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Ice Quivers with Potential Arising from Once-punctured Polygons and Cohen–Macaulay Modules

by

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Abstract

Given a tagged triangulation of a once-punctured polygon P^* with n vertices, we associate an ice quiver with potential such that the frozen part of the associated frozen Jacobian algebra has the structure of a Gorenstein $K[X]$-order Λ. Then we show that the stable category of the category of Cohen–Macaulay Λ-modules is equivalent to the cluster category $\mathcal{C}$ of type D_n. This gives a natural interpretation of the usual indexation of cluster tilting objects of $\mathcal{C}$ by tagged triangulations of P^*. Moreover, it extends naturally the triangulated categorification by $\mathcal{C}$ of the cluster algebra of type D_n to an exact categorification by adding coefficients corresponding to the sides of P. Finally, we lift the previous equivalence of categories to an equivalence between the stable category of graded Cohen–Macaulay Λ-modules and the bounded derived category of modules over a path algebra of type D_n.

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Contents

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§1. Introduction

In a previous paper [9], we constructed ice quivers with potential arising from triangulations of polygons and we proved that the frozen parts of their frozen Jacobian algebras are orders. We proved that the categories of Cohen–Macaulay modules over these orders are stably equivalent to cluster categories of type A. The aim of this paper is to extend these results to tagged triangulations of once-punctured polygons to recover cluster categories of type D. We refer to [9] for a detailed introduction and we will focus here on the tools we specifically need for this new case.

For every bordered surface with marked points, Fomin, Shapiro and Thurston introduced the concept of tagged triangulations and their mutations [12]. Then, they associated to each of these triangulations a quiver $Q(\sigma)$ and showed that the combinatorics of triangulations of the surface correspond to that of the cluster algebra defined by $Q(\sigma)$. Later in [25], Labardini-Fragoso associated a potential $W(\sigma)$ on $Q(\sigma)$. He proved that flips of triangulations are compatible with mutations of quivers with potential. This was generalized to the case of tagged triangulations by Labardini-Fragoso and Cerulli Irelli in [7, 26].

We refer to [3, 8, 30, 31] for a general background on Cohen–Macaulay modules (or lattices) over orders. Recently, strong connections between Cohen–Macaulay representation theory and tilting theory, especially cluster categories, have been established [1, 2, 11, 20, 21, 22, 24]. This paper enlarges some of these connections by dealing with frozen Jacobian algebras associated with tagged triangulations of once-punctured polygons from the viewpoint of Cohen–Macaulay representation theory.

Throughout this paper, K denotes a field and $R = K[X]$. We extend the construction of [12], and associate an ice quiver with potential (Q_σ, W_σ, F) to each tagged triangulation σ of a once-punctured polygon P^* with n vertices by adding a set F of n frozen vertices corresponding to the edges of the polygon and certain arrows (see Definition 2.9). We study the associated frozen Jacobian algebra

$$\Gamma_\sigma := \mathcal{P}(Q_\sigma, W_\sigma, F)$$

(see Definition 2.1). Our main results are the following:

Theorem 1.1 (Theorems 2.19 and 2.30). *Let e_F be the sum of the idempotents of Γ_σ at frozen vertices. Then*

(1) *the frozen Jacobian algebra Γ_σ has the structure of an R-order (see Definition 2.17 and Remark 2.18);*
(2) *the frozen part $e_F\Gamma_\sigma e_F$ is isomorphic to the Gorenstein R-order*

$$\Lambda := \begin{bmatrix} R' & R' & R' & \cdots & R' & X^{-1}(X,Y) \\ (X,Y) & R' & R' & \cdots & R' & R' \\ (X) & (X,Y) & R' & \cdots & R' & R' \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (X) & (X) & (X) & \cdots & R' & R' \\ (X) & (X) & (X) & \cdots & (X,Y) & R' \end{bmatrix}_{n\times n}, \tag{1.2}$$

where $R' = K[X,Y]/(Y(X-Y))$ and each entry of the matrix is an R'-submodule of $R'[X^{-1}]$.

Remark 1.3. In view of the isomorphism of R-algebras

$$R' \cong R - R := \{(P,Q) \in R^2 \mid P - Q \in (X)\}, \qquad Y \mapsto (0,X),$$

we have an isomorphism

$$\Lambda \cong \begin{bmatrix} R-R & R-R & R-R & \cdots & R-R & R\times R \\ (X)\times(X) & R-R & R-R & \cdots & R-R & R-R \\ (X)-(X) & (X)\times(X) & R-R & \cdots & R-R & R-R \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (X)-(X) & (X)-(X) & (X)-(X) & \cdots & R-R & R-R \\ (X)-(X) & (X)-(X) & (X)-(X) & \cdots & (X)\times(X) & R-R \end{bmatrix}_{n\times n}$$

($(X)-(X)$ is the ideal of $R-R$ generated by (X,X)).

This order is part of a wide class of Gorenstein orders, called almost Bass orders, introduced and studied by Drozd–Kirichenko–Roĭter and Hijikata–Nishida [16, 17] (see also [18]). More precisely, Λ is an *almost Bass order of type (III).*

Theorem 1.4 (Theorems 2.30, 3.3, 3.16 and 3.19). *The category* $\mathrm{CM}\,\Lambda$ *has the following properties:*

(1) *For any tagged triangulation σ of P^*, we can map each tagged arc a of σ to the indecomposable Cohen–Macaulay Λ-module $e_F\Gamma_\sigma e_a$, where e_a is the idempotent of Γ_σ at a. This module only depends on a (not on σ) and this*

map induces one-to-one correspondences

$$\{\textit{sides and tagged arcs of } P^*\} \leftrightarrow \{\textit{indecomposable objects of } \mathrm{CM}\,\Lambda\}/\cong,$$
$$\{\textit{sides of } P\} \leftrightarrow \{\textit{indecomposable projectives of } \mathrm{CM}\,\Lambda\}/\cong,$$
$$\{\textit{tagged triangulations of } P^*\} \leftrightarrow \{\textit{basic cluster tilting objects of } \mathrm{CM}\,\Lambda\}/\cong.$$

(2) *For the cluster tilting object* $T_\sigma := e_F\Gamma_\sigma$ *corresponding to a tagged triangulation* σ,

$$\mathrm{End}_{\mathrm{CM}\,\Lambda}(T_\sigma) \cong \Gamma_\sigma^{\mathrm{op}}.$$

(3) *The category* $\underline{\mathrm{CM}}\,\Lambda$ *is* 2*-Calabi–Yau.*

(4) *If* K *is a perfect field, there is a triangle-equivalence* $\mathcal{C}(KQ) \cong \underline{\mathrm{CM}}\,\Lambda$*, where* Q *is a quiver of type* D_n *and* $\mathcal{C}(KQ)$ *is the corresponding cluster category.*

Remark 1.5. To prove Theorem 1.4(3), we establish that

$$\mathrm{CM}\,\Lambda \cong \mathrm{CM}^{\mathbb{Z}/n\mathbb{Z}}(K[x,y]/(x^{n-1}y - y^2)),$$

where x has degree 1 and y has degree -1 (modulo n).

Usually, the cluster category $\mathcal{C}(KQ)$ is constructed as an orbit category of the bounded derived category $\mathcal{D}^{\mathrm{b}}(KQ)$. We can reinterpret this result in this context by studying the category of graded Cohen–Macaulay Λ-modules $\mathrm{CM}^{\mathbb{Z}}\,\Lambda$:

Theorem 1.6 (Theorem 4.5). *With the same notation as before:*

(1) *The Cohen–Macaulay* Λ*-module* T_σ *can be lifted to a tilting object in* $\underline{\mathrm{CM}}^{\mathbb{Z}}\,\Lambda$.

(2) *There exists a triangle-equivalence* $\mathcal{D}^{\mathrm{b}}(KQ) \cong \underline{\mathrm{CM}}^{\mathbb{Z}}\,\Lambda$.

In Section 2, we introduce ice quivers with potential (Q_σ, W_σ, F) associated with tagged triangulations σ of a once-punctured polygon P^*. We also introduce combinatorial and algebraic elementary tools in Subsection 2.3. Finally, we prove in this section that the frozen Jacobian algebra Γ_σ associated with (Q_σ, W_σ, F) is an R-order, and that $\Lambda \cong e_F\Gamma_\sigma e_F$ which is independent of σ. In Section 3, we classify Cohen–Macaulay modules over Λ, we compute homological properties of $\mathrm{CM}\,\Lambda$ and we establish the correspondence between tagged triangulations of P^* and basic cluster tilting objects of $\mathrm{CM}\,\Lambda$. Thus, after proving that $\mathrm{CM}\,\Lambda$ is Frobenius stably 2-Calabi–Yau, we conclude that $\mathrm{CM}\,\Lambda$ is stably triangle-equivalent to a cluster category of type D. In Section 4, we deal with results about $\mathrm{CM}^{\mathbb{Z}}\,\Lambda$.

Notice that the naive generalizations of these results to other surfaces do not hold in general, as shown in Subsection 2.5 for a digon with two punctures.

§2. Ice quivers with potential associated with triangulations

In this section, we introduce ice quivers with potential associated with tagged triangulations of a once-punctured polygon and their frozen Jacobian algebras. We show that in any case, the frozen Jacobian algebra has the structure of an R-order, and its frozen part is isomorphic to a given R-order Λ defined in (1.2).

§2.1. Frozen Jacobian algebras

We refer to [10] for background about quivers with potential. Let Q be a finite connected quiver without loops, with set of vertices $Q_0 = \{1, \dots, n\}$ and set of arrows Q_1. As usual, if $\alpha \in Q_1$, we denote by $s(\alpha)$ its starting vertex and by $e(\alpha)$ its ending vertex. We denote by KQ_i the K-vector space with basis Q_i consisting of paths of length i in Q, and by $KQ_{i,\mathrm{cyc}}$ the subspace of KQ_i spanned by all cycles in KQ_i. Consider the path algebra $KQ = \bigoplus_{i\geq 0} KQ_i$. An element $W \in \bigoplus_{i\geq 1} KQ_{i,\mathrm{cyc}}$ is called a *potential*. Two potentials W and W' are called *cyclically equivalent* if $W - W'$ belongs to $[KQ, KQ]$, the vector space spanned by commutators. A *quiver with potential* is a pair (Q, W) consisting of a quiver Q without loops and a potential W which does not have two cyclically equivalent terms.

For each arrow $\alpha \in Q_1$, the *cyclic derivative* ∂_α is the linear map from $\bigoplus_{i\geq 1} KQ_{i,\mathrm{cyc}}$ to KQ defined on cycles by

$$\partial_\alpha(\alpha_1 \dots \alpha_d) = \sum_{\alpha_i=\alpha} \alpha_{i+1} \dots \alpha_d \alpha_1 \dots \alpha_{i-1}.$$

Definition 2.1 ([5]). An *ice quiver with potential* is a triple (Q, W, F), where (Q, W) is a quiver with potential and F is a subset of Q_0. Vertices in F are called *frozen vertices*.

The *frozen Jacobian algebra* is defined by

$$\mathcal{P}(Q, W, F) = KQ/\mathcal{J}(W, F),$$

where $\mathcal{J}(W, F)$ is the ideal

$$\mathcal{J}(W, F) = \langle \partial_\alpha W \mid \alpha \in Q_1,\ s(\alpha) \notin F \text{ or } e(\alpha) \notin F \rangle$$

of KQ.

Example 2.2. Consider the quiver Q of Figure 2.3 with potential $W = \alpha_1\beta_1\gamma_1 + \alpha_2\beta_2\gamma_2 + \alpha_3\beta_3\gamma_3 - \gamma_1\beta_2\alpha_3$ and set of frozen vertices $F = \{4, 5, 6\}$. Then the Jacobian ideal is

$$\mathcal{J}(W, F) = \langle \beta_1\gamma_1, \gamma_1\alpha_1, \alpha_1\beta_1 - \beta_2\alpha_3, \beta_2\gamma_2, \gamma_2\alpha_2 - \alpha_3\gamma_1, \beta_3\gamma_3 - \gamma_1\beta_2, \gamma_3\alpha_3 \rangle.$$

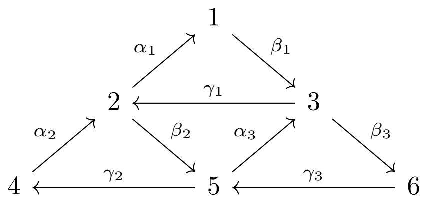


Figure 2.3. Example of iced quiver with potential.

Note that this ice quiver with potential appeared in connection with preprojective algebras [5, 13].

§2.2. Ice quivers with potential arising from triangulations

We recall the definition of triangulations of a polygon with one puncture and introduce our definition of ice quivers with potential arising from tagged triangulations of a polygon with one puncture.

Definition 2.4. Let P be a regular polygon with n vertices and n sides. Fix a marked point inside P. Then the marked point is called a *puncture* and the combination of P and the puncture is called a *(once-)punctured polygon* P^*. We define the *interior of* P^* to be the interior of the polygon P excluding the puncture. We denote by M the set of all the n vertices of the polygon and the puncture.

Definition 2.5 (Tagged arcs [12, Definition 7.1]). A *tagged arc* in the punctured polygon P^* is a curve a in P such that

(1) the endpoints of a are distinct in M;
(2) a does not intersect itself;
(3) except for the endpoints, a is disjoint from M and from the sides of P;
(4) a does not cut out an unpunctured digon. (In other words, a is not contractible onto the sides of P.)

Each arc a is considered up to isotopy inside the class of such curves.

Moreover, each arc incident to the puncture has to be tagged either *plain* or *notched*.

In the figures, the plain tags are omitted while the notched tags are represented by $\bowtie$.

Definition 2.6 (Compatibility of tagged arcs [12, Definition 7.4]). Two tagged arcs a and b are *compatible* if the following conditions are satisfied:

(1) there are curves in their respective isotopy classes whose relative interiors do not intersect;
(2) if a and b are incident to the puncture and not isotopic, they are either both plain, or both notched.

Definition 2.7 ([12]). A *tagged triangulation* of the punctured polygon P^* is the union of the set of sides of P and any maximal collection of pairwise compatible tagged arcs of P^*.

Remark 2.8. The set of all tagged arcs in a punctured polygon is finite. Moreover, any tagged triangulation can be realized up to isotopy as a collection of tagged non-intersecting arcs.

Let us now define ice quivers with potential arising from punctured polygons:

Definition 2.9. Let P^* be a punctured polygon with n sides and let σ be a tagged triangulation of P^*. For convenience, the n sides of P and all the tagged arcs of σ are called the *edges* of σ. A *true triangle* of σ is a triangle consisting of edges of σ such that the puncture is not in its interior.

We assign to σ two ice quivers with potential (Q_σ, W_σ, F) and $(Q'_\sigma, W'_\sigma, F)$ as follows.

The quiver Q'_σ is a quiver whose vertices are indexed by the edges of σ. Whenever two edges a and b are sides of a common true triangle of σ, then Q'_σ contains an internal arrow $a \to b$ in the true triangle if a is a predecessor of b with respect to anticlockwise orientation centred at the joint vertex. For every vertex of the polygon P, there is an external arrow $a \to b$ where a and b are its two incident sides of P, a being a predecessor of b with respect to anticlockwise orientation centred at the joint vertex. Moreover, if the puncture is adjacent to exactly one notched arc and one plain arc of σ, we have the configuration shown in Figure 2.10.

The quiver Q_σ is obtained from Q'_σ by removing external arrows winding around vertices of P with no incident tagged arc in σ.

We say that a cycle of Q_σ (resp. Q'_σ) is *planar* if it does not contain any arrow of Q_σ (resp. Q'_σ) in its interior and each arrow appears at most once. Notice that for the definition of planar, the quivers are not abstract but embedded in the plane (each internal arrow being drawn inside the triangle it is constructed from, and each external arrow winding around the corresponding vertex outside the polygon). We have the following possible different kinds of planar cycles in Q_σ and Q'_σ:

(1) *clockwise triangles* which come from true triangles in σ;

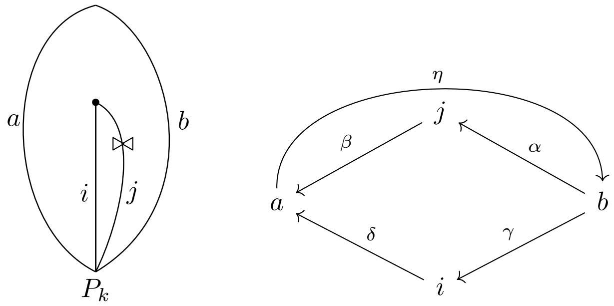


Figure 2.10. Once-punctured digon. We depict the part of Q_σ quiver induced by a once-punctured digon. Notice that there are two other arrows linking a and b if P is the digon itself.

(2) an *anticlockwise punctured cycle* which consists of the arrows connecting arcs incident to the puncture;
(3) *anticlockwise external cycles* which contain exactly one external arrow and each of which is centred at a vertex of P.

We define F as the subset of $(Q_\sigma)_0$ indexed by the n sides of the polygon P. The potential W_σ (resp. W'_σ) is defined as

$$\sum \text{clockwise triangles} - \sum \text{anticlockwise external cycles} \\ - \text{the anticlockwise punctured cycle}$$

in Q_σ (resp. Q'_σ).

When there is a once-punctured digon in the triangulation σ as shown in Figure 2.10, we have to slightly adapt the previous definition. The anticlockwise external cycle centred at P_k which is taken in account is the one containing $\gamma\delta$. On the other hand, both $\eta\alpha\beta$ and $\eta\gamma\delta$ appear as clockwise triangles in W_σ and W'_σ. In this case, there is no anticlockwise punctured cycle. An explicit case involving a once-punctured digon is described in the proof of Lemma 2.27.

Example 2.11. Let us consider the triangulation σ of Figure 2.12. We drew the corresponding quivers (γ is in Q'_σ but not in Q_σ). We have

$$W_\sigma = fgh + abc + ade - \alpha ag - \beta fbc \quad \text{and} \quad W'_\sigma = W_\sigma - \gamma h.$$

From now on, for any tagged triangulation σ of the punctured polygon P^*, we denote $\mathcal{P}(Q_\sigma, W_\sigma, F)$ by Γ_σ.

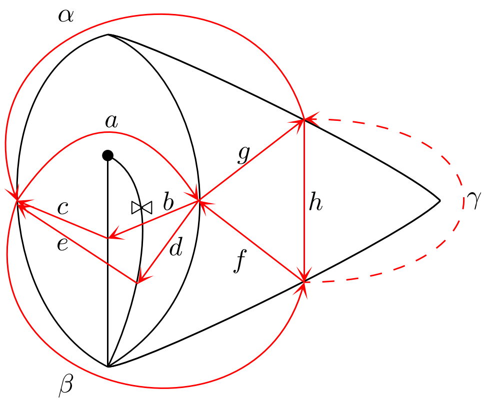


Figure 2.12. Quivers associated with a tagged triangulation.

Remark 2.13. (1) We can also realize Γ_σ as

$$\Gamma_\sigma \cong \frac{KQ'_\sigma}{\mathcal{J}'(W'_\sigma)} \quad \text{where} \quad \mathcal{J}'(W'_\sigma) := \langle \partial_\alpha W'_\sigma \mid \alpha \in Q_1,\ \alpha \text{ is not external}\rangle.$$

We will use both definitions freely depending on convenience.

(2) Notice that (Q'_σ, W'_σ) and (Q_σ, W_σ) are not necessarily reduced, in the sense that oriented 2-cycles can appear in the potential, because some vertices of the polygon have no incident tagged arcs in σ, or because the puncture has exactly two non-isotopic incident tagged arcs. Thus, it is possible that non-admissible relations appear.

(3) All arrows of Q_σ (resp. Q'_σ) appear either once in W_σ (resp. W'_σ), or twice with opposite signs. Thus all relations derived from the potential are either commutativity relations (of the form $w = w'$ for two paths w and w' of length at least 1), or 0 relations (of the form $w = 0$ for a path w of length at least 1).

§2.3. Notation and preliminaries

The vertices of the polygon P are labelled $P_1, \dots, P_n$ in counter-clockwise order. When we do computations on the indices of vertices of P, we compute modulo n. If $r, s \in [\![1, n]\!] := \{1, \dots, n\}$, we denote

$$d(r,s) := \begin{cases} s - r & \text{if } s \geq r, \\ s - r + n & \text{if } s < r. \end{cases}$$

We also denote $[\![r, s]\!] = \{r, r{+}1, \dots, s\}$, $[\![r, s[\![= [\![r, s]\!] \smallsetminus \{s\}$, $]\!]r, s]\!] = [\![r, s]\!] \smallsetminus \{r\}$ and $]\!]r, s[\![= [\![1, n]\!] \smallsetminus [\![s, r]\!]$ (notice that $]\!]r, r[\![= [\![1, n]\!] \smallsetminus \{r\}$). If A is a condition, we define δ_A to be 1 if A is satisfied and 0 if A is not satisfied.

For $r,s,t \in [\![1,n]\!]$, we will use freely the identities $]\!]r,s]\!] = [\![r+1,s+1[\![$, $]\!]r,s]\!] = [\![1,n]\!] \smallsetminus]\!]s,r]\!]$, $[\![r,s[\![= [\![1,n]\!] \smallsetminus [\![s,r[\![$ and $\delta_{r\in]\!]s,t[\![} = \delta_{s\in[\![t,r[\![} = \delta_{t\in]\!]r,s]\!]}$.

If $r,s \in [\![1,n]\!]$, we denote by (P_r,P_s) the arc going from P_r to P_s turning counter-clockwise around the puncture (thus, (P_r,P_{r+1}) is a side of the polygon and (P_{r+1},P_r) is not except if $n=2$). We denote by $(P_r,*)$ the plain tagged arc from P_r to the puncture and by $(P_r,\bowtie)$ the notched tagged arc from P_r to the puncture.

From now on, we always denote $a=(P_{a_1},P_{a_2})$ if a is not incident to the puncture, and $a=(P_{a_1},*)$ or $a=(P_{a_1},\bowtie)$ if a is incident to the puncture (in the latter case, we fix $a_2=a_1$ by convention). We need to fix some geometrical definition. To make it precise, we suppose that P is a regular polygon inscribed in the unit circle and that the puncture is at the origin of the plane. Then $\vec{a}$ is the vector from P_{a_1} to P_{a_2} if $a_1 \neq a_2$, and it is the unit vector tangent at P_{a_1} to the unit circle in the clockwise direction if $a_1=a_2$.

If a and b are tagged arcs of P^*, we define

$$\ell^\theta_{a,b} := d(a_1,b_1) + d(a_2,b_2) + n|\delta_{a_1\in]\!]b_1,a_2[\![} - \delta_{b_2\in]\!]b_1,a_2[\![}|.$$

Lemmas 2.14 and 2.15 are elementary observations about ℓ^θ. Proofs are computational and given for the sake of completeness. We suggest skipping them on a first reading.

Lemma 2.14. *If a and b are two sides or tagged arcs of P^*, the angle from $\vec{a}$ to $\vec{b}$ is*

$$\frac{\pi}{n}\ell^\theta_{a,b},$$

up to a multiple of 2π.

Proof. First of all, in complex coordinates, if $a_1 \neq a_2$,

$$\begin{aligned}\vec{a} &= \exp\left(2\pi i\frac{a_2}{n}\right) - \exp\left(2\pi i\frac{a_1}{n}\right)\\ &= \exp\left(\pi i\frac{a_1+a_2}{n}\right)\left(\exp\left(\pi i\frac{a_2-a_1}{n}\right) - \exp\left(\pi i\frac{a_1-a_2}{n}\right)\right)\\ &= \exp\left(\pi i\frac{a_1+a_2}{n}\right)2i\sin\left(\pi\frac{a_2-a_1}{n}\right),\end{aligned}$$

so the argument of $\vec{a}$ is

$$\frac{\pi}{n}\left(a_1+a_2+\frac{n}{2}+n\delta_{a_1\geq a_2}\right)$$

(note that this formula works also if $a_1 = a_2$). So the angle from $\vec{a}$ to $\vec{b}$ is

$$\frac{\pi}{n}(b_1 - a_1 + b_2 - a_2 + n(\delta_{b_1 \geq b_2} - \delta_{a_1 \geq a_2})).$$

As $\ell^\theta_{a,b}$ is clearly invariant by rotation of the polygon, as also is the angle from $\vec{a}$ to $\vec{b}$, we can suppose that $a_2 = 1$ and the angle from a to b becomes

$$\begin{aligned}\frac{\pi}{n}(d(a_1, b_1) - n\delta_{a_1 > b_1} + d(1, b_2) + n(\delta_{b_1 \geq b_2} - 1))\\ &= \frac{\pi}{n}(d(a_1, b_1) + d(1, b_2) - n\delta_{a_1 > b_1} - n\delta_{b_1 < b_2})\\ &= \frac{\pi}{n}(d(a_1, b_1) + d(1, b_2) - n\delta_{a_1 \in \rrbracket b_1, 1 \llbracket} - n\delta_{b_2 \in \rrbracket b_1, 1 \llbracket}),\end{aligned}$$

which is clearly congruent to $\pi\ell^\theta_{a,b}/n$ modulo 2π. □

Another important point is that ℓ^θ is subadditive:

Lemma 2.15. *If a, b and c are three sides or tagged arcs of P^*, then $\ell^\theta_{a,b} + \ell^\theta_{b,c} \geq \ell^\theta_{a,c}$. More precisely,*

- *if a is a side of P,*

$$\ell^\theta_{a,b} + \ell^\theta_{b,c} = \ell^\theta_{a,c} + 2n(\delta_{c_2 \in \rrbracket c_1, b_2 \llbracket}\delta_{b_1 \in \llbracket b_2, c_1 \rrbracket} + \delta_{a_1 \in \rrbracket b_1, c_1 \rrbracket});$$

- *if b is a side of P,*

$$\ell^\theta_{a,b} + \ell^\theta_{b,c} = \ell^\theta_{a,c} + 2n(\delta_{a_1 \in \rrbracket a_2 - 1, c_1 \rrbracket}\delta_{c_2 \in \rrbracket a_2 - 1, c_1 \rrbracket} + \delta_{b_1 \in \rrbracket c_1, a_2 - 1 \llbracket});$$

- *if c is a side of P,*

$$\ell^\theta_{a,b} + \ell^\theta_{b,c} = \ell^\theta_{a,c} + 2n(\delta_{a_1 \in \rrbracket b_1, a_2 \llbracket}\delta_{b_2 \in \llbracket a_2, b_1 \rrbracket} + \delta_{c_2 \in \llbracket a_2, b_2 \llbracket}).$$

Proof. We have

$$\begin{aligned}\ell^\theta_{a,b} + \ell^\theta_{b,c} - \ell^\theta_{a,c} &= d(a_1, b_1) + d(b_1, c_1) - d(a_1, c_1) + d(a_2, b_2) + d(b_2, c_2) - d(a_2, c_2)\\ &\quad + n(|\delta_{a_1 \in \rrbracket b_1, a_2 \llbracket} - \delta_{b_2 \in \rrbracket b_1, a_2 \llbracket}| + |\delta_{b_1 \in \rrbracket c_1, b_2 \llbracket} - \delta_{c_2 \in \rrbracket c_1, b_2 \llbracket}|\\ &\quad - |\delta_{a_1 \in \rrbracket c_1, a_2 \llbracket} - \delta_{c_2 \in \rrbracket c_1, a_2 \llbracket}|)\\ &= n(\delta_{b_1 \in \rrbracket c_1, a_1 \llbracket} + \delta_{b_2 \in \rrbracket c_2, a_2 \llbracket} + |\delta_{a_1 \in \rrbracket b_1, a_2 \llbracket} - \delta_{b_2 \in \rrbracket b_1, a_2 \llbracket}|\\ &\quad + |\delta_{b_1 \in \rrbracket c_1, b_2 \llbracket} - \delta_{c_2 \in \rrbracket c_1, b_2 \llbracket}| - |\delta_{a_1 \in \rrbracket c_1, a_2 \llbracket} - \delta_{c_2 \in \rrbracket c_1, a_2 \llbracket}|).\end{aligned}$$

If this quantity were negative, we would have $b_1 \in \llbracket a_1, c_1 \rrbracket$ and $b_2 \in \llbracket a_2, c_2 \rrbracket$ and one of the following two posibilities:

- $a_1 \in \rrbracket c_1, a_2 \llbracket$ and $c_2 \in \llbracket a_2, c_1 \rrbracket$. As $c_2 \notin \rrbracket c_1, b_2 \llbracket$, we get $b_1 \notin \rrbracket c_1, b_2 \llbracket$ so $b_1 \in \llbracket b_2, c_1 \rrbracket$. It is then easy to deduce that $a_1 \in \rrbracket b_1, a_2 \llbracket$ and $b_2 \notin \rrbracket b_1, a_2 \llbracket$, contrary to the hypothesis.

