace kernel defines a microlocal Lefschetz class $\mu\mathrm{Le}(\mathrm{TK}_\phi(F,\Phi),\phi)$. We also denote this class by $\mu\mathrm{Le}(F,\Phi,\phi)$. Note that this construction coincides with that of Lefschetz cycles in [MT10].

Graph trace kernels over one point. Let $X = \mathrm{pt}$. In this case, a (graph) trace kernel (K, u, v) is the data of $K \in \mathsf{D}^{\mathrm{b}}(k)$ and morphisms

$$k \xrightarrow{u} K \xrightarrow{v} k. \tag{3.15}$$

The (microlocal) Lefschetz class $\mathrm{Le}(K)$ of K is the image of $1 \in k$ under vu.

Let us denote by $\mathsf{D}^{\mathrm{b}}_f(k)$ the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(k)$ consisting of objects with finite-dimensional cohomology. Let $V \in \mathsf{D}^{\mathrm{b}}_f(k)$ and $f\colon V \to V$ be a k-linear map. Set $K := V^* \otimes V$ where $V^* := \mathrm{RHom}(V, k)$. Let u be the dual of the trace morphism $V \otimes V^* \to k$ and v be the composition of $\mathrm{id}_{V^*} \otimes f\colon V^* \otimes V \to V^* \otimes V$ and the trace morphism. Then

$$\mathrm{Le}(V^* \otimes V) = \mathrm{tr}(f) := \sum_{p \in \mathbb{Z}} (-1)^p \,\mathrm{tr}(H^p(f)). \tag{3.16}$$

§4. Main results

§4.1. Compositions of microlocal Lefschetz classes

Let X_1, X_2, X_3 be manifolds and $\phi_i\colon X_i \to X_i$ $(i = 1, 2, 3)$ be morphisms. For $ij = 12, 23$, let K_{ij} be a ϕ_{ij}-graph trace kernel.

Lemma 4.1. *There are natural morphisms*

$$K_{12} \underset{22}{\circ} (k_{\Gamma_{\phi_2}} \boxtimes \omega_{\Gamma_{\phi_3}}) \to \omega_{\Gamma_{\phi_{13}}}, \tag{4.1}$$

$$k_{\Delta_{13}} \to K_{12} \underset{22}{\ast} (\omega^{\otimes -1}_{\Delta_2} \boxtimes k_{\Delta_3}). \tag{4.2}$$

Proof. (i) By Lemma 3.2(i), we have a morphism

$$\omega_{\Gamma_{\phi_{12}}} \underset{22}{\circ} (k_{\Gamma_{\phi_2}} \boxtimes \omega_{\Gamma_{\phi_3}}) \to \omega_{\Gamma_{\phi_{13}}}. \tag{4.3}$$

Composing it with the morphism $K_{12} \to \omega_{\Gamma_{\phi_{12}}}$, we get (4.1).

(ii) By Lemma 3.2(ii), we have a morphism

$$k_{\Delta_{13}} \to k_{\Delta_{12}} \underset{22}{\ast} (\omega^{\otimes -1}_{\Delta_2} \boxtimes k_{\Delta_3}). \tag{4.4}$$

Composing it with the morphism $k_{\Delta_{12}} \to K_{12}$, we get (4.2). □

Let $\Lambda_{1122} \subset T^*X_{1122}$ be a closed conic subset containing $\mathrm{SS}(K_{12})$, and Λ_{23} be a closed conic subset of T^*X_{23}. Assume that

$$\Lambda_{1122} \overset{a}{\underset{22}{\times}} \delta^a_{T^*X_{23}} \Lambda_{23} \to T^*X_{1133} \text{ is proper.} \tag{4.5}$$

Set

$$\begin{aligned}\Lambda_{12} &:= \Lambda_{1122} \cap T^*_{\Delta_{12}} X_{1122} \cap T^*_{\Gamma_{\phi_{12}}} X_{1122},\\ \Lambda_{1133} &:= (\Lambda_{1122} \cap T^*_{\Gamma_{\phi_{12}}} X_{1122}) \overset{a}{\underset{22}{\circ}} \delta^a_{T^*X_{23}} \Lambda_{23},\\ \Lambda_{13} &:= \Lambda_{1133} \cap T^*_{\Delta_{13}} X_{1133} \cap T^*_{\Gamma_{\phi_{13}}} X_{1133}\\ &= \Lambda_{12} \overset{a}{\underset{2}{\circ}} \Lambda_{23} \cap (\delta^a_{T^*X_{13}})^{-1}(T^*_{\Gamma_{\phi_{13}}} X_{13}).\end{aligned}$$

