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Singularities of Solutions of Quasilinear Partial Differential Equations in a Complex Domain

by

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Abstract

We consider the Cauchy problem for a quasilinear partial differential equation of an arbitrary order in a complex domain. We assume that the initial data have singularities along complex submanifolds. We show that if the characteristic roots are distinct, the singularities of the solution propagate along the characteristic complex submanifolds.

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§1. Introduction

Let $n \geq 1$ and $m \geq 2$. We denote $x = (x_0, x') = (x'', x_n) = (x_0, x''', x_n) = (x_0, x_1, \ldots, x_n) \in \mathbb{C}^{n+1}$. For a function $u(x)$ we denote $\nabla_x^j u(x) = (\partial_x^\alpha u;\, |\alpha| \leq j)$. We consider a quasilinear partial differential equation of the form

$$E(x, \nabla_x^m u) = \sum_{|\alpha|=m} E_\alpha(x, u)\partial_x^\alpha u + E'(x, \nabla_x^{m-1} u) = 0. \tag{1}$$

Let $U_\beta \in \mathbb{C}$ and $U = (U_\beta;\, |\beta| \leq m-1)$. We assume that $E_\alpha(x, u)$ is holomorphic at $(x, u) = (0, u^\circ)$ and $E'(x, U)$ is holomorphic at $(x, U) = (0, U^\circ)$. For functions $u(x)$ and $f(x)$, we denote

$$E[u]f = \sum_{|\alpha|=m} E_\alpha(x, u)\partial_x^\alpha f(x) + E'(x, \nabla_x^{m-1} u).$$

We consider the following Cauchy problem in a complex domain:

$$\begin{cases} E(x, \nabla_x^m u) = 0, \\ \partial_{x_0}^j u(0, x') = u_j(x'), \quad 0 \leq j \leq m-1. \end{cases} \tag{2}$$

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Let $\Omega = \Omega(r) = \{x \in \mathbb{C}^{n+1}; |x| < r\}$, and let

$$\Omega(r) \supset \Omega^{\text{o}} = \Omega^{\text{o}}(r) = \Omega(r) \cap \{x_0 = 0\} \supset Z = Z(r) = \Omega^{\text{o}}(r) \cap \{x_n = 0\}.$$

We assume that the initial values $u_j(x')$ are holomorphic on the universal covering space $\mathcal{R}(\Omega^{\text{o}} \setminus Z)$ of $\Omega^{\text{o}} \setminus Z$. More precisely, we define

$$\mathcal{O}(\mathcal{R}(\Omega^{\text{o}} \setminus Z)) = \{\text{holomorphic functions on } \mathcal{R}(\Omega^{\text{o}} \setminus Z)\},$$
$$\mathcal{O}^{j-q}(\mathcal{R}(\Omega^{\text{o}} \setminus Z)) = \{f(x') \in \mathcal{O}(\mathcal{R}(\Omega^{\text{o}} \setminus Z));\ \partial_{x'}^{\alpha'} f(x') \text{ is bounded if } |\alpha'| \leq j-1, \text{ and } x_n^q \partial_{x'}^{\alpha'} f(x') \text{ is bounded if } |\alpha'| = j\}$$

for $j \in \mathbb{N} = \{1, 2, \dots\}$ and $0 < q < 1$. The set $\mathcal{O}^{j-q}(\mathcal{R}(\Omega^{\text{o}} \setminus Z))$ consists of holomorphic functions on $\mathcal{R}(\Omega^{\text{o}} \setminus Z)$ which can be extended to functions on $\mathcal{R}(\Omega^{\text{o}} \setminus Z) \cup Z$ with Hölder continuity exponent $j - q$ (see [11, Chapter 13, Proposition 8.7]). We remark that if $f(x') \in \mathcal{O}^{1-q}(\mathcal{R}(\Omega^{\text{o}} \setminus Z))$, then we can naturally define the trace $f(x_1, \dots, x_{n-1}, 0)$ on Z.

We assume the following conditions **A1**–**A4** hold:

A1. The initial hypersurface $\{x_0 = 0\}$ is noncharacteristic: $E_\alpha = 1$ for $\alpha = (m, 0, \dots, 0)$.

A2. We have $u_j \in \mathcal{O}^{m-j-q}(\mathcal{R}(\Omega^{\text{o}} \setminus Z))$ for $0 \leq j \leq m-1$.

Therefore $\nabla_x^{m-1} u(0)$ is naturally determined by the initial values of $u(x)$. We define $u^{\text{o}} = u(0)$ and $U^{\text{o}} = \nabla_x^{m-1} u(0)$. We assume the following:

A3. $E_\alpha(x, u)$ is holomorphic near $(x, u) = (0, u(0))$, and $E'(x, U)$ is holomorphic at $(x, U) = (0, \nabla_x^{m-1} u(0))$.

We want to show that the singularities of the solution to (2) propagate along the complex hypersurfaces $Z_1, \dots, Z_m$ which start from Z at $x_0 = 0$. For this purpose, we define the principal symbol $\sigma(E)(x, u, \xi)$ for $\xi = (\xi_0, \dots, \xi_n)$ by $\sigma(E) = \sum_{|\alpha|=m} E_\alpha(x, u)\xi^\alpha$. We finally assume the following:

A4. Let $\xi'^{\text{o}} = (0, \dots, 0, 1) \in \mathbb{C}^n$. The equation $\sigma(E)(x, u, \xi) = 0$ has distinct roots $\xi_0 = \mu_j(x, u, \xi')$, $1 \leq j \leq m$, at $(x, u, \xi') = (0, u^{\text{o}}, \xi'^{\text{o}})$.

We consider the following characteristic function $\kappa_j(x)$ for $1 \leq j \leq m$:

$$\partial_{x_0} \kappa_j(x) = \mu_j(x, u(x), \partial_{x'} \kappa_j(x)), \ \ \kappa_j(0, x') = x_n. \tag{3}$$

We want to assert that the solution u is holomorphic on $\{\kappa_j \neq 0, 1 \leq j \leq m\}$, but we have the following difficulty: The characteristic equation (3) depends on the solution $u(x)$. Since the solution $u(x)$ is multi-valued outside of the characteristic sets $Z_1, \dots, Z_m$, the same should be true for the characteristic function $\kappa_j(x)$ itself.

In other words, the characteristic set Z_j may be different from one of the branches of the solution to another (see [12, Section 6] for an explicit example of such a phenomenon). Taking this difficulty into account, we shall make a precise definition of the characteristic sets and formulate the main results exactly in the next section.

Let us briefly review the history of this problem. For a linear equation, there are many papers studying this problem. We only refer to [1, 6, 7, 14], where one can find further references. For semilinear problems E. Leichtnam [8] and A. Nabaji and C. Wagschal [10] gave a similar result. For a quasilinear equation of order two, this problem was studied in [12, 13]. In the present article we consider a quasilinear equation of an arbitrary order. This is important not only theoretically but also in applications.

Theoretically, a second order equation is very special in that we can apply the hodograph transformation. This means that we can find a coordinate system $y = (y_0, \dots, y_n)$ in which the equation takes the form $\partial_{y_0}^2 - \partial_{y_n}^2 +$ (lower order terms). Using this coordinate system, we can easily calculate the singularities along $\{y_n = \pm y_0\}$ (see [9] for the hodograph transformation). Unfortunately, this method is not applicable for higher order equations. Furthermore, the domains of definition for solutions of higher order equations have complicated geometrical structure. Let D be the domain of definition for a solution. Later we shall see that the Poincaré group $H_1(D)$ is equal to $\mathbb{Z} \times (\mathbb{Z}^{*(m-1)})$. Here $\mathbb{Z}^{*(m-1)}$ denotes the free group generated by $m-1$ elements. If $m = 2$, then $H_1(D) = \mathbb{Z} \times \mathbb{Z}$. If $D = \{y_n \neq y_0,\ y_n \neq -y_0\}$, we can identify $\tilde{y} \in \mathcal{R}(D)$ with $(y, \arg(y_n - y_0), \arg(y_n + y_0)) \in D \times \mathbb{R} \times \mathbb{R}$. Therefore it suffices to calculate the branches of the solution for each $\arg(y_n - y_0)$ and $\arg(y_n + y_0)$. For higher order equations the situation is completely different, and we shall discuss the geometry of the domain of definition.

For applications, some important equations such as the compressible Euler system for perfect fluids reduce to higher order equations. J.-Y. Chemin [2, 3, 4] studied the propagation of singularities of higher order nonlinear equations in a real domain, providing a result covering the compressible Euler system. In the forthcoming article, we shall give an application to the compressible Euler system, giving a new result on singularity propagation. See also [5] for this problem for the incompressible Euler system.

Plan of the paper. In Section 2, we give the precise statement of the main result. In Section 3, we rewrite the equation in terms of a characteristic coordinate system. In Section 4 we consider a complex domain in which we shall solve the equation, and in Section 5 we define two indicators $l(y), d(y)$ describing the geometry of the complex domain. In Section 6 we define some function spaces using these indicators, and in Section 7 we solve the equation in these function spaces. In Section 8 we study the structure of singularity sets.

§2. Main results

Let $r, R > 0$, $1 \le j \le m$, and $a_j = \mu_j(0, u(0), \xi'^{\circ})$. We define

$$V_j = V_j(R,r) = \{x \in \Omega(r);\ |x_n + a_j x_0| < R|x_0|\},$$
$$\bar V_j = \bar V_j(R) = \{x \in \Omega(r);\ |x_n + a_j x_0| \le R|x_0|\},$$
$$\Omega_1 = \Omega_1(R,r) = \Omega(r) \setminus \bigcup_{1\le j\le m} \bar V_j(R,r)$$

(see Figure 1(a)). We assume that $0 < R \ll 1$, and thus $V_j(2R) \cap V_k(2R) = \emptyset$ if $j \ne k$. We shall find the true singularity set Z_k in $V_k(R,r)$ for each k, and thus $V_k(R,r)$ is a neighborhood of the singularity set. Let $k_0 \in \{1, \dots, m\}$. Without loss of generality we discuss the case $k_0 = m$.

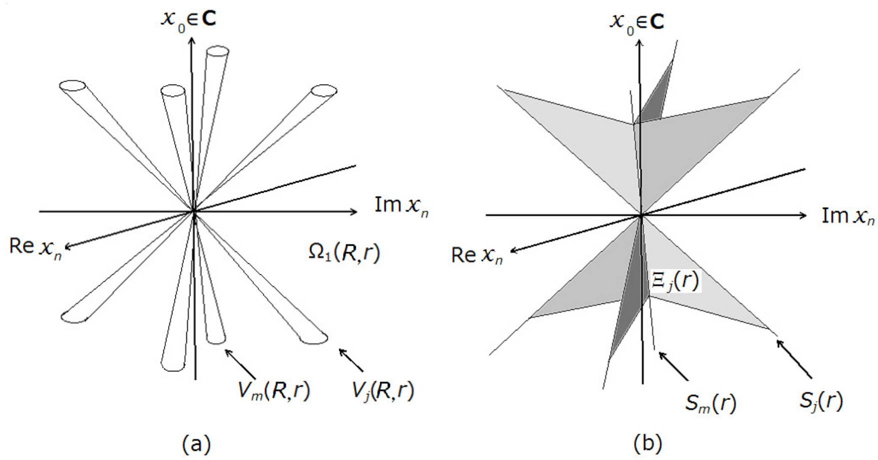

Figure 1. (a) $V_j(R,r)$ and $\Omega_1(R,r)$; (b) $S_j(r)$ and $\Xi_j(r)$.

We next define

$$S_j = S_j(r) = \{x \in \Omega(r);\ x_n + a_j x_0 = 0\},$$
$$\Xi_j = \Xi_j(r) = \text{the convex hull of } S_j \cup S_m$$
$$= \{x \in \Omega(r);\ x_n = -(\theta a_j + (1-\theta)a_m)x_0 \text{ for some } \theta \in [0,1]\},$$
$$\Xi = \Xi(r) = \bigcup_{1\le j\le m-1} \Xi_j(r)$$

(see Figure 1(b)). Let $\Gamma \subset \Omega_1$ be a continuous curve starting at $x^{\circ} = (0, \dots, 0, r/2) \in \Omega_1$ and ending at an arbitrary point $x \in \Omega_1$. We denote the homotopy equivalence class of Γ in Ω_1 by $[\Gamma]$. By definition, $\mathcal{R}(\Omega_1)$ is the set of those homotopy equivalence classes, and consider the natural projection $\pi : \mathcal{R}(\Omega_1) \ni [\Gamma] \mapsto x \in \Omega_1$.

We denote the point $[\Gamma] \in \mathcal{R}(\Omega_1)$ also by $\tilde{x}$ or simply by x, if there is no confusion. For each $N_0 \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$ we denote by $\mathcal{R}_{N_0}(\Omega_1)$ the set of all homotopy equivalence classes $[\Gamma]$ for which we can choose Γ in such a way that $\Gamma \cap \Xi$ consists of at most N_0 points (If $\{x(\theta);\, 0 \le \theta \le \theta^o\}$ is the arc length parametrization of Γ, then $x(\theta) \in \Gamma \cap \Xi$ for at most N_0 values of θ.) Therefore we have $\mathcal{R}_0(\Omega_1) \subset \mathcal{R}_1(\Omega_1) \subset \mathcal{R}_2(\Omega_1) \subset \cdots \subset \mathcal{R}_{N_0}(\Omega_1) \nearrow \mathcal{R}(\Omega_1)$.

Definition. Let $N_0 \in \mathbb{Z}_+$. We say that a function $f(x)$ is of *type* N_0 if there exist $R, r > 0$ such that $f(x)$ is holomorphic on $\mathcal{R}_{N_0}(\Omega_1(R, r))$.

Theorem 1. *For any N_0, there exists a solution to (2) of type N_0.*

We next investigate how to extend the solution inside of the singularity neighborhood $V_m(R, r)$. Roughly speaking, we want to assert that there is a singularity set $Z_m \subset V_m(R, r)$ outside of which the solution is holomorphic (see Figure 2).

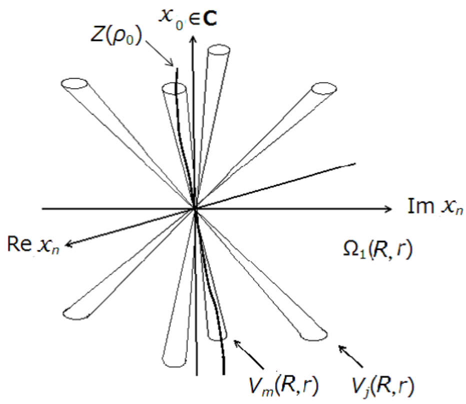

Figure 2. Singularity set.

However, if we consider different branches of the solution, then the singularity set may be different. Assume that $\rho_0(x'')$ is a holomorphic function on $\mathcal{R}(V_m(2R, r))$ satisfying $|\rho_0(x'')| < R|x_0|$. Since $\rho_0(x'')$ is independent of x_n, it is in fact holomorphic on the universal covering space of $\{x'' \in \mathbb{C}^n;\, |x''| < r,\, x_0 \neq 0\}$. We denote $Z(\rho_0) = \{\tilde{x} \in \mathcal{R}(V_m(2R, r));\, x_n = \rho_0(x'')\}$. Note that the projection of $Z(\rho_0)$ onto $\mathbb{C}^{n+1}$ is contained in $V_m(R, r)$. Let $\pi : \mathcal{R}(V_m(2R, r)) \to V_m(2R, r)$ be the natural projection, and let $\tilde{x}^1 \in \mathcal{R}_{N_0}(\Omega_1(R, r)) \cap \pi^{-1}(V_m(2R, r))$. Note that each connected component of $\mathcal{R}(\Omega_1(R, r)) \cap \pi^{-1}(V_m(2R, r))$ is homeomorphic to $\mathcal{R}(V'_m(R, r))$, where

$$\begin{aligned} V'_m(R, r) &= \Omega_1(R, r) \cap V_m(2R, r) \\ &= \{x \in \mathbb{C}^{n+1};\, |x| < r,\, R|x_0| < |x_n + a_m x_0| < 2R|x_0|\}. \end{aligned}$$

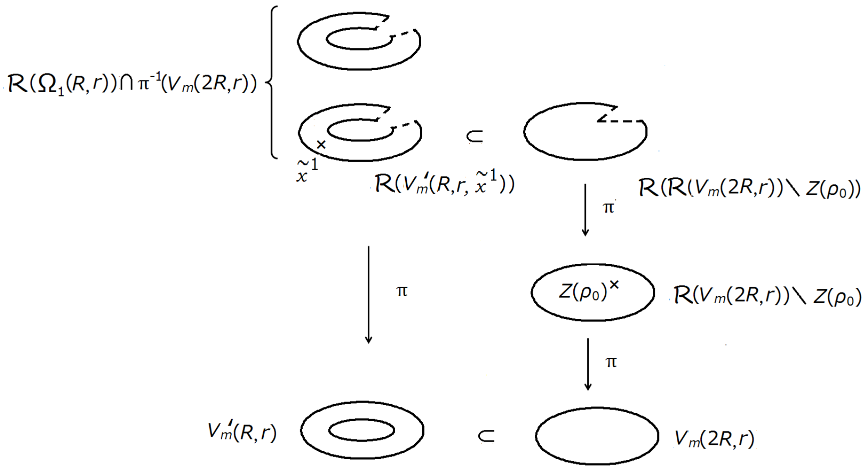

Figure 3. Structure of the universal covering space.