- $a_1 \in [\![a_2, c_1]\!]$ and $c_2 \in]\!]c_1, a_2[\![$. As $a_1 \notin]\!]b_1, a_2[\![$, we get $b_2 \notin]\!]b_1, a_2[\![$ so $b_2 \in [\![a_2, b_1]\!]$. It is then easy to deduce that $b_1 \notin]\!]c_1, b_2[\![$ and $c_2 \in]\!]c_1, b_2[\![$, again contrary to the hypothesis.

Notice that for any $i, j, k, l \in [\![1, n]\!]$, we have the identities

$$\begin{aligned}|\delta_{i\in]\!]k,j[\![} - \delta_{l\in]\!]k,j[\![}| &= \delta_{i\in]\!]k,j[\![}\delta_{l\in[\![j,k]\!]} + \delta_{i\in[\![j,k]\!]}\delta_{l\in]\!]k,j[\![} \\ &= \delta_{i\in]\!]k,j[\![}\delta_{l\in[\![j,k]\!]} + (1 - \delta_{i\in]\!]k,j[\![})(1 - \delta_{l\in[\![j,k]\!]}) \\ &= 2\delta_{i\in]\!]k,j[\![}\delta_{l\in[\![j,k]\!]} + 1 - \delta_{i\in]\!]k,j[\![} - \delta_{l\in[\![j,k]\!]}\end{aligned}$$

and

$$\begin{aligned}|\delta_{i\in]\!]k,j[\![} - \delta_{l\in]\!]k,j[\![}| &= \delta_{i\in]\!]k,j[\![}\delta_{l\in[\![j,k]\!]} + \delta_{i\in[\![j,k]\!]}\delta_{l\in]\!]k,j[\![} \\ &= \delta_{i\in]\!]k,j[\![}(1 - \delta_{l\in]\!]k,j[\![}) + (1 - \delta_{i\in]\!]k,j[\![})\delta_{l\in]\!]k,j[\![} \\ &= \delta_{i\in]\!]k,j[\![} + \delta_{l\in]\!]k,j[\![} - 2\delta_{i\in]\!]k,j[\![}\delta_{l\in]\!]k,j[\![}.\end{aligned}$$

If a is a side of the polygon, we have $a_2 = a_1 + 1$ and the previous difference becomes (up to a factor n)

$$\begin{aligned}&\delta_{b_1\in]\!]c_1,a_1[\![} + \delta_{b_2\in]\!]c_2,a_1+1[\![} + |\delta_{a_1\neq b_1} - \delta_{b_2\in]\!]b_1,a_1+1[\![}| \\ &\qquad\qquad + |\delta_{b_1\in]\!]c_1,b_2[\![} - \delta_{c_2\in]\!]c_1,b_2[\![}| - |\delta_{a_1\neq c_1} - \delta_{c_2\in]\!]c_1,a_1+1[\![}| \\ &= \delta_{a_1\in]\!]b_1,c_1]\!]} + \delta_{b_2\in]\!]c_2,a_1+1[\![} + \delta_{a_1\neq b_1} - \delta_{b_2\in]\!]b_1,a_1+1[\![} \\ &\quad + 2\delta_{b_1\in[\![b_2,c_1]\!]}\delta_{c_2\in]\!]c_1,b_2[\![} + 1 - \delta_{b_1\in[\![b_2,c_1]\!]} - \delta_{c_2\in]\!]c_1,b_2[\![} - \delta_{a_1\neq c_1} + \delta_{c_2\in]\!]c_1,a_1+1[\![} \\ &= \delta_{a_1\in]\!]b_1,c_1]\!]} + 2\delta_{b_1\in[\![b_2,c_1]\!]}\delta_{c_2\in]\!]c_1,b_2[\![} + 1 - \delta_{b_1\in[\![b_2,c_1]\!]} - \delta_{c_2\in]\!]c_1,b_2[\![} \\ &\quad + \delta_{a_1\in]\!]b_2-1,c_2-1]\!]} - \delta_{a_1=b_1} - \delta_{a_1\in]\!]b_2-1,b_1-1]\!]} + \delta_{a_1=c_1} + \delta_{a_1\in]\!]c_2-1,c_1-1]\!]} \\ &= \delta_{a_1\in]\!]b_1,c_1]\!]} + 2\delta_{b_1\in[\![b_2,c_1]\!]}\delta_{c_2\in]\!]c_1,b_2[\![} - \delta_{b_1\in[\![b_2,c_1]\!]} - \delta_{c_2\in]\!]c_1,b_2[\![} \\ &\quad + \delta_{a_1\in]\!]b_2-1,c_2-1]\!]} - \delta_{a_1=b_1} + \delta_{a_1\in]\!]b_1-1,b_2-1]\!]} + \delta_{b_1=b_2} + \delta_{a_1=c_1} + \delta_{a_1\in]\!]c_2-1,c_1-1]\!]} \\ &= \delta_{a_1\in]\!]b_1,c_1]\!]} + 2\delta_{b_1\in[\![b_2,c_1]\!]}\delta_{c_2\in]\!]c_1,b_2[\![} - \delta_{b_1\in]\!]b_2,c_1]\!]} - \delta_{c_2\in]\!]c_1,b_2[\![} \\ &\quad + \delta_{a_1\in]\!]b_1-1,c_1-1]\!]} + \delta_{b_2\in]\!]c_1,b_1[\![} + \delta_{c_2\in]\!]c_1,b_2[\![} - \delta_{a_1=b_1} + \delta_{a_1=c_1} \\ &= 2\delta_{a_1\in]\!]b_1,c_1]\!]} + 2\delta_{b_1\in[\![b_2,c_1]\!]}\delta_{c_2\in]\!]c_1,b_2[\![}.\end{aligned}$$

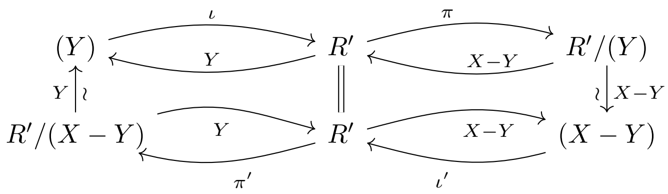
The other computations are analogous. □

Let us introduce the K-algebras that will play an important role in this paper.

As before, $R = K[X]$. We define $R' = K[X, Y]/(YX - Y^2)$. It is in fact an R-order of rank 2 (see Definition 2.17), and we have the following R-isomorphism with a classical Bass order:

$$R' \to R - R := \{(P, Q) \in R^2 \mid P - Q \in (X)\}, \quad Y \mapsto (0, X).$$

The three indecomposable Cohen–Macaulay R'-modules and irreducible morphisms over R appear in each of the two lines of the following commutative diagram:



where Y and $X - Y$ are multiplications, π are projections and ι are natural inclusions.

Finally, we denote by $\mathcal{R}'$ the algebra $K[u^{\pm 1}, v]/(vu - v^2)$ where R' is seen as a subalgebra of $\mathcal{R}'$ through the inclusion

$$R' \hookrightarrow \mathcal{R}', \quad X \mapsto u^{2n}, \quad Y \mapsto v^{2n}. \tag{2.16}$$

§2.4. Frozen Jacobian algebras are R-orders

Let (Q_σ, W_σ, F) be an ice quiver with potential arising from a tagged triangulation σ as defined in Section 2.2, and e_i be the trivial path of length 0 at vertex i. The main result of this section is that $\Gamma_\sigma := \mathcal{P}(Q_\sigma, W_\sigma, F)$ (Definition 2.1) is an R-order.

First, we introduce the definition of orders and Cohen–Macaulay modules over orders.

Definition 2.17. Let S be a commutative Noetherian ring of Krull dimension 1. An S-algebra A is called an *S-order* if it is a finitely generated S-module and $\operatorname{soc}_S A = 0$. For an S-order A, a left A-module M is called a (maximal) *Cohen–Macaulay A-module* if it is finitely generated as an S-module and $\operatorname{soc}_S M = 0$ (or equivalently $\operatorname{soc}_A M = 0$). We denote by $\operatorname{CM} A$ the category of Cohen–Macaulay A-modules. It is a full exact subcategory of $\operatorname{mod} A$.

Remark 2.18. If S is a principal ideal domain (e.g. $S = R$) and $M \in \operatorname{mod} S$, then $\operatorname{soc}_S M = 0$ if and only if M is free as an S-module.

We refer to [3], [8], [30] and [31] for more details about orders and Cohen–Macaulay modules.

The main theorem of this subsection is the following.

Theorem 2.19. *The frozen Jacobian algebra Γ_σ has the structure of an R-order.*

The remaining part of this subsection is devoted to proving Theorem 2.19. The strategy is to define a grading on Γ_σ, to prove that the centre $Z(\Gamma_\sigma)$ of Γ_σ is R', and to give its order structure as an R'-module. Notice that the centre of Jacobian algebras coming from surfaces without boundary was computed by Ladkani [27, Proposition 4.11].

We describe Γ_σ in detail in Proposition 2.26. Let us define a grading on Q_σ (and Q'_σ).

Definition 2.20 (θ-length). Let a and b be sides or tagged arcs of P^*, and $\alpha : a \to b$ be an arrow of Q'_σ. Let θ be the value of the oriented angle from $\vec{a}$ to $\vec{b}$ taken in $[0, 2\pi)$. We define the *θ-length* of α by

$$\ell^\theta(\alpha) = \frac{n}{\pi}\theta.$$

The θ-length of arrows extends additively to a map ℓ^θ from paths to integers, defining a grading on KQ_σ (and KQ'_σ) which will also be denoted by ℓ^θ.

Remark 2.21. Using Lemma 2.14, we easily see that $\ell^\theta(\alpha) = \ell^\theta_{a,b}$ for any arrow $\alpha : a \to b$. Indeed, if a and b share a common endpoint, then $0 \leq \ell^\theta_{a,b} < 2n$.

We now prove that for any tagged arcs or sides a and b of P^*, the possible θ-lengths of paths from a to b in Q_σ do not depend on the triangulation σ containing a and b.

Proposition 2.22. *Let σ and σ' be two different triangulations of the punctured polygon P^*. For any two edges a and b common to σ and σ', the minimal θ-length of paths from a to b in Q_σ is the same as the one in $Q_{\sigma'}$.*

Proof. Any two triangulations can be related by a sequence of flips such that each time we only change one arc in the related triangulation to get another one. Therefore, without losing generality, we can assume that the two triangulations σ and σ' have all arcs the same except one. We show the possible differences between σ and σ' in Figure 2.23.

It is sufficient to prove that for any two vertices common to both triangulations, and for any path between them in one triangulation, we can find a path with the same θ-length in the other triangulation. In each case, certain compositions of two arrows in one of the diagrams have to be replaced by one arrow in the other diagram. We can check case by case that the θ-lengths of both are equal.

For example, suppose that the triangulations only differ in a square not incident to the puncture as shown at the top of Figure 2.23. Considering the given position of the puncture, the arcs are (P_i, P_k), (P_i, P_j), (P_i, P_l), (P_j, P_k), (P_l, P_j)

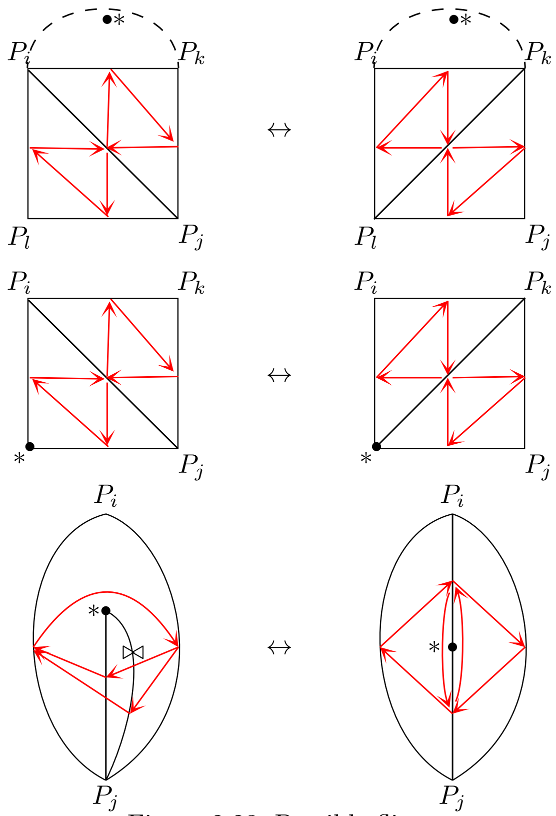


Figure 2.23. Possible flips.

and (P_l, P_k), and we get

$$\begin{aligned}&\ell^\theta((P_j, P_k), (P_i, P_j)) + \ell^\theta((P_i, P_j), (P_l, P_j))\\ &\quad = d(j, i) + d(k, j) + d(i, l) = d(j, l) + d(k, j) = \ell^\theta((P_j, P_k), (P_l, P_j)).\end{aligned}$$

The other cases are handled in the same way. □

Proposition 2.24. *The potential W_σ (resp. W'_σ) on Q_σ (resp. Q'_σ) is homogeneous of θ-length $2n$. Thus, ℓ^θ induces a grading on Γ_σ.*

Proof. Consider a true triangle of σ. Up to cyclic permutation, we can denote its three sides by a, b, c in clockwise order, satisfying $a_1 = b_2$, $a_2 = c_2$. Moreover, they satisfy either $b_1 = c_1$ and $a_1 \in \rrbracket b_1, a_2 \llbracket$ (if the triangle is not incident to the puncture), or $b_1 = b_2$ and $c_1 = c_2$ (if the triangle is incident to the puncture). In any case, the θ-length of the clockwise triangle induced by this true triangle is

$$
\begin{aligned}
&\ell^\theta_{a,b} + \ell^\theta_{b,c} + \ell^\theta_{c,a} \\
&\quad = (d(a_1,b_1) + d(b_1,c_1) + d(c_1,a_1)) + (d(a_2,b_2) + d(b_2,c_2) + d(c_2,a_2)) \\
&\qquad + n(|\delta_{a_1\in\rrbracket b_1,a_2\llbracket} - \delta_{b_2\in\rrbracket b_1,a_2\llbracket}| + |\delta_{b_1\in\rrbracket c_1,b_2\llbracket} - \delta_{c_2\in\rrbracket c_1,b_2\llbracket}| \\
&\qquad + |\delta_{c_1\in\rrbracket a_1,c_2\llbracket} - \delta_{a_2\in\rrbracket a_1,c_2\llbracket}|) \\
&\quad = n + n + n\times 0 = 2n.
\end{aligned}
$$

Using the flips introduced in the proof of Proposition 2.22, we can transform σ to a triangulation τ that consists of the sides of P and the tagged arcs incident to the puncture (as in Figure 2.29, p. 162). Moreover, using the reasoning of Proposition 2.22, flips clearly preserve the θ-length of anticlockwise planar cycles of Q'_σ winding around the puncture or vertices of P^*. Therefore, it is enough to see that W'_τ is homogeneous of θ-length $2n$. This is easy to check by calculation. As terms of W_σ are terms of W'_σ, W_σ is also homogeneous. □

Proposition 2.25. *Let a and b be two edges of σ which are not incident to the puncture or tagged in the same way. The minimal θ-length of paths from a to b in Q_σ is $\ell^\theta_{a,b}$.*

Proof. Let us prove first that there exists a path from a to b with θ-length $\ell^\theta_{a,b}$. We use induction on $\ell^\theta_{a,b}$. If it is 0, then $a = b$ and the result is obvious. If a and b are both incident to the puncture and not isotopic, the result is immediate (consider the triangulation τ consisting of the sides of P and the plain tagged arcs incident to the puncture).

Suppose that a is not incident to the puncture. We consider four cases:

- Suppose that $b_1, b_2 \neq a_2$. Consider an arc c such that $c_1 = a_1$ and $c_2 = a_2 + 1$ and tagged in the same way as b if $b_1 = b_2$ and $c_1 = c_2$. The arc c is either isotopic to the union of a and a side of the polygon (if $a_2 \neq a_1 - 1$), or to a part of a (if $a_2 = a_1 - 1$). In any case, a, b and c are compatible so we can choose a triangulation σ' containing a, b and c. In $Q_{\sigma'}$, there is an arrow α from a to c of θ-length 1, and as

$$
\begin{aligned}
\ell^\theta_{c,b} &= d(c_1,b_1) + d(c_2,b_2) + n|\delta_{c_1\in\rrbracket b_1,c_2\llbracket} - \delta_{b_2\in\rrbracket b_1,c_2\llbracket}| \\
&= d(a_1,b_1) + d(a_2+1,b_2) + n|\delta_{a_1\in\rrbracket b_1,a_2+1\llbracket} - \delta_{b_2\in\rrbracket b_1,a_2+1\llbracket}| \\
&= d(a_1,b_1) + d(a_2,b_2) - 1 + n|\delta_{a_1\in\rrbracket b_1,a_2\llbracket} - \delta_{b_2\in\rrbracket b_1,a_2\llbracket}| = \ell^\theta_{a,b} - 1,
\end{aligned}
$$

  we can apply the induction hypothesis: there is a path w from c to b of θ-length $\ell^\theta_{c,b}$ and the path αw has the expected θ-length $\ell^\theta_{a,b}$. Finally, thanks to Proposition 2.22, there is a path of the same θ-length in σ.

- Suppose that $b_1, b_2 \neq a_1$ and $a_2 \neq a_1 + 1$. The same reasoning works with $c_1 = a_1 + 1$ and $c_2 = a_2$.
- Suppose that $b_1 = a_2$ and $b_2 = a_1$. In this case, we construct σ' by taking a, b and the two tagged arcs connecting the puncture and a_2. In σ', there is an arrow from a to b which has, by definition, θ-length $\ell^\theta_{a,b}$.
- Suppose that $a_2 = a_1 + 1$ and $b_1 \neq a_1$. In this case, set $c = (P_{a_1+1}, P_{a_2+1})$ and complete to a triangulation σ' (containing b). We have an arrow α from a to c of θ-length 2. Moreover,

$$\begin{aligned}\ell^\theta_{c,b} &= d(c_1, b_1) + d(c_2, b_2) + n|\delta_{c_1 \in \rrbracket b_1, c_2 \llbracket} - \delta_{b_2 \in \rrbracket b_1, c_2 \llbracket}| \\ &= d(a_1 + 1, b_1) + d(a_2 + 1, b_2) + n|\delta_{a_1+1 \in \rrbracket b_1, a_2+1 \llbracket} - \delta_{b_2 \in \rrbracket b_1, a_2+1 \llbracket}| \\ &= d(a_1, b_1) + d(a_2, b_2) - 2 + n|\delta_{a_1 \in \rrbracket b_1, a_2 \llbracket} - \delta_{b_2 \in \rrbracket b_1, a_2 \llbracket}| = \ell^\theta_{a,b} - 2\end{aligned}$$

  and the same reasoning as before works.

The case where b is not incident to the puncture is similar.

Let us now prove by induction on the θ-length of any path w from a to b that this θ-length is at least $\ell^\theta_{a,b}$. When $a = b$ or if w is an arrow, it is clear. When a and b are different and w is not an arrow, w is the composition of two nonzero paths w' from a to c and w'' from c to b. By induction hypothesis and Lemma 2.15, we have

$$\ell^\theta(w) = \ell^\theta(w') + \ell^\theta(w'') \geq \ell^\theta_{a,c} + \ell^\theta_{c,b} \geq \ell^\theta_{a,b}. \qquad \square$$

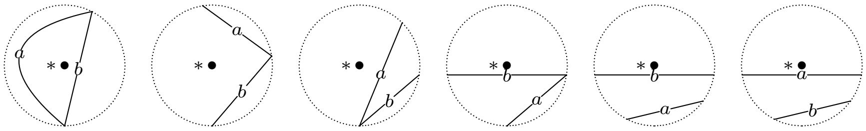
At the end of this section, we will prove that Γ_σ has the structure of an R-order and we will specify its structure. More precisely, we prove that the centre of Γ_σ is isomorphic to R' and we realize Γ_σ as a matrix algebra whose entries are free R-submodules of $\mathcal{R}'$ (see (2.16), p. 153).

For each vertex d of Q'_σ, we will define an element C_d of $e_d \Gamma_\sigma e_d$ as follows.

- If Q'_σ does not contain Figure 2.10 as a subtriangulation, all the planar cycles at d are equivalent (because of commutativity relations). Denote by C_d the common value of these planar cycles in Γ_σ.
- If Q'_σ contains Figure 2.10 as a subtriangulation, if $d \neq j$, all cycles at d that are planar in the full subquiver $Q'_\sigma \smallsetminus \{j\}$ are equivalent. Denote by C_d their common value in Γ_σ. We define $C_j = 0$.

For each frozen vertex a of Q'_σ, we denote by E_a the big cycle at a passing through each external arrow once. Finally, if a and b are two vertices of Q'_σ corresponding to edges which are not incident to the puncture, we write $a \vdash b$ if

$a_2 \in \,]\!]b_2, a_1[\![$ or $b_1 \in \,]\!]b_2, a_1[\![$. We have $a \vdash b$ in the following cases:



Fix a grading of $\mathcal{R}'$ such that u and v have degree 1. Then R' is graded as a subalgebra of $\mathcal{R}'$ (X and Y have degree $2n$). If a and b are two vertices of Q'_σ, we consider a graded R'-submodule $A_{a,b}$ of $\mathcal{R}'$ (that is also free over R) defined in the following way:

- $A_{a,b} = 0$ if a and b are incident to the puncture and tagged differently (as i and j in Figure 2.10);
- $A_{a,b} = u^{\ell^\theta_{a,b}-1} v R'$ if one of a and b is incident to the puncture and plain and the other one either incident to the puncture and plain, or not incident to the puncture;
- $A_{a,b} = u^{\ell^\theta_{a,b}-1}(u-v)R'$ if one of a and b is incident to the puncture and notched and the other one either incident to the puncture and notched, or not incident to the puncture;
- $A_{a,b} = u^{\ell^\theta_{a,b}} R' + v^{\ell^\theta_{a,b}} R'$ if a and b are not incident to the puncture and $a \vdash b$;
- $A_{a,b} = u^{\ell^\theta_{a,b}} R'$ if a and b are not incident to the puncture and $a \nvdash b$.

It is an easy consequence of Lemma 2.15 that $A := (A_{a,b})_{a,b \in Q'_{\sigma,0}}$ is an R-subalgebra of the matrix algebra $\mathrm{M}_{Q'_{\sigma,0}}(\mathcal{R}')$.

Proposition 2.26. *There exists an isomorphism of graded algebras $\phi_\sigma : R' \to Z(\Gamma_\sigma)$ ($Z(\Gamma_\sigma)$ is graded by θ-length). Moreover, for the induced R'-algebra structure of Γ_σ, there is an isomorphism of graded R'-algebras $\psi_\sigma : A \to \Gamma_\sigma$ induced by isomorphisms of graded R'-modules*

$$\psi_\sigma^{a,b} : A_{a,b} \xrightarrow{\sim} e_a \Gamma_\sigma e_b$$

(Γ_σ is graded by θ-length). Finally, the following properties are satisfied:

(i)$_\sigma$ *for each frozen vertex a of Q_σ,*

$$e_a \phi_\sigma(X) = \phi_\sigma(X) e_a = E_a;$$

(ii)$_\sigma$ *for each vertex a of Q_σ,*

$$e_a \phi_\sigma(Y) = \phi_\sigma(Y) e_a = \begin{cases} e_a \phi_\sigma(X) - C_a & \text{if } \sigma \text{ has no plain arc incident to the puncture,} \\ C_a & \text{otherwise;} \end{cases}$$

(iii)$_\sigma$ *for any pair of frozen vertices a and b, $\psi_\sigma^{a,b}(u^{\ell^\theta_{a,b}})$ is equivalent to the shortest path among paths consisting of external arrows from a to b;*

(iv)$_\sigma$ *for any pair of frozen vertices a and b such that a is an immediate successor of b in anticlockwise order ($b_2 = a_1$), let $s_{a,b}$ be the path from a to b whose composition with the external arrow $b \to a$ is the anticlockwise external cycle winding around a_1; then*

$$\psi_\sigma^{a,b}(v^{\ell^\theta_{a,b}}) = \begin{cases} \psi_\sigma^{a,b}(u^{\ell^\theta_{a,b}}) - s_{a,b} & \text{if } \sigma \text{ has no plain arc incident to the puncture,} \\ s_{a,b} & \text{otherwise;} \end{cases}$$

(v)$_\sigma$ *for any external arrow α of Q'_σ and any $w \in \Gamma_\sigma$, if $\alpha w = 0$ then $e_{t(\alpha)}w = 0$, and if $w\alpha = 0$ then $we_{s(\alpha)} = 0$.*

First of all, when the triangulation has only notched arcs incident to the puncture, the situation is similar to the fully plain one. Then, up to applying the R-automorphism of R' given by $Y \mapsto X - Y$ and the $K[u^{\pm 1}]$-automorphism of $\mathcal{R}'$ given by $v \mapsto u - v$, both results are equivalent (note that this pair of automorphisms commutes with the inclusion $R' \subset \mathcal{R}'$). From now on, we will only look at cases where triangulations have at most one notched arc incident to the puncture.

Observe that $(v)_\sigma$ is in fact a consequence of the rest of the proposition. Indeed, suppose that α is an external arrow from a vertex a to a vertex b and suppose that $w \in \Gamma_\sigma$ satisfies $\alpha w = 0$. Without loss of generality, we can suppose that $w = e_b w e_c$ for some vertex c of Q'_σ. Thus, there is $p \in A_{b,c}$ such that $w = \psi_\sigma^{b,c}(p)$. Then, thanks to (i)$_\sigma$ and the multiplicativity of ψ_σ, we have

$$0 = E_b w = \phi_\sigma(X) e_b \psi_\sigma^{b,c}(p) = \phi_\sigma(X)\psi_\sigma^{b,c}(p) = \psi_\sigma^{b,c}(Xp),$$

and as $\psi_\sigma^{b,c}$ is injective, we get $Xp = 0$ in $A_{b,c}$. Since $A_{b,c}$ is free over $R \subset R'$, it follows that $p = 0$ and therefore $w = 0$. The other case is dealt with in the same way.