We define a map

$$\Phi_{K_{12}} \colon \mathbb{MH}_{\Lambda_{23}}(\phi_{23}) \to \mathbb{MH}_{\Lambda_{13}}(\phi_{13}) \tag{4.6}$$

by the chain of morphisms

$$\begin{aligned}\mathbb{MH}_{\Lambda_{23}}(\phi_{23}) &\simeq \mathrm{R}\Gamma_{\delta^a_{T^*X_{23}}\Lambda_{23}}(T^*X_{2233}; \mu hom(k_{\Delta_{23}}, \omega_{\Gamma_{\phi_{23}}}))\\ &\simeq \mathrm{R}\Gamma_{\delta^a_{T^*X_{23}}\Lambda_{23}}(T^*X_{2233}; \mu hom(\omega_{\Delta_2}^{\otimes -1} \boxtimes k_{\Delta_3}, k_{\Gamma_{\phi_2}} \boxtimes \omega_{\Gamma_{\phi_3}}))\\ &\to \mathrm{R}\Gamma_{\Lambda_{1133}}(T^*X_{1133}; \mu hom(K_{12}, \omega_{\Gamma_{\phi_{12}}}) \overset{a}{\underset{22}{\circ}} \mu hom(\omega_{\Delta_2}^{\otimes -1} \boxtimes k_{\Delta_3}, k_{\Gamma_{\phi_2}} \boxtimes \omega_{\Gamma_{\phi_3}}))\\ &\to \mathrm{R}\Gamma_{\Lambda_{1133}}(T^*X_{1133}; \mu hom(K_{12} \underset{22}{*} (\omega_{\Delta_2}^{\otimes -1} \boxtimes k_{\Delta_3}), \omega_{\Gamma_{\phi_{12}}} \underset{22}{\circ} (k_{\Gamma_{\phi_2}} \boxtimes \omega_{\Gamma_{\phi_3}})))\\ &\to \mathrm{R}\Gamma_{\delta^a_{T^*X_{13}}\Lambda_{13}}(T^*X_{1133}; \mu hom(k_{\Delta_{13}}, \omega_{\Gamma_{\phi_{13}}})) \simeq \mathbb{MH}_{\Lambda_{13}}(\phi_{13}).\end{aligned}$$

The first morphism is given by $v \colon K_{12} \to \omega_{\Gamma_{\phi_{12}}}$ as follows:

$$\begin{aligned}&\mathrm{R}\Gamma_{\delta^a_{T^*X_{23}}\Lambda_{23}}(T^*X_{2233}; \mu hom(F, G))\\ &\qquad \to \mathrm{R}\Gamma_{\Lambda_{1122} \cap T^*_{\Gamma_{\phi_{12}}} X_{1122}}(T^*X_{1122}; \mu hom(K, \omega_{\Gamma_{\phi_{12}}}))\\ &\qquad\quad \otimes \mathrm{R}\Gamma_{\delta^a_{T^*X_{23}}\Lambda_{23}}(T^*X_{2233}; \mu hom(F, G))\\ &\qquad \to \mathrm{R}\Gamma_{\Lambda_{1133}}(T^*X_{1133}; \mu hom(K_{12}, \omega_{\Gamma_{\phi_{12}}}) \overset{a}{\underset{22}{\circ}} \mu hom(F, G)).\end{aligned}$$

Here, we set $F := \omega_{\Delta_2}^{\otimes -1} \boxtimes k_{\Delta_3}, G := k_{\Gamma_{\phi_2}} \boxtimes \omega_{\Gamma_{\phi_3}}$, and $K := K_{12}$ for simplicity, and use (4.5). The second morphism comes from Proposition 2.5, and the last one is induced by the morphisms in Lemmas 3.2 and 4.1.