We denote the connected component of $\mathcal{R}(\Omega_1(R,r)) \cap \pi^{-1}(V_m(2R,r))$ containing $\tilde{x}^1$ by $\mathcal{R}(V'_m(R,r,\tilde{x}^1))$. Then we have $\mathcal{R}(V'_m(R,r,\tilde{x}^1)) \subset \mathcal{R}(\mathcal{R}(V_m(2R,r)) \setminus Z(\rho_0))$. We can make the following identifications (see Figure 3):

$$\mathcal{R}(V_m(2R,r)) \ni \tilde{x} \mapsto (x, \arg x_0) \in V_m(2R,r) \times \mathbb{R},$$
$$\mathcal{R}(\mathcal{R}(V_m(2R,r,\tilde{x}^1)) \setminus Z(\rho_0)) \ni \tilde{x} \mapsto (x, \arg x_0, \arg(x_n - \rho_0)) \in V_m(2R,r) \times \mathbb{R}^2.$$

This yields the following natural mappings:

$$\begin{array}{ccccc} \mathcal{R}(\mathcal{R}(V_m(2R,r)) \setminus Z(\rho_0)) & \to & \mathcal{R}(V_m(2R,r)) \setminus Z(\rho_0) & \to & V_m(2R,r) \\ \Cup & & \Cup & & \Cup \\ (x, \arg x_0, \arg(x_n - \rho_0)) & \mapsto & (x, \arg x_0) & \mapsto & x \end{array}$$

If $N_0 \in \mathbb{N}$, we define

$$\mathcal{S}_{N_0}(V_m(R,r,\tilde{x}^1) \setminus Z(\rho_0)) = \Bigg\{ (x, \arg x_0, \arg(x_n - \rho_0))) \in V_m(2R,r,\tilde{x}^1) \times \mathbb{R}^2; \\ \left| \arg\left(\frac{x_0}{x_n - \rho_0(x)}\right) - \arg\left(\frac{x_0^1}{x_n^1 - \rho_0(x^1)}\right) \right| < N_0\pi \Bigg\}.$$

Definition. Let $N_0 \in \mathbb{N}$. We say that a function $f(x)$ is of *type* (N_0, m) if there exist $R, r > 0$ such that the following conditions are satisfied. First, $f(x)$ is holomorphic on $\mathcal{R}_{N_0}(\Omega_1(R,r))$, and for any $\tilde{x}^1 \in \mathcal{R}_{N_0}(\Omega_1(R,r)) \cap \pi^{-1}(V_m(2R,r))$, there exists a holomorphic function $\rho_0(x'')$ on $\mathcal{R}(V_m(2R,r,\tilde{x}^1))$ satisfying $|\rho_0(x'')| < R|x_0|$. Furthermore, we can analytically continue the branch of $f(x)$ at $\tilde{x}^1$ to $\mathcal{S}_{N_0}(\mathcal{R}(V_m(2R,r,\tilde{x}^1)) \setminus Z(\rho_0))$. We have denoted by $Z(\rho_0)$ the singularity set corresponding to $\tilde{x}^1$. Here the constants R, r may depend on N_0.

Definition. Let $N_0, k \in \mathbb{N}$ and $0 < q < 1$. We assume that $f(x)$ is a function of type (N_0, m). We say that $f(x)$ is of *type* $(N_0, m, k-q)$ if there exist $R, r > 0$ satisfying the following additional condition: Let $\tilde{x}^1 \in \mathcal{R}_{N_0}(\Omega_1(R,r)) \cap \pi^{-1}(V_m(2R,r))$. We consider the corresponding branch of $f(x)$ extended to $\mathcal{S}_{N_0}(\mathcal{R}(V_m(2R,r,\tilde{x}^1)) \setminus Z(\rho_0))$. If $|\alpha| \le k$, then

$$|\partial_x^\alpha f(x)| \le C|x_n - \rho_0(x'')|^{-(|\alpha|-k+q)_+} \qquad (z_+ = \max(|\alpha|-k+q, 0))$$

for some $C > 0$ on $\mathcal{S}_{N_0}(\mathcal{R}(V_m(R',r,\tilde{x}^1))\setminus Z(\rho_0)$. Here $\rho_0(x'')$ is the function defined above.

Remark. From now on, we denote $\kappa_m(x)$ simply by $\kappa(x)$. If $f(x)$ is of type $(N_0, m, 1-q)$, we can define the trace of $f(x)|_{Z(\rho_0)}$ on the singularity set $Z(\rho_0)$.

Theorem 2. *For each $N_0 \in \mathbb{N}$, there exists a holomorphic solution u of (2) and κ of (3) for $j = m$ of type $(N_0, m, m-q)$. Furthermore, κ does not vanish at any point in its domain of definition, and $\kappa(x)|_{Z(\rho_0)} = 0$. In this sense we may write $Z_m = Z(\rho_0) = \{\tilde{x} \in \mathcal{R}(V_m(R', r, \tilde{x}^1)); \kappa(\tilde{x}) = 0\}$.*

Remark. If $E(x, \nabla_x^m u)$, the initial values $u_j(x')$, and all the characteristic roots $\mu_j(x, u(x), \xi')$ are real functions, then the solution $u(x)$ and the characteristic functions $\kappa_j(x)$ are also real valued. Considering $u_j^\pm(x') = \lim_{\varepsilon\to+0} u_j(x''', x_n \pm \varepsilon\sqrt{-1})$ for real x, we can define the natural branch $u^\pm(x)$ and $\kappa_j^\pm(x)$ for real x. Then the solution $u^\pm(x)$ is real analytic on $\{x \in \mathbb{R}^{n+1}; |x| < r, \kappa_j^\pm(x) \ne 0$ for $1 \le j \le m\}$.

§3. Deformation of the equation

If $u(x)$ satisfies condition (2) and $\kappa(x)$ satisfies condition (3) for $k = m$, we define $y = (y'', y_n) = (x'', \kappa(x))$, which we regard as a new coordinate system. We next rewrite the equation $E(x, \nabla_x^m u) = 0$ using this coordinate system. If we can define the inverse function $x(y) = (y'', x_n(y))$, then it is easy to see that

$$\partial_{x_j}\kappa = \begin{cases} -\partial_{y_j}x_n/\partial_{y_n}x_n, & j \ne n, \\ 1/\partial_{y_n}x_n, & j = n, \end{cases} \tag{4}$$

and thus

$$\partial_{x_j} f = \begin{cases} \partial_{y_j} f - \dfrac{\partial_{y_j}x_n}{\partial_{y_n}x_n}\partial_{y_n} f, & j \ne n, \\ \dfrac{1}{\partial_{y_n}x_n}\partial_{y_n} f, & j = n. \end{cases} \tag{5}$$

Therefore, considering $x_n = x_n(y)$ as a function of y, we can rewrite (3) as

$$-\partial_{y_0} x_n = \mu_m(y, u(y), \xi')|_{\xi'=(-\partial_{y_1}x_n(y),\dots,-\partial_{y_{n-1}}x_n(y),1)}, \qquad x_n(0,y') = y_n. \tag{6}$$

We denote $u(x(y))$ also by $u(y)$. We set $W = (W_{k,\alpha};\ 1 \le k \le 2,\ |\alpha| \le k-1) \in \mathbb{C}^{n+3}$. We define $W^\circ = (W^\circ_{k,\alpha};\ 1 \le k \le 2,\ |\alpha| \le k-1) \in \mathbb{C}^{n+3}$, where

$$W^\circ_{k,\alpha} = \begin{cases} u(0), & k=1,\ |\alpha|=0,\\ \partial_y^\alpha x_n(0), & k=2,\ |\alpha| \le 1. \end{cases}$$

Let $u(x)$ satisfy (2) and $x_n(y)$ satisfy (6). Let $F(y,W)$ be a holomorphic function on a neighborhood of $(y,W) = (0,W^\circ)$, and set $c(y) = F(y, u(y), \nabla_y^1 x_n(y))$. We denote the set of such functions $c(y)$ by $\mathcal{P}^0$. If $j \in \mathbb{N}$, we denote by $\mathcal{P}^j$ the set of polynomials of the components of $(\nabla_y^j u, \nabla_y^j x_n)$ with coefficients belonging to $\mathcal{P}^0$. For a function $f(y)$ we define $\mathcal{P}^j(f) = \{\sum_{|\alpha|\le j} c_\alpha(y)\partial_y^\alpha f;\ c_\alpha(x) \in \mathcal{P}^j\}$.

If $j \ge 1$, we denote $X^j = (X_{k,\alpha};\ 1 \le k \le 2,\ |\alpha| \le j)$. We define $X^\circ = (X^\circ_{k,\alpha};\ 1 \le k \le 2,\ |\alpha| \le j)$ where $X^\circ_{k,\alpha} = \partial_y^\alpha v_k(0)$. Let $F(y,X)$ be a holomorphic function on a neighborhood of $(y,X) = (0,X^\circ)$, and set $c(y) = F(y, \nabla_y^j v(y))$. We denote the set of such functions $c(y)$ by $\mathcal{Q}^j$. Therefore $\mathcal{P}^j \subset \mathcal{Q}^j$.

Let $\Lambda = \Lambda_1 \cdots \Lambda_m$, where $\Lambda_j = \partial_{y_0} - (a_j - a_m)\partial_{y_n}$. The aim of this section is to prove that we can approximate $E(x, \nabla_x^m u)$ by Λ in the following sense:

Proposition 1. *There exist $Q_{j,\alpha} \in \mathcal{P}^0$ for $j = 1,2$ and $|\alpha| = m$ such that*

$$E(x, \nabla_x^m u) = \Lambda u + \sum_{|\alpha|=m} Q_{1,\alpha}\partial_y^\alpha u + \sum_{|\alpha|=m} Q_{2,\alpha}\partial_y^\alpha x_n + Q'(y, \nabla_y^{m-1}u, \nabla_y^{m-1}x_n),$$

where $Q' \in \mathcal{Q}^{m-1}$ and

$$\begin{cases} Q_{1,\alpha} = 0, & \alpha = (m,0,\dots,0) \text{ or } (0,\dots,0,m),\\ [Q_{1,\alpha}(y, u(y), \nabla_y^1 x_n(y))]_{y=0} = 0, & \alpha_0 + \alpha_n = m,\\ Q_{2,\alpha} = 0, & \alpha_0 + \alpha_n = m. \end{cases} \tag{7}$$

We prepare some results.

Lemma 1. *If $|\alpha| \ge 2$, then*

$$\partial_x^\alpha \kappa \equiv \sum_{|\beta|=|\alpha|} A_{\alpha\beta}\partial_y^\beta x_n \text{ modulo } \mathcal{P}^{|\alpha|-1}$$

for some $A_{\alpha\beta} \in \mathcal{P}^0$. Furthermore, if $\beta = (0,\dots,0,|\alpha|)$, then $A_{\alpha\beta} = -(\partial_x\kappa)^\alpha/\partial_{y_n}x_n$.

Proof. We first consider the case $|\alpha| = 2$. Let $\partial_x^\alpha = \partial_{x_j}\partial_{x_k}$, and assume $0 \le j, k \le n-1$. From (4) and (5), we have

$$\begin{aligned}\partial_x^\alpha \kappa = \partial_{x_j}(\partial_{x_k}\kappa) &= -\left(\partial_{y_j} - \frac{\partial_{y_j} x_n}{\partial_{y_n} x_n}\partial_{y_n}\right)\frac{\partial_{y_k} x_n}{\partial_{y_n} x_n} \\ &= -\frac{\partial_{y_j}\partial_{y_k} x_n}{\partial_{y_n} x_n} + \frac{\partial_{y_j} x_n \cdot \partial_{y_k}\partial_{y_n} x_n}{(\partial_{y_n} x_n)^2} + \frac{\partial_{y_k} x_n \cdot \partial_{y_j}\partial_{y_n} x_n}{(\partial_{y_n} x_n)^2} - \frac{\partial_{y_j} x_n \cdot \partial_{y_k} x_n \cdot \partial_{y_n}^2 x_n}{(\partial_{y_n} x_n)^3} \\ &= -\frac{\partial_{y_j}\partial_{y_k} x_n}{\partial_{y_n} x_n} - \frac{\partial_{x_j}\kappa \cdot \partial_{y_k}\partial_{y_n} x_n}{\partial_{y_n} x_n} - \frac{\partial_{x_k}\kappa \cdot \partial_{y_j}\partial_{y_n} x_n}{\partial_{y_n} x_n} - \frac{\partial_{x_j}\kappa \cdot \partial_{x_k}\kappa \cdot \partial_{y_n}^2 x_n}{\partial_{y_n} x_n},\end{aligned}$$

which proves the lemma in this case. We can handle the case $j = n$ or $k = n$ similarly.

We next assume that $l \ge 3$ and the conclusion is true if $|\alpha| \le l-1$. We consider the case $|\alpha| = l$. We assume that $\partial_x^\alpha = \partial_{x_k}\partial_x^\gamma$ where $|\gamma| = l-1$. Let $0 \le k \le n-1$. Then

$$\begin{aligned}\partial_x^\alpha \kappa = \partial_{x_k}(\partial_x^\gamma \kappa) &\equiv \sum_{|\beta|=|\gamma|}\left(\partial_{y_k} - \frac{\partial_{y_k} x_n}{\partial_{y_n} x_n}\partial_{y_n}\right)(A_{\gamma\beta}\partial_y^\beta x_n) \\ &\equiv \sum_{|\beta|=|\gamma|}\left(A_{\gamma\beta}\partial_{y_k}\partial_y^\beta x_n - \frac{\partial_{y_k} x_n}{\partial_{y_n} x_n}\cdot A_{\gamma\beta}\partial_{y_n}\partial_y^\beta x_n\right)\end{aligned}$$

modulo $\mathcal{P}^{l-1}$, from which we obtain the lemma. We can handle the case $k = n$ similarly. □

Lemma 2. *We have* $E[u]\kappa \equiv \sum_{|\alpha|=m} A'_\alpha \partial_y^\alpha x_n(y)$ *modulo* $\mathcal{Q}^{m-1}$ *for some* $A'_\alpha \in \mathcal{P}^0$. *Furthermore, if* $\alpha_0 \ne 0$ *or* $\alpha_n = m$, *then* $A'_\alpha = 0$.

Proof. Using the notation of Lemma 1, we have

$$E[u]\kappa \equiv \sum_{|\alpha|=|\beta|=m} E_\alpha(x,u)A_{\alpha\beta}\partial_y^\beta x_n = \sum_{|\beta|=m} A'_\beta \partial_y^\beta x_n$$

modulo $\mathcal{Q}^{m-1}$, where $A'_\beta = \sum_{|\alpha|=m} E_\alpha(x,u)A_{\alpha\beta}$. If $\beta = (0,\dots,0,m)$, then we have $A'_\beta = -\sum_{|\alpha|=m} E_\alpha(x,u)(\partial_x\kappa)^\alpha/\partial_{y_n} x_n = 0$. If $|\beta| = m$ and $\beta_0 \ne 0$, using (6) we can rewrite $\partial_y^\beta x_n$ in the form

$$\partial_y^\beta x_n \equiv \sum_{\substack{|\gamma|=m \\ \gamma_0=0 \\ \gamma_n \ne m}} A''_{\beta\gamma}\partial_y^\gamma x_n \text{ modulo } \mathcal{P}^{m-1}$$

for some $A''_{\beta\gamma} \in \mathcal{P}^0$. Therefore we can delete $\partial_y^\beta x_n$ with $\beta_0 \ne 0$, and we obtain the conclusion of Lemma 2. □

Lemma 3. *If $|\alpha| \geq 2$, then*

$$\partial_x^\alpha f(x) \equiv \sum_{\substack{\beta+\gamma=\alpha \\ \beta_n=0}} \frac{\alpha!}{\beta!\gamma!}(\partial_x\kappa)^\gamma \partial_y^\beta \partial_{y_n}^{|\gamma|} f + \partial_x^\alpha \kappa \cdot \partial_{y_n} f \ \textit{modulo}\ \mathcal{P}^{|\alpha|-1}(f).$$

The proof is almost the same as that of Lemma 3, and we omit it.