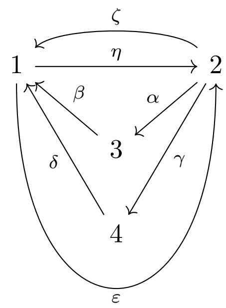
If we suppose that the existence of ϕ_σ as a morphism of algebras and (ii)$_\sigma$ are proved, we can easily deduce that ϕ_σ is graded. Indeed, we proved in Proposition 2.24 that $\ell^\theta(C_k) = 2n$. Thus, thanks to (ii)$_\sigma$, if σ has at most one notched edge incident to the puncture, then $\phi_\sigma(Y)$ is homogeneous of degree $2n$. Moreover, as ϕ_σ is a morphism of algebras, we have $\phi_\sigma(X)\phi_\sigma(Y) = \phi_\sigma(Y)\phi_\sigma(Y)$, so $\phi_\sigma(X)$ is also homogeneous of degree $2n$.

It is then automatic that $\psi_\sigma^{a,b}$ is graded for any vertices a and b (under the hypothesis that $\psi_\sigma^{a,b}$ is an isomorphism of R'-modules). Indeed, $\ell^\theta_{a,b}$ is by definition the minimal θ-length of a path from a to b.

The strategy for the rest of the proposition is to use induction on n. We start by proving the proposition for two families of initial cases.

Lemma 2.27. *Suppose that $n = 2$ and the triangulation σ consists of one plain and one notched arc incident to the puncture as in Figure 2.10. Then the conclusions of Proposition 2.26 are satisfied.*

Proof. In this case, the quiver Q'_σ is



and the potential is $W'_\sigma = \eta\alpha\beta + \eta\gamma\delta - \gamma\delta\varepsilon - \eta\zeta$, so the relations are $\beta\eta = \eta\alpha = 0$, $\delta\eta = \delta\varepsilon$, $\eta\gamma = \varepsilon\gamma$ and $\zeta = \alpha\beta + \gamma\delta$ (3 corresponds to the notched arc, and 4 to the plain one). Using these relations we get

Claim 1. *Any path different from η can be, up to the relations, expressed in a unique way without subpaths of the form η, ζ, $\beta\varepsilon\gamma$ or $\delta\varepsilon\alpha$ (the last two are 0 in Γ_σ).*

The θ-lengths are given by $\ell^\theta(\alpha) = \ell^\theta(\beta) = \ell^\theta(\gamma) = \ell^\theta(\delta) = 1$ and $\ell^\theta(\varepsilon) = \ell^\theta(\zeta) = \ell^\theta(\eta) = 2$.

Let us prove that there is an isomorphism

$$\phi_\sigma : R' \to Z(\Gamma_\sigma), \qquad X \mapsto \varepsilon\zeta + \zeta\varepsilon + \beta\varepsilon\alpha + \delta\varepsilon\gamma, \qquad Y \mapsto \varepsilon\gamma\delta + \gamma\delta\varepsilon + \delta\varepsilon\gamma.$$

It is easy to see that $\phi_\sigma(X)$ and $\phi_\sigma(Y)$ commute with all arrows, so the image is included in the centre. Moreover, we have

$$\begin{aligned}\phi_\sigma(X)\phi_\sigma(Y) &= \varepsilon\alpha\beta\varepsilon\gamma\delta + \varepsilon\gamma\delta\varepsilon\gamma\delta + \alpha\beta\varepsilon\gamma\delta\varepsilon + \gamma\delta\varepsilon\gamma\delta\varepsilon + \delta\varepsilon\gamma\delta\varepsilon\gamma \\ &= \varepsilon\gamma\delta\varepsilon\gamma\delta + \gamma\delta\varepsilon\gamma\delta\varepsilon + \delta\varepsilon\gamma\delta\varepsilon\gamma = \phi_\sigma(Y)^2,\end{aligned}$$

so ϕ_σ is a morphism. Notice that

$$\phi_1 = e_1\phi_\sigma e_1 : R' \to e_1 Z(\Gamma_\sigma)e_1, \qquad X \mapsto \varepsilon\alpha\beta + \varepsilon\gamma\delta, \qquad Y \mapsto \varepsilon\gamma\delta,$$

is an isomorphism. Indeed, surjectivity comes from Claim 1. For injectivity, notice that every element of R' can be (uniquely) written in the form $P + YQ$ where P

and Q are polynomials in X. Then, as $\varepsilon\alpha\beta\varepsilon\gamma\delta = \varepsilon\gamma\delta\varepsilon\alpha\beta = 0$,

$$\phi_1(P + YQ) = P(\varepsilon\alpha\beta) + P(\varepsilon\gamma\delta) - P(0)e_1 + \varepsilon\gamma\delta Q(\varepsilon\gamma\delta).$$

If $\phi_1(P + YQ) = 0$ then $\beta\phi_1(P + YQ) = \beta P(\varepsilon\alpha\beta) = 0$. As $\Gamma_\sigma/(e_4, \eta)$ is a path algebra (all relations are in the ideal (e_4, η) of KQ'_σ except $\zeta = \alpha\beta + \gamma\delta$), we get $P = 0$. Then $\varepsilon\gamma\delta Q(\varepsilon\gamma\delta) = 0$. As $\Gamma_\sigma/(e_3, \eta - \varepsilon)$ is a path algebra (all relations are in the ideal $(e_3, \eta - \varepsilon)$ of KQ'_σ except $\zeta = \alpha\beta + \gamma\delta$), we get $Q = 0$. Thus ϕ_1 is injective. We deduce immediately that ϕ_σ is also injective.

For the surjectivity of ϕ_σ, take an element z of $Z(\Gamma_\sigma)$. Using Claim 1, it is immediate that we can write

$$z = P_1(\varepsilon\alpha\beta) + Q_1(\varepsilon\gamma\delta) + P_2(\alpha\beta\varepsilon) + Q_2(\gamma\delta\varepsilon) + P_3(\beta\varepsilon\alpha) + Q_4(\delta\varepsilon\gamma)$$

where Q_1 and Q_2 have no constant terms (we make the convention that $(\varepsilon\alpha\beta)^0 = e_1$, $(\alpha\beta\varepsilon)^0 = e_2$, $(\beta\varepsilon\alpha)^0 = e_3$ and $(\delta\varepsilon\gamma)^0 = e_4$). Using the identity $\alpha z = z\alpha$, as $\delta\varepsilon\alpha = 0$, we get $P_2(\alpha\beta\varepsilon)\alpha = \alpha P_3(\beta\varepsilon\alpha)$ and, thanks to the grading by ℓ^θ, $P_2 = P_3$. In the same way, $\beta z = z\beta$ implies $P_1 = P_3$, $\gamma z = z\gamma$ implies $Q_2 = Q_4$, and $\delta z = z\delta$ implies $Q_1 = Q_4$. So $z = P_1(\varepsilon\alpha\beta + \alpha\beta\varepsilon + \beta\varepsilon\alpha) + Q_1(\varepsilon\gamma\delta + \gamma\delta\varepsilon + \delta\varepsilon\gamma) = \phi_\sigma(P_1(X - Y) + Q_1(Y))$.

It is an easy observation, using Claim 1, that $e_1 Z(\Gamma_\sigma)e_i = e_i\Gamma_\sigma e_i$ for every $i \in Q'_{\sigma,0}$. This permits one to compute, together with the θ-lengths given at the beginning, the following isomorphisms of R'-modules (denoted $\psi_\sigma^{a,b}$) from $A_{a,b}$ to $e_a\Gamma_\sigma e_b$ where $a, b \in [\![1, 4]\!]$:

a \ b	1	2	3	4
1	$1 \mapsto e_1$	$u^2 \mapsto \varepsilon, v^2 \mapsto \eta$	$(u-v)^3 \mapsto \varepsilon\alpha$	$v^3 \mapsto \varepsilon\gamma$
2	$u^2 \mapsto \zeta, v^2 \mapsto \gamma\delta$	$1 \mapsto e_2$	$u - v \mapsto \alpha$	$v \mapsto \gamma$
3	$u - v \mapsto \beta$	$(u-v)^3 \mapsto \beta\varepsilon$	$u^{-1}(u-v) \mapsto e_3$	0
4	$v \mapsto \delta$	$v^3 \mapsto \delta\varepsilon$	0	$u^{-1}v \mapsto e_4$

(note that $1 \vdash 2$ and $2 \vdash 1$). Points (i)$_\sigma$ to (iv)$_\sigma$ are easy to check. Multiplicativity can be easily checked case by case. □

Lemma 2.28. *Suppose that the triangulation consists of the (plain) arcs connecting each vertex of the polygon to the puncture. Then the conclusions of Proposition 2.26 are satisfied.*

Proof. For each i from 1 to n, denote by i the arc from P_i to P_{i+1} and by i' the arc from P_i to the puncture. Let α_i be the arrow of Q_σ from i' to $(i+1)'$, β_i the arrow from $(i+1)'$ to i, γ_i the arrow from i to i' and δ_i the arrow from $i-1$ to i (modulo n)

(Figure 2.29). Note that $\ell^\theta(\alpha_i) = \ell^\theta(\delta_i) = 2$, $\ell^\theta(\beta_i) = \ell^\theta(\gamma_i) = n-1$. The relations in Γ_σ are $\beta_i\gamma_i = \alpha_{i+1}\alpha_{i+2}\dots\alpha_{i-2}\alpha_i$, $\gamma_i\alpha_i = \delta_{i+1}\gamma_{i+1}$ and $\alpha_i\beta_i = \beta_{i-1}\delta_i$ for all $i \in [\![1, n]\!]$.

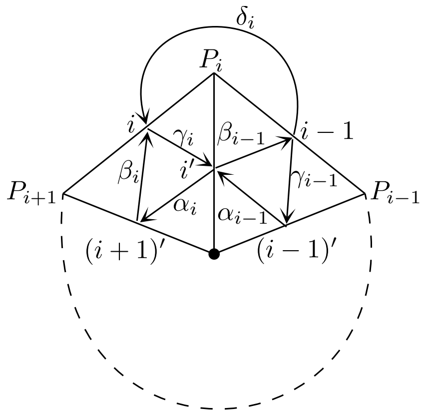


Figure 2.29. Initial case.

Notice that any path is equivalent to a path containing only arrows of type δ or to a path containing no arrow of type δ. Then, up to equivalence, a path containing no arrow of type δ can be supposed not to contain any arrow of type γ except maybe at the beginning, and not to contain any arrow of type β except maybe at the end. To summarize:

Claim 2. *Any path of Q'_σ is equivalent to a path of the form*

$$\delta_i\delta_{i+1}\dots\delta_j \quad \text{or} \quad \gamma_i^\mu\alpha_i\alpha_{i+1}\dots\alpha_j\beta_j^\nu,$$

where $\mu, \nu \in \{0, 1\}$.

For $i \in [\![1, n]\!]$, we denote $E_i = \delta_{i+1}\delta_{i+2}\dots\delta_i$, $C_i = \gamma_i\beta_{i-1}\delta_i = \gamma_i\alpha_i\beta_i = \delta_{i+1}\gamma_{i+1}\beta_i$ and $C_{i'} = \beta_{i-1}\delta_i\gamma_i = \beta_{i-1}\gamma_{i-1}\alpha_{i-1} = \alpha_i\alpha_{i+1}\dots\alpha_{i-1} = \alpha_i\beta_i\gamma_i$. Finally, we set

$$E = \sum_{i=1}^n (E_i + C_{i'}) \quad \text{and} \quad C = \sum_{i=1}^n (C_i + C_{i'}).$$

Let us prove that there is an isomorphism of algebras given by

$$\phi_\sigma : R' = K[X, Y]/(YX - Y^2) \to Z(\Gamma_\sigma), \qquad X \mapsto E, \qquad Y \mapsto C.$$

We easily see from the relations that $C_{i'}\alpha_i = \alpha_i C_{(i+1)'}$, $C_{(i+1)'}\beta_i = \beta_i C_i = \beta_i E_i$, $C_i\gamma_i = E_i\gamma_i = \gamma_i C_{i'}$, $\delta_i E_i = E_{i-1}\delta_i$, $\delta_i C_i = C_{i-1}\delta_i$, $\beta_i C_i = \beta_i E_i$, $C_i\gamma_i = E_i\gamma_i$ and $C_i E_i = C_i^2$. Therefore C and E are in the centre of Γ_σ, and ϕ_σ is a morphism of algebras.

Any element of R' can be written as $P(X) + YQ(Y)$ where P and Q are polynomials. If this element is in $\ker \phi_\sigma$, then $P(E) + CQ(C) = 0$. Notice now that there are no relations between a path which contains only arrows of type δ and any other path. Thus, we should have $P(E) = 0$ and then $P = 0$ as a polynomial (paths appearing in $CQ(C)$ contain arrows which are not of type δ). Then $CQ(C) = 0$. Powers of C have different θ-lengths, so $Q = 0$ and finally $\ker \phi_\sigma = 0$.

Let us prove that ϕ_σ is surjective. Let $z \in Z(\Gamma_\sigma)$. Using Claim 2 and $E_iC_i = C_i^2$, we can write

$$z = \sum_{i=1}^{n} [P_i(E_i) + Q_i(C_i) + S_i(C_{i'})]$$

for some polynomials P_i, Q_i and S_i where Q_i has no constant term (we make the convention that $E_i^0 = e_i$ and $C_{i'}^0 = e_{i'}$). For any i, $z\alpha_i = \alpha_i z$ implies that $S_i(C_{i'})\alpha_i = \alpha_i S_{i+1}(C_{(i+1)'})$, so using the grading by θ-length, $S_i = S_{i+1}$. In the same way, $z\beta_i = \beta_i z$ implies $S_{i+1}(C_{(i+1)'})\beta_i = \beta_i(P_i(E_i)+Q_i(C_i)) = (P_i(C_{(i+1)'})+ Q_i(C_{(i+1)'})\beta_i$ because $\beta_i E_i = \beta_i C_i = C_{(i+1)'}\beta_i$, so we get $S_{i+1} = P_i + Q_i$. Finally, $z\delta_i = \delta_i z$ implies $(P_{i-1}(E_{i-1}) + Q_{i-1}(C_{i-1}))\delta_i = \delta_i(P_i(E_i) + Q_i(C_i))$. As already observed before, there are no relations between paths containing only δ's and other paths. Thus, $P_{i-1}(E_{i-1})\delta_i = \delta_i P_i(E_i)$ and $Q_{i-1}(C_{i-1})\delta_i = \delta_i Q_i(C_i)$, and using θ-length, $P_{i-1} = P_i$ and $Q_{i-1} = Q_i$. Finally, we get

$$z = P_1\Big(\sum_{i=1}^{n} E_i\Big) + Q_1\Big(\sum_{i=1}^{n} C_i\Big) + (P_1 + Q_1)\Big(\sum_{i=1}^{n} C_{i'}\Big) = \phi_\sigma(P_1(X) + Q_1(Y)),$$

so ϕ_σ is surjective.

Let now i, j be two frozen vertices. Notice that

$$i \vdash j \;\Leftrightarrow\; i+1 \in \,]\!]j+1, i[\![\text{ or } j \in \,]\!]j+1, i[\![\;\Leftrightarrow\; j = i-1.$$

The following maps are isomorphisms of graded R'-modules:

$$\begin{aligned}
&\psi_\sigma^{i,j} : u^{2d(i,j)}R' \to e_i\Gamma_\sigma e_j, \quad u^{2d(i,j)} \mapsto \delta_{i+1}\delta_{i+2}\dots\delta_j \quad (j \neq i-1);\\
&\psi_\sigma^{i,i-1} : u^{2(n-1)}R' + v^{2(n-1)}R' \to e_i\Gamma_\sigma e_{i-1},\\
&\qquad\qquad u^{2(n-1)} \mapsto \delta_{i+1}\delta_{i+2}\dots\delta_{i-1},\; v^{2(n-1)} \mapsto \gamma_i\beta_{i-1};\\
&\psi_\sigma^{i,j'} : v^{2d(i,j)+n-1}R' \to e_i\Gamma_\sigma e_{j'}, \quad v^{2d(i,j)+n-1} \mapsto \gamma_i\alpha_i\dots\alpha_{j-1};\\
&\psi_\sigma^{i',j} : v^{2d(i,j+1)+n-1}R' \to e_{i'}\Gamma_\sigma e_j, \quad v^{2d(i,j+1)+n-1} \mapsto \alpha_i\dots\alpha_j\beta_j;\\
&\psi_\sigma^{i',j'} : v^{2d(i,j)}R' \to e_{i'}\Gamma_\sigma e_{j'}, \quad v^{2d(i,j)} \mapsto \alpha_i\dots\alpha_{j-1} \quad (j \neq i);\\
&\psi_\sigma^{i',i'} : u^{-1}vR' \to e_{i'}\Gamma_\sigma e_{i'}, \quad u^{-1}v \mapsto e_{i'}.
\end{aligned}$$

The argument mainly relies on Claim 2. Let us for example look at the second case. It is easy to check that

$$\psi_\sigma^{i,i-1}(v^{2(n-1)})(E - C) = 0 \quad \text{and} \quad \psi_\sigma^{i,i-1}(u^{2(n-1)} - v^{2(n-1)})C = 0,$$

so $\psi_\sigma^{i,i-1}$ is a morphism. Moreover, if an element $u^{2(n-1)}P(X) + v^{2(n-1)}Q(Y)$ is mapped to 0 by this map, using the same kind of analysis as before, we prove that $P = Q = 0$, so the map is injective. For surjectivity, notice that, for any path that does not contain δ from i to $i-1$, different from $\gamma_i\beta_{i-1}$, in the form given by Claim 2, we can write

$$\begin{aligned}\gamma_i\alpha_i \dots \alpha_{i-1}\beta_{i-1} &= \delta_{i+1}\gamma_{i+1}\alpha_{i+1}\dots\alpha_{i-1}\beta_{i-1} = \cdots \\ &= \delta_{i+1}\dots\delta_{i-1}\gamma_{i-1}C_{i-1}^k\alpha_{i-1}\beta_{i-1} = \delta_{i+1}\dots\delta_{i-1}C_{i-1}^{k+1},\end{aligned}$$

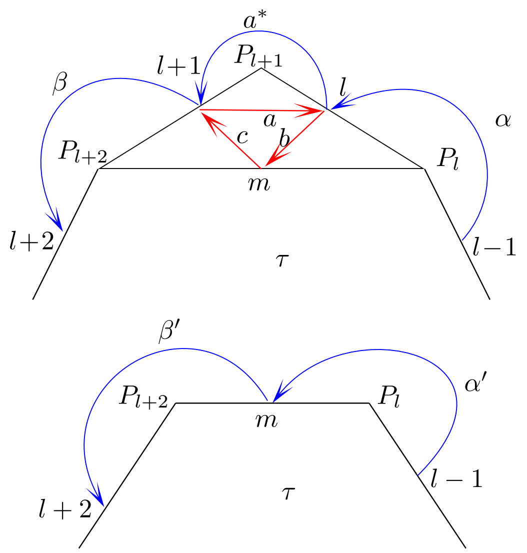
so the map is surjective.

Multiplicativity can be checked case by case. For example, if $i, j, k \in [\![1, n]\!]$ and $j \in]\!]i, k[\![$, we have

$$\begin{aligned}\psi_\sigma^{i,j'}(v^{2d(i,j)+n-1})\psi_\sigma^{j',k}(v^{2d(j,k+1)+n-1}) &= \gamma_i\alpha_i\alpha_{i+1}\dots\alpha_{j-1}\alpha_j\dots\alpha_k\beta_k \\ &= \delta_{i+1}\gamma_{i+1}\alpha_{i+1}\dots\alpha_k\beta_k = \cdots = \delta_{i+1}\dots\delta_k\gamma_k\alpha_k\beta_k = \delta_{i+1}\dots\delta_k C \\ &= \psi_\sigma^{i,k}(u^{2d(i,k)}Y) = \psi_\sigma^{i,k}(v^{2d(i,k)+2n}) = \psi_\sigma^{i,k}(v^{2d(i,j)+n-1}v^{2d(j,k+1)+n-1}).\end{aligned}$$

Points $(\mathrm{i})_\sigma$ to $(\mathrm{iv})_\sigma$ are easy to check (and recall that $(\mathrm{v})_\sigma$ is a consequence of them). □

Proof of Proposition 2.26. Suppose that the result is proved for all triangulations of polygons with $n-1$ vertices and that there is a *corner triangle* $P_lP_{l+1}P_{l+2}$ in the triangulation, as follows:



(if there is no corner triangle, we are either in the case of Lemma 2.27, or in the case of Lemma 2.28). By induction hypothesis, the expected results hold for τ. As the θ-length depends on the size of the polygon, we will denote by $\ell^{\theta,\tau}$ (resp. ℓ^{θ}) the θ-length in τ (resp. σ). In the same way, as the inclusion $R' \subset \mathcal{R}'$ depends on the triangulation, we will call u_τ and v_τ the generators of $\mathcal{R}'$ when we consider this inclusion for τ.

Then there is a nonunital monomorphism

$$\xi : KQ'_\tau \hookrightarrow KQ'_\sigma, \qquad \alpha' \mapsto \alpha b,\ \beta' \mapsto c\beta,\ \gamma \mapsto \gamma \text{ for } \gamma \in (Q'_\tau)_1 \smallsetminus \{\alpha', \beta'\}.$$

We have $W'_\sigma = \xi(W'_\tau) + abc - aa^*$, so for any $\gamma \in Q'_\tau$ which is not external, $\partial_\gamma W'_\sigma = \xi(\partial_\gamma W'_\tau)$. Thus ξ induces a morphism $\bar{\xi} : \Gamma_\tau \to \Gamma_\sigma$. Notice that by using the relations, any path of Q'_σ is equivalent to an element which does not contain a^*, ca or ab. It is then easy to see that:

Claim 3. *Any path of Q'_σ is equivalent to a path of one of the following forms where ω is a path of Q'_τ:*

$$\xi(\omega),\quad \xi(\omega)\alpha,\quad \xi(\omega)c,\quad b\xi(\omega),\quad \beta\xi(\omega),\quad b\xi(\omega)\alpha,\quad b\xi(\omega)c,\quad a,\quad \beta\xi(\omega)\alpha,\quad \beta\xi(\omega)c.$$

Let us prove that $\bar{\xi}$ is in fact injective. Let π_τ be the canonical projection $KQ'_\tau \to \Gamma_\tau$. Thanks to (iv)$_\tau$, we get

$$\begin{aligned}
\ker\bar{\xi} &= \pi_\tau\big(\xi^{-1}(\langle \partial_b W'_\sigma, \partial_c W'_\sigma\rangle + \xi(\mathcal{J}'(W'_\tau)))\big)\\
&= \pi_\tau\big(\xi^{-1}(\langle \partial_b W'_\sigma, \partial_c W'_\sigma\rangle) + \xi^{-1}(\xi(\mathcal{J}'(W'_\tau)))\big)\\
&= \pi_\tau(\xi^{-1}(\langle \partial_b W'_\sigma, \partial_c W'_\sigma\rangle)) + \pi_\tau(\mathcal{J}'(W'_\tau))\\
&= \pi_\tau\big(\xi^{-1}(\langle ca - \omega_{m,l-1}\alpha, ab - \beta\omega_{l+2,m}\rangle)\big)
\end{aligned}$$

where $\omega_{m,l-1} = \psi_\tau^{m,l-1}\big(v_\tau^{\ell^{\theta,\tau}_{m,l-1}}\big)$ and $\omega_{l+2,m} = \psi_\tau^{l+2,m}\big(v_\tau^{\ell^{\theta,\tau}_{l+2,m}}\big)$.

Up to equivalence, a path of Q'_σ can always be supposed not to contain a^*. Moreover, $e_m \operatorname{Im}(\xi) e_l = 0 = e_{l+1} \operatorname{Im}(\xi) e_m$, so

$$\xi^{-1}(\langle ca - \omega_{m,l-1}\alpha, ab - \beta\omega_{l+2,m}\rangle) = \xi^{-1}(\langle cab - \omega_{m,l-1}\alpha b, cab - c\beta\omega_{l+2,m}\rangle)$$

and

$$\begin{aligned}
\ker\bar{\xi} &= \pi_\tau\big(\xi^{-1}(\langle cab - \omega_{m,l-1}\alpha b, cab - c\beta\omega_{l+2,m}\rangle)\big)\\
&= \pi_\tau\big(\xi^{-1}(\langle c\beta\omega_{l+2,m} - \omega_{m,l-1}\alpha b, cab - c\beta\omega_{l+2,m}\rangle)\big)\\
&= \pi_\tau\big(\langle \beta'\omega_{l+2,m} - \omega_{m,l-1}\alpha'\rangle + \xi^{-1}(\langle cab - c\beta\omega_{l+2,m}\rangle)\big)\\
&= \pi_\tau\big(\xi^{-1}(\langle cab - c\beta\omega_{l+2,m}\rangle)\big) = \pi_\tau(0) = 0.
\end{aligned}$$

Let us prove that the following map is an isomorphism:

$$\phi_\sigma : Z(\Gamma_\tau) \cong R' \to Z(\Gamma_\sigma), \qquad \begin{aligned} X &\mapsto \bar{\xi}(\phi_\tau(X)) + E_l + E_{l+1}, \\ Y &\mapsto \bar{\xi}(\phi_\tau(Y)) + C_l + C_{l+1}. \end{aligned}$$

First of all, $\phi_\sigma(X), \phi_\sigma(Y) \in Z(\Gamma_\sigma)$. Indeed, for each arrow γ in Q_τ, we have, by using induction hypothesis,

$$\gamma\phi_\sigma(X) = \gamma\bar{\xi}(\phi_\tau(X)) = \bar{\xi}(\gamma\phi_\tau(X)) = \bar{\xi}(\phi_\tau(X)\gamma) = \bar{\xi}(\phi_\tau(X))\gamma = \phi_\sigma(X)\gamma,$$

and the same for Y. For arrows which are not in Q_τ we have

$$\begin{aligned} a\phi_\sigma(X) = aE_l = aa^*\beta \overbrace{\cdots}^{\text{ext.}} \alpha &= C_{l+1}\beta \overbrace{\cdots}^{\text{ext.}} \alpha \\ &= \beta C_{l+2} \overbrace{\cdots}^{\text{ext.}} \alpha = \cdots = \beta \overbrace{\cdots}^{\text{ext.}} C_l = E_{l+1}a = \phi_\sigma(X)a \end{aligned}$$

where ext. denote products of external arrows and, thanks to $(\mathrm{i})_\tau$,

$$\begin{aligned} b\phi_\sigma(X) &= bc\beta \overbrace{\cdots}^{\text{ext.}} \alpha b = E_l b = \phi_\sigma(X)b, \\ c\phi_\sigma(X) &= cE_{l+1} = c\beta \overbrace{\cdots}^{\text{ext.}} \alpha bc = \phi_\sigma(X)c, \\ \beta\phi_\sigma(X) &= \beta E_{l+2} = E_{l+1}\beta = \phi_\sigma(X)\beta, \\ \alpha\phi_\sigma(X) &= \alpha E_l = E_{l-1}\alpha = \phi_\sigma(X)\alpha. \end{aligned}$$

For $\phi_\sigma(Y)$, by using $(\mathrm{ii})_\tau$, all the computations are immediate.