Lemma 4.2. *In the situation as above, we have*

$$\Phi_{K_{12}} = \mu\mathrm{Le}(K_{12}, \phi_{12}) \overset{a}{\underset{2}{\circ}} \colon \mathbb{MH}_{\Lambda_{23}}(\phi_{23}) \to \mathbb{MH}_{\Lambda_{13}}(\phi_{13}), \tag{4.7}$$

where the right hand side is the map given by Corollary 3.4.

Proof. Consider the following commutative diagram, where we use the same notation as above: $F := \omega_{\Delta_2}^{\otimes -1} \boxtimes k_{\Delta_3}, G := k_{\Gamma_{\phi_2}} \boxtimes \omega_{\Gamma_{\phi_3}}$ and $K := K_{12}$.

$$
\begin{array}{ccc}
\mathrm{R}\Gamma_{\delta^a_{T^*X_{23}}\Lambda_{23}}(T^*X_{2233};\mu hom(F,G)) & \xrightarrow{d} & \\
\downarrow a & & \mathrm{R}\Gamma_{\Lambda_{1133}}(T^*X_{1133};\mu hom(k_{\Delta_{12}},\omega_{\Gamma_{\phi_{12}}}) \overset{a}{\underset{22}{\circ}} \mu hom(F,G)) \\
\mathrm{R}\Gamma_{\Lambda_{1133}}(T^*X_{1133};\mu hom(K,\omega_{\Gamma_{\phi_{12}}}) \overset{a}{\underset{22}{\circ}} \mu hom(F,G)) & \nearrow & \downarrow e \\
\downarrow b & & \mathrm{R}\Gamma_{\Lambda_{1133}}(T^*X_{1133};\mu hom(k_{\Delta_{12}} \underset{22}{*} F,\omega_{\Gamma_{\phi_{12}}} \underset{22}{\circ} G)) \\
\mathrm{R}\Gamma_{\Lambda_{1133}}(T^*X_{1133};\mu hom(K \underset{22}{*} F,\omega_{\Gamma_{\phi_{12}}} \underset{22}{\circ} G)) & \xrightarrow{c} &
\end{array}
$$

By using the morphisms in Lemma 3.2, we get a morphism

$$
\begin{aligned}
(4.8)\qquad w\colon \mathrm{R}\Gamma_{\Lambda_{1133}}(T^*X_{1133};\mu hom(k_{\Delta_{12}} \underset{22}{*} F,\omega_{\phi_{12}} \underset{22}{\circ} G)) \\
\to \mathrm{R}\Gamma_{\delta^a_{T^*X_{13}}\Lambda_{13}}(T^*X_{1133};\mu hom(k_{\Delta_{13}},\omega_{\phi_{13}})).
\end{aligned}
$$

By construction, the morphism $\Phi_{K_{12}}$ is obtained as the composition of a, b, c, and w. On the other hand, the morphism $\mu\mathrm{Le}(K_{12},\phi_{12}) \overset{a}{\underset{2}{\circ}}$ is obtained as the composition of d, e, and w. Hence, the result follows from the commutativity of the above diagram. □

For $ij = 12, 23$, let $\Lambda_{iijj} \subset T^*X_{iijj}$ be a closed conic subset containing $\mathrm{SS}(K_{ij})$. Assume that

$$
(4.9)\qquad \Lambda_{1122} \overset{a}{\underset{22}{\times}} \Lambda_{2233} \to T^*X_{1133} \text{ is proper.}
$$

Set $\Lambda_{1133} = \Lambda_{1122} \overset{a}{\underset{22}{\circ}} \Lambda_{2233}$ and $\Lambda_{13} = \Lambda_{1133} \cap T^*_{\Delta_{13}}X_{1133} \cap T^*_{\Gamma_{\phi_{13}}}X_{1133}$.

Theorem 4.3. *Let K_{ij} be a ϕ_{ij}-graph trace kernel with $\mathrm{SS}(K_{ij}) \subset \Lambda_{iijj}$. Assume that (4.9) holds and set $\widetilde{K}_{23} := (\omega_2^{\otimes -1} \boxtimes k_{233}) \otimes K_{23}$. Then the following hold.*

(i) *The object $K_{12} \underset{22}{\circ} \widetilde{K}_{23}$ is a ϕ_{13}-graph trace kernel.*