Proof of Proposition 1. From Lemmas 2 and 3, we have

$$\begin{aligned} E(x, \nabla_x^m u) &\equiv \sum_{|\alpha|=m} E_\alpha(x,u)\partial_x^\alpha u \\ &\equiv \sum_{\substack{|\beta|+|\gamma|=m \\ \beta_n=0}} \frac{(\beta+\gamma)!}{\beta!\gamma!} E_{\beta+\gamma}(x,u)(\partial_x\kappa)^\gamma \partial_y^\beta \partial_{y_n}^{|\gamma|} u + E[u]\kappa \cdot \partial_{y_n} u \end{aligned}$$

modulo $\mathcal{Q}^{m-1}$. From Lemma 3 we have

$$E(x, \nabla_x^m u) \equiv \sum_{|\alpha|=m} Q''_{1,\alpha} \partial_y^\alpha u + \sum_{|\alpha|=m} Q_{2,\alpha} \partial_y^\alpha x_n.$$

Here $Q_{2,\alpha}$ denotes A'_α of Lemma 2, and

$$Q''_{1,\alpha} = \sum_{S(\alpha)} \frac{(\beta+\gamma)!}{\beta!\gamma!} E_{\beta+\gamma}(x,u)(\partial_x\kappa)^\gamma,$$

where $S(\alpha) = \{(\beta,\gamma) \in \mathbb{Z}_+^{n+1} \times \mathbb{Z}_+^{n+1};\ \beta = (\alpha'', 0),\ |\gamma| = \alpha_n\}$. If $\alpha = (m, 0, \ldots, 0)$, then $S(\alpha)$ has a unique element $(\beta,\gamma) = (\alpha, (0,\ldots,0))$, and we have $Q''_{1,\alpha} = 1$. Similarly we can prove $Q''_{1,\alpha} = 0$ if $\alpha = (0,\ldots,0,m)$ and $Q_{2,\alpha} = 0$ if $\alpha_0 + \alpha_n = m$. Furthermore, we have

$$\begin{aligned} \sum_{j+k=m} [Q''_{1,(j,0,\ldots,0,k)} \eta_0^j \eta_n^k]_{y=0} &= \sum_{|\alpha|=m} [E_\alpha(\eta_0 + \mu_m(x,\xi'), \xi')^\alpha]_{x=0,\xi'=(0,\ldots,0,\eta_n)} \\ &= \prod_{1\leq j\leq m} [\eta_0 - \mu_j(x,\xi') + \mu_m(x,\xi')]_{x=0,\xi'=(0,\ldots,0,\eta_n)} = 0. \end{aligned}$$

Comparing the coefficients, we get $[Q''_{1,\alpha}(y, u(y), \nabla_y^1 x_n(y))]_{y=0} = 0$ if $\alpha_0 + \alpha_n = m$. Denoting $\sum_{|\alpha|=m} Q''_{1,\alpha}\partial_y^\alpha u = \Lambda u + \sum_{|\alpha|=m} Q_{1,\alpha}\partial_y^\alpha u$, we obtain the conclusion of Proposition 1. □

§4. Characteristic coordinate system

We assume that $M > 0$ is large, $0 < R \ll 1/M$, and each of $R_1, \ldots, R_m$ denotes either R or 0. We define $a_j = \mu_j(0, u(0), \xi'^{\circ})$ as before, and $b_j = a_j - a_m$. We set

$$\Omega_2 = \{y \in \mathbb{C}^{n+1};\ |y_n + b_j y_0| > R_k|y_0| \text{ for } 1 \leq j \leq m\}.$$

We want to solve (1) for $y \in \Omega_2$, and we need to study some of its geometrical properties. For $1 \le k \le m$ we define

$$\Psi^k : \mathbb{C}^{n+1} \ni y \mapsto (y'', y_n + b_k y_0) \in \mathbb{C}^{n+1}.$$

We denote $z = \Psi^k(y)$ and $\tau = z_0/z_n = y_0/(y_n + b_k y_0)$. Then

$$\psi^k : \Omega_2 \ni y \overset{\sim}{\mapsto} (\tau, z') \in \omega^k \times \mathbb{C}^{n-1} \times (\mathbb{C} \setminus \{0\}), \tag{8}$$

where $\omega^k = \{\tau \in \mathbb{C};\, |1 + (b_j - b_k)\tau| > R_j|\tau|,\, 1 \le j \le m\}$. For $k = m$, we denote τ also by $\sigma = y_0/y_n$.

We can illustrate these domains as follows. If $\zeta \in \mathbb{C}$ and $\rho \ge 0$, we define $\bar{B}(\zeta, \rho) = \{z \in \mathbb{C};\, |z - \zeta| \le \rho\}$ and $B(\zeta, r) = \{z \in \mathbb{C};\, |z - \zeta| < r\}$. If $r = \infty$, we set $\bar{B}(\zeta, \rho) = B(\zeta, \rho) = \mathbb{C}$. If $s = 0$, we define $\bar{B}(\zeta, 1/s) = B(\zeta, 1/s) = \mathbb{C}$. Finally, we set

$$B_j^k = B\left(\frac{\bar{b}_j - \bar{b}_k}{|b_j - b_k|^2 - R_j^2}, \frac{R_j}{|b_j - b_k|^2 - R_j^2}\right) \quad \text{for } j \ne k,$$

and denote its closure by $\bar{B}_j^k$. It is easy to see that $-1/(b_j - b_k) \in B_j^k$ and we may assume $B_i^k \cap B_j^k = \emptyset$ if $i \ne j$. By a direct calculation, we have

$$\omega^k = B(0, 1/R_k) \setminus \bigcup_{j \ne k} \bar{B}_j^k.$$

In Figure 4(a) we illustrate the following case: $k = m = 4$, $b_j - b_k = e^{2i\sqrt{-1}\pi/3}$ $(1 \le j \le 3)$, $R_1 = \cdots = R_4 = R$. We define

$$\varphi^k : \Omega_2 \ni y \mapsto y_0/(y_n + b_k y_0) \in \omega^k,$$
$$\varphi^{k'k} : \omega^k \ni \varphi^k(y) \mapsto \varphi^{k'}(y) \in \omega^{k'}.$$

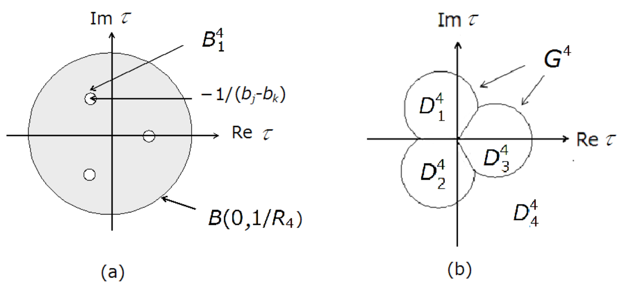

Figure 4. (a) The domain ω^k; (b) the partition of $\mathbb{C}$.

Denoting $\tau = \varphi^k(y)$, we have $\varphi^{k',k}(\tau) = \tau/(1 + (b_{k'} - b_k)\tau)$. We define

$$D_{ij} = \{y \in \Omega_2;\ |y_n + a_i y_0| < |y_n + a_j y_0|\}, \quad i \neq j,$$
$$D_i = \bigcap_{j \neq i} D_{ij},$$
$$G = \Omega_2 \setminus \bigcup_{1 \leq j \leq m} D_j,$$

and

$$D^k_{ij} = \varphi^k(D_{ij}) = \{\tau \in \mathbb{C};\ |1 + (b_i - b_k)\tau| < |1 + (b_j - b_k)\tau|\}, \quad j \neq i,$$
$$D^k_i = \varphi^k(D_i) = \bigcap_{j \neq i} D^k_{ij},$$
$$G^k = \varphi^k(G) = \mathbb{C} \setminus \bigcup_{1 \leq j \leq m} D^k_j.$$

We have the partition $D^k_1 \sqcup \cdots \sqcup D^k_m \sqcup G^k = \mathbb{C}$ (see Figure 4(b)). Roughly speaking, $\tau \in D^k_j$ means that τ is near $-1/(b_j - b_k)$, and $\tau \in G^k$ means that τ is near none of $-1/(b_j - b_k)$. In Figure 4(b) we illustrate the following case: $k = m = 4$, $b_j - b_k = e^{2j\sqrt{-1}\pi/3}$ $(1 \leq j \leq 3)$. We also note $0 \in \omega^k$.

Lemma 4. (i) *If* $|b_i - b_k| > |b_j - b_k|$, *then*

$$D^k_{ij} = B\left(\frac{-\bar{b}_i + \bar{b}_j}{|b_i - b_k|^2 - |b_j - b_k|^2}, \frac{|\bar{b}_i + \bar{b}_j|}{|b_i - b_k|^2 - |b_j - b_k|^2}\right).$$

(ii) *If* $|b_i - b_k| = |b_j - b_k|$, *then* $D^k_{ij} = \{\tau \in \mathbb{C};\ \mathrm{Re}\,((b_i - b_j)\tau) > 0\}$.

(iii) *If* $|b_i - b_k| < |b_j - b_k|$, *then*

$$D^k_{ij} = \mathbb{C} \setminus \bar{B}\left(\frac{-\bar{b}_i + \bar{b}_j}{|b_i - b_k|^2 - |b_j - b_k|^2}, \frac{|\bar{b}_i + \bar{b}_j|}{|b_j - b_k|^2 - |b_j - b_k|^2}\right).$$

Proof. We have

$$\begin{aligned} \tau \in D_{ij} &\Leftrightarrow |1 + (b_i - b_k)\tau| < |1 + (b_j - b_k)\tau| \\ &\Leftrightarrow (|b_i - b_k|^2 - |b_j - b_k|^2)\tau\bar{\tau} + (b_i - b_j)\tau + (\bar{b}_i - \bar{b}_j)\bar{\tau} < 0, \end{aligned}$$

and we can easily prove the lemma. □

Lemma 5. *If* $j \neq k$, *then* $-1/(b_j - b_k) \in \bar{B}^k_j \subset D^k_j$, *and* D^k_j *is a bounded domain. Its boundary* ∂D^k_j *is the union of finitely many circular arcs or line segments, and it is a continuous curve surrounding* $-1/(b_j - b_k)$.

Proof. Since R is small, we deduce the first statement from Lemma 4(i). Trivially we have $\partial D^k_j = \bigcup_{i \neq j}\{\tau \in \mathbb{C};\ |1 + (b_i - b_k)\tau| = |1 + (b_j - b_k)\tau|,\ |1 + (b_j - b_k)\tau| \leq |1 + (b_l - b_k)\tau|$ for any $l \neq i\}$. This yields the second statement. □

Lemma 6. *Let $\tau \in D_k^k$ and let $X(\tau) = \{\theta\tau;\, \theta \geq 0\}$. Then there exists $\theta_0 \geq 0$ such that $X(\tau) \cap D_k^k = \{\theta\tau;\, \theta > \theta_0\}$.*

Proof. We have $D_k^k = \bigcap_{j \neq k} D_{kj}^k$. By Lemma 4(iii), each D_{kj}^k is the complement of a disk whose boundary contains the origin. Therefore there exists some θ_j such that $X(\tau) \cap D_{kj}^k = \{\theta\tau;\, \theta > \theta_j\}$. Hence $X(\tau) \cap D_k^k = \{\theta\tau;\, \theta > \theta_0\}$ with $\theta_0 = \min_{j \neq k} \theta_j$. □

Definition. If $\tau \in D_k^k$, we define $\tau_G = \theta_0\tau$ for the above θ_0 (see Figure 5(a)). If $\tau \in D_j^k$ for $j \neq k$, we denote $\iota = \varphi^{jk}(\tau) \in D_k^k$. Then we have defined ι_G for this $\iota \in D_k^k$, and we set $\tau_G = \varphi^{k,j}(\iota_G)$. If $\tau \in G^k$, we define $\tau_G = \tau$.

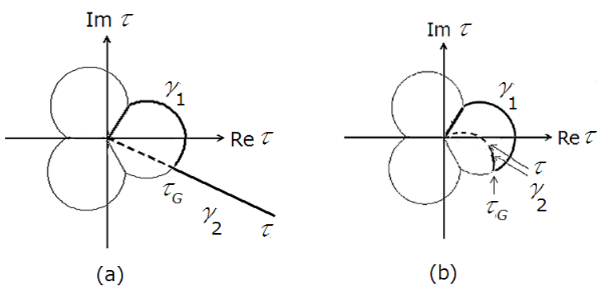

Figure 5. τ_C and γ^k.

Let $\tau \in D_j^k$ for $j \neq k$. Since $\varphi^{jk}(\tau) \in D_k^k$, the four points $0, \varphi^{k,j}(\tau_G), \varphi^{k,j}(\tau)$ and ∞ lie on the same line in this order. This means that the four points $0, \tau_G, \tau$, $1/(b_k - b_j)$ lie on the same circular arc (or the same line) in this order (see Figure 5(b)). If $k = m$, then we denote τ_G also by σ_G.

We say that a continuous curve $\gamma \subset \omega^m$ from 0 to σ is a *canonical curve* if it satisfies the following conditions:

(i) $\gamma = \gamma_1 + \gamma_2$, where $\gamma_1 \subset G^m$ is a continuous curve from 0 to σ_G. We assume that γ_1 is the shortest curve among the curves contained in G^m belonging to the same homotopy class.

(ii) If $\sigma \in G^m$, then $\gamma_2 = \emptyset$.

(iii) If $\sigma \in D_j^m$, then γ_2 is the segment from σ_G to σ of the circle (or the line) passing through $0, \sigma_G, \sigma, -1/(b_j - b_m)$.

If γ is a canonical curve from 0 to σ, then:

(i) $\varphi^{k,m}(\gamma) = \gamma^k = \gamma_1^k + \gamma_2^k$, and $\gamma_1^k \subset G^k$ is a continuous curve from 0 to $\tau_G = \varphi^{k,m}(\sigma_G)$.

(ii) If $\tau = \varphi^{k,m}(\sigma) \in G^k$, then $\gamma_2^k = \emptyset$.

(iii) If $\tau \in D_j^k$, then γ_2^k is the segment from τ_G to τ of the circle (or the line) passing through $0, \tau_G, \tau, -1/(b_j - b_k)$.

In any case, $\gamma_1^k \subset G^k$ is a continuous curve from 0 to τ_G, and γ_2^k is a circular arc or a line segment from τ_G to τ.

We next explain the meaning of these curves. From (8), we have

$$H_1(\Omega_2) = H_1(\omega^k) \times H_1(\mathbb{C} \setminus \{0\}) = (\overbrace{\mathbb{Z} * \cdots * \mathbb{Z}}^{m-1 \text{ times}}) \times \mathbb{Z},$$

for each k (here $*$ denotes free product). This means the following: Let $\Gamma \subset \Omega_2$ be a continuous curve from $y^{\mathrm{o}} \in \Omega_2 \cap (\{0\} \times \mathbb{C}^n)$ to $y^* \in \Omega_2$ and let $z^{\mathrm{o}} = \Psi^k(y^{\mathrm{o}})$, $z^* = \Psi^k(y^*)$. Then Γ is homotopically equivalent to $\Gamma^{k,0} + \Gamma^{k,1}$ in Ω_2, where

(a) $\Psi^k(\Gamma^{k,0})$ is a continuous curve in the initial hyperplane $\Psi^k(\Omega_2) \cap (\{0\} \times \mathbb{C}^n)$ from z^{o} to $(0, z_1^*, \ldots, z_n^*)$;

(b) $\Psi^k(\Gamma^{k,1}) \subset \mathbb{C} \times \{(z_1^*, \ldots, z_n^*)\}$ is a continuous curve from $(0, z_1^*, \ldots, z_n^*)$ to z^*. We may assume $\Psi^k(\Gamma^{k,1}) = \{(\tau z_n^*, z_1^*, \ldots, z_n^*);\ \tau \in \varphi^k(\gamma)\}$, where γ is a canonical curve.

In this sense, the homotopy equivalence class $[\Gamma]$ is essentially described by a canonical curve γ, neglecting $\Gamma^{k,0}$ in the initial hyperplane. We also note that $\Gamma^{k,1}$ is a characteristic curve for the operator Λ_k. There are m ways of this description corresponding to the choice of $k \in \{1, \ldots, m\}$. It is not clear whether specifying a point y^*, we can use the same curve γ in the above discussion for $1 \le k \le m$. We have the following result.