Notice now that

$$\begin{aligned} C_l^2 &= bc(ab)(ca) = bc\beta\bar{\xi}\big(\psi_\tau^{l+2,m}\big(v_\tau^{\ell^{\theta,\tau}_{l+2,m}}\big)\big)\bar{\xi}\big(\psi_\tau^{m,l-1}\big(v_\tau^{\ell^{\theta,\tau}_{m,l-1}}\big)\big)\alpha \\ &= bc\beta\bar{\xi}\big(\psi_\tau^{l+2,l-1}\big(v_\tau^{\ell^{\theta,\tau}_{l+2,l-1}+2n}\big)\big)\alpha = bc\beta\bar{\xi}\big(\psi_\tau^{l+2,l-1}\big(Yu_\tau^{\ell^{\theta,\tau}_{l+2,l-1}}\big)\big)\alpha \\ &= bc\beta C_{l+2}\bar{\xi}\big(\psi_\tau^{l+2,l-1}\big(u_\tau^{\ell^{\theta,\tau}_{l+2,l-1}}\big)\big)\alpha \\ &= C_l bc\beta\bar{\xi}\big(\psi_\tau^{l+2,l-1}\big(u_\tau^{\ell^{\theta,\tau}_{l+2,l-1}}\big)\big)\alpha = C_l E_l \end{aligned}$$

and $C_{l+1}^2 = C_{l+1}E_{l+1}$ by the same method. Therefore

$$\begin{aligned} \phi_\sigma(YX) &= \bar{\xi}(\phi_\tau(Y))\bar{\xi}(\phi_\tau(X)) + C_l E_l + C_{l+1}E_{l+1} \\ &= \bar{\xi}(\phi_\tau(Y^2)) + C_l^2 + C_{l+1}^2 = \phi_\sigma(Y^2), \end{aligned}$$

so ϕ_σ is a morphism. As $\bar{\xi}$ and ϕ_τ are injective, ϕ_σ is also injective. The last thing to show is that ϕ_σ is surjective.

Using Claim 3 and the fact that, as it commutes with idempotents, every element $z \in Z(\Gamma_\sigma)$ is a linear combination of cycles, we can write

$$z = \bar{\xi}(z') + b\bar{\xi}(z'')\alpha + \beta\bar{\xi}(z''')c.$$

Then, as z is in the centre, for any $x \in \Gamma_\tau$,

$$\bar{\xi}(xz') = \bar{\xi}(x)\bar{\xi}(z') = \bar{\xi}(x)z = z\bar{\xi}(x) = \bar{\xi}(z')\bar{\xi}(x) = \bar{\xi}(z'x),$$

and, as $\bar{\xi}$ is injective, $xz' = z'x$ and $z' \in Z(\Gamma_\tau)$. Therefore, up to subtracting $\phi_\sigma(z')$, we can suppose that $z = b\bar{\xi}(z'')\alpha + \beta\bar{\xi}(z''')c$. Hence, we have

$$0 = z\alpha b = \alpha z b = \bar{\xi}(\alpha' z'' \alpha'),$$

and, as $\bar{\xi}$ is injective, $\alpha' z'' \alpha' = 0$. Finally, $e_m z'' e_{l-1} = 0$ thanks to $(\mathrm{v})_\tau$. In the same way $e_{l+2} z''' e_m = 0$, so $z = 0$. Therefore ϕ_σ is surjective.

We will prove the existence of ψ_σ as a family of morphisms of R'-modules. Then these morphisms are automatically graded by looking at homogeneous generators. If $i, j \notin \{l, l+1\}$, $\bar{\xi}$ induces an isomorphism of R'-modules from $e_i\Gamma_\tau e_j$ to $e_i\Gamma_\sigma e_j$. This proves the existence of $\psi_\sigma^{i,j}$ in this case.

Recall that $l = (P_l, P_{l+1})$, $l+1 = (P_{l+1}, P_{l+2})$ and $m = (P_l, P_{l+2})$. Thus, $l+1 \vdash l$, $l \nvdash l+1$, and for any $i \in Q'_{\sigma,0} \smallsetminus \{l, l+1\}$ which is not incident to the puncture, we have $i_1, i_2 \neq l+1$, so

$$\begin{aligned}
i \vdash l \;&\Leftrightarrow\; i_2 \in \rrbracket l+1, i_1 \llbracket \text{ or } l \in \rrbracket l+1, i_1 \llbracket \\
&\Leftrightarrow\; i_2 \in \rrbracket l, i_1 \llbracket \text{ or } l-1 \in \rrbracket l, i_1 \llbracket \;\Leftrightarrow\; i \vdash l-1; \\
l \vdash i \;&\Leftrightarrow\; l+1 \in \rrbracket i_2, l \llbracket \text{ or } i_1 \in \rrbracket i_2, l \llbracket \\
&\Leftrightarrow\; l+2 \in \rrbracket i_2, l \llbracket \text{ or } i_1 \in \rrbracket i_2, l \llbracket \;\Leftrightarrow\; m \vdash i; \\
i \vdash l+1 \;&\Leftrightarrow\; i_2 \in \rrbracket l+2, i_1 \llbracket \text{ or } l+1 \in \rrbracket l+2, i_1 \llbracket \\
&\Leftrightarrow\; i_2 \in \rrbracket l+2, i_1 \llbracket \text{ or } l \in \rrbracket l+2, i_1 \llbracket \;\Leftrightarrow\; i \vdash m; \\
l+1 \vdash i \;&\Leftrightarrow\; l+2 \in \rrbracket i_2, l+1 \llbracket \text{ or } i_1 \in \rrbracket i_2, l+1 \llbracket \\
&\Leftrightarrow\; l+3 \in \rrbracket i_2, l+2 \llbracket \text{ or } i_1 \in \rrbracket i_2, l+2 \llbracket \;\Leftrightarrow\; l+2 \vdash i.
\end{aligned}$$

Let $i \notin \{l, l+1\}$. There are isomorphisms of R'-modules

$$e_i\Gamma_\tau e_{l-1} \to e_i\Gamma_\sigma e_l,\ \omega \mapsto \bar{\xi}(\omega)\alpha; \quad e_{l+2}\Gamma_\tau e_i \to e_{l+1}\Gamma_\sigma e_i,\ \omega \mapsto \beta\bar{\xi}(\omega);$$
$$e_m\Gamma_\tau e_i \to e_l\Gamma_\sigma e_i,\ \omega \mapsto b\bar{\xi}(\omega); \qquad e_i\Gamma_\tau e_m \to e_i\Gamma_\sigma e_{l+1},\ \omega \mapsto \bar{\xi}(\omega)c.$$

Injectivity comes from $(\mathrm{v})_\tau$. For example, if $\bar{\xi}(\omega)\alpha = 0$ then $\bar{\xi}(\omega)\alpha b = 0$ and therefore $\bar{\xi}(\omega\alpha') = 0$, so $\omega\alpha' = 0$ and finally $\omega = 0$. For surjectivity, it is enough to use Claim 3. In the same way, there is an isomorphism of R'-modules

$$e_m\Gamma_\tau e_m \to e_l\Gamma_\sigma e_{l+1}, \qquad \omega \mapsto b\bar{\xi}(\omega)c.$$

Thus we get the expected R'-module structure for $e_i\Gamma_\sigma e_j$ except when $i = j \in \{l, l+1\}$ or $i = l+1$ and $j = l$.

Suppose that $i = j = l$. The elements of $e_l\Gamma_\sigma e_l$ are of the form $\lambda e_l + b\omega\alpha$ for $\lambda \in K$ and $\omega \in e_m\Gamma_\sigma e_{l-1}$. We already know that

$$e_m\Gamma_\sigma e_{l-1} \cong u^{\ell^\theta_{m,l-1}} R' + v^{\ell^\theta_{m,l-1}} R',$$

and we get the following isomorphism of R'-modules:

$$R' \cong_K K \oplus u^3(u^{\ell^\theta_{m,l-1}} R' + v^{\ell^\theta_{m,l-1}} R') \to e_l\Gamma_\sigma e_l, \qquad (\lambda, u^3 p) \mapsto \lambda e_l + b\psi_\sigma^{m,l-1}(p)\alpha$$

(injectivity comes from $(v)_\tau$ and the injectivity of $\bar{\xi}$).

In the same way, if $i = j = l + 1$, there is an isomorphism of R'-modules

$$\begin{aligned} R' \cong_K K \oplus u^3(u^{\ell^\theta_{l+2,m}} R' + v^{\ell^\theta_{l+2,m}} R') &\to e_{l+1}\Gamma_\sigma e_{l+1}, \\ (\lambda, u^3 p) &\mapsto \lambda e_{l+1} + \beta\psi_\sigma^{l+2,m}(p)c. \end{aligned}$$

Finally, suppose that $i = l + 1$ and $j = l$. The elements of $e_{l+1}\Gamma_\sigma e_l$ are of the form $\lambda a + \beta\omega\alpha$ and there is an isomorphism of R'-modules

$$\begin{aligned} u^{\ell^\theta_{l+1,l}} R' + v^{\ell^\theta_{l+1,l}} R' \cong_K K v^{\ell^\theta_{l+1,l}} \oplus u^4 u^{\ell^\theta_{l+2,l-1}} R' &\to e_{l+1}\Gamma_\sigma e_l, \\ (\lambda v^{\ell^\theta_{l+1,l}}, u^4 p) &\mapsto \lambda a + \beta\psi_\sigma^{l+2,l-1}(p)\alpha. \end{aligned}$$

Indeed, the only nonimmediate thing to check is $aE_l = aC_l$. By induction hypothesis (in particular $(\mathrm{v})_\tau$),

$$\begin{aligned} aE_l &= abc\beta\bar{\xi}\big(\psi_\tau^{l+2,l-1}\big(u_\tau^{\ell^{\theta,\tau}_{l+2,l-1}}\big)\big)\alpha = \beta\bar{\xi}\big(\psi_\tau^{l+2,l-1}\big(Y u_\tau^{\ell^{\theta,\tau}_{l+2,l-1}}\big)\big)\alpha \\ &= \beta\bar{\xi}\big(\psi_\tau^{l+2,l-1}\big(Y v_\tau^{\ell^{\theta,\tau}_{l+2,l-1}}\big)\big)\alpha = \beta\bar{\xi}\big(\psi_\tau^{l+2,m}\big(v_\tau^{\ell^{\theta,\tau}_{l+2,m}}\big)\big)\bar{\xi}\big(\psi_\tau^{m,l-1}\big(v_\tau^{\ell^{\theta,\tau}_{m,l-1}}\big)\big)\alpha \\ &= ab\bar{\xi}\big(\psi_\tau^{m,l-1}\big(v_\tau^{\ell^{\theta,\tau}_{m,l-1}}\big)\big)\alpha = aC_l. \end{aligned}$$

For the multiplicativity of the ψ_σ, let us start by noticing that, thanks to the beginning of the proof of Lemma 2.15, for any vertices i, j and k of Q'_τ, we have the identity

$$\frac{\ell^\theta_{i,j} + \ell^\theta_{j,k} - \ell^\theta_{i,k}}{n} = \frac{\ell^{\theta,\tau}_{i,j} + \ell^{\theta,\tau}_{j,k} - \ell^{\theta,\tau}_{i,k}}{n-1}.$$

It implies that $\psi_\sigma^{i,j}(w)\psi_\sigma^{j,k}(w') = \psi_\sigma^{i,k}(ww')$ for any $(w, w') \in A_{i,j} \times A_{j,k}$. Indeed, it is enough to prove this when w and w' are generators as R'-modules. Suppose for example that $w = u^{\ell^\theta_{i,j}}$ and $w' = u^{\ell^\theta_{j,k}}$. Then, by the induction hypothesis,

$$\begin{aligned}\psi_\sigma^{i,j}\big(u^{\ell^\theta_{i,j}}\big)\psi_\sigma^{j,k}\big(u^{\ell^\theta_{j,k}}\big) &= \bar{\xi}\big(\psi_\tau^{i,j}\big(u_\tau^{\ell^{\theta,\tau}_{i,j}}\big)\psi_\tau^{j,k}\big(u_\tau^{\ell^{\theta,\tau}_{j,k}}\big)\big) = \bar{\xi}\big(\psi_\tau^{i,k}\big(u_\tau^{\ell^{\theta,\tau}_{i,j}+\ell^{\theta,\tau}_{j,k}}\big)\big)\\ &= \bar{\xi}\big(\phi_\tau\big(X^{(\ell^{\theta,\tau}_{i,j}+\ell^{\theta,\tau}_{j,k}-\ell^{\theta,\tau}_{i,k})/2(n-1)}\big)\psi_\tau^{i,k}\big(u_\tau^{\ell^{\theta,\tau}_{i,k}}\big)\big)\\ &= \phi_\sigma\big(X^{(\ell^{\theta,\tau}_{i,j}+\ell^{\theta,\tau}_{j,k}-\ell^{\theta,\tau}_{i,k})/2(n-1)}\big)\psi_\sigma^{i,k}\big(u_\tau^{\ell^{\theta}_{i,k}}\big)\\ &= \phi_\sigma\big(X^{(\ell^{\theta}_{i,j}+\ell^{\theta}_{j,k}-\ell^{\theta}_{i,k})/2n}\big)\psi_\sigma^{i,k}\big(u^{\ell^{\theta}_{i,k}}\big) = \psi_\sigma^{i,k}\big(u^{\ell^\theta_{i,j}+\ell^\theta_{j,k}}\big).\end{aligned}$$

Multiplicativity for paths starting or ending at l or $l+1$ can be deduced easily from that. For example, if i and k are vertices of τ,

$$\begin{aligned}\psi_\sigma^{i,l}\big(u^{\ell^\theta_{i,l}}\big)\psi_\sigma^{l,k}\big(u^{\ell^\theta_{l,k}}\big) &= \psi_\sigma^{i,l-1}\big(u^{\ell^\theta_{i,l-1}}\big)\alpha b\psi_\sigma^{m,k}\big(u^{\ell^\theta_{m,k}}\big)\\ &= \psi_\sigma^{i,l-1}\big(u^{\ell^\theta_{i,l-1}}\big)\psi_\sigma^{l-1,m}(u^3)\psi_\sigma^{m,k}\big(u^{\ell^\theta_{m,k}}\big)\\ &= \psi_\sigma^{i,k}\big(u^{\ell^\theta_{i,l-1}+3+\ell^\theta_{m,k}}\big) = \psi_\sigma^{i,k}\big(u^{\ell^\theta_{i,l}+\ell^\theta_{l,k}}\big).\end{aligned}$$

The last thing to check are the five additional conditions. Points (i)$_\sigma$ to (iv)$_\sigma$ are easy to check and (v)$_\sigma$ is a consequence of them. ☐

Theorem 2.30. *There is an isomorphism of R-orders (and R'-algebras)*

$$e_F\Gamma_\sigma e_F = [u^{2d(i,j)}R' + v^{2d(i,j)}R'^{\delta_{j=i-1}}]_{i,j\in[\![1,n]\!]} \cong \Lambda$$

where the entries of $e_F\Gamma_\sigma e_F$ are R'-submodules of $\mathcal{R}'$ and Λ is defined at (1.2).

For each edge a of σ, the $e_F\Gamma_\sigma e_F$-module

$$M_a := e_F\Gamma_\sigma e_a = [A_{1,a}, A_{2,a}, \ldots, A_{n,a}]^{\mathrm{t}}$$

is, as a Λ-module, isomorphic to

$$\begin{array}{ll} [\overbrace{(Y)\ldots(Y)}^{a_1}\overbrace{(Y^2)\ldots(Y^2)}^{n-a_1}]^{\mathrm{t}} & \textit{if } a=(P_{a_1},*);\\ [\overbrace{(X-Y)\ldots(X-Y)}^{a_1}\overbrace{(X^2-Y^2)\ldots(X^2-Y^2)}^{n-a_1}]^{\mathrm{t}} & \textit{if } a=(P_{a_1},\bowtie);\\ [\overbrace{R'\ldots R'}^{a_1}\overbrace{(X,Y)\ldots(X,Y)}^{a_2-a_1}\overbrace{(X)\ldots(X)}^{n-a_2}]^{\mathrm{t}} & \textit{if } a_1<a_2;\\ [\overbrace{(X,Y)\ldots(X,Y)}^{a_2}\overbrace{(X)\ldots(X)}^{a_1-a_2}\overbrace{(X^2,Y^2)\ldots(X^2,Y^2)}^{n-a_1}]^{\mathrm{t}} & \textit{if } a_1>a_2.\end{array}$$

Proof. If we arrange the sides of the polygon in the order (P_1,P_2), (P_2,P_3), $\ldots$, (P_{n-1},P_n), (P_n,P_1), the first equality is a direct application of Proposition 2.26. Notice that for sides i and j of the polygon, we can rewrite $A_{i,j}$ as

$$A_{i,j} = u^{2d(i,j)}R' + v^{2d(i,j)}R'^{\delta_{j=i-1}}.$$

We conjugate by the diagonal matrix with diagonal entries $u^{2d(1,i)}$ for $i \in [\![1,n]\!]$; then the matrix we obtain has entries

$$\begin{aligned} u^{2d(1,i)}&\big(u^{2d(i,j)}R' + v^{2d(i,j)}R'^{\delta_{j=i-1}}\big)u^{-2d(1,j)} \\ &= u^{2d(i,j)-1+2d(1,i)-2d(1,j)}\big(uR' + vR'^{\delta_{j=i-1}}\big) \\ &= u^{2n\delta_{i\in]\!]j,1[\![}-1}\big(uR' + vR'^{\delta_{j=i-1}}\big) = X^{\delta_{i>j}}R' + X^{-\delta_{i\le j}}YR'^{\delta_{j=i-1}} \end{aligned}$$

for $i, j \in [\![1,n]\!]$. It is Λ.

We obtain the structure of M_a, up to some degree shift, by multiplying on the left by the same diagonal matrix. □

Remark 2.31. Notice that in Theorem 2.30, the module M_a depends only on the edge a and not on the triangulation σ.

§2.5. Counterexample with more than one puncture

In this subsection, we show that we cannot expect to generalize these results to polygons with more than one puncture.

We take the triangulation σ of a twice-punctured digon of Figure 2.32. It induces the quiver Q_σ on the right and using the same definition as in Section 2, we find that the natural analogue of Γ_σ is the path algebra of the quiver modulo all obvious commutativity relations. We still call it Γ_σ. Suppose that Γ_σ is a $K[U]$-order, for U in the centre of Γ_σ. We can write $e_1U = P(\alpha\beta) + a\omega$ where P is a polynomial and $\omega \in e_6\Gamma_\sigma e_1$. As, for $\ell > 0$, $c(\alpha\beta)^\ell$ is clearly not divisible by c on the right, and as $cU = Uc$, we infer that P is a constant polynomial. So, if we denote by $\pi : e_1\Gamma_\sigma e_1 \to K[\alpha\beta]$ the canonical projection, we get $\pi(U) = P(0) \in K$. Hence $K[\alpha\beta]$ is not a finitely generated module over $K[\pi(U)] = K$ and therefore $e_1\Gamma_\sigma e_1$ is not a finitely generated module over $K[U]$, a contradiction.

This counterexample is easy to generalize to any polygon with at least two punctures.

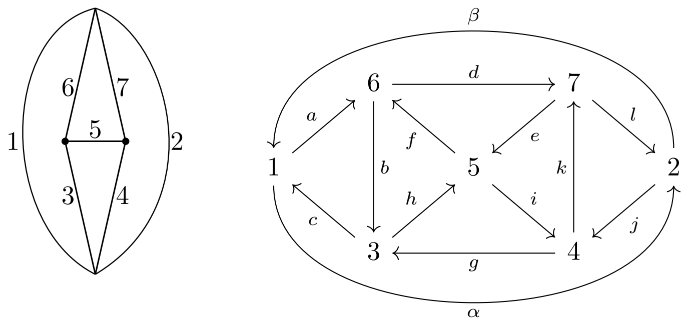


Figure 2.32. Twice-punctured digon.

§3. Cohen–Macaulay modules over Λ

The aim of this section is to study the representation theory of Λ and its connection to tagged triangulations of the punctured polygon P^* and the cluster category of type D_n. In particular, we classify all Cohen–Macaulay Λ-modules and construct a bijection between the set of isomorphism classes of all indecomposable Cohen–Macaulay Λ-modules and the set of all sides and tagged arcs of P^*. We then show that the stable category $\underline{\mathrm{CM}}\,\Lambda$ of Cohen–Macaulay Λ-modules is 2-Calabi–Yau and $\underline{\mathrm{CM}}\,\Lambda$ is triangle-equivalent to the cluster category of type D_n. To summarize, we will prove that CM Λ admits the Auslander–Reiten quiver of Figures 3.1 and 3.2.

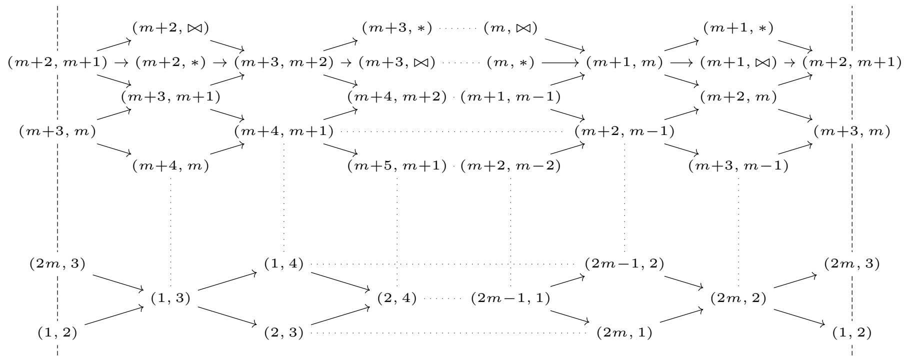


Figure 3.1. CM Λ for $n = 2m$.

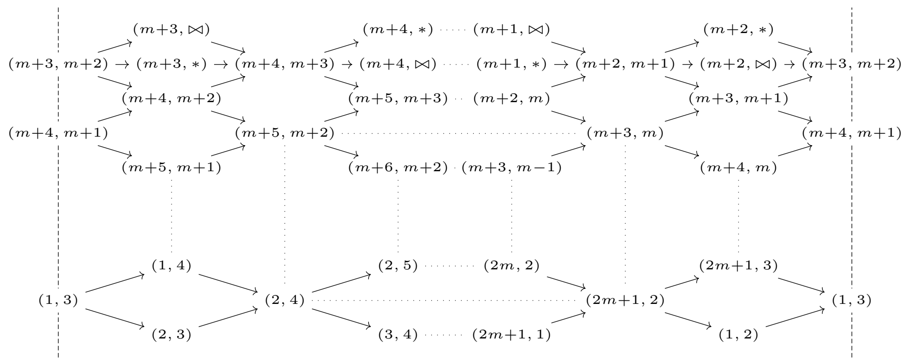


Figure 3.2. CM Λ for $n = 2m + 1$.

§3.1. Classification of Cohen–Macaulay Λ-modules

Let $\mathcal{S}$ be the set of tagged arcs and sides of the once-punctured polygon P^*. In this subsection, we prove the following theorem:

Theorem 3.3. (1) *There is a bijection between $\mathcal{S}$ and the set of isomorphism classes of indecomposable Cohen–Macaulay Λ-modules given by $a \mapsto M_a$ (M_a is defined in Theorem 2.30).*

(2) *Any Cohen–Macaulay Λ-module is isomorphic to $\bigoplus_{a\in\mathcal{S}} M_a^{l_a}$ for some non-negative integers l_a. Moreover, the l_a are uniquely determined.*

Remark 3.4. Theorem 3.3 shows that the Krull–Schmidt–Azumaya property is valid in this case. This is interesting in its own right since our base ring $R = K[X]$ is not even local, and in such a case, this property usually fails.

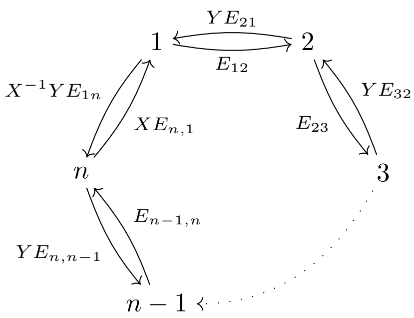
First of all, it is immediate that the M_a are nonisomorphic indecomposable Cohen–Macaulay Λ-modules. To prove this theorem, let us define the following elements of Λ:

$$\alpha_i = E_{i,i+1}, \quad \alpha_n = XE_{n,1}, \quad \beta_i = YE_{i+1,i}, \quad \beta_n = X^{-1}YE_{1,n}.$$

Together with the idempotents E_{ii}, they generate Λ as an R'-algebra and satisfy the relations

$$\begin{cases} \alpha_i\alpha_{i+1}\dots\alpha_{i-1} = XE_{ii}, \\ \beta_{i-1}\beta_{i-2}\dots\beta_i = Y^{n-1}E_{ii}, \\ \alpha_i\beta_i = \beta_{i-1}\alpha_{i-1} = YE_{ii}, \end{cases}$$

for $i \in [\![1, n]\!]$. In fact, the quiver of Λ is



Lemma 3.5. *Let $r \in (Y)^{\oplus m}$, $s \in R'^{\oplus p}$ and $t \in (X-Y)^{\oplus q}$ be vectors such that the ideal I generated by their entries includes the ideal (X, Y). Then there exists an invertible $(m+p+q)\times(m+p+q)$ matrix*

$$G = \begin{pmatrix} A & B & 0 \\ C & D & E \\ 0 & F & G \end{pmatrix}$$

with coefficients in R' where B has coefficients in (Y) and F has coefficients in $(X-Y)$ such that

- *either* $G[r\,s\,t]^{\mathrm{t}}$ *contains one* 1 *in its second block and* 0 *everywhere else;*
- *either* $G[r\,s\,t]^{\mathrm{t}}$ *contains one* Y *in the first or second block, one* $X-Y$ *in the second or third block and* 0 *everywhere else.*

Proof. The proof mainly relies on the Euclidean algorithm. We can write $r = r'Y$, $s = s' + s''Y$ and $t = t'(X-Y)$ where $r' \in R^{\oplus m}$, $s', s'' \in R^{\oplus p}$ and $t' \in R^{\oplus q}$. Up to applying the Euclidean algorithm on the entries of s' and then on the entries of s'' (which is multiplying by an invertible matrix on the left), we can suppose that

$$s = [Q_1 + Q_2Y \quad Q_3Y]^{\mathrm{t}}$$

for some $Q_1, Q_2, Q_3 \in R$ (we can ignore 0 entries). With the same method, we can suppose that r has only one (nonzero) entry PY and t has only one (nonzero) entry $S(X-Y)$. Using a sequence of (feasible) matrix multiplications

$$\begin{bmatrix} 1 & 0 \\ T(X-Y) & 1 \end{bmatrix}\begin{bmatrix} Q_1+Q_2Y \\ S(X-Y) \end{bmatrix} = \begin{bmatrix} Q_1+Q_2Y \\ (S+TQ_1)(X-Y) \end{bmatrix},$$
$$\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}\begin{bmatrix} Q_1+Q_2Y \\ S(X-Y) \end{bmatrix} = \begin{bmatrix} (Q_1+TSX)+(Q_2-TS)Y \\ S(X-Y) \end{bmatrix},$$

thanks to the Euclidean algorithm, we can assume that $Q_1 = 0$ or $S = 0$. In the same way, we can assume that $Q_3 = 0$ or $P = 0$. If $Q_1 = 0$, we can ensure that at most one of Q_2, Q_3 and P is nonzero. To summarize, and forgetting about 0 entries, we can assume to be in one of the following four cases:

1. $r = PY$, $s = 0$, $t = S(X-Y)$: In this case, by our assumption,

   $$(Y) \oplus (X-Y) = (X,Y) \subset I = (PY, S(X-Y)) = (PY) \oplus (S(X-Y)),$$

   so we deduce that $(Y) = (PY)$ and $(X-Y) = (S(X-Y))$, so up to scalar multiplication $P = S = 1$. This is one of the expected cases.
2. $r = 0$, $s = QY$, $t = S(X-Y)$: In this case, by the same reasoning as before, up to scalar multiplication $Q = S = 1$. This is one of the expected cases.
3. $r = PY$, $s = Q_1 + Q_2Y$, $t = 0$: In this case, we rewrite $s = Q_1' + Q_2'(X-Y)$ where $Q_2' = -Q_2$ and $Q_1' = Q_1 + Q_2X$. Up to a sequence of feasible matrix multiplications

   $$\begin{bmatrix} 1 & 0 \\ T & 1 \end{bmatrix}\begin{bmatrix} PY \\ Q_1'+Q_2'(X-Y) \end{bmatrix} = \begin{bmatrix} PY \\ (Q_1'+TPX)+(Q_2'-TP)(X-Y) \end{bmatrix},$$
   $$\begin{bmatrix} 1 & TY \\ 0 & 1 \end{bmatrix}\begin{bmatrix} PY \\ Q_1'+Q_2'(X-Y) \end{bmatrix} = \begin{bmatrix} (P+TQ_1')Y \\ Q_1'+Q_2'(X-Y) \end{bmatrix},$$

we can suppose that $P = 0$ or $Q'_1 = 0$. If $Q'_1 = 0$, we are in the previous case and we can conclude. If $P = 0$, then I is a principal ideal containing (X, Y), so $I = R'$ and, up to scalar multiplication, $Q'_1 + Q'_2(X - Y) = 1$. We are again in an expected case.