(ii) *We have $\mu\mathrm{Le}(K_{12} \underset{22}{\circ} \widetilde{K}_{23}, \phi_{13}) = \mu\mathrm{Le}(K_{12},\phi_{12}) \overset{a}{\underset{2}{\circ}} \mu\mathrm{Le}(K_{23},\phi_{23})$ as an element of $\mathbb{MH}^0_{\Lambda_{13}}(\phi_{13})$.*

Proof. (i) Tensoring the sequence

$$
(4.10)\qquad k_{\Delta_{23}} \to K_{23} \to \omega_{\Gamma_{\phi_{23}}}
$$

with $\omega_2^{\otimes -1} \boxtimes k_{233}$, we get a sequence

$$\omega_{\Delta_2}^{\otimes -1} \boxtimes k_{\Delta_3} \to \widetilde{K}_{23} \to k_{\Gamma_{\phi_2}} \boxtimes \omega_{\Gamma_{\phi_3}}. \tag{4.11}$$

Combining this with Lemma 4.1, we obtain sequences

$$k_{\Delta_{13}} \to K_{12} \underset{22}{*} (\omega_{\Delta_2}^{\otimes -1} \boxtimes k_{\Delta_3}) \to K_{12} \underset{22}{*} \widetilde{K}_{23} \tag{4.12}$$

and

$$K_{12} \underset{22}{\circ} \widetilde{K}_{23} \to K_{12} \underset{22}{\circ} (k_{\Gamma_{\phi_2}} \boxtimes \omega_{\Gamma_{\phi_3}}) \to \omega_{\Gamma_{\phi_{13}}}. \tag{4.13}$$

By the assumption (4.9), we have an isomorphism

$$\alpha\colon K_{12} \underset{22}{\circ} \widetilde{K}_{23} \xrightarrow{\sim} K_{12} \underset{22}{*} \widetilde{K}_{23}. \tag{4.14}$$

Using (4.12)–(4.14), we find that $K_{12} \underset{22}{\circ} \widetilde{K}_{23}$ is a ϕ_{13}-graph trace kernel.

(ii) By Proposition 2.5, under the assumption (4.9), $\mathrm{id}_{K_{12}}$ and $\mathrm{id}_{\widetilde{K}_{23}}$ define a morphism

$$\beta\colon K_{12} \underset{22}{*} \widetilde{K}_{23} \to K_{12} \underset{22}{\circ} \widetilde{K}_{23}. \tag{4.15}$$

This morphism is the inverse of the morphism α of (4.14).

Now let us consider the following commutative diagram:

$$\begin{array}{ccccccc}
k_{\Delta_{13}} & \longrightarrow & K_{12} \underset{22}{*} (\omega_{\Delta_2}^{\otimes -1} \boxtimes k_{\Delta_3}) & \longrightarrow & K_{12} \underset{22}{\circ} (k_{\Gamma_{\phi_2}} \boxtimes \omega_{\Gamma_{\phi_3}}) & \longrightarrow & \omega_{\Gamma_{\phi_{12}}} \underset{22}{\circ} (k_{\Gamma_{\phi_2}} \boxtimes \omega_{\Gamma_{\phi_3}}) \\
& \searrow & \downarrow & & \uparrow & \nearrow & \\
& & K_{12} \underset{22}{*} \widetilde{K}_{23} & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & K_{12} \underset{22}{\circ} \widetilde{K}_{23} & &
\end{array}$$

By the graph trace kernel structure of $K_{12} \underset{22}{\circ} \widetilde{K}_{23}$, the composition of the bottom arrows and $\gamma\colon \omega_{\Gamma_{\phi_{12}}} \underset{22}{\circ} (k_{\Gamma_{\phi_2}} \boxtimes \omega_{\Gamma_{\phi_3}}) \to \omega_{\Gamma_{\phi_{13}}}$ defines $\mu\mathrm{Le}(K_{12} \underset{22}{\circ} \widetilde{K}_{23}, \phi_{13})$. By the construction of the map $\Phi_{K_{12}}$, the composition of the top arrows and γ defines $\Phi_{K_{12}}(\mu\mathrm{Le}(K_{23}, \phi_{23}))$. Hence, the result follows from Lemma 4.2. □

§4.2. Operations on microlocal Lefschetz classes

Let X_1 and X_2 be manifolds and $\phi_1\colon X_1 \to X_1$ and $\phi_2\colon X_2 \to X_2$ be morphisms of manifolds. For $i = 1, 2$, let K_i be a ϕ_i-graph trace kernel and let Λ_{ii} be a closed conic subset of T^*X_{ii} with $\mathrm{SS}(K_i) \subset \Lambda_{ii}$.