Lemma 7. *In the above discussion, each $\Gamma^{k,1}$ is described by the same canonical curve γ.*

Proof. For $k = m$, let us choose a canonical curve γ as above. Then $\Gamma^m = \Gamma^{m,0} + \Gamma^{m,1}$, where $\Gamma^{m,0} \subset \Omega_2 \cap (\{0\} \times \mathbb{C}^n)$ and $\Gamma^{m,1} = \{(\sigma y_n^*, y_1^*, \ldots, y_n^*);\ \sigma \in \gamma\}$. It follows that $\Psi^k(\Gamma^m) = \Psi^k(\Gamma^{m,0}) + \Psi^k(\Gamma^{m,1})$, where

$$\begin{aligned}\Psi^k(\Gamma^{m,1}) &= \{\Psi^k(\sigma y_n^*, y_1^*, \ldots, y_{n-1}^*, y_n^*);\ \sigma \in \gamma\}\\ &= \{(\sigma y_n^*/(1 + b_k\sigma), y_1^*, \ldots, y_{n-1}^*, (1 + b_k\sigma)y_n^*);\ \sigma \in \gamma\}\\ &= \{(\tau y_n^*, y_1^*, \ldots, y_{n-1}^*, y_n^*/(1 - b_k\tau));\ \tau \in \varphi^{k,m}(\gamma)\}.\end{aligned}$$

Here $\tau = \varphi^{k,m}(\sigma)$ varies from 0 to τ^*. Since we can describe Ω_2 in the form of a direct product as in (8), $\Gamma^{m,1}$ is homotopically equivalent to $\Gamma^{m,2} + \Gamma^{m,3}$ in Ω_2, where

$$\begin{aligned}\Psi^k(\Gamma^{m,2}) &= \{(0, y_1^*, \ldots, y_{n-1}^*, y_n^*/(1 - b_k\tau));\ \tau \in \varphi^{k,m}(\gamma)\},\\ \Psi^k(\Gamma^{m,3}) &= \{(\tau y_n^*, y_1^*, \ldots, y_{n-1}^*, y_n^*/(1 - b_k\tau^*));\ \tau \in \varphi^{k,m}(\gamma)\}.\end{aligned}$$

We have $y_n^*/(1-b_k\tau^*) = z_n^*$, and therefore $\Gamma^{m,3} = \Gamma^{k,1}$. On the other hand, $\Gamma^{m,2}$ is contained in the initial hyperplane. Therefore $\Gamma^{k,0}$ is homotopically equivalent to $\Gamma^{m,0} + \Gamma^{m,2}$ in the initial hyperplane. Neglecting curves in the initial hyperplane, $\Gamma^{m,1}$ and $\Gamma^{k,1}$ are described by the same canonical curve γ. □

§5. Properties of a canonical curve

In this section, we always assume the following condition. Let $y^* \in \Omega_2, \sigma^* = y_0^*/y_n^*$, and let σ be a point on a canonical curve γ from 0 to σ^*. Let $1 \le k \le m$, and $\tau^* = \varphi^{k,m}(\sigma^*)$, $\tau = \varphi^{k,m}(\sigma)$. Furthermore, we assume

$$\begin{aligned} z^* &= \Psi^k(y^*) = (z_0^*, z_1^*, \ldots, z_{n-1}^*, z_n^*) = (y_0^*, y_1^*, \ldots, y_{n-1}^*, y_n^* + b_k y_0^*), \\ z &= (\tau z_n^*, z_1^*, \ldots, z_{n-1}^*, z_n^*), \\ y &= (\Psi^k)^{-1}(z) = (\tau z_n^*, z_1^*, \ldots, z_{n-1}^*, (1-b_k\tau) z_n^*). \end{aligned}$$

This means that we regard y^* as a fixed point, and y as a point moving along a characteristic curve $\Gamma^{k,1}$ in Lemma 7.

Let $M > 0$ be large. For $1 \le j \le m$, we define

$$\begin{aligned} \gamma^j &= \varphi^{j,m}(\gamma), \\ \gamma_M^j &= \varphi^{j,m}(\gamma) \cap \{\tau \in \gamma^j;\, |\tau| \le M\}, \\ \lambda^j(\sigma^*) &= \text{the length of } \gamma^j, \\ \lambda_M^j(\sigma^*) &= \text{the length of } \gamma_M^j, \\ \lambda_M(\sigma^*) &= \sum_{1 \le j \le m} \lambda_M^j(\sigma^*). \end{aligned}$$

For each j, $|y_n^* + b_j y_0^*|\lambda^j(\sigma^*)$ denotes the length of the characteristic curve $\Psi^j(\Gamma^{j,1})$ in Lemma 7, and thus the distance (not in mathematically strict sense) between $y_0 = 0$ and $y_0 = y_0^*$, considered in the universal covering space.

Remark. We have defined the canonical curve expecting that if y is moving backwards from $y_0 = y_0^*$ to $y_0 = 0$ along $\Gamma^{k,1}$, then the solution behaves in a good manner, in the following sense:

(i) The distance $\max_{1 \le j \le m} |y_n + b_j y_0|\lambda^j(\sigma)$ between the initial hyperplane and y is decreasing (from σ^* to 0).

(ii) The distance $\min_{1 \le j \le m} |y_n + b_j y_0|$ between the singularity set and y is increasing.

Unfortunately, these are not literally true. For example, let us consider (i). If $j = k$, then by assumption $|y_n + b_k y_0| = |y_n^* + b_k y_0^*|$ and trivially $\lambda^j(\sigma) \le \lambda^j(\sigma^*)$.

Therefore (i) is true for $j = k$. However, if $j \neq k$, then it may happen that $|y_n + b_j y_0| > |y_n^* + b_j y_0^*|$ and (i) may be false. We shall see that (i) and (ii) are true after modifying these indicators (see Propositions 2 and 3 below):

Definition. We define the modified indicators by

$$l^j(y) = e^{M^5 \lambda_M(\sigma)} |y_n + b_j y_0| \lambda^j(\sigma), \qquad l(y) = \max_{1 \le j \le m} l^j(y),$$

$$d^j(y) = e^{-M^5 \lambda_M(\sigma)} |y_n + b_j y_0| - R_j l(y), \quad d(y) = \min_{1 \le j \le m} d^j(y).$$

Then we have the following result.

Proposition 2. *We have* $l(y^*) - l(y) \ge M^{-5} |y_n^* + b_k y_0^*| (\lambda^k(\sigma^*) - \lambda^k(\sigma))$.

To prove Proposition 2, we first prepare two lemmas.

Lemma 8. *Assume* $|\varphi^{j_0,m}(\sigma^*)| \ge M$ *for some* j_0. *Then there exists some* $j \neq j_0$ *for which* $l^j(\sigma^*) \ge l^{j_0}(\sigma^*)$.

Proof. From $|\varphi^{j_0,m}(\sigma^*)| \ge M$ we obtain $\varphi^{j_0,m}(\sigma^*) \in D_{j_0}^{j_0}$ (see Figure 4(b)). Let σ_G^* be the point defined in Section 3 (see Figure 5). Then $\varphi^{j_0,m}(\sigma_G^*)$ belongs to the closure of $D_{j_0}^{j_0} \cap D_j^{j_0}$ for some $j \neq j_0$. This means

$$|1 + (b_j - b_{j_0}) \varphi^{j_0,m}(\sigma_G^*)| = 1. \tag{9}$$

It suffices to show the following two inequalities for this j:

$$(\lambda^j(\sigma^*) - \lambda^j(\sigma_G^*)) |y_n^* + b_j y_0^*| \ge (\lambda^{j_0}(\sigma^*) - \lambda^{j_0}(\sigma_G^*)) |y_n^* + b_{j_0} y_0^*|, \tag{10}$$

$$\lambda^j(\sigma_G^*) |y_n^* + b_j y_0^*| \ge \lambda^{j_0}(\sigma_G^*) |y_n^* + b_{j_0} y_0^*|. \tag{11}$$

We denote $\varphi^{j_0,m}(\sigma) = \iota$, $\varphi^{j_0,m}(\sigma^*) = \iota^*$, $\varphi^{j_0,m}(\sigma_G^*) = \iota_G^*$. From (9) we obtain

$$\lambda^j(\sigma^*) - \lambda^j(\sigma_G^*) \ge |\varphi^{j,m}(\sigma^*) - \varphi^{j,m}(\sigma_G^*)| = \left| \frac{\iota^*}{1 + (b_j - b_{j_0})\iota^*} - \frac{\iota_G^*}{1 + (b_j - b_{j_0})\iota_G^*} \right|$$
$$= \left| \frac{\iota^* - \iota_G^*}{1 + (b_j - b_{j_0})\iota^*} \right|.$$

Since $y_n^* + b_j y_0^* = (1 + (b_j - b_{j_0})\iota^*)(y_n^* + b_{j_0} y_0^*)$, we have

$$(\lambda^j(\sigma^*) - \lambda^j(\sigma_G^*)) |y_n^* + b_j y_0^*| \ge |\iota^* - \iota_G^*| \cdot |y_n^* + b_{j_0} y_0^*|.$$

From the assumption $|\iota^*| \ge M$ we have $|\iota^* - \iota_G^*| = \lambda^{j_0}(\sigma^*) - \lambda^{j_0}(\sigma_G^*)$ (see Figure 5(a)), and we obtain (10).

We next show (11). We have $\lambda^j(\sigma_G^*) = \int_0^{\iota_G^*} |d\varphi^{j,j_0}(\iota)|$. Here ι moves on G^{j_0} and so $|1 + (b_{j_0} - b_j)\iota| \le M^{1/4}$ (see Figure 5). It follows that

$$\left| \frac{d\varphi^{j,j_0}(\iota)}{d\iota} \right| = \left| \frac{d}{d\iota} \left(\frac{\iota}{1 + (b_j - b_{j_0})\iota} \right) \right| = \frac{1}{|1 + (b_{j_0} - b_j)\iota|^2} \ge M^{-1/2}.$$

Therefore $\lambda^j(\sigma_G^*) \geq M^{-1/2} \int_0^{\iota_G^*} |d\iota| = M^{-1/2}\lambda^{j_0}(\sigma_G^*)$. On the other hand,

$$|y_n^* + b_j y_0^*| = |(1 + (b_j - b_{j_0})\iota)(y_n^* + b_{j_0} y_0^*)| \geq M^{1/2}|y_n^* + b_{j_0} y_0^*|.$$

Therefore (11) holds. □

Lemma 9. (i) *If* $|\varphi^{j,m}(\sigma)| \leq M$, *then* $|1 + (b_i - b_j)\varphi^{j,m}(\sigma)| \leq M^{5/4}$.
(ii) *If* $|\varphi^{i,m}(\sigma)| \leq M$, *then* $|1 + (b_i - b_j)\varphi^{j,m}(\sigma)| \geq M^{-5/4}$.

Proof. (i) If $|\varphi^{j,m}(\sigma)| \leq M$, we have

$$|1 + (b_i - b_j)\varphi^{j,m}(\sigma)| \leq 1 + M|b_i - b_j| \leq M^{5/4}.$$

(ii) If $|\varphi^{i,m}(\sigma)| \leq M$, from (i) we have

$$|1 + (b_i - b_j)\varphi^{j,m}(\sigma)| = |1 + (b_j - b_i)\varphi^{i,m}(\sigma)|^{-1} \geq M^{-5/4}.$$ □

Proof of Proposition 2. We need to consider the following cases:

(a) $|\varphi^{j,m}(\sigma^*)| \leq M$ for any j.
(b) $|\varphi^{j_0,m}(\sigma)| \geq M$ for some j_0.
(c) $|\varphi^{j_0,m}(\sigma^*)| \geq M$ for some j_0 and $|\varphi^{j,m}(\sigma)| \leq M$ for any j.

We first consider case (a). Let $1 \leq j \leq m$. In this case σ also satisfies the same condition $|\varphi^{j,m}(\sigma)| \leq M$ automatically (see Figure 5). It suffices to show

$$l^j(y) \leq l^j(y^*) - |y_n^* + b_k y_0^*|(\lambda^k(\sigma^*) - \lambda^k(\sigma)). \tag{12}$$

From Lemma 9 we obtain

$$\left|\frac{y_n + b_j y_0}{y_n^* + b_j y_0^*}\right| = \left|1 + \frac{(b_j - b_k)(\tau - \tau^*)}{1 + (b_j - b_k)\tau^*}\right| r \leq 1 + M^5|\tau^* - \tau| \leq e^{M^5(\lambda_M(\sigma^*) - \lambda_M(\sigma))}.$$

Therefore $e^{M^5\lambda_M(\sigma)}|y_n + b_j y_0| \leq e^{M^5\lambda_M(\sigma^*)}|y_n^* + b_j y_0^*|$, and it follows that

$$\begin{aligned} l^j(y) &= e^{M^5\lambda_M(\sigma)}|y_n + b_j y_0|\lambda^j(\sigma) \leq e^{M^5\lambda_M(\sigma^*)}|y_n^* + b_j y_0^*|\lambda^j(\sigma) \\ &\leq l^j(y^*) - |y_n^* + b_j y_0^*|(\lambda^j(\sigma^*) - \lambda^j(\sigma)). \end{aligned} \tag{13}$$

On the other hand, from Lemma 9 we have

$$\left|\frac{d}{d\tau}\varphi^{jk}(\tau)\right| = |1 + (b_j - b_k)\tau|^{-2} \geq M^{-5/2}.$$

Therefore

$$\lambda^j(\sigma^*) - \lambda^j(\sigma) = \int_\tau^{\tau^*} |d\varphi^{jk}(\tau')| \geq M^{-5/2}(\lambda^k(\sigma^*) - \lambda^k(\sigma)). \tag{14}$$

We also have

$$|y_n^* + b_j y_0^*| = |(1 + (b_j - b_k)\tau^*)(y_n^* + b_k y_0^*)| \geq M^{-5/2}|y_n^* + b_k y_0^*|. \tag{15}$$

From (14) and (15), we get

$$(\lambda^j(\sigma^*) - \lambda^j(\sigma))|y_n^* + b_j y_0^*| \geq M^{-5}(\lambda^k(\sigma^*) - \lambda^k(\sigma))|y_n^* + b_k y_0^*|. \tag{16}$$

From (13) and (16) we obtain (12) in case (a).

We next consider case (b). In this case, the number j_0 is uniquely determined and σ^* automatically satisfies the same condition $|\varphi^{j_0,m}(\sigma^*)| \geq M$ for the same j_0. By Lemma 8, we have $l(y) = \max_{j \neq j_0} l^j(y)$, $l(y^*) = \max_{j \neq j_0} l^j(y^*)$. Therefore it suffices to show (12) for $j \neq j_0$. We consider the following subcases:

(b1) $j_0 \neq k$.
(b2) $j_0 = k$.

In subcase (b1), if $j \neq j_0, k$, we can apply the same reasoning as in case (a), and we obtain (12). If $j = k$, then (12) is trivially true (see the remark at the beginning of this section). Thus we obtain (12) if $j \neq j_0$.

In subcase (b2), assume $j \neq j_0$ $(= k)$; we will show (12). We define $(b_j - b_k)\tau = re^{\sqrt{-1}\,\alpha}$ with $r > M^{1/2}$ and $0 \leq \theta < 2\pi$, and we set $f(r) = |1 + (b_j - b_k)\tau|$. We have $f(r) = |1 + re^{\sqrt{-1}\,\alpha}| = (1 + 2r\cos\alpha + r^2)^{1/2}$, and thus $f'(r) = (r + \cos\alpha)/f(r) \geq 1/2$. It follows that

$$\begin{aligned} |y_n^* + b_j y_0^*| - |y_n + b_j y_0| &= \big(|1 + (b_j - b_k)\tau^*| - |1 + (b_j - b_k)\tau|\big)|y_n^* + b_k y_0^*| \\ &\geq |b_j - b_k| \cdot |\tau^* - \tau| \cdot |y_n^* + b_k y_0^*|/2 \\ &\geq (\lambda^k(\sigma^*) - \lambda^k(\sigma))|y_n^* + b_k y_0^*|/M. \end{aligned}$$

Therefore

$$\begin{aligned} l^j(y^*) &= e^{M^5\lambda_M(\sigma^*)}|y_n^* + b_j y_0^*|\lambda^j(\sigma^*) \\ &\geq e^{M^5\lambda_M(\sigma^*)}|y_n + b_j y_0|\lambda^j(\sigma^*) + (\lambda^k(\sigma^*) - \lambda^k(\sigma))|y_n^* + b_k y_0^*|\lambda^j(\sigma^*)/M \\ &\geq l^j(y) + (\lambda^k(\sigma^*) - \lambda^k(\sigma))|y_n^* + b_k y_0^*|\lambda^j(\sigma^*)/M. \end{aligned}$$

Here $\lambda^j(\sigma^*) \geq |\varphi^{j,m}(\sigma^*)| = |(\varphi^{k,m}(\sigma^*))^{-1} + b_j - b_k|^{-1} \geq M^{-1}$. Thus we obtain (12) in case (b2).

In case (c), there exists a unique point $\sigma^1 \in \gamma$ which satisfies $|\varphi^{j_0,m}(\sigma^1)| = M$ for the number j_0 in the statement of (c). Let $y^1 \in \Omega_2$ correspond to τ^1. Considering (y, y^1) instead of (y, y^*), we can apply case (a) to obtain

$$l^j(y) \leq l^j(y^1) - M^{-5}(\lambda^k(\tau^1) - \lambda(\tau))|y_n^* + b_k y_0^*|.$$

Considering (y^1, y^*) instead of (y, y^*), we can apply case (b) to get

$$l^j(y^1) \le l^j(y^*) - M^{-5}(\lambda^k(\tau^*) - \lambda(\tau^1))|y_n^* + b_k y_0^*|.$$

Therefore we obtain (12) in case (c). $\square$

Proposition 3. *If $\lambda_M(\sigma^*) \le M^2$, then*

$$d(y) - d(y^*) \ge R_k |y_n^* + b_k y_0^*|(\lambda^k(\sigma^*) - \lambda^k(\sigma)).$$

Proof. We consider cases (a)–(c), listed in the proof of Proposition 2.