4. $r = 0$, $s = [Q_1 + Q_2Y \;\; Q_3Y]$, $t = 0$: This case in similar to the previous one. ☐

Lemma 3.6. *Let $M = R' \oplus M_2$ and N be Cohen–Macaulay R'-modules and $f : M \to N$ and $g, g' : N \to M$ be morphisms satisfying $gf = X\,\mathrm{Id}_M$, $fg = X\,\mathrm{Id}_N$, $g'f = Y\,\mathrm{Id}_M$ and $fg' = Y\,\mathrm{Id}_N$. There exists an isomorphism $\phi : N \to N_1 \oplus N_2$ such that*

$$\phi f = \begin{bmatrix} \psi_{11} & \psi_{12} \\ 0 & \psi_{22} \end{bmatrix}, \qquad g\phi^{-1} = \begin{bmatrix} \chi_{11} & \chi_{12} \\ 0 & \chi_{22} \end{bmatrix}, \qquad g'\phi^{-1} = \begin{bmatrix} \chi'_{11} & \chi'_{12} \\ 0 & \chi'_{22} \end{bmatrix},$$

where either

- *$N_1 = R'$, $\psi_{11} = 1$, $\chi_{11} = X$ and $\chi'_{11} = Y$, or*
- *$N_1 = (X, Y)$, $\psi_{11} = X$, χ_{11} is the inclusion and χ'_{11} maps both X and Y to Y.*

Proof. Let $f_1 : R' \to N$ be

$$f_1 = f \circ \begin{bmatrix} \mathrm{Id}_{R'} \\ 0 \end{bmatrix}.$$

As (Y), R' and $(X - Y)$ are the only isomorphism classes of indecomposable Cohen–Macaulay R'-modules, we can decompose, up to isomorphism of N,

$$N = (Y)^{\oplus m} \oplus R'^{\oplus p} \oplus (X - Y)^{\oplus q}, \qquad f_1 = \begin{bmatrix} r \\ s \\ t \end{bmatrix},$$

where r is a vector with entries in (Y), s is a vector with entries in R' and t is a vector with entries in $(X - Y)$. Using $gf = X\,\mathrm{Id}_N$ and $g'f = Y\,\mathrm{Id}_N$, we conclude that the ideal generated by the entries of r, s and t contains (X, Y), so, thanks to Lemma 3.5, up to multiplying f on the left by an invertible matrix and reordering the rows, we can suppose that we are in one of the following cases:

1. $N = R' \oplus N_2$ and

$$f = \begin{bmatrix} 1 & * \\ 0 & * \end{bmatrix}.$$

   In this case, we can write

$$g = \begin{bmatrix} X & * \\ 0 & * \end{bmatrix} \quad \text{and} \quad g' = \begin{bmatrix} Y & * \\ 0 & * \end{bmatrix}$$

   using the identities $gf = X\,\mathrm{Id}_N$ and $g'f = Y\,\mathrm{Id}_N$. We are in the first expected case.

2. $N = (Y) \oplus R' \oplus N_2$ and
$$f = \begin{bmatrix} Y & * \\ X-Y & * \\ 0 & * \end{bmatrix}.$$
In this case, we can write
$$g = \begin{bmatrix} \iota_Y & 1 & * \\ * & * & * \end{bmatrix} \quad \text{and} \quad g' = \begin{bmatrix} \iota_Y & 0 & * \\ * & * & * \end{bmatrix}$$
Up to column operations on g and corresponding row operations on f, we can write
$$f = \begin{bmatrix} Y & * \\ X & 0 \\ 0 & * \end{bmatrix}, \quad g = \begin{bmatrix} 0 & 1 & 0 \\ * & * & * \end{bmatrix}, \quad g' = \begin{bmatrix} \iota_Y & 0 & * \\ * & * & * \end{bmatrix}$$
(the 0 in the second column of f comes from $gf = X \operatorname{Id}_N$). It is now easy to see that we cannot get $fg' = Y \operatorname{Id}_M$. So this case is excluded.
3. $N = (Y) \oplus (X - Y) \oplus N_2$ and
$$f = \begin{bmatrix} Y & * \\ X-Y & * \\ 0 & * \end{bmatrix}.$$
In this case, we can write
$$g = \begin{bmatrix} \iota_Y & \iota_{X-Y} & * \\ 0 & 0 & * \end{bmatrix} \quad \text{and} \quad g' = \begin{bmatrix} \iota_Y & 0 & * \\ 0 & 0 & * \end{bmatrix}$$
(once again, we use the fact that (Y) and $(X - Y)$ are in a direct sum). Using the equality $(Y) \oplus (X - Y) = (X, Y)$, we are in the second expected case.
4. $N = R' \oplus R' \oplus N_2$ and
$$f = \begin{bmatrix} Y & * \\ X-Y & * \\ 0 & * \end{bmatrix}.$$
We can write
$$g = \begin{bmatrix} 1 & 1 & * \\ * & * & * \end{bmatrix} \quad \text{and} \quad g' = \begin{bmatrix} 1 & 0 & * \\ * & * & * \end{bmatrix}$$
As in (ii), using column operations on g, we can rewrite
$$f = \begin{bmatrix} Y & * \\ X & 0 \\ 0 & * \end{bmatrix}, \quad g = \begin{bmatrix} 0 & 1 & 0 \\ * & * & * \end{bmatrix}, \quad g' = \begin{bmatrix} 1 & 0 & * \\ * & * & * \end{bmatrix},$$
and this contradicts $fg' = Y \operatorname{Id}_M$.

5. $N = R' \oplus (X-Y) \oplus N_2$ and

$$f = \begin{bmatrix} Y & * \\ X-Y & * \\ 0 & * \end{bmatrix}.$$

We can write

$$g = \begin{bmatrix} 1 & \iota_{X-Y} & * \\ * & * & * \end{bmatrix} \quad \text{and} \quad g' = \begin{bmatrix} 1 & 0 & * \\ * & * & * \end{bmatrix}$$

Using column operations on g and g', we can rewrite

$$f = \begin{bmatrix} Y & 0 \\ X-Y & * \\ 0 & * \end{bmatrix}, \quad g = \begin{bmatrix} 1 & \iota_{X-Y} & * \\ * & * & * \end{bmatrix}, \quad g' = \begin{bmatrix} 1 & 0 & 0 \\ * & * & * \end{bmatrix}$$

(the 0 in the second column of f comes from $g'f = Y\,\mathrm{Id}_N$). We cannot have $fg = X\,\mathrm{Id}_M$, so this case is excluded. □

We can easily dualize the previous lemma (over R'):

Lemma 3.7. *Let $M = R' \oplus M_2$ and N be Cohen–Macaulay R'-modules and $f : N \to M$ and $g, g' : M \to N$ be morphisms satisfying $gf = X\,\mathrm{Id}_N$, $fg = X\,\mathrm{Id}_M$, $g'f = Y\,\mathrm{Id}_N$ and $fg' = Y\,\mathrm{Id}_M$. There exists an isomorphism $\phi : N \to N_1 \oplus N_2$ such that*

$$f\phi^{-1} = \begin{bmatrix} \psi_{11} & 0 \\ \psi_{21} & \psi_{22} \end{bmatrix}, \quad \phi g = \begin{bmatrix} \chi_{11} & 0 \\ \chi_{21} & \chi_{22} \end{bmatrix}, \quad \phi g' = \begin{bmatrix} \chi'_{11} & 0 \\ \chi'_{21} & \chi'_{22} \end{bmatrix},$$

where either

- $N_1 = R'$, $\psi_{11} = 1$, $\chi_{11} = X$ *and* $\chi'_{11} = Y$, *or*
- $N_1 = (X, Y)$, ψ_{11} *is the inclusion*, $\chi_{11} = X$ *and* $\chi'_{11} = Y$.

Lemma 3.8. *Let M be a Cohen–Macaulay Λ-module. If M, as an R'-module, has a direct summand isomorphic to R', then M has a direct summand isomorphic to M_a for some tagged arc or side a of P^* which is not incident to the puncture.*

Proof. For $i \in [\![1, n]\!]$, let $M_i = E_{ii}M$. By abuse of notation, we denote by $\alpha_i : M_{i+1} \to M_i$ and $\beta_i : M_i \to M_{i+1}$ the morphisms of R'-modules corresponding to the elements with the same names in Λ.

Let $i, j \in [\![1, n]\!]$ be such that $\alpha_i \alpha_{i+1} \dots \alpha_{j-1} \alpha_j$ has a direct summand isomorphic to

$$R' \xrightarrow{X} R'$$

(such a pair exists as M contains R' as a direct summand, and $\alpha_i\alpha_{i+1}\dots\alpha_{i-1} = X\,\mathrm{Id}_{M_i}$ for any $i \in [\![1,n]\!]$). If $j < i$, note that $\alpha_n\alpha_1$ appears in the previous composition. The number k of factors of this composition is $d(i,j)+1$. We make the additional assumption that k is as small as possible. Without loss of generality, we can suppose that $i = 1 \leq j$ (the problem is invariant under cyclic permutation). Using Lemma 3.6 for $f = \alpha_j$, $g = \alpha_{j+1}\alpha_{j+2}\dots\alpha_{j-1}$ and $g' = \beta_j$, we find that actually $j > 1$ and we can suppose that $M_{j+1} = R' \oplus M'_{j+1}$, $M_j = (X,Y) \oplus M'_j$ and

$$\alpha_j = \begin{bmatrix} X & \alpha_{j,12} \\ 0 & \alpha_{j,22} \end{bmatrix}$$

(the other possibility of Lemma 3.6 would contradict the minimality of k). Then, we easily see that we can write $M_1 = R' \oplus M'_1$ and

$$\gamma = \alpha_1 \dots \alpha_{j-1} = \begin{bmatrix} \iota_{(X,Y)} & \gamma_{12} \\ 0 & \gamma_{22} \end{bmatrix}$$

where $\iota_{(X,Y)} : (X,Y) \to R'$ is the inclusion. Note that by hypothesis

$$\gamma\alpha_j = \begin{bmatrix} X & 0 \\ 0 & * \end{bmatrix}.$$

As all morphisms to R' which are in the radical of $\mathrm{CM}\,R'$ factor through $\iota_{(X,Y)} : (X,Y) \to R'$, by column operations on γ which do not affect the previous shapes we can suppose that one of the following holds:

- $\gamma_{12} = 0$;
- $M'_j = R' \oplus M''_j$ and

  $$\gamma = \begin{bmatrix} \iota_{(X,Y)} & 1 & 0 \\ 0 & * & * \end{bmatrix} \quad \text{and} \quad \alpha_j = \begin{bmatrix} X & \alpha_{j,12} \\ 0 & * \\ 0 & * \end{bmatrix};$$

  then by a column operation on γ, we get

  $$\gamma = \begin{bmatrix} 0 & 1 & 0 \\ * & * & * \end{bmatrix} \quad \text{and} \quad \alpha_j = \begin{bmatrix} X & \alpha_{j,12} \\ X & 0 \\ 0 & * \end{bmatrix}$$

  (the 0 in the second column of α_j comes from the shape of $\gamma\alpha_j$). But this contradicts the existence of β_j such that $\alpha_j\beta_j = Y\,\mathrm{Id}_{M_j}$.

Finally, we get the situation

$$\gamma = \begin{bmatrix} \iota_{(X,Y)} & 0 \\ 0 & \gamma_{22} \end{bmatrix} \quad \text{and} \quad \alpha_j = \begin{bmatrix} X & 0 \\ 0 & \alpha_{j,22} \end{bmatrix}.$$

Now, using Lemma 3.7 for $j > 2$ permits us to suppose that

$$\alpha_1 = \begin{bmatrix} \iota_{(X,Y)} & 0 \\ \alpha_{1,21} & \alpha_{1,22} \end{bmatrix}.$$

Then we easily get

$$\gamma' = \alpha_2 \dots \alpha_{j-1} = \begin{bmatrix} 1 & 0 \\ \gamma'_{21} & \gamma'_{22} \end{bmatrix}.$$

By row operations on γ' (and the corresponding ones on α_1), we can suppose that

$$\alpha_1 = \begin{bmatrix} \iota_{(X,Y)} & 0 \\ 0 & \alpha_{1,22} \end{bmatrix} \quad \text{and} \quad \gamma' = \begin{bmatrix} 1 & 0 \\ 0 & \gamma'_{22} \end{bmatrix}.$$

We also easily get

$$\gamma'' = \alpha_{j+1} \dots \alpha_n = \begin{bmatrix} 1 & 0 \\ 0 & \gamma''_{22} \end{bmatrix}.$$

Acting by automorphisms on $M_3, \dots, M_{j-1}$ if $j > 3$ and on $M_{j+2}, \dots, M_n$ if $j < n-1$ permits us easily to suppose that

$$\alpha_\ell = \begin{bmatrix} 1 & 0 \\ 0 & \alpha_{\ell,22} \end{bmatrix}$$

for any $\ell \in [\![2, j-1]\!] \cup [\![j+1, n]\!]$. Then we conclude that M has a direct summand isomorphic to

$$[R' \overbrace{(X,Y) \dots (X,Y)}^{j-1} \overbrace{(X) \dots (X)}^{n-j}]^{\mathrm{t}} \cong M_{(P_1,P_j)}. \qquad \square$$

Lemma 3.9. *Let M be a Cohen–Macaulay Λ-module. If M, as an R'-module, has no direct summand isomorphic to R' then M has a direct summand isomorphic to some M_a where a is a tagged arc of P^* incident to the puncture.*

Proof. Denote as before $M_i = E_{ii}M$. As an R'-module, M is a direct sum of copies of (Y) and $(X-Y)$. As there are no morphisms between (Y) and $(X-Y)$, we can suppose that only one of them appears as a summand of M and therefore the matrix coefficients of the α_i are just elements of R. Up to circular permutation, we can suppose that α_n is not invertible. Choose an R-basis $\{e_1, \dots, e_\ell\}$ of M_n such that e_1 is not in the image of α_n. By the usual Euclidean algorithm applied on the right of α_n, we can suppose that

$$\alpha_n = \begin{bmatrix} \lambda & 0 \\ * & * \end{bmatrix} \quad \text{and} \quad \alpha_1 \dots \alpha_{n-1} = \begin{bmatrix} \lambda' & 0 \\ * & * \end{bmatrix}.$$

As $\lambda'\lambda = X$ and e_1 is not reached by α_n, we can suppose up to a scalar change of basis that $\lambda = X$ and $\lambda' = 1$. Hence, by row operations on $\alpha_1 \dots \alpha_{n-1}$, we can suppose that

$$\alpha_n = \begin{bmatrix} X & 0 \\ 0 & * \end{bmatrix} \quad \text{and} \quad \alpha_1 \dots \alpha_{n-1} = \begin{bmatrix} 1 & 0 \\ 0 & * \end{bmatrix}$$

(the lower left 0 of α_n comes from $\alpha_n \alpha_1 \alpha_2 \dots \alpha_{n-1} = X \operatorname{Id}_{M_n}$). Therefore, by changes of basis of $M_2, \dots, M_{n-1}$, we can suppose that

$$\alpha_\ell = \begin{bmatrix} 1 & 0 \\ 0 & * \end{bmatrix}$$

for $\ell \in [\![1, n-1]\!]$. Finally, M has a direct summand isomorphic to

$$\overbrace{[(Y)\cdots(Y)]}^{n}{}^{\mathrm{t}} \cong M_{(P_n,*)} \quad \text{or} \quad \overbrace{[(X-Y)\cdots(X-Y)]}^{n}{}^{\mathrm{t}} \cong M_{(P_n,\bowtie)}. \qquad \square$$

Proof of Theorem 3.3. First of all, thanks to Lemmas 3.8 and 3.9, any Cohen–Macaulay Λ-module can be decomposed as expected.

For the uniqueness of the decomposition, we need to use Proposition 3.10 (notice that we do not use Theorem 3.3 in its proof). The endomorphism algebra of M_a is isomorphic to R' if a is not incident to the puncture and isomorphic to R if a is incident to the puncture. Moreover, any endomorphism factorizing through another indecomposable is in the ideal (X, Y) in the first case and in (X) in the second case. Thus, if we denote $\hat{\Lambda} = K[\![X]\!] \otimes_R \Lambda$, and consider the functor $K[\![X]\!] \otimes_R - : \operatorname{CM} \Lambda \to \operatorname{CM} \hat{\Lambda}$, nonisomorphic indecomposable objects are mapped to nonisomorphic objects, which are also indecomposable. Moreover, the endomorphism rings of the objects $K[\![X]\!] \otimes_R M_a$ are local, so we get the uniqueness of the decomposition of objects in the essential image of the functor $K[\![X]\!] \otimes_R -$. This permits us to conclude the proof. $\square$

§3.2. Homological structure of $\operatorname{CM} \Lambda$

The aim of this subsection is to compute spaces of morphisms and extensions in the category $\operatorname{CM} \Lambda \cong \operatorname{CM} \Lambda'$ where $\Lambda' = e_F \Gamma_\sigma e_F$. For convenience of notation, we will work with Λ'. Notice that the definitions of $a \vdash b$ and $A_{a,b}$ given for two tagged arcs a and b before Proposition 2.26 make sense even when a and b are not compatible (there are cases where $a \vdash b$ other than the one depicted there).

Proposition 3.10. *Let a and b be two tagged arcs or sides of P^*. In the notation of Proposition 2.26, we have*

$$\operatorname{Hom}_{\Lambda'}(M_a, M_b) \cong A_{a,b}.$$

Moreover, these morphisms are realized by right multiplication in $\mathcal{R}'$, and therefore composition of morphisms corresponds to multiplication in $\mathcal{R}'$.

Proof. First of all, for $i \in [\![1, n]\!]$, using Theorem 2.30, recall that $E_{ii}M_a \cong A_{i,a}$ and $E_{ii}M_b \cong A_{i,b}$ (in a compatible way with the Λ'-module structure). Thus, we know that for any i, $\mathrm{Hom}_{R'}(A_{i,a}, A_{i,b})$ can be realized as an R'-submodule of $\mathcal{R}'$ through multiplication. Namely,

$A_{i,a}$ \ $A_{i,b}$	$u^{j'-1}vR'$	$u^{j'-1}(u-v)R'$	$u^{j'}R'$
$u^{j-1}vR'$	$u^{j'-j-1}vR'$	0	$u^{j'-j+2n-1}vR'$
$u^{j-1}(u-v)R'$	0	$u^{j'-j-1}(u-v)R'$	$u^{j'-j+2n-1}(u-v)R'$
$u^{j}R'$	$u^{j'-j-1}vR'$	$u^{j'-j-1}(u-v)R'$	$u^{j'-j}R'$

where $j = \ell^{\theta}_{i,a}$ and $j' = \ell^{\theta}_{i,b}$ (the only other kind of $A_{i,a}$ or $A_{i,b}$ which can appear is $u^jR' + u^{j-1}vR' = u^{j-1}((u-v)R' \oplus vR')$, which can be realized as the direct sum of the first two rows, and the first two columns; in any of these cases, the sum is direct inside $\mathcal{R}'$).

If $f \in \mathrm{Hom}_{\Lambda'}(M_a, M_b)$, let $f_i \in \mathrm{Hom}_{R'}(A_{i,a}, A_{i,b})$ be its ith component. As $u^2E_{i,i+1} \in \Lambda'$, for any $m \in M_a$ we find that $f(u^2E_{i,i+1}m) = u^2E_{i,i+1}f(m)$. This can be rewritten as $f_i(u^2m_{i+1}) = u^2f_{i+1}(m_{i+1})$ or again $f_iu^2m_{i+1} = u^2f_{i+1}m_{i+1}$ if f_i, f_{i+1} are considered as elements of $\mathcal{R}'$. As u^2 is invertible in $\mathcal{R}'$, we get $f_im_{i+1} = f_{i+1}m_{i+1}$. This is true for any $m_{i+1} \in A_{i+1,a}$, so $f_i - f_{i+1}$ is in the annihilator of $A_{i+1,a}$. By Theorem 2.30, the annihilators of $A_{i+1,a}$ and $A_{i,a}$ are the same and included in

- $(u-v)\mathcal{R}'$ if a is incident to the puncture and plain;
- $v\mathcal{R}'$ if a is incident to the puncture and notched;
- 0 if a is not incident to the puncture.

Moreover, looking at the previous table, we find that

- $\mathrm{Hom}_{R'}(A_{i,a}, A_{i,b}) + \mathrm{Hom}_{R'}(A_{i+1,a}, A_{i+1,b}) \subset v\mathcal{R}'$ if a is incident to the puncture and plain;
- $\mathrm{Hom}_{R'}(A_{i,a}, A_{i,b}) + \mathrm{Hom}_{R'}(A_{i+1,a}, A_{i+1,b}) \subset (u-v)\mathcal{R}'$ if a is incident to the puncture and notched,

so $\mathrm{Hom}_{R'}(A_{i,a}, A_{i,b}) + \mathrm{Hom}_{R'}(A_{i+1,a}, A_{i+1,b})$ intersects the annihilator of $A_{i,a}$ at 0 and we obtain $f_i = f_{i+1}$.

Finally, we get

$$\mathrm{Hom}_{\Lambda'}(M_a, M_b) = \bigcap_{i=1}^{n} \mathrm{Hom}_{R'}(A_{i,a}, A_{i,b})$$

as R'-submodules of $\mathcal{R}'$.

(a) Suppose now that neither a nor b is incident to the puncture. For $i \in [\![1,n]\!]$, we have

$$A_{i,a} = \begin{cases} u^{\ell^\theta_{(P_i,P_{i+1}),a}} R' & \text{if } (P_i,P_{i+1}) \nvdash a, \\ u^{\ell^\theta_{(P_i,P_{i+1}),a}} (R' + u^{-1}vR') & \text{if } (P_i,P_{i+1}) \vdash a, \end{cases}$$

and

$$A_{i,b} = \begin{cases} u^{\ell^\theta_{(P_i,P_{i+1}),b}} R' & \text{if } (P_i,P_{i+1}) \nvdash b, \\ u^{\ell^\theta_{(P_i,P_{i+1}),b}} (R' + u^{-1}vR') & \text{if } (P_i,P_{i+1}) \vdash b. \end{cases}$$

Therefore,

$$\mathrm{Hom}_{R'}(A_{i,a},A_{i,b}) = \begin{cases} u^{\ell^\theta_{(P_i,P_{i+1}),b} - \ell^\theta_{(P_i,P_{i+1}),a}} R' & \text{if } (P_i,P_{i+1}) \nvdash a \text{ and } (P_i,P_{i+1}) \nvdash b, \\ u^{\ell^\theta_{(P_i,P_{i+1}),b} - \ell^\theta_{(P_i,P_{i+1}),a} + 2n} (R' + u^{-1}vR') & \text{if } (P_i,P_{i+1}) \vdash a \text{ and } (P_i,P_{i+1}) \nvdash b, \\ u^{\ell^\theta_{(P_i,P_{i+1}),b} - \ell^\theta_{(P_i,P_{i+1}),a}} (R' + u^{-1}vR') & \text{if } (P_i,P_{i+1}) \vdash b. \end{cases}$$

Using Lemma 2.15, we obtain

$$\mathrm{Hom}_{R'}(A_{i,a},A_{i,b}) = \begin{cases} u^{\ell^\theta_{a,b} - 2n(\delta_{b_2 \in]\!]b_1,a_2[\![}\delta_{a_1 \in [\![a_2,b_1]\!]} + \delta_{i \in]\!]a_1,b_1]\!]})} R' & \text{if } (P_i,P_{i+1}) \nvdash a \text{ and } (P_i,P_{i+1}) \nvdash b, \\ u^{\ell^\theta_{a,b} - 2n(\delta_{b_2 \in]\!]b_1,a_2[\![}\delta_{a_1 \in [\![a_2,b_1]\!]} + \delta_{i \in]\!]a_1,b_1]\!]} - 1)} (R' + u^{-1}vR') & \\ \qquad\qquad \text{if } (P_i,P_{i+1}) \vdash a \text{ and } (P_i,P_{i+1}) \nvdash b, \\ u^{\ell^\theta_{a,b} - 2n(\delta_{b_2 \in]\!]b_1,a_2[\![}\delta_{a_1 \in [\![a_2,b_1]\!]} + \delta_{i \in]\!]a_1,b_1]\!]})} (R' + u^{-1}vR') & \text{if } (P_i,P_{i+1}) \vdash b. \end{cases}$$

Notice that $(P_i,P_{i+1}) \vdash a$ if and only if $i \in]\!]a_1,a_2]\!]$.

(a-1) Suppose that $a \nvdash b$. This means that $a_2 \notin]\!]b_2,a_1[\![$ and $b_1 \notin]\!]b_2,a_1[\![$. In this case, $\delta_{b_2 \in]\!]b_1,a_2[\![}\delta_{a_1 \in [\![a_2,b_1]\!]} = 0$. Taking $i = a_1$, we have $(P_i,P_{i+1}) \nvdash a$ and $(P_i,P_{i+1}) \nvdash b$ and an easy computation gives $\mathrm{Hom}_{R'}(A_{i,a},A_{i,b}) = u^{\ell^\theta_{a,b}} R'$. The only way to get a smaller module would be in the case $(P_i,P_{i+1}) \vdash a$ and $(P_i,P_{i+1}) \nvdash b$, that is, $i \in]\!]a_1,a_2]\!] \cap]\!]b_2,b_1]\!]$. With the current hypotheses, we get $]\!]a_1,a_2]\!] \cap]\!]b_2,b_1]\!] \subset]\!]a_1,b_1]\!]$, so actually we cannot get a smaller module.

(a-2) Suppose now that $a \vdash b$. This means that $a_2 \in]\!]b_2,a_1[\![$ or $b_1 \in]\!]b_2,a_1[\![$. Let us consider two cases:

- $b_2 \in]\!]b_1,a_2[\![$ and $a_1 \in [\![a_2,b_1]\!]$: Taking $i = b_2$, we have $(P_i,P_{i+1}) \vdash a$ and $(P_i,P_{i+1}) \nvdash b$, and an easy computation gives

  $$\mathrm{Hom}_{R'}(A_{i,a},A_{i,b}) = u^{\ell^\theta_{a,b}}(R' + u^{-1}vR').$$

  Thanks to the term $\delta_{b_2 \in]\!]b_1,a_2[\![}\delta_{a_1 \in [\![a_2,b_1]\!]}$, submodules that appear for any other i are bigger.