Let $f\colon X_1 \to X_2$ be a morphism of manifolds. Assume that $\phi_2 f = f\phi_1$, that is, the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ {\scriptstyle \phi_1}\downarrow & & \downarrow{\scriptstyle \phi_2} \\ X_1 & \xrightarrow{f} & X_2 \end{array}$$

commutes. Since $\Gamma_f \subset (\phi_{12})^{-1}(\Gamma_f)$, we have a natural morphism

$$\Phi\colon (\phi_{12})^{-1}k_{\Gamma_f} \to k_{\Gamma_f}. \tag{4.16}$$

Then the pair (k_{Γ_f}, Φ) defines naturally a ϕ_{12}-graph trace kernel $\mathrm{TK}_{\phi_{12}}(k_{\Gamma_f}, \Phi)$ by Lemma 3.7.

Set $\widetilde{f} := f \times f\colon X_{11} \to X_{22}$. We identify X_{1212} with X_{1122}. Then we have

$$\begin{aligned} (\omega_1^{\otimes -1} \boxtimes k_{122}) \otimes \mathrm{TK}_{\phi_{12}}(k_{\Gamma_f}, \Phi) &\simeq (\omega_1^{\otimes -1} \boxtimes k_{122}) \otimes \omega_{\Gamma_f} \boxtimes k_{\Gamma_f} \\ &\simeq k_{\Gamma_f} \boxtimes k_{\Gamma_f} \simeq k_{\Gamma_{\widetilde{f}}}. \end{aligned}$$

We also note that

$$\mathrm{R}\widetilde{f}_! K_1 \simeq K_1 \underset{11}{\circ} k_{\Gamma_{\widetilde{f}}}, \quad \widetilde{f}^{-1} K_2 \simeq k_{\Gamma_{\widetilde{f}}} \underset{22}{\circ} K_2. \tag{4.17}$$

External product. Let $X_2 = \mathrm{pt}$. We then write X_2 instead of X_3. For $i = 1, 2$, let Λ_i be a closed conic subset of T^*X_i. In this case, we have the composition morphism

$$\mathbb{MH}_{\Lambda_1}(\phi_1) \otimes \mathbb{MH}_{\Lambda_2}(\phi_2) \xrightarrow{\circ} \mathbb{MH}_{\Lambda_1 \times \Lambda_2}(\phi_{12}). \tag{4.18}$$

Taking the 0-th cohomology, we have a morphism

$$\mathbb{MH}^0_{\Lambda_1}(\phi_1) \otimes \mathbb{MH}^0_{\Lambda_2}(\phi_2) \xrightarrow{\circ} \mathbb{MH}^0_{\Lambda_1 \times \Lambda_2}(\phi_{12}). \tag{4.19}$$

In this case, we shall write $\lambda_1 \times \lambda_2$ instead of $\lambda_1 \circ \lambda_2$.

Set

$$\Lambda_i := \Lambda_{ii} \cap T^*_{\Delta_i} X_{ii} \quad (i = 1, 2). \tag{4.20}$$

Then by Theorem 4.3, we obtain the following.

Proposition 4.4. *The object $K_1 \boxtimes K_2$ is a ϕ_{12}-graph trace kernel and*

$$\mu\mathrm{Le}(K_1 \boxtimes K_2, \phi_{12}) = \mu\mathrm{Le}(K_1, \phi_1) \times \mu\mathrm{Le}(K_2, \phi_2). \tag{4.21}$$