In case (a), let $1 \le j \le m$. Then σ satisfies $|\varphi^{j,m}(\sigma)| \le M$. From Lemma 9,

$$\left|\frac{y_n^* + b_j y_0^*}{y_n + b_j y_0}\right| = \left|1 + \frac{(b_j - b_k)(\tau^* - \tau)}{1 + (b_j - b_k)\tau}\right| \le 1 + M^5|\tau^* - \tau| \le e^{M^5(\lambda_M(\sigma^*) - \lambda_M(\sigma))/2}.$$

Therefore $e^{-M^5\lambda_M(\sigma)/2}|y_n + b_j y_0| \ge e^{-M^5\lambda_M(\sigma^*)/2}|y_n^* + b_j y_0^*|$, and thus

$$\begin{aligned} e^{-M^5\lambda_M(\sigma)}|y_n + b_j y_0| &\ge e^{-M^5\lambda_M(\sigma^*)} e^{M^5(\lambda_M(\sigma^*) - \lambda_M(\sigma))/2}|y_n^* + b_j y_0^*| \\ &\ge e^{-M^5\lambda_M(\sigma^*)}|y_n^* + b_j y_0^*| + e^{-M^6}(\lambda_M(\sigma^*) - \lambda_M(\sigma))|y_n^* + b_j y_0^*|. \end{aligned}$$

From Lemma 9 we have $|y_n^* + b_j y_0^*| = |(1 + (b_j - b_k)\tau^*)(y_n^* + b_k y_0^*)| \ge M^{-2}|y_n^* + b_k y_0^*|$, and it follows that

$$\begin{aligned} &e^{-M^5\lambda_M(\sigma)}|y_n + b_j y_0| \\ &\quad \ge e^{-M^5\lambda_M(\sigma^*)}|y_n^* + b_j y_0^*| + M^{-8}e^{-M^6}|y_n^* + b_k y_0^*|(\lambda_M(\sigma^*) - \lambda_M(\sigma)). \end{aligned}$$

This means

$$d^j(y) \ge d^j(y^*) + R_k|y_n^* + b_k y_0^*|(\lambda_M(\sigma^*) - \lambda_M(\sigma)).$$

Thus we obtain the conclusion in case (a).

In case (b), the number j_0 is uniquely determined. We need to consider subcases (b1) and (b2), as before. For (b1), we have

$$|1 + (b_{j_0} - b_k)\tau| = 1/|1 + (b_k - b_{j_0})\varphi^{j_0,m}(\sigma)| \le M^{-2/3}, \tag{17}$$

and therefore

$$M^{-1/6} \le |\tau| \le M^{1/6}. \tag{18}$$

For any j, we have $\lambda^j(\sigma) \le \lambda_M^j(\sigma) + |\varphi^{j,m}(\sigma)| \le M^2 + |\varphi^{j,m}(\sigma)|$ by assumption. Therefore

$$\begin{aligned} l^j(y) &= e^{M^5\lambda_M(\sigma)}|y_n + b_j y_0|\lambda^j(\sigma) \le e^{M^7}(M^2 + |\varphi^{j,m}(\sigma)|)|y_n + b_j y_0| \\ &= e^{M^7}\big(M^2|1 + (b_j - b_k)\tau| + |\tau|\big)|y_n^* + b_k y_0^*| \le e^{M^7}M^3|y_n^* + b_k y_0^*|, \end{aligned}$$

and thus

$$l(y) \le e^{M^7} M^3 |y_n^* + b_k y_0^*|. \tag{19}$$

We next show

$$d^j(y) \ge d^{j_0}(y) \quad \text{ if } j \ne j_0. \tag{20}$$

To see this, from (19) we note that

$$\begin{aligned} d^j(y) &= e^{-M^5 \lambda_M(\sigma)} |y_n + b_j y_0| - R_j l(y) \\ &\ge \{e^{-M^5 \lambda_M(\sigma)} |1 + (b_j - b_k)\tau| - R_j e^{M^7} M^3\} |y_n^* + b_k y_0^*|. \end{aligned}$$

From (17) and (18),

$$|1 + (b_j - b_k)\tau| \ge |(b_j - b_{j_0})\tau| - |1 + (b_{j_0} - b_k)\tau| \ge 2M^{-1/3}$$

and thus

$$d^j(y) \ge M^{-1/3} e^{-M^5 \lambda_M(\sigma)} |y_n^* + b_k y_0^*| \ge \frac{2M^{-1/3} e^{-M^5 \lambda_M(\sigma)} |y_n + j_0 y_0|}{|1 + (b_{j_0} - b_k)\tau|} \ge d^{j_0}(y).$$

Hence we obtain (20), and this means

$$d(y) = d^{j_0}(y). \tag{21}$$

Here the four points $0, \tau, \tau^*, 1/(b_{j_0} - b_k)$ are located on the same circle (or line) whose curvature radius is at least $1/(2|b_{j_0} - b_k|)$. On the other hand, τ and τ^* are sufficiently near, and thus

$$\begin{aligned} |y_n + j_0 y_0| &= |1 + (b_{j_0} - b_k)\tau| \cdot |y_n^* + b_k y_0^*| \\ &\ge \{|1 + (b_{j_0} - b_k)\tau^*| + M^{-1}(\lambda^k(\sigma^*) - \lambda^k(\sigma))\} |y_n^* + b_k y_0^*| \\ &= |y_n^* + b_{j_0} y_0^*| + M^{-1}(\lambda^k(\sigma^*) - \lambda^k(\sigma)) |y_n^* + b_k y_0^*|. \end{aligned} \tag{22}$$

From (22) we have

$$\begin{aligned} d^{j_0}(y) &= e^{-M^5 \lambda_M(\sigma)} |y_n + b_{j_0} y_0| - R_{j_0} l(y) \\ &\ge e^{-M^5 \lambda_M(\sigma)} |y_n^* + b_{j_0} y_0^*| + M^{-1} e^{-M^7} (\lambda^k(\sigma^*) - \lambda^k(\sigma)) |y_n^* + b_k y_0^*| - R_{j_0} l(y) \\ &\ge d^{j_0}(y^*) + R_k(\lambda^k(\sigma^*) - \lambda^k(\sigma)) |y_n^* + b_k y_0^*|. \end{aligned}$$

Therefore $d(y) = d^{j_0}(y)$ satisfies the conclusion for case (b1).

In case (b2), we can similarly prove (20) for $j \ne j_0$ $(= k)$. Then $d = d^k$ satisfies

$$\begin{aligned} d^k(y^*) &= e^{-M^5 \lambda_M(\sigma^*)} |y_n^* + b_k y_0^*| - R_k l(y^*) \\ &\le e^{-M^5 \lambda_M(\sigma)} |y_n^* + b_k y_0^*| - R_k l(y) - R_k(\lambda^k(\sigma^*) - \lambda^k(\sigma)) |y_n^* + b_k y_0^*| \\ &= d^k(y) - R_k(\lambda^k(\sigma^*) - \lambda^k(\sigma)) |y_n^* + b_k y_0^*|. \end{aligned}$$

This proves case (b2). We can handle case (c) as in Proposition 2. □

From now on, we assume $0 < r \ll R \ll 1/M$. We define

$$\mathcal{L}_0(y^*) = r - r^{-1}l(y^*) - |y'^*|, \quad \mathcal{L}_1(y^*) = \min(d(y^*), \mathcal{L}_0(y^*)).$$

Roughly speaking, these functions describe the following quantities. We have already explained that $l(y^*)$ describes the distance from $y_0 = 0$ to $y_0 = y_0^*$ in the universal covering space. Hence $r^{-1}l(y^*) + |y'^*|$ describes the distance from $y = 0$ to $y = y^*$. Moreover $\mathcal{L}_0(y^*)$ is the distance from y^* to the set $\{r^{-1}l(y) + |y'| = r\}$. Finally, $d(y^*)$ describes the distance from y^* to the singularity set.

Proposition 4. (i) *We have*

$$\mathcal{L}_0(y) \geq \mathcal{L}_0(y^*) + \frac{1}{2r}|y_n^* + b_k y_0^*|(\lambda^k(\sigma^*) - \lambda^k(\sigma)).$$

(ii) *If* $\lambda_M(y^*) \leq M^2$, *then*

$$\mathcal{L}_1(y) \geq \mathcal{L}_1(y^*) + R_k|y_n^* + b_k y_0^*|(\lambda^k(\sigma^*) - \lambda^k(\sigma)).$$

Proof. We have

$$\begin{aligned}\mathcal{L}_0(y) - \mathcal{L}_0(y^*) &= r^{-1}l(y) + r^{-1}l(y^*) - |y'| + |y'^*| \\ &\geq r^{-1}|y_n^* + b_k y_0^*|(\lambda^k(\sigma^*) - \lambda^k(\sigma)) - |y_n^* - y_n|.\end{aligned}$$

Here $y_n = (1 - b_k\tau)(y_n + b_k y_0)$, and

$$|y_n^* - y_n| \leq |b_k(\tau^* - \tau)(y_n^* + b_k y_0^*)| \leq M|y_n^* + b_k y_0^*|(\lambda^k(\sigma^*) - \lambda^k(\sigma)).$$

Thus we obtain (i). Statement (ii) is trivial. □

Remark. Thus far we have assumed that each of $R_1, \ldots, R_m$ is either R or 0. From now on, we assume $R_m = 0$ and $R_k = R$ for $k \neq m$. Correspondingly, we modify the definition of Ω_2 as follows:

$$\Omega_2 = \{y \in \mathbb{C}^{n+1};\ y_n + my_0 \neq 0,\ |y_n + b_j y_0| > R|y_0|\ (j \neq m)\}.$$

§6. Function spaces

In this section, we define some function spaces which we shall use to solve the Cauchy problem. We define

$$\tilde{\Omega}_2 = \{y \in \mathcal{R}(\Omega_2);\ \mathcal{L}_1(y) > 0,\ \lambda_M(y) < M^2\}.$$

Remark. (i) We denote the universal covering space of Ω_2 by $\mathcal{R}(\Omega_2)$, and $\tilde{\Omega}_2$ is a subset of $\mathcal{R}(\Omega_2)$.

(ii) If we consider finitely many branches of the solution, we may assume that the corresponding value of λ_M is less than $2M$, and thus the corresponding point $\tilde{y}$ belongs to $\tilde{\Omega}_2$ if $\mathcal{L}_1(y) > 0$.

Definition. Let $i, j \in \mathbb{N}$ and $f(y) \in \mathcal{O}(\tilde{\Omega}_2)$. We define

$$\|f\|_{1-q,1-q} = \sum_{\alpha' \in \mathbb{Z}_+^n} \frac{R^{|\alpha'|}}{\alpha'!} \sup_{y \in \tilde{\Omega}_2} \bigl(|\partial_{y'}^{\alpha'} f(y)| \mathcal{L}_1(y)^{(\alpha_n - 1 + q)_+} \mathcal{L}_0(y)^{(|\alpha'''| - 1 + q)_+}\bigr),$$

$$\|f\|_{i-q,\,j-q} = \sum_{\substack{|\alpha'| \le i-1 \\ |\beta'''| \le j-1}} \frac{1}{\alpha'!\beta'''!} \|\partial_{y'}^{\alpha'} \partial_{y'''}^{\beta'''} f\|_{1-q,1-q}.$$

Here we have set $\alpha' = (\alpha''', \alpha_n) = (\alpha_1, \dots, \alpha_{n-1}, \alpha_n)$ and $s_+ = \max(s, 0)$. Finally, we denote by $\mathcal{O}^{i-q,\,j-q}(\tilde{\Omega}_2)$ the set of holomorphic functions on $\tilde{\Omega}_2$ which satisfy $\|f\|_{i-q,\,j-q} < \infty$.

Remark. We have $\mathcal{O}^{i+1-q,\,j-q}(\tilde{\Omega}_2) \subset \mathcal{O}^{i-q,\,j+1-q}(\tilde{\Omega}_2) \subset \mathcal{O}^{i-q,\,j-q}(\tilde{\Omega}_2)$. If $f(x') \in \mathcal{O}^{i-q}(\mathcal{R}(\Omega^{\rm o} \setminus Z))$ and $R > 0$ is small, then $f(x_1, \dots, x_{n-1}, x + b_k x_0) \in \mathcal{O}^{i-q,2-q}(\tilde{\Omega}_2)$, after modifying r if necessary. If $f(x) \in \mathcal{O}^{i-q,j-q}(\tilde{\Omega}_2)$, then

$$|\partial_{y'}^{\alpha'} f| \le \|f\|_{i-q,\,j-q} R^{-|\alpha'|} \alpha'! \mathcal{L}_1(y)^{-(\alpha_n - i + q)_+} \mathcal{L}_0(y)^{-(|\alpha'''| - j + q)_+}$$

on $\tilde{\Omega}_2$. Therefore $f(y)$ is Hölder continuous of exponent $i - q$ along $\{y_n = 0\}$. The second index $j - q$ does not have an important meaning.

We explain some properties of these function spaces.

Lemma 10. *If $f, g \in \mathcal{O}^{i-q,\,j-q}(\tilde{\Omega}_2)$, then $fg \in \mathcal{O}^{i-q,\,j-q}(\tilde{\Omega}_2)$ and*

$$\|fg\|_{i-q,\,j-q} \le \|f\|_{i-q,\,j-q} \|g\|_{i-q,\,j-q}, \tag{23}$$

$$\|fg\|_{i-q,\,j-q} \le \sum_{\substack{i_1+i_2=i+1 \\ j_1+j_2=j+1}} \|f\|_{i_1-q,\,j_1-q} \|g\|_{i_2-q,\,j_2-q}. \tag{24}$$

Proof. By definition we have

$$\|fg\|_{1-q,1-q} = \sum_{\alpha', \beta' \in \mathbb{Z}_+^n} \frac{R^{|\alpha'|+|\beta'|}}{\alpha'!\beta'!} \sup_{y \in \tilde{\Omega}_2} \bigl(|\partial_{y'}^{\alpha'} f \partial_{y'}^{\beta'} g| \mathcal{L}_1^{(\alpha_n + \beta_n - 1 + q)_+} \mathcal{L}_0^{(|\alpha''' + \beta'''| - 1 + q)_+}\bigr).$$

From $(\alpha_n + \beta_n - 1 + q)_+ \ge (\alpha_n - 1 + q)_+ + (\beta_n - 1 + q)_+$ it follows that $\|fg\|_{1-q,1-q} \le \|f\|_{1-q,1-q} \|g\|_{1-q,1-q}$. If $i, j \in \mathbb{N}$, then

$$\|fg\|_{i-q,\,j-q} \le \sum_{\substack{|\alpha'+\beta'|\le i-1\\ |\gamma'''+\delta'''|\le j-1}} \frac{1}{\alpha'!\beta'!\gamma'''!\delta'''!}\|\partial_{y'}^{\alpha'}\partial_{y'''}^{\gamma'''}f\|_{1-q,1-q}\|\partial_{y'}^{\beta'}\partial_{y'''}^{\delta'''}g\|_{1-q,1-q}$$

$$\le \sum_{\substack{i_1+i_2=i+1\\ j_1+j_2=j+1}} \sum_{\substack{|\alpha'|\le i_1-1,\,|\beta'|\le i_2-1\\ |\gamma'''|\le j_1-1,\,|\delta'''|\le j_2-1}} \frac{1}{\alpha'!\beta'!\gamma'''!\delta'''!}\|\partial_{y'}^{\alpha'}\partial_{y'''}^{\gamma'''}f\|_{1-q,1-q}\|\partial_{y'}^{\beta'}\partial_{y'''}^{\delta'''}g\|_{1-q,1-q}$$

$$\le \sum_{\substack{i_1+i_2=i+1\\ j_1+j_2=j+1}} \|f\|_{i_1-q,\,j_1-q}\|g\|_{i_2-q,\,j_2-q}.$$

This proves (24). The proof of (23) is similar. □

Lemma 11. *Let $N \in \mathbb{N}$ and $X \subset \mathbb{C}^N$ be a connected and simply connected domain. Let $F(\zeta)$ be a holomorphic function defined for $\zeta = (\zeta_1, \dots, \zeta_N) \in X$. Assume that $f_1(y), \dots, f_N(y) \in \mathcal{O}(\tilde{\Omega}_2)$ satisfy $(f_1(y), \dots, f_N(y)) \in X$ for any $y \in \tilde{\Omega}_2$. Then $g(y) = F(f_1(y), \dots, f_N(y))$ satisfies*

$$\|g\|_{1-q,1-q} \le \sum_{\Gamma\in\mathbb{Z}_+^N} \frac{1}{\Gamma!}\sup_{\zeta\in X}|\partial_\zeta^\Gamma F(\zeta)| \cdot \prod_{1\le k\le N}(\|f_k\|_{1-q,1-q})^{\Gamma_k}.$$