- $b_2 \notin \rrbracket b_1, a_2 \llbracket$ or $a_1 \notin \llbracket a_2, b_1 \rrbracket$: In this case, as $b_1 \neq b_2$, we see that in fact $b_1 \in \rrbracket b_2, a_1 \llbracket$. Taking $i = a_1$, we obtain $(P_i, P_{i+1}) \vdash b$ and

$$\operatorname{Hom}_{R'}(A_{i,a}, A_{i,b}) = u^{\ell^\theta_{a,b}}(R' + u^{-1}vR').$$

  Other submodules are bigger, because if $(P_i, P_{i+1}) \nvdash b$, that is, $i \in \rrbracket b_2, b_1 \rrbracket$, we would have $i \in \rrbracket a_1, b_1 \rrbracket$.

We finished the case where neither a nor b is incident to the puncture.

(b) Suppose now that both a and b are incident to the puncture. For $i \in \llbracket 1, n \rrbracket$, we have

$$A_{i,a} = \begin{cases} u^{\ell^\theta_{(P_i,P_{i+1}),a}-1}vR' & \text{if } a \text{ is plain,} \\ u^{\ell^\theta_{(P_i,P_{i+1}),a}-1}(u-v)R' & \text{if } a \text{ is notched,} \end{cases}$$

$$A_{i,b} = \begin{cases} u^{\ell^\theta_{(P_i,P_{i+1}),b}-1}vR' & \text{if } b \text{ is plain,} \\ u^{\ell^\theta_{(P_i,P_{i+1}),b}-1}(u-v)R' & \text{if } b \text{ is notched.} \end{cases}$$

As there are no morphisms if the tags are different, we can suppose that both a and b are plain, and we obtain, for any $i \in \llbracket 1, n \rrbracket$,

$$\operatorname{Hom}_{R'}(A_{i,a}, A_{i,b}) = u^{\ell^\theta_{(P_i,P_{i+1}),b} - \ell^\theta_{(P_i,P_{i+1}),a} - 1}vR'.$$

Using Lemma 2.15 and the fact that $b_1 = b_2$, we deduce that

$$\operatorname{Hom}_{R'}(A_{i,a}, A_{i,b}) = u^{\ell^\theta_{a,b} - 2n\delta_{i \in \rrbracket a_1, b_1 \rrbracket} - 1}vR',$$

and for $i = a_1$, $\operatorname{Hom}_{R'}(A_{i,a}, A_{i,b}) = u^{\ell^\theta_{a,b}-1}vR'$, which is of course the smallest possible.

(c) Suppose now that a is incident to the puncture and b is not. Without loss of generality, we can suppose a is plain. We have $(X - Y)M_a = 0$. Therefore, for any $f \in \operatorname{Hom}_{\Lambda'}(M_a, M_b)$, $(X - Y) \operatorname{Im} f = 0$. Notice now that

$$M'_b = \{m \in M_b \mid (X - Y)m = 0\}$$

satisfies

$$E_{ii}M'_b = \begin{cases} u^{\ell^\theta_{(P_i,P_{i+1}),b}+2n-1}vR' & \text{if } (P_i, P_{i+1}) \nvdash b \\ u^{\ell^\theta_{(P_i,P_{i+1}),b}-1}vR' & \text{if } (P_i, P_{i+1}) \vdash b \end{cases} = u^{\ell^\theta_{(P_i,P_{i+1}),b} + 2n\delta_{i \in \rrbracket b_2, b_1 \rrbracket} - 1}vR'.$$

Let $b' = (P_{b_2}, *)$. Thanks to Lemma 2.15, we can rewrite

$$\begin{aligned} \ell^\theta_{(P_i,P_{i+1}),b} + 2n\delta_{i \in \rrbracket b_2, b_1 \rrbracket} &= \ell^\theta_{(P_i,P_{i+1}),b'} + \ell^\theta_{b',b} - 2n\delta_{i \in \rrbracket b_2, b_1 \rrbracket} + 2n\delta_{i \in \rrbracket b_2, b_1 \rrbracket} \\ &= \ell^\theta_{(P_i,P_{i+1}),b'} + \ell^\theta_{b',b}. \end{aligned}$$

Thus $M'_b = u^{\ell^\theta_{b',b}} M_{b'}$ and

$$\operatorname{Hom}_{\Lambda'}(M_a, M_b) = u^{\ell^\theta_{b',b}} \operatorname{Hom}_{\Lambda'}(M_a, M_{b'}) = u^{\ell^\theta_{b',b}+\ell^\theta_{a,b'}-1} vR',$$

and we have (using $a_1 = a_2$)

$$\ell^\theta_{b',b} + \ell^\theta_{a,b'} = d(b_2, b_1) + 2d(a_1, b_2) = d(a_1, b_1) + n\delta_{b_2 \in \rrbracket b_1, a_1 \llbracket} + d(a_1, b_2) = \ell^\theta_{a,b},$$

which concludes this case.

(d) Finally, suppose that b is incident to the puncture and a is not. Without loss of generality, we can suppose b is plain. As $(X - Y)M_b = 0$,

$$\operatorname{Hom}_{\Lambda'}(M_a, M_b) = \operatorname{Hom}_{\Lambda'}(M'_a, M_b)$$

where $M'_a = M_a/(X - Y)M_a$. Using the same idea as before, we find that

$$E_{ii} M'_a = u^{\ell^\theta_{(P_i,P_{i+1}),a}-1} vR',$$

and thanks to Lemma 2.15, if $a' = (P_{a_1}, *)$,

$$\begin{aligned}\ell^\theta_{(P_i,P_{i+1}),a} &= \ell^\theta_{(P_i,P_{i+1}),a'} + \ell^\theta_{a',a} - 2n(\delta_{a_2 \in \rrbracket a_1, a_1 \llbracket} \delta_{a_1 \in \llbracket a_1, a_1 \rrbracket} + \delta_{i \in \rrbracket a_1, a_1 \rrbracket}) \\ &= \ell^\theta_{(P_i,P_{i+1}),a'} + \ell^\theta_{a',a} - 2n,\end{aligned}$$

and therefore $M'_a = u^{\ell^\theta_{a',a}-2n} M_{a'}$. Thus

$$\operatorname{Hom}_{\Lambda'}(M_a, M_b) = u^{2n-\ell^\theta_{a',a}} \operatorname{Hom}_{\Lambda'}(M_{a'}, M_b) = u^{2n-\ell^\theta_{a',a}+\ell^\theta_{a',b}-1} vR',$$

and, since $b_1 = b_2$,

$$\begin{aligned}2n - \ell^\theta_{a',a} + \ell^\theta_{a',b} &= 2n - (d(a_1, a_2) + n) + 2d(a_1, b_1) \\ &= 2n - (n - d(a_2, a_1) + n) + 2d(a_1, b_1) \\ &= d(a_1, b_1) + d(a_2, b_1) + n\delta_{a_1 \in \rrbracket b_1, a_2 \llbracket} = \ell^\theta_{a,b}.\end{aligned}$$

This concludes the proof. □

Proposition 3.11. *Let a and b be two tagged arcs or sides. Let M_a and M_b be the corresponding indecomposable Λ'-modules. We have the following isomorphisms of graded R'-modules:*

- $\underline{\operatorname{Hom}}_{\Lambda'}(M_a, M_b) = 0$ *if a and b are both incident to the puncture with different tags;*
- $\underline{\operatorname{Hom}}_{\Lambda'}(M_a, M_b) = u^{\ell^\theta_{a,b}}(R'/(X,Y))^{\oplus\varepsilon}$, *where*

$$\varepsilon = \delta_{a_1-1 \in \rrbracket b_1, b_2 \llbracket} \delta_{b_2+1 \in \rrbracket a_1, a_2 \llbracket} = \delta_{a_2-1 \in \rrbracket b_1, b_2 \llbracket} \delta_{b_1+1 \in \rrbracket a_1, a_2 \llbracket},$$

if either a and b are both incident to the puncture with the same tag, or exactly one of them is incident to the puncture;

- $\underline{\mathrm{Hom}}_{\Lambda'}(M_a, M_b) = u^{\ell^\theta_{a,b}}(R'/(X,Y))^{\oplus\varepsilon}$, *where*

$$\varepsilon = \delta_{a_1-1\in\rrbracket b_1,b_2\llbracket}\delta_{b_2+1\in\rrbracket a_1,a_2\llbracket} + \delta_{a_2-1\in\rrbracket b_1,b_2\llbracket}\delta_{b_1+1\in\rrbracket a_1,a_2\llbracket},$$

if neither a nor b is incident to the puncture.

Proof. (a) Suppose first that neither a nor b is incident to the puncture. For any $i \in \llbracket 1, n\rrbracket$, let P_i be the projective module corresponding to the arc (P_i, P_{i+1}) of the polygon. Thanks to Proposition 3.10, we have

$$\mathrm{Hom}_{\Lambda'}(M_a, P_i) = \begin{cases} u^{\ell^\theta_{a,(P_i,P_{i+1})}}R' & \text{if } a \nvdash (P_i, P_{i+1}), \\ u^{\ell^\theta_{a,(P_i,P_{i+1})}}(R' + u^{-1}vR') & \text{if } a \vdash (P_i, P_{i+1}), \end{cases}$$

and

$$\mathrm{Hom}_{\Lambda'}(P_i, M_b) = \begin{cases} u^{\ell^\theta_{(P_i,P_{i+1}),b}}R' & \text{if } (P_i, P_{i+1}) \nvdash b, \\ u^{\ell^\theta_{(P_i,P_{i+1}),b}}(R' + u^{-1}vR') & \text{if } (P_i, P_{i+1}) \vdash b. \end{cases}$$

As a consequence,

$$\begin{aligned}&\mathrm{Hom}_{\Lambda'}(P_i, M_b) \circ \mathrm{Hom}_{\Lambda'}(M_a, P_i) \\ &\quad= \begin{cases} u^{\ell^\theta_{a,(P_i,P_{i+1})}+\ell^\theta_{(P_i,P_{i+1}),b}}R' & \text{if } a \nvdash (P_i, P_{i+1}) \text{ and } (P_i, P_{i+1}) \nvdash b, \\ u^{\ell^\theta_{a,(P_i,P_{i+1})}+\ell^\theta_{(P_i,P_{i+1}),b}}(R' + u^{-1}vR') & \text{if } a \vdash (P_i, P_{i+1}) \text{ or } (P_i, P_{i+1}) \vdash b. \end{cases}\end{aligned}$$

Using Lemma 2.15, we get

$$\ell^\theta_{a,(P_i,P_{i+1})} + \ell^\theta_{(P_i,P_{i+1}),b} = \ell^\theta_{a,b} + 2n(\delta_{a_1\in\rrbracket a_2-1,b_1\rrbracket}\delta_{b_2\in\rrbracket a_2-1,b_1\rrbracket} + \delta_{i\in\rrbracket b_1,a_2-1\llbracket}).$$

The minimum is reached for $i \in \llbracket a_2 - 1, b_1\rrbracket$ and is

$$\ell^\theta_{a,b} + 2n\delta_{b_2\in\rrbracket a_2-1,b_1\rrbracket}\delta_{a_1\in\rrbracket a_2-1,b_1\rrbracket} = \ell^\theta_{a,b} + 2n\delta_{a_2-1\in\rrbracket b_1,b_2\llbracket}\delta_{b_1+1\in\rrbracket a_1,a_2\llbracket}.$$

Recall now that $a \vdash (P_i, P_{i+1})$ if and only if $a_2 \in \rrbracket i+1, a_1\llbracket$ if and only if $i \in \llbracket a_1-1, a_2-1\llbracket$, and $(P_i, P_{i+1}) \vdash b$ if and only if $b_1 \in \rrbracket b_2, i\llbracket$ if and only if $i \in \rrbracket b_1, b_2\rrbracket$. So $a \vdash (P_i, P_{i+1})$ or $(P_i, P_{i+1}) \vdash b$ if and only if $i \in \llbracket a_1 - 1, a_2 - 1\llbracket \cup \rrbracket b_1, b_2\rrbracket$. If $\llbracket a_1 - 1, a_2 - 1\llbracket \cup \rrbracket b_1, b_2\rrbracket$ intersects $\llbracket a_2 - 1, b_1\rrbracket$, we deduce that

$$\begin{aligned}\mathcal{P}(M_a, M_b) &= \sum_{i=1}^{n} \mathrm{Hom}_{\Lambda'}(P_i, M_b) \circ \mathrm{Hom}_{\Lambda'}(M_a, P_i) \\ &= u^{\ell^\theta_{a,b}+2n\delta_{a_2-1\in\rrbracket b_1,b_2\llbracket}\delta_{b_1+1\in\rrbracket a_1,a_2\llbracket}}(R' + u^{-1}vR')\end{aligned}$$

and otherwise

$$\begin{aligned}\mathcal{P}(M_a, M_b) &= \sum_{i=1}^{n} \mathrm{Hom}_{\Lambda'}(P_i, M_b) \circ \mathrm{Hom}_{\Lambda'}(M_a, P_i) \\ &= u^{\ell^\theta_{a,b} + 2n\delta_{a_2-1 \in]\!]b_1,b_2[\![}\delta_{b_1+1\in]\!]a_1,a_2[\![}} R'.\end{aligned}$$

Notice that $[\![a_1 - 1, a_2 - 1[\![\cup]\!]b_1, b_2]\!]$ intersects $[\![a_2 - 1, b_1]\!]$ if and only if $b_1+1 \in [\![a_1, a_2[\![$ or $a_2-1 \in]\!]b_1, b_2]\!]$, if and only if $b_1+1 \in]\!]a_1, a_2[\![$ or $a_2-1 \in]\!]b_1, b_2[\![$ or $a_1 = b_1 + 1$ or $a_2 = b_2 + 1$.

Then we can simplify $\mathcal{P}(M_a, M_b)$ in the following way:

Case 1: $\mathcal{P}(M_a, M_b) = u^{\ell^\theta_{a,b}+2n}(R' + u^{-1}vR')$ if $b_1+1 \in]\!]a_1, a_2[\![$ and $a_2-1 \in]\!]b_1, b_2[\![$,
Case 2: $\mathcal{P}(M_a, M_b) = u^{\ell^\theta_{a,b}} R'$ if $b_1 + 1 \notin [\![a_1, a_2[\![$ and $a_2 - 1 \notin]\!]b_1, b_2]\!]$,
Case 3: $\mathcal{P}(M_a, M_b) = u^{\ell^\theta_{a,b}}(R' + u^{-1}vR')$ otherwise.

Recall also that

$$\mathcal{P}(M_a, M_b) \subset \mathrm{Hom}_{\Lambda'}(M_a, M_b) = \begin{cases} u^{\ell^\theta_{a,b}} R' & \text{if } a \nvdash b, \\ u^{\ell^\theta_{a,b}}(R' + u^{-1}vR') & \text{if } a \vdash b, \end{cases}$$

and therefore, in Case 3, we will always get $\underline{\mathrm{Hom}}_{\Lambda'}(M_a, M_b) = 0$. In Case 1, if $a \nvdash b$, we get $\underline{\mathrm{Hom}}_{\Lambda'}(M_a, M_b) \cong u^{\ell^\theta_{a,b}} R'/(X, Y)$ (as graded R'-modules); if $a \vdash b$, we get $\underline{\mathrm{Hom}}_{\Lambda'}(M_a, M_b) \cong u^{\ell^\theta_{a,b}}(R'/(X, Y) \oplus R'/(X, Y))$. In Case 2, if $a \nvdash b$, we get $\underline{\mathrm{Hom}}_{\Lambda'}(M_a, M_b) = 0$; if $a \vdash b$, we get $\underline{\mathrm{Hom}}_{\Lambda'}(M_a, M_b) \cong u^{\ell^\theta_{a,b}} R'/(X, Y)$.

Notice that $a \vdash b$ if and only if $a_1 - 1 \in [\![b_1, b_2[\![$ or $b_2 + 1 \in]\!]a_1, a_2]\!]$. Then an easy case by case analysis concludes the case where neither a nor b is incident to the puncture.

(b) Suppose now that at least one of a and b is incident to the puncture. Without loss of generality, we can suppose that no notched tag appears. An easy computation shows that, in any case,

$$\mathrm{Hom}_{\Lambda'}(P_i, M_b) \circ \mathrm{Hom}_{\Lambda'}(M_a, P_i) = u^{\ell^\theta_{a,(P_i,P_{i+1})} + \ell^\theta_{(P_i,P_{i+1}),b} - 1} vR'.$$

By Lemma 2.15,

$$\ell^\theta_{a,(P_i,P_{i+1})} + \ell^\theta_{(P_i,P_{i+1}),b} = \ell^\theta_{a,b} + 2n(\delta_{a_1 \in]\!]a_2-1,b_1]\!]}\delta_{b_2 \in]\!]a_2-1,b_1]\!]} + \delta_{i \in]\!]b_1,a_2-1[\![}),$$

which implies, as before, that the minimum is reached for $i \in [\![a_2 - 1, b_1]\!]$ and is

$$\ell^\theta_{a,b} + 2n\delta_{a_2-1 \in]\!]b_1,b_2[\![}\delta_{b_1+1 \in]\!]a_1,a_2[\![}.$$

Therefore, $\underline{\mathrm{Hom}}_{\Lambda'}(M_a, M_b) \cong u^{\ell^\theta_{a,b}} R'/(X, Y)$ if $a_2 - 1 \in]\!]b_1, b_2[\![$ and $b_1 + 1 \in]\!]a_1, a_2[\![$, and $\underline{\mathrm{Hom}}_{\Lambda'}(M_a, M_b) = 0$ otherwise. □

Proposition 3.12. *The category* $\mathrm{CM}\,\Lambda'$ *admits the following Auslander–Reiten sequences for* $j \neq i, i+1$*:*

$$0 \to M_{(P_i,P_j)} \xrightarrow{\left[\begin{smallmatrix} u \\ u \end{smallmatrix}\right]} M_{(P_{i+1},P_j)} \oplus M_{(P_i,P_{j+1})} \xrightarrow{[-u\ \ u]} M_{(P_{i+1},P_{j+1})} \to 0,$$
$$0 \to M_{(P_i,*)} \xrightarrow{v} M_{(P_{i+1},P_i)} \xrightarrow{u-v} M_{(P_{i+1},\bowtie)} \to 0,$$
$$0 \to M_{(P_i,\bowtie)} \xrightarrow{u-v} M_{(P_{i+1},P_i)} \xrightarrow{v} M_{(P_{i+1},*)} \to 0.$$

Notice that $M_{(P_i,P_i)}$*, if it appears, has to be interpreted as* $M_{(P_i,*)} \oplus M_{(P_i,\bowtie)}$*. Thus,* $\mathrm{CM}\,\Lambda'$ *admits an Auslander–Reiten translation* τ *defined by*

$$\tau(M_{(P_i,P_j)}) = M_{(P_{i-1},P_{j-1})} \quad \text{if } j \neq i, i+1,$$
$$\tau(M_{(P_i,*)}) = M_{(P_{i-1},\bowtie)}, \qquad \tau(M_{(P_i,\bowtie)}) = M_{(P_{i-1},*)}.$$

Proof. (a) Consider the first case. Let a be a side or an arc of P^* which is not incident to the puncture or a formal sum $(P_{a_1},*)\oplus(P_{a_1},\bowtie)$, and $f : M_{(P_i,P_j)} \to M_a$ be a morphism which is not a split monomorphism. According to Proposition 3.10, the degree $\deg(f)$ of f is at least

$$\ell^\theta_{(P_i,P_j),a} + 2n\delta_{a=(P_i,P_j)}.$$

Moreover, using the beginning of the proof of Lemma 2.15, we get

$$\begin{aligned}
&\ell^\theta_{(P_i,P_j),(P_{i+1},P_j)} + \ell^\theta_{(P_{i+1},P_j),a} - \ell^\theta_{(P_i,P_j),a} \\
&\qquad = n\big(\delta_{i=a_1} + 0 + 0 + |\delta_{i+1\in\rrbracket a_1,j\llbracket} - \delta_{a_2\in\rrbracket a_1,j\llbracket}| - |\delta_{i\in\rrbracket a_1,j\llbracket} - \delta_{a_2\in\rrbracket a_1,j\llbracket}|\big) \\
&\qquad = n\big(\delta_{i=a_1} + (\delta_{a_2\in\llbracket j,a_1\rrbracket} - \delta_{a_2\in\rrbracket a_1,j\llbracket})(\delta_{i+1\in\rrbracket a_1,j\llbracket} - \delta_{i\in\rrbracket a_1,j\llbracket})\big) \\
&\qquad = n\big(\delta_{i=a_1} + (2\delta_{a_2\in\llbracket j,a_1\rrbracket} - 1)\delta_{i=a_1}\big) = 2n\delta_{i=a_1}\delta_{a_2\in\llbracket j,a_1\rrbracket},
\end{aligned}$$

so $\deg(f) \geq \ell^\theta_{(P_i,P_j),(P_{i+1},P_j)} + \ell^\theta_{(P_{i+1},P_j),a} + 2n(\delta_{a=(P_i,P_j)} - \delta_{i=a_1}\delta_{a_2\in\llbracket j,a_1\rrbracket})$, and

$$\begin{aligned}
&\ell^\theta_{(P_i,P_j),(P_i,P_{j+1})} + \ell^\theta_{(P_i,P_{j+1}),a} - \ell^\theta_{(P_i,P_j),a} \\
&\qquad = n\big(0 + \delta_{j=a_2} + 0 + |\delta_{i\in\rrbracket a_1,j+1\llbracket} - \delta_{a_2\in\rrbracket a_1,j+1\llbracket}| - |\delta_{i\in\rrbracket a_1,j\llbracket} - \delta_{a_2\in\rrbracket a_1,j\llbracket}|\big) \\
&\qquad = n\big(\delta_{j=a_2} - (\delta_{i\in\rrbracket a_1,j\llbracket} - \delta_{i\in\llbracket j,a_1\rrbracket})\delta_{j=a_2}\big) = 2n\delta_{j=a_2}\delta_{i\in\llbracket j,a_1\rrbracket},
\end{aligned}$$

so $\deg(f) \geq \ell^\theta_{(P_i,P_j),(P_i,P_{j+1})} + \ell^\theta_{(P_i,P_{j+1}),a} + 2n(\delta_{a=(P_i,P_j)} - \delta_{j=a_2}\delta_{i\in\llbracket j,a_1\rrbracket})$. As at least one of $\delta_{a=(P_i,P_j)} - \delta_{i=a_1}\delta_{a_2\in\llbracket j,a_1\rrbracket}$ and $\delta_{a=(P_i,P_j)} - \delta_{j=a_2}\delta_{i\in\llbracket j,a_1\rrbracket}$ is non-negative, we get

$$\deg(f) \geq \min\big(\ell^\theta_{(P_i,P_j),(P_{i+1},P_j)} + \ell^\theta_{(P_{i+1},P_j),a}, \ell^\theta_{(P_i,P_j),(P_i,P_{j+1})} + \ell^\theta_{(P_i,P_{j+1}),a}\big).$$

Suppose that $a_1 \neq i$ and $a_2 \neq j$. In this case,

$$\ell^\theta_{(P_i,P_j),(P_{i+1},P_j)} + \ell^\theta_{(P_{i+1},P_j),a} = \ell^\theta_{(P_i,P_j),(P_i,P_{j+1})} + \ell^\theta_{(P_i,P_{j+1}),a}.$$

Notice that if $(P_i, P_j) \vdash a$, i.e. $j \in \rrbracket a_2, i\llbracket$ or $a_1 \in \rrbracket a_2, i\llbracket$, then $j \in \rrbracket a_2, i+1\llbracket$ or $a_1 \in \rrbracket a_2, i+1\llbracket$ or $j+1 \in \rrbracket a_2, i\llbracket$ or $a_1 \in \rrbracket a_2, i\llbracket$, i.e. $(P_{i+1}, P_j) \vdash a$ or $(P_i, P_{j+1}) \vdash a$. From that fact and easy observations, we get

$$f \in \mathrm{Hom}_{\Lambda'}(M_{(P_{i+1},P_j)} \oplus M_{(P_i,P_{j+1})}, M_a)u.$$

Suppose that $a_1 = i$. Then $\deg(f) \geq \ell^\theta_{(P_i,P_j),(P_i,P_{j+1})} + \ell^\theta_{(P_i,P_{j+1}),a}$.

If $(P_i, P_j) \vdash a$, i.e. $j \in \rrbracket a_2, i\llbracket$, then we have $j+1 \in \rrbracket a_2, i\llbracket$ or $j+1 = i$, i.e. $(P_i, P_{j+1}) \vdash a$ or $(P_i, P_{j+1}) = (P_i, *) \oplus (P_i, \bowtie)$. From that fact and easy observations, we get

$$f \in \mathrm{Hom}_{\Lambda'}(M_{(P_i,P_{j+1})}, M_a)u.$$

Suppose that $a_2 = j$. Then $\deg(f) \geq \ell^\theta_{(P_i,P_j),(P_{i+1},P_j)} + \ell^\theta_{(P_{i+1},P_j),a}$.

If $(P_i, P_j) \vdash a$, i.e. $a_1 \in \rrbracket a_2, i\llbracket$ then we have $a_1 \in \rrbracket a_2, i+1\llbracket$, i.e. $(P_{i+1}, P_j) \vdash a$. From that fact and easy observations, we get

$$f \in \mathrm{Hom}_{\Lambda'}(M_{(P_{i+1},P_j)}, M_a)u.$$

Thus in the first case, we have an almost split sequence.

(b) Let us consider the second case (the third case is similar to the second). Let $f : M_{(P_i,*)} \to M_a$ be a morphism which is not a split monomorphism. As before, $\deg(f) \geq \ell^\theta_{(P_i,*),a} + 2n\delta_{a=(P_i,*)}$.