Direct image. Let $X_1 = \mathrm{pt}$. We then write X_1, X_2 instead of X_2, X_3. Let $\Lambda_1 \subset T^*X_1$ be a closed conic subset. Assume that

$$f \text{ is proper on } \Lambda_1 \cap T^*_{X_1}X_1. \tag{4.22}$$

We set

$$\begin{aligned} f_{\mu,\phi_1\to\phi_2}(\Lambda_1) &:= \Lambda_1 \overset{a}{\underset{1}{\circ}} T^*_{\Gamma_f}X_{12} \cap (\delta^a_{T^*X_2})^{-1}(T^*_{\Gamma_{\phi_2}}X_{22}) \\ &= f_d f_\pi^{-1}(\Lambda_1) \cap (\delta^a_{T^*X_2})^{-1}(T^*_{\Gamma_{\phi_2}}X_{22}) \subset T^*X_2, \end{aligned}$$

and

$$f_{\mu,\phi_1\to\phi_2} := \overset{a}{\underset{1}{\circ}} \mu\mathrm{Le}(k_{\Gamma_f}, \Phi, \phi_{12}) \colon \mathbb{MH}^0_{\Lambda_1}(\phi_1) \to \mathbb{MH}^0_{f_{\mu,\phi_1\to\phi_2}(\Lambda_1)}(\phi_2). \tag{4.23}$$

Proposition 4.5. *Assume that $\widetilde{f}$ is proper on $\Lambda_{11} \cap T^*_{X_{11}}X_{11}$ and set $\Lambda_1 := \Lambda_{11} \cap T^*_{\Delta_1}X_{11}$. Then the object $\mathrm{R}\widetilde{f}_! K_1$ is a ϕ_2-graph trace kernel and*

$$\mu\mathrm{Le}(\mathrm{R}\widetilde{f}_! K_1, \phi_2) = f_{\mu,\phi_1\to\phi_2}(\mu\mathrm{Le}(K_1, \phi_1)). \tag{4.24}$$

Proof. The assumption implies that $\Lambda_{11} \overset{a}{\underset{11}{\times}} T^*_{\Gamma_{\tilde f}}X_{1122} \to T^*X_{22}$ is proper. By Theorem 4.3, $K_1 \underset{11}{\circ} (\omega_1^{\otimes -1} \boxtimes k_{122}) \otimes \mathrm{TK}_{\phi_{12}}(k_{\Gamma_f}, \Phi) \simeq K_1 \underset{11}{\circ} k_{\Gamma_{\tilde f}} \simeq \mathrm{R}\widetilde{f}_! K_1$ is a ϕ_2-graph trace kernel and we have

$$\mu\mathrm{Le}(\mathrm{R}\widetilde{f}_! K_1, \phi_2) = \mu\mathrm{Le}(K_1, \phi_1) \overset{a}{\underset{1}{\circ}} \mu\mathrm{Le}(k_{\Gamma_f}, \Phi, \phi_{12}). \tag{4.25}$$

□

Inverse image. Let $X_3 = \mathrm{pt}$. Let $\Lambda_2 \subset T^*X_2$ be a closed conic subset. Assume that

$$f \text{ is non-characteristic for } \Lambda_2. \tag{4.26}$$

We set

$$\begin{aligned} f^{\mu,\phi_1\to\phi_2}(\Lambda_2) &:= T^*_{\Gamma_f}X_{12} \overset{a}{\underset{2}{\circ}} \Lambda_2 \cap (\delta^a_{T^*X_1})^{-1}(T^*_{\Gamma_{\phi_1}}X_{11}) \\ &= f_d f_\pi^{-1}(\Lambda_2) \cap (\delta^a_{T^*X_1})^{-1}(T^*_{\Gamma_{\phi_1}}X_{11}) \subset T^*X_1, \end{aligned}$$

and

$$f^{\mu,\phi_1\to\phi_2} := \mu\mathrm{Le}(k_{\Gamma_f}, \Phi, \phi_{12}) \overset{a}{\underset{2}{\circ}} \colon \mathbb{MH}^0_{\Lambda_2}(\phi_2) \to \mathbb{MH}^0_{f^{\mu,\phi_1\to\phi_2}(\Lambda_2)}(\phi_1). \tag{4.27}$$

Proposition 4.6. *Assume that $\widetilde{f}$ is non-characteristic for Λ_{22} and set $\Lambda_2 := \Lambda_{22} \cap T^*_{\Delta_2}X_{22}$. Then the object $(\omega_{X_1/X_2} \boxtimes k_1) \otimes \widetilde{f}^{-1}K_2$ is a ϕ_1-graph trace kernel and*

$$\mu\mathrm{Le}((\omega_{X_1/X_2} \boxtimes k_1) \otimes \widetilde{f}^{-1}K_2, \phi_1) = f^{\mu,\phi_1\to\phi_2}(\mu\mathrm{Le}(K_2, \phi_2)). \tag{4.28}$$