Proof. Let $h \ge 0$ and fix $y^\circ \in X$. We denote the set of holomorphic functions defined in a neighborhood of y° by $\mathcal{O}_{\mathbb{C}^{n+1},y^\circ}$. We define $Y = \sum_{|\delta'|=h+1}(y-y^\circ)^{\delta'}\mathcal{O}_{\mathbb{C}^{n+1},y^\circ}$. We enumerate the elements of $\mathbb{Z}_+^n \setminus \{(0,\dots,0)\}$ as $\delta'(1), \delta'(2), \dots$ in such a way that $|\delta'(1)| \le |\delta'(2)| \le \cdots$. Then there exists $h' \in \mathbb{Z}$ such that $|\delta'(j)| \le h \Leftrightarrow 1 \le j \le h'$. We have

$$g(y_0^\circ, y') \equiv \sum_{\Gamma\in\mathbb{Z}_+^N}\left(\prod_{1\le k\le N}(f_k(y_0^\circ,y')-f_k(y^\circ))^{\Gamma_k}\right)\frac{\partial_\zeta^\Gamma F(f_1(y^\circ),\dots,f_N(y^\circ))}{\Gamma!} \equiv$$

$$\sum_{\substack{\Gamma^1,\dots,\Gamma^{h'}\\ \in\mathbb{Z}_+^n}}\left(\prod_{\substack{1\le k\le N\\ 1\le j\le h'}}\left(\frac{(y'-y'^\circ)^{\delta'(j)}}{\delta'(j)!}\partial_{y'}^{\delta'(j)}f_k(y^\circ)\right)^{\Gamma_k^j}\right)\frac{\partial_\zeta^{\Gamma^1+\cdots+\Gamma^{h'}}F(f_1(y^\circ),\dots,f_N(y^\circ))}{\Gamma^1!\cdots\Gamma^{h'}!}$$

modulo Y. It follows that if α' satisfies $|\alpha'| \le h$, then we have

$$\partial_{y'}^{\alpha'}g(y^\circ)/\alpha'! =$$

$$\sum_{\Gamma^1,\dots,\Gamma^{h'}\in A(\alpha')}\left(\prod_{\substack{1\le k\le N\\ 1\le j\le h'}}\left(\frac{1}{\delta'(j)!}\partial_{y'}^{\delta'(j)}f_k(y^\circ)\right)^{\Gamma_k^j}\right)\frac{\partial_\zeta^{\Gamma^1+\cdots+\Gamma^{h'}}F(f_1(y^\circ),\dots,f_N(y^\circ))}{\Gamma^1!\cdots\Gamma^{h'}!},$$

where $A(\alpha') = \{(\Gamma^1,\dots,\Gamma^{h'}) \in \mathbb{Z}_+^{N\times h'};\ |\Gamma^1|\delta'(1) + \cdots + |\Gamma^{h'}|\delta'(h') = \alpha'\}$. From now on, we identify an element $(\Gamma^1,\dots,\Gamma^{h'})$ of $A(\alpha')$ with the infinite sequence

$\Gamma^\infty = (\Gamma^1, \dots, \Gamma^{h'}, (0,\dots,0), (0,\dots,0), \dots)$ of elements belonging to $\mathbb{Z}_+^N$. We define $\Gamma^\infty ! = \Gamma^1! \cdots \Gamma^{h'}!$ and $\partial_\zeta^{\Gamma^\infty} F = \partial_\zeta^{\Gamma^1+\cdots+\Gamma^{h'}} F$. Then for each $\alpha' \in \mathbb{Z}_+^n$,

$$\begin{aligned}
&\frac{R^{|\alpha'|}}{\alpha'!} |\partial_{y'}^{\alpha'} g(y)| \mathcal{L}_1(y)^{(\alpha_n - 1+q)_+} \mathcal{L}_0(y)^{(|\alpha'''|-1+q)_+} \\
&\qquad \le \sum_{\Gamma^\infty \in A(\alpha')} \frac{|\partial_\zeta^{\Gamma^\infty} F(f_1(y), \dots, f_N(y))|}{\Gamma^\infty !} \\
&\qquad\quad \cdot \Bigg(\prod_{\substack{1\le k\le N\\ 1\le j\le h'}} \bigg(\frac{R^{|\delta'(j)|}}{\delta'(j)!} |\partial_{y'}^{\delta'(j)} f_k(y)| \mathcal{L}_1(y)^{(\delta_n(j)-1+q)_+} \mathcal{L}_0(y)^{(|\delta'''(j)|-1+q)_+} \bigg)^{\Gamma_k^j} \Bigg).
\end{aligned}$$

Considering the supremum over $\tilde\Omega_2$ and the summation for $\alpha' \in \mathbb{Z}_+^n$, we obtain the lemma. □

Lemma 12. *We have*

$$\|\partial_{y'}^{\alpha'} \partial_{y'''}^{\beta'''} f\|_{i-q,\, j-q} \le 2^{|\alpha'|+|\beta'''|+i+j} \alpha'! \beta'''! \|f\|_{i+|\alpha'|-q,\, j+|\beta'''|-q}.$$

Proof. By definition we have

$$\begin{aligned}
\|\partial_{y'}^{\alpha'} \partial_{y'''}^{\beta'''} f\|_{i-q,\, j-q} &\le \sum_{\substack{|\gamma'|\le i-1\\ |\delta'''|\le j-1}} \frac{1}{\gamma'!\delta'''!} \|\partial_{y'}^{\alpha'+\gamma'} \partial_{y'''}^{\beta'''+\delta'''} f\|_{i-q,\, j-q} \\
&\le \sum_{\substack{|\gamma'|\le i-1\\ |\delta'''|\le j-1}} \frac{2^{|\alpha'+\gamma'|+|\beta'''+\delta'''|} \alpha'! \beta'''!}{(\alpha'+\gamma')!(\beta'''+\delta''')!} \|\partial_{y'}^{\alpha'+\gamma'} \partial_{y'''}^{\beta'''+\delta'''} f\|_{1-q, 1-q} \\
&\le 2^{|\alpha'|+|\beta'''|+i+j} \alpha'! \beta'''! \|f\|_{i+|\alpha'|-q,\, j+|\beta'''|-q}.
\end{aligned}$$ □

We next consider $\Lambda_k = \partial_{y_0} - b_k \partial_{y_n}$ for $1 \le k \le m$. If $u, f \in \mathcal{O}(\tilde\Omega_2)$ satisfy $\Lambda_k u = f$, $u(0, y') = 0$, we denote $u = \Lambda_k^{-1} f$. Then

$$\Lambda_k \Lambda_k^{-1} f = f, \qquad \Lambda_k^{-1} \Lambda_k u = u(y) - u(0, y''', y_n + b_k y_0), \qquad \partial_{y'}^{\alpha'} \Lambda_k^{-1} f = \Lambda_k^{-1} \partial_{y'}^{\alpha'} f.$$

Let $y^* \in \Omega_2$, $\sigma^* = y_0^*/y_n^*$, and let γ be a canonical curve from 0 to σ^*. We denote its length by θ^*, and let $\gamma = \{\sigma(\theta);\, 0 \le \theta \le \theta^*\}$ be the arc length parametrization of γ. We denote $\tau(\theta) = \varphi^{k,m}(\sigma(\theta))$ and

$$\begin{aligned}
z^* &= \Psi^k(y^*) = (y_0^*, y_1^*, \dots, y_{n-1}^*, y_n^* + b_k y_0^*), \\
z(\theta) &= (\tau(\theta) z_n^*, z_1^*, \dots, z_{n-1}^*, z_n^*), \\
y(\theta) &= (\Psi^k)^{-1}(z(\theta)) = \big(\tau(\theta) z_n^*, z_1^*, \dots, z_{n-1}^*, (1 - b_k \tau(\theta)) z_n^*\big).
\end{aligned}$$

We can regard $y(\theta)$ and y^* as points of $\mathcal{R}(\Omega_2)$. Then $\Lambda_k^{-1} f(y^*) = \int_0^\theta f(y(\theta)) z_n^* \, d\tau(\theta)$ for $1 \le k \le m$. We have the following result.

Lemma 13. (i) *If* $1 \le k \le m$, *then* $\|\Lambda_k^{-1} f\|_{1-q,2-q} \le r^{1/2}\|f\|_{1-q,1-q}$.
(ii) *If* $k \ne m$, *then* $\|\Lambda_k^{-1} f\|_{2-q,1-q} \le R^{-3}\|f\|_{1-q,1-q}$.

Proof. We define

$$\begin{aligned} C_{\alpha'} &= \sup_{y\in\tilde{\Omega}_2}\big(|\partial_{y'}^{\alpha'} f(y)|\mathcal{L}_1(y)^{(\alpha_n-1+q)_+}\mathcal{L}_0(y)^{(|\alpha'''|-1+q)_+}\big),\\ C_{\alpha'}^j &= \sup_{y\in\tilde{\Omega}_2}\big(|\partial_{y_j}\partial_{y'}^{\alpha'} f(y)|\mathcal{L}_1(y)^{(\alpha_n-1+q)_+}\mathcal{L}_0(y)^{(|\alpha'''|+q)_+}\big), \quad 1\le j\le n-1,\\ C_{\alpha'}^n &= \sup_{y\in\tilde{\Omega}_2}\big(|\partial_{y_n}\partial_{y'}^{\alpha'} f(y)|\mathcal{L}_1(y)^{(\alpha_n+q)_+}\mathcal{L}_0(y)^{(|\alpha'''|-1+q)_+}\big). \end{aligned}$$

To prove (i), it suffices to show

$$\|\partial_{y'''}^{\beta'''}\Lambda_k^{-1} f\|_{1-q,1-q} \le r^{3/4}\|f\|_{1-q,1-q}, \quad |\beta'''|\le 1,\ 1\le k\le m. \tag{25}$$

We only give the proof for $|\beta'''| = 1$ (the case $|\beta'''| = 0$ is easier). We assume $1 \le k \le m$, $1 \le j \le n-1$. Then

$$\|\partial_{y_j}\Lambda_k^{-1} f\|_{1-q,1-q} \le \sum_{\alpha'}\frac{R^{|\alpha'|}}{\alpha'!}C''_{\alpha'},$$

where

$$C''_{\alpha'} = \sup_{y^*\in\tilde{\Omega}_2}\left(|z_n^*|\mathcal{L}_1(y^*)^{(\alpha_n-1+q)_+}\mathcal{L}_0(y^*)^{(|\alpha'''|-1+q)_+}\int_0^\theta |\partial_{y_j}\partial_{y'}^{\alpha'} f(y(\theta))|\,|d\tau(\theta)|\right).$$

Using Proposition 4, we obtain

$$\begin{aligned} |\partial_{y_j}\partial_{y'}^{\alpha'} f)y(\theta))| &\le C_{\alpha'}^j\mathcal{L}_1(y(\theta))^{-(\alpha_n-1+q)_+}\mathcal{L}_0(y(\theta))^{-|\alpha'''|-q}\\ &\le C_{\alpha'}^j\mathcal{L}_1(y^*)^{-(\alpha_n-1+q)_+}\left(\mathcal{L}_0(y^*)+\frac{|z_n^*|}{2r}\big(\lambda^k(\sigma^*)-\lambda^k(\sigma(\theta))\big)\right)^{-|\alpha'''|-q}. \end{aligned}$$

It follows that

$$C''_{\alpha'} \le \sup_{y^*\in\tilde{\Omega}_2}\big(C_{\alpha'}^j|z_n^*|\mathcal{L}_0(y^*)^{(|\alpha'''|-1+q)_+}I(y^*)\big),$$

where

$$I(y^*) = \int_0^\theta\left(\mathcal{L}_0(y^*)+\frac{|z_n^*|}{2r}\big(\lambda^k(\sigma^*)-\lambda^k(\sigma(\theta))\big)\right)^{-|\alpha'''|-q}|d\tau(\theta)|.$$

By a direct calculation, we obtain

$$I(y^*) \le \begin{cases} \dfrac{2r}{(1-q)|z_n^*|}\left(\mathcal{L}_0(y^*)+\dfrac{|z_n^*|}{2r}\lambda^k(\sigma^*)\right)^q \le \dfrac{2r}{1-q}, & |\alpha'''| = 0,\\ \dfrac{2r}{(|\alpha'''|+1-q)|z_n^*|}\mathcal{L}_0(y^*)^{-|\alpha'''|+1-q}, & |\alpha'''|\ge 1. \end{cases}$$

Anyway, we have

$$I(y^*) \le \frac{2r}{(|\alpha'''|+1)(1-q)|z_n^*|\mathcal{L}_0(y^*)^{(|\alpha'''|-1+q)_+}},$$

and

$$C''_{\alpha'} \le \frac{2rC^j_{\alpha'}}{(1-q)(|\alpha'''|+1)}.$$

This means

$$\|\partial^{\beta'''}_{y'''}\Lambda_k^{-1}f\|_{1-q,1-q} \le \frac{2r}{R(1-q)}\|f\|_{1-q,1-q},$$

and we obtain (25) for $|\beta'''|=1$.

To prove (ii), it suffices to show

$$\|\partial_{y_n}\Lambda_k^{-1}f\|_{1-q,1-q} \le \frac{1}{qR^2}\|f\|_{1-q,1-q}\ , \quad k \ne m. \tag{26}$$

We have

$$\|\partial_{y_n}\Lambda_k^{-1}f\|_{1-q,1-q} \le \sum_{\alpha'}\frac{R^{|\alpha'|}}{\alpha'!}C'''_{\alpha'},$$

where

$$C'''_{\alpha'} = \sup_{y^*\in\tilde{\Omega}_2}\left(|z_n^*|\mathcal{L}_1(y^*)^{(\alpha_n-1+q)_+}\mathcal{L}_0(y^*)^{(|\alpha'''|-1+q)_+}\int_0^\theta |(\partial_{y_n}\partial^{\alpha'}_{y'}f)(y(\theta))|\,|d\tau(\theta)|\right).$$

Using Proposition 4, we obtain

$$\begin{aligned}|(\partial_{y_n}\partial^{\alpha'}_{y'}f)(y(\theta))| &\le C^n_{\alpha'}\mathcal{L}_1(y(\theta))^{-\alpha_n-q}\mathcal{L}_0(y(\theta))^{-(|\alpha'''|-1+q)_+}\\ &\le C^n_{\alpha'}\big(\mathcal{L}_1(y^*)+R|z_n^*|(\lambda^k(\sigma^*)-\lambda^k(\sigma(\theta)))\big)^{-\alpha_n-1+q}\mathcal{L}_0(y^*)^{(|\alpha'''|-1+q)_+}.\end{aligned}$$

It follows that

$$C'''_{\alpha'} \le \sup_{y^*\in\tilde{\Omega}_2}\big(C^n_{\alpha'}|z_n^*|\mathcal{L}_1(y^*)^{(\alpha_n-1+q)_+}J(y^*)\big),$$

where

$$J(y^*) = \int_0^\theta \big(\mathcal{L}_1(y^*)+R|z_n^*|(\lambda^k(\sigma^*)-\lambda^k(\sigma(\theta)))\big)^{-\alpha_n-1+q}\,|d\tau(\theta)|.$$

As above, we have

$$J(y^*) \le \frac{\mathcal{L}_1(y^*)^{-(\alpha_n-1+q)_+}}{(\alpha_n+1)R(1-q)|z_n^*|},$$

and thus

$$C'''_{\alpha'} \le \frac{C^n_{\alpha'}}{R(1-q)(\alpha_n+1)}.$$

Therefore

$$\|\partial_{y_n}\Lambda_k^{-1}f\|_{1-q,1-q} \le \frac{1}{R^2 q}\|f\|_{1-q,1-q}.$$

This proves (26). □

Corollary. (i) *If* $1 \le k \le m$, *then* $\|\Lambda_k^{-1}f\|_{i-q,\,j+1-q} \le r^{1/2}\|f\|_{i-q,\,j-q}$.
(ii) *If* $k \ne m$, *then* $\|\Lambda_k^{-1}f\|_{i+1-q,\,j-q} \le R^{-3}\|f\|_{i-q,\,j-q}$.

Proof. We have

$$\begin{aligned}
\|\Lambda_k^{-1}f\|_{i-q,\,j+1-q} &= \sum_{\substack{|\alpha'|\le i-1\\ |\beta'''|\le j}} \frac{1}{\alpha'!\beta'''!}\|\partial_{y'}^{\alpha'}\partial_{y'''}^{\beta'''}\Lambda_k^{-1}f\|_{1-q,1-q}\\
&\le \sum_{\substack{|\alpha'|\le i-1\\ |\beta'''|\le j-1}} \frac{1}{\alpha'!\beta'''!}\|\Lambda_k^{-1}(\partial_{y'}^{\alpha'}\partial_{y'''}^{\beta'''}f)\|_{1-q,2-q}\\
&\le r^{1/2}\sum_{\substack{|\alpha'|\le i-1\\ |\beta'''|\le j-1}} \frac{1}{\alpha'!\beta'''!}\|\partial_{y'}^{\alpha'}\partial_{y'''}^{\beta'''}f\|_{1-q,1-q} = r^{1/2}\|f\|_{i-q,\,j-q},
\end{aligned}$$

and we obtain (i). We can prove (ii) similarly. □

Lemma 14. *We have* $\|f(y)-f(0)\|_{1-q,1-q} \le r^{q/2}(\|f\|_{2-q,1-q} + \|\Lambda_m f\|_{1-q,1-q})$.