Notice that, thanks to the beginning of the proof of Lemma 2.15,

$$\begin{aligned}\ell^\theta_{(P_i,*),(P_{i+1},P_i)} &+ \ell^\theta_{(P_{i+1},P_i),a} - \ell^\theta_{(P_i,*),a} \\ &= n\big(\delta_{i=a_1} + 0 + 0 + |\delta_{i=a_1} - \delta_{a_2 \in \rrbracket a_1, i\llbracket}| - \delta_{a_2 \in \rrbracket a_1, i\llbracket}\big) \\ &= 2n\delta_{i=a_1}\delta_{a_2 \in \llbracket i, a_1\rrbracket} = 2n\delta_{a=(P_i,*)},\end{aligned}$$

so $f \in \mathrm{Hom}_{\Lambda'}(M_{(P_{i+1},P_i)}, M_a)v$. □

Proposition 3.13. *Denote $v' = u - v$. The nonsplit extensions between indecomposable objects of $\mathrm{CM}\,\Lambda'$ are, up to isomorphism,*

$$0 \to M_{(P_i,P_j)} \xrightarrow{\left[\begin{smallmatrix} u^{d(j,l)} \\ u^{d(i,k)} \end{smallmatrix}\right]} M_{(P_i,P_l)} \oplus M_{(P_k,P_j)} \xrightarrow{[u^{d(i,k)} \ \ -u^{d(j,l)}]} M_{(P_k,P_l)} \to 0$$

if $k \in \rrbracket i, j\llbracket$ and $l \in \rrbracket j, i\rrbracket$;

$$0 \to M_{(P_i,P_j)} \xrightarrow{\left[\begin{smallmatrix} u^{d(j,k)+n} \\ u^{d(i,l)} \end{smallmatrix}\right]} M_{(P_k,P_i)} \oplus M_{(P_l,P_j)} \xrightarrow{[u^{d(i,l)} \ \ -u^{d(j,k)+n}]} M_{(P_k,P_l)} \to 0$$

if $k \in \llbracket j, i\llbracket$ and $l \in \rrbracket i, j\llbracket$;

$$0 \to M_{(P_i,P_j)} \xrightarrow{\left[\begin{smallmatrix} u^{d(j,k)} \\ u^{d(i,l)} \end{smallmatrix}\right]} M_{(P_k,P_i)} \oplus M_{(P_l,P_j)} \xrightarrow{[u^{d(i,l)} \ \ -u^{d(j,k)}]} M_{(P_k,P_l)} \to 0,$$

$$0 \to M_{(P_i,P_j)} \xrightarrow{\left[\begin{smallmatrix} u^{d(j,l)} \\ u^{d(i,k)} \end{smallmatrix}\right]} M_{(P_l,P_i)} \oplus M_{(P_k,P_j)} \xrightarrow{[u^{d(i,k)} \ \ -u^{d(j,l)}]} M_{(P_k,P_l)} \to 0$$

if $l \in \rrbracket i,k \llbracket$ *and* $j \in \rrbracket k,i \llbracket$;

$$0 \to M_{(P_i,P_j)} \xrightarrow{\left[\begin{smallmatrix} u^{d(i,k)} \\ v^{d(j,i)} \end{smallmatrix}\right]} M_{(P_k,P_j)} \oplus M_{(P_i,*)} \xrightarrow{[v^{d(j,k)} \ \ -v^{2d(i,k)}]} M_{(P_k,*)} \to 0,$$

$$0 \to M_{(P_i,P_j)} \xrightarrow{\left[\begin{smallmatrix} u^{d(i,k)} \\ v'^{d(j,i)} \end{smallmatrix}\right]} M_{(P_k,P_j)} \oplus M_{(P_i,\bowtie)} \xrightarrow{[v'^{d(j,k)} \ \ -v'^{2d(i,k)}]} M_{(P_k,\bowtie)} \to 0$$

if $k \in \rrbracket i,j \llbracket$;

$$0 \to M_{(P_i,*)} \xrightarrow{\left[\begin{smallmatrix} v^{d(i,k)} \\ v^{2d(i,l)} \end{smallmatrix}\right]} M_{(P_k,P_i)} \oplus M_{(P_l,*)} \xrightarrow{[u^{d(i,l)} \ \ -v^{d(l,k)}]} M_{(P_k,P_l)} \to 0,$$

$$0 \to M_{(P_i,\bowtie)} \xrightarrow{\left[\begin{smallmatrix} v'^{d(i,k)} \\ v'^{2d(i,l)} \end{smallmatrix}\right]} M_{(P_k,P_i)} \oplus M_{(P_l,\bowtie)} \xrightarrow{[u^{d(i,l)} \ \ -v'^{d(l,k)}]} M_{(P_k,P_l)} \to 0$$

if $i \in \rrbracket k,l \llbracket$;

$$0 \to M_{(P_i,*)} \xrightarrow{v^{d(i,k)}} M_{(P_k,P_i)} \xrightarrow{v'^{d(i,k)}} M_{(P_k,\bowtie)} \to 0,$$

$$0 \to M_{(P_i,\bowtie)} \xrightarrow{v'^{d(i,k)}} M_{(P_k,P_i)} \xrightarrow{v^{d(i,k)}} M_{(P_k,*)} \to 0$$

if $i \neq k$.

Moreover, in each case, fixing representatives of these isomorphism classes of short exact sequences induces a basis of the corresponding extension group.

Proof. First of all, it is easy to check that all these nonsplit extensions exist (to prove exactness, the easiest way is to project the sequence on each idempotent) and they are nonsplit and not isomorphic to each other (and therefore linearly independent). Let $i,j,k,l \in \llbracket 1,n \rrbracket$ with $j \neq i, i+1$ and $l \neq k, k+1$. Thanks to Proposition 3.12, we know that $\mathrm{CM}\,\Lambda'$ admits an Auslander–Reiten duality

$$\mathrm{Ext}^1_{\Lambda'}(X,Y) \cong \mathrm{Hom}_K(\underline{\mathrm{Hom}}_{\Lambda'}(Y,\tau(X)),K).$$

Then, using Proposition 3.11, we get

$$\begin{aligned}
\dim \mathrm{Ext}^1_{\Lambda'}((M_{(P_k,P_l)}, M_{(P_i,P_j)}) &= \dim \underline{\mathrm{Hom}}_{\Lambda'}(M_{(P_i,P_j)}, M_{(P_{k-1},P_{l-1})}) \\
&= \delta_{i-1\in \rrbracket k-1,l-1\llbracket}\delta_{l\in\rrbracket i,j\llbracket} + \delta_{j-1\in\rrbracket k-1,l-1\llbracket}\delta_{k\in\rrbracket i,j\llbracket} \\
&= \delta_{i\in\rrbracket k,l\llbracket}\delta_{l\in\rrbracket i,j\llbracket} + \delta_{j\in\rrbracket k,l\llbracket}\delta_{k\in\rrbracket i,j\llbracket} \\
&= \begin{cases} 2 & \text{if } l \in \rrbracket i,k\llbracket \text{ and } j \in \rrbracket k,i\llbracket, \\ 1 & \text{if } k \in \rrbracket i,j\llbracket \text{ and } l \in \rrbracket j,i\rrbracket, \\ & \text{or } k \in \llbracket j,i\llbracket \text{ and } l \in \rrbracket i,j\llbracket, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

We also get

$$\begin{aligned}\dim \operatorname{Ext}^1_{\Lambda'}(M_{(P_k,*)}, M_{(P_i,P_j)}) &= \dim \underline{\operatorname{Hom}}_{\Lambda'}(M_{(P_i,P_j)}, M_{(P_{k-1},\bowtie)})\\ &= \delta_{i-1\in]\!]k-1,k-1[\![}\delta_{k\in]\!]i,j[\![} = \delta_{k\in]\!]i,j[\![};\end{aligned}$$

$$\begin{aligned}\dim \operatorname{Ext}^1_{\Lambda'}(M_{(P_k,P_l)}, M_{(P_i,*)}) &= \dim \underline{\operatorname{Hom}}_{\Lambda'}(M_{(P_i,*)}, M_{(P_{k-1},P_{l-1})})\\ &= \delta_{i-1\in]\!]k-1,l-1[\![}\delta_{l\in]\!]i,i[\![} = \delta_{i\in]\!]k,l[\![};\end{aligned}$$

$$\dim \operatorname{Ext}^1_{\Lambda'}(M_{(P_k,*)}, M_{(P_i,*)}) = \dim \underline{\operatorname{Hom}}_{\Lambda'}(M_{(P_i,*)}, M_{(P_{k-1},\bowtie)}) = 0;$$

$$\begin{aligned}\dim \operatorname{Ext}^1_{\Lambda'}(M_{(P_k,*)}, M_{(P_i,\bowtie)}) &= \dim \underline{\operatorname{Hom}}_{\Lambda'}(M_{(P_i,\bowtie)}, M_{(P_{k-1},\bowtie)})\\ &= \delta_{i-1\in]\!]k-1,k-1[\![}\delta_{k\in]\!]i,i[\![} = \delta_{i\neq k}.\end{aligned}$$

The other cases are realized by swapping $*$ and $\bowtie$. In any case, we exhausted the dimensions with the short exact sequences that we provided. □

Corollary 3.14. *If a and b are two tagged arcs of P^*, then $\dim \operatorname{Ext}^1_{\Lambda'}(M_a, M_b)$ is the minimal number of intersection points between representatives of their isotopy classes (where $(P_i,*)$ and $(P_j,\bowtie)$ intersect once for $i \neq j$ by convention).*

Proof. It is an easy case by case argument. □

§3.3. Cluster tilting objects of CM Λ and relation to the cluster category

Let us recall the definition of cluster tilting objects.

Definition 3.15. Let $\mathcal{C}$ be a triangulated or exact category. An object T in $\mathcal{C}$ is said to be *cluster tilting* if

$$\operatorname{add} T = \{Z \in \mathcal{C} \mid \operatorname{Ext}^1_{\mathcal{C}}(T,Z) = 0\} = \{Z \in \mathcal{C} \mid \operatorname{Ext}^1_{\mathcal{C}}(Z,T) = 0\},$$

where $\operatorname{add} T$ is the set of finite direct sums of direct summands of T.

For any tagged triangulation σ of the once-punctured polygon P^*, we denote

$$T_\sigma = \bigoplus_{a\in\sigma} M_a \cong e_F \Gamma_\sigma.$$

Theorem 3.16. *The map $\sigma \mapsto T_\sigma$ gives a one-to-one correspondence between the set of tagged triangulations of P^* and the set of isomorphism classes of basic cluster tilting objects in CM Λ. Moreover, for any tagged triangulation σ, $\operatorname{End}_\Lambda(T_\sigma) \cong \Gamma_\sigma^{\mathrm{op}}$ via right multiplication.*

Proof. Let E be a set of tagged arcs and sides of P^* and $M_E = \bigoplus_{a\in E} M_a$ the corresponding object in $\mathrm{CM}\,\Lambda$. By Corollary 3.14, any two arcs in E are compatible if and only if $\mathrm{Ext}^1_\Lambda(M_E, M_E) = 0$. Thus, M_E is cluster tilting if and only if it is a maximal set of compatible tagged arcs and sides of P^* if and only if $E = \sigma$ is a tagged triangulation of P^*. Thus, $M_E = T_\sigma$.

For the second part, thanks to Propositions 2.26 and 3.10, for any $a, b \in \sigma$, $\mathrm{Hom}_{\Lambda'}(M_a, M_b) \cong A_{a,b} \cong e_a\Gamma_\sigma e_b$. Therefore,

$$\mathrm{End}_{\Lambda'}(T_\sigma) = \bigoplus_{a,b\in\sigma} e_a\Gamma_\sigma e_b = \Gamma_\sigma.$$

Moreover, composition on the left coincides with multiplication on the right by Propositions 2.26 and 3.10. Notice that we get the opposite algebra because we make endomorphism rings act on the left. □

Theorems 3.3 and 3.16 show that the category $\mathrm{CM}\,\Lambda$ is very similar to the cluster category of type D_n. In the rest of this section, we give an explicit connection. First, we recall some basic facts about cluster categories. The cluster category is defined in [6] as follows.

Definition 3.17. For an acyclic quiver Q, the *cluster category* $\mathcal{C}(KQ)$ is the orbit category $\mathcal{D}^{\mathrm{b}}(KQ)/F$ of the bounded derived category $\mathcal{D}^{\mathrm{b}}(KQ)$ by the functor $F = \tau^{-1}[1]$, where τ denotes the Auslander–Reiten translation and $[1]$ denotes the shift functor. The objects in $\mathcal{C}(KQ)$ are the same as in $\mathcal{D}^{\mathrm{b}}(KQ)$, and the morphisms are given by

$$\mathrm{Hom}_{\mathcal{C}(KQ)}(X, Y) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(KQ)}(F^iX, Y),$$

where X and Y are objects in $\mathcal{D}^{\mathrm{b}}(KQ)$. For $f \in \mathrm{Hom}_{\mathcal{C}(KQ)}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}(KQ)}(Y, Z)$, the composition is defined by

$$(g \circ f)_i = \sum_{i_1+i_2=i} g_{i_1} \circ F^{i_1}(f_{i_2})$$

for all $i \in \mathbb{Z}$.

In [15], Happel proved that $\mathcal{D}^{\mathrm{b}}(KQ)$ has Auslander–Reiten triangles. For a Dynkin quiver Q, he showed in [14] that the Auslander–Reiten quiver of $\mathcal{D}^{\mathrm{b}}(KQ)$ is $\mathbb{Z}\Delta$, where Δ is the underlying Dynkin diagram of Q. Then the Auslander–Reiten quiver of $\mathcal{C}(KQ)$ is $\mathbb{Z}\Delta/\varphi$, where φ is the graph automorphism induced by $\tau^{-1}[1]$. In type D_n, the Auslander–Reiten quiver of $\mathcal{C}$ has the shape of a cylinder with n τ-orbits. As a quiver, it is the same as the quiver of $\underline{\mathrm{CM}}\,\Lambda$ (see Figures 3.1 and 3.2).

Recall that a triangulated category is said to be *algebraic* if it is the stable category of a Frobenius category. Let us state the following result due to Keller and Reiten.

Theorem 3.18 ([24, Introduction and Appendix]). *If K is a perfect field and $\mathcal{C}$ an algebraic 2-Calabi–Yau triangulated category containing a cluster tilting object T with $\mathrm{End}_{\mathcal{C}}(T) \cong KQ$ hereditary, then there is a triangle-equivalence $\mathcal{C}(KQ) \to \mathcal{C}$.*

By using the above statements, we show the following triangle-equivalences between cluster categories of type D and stable categories of Cohen–Macaulay modules.

Theorem 3.19. (1) *The stable category $\underline{\mathrm{CM}}\,\Lambda$ is 2-Calabi–Yau.*

(2) *If K is perfect, then there is a triangle-equivalence $\mathcal{C}(KQ) \cong \underline{\mathrm{CM}}\,\Lambda$ for a quiver Q of type D_n.*

Proof. We will prove (1) in the next subsection independently.

Let σ be the triangulation of P^* whose set of tagged arcs is

$$\{(P_1, P_3), (P_1, P_4), \ldots, (P_1, P_n), (P_1, *), (P_1, \bowtie)\}.$$

The full subquiver Q of Q_σ with set of vertices $Q_{\sigma,0} \setminus F$ is a quiver of type D_n. Thus, we have

$$\Gamma_\sigma^{\mathrm{op}}/(e_F) \cong (KQ)^{\mathrm{op}}.$$

By Theorem 3.16, for the cluster tilting object T_σ, we have the isomorphism

$$\underline{\mathrm{End}}_\Lambda(T_\sigma) \cong \Gamma_\sigma^{\mathrm{op}}/(e_F).$$

Then, by Theorem 3.18, we obtain $\mathcal{C}((KQ)^{\mathrm{op}}) \cong \underline{\mathrm{CM}}\,\Lambda$. □

§3.4. Proof of Theorem 3.19(1)

Here, we prove that the stable category $\underline{\mathrm{CM}}\,\Lambda$ is 2-Calabi–Yau. Throughout, we denote $\mathrm{D}_K := \mathrm{Hom}_K(-, K)$, $\mathrm{D}_R := \mathrm{Hom}_R(-, R)$ and $(-)^* := \mathrm{Hom}_\Lambda(-, \Lambda)$.

Let us recall some general definitions and facts about Cohen–Macaulay modules. Let A be an R-order.

Definition 3.20. We say that X is an *injective* Cohen–Macaulay A-module if $\mathrm{Ext}^1_A(Y, X) = 0$ for any $Y \in \mathrm{CM}\,A$, or equivalently, $X \in \mathrm{add}(\mathrm{Hom}_R(A^{\mathrm{op}}, R))$. Denote by $\mathrm{inj}\,A$ the category of injective Cohen–Macaulay A-modules.

An R-order A is *Gorenstein* if $\mathrm{Hom}_R(A_A, R)$ is projective as a left A-module, or equivalently, $\mathrm{Hom}_R({}_A A, R)$ is projective as a right A-module. We have an exact duality $\mathrm{D}_R : \mathrm{CM}\,A^{\mathrm{op}} \to \mathrm{CM}\,A$.

The Nakayama functor is defined here by $\nu : \operatorname{proj} A \xrightarrow{(-)^*} \operatorname{proj} A^{\mathrm{op}} \xrightarrow{\mathrm{D}_R} \operatorname{inj} A$, which is isomorphic to $(\mathrm{D}_R A) \otimes_A -$. For any Cohen–Macaulay A-module X, consider a projective presentation

$$P_1 \xrightarrow{f} P_0 \to X \to 0$$

and apply $(-)^* : \operatorname{mod} A \to \operatorname{mod} A^{\mathrm{op}}$ to get the exact sequence

$$0 \to X^* \to P_0^* \xrightarrow{f^*} P_1^* \to \operatorname{coker}(f^*) \to 0.$$

We denote $\operatorname{coker}(f^*)$ by $\operatorname{Tr} X$ and we get $\operatorname{Im}(f^*) = \Omega \operatorname{Tr} X$, where Ω is the syzygy functor $\underline{\operatorname{mod}}\, A^{\mathrm{op}} \to \underline{\operatorname{mod}}\, A^{\mathrm{op}}$. Then we apply $\mathrm{D}_R : \mathrm{CM}\, A^{\mathrm{op}} \to \mathrm{CM}\, A$ to

$$0 \to X^* \to P_0^* \xrightarrow{f^*} \Omega \operatorname{Tr} X \to 0$$

and denoting $\tau X := \mathrm{D}_R\, \Omega \operatorname{Tr} X$, we get the exact sequence

$$0 \to \tau X \to \nu P_0 \to \nu X \to 0. \tag{3.21}$$

For an R-order A, if $K(x) \otimes_R A$ is a semisimple $K(x)$-algebra, then we call A an *isolated singularity*. By using the notions above, we have the following well-known results in Auslander–Reiten theory.

Theorem 3.22 ([3, 28, 29]). *Let A be an R-order that is an isolated singularity. Then*

(1) [3, Chapter I, Proposition 8.3] *The construction τ gives an equivalence of categories $\underline{\mathrm{CM}}\, A \to \overline{\mathrm{CM}}\, A$, where $\overline{\mathrm{CM}}\, A$ is the quotient of $\mathrm{CM}\, A$ by the subgroup of maps which factor through an injective object.*

(2) [3, Chapter I, Proposition 8.7] *For $X, Y \in \underline{\mathrm{CM}}\, A$, there is a functorial isomorphism*

$$\underline{\operatorname{Hom}}_A(X, Y) \cong \mathrm{D}_K \operatorname{Ext}^1_A(Y, \tau X).$$

For Gorenstein orders, we have the following nice properties.

Proposition 3.23. *Assume that A is a Gorenstein isolated singularity. Then*

(1) $\mathrm{CM}\, A$ *is a Frobenius category.*

(2) $\underline{\mathrm{CM}}\, A$ *is a K-linear Hom-finite triangulated category.*

(3) $\tau = \Omega\nu = [-1] \circ \nu$.

Proof. (1) The projective objects in $\mathrm{CM}\, A$ are just projective A-modules. They are also injective objects. Since each finitely generated A-module is a quotient of a projective Λ-module, it follows that $\mathrm{CM}\, A$ is a Frobenius category; (2) is due to [14] and [31, Lemma 3.3]; (3) is a direct consequence of (3.21). □

The order Λ is Gorenstein. Indeed, as a graded left Λ-module,

$$\mathrm{D}_R(\Lambda_\Lambda) = \mathrm{Hom}_R\left(\begin{bmatrix} R' & R' & R' & \cdots & R' & R' & X^{-1}(X,Y) \\ (X,Y) & R' & R' & \cdots & R' & R' & R' \\ (X) & (X,Y) & R' & \cdots & R' & R' & R' \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ (X) & (X) & (X) & \cdots & R' & R' & R' \\ (X) & (X) & (X) & \cdots & (X,Y) & R' & R' \\ (X) & (X) & (X) & \cdots & (X) & (X,Y) & R' \end{bmatrix}, R\right)$$

can be identified with

$$X^{-1}\begin{bmatrix} R' & X^{-1}(X,Y) & X^{-1}R' & \cdots & X^{-1}R' & X^{-1}R' & X^{-1}R' \\ R' & R' & X^{-1}(X,Y) & \cdots & X^{-1}R' & X^{-1}R' & X^{-1}R' \\ R' & R' & R' & \cdots & X^{-1}R' & X^{-1}R' & X^{-1}R' \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ R' & R' & R' & \cdots & R' & X^{-1}(X,Y) & X^{-1}R' \\ R' & R' & R' & \cdots & R' & R' & X^{-1}(X,Y) \\ (X,Y) & R' & R' & \cdots & R' & R' & R' \end{bmatrix}$$
$$= \Lambda V^{-1} \subset \mathrm{M}_n(R'[X^{-1}]),$$

where

$$V = \begin{bmatrix} 0 & 0 & \dots & 0 & X & 0 \\ 0 & 0 & \dots & 0 & 0 & X \\ X^2 & 0 & \dots & 0 & 0 & 0 \\ 0 & X^2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & X^2 & 0 & 0 \end{bmatrix}.$$

Therefore $\mathrm{D}_R(\Lambda_\Lambda)$ is a projective (left) Λ-module.

According to Theorem 3.22 and Proposition 3.23, we have

$$\underline{\mathrm{Hom}}_\Lambda(X,Y) \cong \mathrm{D}_K\, \underline{\mathrm{Hom}}_\Lambda(Y, \nu X)$$

for $X, Y \in \mathrm{CM}\,\Lambda$. Thus $\nu = (\mathrm{D}_R\,\Lambda) \otimes_\Lambda -$ is a Serre functor. We want to prove that

$$(\mathrm{D}_R\,\Lambda) \otimes_\Lambda - \cong \Omega^{-2}(-).$$

Thanks to the previous discussion, there is an isomorphism of Λ-modules

$$f : \Lambda \to \mathrm{D}_R(\Lambda_\Lambda), \qquad \mu \mapsto \mu V^{-1}.$$

We define an automorphism α of Λ by $\alpha(\lambda) = V^{-1}\lambda V$ for $\lambda \in \Lambda$. The automorphism α corresponds to a $4\pi/n$ counter-clockwise rotation of the quiver of Λ shown on page 172. In fact, if

$$\lambda = \begin{bmatrix} \lambda_{1,1} & \lambda_{1,2} & \dots & \lambda_{1,n-2} & \lambda_{1,n-1} & \lambda_{1,n} \\ \lambda_{2,1} & \lambda_{2,2} & \dots & \lambda_{2,n-2} & \lambda_{2,n-1} & \lambda_{2,n} \\ \lambda_{3,1} & \lambda_{3,2} & \dots & \lambda_{3,n-2} & \lambda_{3,n-1} & \lambda_{3,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \lambda_{n-2,1} & \lambda_{1,2} & \dots & \lambda_{n-2,n-2} & \lambda_{n-2,n-1} & \lambda_{n-2,n} \\ \lambda_{n-1,1} & \lambda_{n-1,2} & \dots & \lambda_{n-1,n-2} & \lambda_{n-1,n-1} & \lambda_{n-1,n} \\ \lambda_{n,1} & \lambda_{n,2} & \dots & \lambda_{n,n-2} & \lambda_{n,n-1} & \lambda_{n,n} \end{bmatrix}$$

is an element in Λ, then

$$\alpha(\lambda) = \begin{bmatrix} \lambda_{3,3} & \lambda_{3,4} & \dots & \lambda_{3,n} & X^{-1}\lambda_{3,1} & X^{-1}\lambda_{3,2} \\ \lambda_{4,3} & \lambda_{4,4} & \dots & \lambda_{4,n} & X^{-1}\lambda_{4,1} & X^{-1}\lambda_{4,2} \\ \lambda_{5,3} & \lambda_{5,4} & \dots & \lambda_{5,n} & X^{-1}\lambda_{5,1} & X^{-1}\lambda_{5,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \lambda_{n,3} & \lambda_{n,4} & \dots & \lambda_{n,n} & X^{-1}\lambda_{n,1} & X^{-1}\lambda_{n,2} \\ X\lambda_{1,3} & X\lambda_{1,4} & \dots & X\lambda_{1,n} & \lambda_{1,1} & \lambda_{1,2} \\ X\lambda_{2,3} & X\lambda_{2,4} & \dots & X\lambda_{2,n} & \lambda_{2,1} & \lambda_{2,2} \end{bmatrix}.$$

Let A and B be two R-orders. For an (A,B)-bimodule M, $\vartheta \in \mathrm{Aut}(A)$ and $\varsigma \in \mathrm{Aut}(B)$, we define ${}_{\vartheta}M_{\varsigma} := M$ as a vector space, and the (A,B)-bimodule structure is given by

$$a \times m \times b = \vartheta(a)m\varsigma(b)$$

for $m \in {}_{\vartheta}M_{\varsigma}$ and $a \in A$, $b \in B$. Since $\vartheta \in \mathrm{Aut}(A)$, ${}_{\vartheta}(-)$ is an automorphism of $\operatorname{mod} A$.

Proposition 3.24. *The above $f : \Lambda \to \mathrm{D}_R\,\Lambda$ gives an isomorphism of Λ-bimodules*

$${}_1\Lambda_{\alpha} \cong \mathrm{D}_R\,\Lambda.$$

Proof. Clearly, f preserves the left action of Λ. Moreover, it preserves the right action since for $\lambda, \mu \in \Lambda$, we have

$$f(\mu\alpha(\lambda)) = f(\mu(V^{-1}\lambda V)) = \mu(V^{-1}\lambda V)V^{-1} = \mu V^{-1}\lambda = f(\mu)\lambda. \qquad \square$$

By using the isomorphism of Proposition 3.24, we find the following description of the Nakayama functor ν.

Lemma 3.25. *We have an isomorphism $\nu \cong {}_{\alpha^{-1}}(-)$ of endofunctors of $\underline{\mathrm{CM}}\,\Lambda$.*

Proof. Since $\mathrm{D}_R\Lambda \cong {}_1\Lambda_\alpha$, it follows that $\nu \cong {}_1\Lambda_\alpha \otimes_\Lambda -$. On the other hand, we have an isomorphism $H : {}_1\Lambda_\alpha \otimes_\Lambda - \cong {}_{\alpha^{-1}}(-)$ given by $\lambda \otimes - \mapsto \alpha^{-1}(\lambda)(-)$. □

Let $T = K[x, y]$ and $S := K[x, y]/(p)$ for some $p \in T$.

We define a $\mathbb{Z}/n\mathbb{Z}$-grading on T by setting $\deg(x) = 1$ and $\deg(y) = -1$. This makes T a $\mathbb{Z}/n\mathbb{Z}$-graded algebra

$$T = \bigoplus_{\bar{i}\in\mathbb{Z}/n\mathbb{Z}} T_{\bar{i}} = T_{\bar{0}} \oplus T_{\bar{1}} \oplus \cdots \oplus T_{\overline{n-1}}.$$

Suppose that p is homogeneous of degree d with respect to this grading. Then the quotient ring

$$S = K[x, y]/(p) = S_{\bar{0}} \oplus S_{\bar{1}} \oplus \cdots \oplus S_{\overline{n-1}}$$

has a natural structure of a $\mathbb{Z}/n\mathbb{Z}$-graded algebra. The following result can be easily established from classical results about matrix factorization (see [9, Theorem 3.22] for a detailed proof).

Theorem 3.26 ([31]). *In the category* $\underline{\mathrm{CM}}^{\mathbb{Z}/n\mathbb{Z}} S$*, there is an isomorphism of autoequivalences* $[2] \cong (-d)$*.*

Setting $p := x^{n-1}y - y^2$, we have $S = K[x, y]/(x^{n-1}y - y^2)$. Identifying $R' = K[X, Y]/(XY - Y^2)$ as a subalgebra of S via $X \mapsto x^n$ and $Y \mapsto xy$, we regard S as an R'-algebra. We obtain the following lemma.