Proof. The assumption implies that $T^*_{\Gamma_{\widetilde{f}}} X_{1122} \overset{a}{\underset{22}{\times}} \Lambda_{22} \to T^*X_{11}$ is proper. By Theorem 4.3, $\mathrm{TK}_{\phi_{12}}(k_{\Gamma_f}, \Phi) \underset{22}{\circ} (\omega_2^{\otimes -1} \boxtimes k_2) \otimes K_2$ is a ϕ_1-graph trace kernel. Here, we have isomorphisms

$$\begin{aligned}\mathrm{TK}_{\phi_{12}}(k_{\Gamma_f}, \Phi) \underset{22}{\circ} (\omega_2^{\otimes -1} \boxtimes k_2) \otimes K_2 &\simeq (\omega_1 \boxtimes k_1) \otimes (k_{\Gamma_{\widetilde{f}}} \underset{22}{\circ} (\omega_2^{\otimes -1} \boxtimes k_2) \otimes K_2) \\ &\simeq (\omega_1 \boxtimes k_1) \otimes \widetilde{f}^{-1}((\omega_2^{\otimes -1} \boxtimes k_2) \otimes K_2) \\ &\simeq (\omega_1 \boxtimes k_1) \otimes (f^{-1}\omega_2^{\otimes -1} \boxtimes f^{-1}k_2) \otimes \widetilde{f}^{-1}K_2 \\ &\simeq (\omega_{X_1/X_2} \boxtimes k_1) \otimes \widetilde{f}^{-1}K_2.\end{aligned}$$

Applying Theorem 4.3 again, we obtain

$$\mu\mathrm{Le}((\omega_{X_1/X_2} \boxtimes k_1) \otimes \widetilde{f}^{-1}K_2, \phi_1) = \mu\mathrm{Le}(k_{\Gamma_f}, \Phi, \phi_{12}) \overset{a}{\underset{2}{\circ}} \mu\mathrm{Le}(K_2, \phi_2). \tag{4.29}$$ □

Tensor product. Let $X_1 = X_2 = X$ and $\phi_1 = \phi_2 = \phi$. For $i = 1, 2$, let K_i be a ϕ-graph trace kernel and $\Lambda_{ii} \subset T^*(X \times X)$ be a closed conic subset satisfying $\mathrm{SS}(K_i) \subset \Lambda_{ii}$. Assume that

$$\Lambda_{11} \cap \Lambda_{22}^a \subset T^*_{X \times X}(X \times X), \tag{4.30}$$

and set

$$\Lambda_i := \Lambda_{ii} \cap T^*_\Delta(X \times X) \quad (i = 1, 2). \tag{4.31}$$

Recall that for a morphism $f \colon X \to Y$, we set $\widetilde{f} := f \times f \colon X \times X \to Y \times Y$. Since $\widetilde{\phi}\delta = \delta\phi$, we have a morphism

$$\delta^{\mu, \phi \to \widetilde{\phi}} \colon \mathbb{MH}^0_{\Lambda_1 \times \Lambda_2}(\widetilde{\phi}) \to \mathbb{MH}^0_{\Lambda_1 + \Lambda_2}(\phi). \tag{4.32}$$

Composing it with the morphism of external product

$$\times \colon \mathbb{MH}^0_{\Lambda_1}(\phi) \otimes \mathbb{MH}^0_{\Lambda_2}(\phi) \to \mathbb{MH}^0_{\Lambda_1 \times \Lambda_2}(\widetilde{\phi}), \tag{4.33}$$

we get a convolution morphism

$$\star \colon \mathbb{MH}^0_{\Lambda_1}(\phi) \otimes \mathbb{MH}^0_{\Lambda_2}(\phi) \to \mathbb{MH}^0_{\Lambda_1 + \Lambda_2}(\phi). \tag{4.34}$$

Proposition 4.7. *Assume that* (4.30) *holds. Then the object* $(\omega_X^{\otimes -1} \boxtimes k_X) \otimes K_1 \otimes K_2$ *is a* ϕ*-graph trace kernel and*

$$\mu\mathrm{Le}((\omega_X^{\otimes -1} \boxtimes k_X) \otimes K_1 \otimes K_2, \phi) = \mu\mathrm{Le}(K_1, \phi) \star \mu\mathrm{Le}(K_2, \phi). \tag{4.35}$$