Proof. We can write $f(y)-f(0) = f_1(y)+f_2(y)$, where

$$\begin{aligned}
f_1(y) &= f(y) - f(0, y_1, \dots, y_{n-1}, y_n + b_m y_0),\\
f_2(y) &= f(0, y_1, \dots, y_{n-1}, y_n + b_m y_0) - f(0).
\end{aligned}$$

We have $f_1(y) = \Lambda_m^{-1}\Lambda_m f$. From Lemma 13 it follows that

$$|f_1(y)| \le \|f_1\|_{1-q,1-q} \le r^{1/2}\|\Lambda_m f\|_{1-q,1-q}.$$

On the other hand,

$$|f_2(y)| = \left|\int_0^1 \frac{d}{d\theta} f(0,\theta y')\,d\theta\right| \le \sum_{1\le j\le n} |y_j| \int_0^1 |\partial_{y_j} f(0,\theta y')|\,d\theta \le nr\|f\|_{2-q,1-q}.$$

It follows that

$$|f(y)-f(0)| \le |f_1(y)| + |f_2(y)| \le nr\|f\|_{2-q,1-q} + r^{1/2}\|\Lambda_m f\|_{1-q,1-q}.$$

We have

$$\begin{aligned}
&\|f(y)-f(0)\|_{1-q,1-q}-\sup_{y\in\tilde\Omega_2}|f(y)-f(0)|\\
&=\sum_{|\alpha'|\ge 1}\frac{R^{|\alpha'|}}{\alpha'!}\sup_{y\in\tilde\Omega_2}\big(|\partial_{y'}^{\alpha'}(f(y)-f(0))|\mathcal{L}_1(y)^{(\alpha_n-1+q)_+}\mathcal{L}_0(y)^{(|\alpha'''|-1+q)_+}\big)\\
&\le\sum_{|\alpha'|\ge 0}\frac{R^{|\alpha'|+1}}{\alpha'!(\alpha_n+1)}\sup_{y\in\tilde\Omega_2}\big(|\partial_{y'}^{\alpha'}\partial_{y_n}f(y)|\mathcal{L}_1(y)^{\alpha_n+q}\mathcal{L}_0(y)^{(|\alpha'''|-1+q)_+}\big)\\
&\quad+\sum_{\substack{|\alpha'|\ge 0\\1\le k\le n-1}}\frac{R^{|\alpha'|+1}}{\alpha'!(\alpha_k+1)}\sup_{y\in\tilde\Omega_2}\big(|\partial_{y'}^{\alpha'}\partial_{y_k}f(y)|\mathcal{L}_1(y)^{(\alpha_n-1+q)_+}\mathcal{L}_0(y)^{|\alpha'''|+q}\big).
\end{aligned}$$

Moreover,

$$\mathcal{L}_1(y)^{\alpha_n+q}\le r^q\mathcal{L}_1(y)^{(\alpha_n-1+q)_+}\quad\text{and}\quad\mathcal{L}_0(y)^{(|\alpha'''|+q)_+}+\le r^q\mathcal{L}_0(y)^{(|\alpha'''|-1+q)_+}.$$

It follows that

$$\begin{aligned}
\|f(y)-f(0)\|_{1-q,1-q}&\le(r^{1/2}+nr+nRr^q)(\|f\|_{2-q,1-q}+\|\Lambda_m f\|_{1-q,1-q})\\
&\le r^{q/2}(\|f\|_{q+1,q}+\|\Lambda_m f\|_{1-q,1-q}).
\end{aligned}\qquad\square$$

Lemma 14 means that if $f\in\mathcal{O}^{2-q,1-q}(\tilde\Omega_2)$, $\Lambda_m f\in\mathcal{O}^{1-q,1-q}(\tilde\Omega_2)$ and $f(0)=0$, then $\|f(y)\|_{1-q,1-q}$ is small.

§7. Construction of the solution

We denote $v(y)=(v_1(y),v_2(y))=(u(y),x_n(y))$. We define

$$\mathcal{T}(\tilde\Omega_2)=\{f(y)\in\mathcal{O}^{m-q,2-q}(\tilde\Omega_2);\ \partial_{y_0}^l f\in\mathcal{O}^{m-l-q,2-q}(\tilde\Omega_2),\ 0\le l\le m-1\}.$$

Proposition 1 and Lemma 2 mean that it suffices to consider the following initial value problem for $v(y)\in\mathcal{T}(\tilde\Omega_2)\times\mathcal{T}(\tilde\Omega_2)$:

$$\begin{cases}\Lambda v_j(y)=f_j(v), & 1\le j\le 2,\\ \partial_{y_0}^l v_j(0,y')=v_{jl}(y')\in\mathcal{O}^{m-l-q,2-q}(\tilde\Omega_2), & 1\le j\le 2,\ 0\le l\le m-1.\end{cases}\tag{27}$$

Here we have denoted

$$f_j(v)=-\sum_{\substack{1\le k\le 2\\|\alpha|=m}}Q_{j,k,\alpha}(v)\partial_y^\alpha v_k+g_j(v)$$

for some $Q_{j,k,\alpha} \in \mathcal{P}^0$ and $g_j \in \mathcal{Q}^{m-1}$. They satisfy

$$\text{(28)} \quad \begin{cases} Q_{1,1,\alpha} = 0, & \alpha = (m, 0, \ldots, 0) \text{ or } (0, \ldots, 0, m), \\ [Q_{1,1,\alpha}(y, u(y), \nabla_y^1 x_n(y))]_{y=0} = 0, & \alpha_0 + \alpha_n = m, \\ Q_{1,2,\alpha} = Q_{2,1,\alpha} = Q_{2,2,\alpha} = 0, & \alpha_0 + \alpha_n = m. \end{cases}$$

We solve (27) by successive approximation. For $i \geq 0$, we consider the following initial value problem:

$$\text{(29)} \quad \begin{cases} \Lambda v_j^{(i)}(y) = f_j^{(i)}(y), & 1 \leq j \leq 2, \\ \partial_{y_0}^l v_j^{(i)}(0, y') = v_{jl}(y'), & 1 \leq j \leq 2,\ 0 \leq l \leq m-1. \end{cases}$$

Here we have denoted

$$f_j^{(i)}(y) = \begin{cases} 0 & i = 0, \\ f_j(v^{(i-1)}), & i \geq 1. \end{cases}$$

If $i = 0$, we can find the solution of (29) in the form

$$\text{(30)} \quad v_j^{(0)}(y) = \sum_{1 \leq k \leq m} v_{jk}^{(0)}(y_1, \ldots, y_{n-1}, y_n + b_k y_0)$$

for some $v_{jk}^{(0)}(y')$. In fact, substituting (30) into (29), we have $\sum_{1 \leq k \leq m} b_k^l \partial_{y_n}^l v_{jk}^{(0)}(y') = v_{jl}(y')$. It suffices to define $v_{jk}^{(0)}(y')$ by

$$\begin{pmatrix} 1 & \cdots & 1 \\ b_1 & \cdots & b_m \\ \vdots & & \vdots \\ b_1^{m-1} & \cdots & b_m^{m-1} \end{pmatrix} \begin{pmatrix} v_{j1}^{(0)}(y') \\ \vdots \\ v_{jm}^{(0)}(y') \end{pmatrix} = \begin{pmatrix} v'_{j0}(y') \\ \vdots \\ v'_{j,m-1}(y') \end{pmatrix}.$$

Here we have set

$$v'_{j0}(y') = v_{j0}(y') \quad \text{and} \quad v'_{jl}(y') = \int_0^{y_n} \frac{(y_n - t)^{l-1}}{(l-1)!}\, dt$$

for $1 \leq l \leq m-1$. These functions $v_{jk}^{(0)}(y')$ belong to $\mathcal{T}$, and we have $\partial_{y_0}^h v_j^{(0)}(0) = \partial_{y_0}^h v_j(0)$ for $0 \leq h \leq m-1$. If $i \geq 1$, we define $\bar{v}_j^{(i)} = v_j^{(i)} - v_j^{(i-1)}$ and $\bar{f}_j^{(i)} = f_j^{(i)} - f_j^{(i-1)}$. Then we need to solve

$$\text{(31)} \quad \begin{cases} \Lambda \bar{v}_j^{(i)}(y) = \bar{f}_j^{(i)}(y), & 1 \leq j \leq 2, \\ \partial_{y_0}^l v_j^{(i)}(0, y') = 0, & 1 \leq j \leq 2,\ 0 \leq l \leq m-1, \end{cases}$$

and the solution is $\bar{v}_j^{(i)}(y) = \Lambda_1^{-1} \cdots \Lambda_m^{-1} \bar{f}_j^{(i)}(y)$. We define $r_1 = r^{q(1-q)/6}$ $(\geq r^{1/24})$. We have the following result.

Proposition 5. *If $i \geq 1$ and $0 \leq h \leq m-1$, then*

$$\|\partial_{y_0}^h \bar{v}_j^{(i)}\|_{m-h-q,1-q} \leq r_1^{2i+3j-2}, \quad \|\partial_{y_0}^h \bar{v}_j^{(i)}\|_{m-h-q,2-q} \leq r_1^{2i-3j-6}.$$

Proof. We define

$$\begin{aligned}
f_{j0}(v) &= -\sum_{\alpha \in X_1} Q_{j,j,\alpha}(v)\partial_{y_0}^{\alpha_0-1}\partial_{y'}^{\alpha'} v_1,\\
f_{j1}(v) &= \partial_{y_0} f_{j0}(v),\\
f_{j2}(v) &= -\sum_{\substack{1\leq k\leq 2\\ \alpha\in X_2}} Q_{j,k,\alpha}(v)\partial_y^\alpha v_k,\\
f_{j3}(v) &= \sum_{\alpha\in X_1} \partial_{y_0} Q_{j,j,\alpha}(v)\cdot \partial_{y_0}^{\alpha_0-1}\partial_{y'}^{\alpha'} v_j + g_j(v),
\end{aligned}$$

where $X_1 = \{\alpha \in \mathbb{Z}_+^{n+1};\ \alpha_0+\alpha_n = m,\ 1 \leq \alpha_0 \leq m-1,\ |\alpha'''| = 0\}$ and $X_2 = \{\alpha \in \mathbb{Z}_+^{n+1};\ |\alpha| = m,\ |\alpha'''| \neq 0\}$. It is easy to see that $f_j(v) = \sum_{1\leq j'\leq 3} f_{jj'}(v)$. If $i \geq 1$ and $0 \leq j \leq 3$, we define

$$f_{jj'}^{(i)}(y) = f_{jj'}(v^{(i-1)}), \quad \bar{f}_{jj'}^{(i)}(y) = f_{jj'}^{(i)} - f_{jj'}^{(i-1)} \quad (f_{jj'}^{(0)}(y) = 0).$$

We define $\bar{v}_{jj'}^{(i)}(y) = \Lambda_1^{-1}\cdots\Lambda_m^{-1}\bar{f}_{jj'}^{(i)}(y)$ for $i \geq 1$ and $1 \leq j \leq 3$.

The proof of Proposition 5 consists of two steps:

STEP 1. We prove Proposition 5 for $i = 1$.

STEP 2. We assume that $i_0 \geq 2$ and Proposition 5 is true if $1 \leq i \leq i_0 - 1$. We prove the case of $i = i_0$ under this assumption.

These steps are very similar, and we only explain Step 2. By the assumption of induction, we have $|\partial_y^\alpha v_j^{(i)}(y) - \partial_y^\alpha v_j(0)| \ll 1$ on $\tilde{\Omega}_2$ if $0 \leq i \leq i_0-1$, $|\alpha| \leq m-1$. Therefore $Q_{j,k,\alpha}(v^{(i)}(y))$ and $g_i(v^{(i)})$ are well-defined.

Let $i = i_0$. We first show

$$\|\bar{f}_{j0}^{(i)}\|_{1-q,1-q} \leq 2M^3 r_1^{2i+3j}. \tag{32}$$

If $j = 2$, then $f_{j0} = 0$, and thus we may assume $j = 1$. We define

$$\begin{aligned}
\bar{f}_{j00}^{(i)} &= -\sum_{\alpha\in X_1} Q_{j,j,\alpha}(v^{(i-2)})\partial_{y_0}^{\alpha_0-1}\partial_{y'}^{\alpha'}\bar{v}_j^{(i-1)},\\
\bar{f}_{j01}^{(i)} &= -\sum_{\alpha\in X_1} (Q_{j,j,\alpha}(v^{(i-1)}) - Q_{j,j,\alpha}(v^{(i-2)}))\partial_{y_0}^{\alpha_0-1}\partial_{y'}^{\alpha'} v_j^{(i-1)}.
\end{aligned}$$

Then $\bar{f}_{j0}^{(i)} = f_{j0}(v^{(i)}) - f_{j0}(v^{(i-1)}) = \bar{f}_{j00}^{(i)} + \bar{f}_{j01}^{(i)}$. We prove the following inequality for $j' = 0, 1$:

$$\|\bar{f}_{j0j'}^{(i)}\|_{1-q,1-q} \leq M^3 r_1^{2i+3j}. \tag{33}$$

We first consider $\bar{f}^{(i)}_{j00}$. We can prove

$$\|\partial^h_{y_0} Q_{j,j,\alpha}(v^{(i-2)})\|_{2-h-q,1-q} \le nM^2 \quad \text{for } h = 0, 1. \tag{34}$$

In fact, from Lemma 11 we first obtain $\|Q_{j,j,\alpha}(v^{(i-2)})\|_{1-q,1-q} \le M$. Furthermore,

$$\partial_{y_k} Q_{j,j,\alpha}(v^{(i-2)}) = \sum_{\substack{|\beta|\le 1\\ 1\le l\le 2}} Q_{j,j,\alpha,\beta}(v^{(i-2)})\partial_{y_k}\partial^\beta_y v^{(i-2)}_l$$

for some $Q_{j,j,\alpha,\beta} \in \mathcal{P}^0$, and we obtain (34). Here $Q_{j,j,\alpha}(v^{(i-2)})$ vanishes when $y = 0$, and from Lemma 14 and (34) we have $\|Q_{j,j,\alpha}(v^{(i-2)})\|_{1-q,1-q} \le 2nM^3 r_1^3$. Therefore

$$\begin{aligned}\|\bar{f}^{(i)}_{j00}\|_{1-q,1-q} &\le \sum_{\alpha\in X_1} \|Q_{j,j,\alpha}(v^{(i-2)})\|_{1-q,1-q}\|\partial^{\alpha_0-1}_{y_0}\partial^{\alpha'}_{y'}\bar{v}^{(i-1)}_j\|_{1-q,1-q}\\ &\le M^4 r_1^{2i+3j}.\end{aligned}$$

This means (33) for $j' = 0$.

On the other hand,

$$\begin{aligned}&\|\bar{f}^{(i)}_{j01}\|_{1-q,1-q}\\ &\le \sum_{\alpha\in X_1} \|Q_{j,j,\alpha}(v^{(i-1)}) - Q_{j,j,\alpha}(v^{(i-2)})\|_{1-q,1-q}\|\partial^{\alpha_0-1}_{y_0}\partial^{\alpha'}_{y'} v^{(i-1)}_j\|_{1-q,1-q}\\ &\le \sum_{\alpha\in X_1} M\|Q_{j,j,\alpha}(v^{(i-1)}) - Q_{j,j,\alpha}(v^{(i-2)})\|_{1-q,1-q}.\end{aligned}$$

In the same way as above, $\|Q_{j,j,\alpha}(v^{(i-1)}) - Q_{j,j,\alpha}(v^{(i-2)})\|_{1-q,1-q} \le M^4 r_1^{2i+3j}$, and we get (33) for $j' = 1$ (and also (32)). We can similarly prove

$$\|\bar{f}^{(i)}_{j0}\|_{1-q,2-q} \le 3M^4 r_1^{2i+3j-5}. \tag{35}$$

We next prove

$$\|\bar{f}^{(i)}_{jj'}\|_{1-q,1-q} \le M^3 r_1^{2i+3j-8} \tag{36}$$

for $2 \le j' \le 3$. If $j' = 2$, we have

$$\begin{aligned}\|\bar{f}^{(i)}_{j2}\|_{1-q,1-q} \le &\sum_{\substack{1\le k\le 2\\ \alpha\in X_2}} \|Q_{j,k,\alpha}(v^{(i-1)}) - Q_{j,k,\alpha}(v^{(i-2)})\|_{1-q,1-q}\|\partial^\alpha_y v^{(i-1)}_j\|_{1-q,1-q}\\ &+ \sum_{\substack{1\le k\le 2\\ \alpha\in X_2}} \|Q_{j,k,\alpha}(v^{(i-2)})\|_{1-q,1-q}\|\partial^{\alpha_0}_{y_0}\partial^{\alpha'}_{y'}\bar{v}^{(i-1)}_j\|_{1-q,1-q}.\end{aligned}$$

We have $\alpha''' \neq 0$ for $\alpha \in X_2$, and it follows that

$$\begin{aligned}
&\|\bar{f}_{j2}^{(i)}\|_{1-q,1-q} \\
&\quad \leq \sum_{\substack{1\leq k\leq 2\\ \alpha\in X_2}} M \sum_{\substack{1\leq l\leq 2\\ 0\leq h\leq 1}} \|\partial_{y_0}^h \bar{v}_k^{(i-1)}\|_{m-h-q,1-q} + \sum_{\substack{1\leq k\leq 2\\ \alpha\in X_2}} M\|\partial_{y_0}^{\alpha_0}\bar{v}_j^{(i-1)}\|_{m-\alpha_0-q,2-q} \\
&\quad \leq M^3 r_1^{2i+3j-5}
\end{aligned}$$

Thus we obtain (36) for $j' = 2$, and similarly for $j' = 3$. We have

$$\begin{aligned}
\bar{v}_{j1}^{(i)} &= \Lambda_1^{-1}\cdots\Lambda_{m-1}^{-1}\bar{f}_{j0}^{(i)}(y) - \Lambda_1^{-1}\cdots\Lambda_{m-1}^{-1}\bar{f}_{j0}^{(i)}(0, y_1, \ldots, y_{n-1}, y_n), \\
\Lambda_1\cdots\Lambda_k\bar{v}_{j1}^{(i)} &= \Lambda_{k+1}^{-1}\cdots\Lambda_{m-1}^{-1}\bar{f}_{j0}^{(i)}(y) - \Lambda_{k+1}^{-1}\cdots\Lambda_{m-1}^{-1}\bar{f}_{j0}^{(i)}(0, y_1, \ldots, y_{n-1}, y_n)
\end{aligned}$$

for $0 \leq k \leq m-1$. Using (33) and Lemma 13, we get

$$\|\Lambda_1\cdots\Lambda_k\bar{v}_{j1}^{(i)}\|_{1-q,1-q} \leq 4M^3R^{-3m+3k}r_1^{2i+3j-1}.$$

Using (35) instead of (32), we obtain

$$\|\Lambda_1\cdots\Lambda_k\bar{v}_{j1}^{(i)}\|_{1-q,2-q} \leq 6M^3R^{-3m+3k}r_1^{2i+3j-5}.$$

Similarly we can prove the same estimates for $\bar{v}_{j2}^{(i)}$ and $\bar{v}_{j3}^{(i)}$, which completes the proof of Proposition 5. □

Corollary. *At each point $y \in \tilde{\Omega}_2$, we can define $v_j(y) = \lim_{j\to\infty} v_j^{(i)}(y)$. Then $\partial_{y_0}^h v_j \in \mathcal{O}^{m-h-q,2-q}(\tilde{\Omega}_2)$ for $1 \leq j \leq 2$, $0 \leq h \leq m-1$. Furthermore, if $|\alpha| \leq m-1$, then $|\partial_t^\alpha v_j(y) - \partial_t^\alpha v_j(0)| \ll 1$ on $\tilde{\Omega}_2$.*

We define

$$\begin{aligned}
\Omega_3 &= \{y \in \mathbb{C}^{n+1};\ |y| < r^3,\ |y_n + b_k y_0| > \sqrt{R_k}\,|y_0| \text{ for } 1 \leq k \leq m\}, \\
\tilde{\Omega}_3 &= \{\tilde{y} \in \mathcal{O}(\tilde{\Omega}_3);\ \lambda_M(\sigma) < M^2\}.
\end{aligned}$$

If $\tilde{y} \in \tilde{\Omega}_3$, then it is easy to see that $\tilde{y} \in \tilde{\Omega}_2$ and

$$\mathcal{L}_0(\tilde{t}) \geq r/2, \qquad \mathcal{L}_1(\tilde{t}) \geq \frac{e^{-2M^7}}{2}\min_{1\leq k\leq m}(|y_n + b_k y_0| - \sqrt{R_k}\,|y_0|).$$

Thus we obtain the following result.

Proposition 6. *If $1 \leq j \leq 2$, then $v_j \in \mathcal{O}(\tilde{\Omega}_3)$, and*

$$|\partial_y^\alpha v_j| \leq r^{-2}\Big(\min_{1\leq k\leq m}(|y_n + b_k y_0| - \sqrt{R_k}\,|y_0|)\Big)^{-(m-q-\alpha_0-\alpha_n)_+}, \quad |\alpha| \leq m.$$

§8. Characteristic hypersurfaces

In this section we transform the solution $u(y)$ of (27) to a function $u(x)$ of x satisfying (2). Let

$$\Omega_4 = \{x \in \mathbb{C}^{n+1};\ |x| < r^5,\ |x_n + a_k x_0| > M\sqrt{R}\,|x_0|,\ 1 \le k \le m\},$$
$$U_4 = \{x \in \mathbb{C}^{n+1};\ |x| < r^5,\ |x_n + a_m x_0| < 2M\sqrt{R}\,|x_0|\}.$$

In Section 1, we have denoted these domains by $\Omega_1(M\sqrt{R}, r^5)$ and $V_m(2M\sqrt{R}, r^5)$, respectively. We have $y'' = x''$ and we write $x(y) = (y'', x_n(y))$. We need to find the inverse function $x_n = x_n(y)$ of $y_n = y_n(x)$. We shall do this in two steps. First we argue in Ω_4, and next we refine this result in U_4. Note that $x_n(y) = v_2(y)$ is a solution of (6). We define

$$f(y) = \mu_m(y, u(y), \xi')|_{\xi' = (-\partial_{y_1} x_n(y), \dots, -\partial_{y_{n-1}} x_n(y), 1)},$$
$$g(y) = x_n(y) - y_n + a_m x_0.$$

Then $\partial_{y_0} g(y) = f(y) - f(0)$ and $g(0, y') = 0$. Hence $|g(y)| \le M^3 |(y_0, y_n)| \cdot |y|$.

Let $T(x) = \lambda_M(x_0/(x_n + a_m x_0))$. We first want to find the inverse function $x_n = x_n(y)$ on $\tilde{\Omega}_4 = \{\tilde{x} \in \mathcal{R}(\Omega_4);\ T(x) < 3M\}$. By definition,

$$y_n = x_n(y) + a_m x_0 - g(y), \tag{37}$$

and we want to define the inverse function $y_n = h(x)$ which satisfies

$$h(x) = x_n + a_m x_0 - g(x'', h(x)) \tag{38}$$

on $\tilde{\Omega}_4$. For this purpose, we define $h^{(i)}(x)$ inductively by

$$h^{(i)}(x) = x_n + a_m x_0 - g^{(i)}(x), \qquad i \ge 0, \tag{39}$$

where

$$g^{(i)}(x) = \begin{cases} 0, & i = 0, \\ g(x'', h^{(i-1)}(x)), & i \ge 1. \end{cases}$$

By induction on $i \ge 1$, we can easily prove that $h^{(i)}(x)$ is well-defined and satisfies

$$|h^{(i)}(x) - h^{(i-1)}(x)| \le (r^{-1}|x|)^i |y^{(0)}(x) + b_k x_0|, \qquad 1 \le k \le m, \tag{40}$$

on $\tilde{\Omega}_4$. Therefore, for each $x \in \tilde{\Omega}_4$ we can define $h(x) = \lim_{i\to\infty} h^{(i)}(x)$. It satisfies (38), and for each $x \in \tilde{\Omega}_4$ we have $(x'', h(x)) \in \tilde{\Omega}_3$. Substituting $y_n = h(x)$ to (37) and comparing it with (38), we have $x_n(x'', h(x)) = x_n$. Therefore we obtain an injection

$$\chi : \tilde{\Omega}_4 \ni x \mapsto (x'', h(x)) \in \tilde{\Omega}_3. \tag{41}$$

Remark. Let $1 \le k \le m$. From (39) and (40) we deduce that $|h^{(0)}(x) + b_k y_0| = |x_n + a_k x_0|$, and

$$|h(x) + b_k y_0| \ge |x_n + a_k x_0|/2 \ge 2\sqrt{R}\,|y_0| \tag{42}$$

on $\tilde{\Omega}_4$.

We next study the problem in the set U_4. We first define $U_3 = \{y \in \Omega_3; |y_n| < M^3 \cdot \sqrt{R}\,|y_0|\}$. Let $\pi_j : \mathcal{R}(\Omega_j) \to \Omega_j$ be the canonical projection, and $\tilde{U}_j = \pi_j^{-1}(U_j) \cap \tilde{\Omega}_j$ for $j = 3, 4$. Let $\tilde{x}^1 \in \tilde{U}_4$. We can easily prove $\chi(\tilde{U}_4) \subset \tilde{U}_3$, and thus $\tilde{y}^1 = \chi(\tilde{x}^1) \in \tilde{U}_3$.

We can regard $\mathcal{R}(U_3) \subset \mathcal{R}(\Omega_3)$. We next prove that we can define a function $\rho_0(x'')$ on $\mathcal{R}(U_4)$ and a mapping of $\{x \in \mathcal{R}(U_4); x_n \ne \rho_0(x'')\}$ into $\{y \in \mathcal{R}(U_3); y_n \ne 0\}$. Since $U_3 = \{y \in \mathbb{C}^{n+1}; |y| < r^3, 0 < |y_n| < M^2\sqrt{R}\,|y_0|\}$, we may identify $\tilde{y} \in \mathcal{R}(U_3)$ with $(y, \arg y_n, \arg y_0) \in U_3 \times \mathbb{R} \times \mathbb{R}$.

Let $(y, \arg y_n, \arg y_0) \in U_3 \times \mathbb{R} \times \mathbb{R}$. Then we have $(y'', \arg y_0) \in U_2 \times \mathbb{R}$ where $U_2 = \{y'' \in \mathbb{C}^n; |y''| < r^3, y_0 \ne 0\}$. We can identify $\tilde{y}'' \in \mathcal{R}(U_2)$ with $(y'', \arg y_0) \in U_2 \times \mathbb{R}$. Therefore if $\tilde{y} \in \mathcal{R}(U_3)$, we can fix $\tilde{y}'' \in \mathcal{R}(U_2)$ and let $y_n \to 0$. In this sense we define $\rho_0(\tilde{y}'') = \lim_{y_n \to 0} x_n(\tilde{y})$ for $\tilde{y}'' = \tilde{x}'' \in \mathcal{R}(U_2)$. It is easy to see $|\rho_0(y'')| \le M|y_0|$. We define

$$\rho'(y) = \int_0^1 \partial_{y_n} x_n(y'', \theta y_n)\, d\theta = \frac{x_n(y) - \rho_0(y'')}{y_n}$$

on $\tilde{U}_3$. Then

$$y_n = (x_n(y) - \rho_0(y''))/\rho'(y), \tag{43}$$

and $|\rho'(y) - 1| \le r$, $|\partial_{y_n}\rho'| \le M$ on $\tilde{U}_3$. We define

$$U_4' = \{(\tilde{x}'', x_n) \in \mathcal{R}(U_2) \times \mathbb{C}; |x| < r^5, 0 < |x_n - \rho_0(x')| < M^2 \cdot \sqrt{R}\,|x_0|\}.$$

Let $\pi_4 : \mathcal{R}(U_4') \to U_4'$ be the canonical map, and let $\tilde{x} \in \mathcal{R}(U_4')$, $x = \pi_4(\tilde{x}) \in U_4'$. We can identify $\tilde{x}$ with $(x, \arg(x_n - \rho_0(x'')), \arg x_0) \in U_4' \times \mathbb{R} \times \mathbb{R}$. We define

$$\tilde{U}_4'(\tilde{x}^1) = \left\{\tilde{x} \in \mathcal{R}(U_4'); \left|\arg\left(\frac{x_0}{x_n - \rho_0(x'')}\right) - \arg\left(\frac{x_0^1}{x_n^1 - \rho_0(x''^1)}\right)\right| < \frac{M}{2}\right\}.$$

We want to define a function $y_n = h'(x)$ which satisfies

$$y_n(x) = (x_n - \rho_0(x''))/\rho'(x'', h'(x)) \tag{44}$$

on $\tilde{U}_4'(\tilde{x}^1)$. It will turn out that $h'(x)$ coincides with the previous function $h(x)$

defined by (41). For the moment we do not discuss this coincidence. We define

$$h'^{(i)}(x) = \begin{cases} (x_n - \rho_0(x''))/\rho'(y^{\circ}), & i = 0, \\ (x_n - \rho_0(x''))/\rho'(x'', h'^{(i-1)}(x)), & i \geq 1. \end{cases} \tag{45}$$

By induction on $i \geq 1$, we can prove that $h'^{(i)}(x)$ is well-defined on $\tilde{U}'_4(\tilde{x}^1)$, and $|h'^{(i)}(x) - h'^{(i-1)}(x)| \leq r^{i/2}|h'^{(0)}(x)|$ on $\tilde{U}'_4(\tilde{x}^1)$. Therefore, for each $x \in \tilde{\Omega}_4$ we can define $h'(x) = \lim_{i\to\infty} h'^{(i)}(x)$. Comparing (43) and (44) as before, we have $y_n = h'(x(y))$. We obtain an injection

$$\chi' : \tilde{U}'_4(\tilde{x}^1) \ni x \mapsto (x'', h'(x)) \in \tilde{\Omega}_3.$$

Both χ and χ' have the same right inverse mapping $\chi'' : y \ \mapsto x(y)$. Furthermore, the Jacobian matrix $\partial x(y)/\partial y$ is nondegenerate, and for any $y \in \tilde{\Omega}_3$, the mapping χ'' is a local isomorphism. Therefore χ and χ' coincide on $\tilde{\Omega}_4 \cap \tilde{U}'_4(\tilde{x}^1)$, and we obtain an injection

$$\chi : \tilde{\Omega}_4 \cup \tilde{U}'_4(\tilde{x}^1) \ni x \mapsto y(x) \in \tilde{\Omega}_3.$$

We have the following result.

Proposition 7. *Let $\tilde{x}^1 \in \tilde{U}_4$, and define $\tilde{U}'_4(\tilde{x}^1)$ as above. Assume that $|\alpha| \leq m$ and M' is large enough.*

(i) *The solution $u(x)$ of (2) is holomorphic on $\tilde{\Omega}_4 \cup \tilde{U}'_4(\tilde{x}^1)$.*
(ii) $|\partial_x^\alpha u(x)| \leq M'\mathcal{L}_1(y(x))^{-(|\alpha|-m+q)_+}$ *on $\tilde{\Omega}_4 \cup \tilde{U}'_4(\tilde{x}^1)$.*
(iii) $|\partial_x^\alpha u(x)| \leq M' \min_{1\leq k\leq m} |x_n + a_k x_0|^{-(|\alpha|-m+q)_+}$ *on $\tilde{\Omega}_4$.*
(iv) $|\partial_x^\alpha u(x)| \leq M'|x_n - \rho_0(x'')|^{-(|\alpha|-m+q)_+}$ *on $\tilde{U}'_4(\tilde{x}^1)$.*

Proof. Denoting $u(y(x))$ also by $u(x)$, we obtain (i). Let us prove (ii). Let $|\alpha| = k \geq 1$. From (5), we have

$$\partial_x^\alpha f(x) = (\partial_{y_n} x_n)^{-2k-1}\Big(\sum_{|\beta|=k} A_{\alpha\beta}\partial_y^\beta f + \sum_{|\beta|=k} B_{\alpha\beta}\partial_y^\beta x_n + C_\alpha\Big),$$

where $A_{\alpha\beta}, B_{\alpha\beta}, C_\alpha$ are polynomials of $(\partial_y^\gamma f,\, \partial_y^\gamma x_n;\, |\gamma| \leq k-1)$. By Proposition 7, we have (ii).

To see (iii), we assume that $\tilde{x} \in \tilde{\Omega}_4$ and $1 \leq k \leq m$. From (42) we have

$$\min_{1\leq k\leq m} |y_n(x) + a_k x_0| \geq \lim_{1\leq k\leq m} |x_n + a_k x_0|/2.$$

It follows that

$$\mathcal{L}_1(y) \geq \min_{1\leq k\leq m} |x_n + a_k x_0|/4,$$

and we obtain (iii). Using (45), we can similarly prove (iv). $\square$

Proof of Theorems 1 and 2. We use the notation Ω_1, V_m of Section 1, and Ω_4, U_4 of Section 7. Let $N_0 \in \mathbb{N}$ and $0 < r_0 \ll R_0 \ll 1/M \ll 1$. Let $\tilde{x} \in \mathcal{R}_{N_0}(\Omega_1(R_0, r_0))$. Choosing $r = r_0^{1/5}, R = R_0^2 M^{-2}$, we find that $(x'', x_n + a_m x_0) \in \Omega_2$, and $\psi^m(x'', x_n + a_m x_0) = x_0/(x_n + a_m x_0) \in \omega^m$. Now

$$\begin{aligned} x \in \Sigma \;&\Leftrightarrow\; x_n + a_m x_0 = -\theta(a_j - a_m)x_0 \text{ for some } \theta \in (0,1] \text{ and } j \\ &\Leftrightarrow\; x_0/(x_n + a_m x_0) = -\theta'/(a_j - a_m) \text{ for some } \theta' \geq 1 \text{ and } j. \end{aligned}$$

It is clear from Figure 6 that the length of the canonical curve corresponding to $(x'', x_n + b_m x_0)$ does not exceed $3M$, i.e., $T(x) < 3M$. Therefore $\mathcal{R}_{N_0}(\Omega_1(R_0, r_0)) \subset \tilde{\Omega}_4(R, r)$ with $R = R_0^2 M^{-2}$, $r = r_0^{1/5}$.

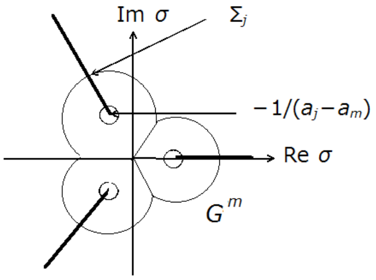

Figure 6. The set Σ illustrated in the σ space.

If $\tilde{x}^1 \in \mathcal{R}_{N_0}(\Omega_1(R, r)) \cap \pi^{-1}(V_m(2R, r))$, then $\tilde{x}^1 \in \tilde{U}_4$. Thus we can define $\rho_0(x'')$ on $\mathcal{R}(U_4)$ and extend the solution on $\tilde{U}_4$. This immediately yields Theorem 2. □

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