Proof. Since we regard $\widetilde{\delta} \colon X \times X \to X \times X \times X \times X$ as the map $(x_1, x_2) \mapsto (x_1, x_2, x_1, x_2)$, we have $\widetilde{\delta}^{-1}(K_1 \boxtimes K_2) \simeq K_1 \otimes K_2$. The assumption implies δ is non-characteristic for $\Lambda_1 \times \Lambda_2$. Thus, the result follows from Propositions 4.4 and 4.6, since $\omega_{X/X \times X} \simeq \omega_X^{\otimes -1}$. □

§4.3. Application to the Lefschetz fixed point formula for constructible sheaves

Let X be a real analytic manifold and $\phi_X\colon X \to X$ be a morphism of manifolds. We denote by $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$ the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(k_X)$ consisting of $\mathbb{R}$-constructible complexes. Since $\mathbb{R}$-constructible complexes are cohomologically constructible, a pair (F, Φ) of an object $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$ and a morphism $\Phi\colon \phi_X^{-1}F \to F$ gives naturally a ϕ_X-graph trace kernel $\mathrm{TK}_{\phi_X}(F, \Phi)$.

Let Y be another real analytic manifold and $\phi_Y : Y \to Y$ be a morphism. Let $f\colon X \to Y$ be a morphism of manifolds which satisfies $\phi_Y f = f\phi_X$. Then we have a natural morphism

$$\phi_Y^{-1}\mathrm{R}f_*F \to \phi_Y^{-1}\mathrm{R}f_*\mathrm{R}\phi_{X*}\phi_X^{-1}F \xrightarrow{\Phi} \phi_Y^{-1}\mathrm{R}\phi_{Y*}\mathrm{R}f_*F \to \mathrm{R}f_*F.$$

We denote this morphism by $\mathrm{R}f_*\Phi$.

Proposition 4.8. *Assume that f is proper on* $\mathrm{Supp}(F)$*. Then*

$$\mu\mathrm{Le}(\mathrm{R}f_*F, \mathrm{R}f_*\Phi, \phi_Y) = f_{\mu,\phi_X\to\phi_Y}(\mu\mathrm{Le}(F, \Phi, \phi_X)). \tag{4.36}$$

Proof. By assumption, $\mathrm{R}f_*F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_Y)$. Hence, the pair $(\mathrm{R}f_*F, \mathrm{R}f_*\Phi)$ gives a ϕ_Y-trace kernel $\mathrm{TK}_{\phi_Y}(\mathrm{R}f_*F, \mathrm{R}f_*\Phi)$. Then we have an isomorphism

$$\mathrm{TK}_{\phi_Y}(\mathrm{R}f_*F, \mathrm{R}f_*\Phi) \simeq \mathrm{R}\widetilde{f}_!\mathrm{TK}_{\phi_X}(F, \Phi). \tag{4.37}$$

Hence, by Proposition 4.5,

$$\begin{aligned}\mu\mathrm{Le}(\mathrm{R}f_*F, \mathrm{R}f_*\Phi, \phi_Y) &= \mu\mathrm{Le}(\mathrm{R}\widetilde{f}_!\mathrm{TK}_{\phi_X}(F, \Phi), \phi_Y)\\ &= f_{\mu,\phi_X\to\phi_Y}(\mu\mathrm{Le}(F, \Phi, \phi_X)).\end{aligned}$$

□

Note that the above formula is similar to that of [MT10].

Applying Proposition 4.8 for $Y = \mathrm{pt}$ and the natural morphism $f = \mathbf{a}\colon X \to \mathrm{pt}$, we obtain

Corollary 4.9. *Assume that* $\mathrm{Supp}(F)$ *is compact. Then*

$$\mathrm{tr}(F, \Phi) = \mathbf{a}_\mu(\mu\mathrm{Le}(F, \Phi, \phi_X)), \tag{4.38}$$

where the left hand side is defined by

$$\mathrm{tr}(F, \Phi) := \sum_{p\in\mathbb{Z}}(-1)^p\,\mathrm{tr}\big(H^p(X; F) \to H^p(X; \phi_X^{-1}F) \xrightarrow{\Phi} H^p(X; F)\big). \tag{4.39}$$

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