Lemma 3.27. *For* $i \in [\![0, n-1]\!]$ *we have, as* R'*-modules,*

$$S_{\bar{i}} \cong \begin{cases} R'x^i & \text{if } i \in [\![0, n-2]\!], \\ (1, X^{-1}Y)x^{n-1} & \text{if } i = n-1. \end{cases}$$

Proof. Let $i \in [\![0, n-2]\!]$. Over $R = K[X]$, $S_{\bar{i}}$ is generated by x^i and $x^{i+1}y$. Thus, we have $S_{\bar{i}} \cong R'x^i$. Over R, $S_{\overline{n-1}}$ is generated by x^{n-1} and y. So $S_{\overline{n-1}} \cong (1, X^{-1}Y)x^{n-1}$. □

From the $\mathbb{Z}/n\mathbb{Z}$-graded algebra S we define an R-order $S^{[n]}$ which is a subalgebra of $\mathrm{M}_n(S)$ as follows:

$$S^{[n]} = \begin{bmatrix} S_{\bar{0}} & S_{\bar{1}} & S_{\bar{2}} & \cdots & S_{\overline{n-2}} & S_{\overline{n-1}} \\ S_{\overline{n-1}} & S_{\bar{0}} & S_{\bar{1}} & \cdots & S_{\overline{n-3}} & S_{\overline{n-2}} \\ S_{\overline{n-2}} & S_{\overline{n-1}} & S_{\bar{0}} & \cdots & S_{\overline{n-4}} & S_{\overline{n-3}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ S_{\bar{2}} & S_{\bar{3}} & S_{\bar{4}} & \cdots & S_{\bar{0}} & S_{\bar{1}} \\ S_{\bar{1}} & S_{\bar{2}} & S_{\bar{3}} & \cdots & S_{\overline{n-1}} & S_{\bar{0}} \end{bmatrix}.$$

Proposition 3.28. *We have an isomorphism $S^{[n]} \cong \Lambda$ of R'-algebras.*

Proof. According to Lemma 3.27, $S^{[n]}$ is the matrix order

$$\begin{bmatrix} R' & R'x & R'x^2 & \cdots & R'x^{n-2} & (1, X^{-1}Y)x^{n-1} \\ (1, X^{-1}Y)x^{n-1} & R' & R'x & \cdots & R'x^{n-3} & R'x^{n-2} \\ R'x^{n-2} & (1, X^{-1}Y)x^{n-1} & R' & \cdots & R'x^{n-4} & R'x^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ R'x^2 & R'x^3 & R'x^4 & \cdots & R' & R'x \\ R'x & R'x^2 & R'x^3 & \cdots & (1, X^{-1}Y)x^{n-1} & R' \end{bmatrix}.$$

Taking the conjugation by the diagonal matrix $B = \operatorname{diag}(x^i)_{i \in [\![0,n]\!]}$, we get

$$BS^{[n]}B^{-1} = \Lambda. \qquad \square$$

From now on, we identify Λ and $S^{[n]}$. Consider the matrix

$$U = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 \end{bmatrix}.$$

The automorphism β of $S^{[n]}$ given by $\beta(s) = U^{-1}sU$ for $s \in S^{[n]}$ corresponds to the automorphism α of Λ. Thus we have an isomorphism ${}_1S^{[n]}_{\beta^{-1}} \cong {}_1\Lambda_{\alpha^{-1}}$ of $S^{[n]}$-bimodules.

Using the notation above, we have the following lemma.

Lemma 3.29. (1) [19, Theorem 3.1] *The functor*

$$F : \operatorname{mod}^{\mathbb{Z}/n\mathbb{Z}} S \to \operatorname{mod} S^{[n]},$$
$$M_{\overline{0}} \oplus M_{\overline{1}} \oplus \cdots \oplus M_{\overline{n-1}} \mapsto \begin{bmatrix} M_{\overline{0}} & M_{\overline{1}} & \dots & M_{\overline{n-1}} \end{bmatrix}^{\mathrm{t}},$$

is an equivalence of categories.

(2) *For $i \in \mathbb{Z}$, we denote by $(i) : \operatorname{mod}^{\mathbb{Z}/n\mathbb{Z}} S \to \operatorname{mod}^{\mathbb{Z}/n\mathbb{Z}} S$ the grade shift functor defined by $M(i)_{\overline{j}} := M_{\overline{i+j}}$ for $M \in \operatorname{mod}^{\mathbb{Z}/n\mathbb{Z}} S$. The functor (i) induces an autofunctor (denoted by γ_i) of $\operatorname{mod} S^{[n]}$ which makes the following diagram commute:*

$$\begin{array}{ccc} \operatorname{mod}^{\mathbb{Z}/n\mathbb{Z}} S & \xrightarrow{F} & \operatorname{mod} S^{[n]} \\ {\scriptstyle (i)}\downarrow & & \downarrow{\scriptstyle \gamma_i} \\ \operatorname{mod}^{\mathbb{Z}/n\mathbb{Z}} S & \xrightarrow{F} & \operatorname{mod} S^{[n]} \end{array}$$

More precisely, for any left $S^{[n]}$-module $\begin{bmatrix} M_{\overline{0}} & M_{\overline{1}} & \dots & M_{\overline{n-1}} \end{bmatrix}^{\mathrm{t}}$, we have

$$\gamma_i\big(\begin{bmatrix} M_{\overline{0}} & M_{\overline{1}} & \dots & M_{\overline{n-1}} \end{bmatrix}^{\mathrm{t}}\big) = \begin{bmatrix} M_{\overline{i}} & M_{\overline{i+1}} & \dots & M_{\overline{i+n-1}} \end{bmatrix}^{\mathrm{t}}.$$

Now we can prove the 2-Calabi–Yau property of $\underline{\mathrm{CM}}\,\Lambda$.

Proof of Theorem 3.19(1). The equivalence $\mathrm{mod}^{\mathbb{Z}/n\mathbb{Z}} S \cong \mathrm{mod}\, S^{[n]} = \mathrm{mod}\,\Lambda$ induces an equivalence

$$\mathrm{CM}^{\mathbb{Z}/n\mathbb{Z}}\, S \cong \mathrm{CM}\, S^{[n]} = \mathrm{CM}\,\Lambda.$$

In the category $\underline{\mathrm{CM}}^{\mathbb{Z}/n\mathbb{Z}}\, S$, according to Theorem 3.26, we have an isomorphism of functors

$$[2] \cong (-\deg\,(x^{n-2} - y^2)) = (2).$$

By Lemma 3.25, we have $\nu \cong {}_{\alpha^{-1}}(-)$. Therefore, it is enough to prove ${}_{\alpha^{-1}}(-) \cong (2)$, or equivalently ${}_{\beta}M \cong \gamma_{-2}(M)$ for any $M \in \mathrm{CM}\, S^{[n]}$.

Let s_i be the row matrix which has 1 in the ith column. Since

$$\begin{aligned} {}_{\beta}M &= \begin{bmatrix} s_0 \times {}_{\beta}M & s_1 \times {}_{\beta}M & s_2 \times {}_{\beta}M & \dots & s_{n-1} \times {}_{\beta}M \end{bmatrix}^{\mathrm{t}} \\ &= \begin{bmatrix} \beta(s_0)M & \beta(s_1)M & \beta(s_2)M & \dots & \beta(s_{n-1})M \end{bmatrix}^{\mathrm{t}} \\ &= \begin{bmatrix} s_{n-2}M & s_{n-1}M & s_0M & \dots & s_{n-3}M \end{bmatrix}^{\mathrm{t}}, \end{aligned}$$

it follows that ${}_{\beta}M \cong \gamma_{-2}(M)$. Therefore, the category $\underline{\mathrm{CM}}\,\Lambda$ is 2-Calabi–Yau. □

§4. Graded Cohen–Macaulay Λ-modules

In this section, we prove a graded version of Theorem 3.19 which gives a relationship between the category $\mathrm{CM}^{\mathbb{Z}}\,\Lambda$ of graded Cohen–Macaulay Λ-modules and the bounded derived category $\mathcal{D}^{\mathrm{b}}(KQ)$ of type D_n.

Let Q be an acyclic quiver. We denote by $\mathcal{K}^{\mathrm{b}}(\mathrm{proj}\,KQ)$ the bounded homotopy category of finitely generated projective KQ-modules, and by $\mathcal{D}^{\mathrm{b}}(KQ)$ the bounded derived category of finitely generated KQ-modules. These are triangulated categories and the canonical embedding $\mathcal{K}^{\mathrm{b}}(\mathrm{proj}\,KQ) \to \mathcal{D}^{\mathrm{b}}(KQ)$ is a triangle functor.

We define a grading on Λ by $\Lambda_i = \Lambda \cap \mathrm{M}_n(KX^i + KX^{i-1}Y)$ for $i \in \mathbb{Z}$. This makes $\Lambda = \bigoplus_{i\in\mathbb{Z}} \Lambda_i$ a $\mathbb{Z}$-graded algebra. The category of graded Cohen–Macaulay Λ-modules, $\mathrm{CM}^{\mathbb{Z}}\,\Lambda$, is defined as follows. The objects are graded Λ-modules which are Cohen–Macaulay, and the morphisms in $\mathrm{CM}^{\mathbb{Z}}\,\Lambda$ are Λ-morphisms preserving the degree. The category $\mathrm{CM}^{\mathbb{Z}}\,\Lambda$ is a Frobenius category. Its stable category is denoted by $\underline{\mathrm{CM}}^{\mathbb{Z}}\,\Lambda$. For $i \in \mathbb{Z}$, we denote by $(i) : \mathrm{CM}^{\mathbb{Z}}\,\Lambda \to \mathrm{CM}^{\mathbb{Z}}\,\Lambda$ the grade shift functor: Given a graded Cohen–Macaulay Λ-module X, we define $X(i)$ to be X as a Λ-module, with the grading $X(i)_j = X_{i+j}$ for any $j \in \mathbb{Z}$.

Remark 4.1. We show that this grading of Λ is analogous to the grading of Λ' given by the θ-length. Let $i, j \in F$. By Theorem 2.30, $e_i\Lambda' e_j \cong e_i \Lambda e_j$. Let $\lambda \in e_i\Lambda' e_j \cong e_i\Lambda e_j$. Using a similar argument to the proof of Theorem 2.30, we get

$$\frac{\ell^\theta(\lambda) + 2d(1,i) - 2d(1,j)}{2n} = \deg(\lambda)$$

where $\deg(\lambda)$ is the degree of λ as a member of Λ. Consider the two graded algebras

$$\Lambda' = \bigoplus_{i=1}^{n} \Lambda' e_i \quad \text{and} \quad \Lambda'' := \operatorname{End}\Big(\bigoplus_{i=1}^{n} u^{2d(1,i)}\Lambda' e_i\Big).$$

By graded Morita equivalence, we have $\operatorname{CM}^{\mathbb{Z}}\Lambda' \cong \operatorname{CM}^{\mathbb{Z}}\Lambda''$. Since $\Lambda \cong \Lambda''$ as R-orders and $\deg(X) = 2n$ in Λ'', it follows that the Auslander–Reiten quiver of $\operatorname{CM}^{\mathbb{Z}}\Lambda''$ has $2n$ connected components each of which is a degree shift of the Auslander–Reiten quiver of $\operatorname{CM}^{\mathbb{Z}}\Lambda$.

We introduce the properties of $\operatorname{CM}^{\mathbb{Z}}\Lambda$ in the following theorems.

Theorem 4.2. (1) *The set of isomorphism classes of indecomposable objects of* $\operatorname{CM}^{\mathbb{Z}}\Lambda$ *is*

$$\{(i,j) \mid i,j \in \mathbb{Z}, 0 < j-i < n\} \cup \{(i,*) \mid i \in \mathbb{Z}\} \cup \{(i,\bowtie) \mid i \in \mathbb{Z}\},$$

where

$$(i,j) = [\overbrace{(X)\cdots(X)}^{i}\,\overbrace{(X^2,Y^2)\cdots(X^2,Y^2)}^{j-i}\,\overbrace{(X^2)\cdots(X^2)}^{n-j}]^{\mathrm{t}} \quad \text{if } 0 < i < j \le n;$$

$$(i,j) = [\overbrace{(X,Y)\cdots(X,Y)}^{j-n}\,\overbrace{(X)\cdots(X)}^{n-j+i}\,\overbrace{(X^2,Y^2)\cdots(X^2,Y^2)}^{n-i}]^{\mathrm{t}} \quad \text{if } i \le n < j;$$

$$(i,*) = [\overbrace{(Y)\cdots(Y)}^{i}\,\overbrace{(Y^2)\cdots(Y^2)}^{n-i}]^{\mathrm{t}} \quad \text{if } 0 < i \le n;$$

$$(i,\bowtie) = [\overbrace{(X-Y)\cdots(X-Y)}^{i}\,\overbrace{(X^2-Y^2)\cdots(X^2-Y^2)}^{n-i}]^{\mathrm{t}} \quad \text{if } 0 < i \le n,$$

and the other (i,j) *are obtained by shift:*

$$\begin{aligned}(i+kn, j+kn) &= (i,j)(k),\\ (i+kn,*) &= (i,*)(k),\\ (i+kn,\bowtie) &= (i,\bowtie)(k),\end{aligned}$$

for $k \in \mathbb{Z}$. *The projective-injective objects are of the form* $(i, i+1)$ *for* $i \in \mathbb{Z}$.

(2) *The nonsplit extensions of indecomposable objects of* $\mathrm{CM}^{\mathbb{Z}}\Lambda$ *are of the form*

$$0 \to (i,j) \to (i,l) \oplus (k,j) \to (k,l) \to 0 \qquad \textit{if } i<k<j<l<i+n;$$
$$0 \to (i,j) \to (k,i+n) \oplus (l,j) \to (k,l+n) \to 0 \qquad \textit{if } i<l<j\leq k<i+n;$$
$$\left.\begin{array}{r} 0 \to (i,j) \to (k,i+n) \oplus (l,j) \to (k,l+n) \to 0 \\ 0 \to (i,j) \to (l,i+n) \oplus (k,j) \to (k,l+n) \to 0 \end{array}\right\} \qquad \textit{if } i<l<k<j<i+n;$$
$$\left.\begin{array}{r} 0 \to (i,j) \to (k,j) \oplus (i,*) \to (k,*) \to 0 \\ 0 \to (i,j) \to (k,j) \oplus (i,\bowtie) \to (k,\bowtie) \to 0 \end{array}\right\} \qquad \textit{if } i<k<j<i+n;$$
$$\left.\begin{array}{r} 0 \to (i,*) \to (k,i+n) \oplus (l,*) \to (k,l+n) \to 0 \\ 0 \to (i,\bowtie) \to (k,i+n) \oplus (l,\bowtie) \to (k,l+n) \to 0 \end{array}\right\} \qquad \textit{if } i<l<k<i+n;$$
$$\left.\begin{array}{r} 0 \to (i,*) \to (k,i+n) \to (k,\bowtie) \to 0 \\ 0 \to (i,\bowtie) \to (k,i+n) \to (k,*) \to 0 \end{array}\right\} \qquad \textit{if } i<k<i+n.$$

Moreover, fixing representatives of these isomorphism classes of short exact sequences induces bases of the corresponding extension groups.

(3) *The exact category* $\mathrm{CM}^{\mathbb{Z}}\Lambda$ *admits the Auslander–Reiten sequences*

$$0 \to (i,j) \to (i,j+1) \oplus (i+1,j) \to (i+1,j+1) \to 0$$

for $i+1<j<i+n$ (*with the convention that* $(i,i+n) = (i,*) \oplus (i,\bowtie)$)*;*

$$0 \to (i,*) \to (i+1,i+n) \to (i+1,\bowtie) \to 0,$$
$$0 \to (i,\bowtie) \to (i+1,i+n) \to (i+1,*) \to 0$$

for any $i \in \mathbb{Z}$.

(4) *The Auslander–Reiten quiver of* $\mathrm{CM}^{\mathbb{Z}}\Lambda$ *is the repetitive quiver of type* D_{n+1} (*unfolded version of Figures* 3.1 *and* 3.2).

(5) *The syzygy in* $\underline{\mathrm{CM}}^{\mathbb{Z}}\Lambda$ *is defined on indecomposable objects by*

$$\Omega((i,j)) = (i+1-n, j+1-n),$$
$$\Omega((i,*)) = (i+1-n,\bowtie), \qquad \Omega((i,\bowtie)) = (i+1-n,*).$$

Proof. (1) First of all, it is immediate that the graded modules (i,j) for $0 < j-i < n$, $(i,*)$ and $(i,\bowtie)$ for $i \in \mathbb{Z}$ are not isomorphic. Therefore, we need to prove that there are no other isomorphism classes. We consider the degree forgetful functor $F : \mathrm{CM}^{\mathbb{Z}}\Lambda \to \mathrm{CM}\,\Lambda$. Let $X \in \mathrm{CM}^{\mathbb{Z}}\Lambda$ be indecomposable and M be an indecomposable direct summand of FX in $\mathrm{CM}\,\Lambda$. By Theorem 3.3, there exists a tagged arc or a side a of P^* such that $M \cong M_a$. Then it is immediate that $M \cong FY$ where $Y \in \mathrm{CM}^{\mathbb{Z}}\Lambda$ is (i,j) for $0<j-i<n$ or $(i,*)$ or $(i,\bowtie)$ for $i \in \mathbb{Z}$. There are two morphisms $f : FY \to FX$ and $g : FX \to FY$ such that $gf = \mathrm{Id}_{FY}$.

Let us write

$$f = \sum_{m \in \mathbb{Z}} f_m \quad \text{and} \quad g = \sum_{m \in \mathbb{Z}} g_m$$

where f_m is a graded morphism from Y to $X(m)$ and g_m a graded morphism from $X(m)$ to Y. Thus, we have

$$\sum_{k \in \mathbb{Z}} g_k f_k = \mathrm{Id}_{FY}$$

and, as the endomorphism ring of Y is K, there exists $k \in \mathbb{Z}$ such that $g_k f_k$ is a nonzero multiple of Id_Y. In other terms, we found two graded morphisms $\tilde{f} : Y \to X(k)$ and $\tilde{g} : X(k) \to Y$ such that $\tilde{g}\tilde{f} = \mathrm{Id}_Y$. Thus, in $\mathrm{mod}^{\mathbb{Z}}(\Lambda)$, we have an isomorphism $X \cong Y(-k) \oplus X'$. As $\mathrm{mod}^{\mathbb{Z}}(R)$ is Krull–Schmidt, X' is necessarily a graded Cohen–Macaulay module. Finally, as X is indecomposable in $\mathrm{CM}^{\mathbb{Z}}\,\Lambda$, we get $X \cong Y(-k)$.

Therefore, the set of isomorphism classes of indecomposable graded Cohen–Macaulay Λ-modules is

$$\{(i,j) \mid i,j \in \mathbb{Z}, 0 < j - i < n\} \cup \{(i,*) \mid i \in \mathbb{Z}\} \cup \{(i,\bowtie) \mid i \in \mathbb{Z}\}.$$

Statements (2) and (3) are direct consequences through F of the ungraded versions of Propositions 3.12 and 3.13. Statement (4) is a direct consequence of (1) and (3).

For (5), using (2), the short exact sequences constructed from projective covers are:

$$\begin{gathered} 0 \to (i+1-n, j+1-n) \to (i,i+1) \oplus (j-n, j-n+1) \to (i,j) \to 0, \\ 0 \to (i+1-n, \bowtie) \to (i,i+1) \to (i,*) \to 0, \\ 0 \to (i+1-n, *) \to (i,i+1) \to (i,\bowtie) \to 0. \end{gathered}$$ □

For any indecomposable graded Cohen–Macaulay Λ-module A, if A is of the form (i,j) for two integers i and j, we write $A_1 = i$ and $A_2 = j$, and if A is of the form $(i,*)$ or $(i,\bowtie)$, we write $A_1 = i$ and $A_2 = i+n$. In this way, all morphisms in $\mathrm{CM}^{\mathbb{Z}}\,\Lambda$ are going in the increasing direction in terms of these pairs of integers.

Definition 4.3. Let $\mathcal{C}$ be a triangulated category. An object T is said to be *tilting* if $\mathrm{Hom}_{\mathcal{C}}(T, T[k]) = 0$ for any $k \neq 0$ and $\mathrm{thick}(T) = \mathcal{C}$, where $\mathrm{thick}(T)$ is the smallest full triangulated subcategory of $\mathcal{C}$ containing T and closed under isomorphisms and direct summands.

Theorem 4.4 ([23, Theorem 4.3], [20, Theorem 2.2], [4]). *Let $\mathcal{C}$ be an algebraic triangulated Krull–Schmidt category. If $\mathcal{C}$ has a tilting object T, then there exists a triangle-equivalence*

$$\mathcal{C} \to \mathcal{K}^{\mathrm{b}}(\mathrm{proj}\,\mathrm{End}_{\mathcal{C}}(T)).$$

Now we get the following theorem which is analogous to Theorems 3.16 and 3.19(2).

Theorem 4.5. *Let Q be a quiver of type D_n. Then*

(1) *for a tagged triangulation σ of the once-punctured polygon P^*, the cluster tilting object $e_F\Gamma_\sigma$ can be lifted to a tilting object in $\underline{\mathrm{CM}}^{\mathbb{Z}}\,\Lambda$;*

(2) *there exists a triangle-equivalence $\mathcal{D}^{\mathrm{b}}(KQ) \cong \underline{\mathrm{CM}}^{\mathbb{Z}}\,\Lambda$.*

Proof. (1) First, we have $e_F\Gamma_\sigma \cong \bigoplus_{a\in\sigma} M_a$. So we need to choose some degree shift of each M_a.

Suppose that all tagged arcs of σ are incident to the puncture. Suppose without loss of generality that they are tagged plain. We can lift σ to the set σ' of indecomposable objects of $\mathrm{CM}^{\mathbb{Z}}\,\Lambda$ of the form $(i,*)$ for $1 \le i \le n$. Let us prove that the graded module $T'_{\sigma'} = \bigoplus_{A\in\sigma'} A$ is tilting (it is T_σ if we forget the grading). Let us check that

$$\mathrm{Hom}_{\underline{\mathrm{CM}}^{\mathbb{Z}}\,\Lambda}((i,*),\Omega^k(j,*)) = 0$$

for any $i,j \in [\![1,n]\!]$ and $k \neq 0$. Thanks to Theorem 4.2, it is easy to compute projective covers of modules and we know that $\Omega^k(j,*) = (j+k(1-n),*)$ if k is even, and $\Omega^k(j,*) = (j+k(1-n),\bowtie)$ if k is odd. Therefore, if k is odd, $\mathrm{Hom}_{\underline{\mathrm{CM}}^{\mathbb{Z}}\,\Lambda}((i,*),\Omega^k(j,*)) = 0$.

Moreover, if $k \ge 2$, we get $j + k(1-n) \le j + 2 - 2n \le 0 < i$. So

$$\mathrm{Hom}_{\underline{\mathrm{CM}}^{\mathbb{Z}}\,\Lambda}((i,*),\Omega^k(j,*)) = 0.$$

If $k \le -2$ is even, we want to prove that

$$\mathrm{Hom}_{\underline{\mathrm{CM}}^{\mathbb{Z}}\,\Lambda}((i,*),\Omega^k(j,*)) = \mathrm{Ext}^1_{\mathrm{CM}^{\mathbb{Z}}\,\Lambda}((i,*),\Omega^{k+1}(j,*)) = 0.$$

We have $\Omega^{k+1}(j,*) = (j+(k+1)(1-n),\bowtie)$ and $j+(k+1)(1-n) \ge j+n-1 \ge n \ge i$ and clearly $\mathrm{Ext}^1_{\mathrm{CM}^{\mathbb{Z}}\,\Lambda}((i,*),\Omega^{k+1}(j,*)) = 0$.

Let us now prove that $\mathrm{thick}(T'_{\sigma'}) = \underline{\mathrm{CM}}^{\mathbb{Z}}\,\Lambda$. First of all, for any $i \in \mathbb{Z}$ such that $n \le i < 2n-2$, considering the short exact sequence

$$0 \to (i-n+1,*) \to (i,i+1) \oplus (n-1,*) \to (i,2n-1) \to 0,$$

as $(i-n+1,*)$ and $(n-1,*)$ are in σ' and $(i,i+1)$ is projective, we see that $(i,2n-1) \in \mathrm{thick}(T'_{\sigma'})$. Now, for any $i \in \mathbb{Z}$ such that $n < i < 2n-1$, using the short exact sequence

$$0 \to (n,2n-1) \to (i,2n-1) \oplus (n,*) \to (i,*) \to 0$$

we find that $(i,*) \in \mathrm{thick}(T'_{\sigma'})$. Thus, as $\Omega^{2k}(j,*) = (j+2k(1-n),*)$, all the $(j,*)$ for $j \in \mathbb{Z}$ are in $\mathrm{thick}(T'_{\sigma'})$. Consider $i,j \in \mathbb{Z}$ such that $1 < j-i < n$. We then have a short exact sequence

$$0 \to (i+1-n,*) \to (i,i+1) \oplus (j-n,*) \to (i,j) \to 0,$$

and, as $(i+1-n,*)$ and $(j-n,*)$ are in $\mathrm{thick}(T'_{\sigma'})$ and $(i,i+1)$ is projective, we deduce that $(i,j) \in \mathrm{thick}(T'_{\sigma'})$. Finally, as $\Omega(i,*) = (i-n+1,\bowtie)$, all the $(i,\bowtie)$ are in $\mathrm{thick}(T'_{\sigma'})$. We have thus proved that $T'_{\sigma'}$ is tilting in this case.

Suppose now that there is at least one tagged arc of σ which is not incident to the puncture. Then there exists a vertex i_0 of P such that i_0 does not have any incident internal edge in σ. Therefore, we can lift the tagged arcs of σ to a set σ' of indecomposable objects of $\mathrm{CM}^{\mathbb{Z}}\Lambda$ such that for any $A \in \sigma'$, we have $i_0 < A_1 < i_0+n$ and $i_0+1 < A_2 < i_0+2n$. Let us prove that the graded module $T'_{\sigma'} = \bigoplus_{A\in\sigma'} A$ is tilting (it is T_σ if we forget the grading). Let us check that

$$\mathrm{Hom}_{\underline{\mathrm{CM}}^{\mathbb{Z}}\Lambda}(A, \Omega^k B) = 0$$

for any $A, B \in \sigma'$ and $k \neq 0$. Let $B' = \Omega^k B$. Thanks to Theorem 4.2, $B'_1 = B_1 + k(1-n)$ and $B'_2 = B_2 + k(1-n)$.

Therefore, if $k>0$, we get $B'_1 \le B_1+1-n \le i_0 < A_1$. So $\mathrm{Hom}_{\underline{\mathrm{CM}}^{\mathbb{Z}}\Lambda}(A,B')=0$.

If $k<-1$, we want to prove that $\mathrm{Hom}_{\underline{\mathrm{CM}}^{\mathbb{Z}}\Lambda}(A,B') = \mathrm{Ext}^1_{\mathrm{CM}^{\mathbb{Z}}\Lambda}(A,\Omega B') = 0$. If we denote $B'' = \Omega B'$, we have $B''_1 = B'_1+1-n \ge B_1+n-1 \ge i_0+n > A_1$. Then as the morphisms are positively directed, $\mathrm{Ext}^1_{\mathrm{CM}^{\mathbb{Z}}\Lambda}(A,\Omega B') = 0$.

For $k=-1$, by Theorem 3.16, we get

$$\mathrm{Hom}_{\underline{\mathrm{CM}}^{\mathbb{Z}}\Lambda}(T'_{\sigma'}, \Omega^{-1}T'_{\sigma'}) \subset \mathrm{Hom}_{\underline{\mathrm{CM}}\Lambda}(T_\sigma, \Omega^{-1}T_\sigma) = \mathrm{Ext}^1_{\mathrm{CM}\Lambda}(T_\sigma, T_\sigma) = 0.$$

Let us now prove that $\mathrm{thick}(T'_{\sigma'}) = \underline{\mathrm{CM}}^{\mathbb{Z}}\Lambda$. Consider an indecomposable object $A \in \mathrm{CM}^{\mathbb{Z}}\Lambda$ with $A_1 = i_0+n$. Let $A' \in \mathrm{CM}\Lambda$ be its image through the forgetful functor. It is a classical lemma about cluster tilting objects that there exists a short exact sequence

$$0 \to T'_1 \to T'_0 \to A' \to 0$$

of Cohen–Macaulay Λ-modules such that $T'_0, T'_1 \in \mathrm{add}(T_\sigma)$.

Let X' be an indecomposable summand of T'_1. For any lift X of X' such that $\mathrm{Ext}^1_{\mathrm{CM}^{\mathbb{Z}}\Lambda}(A,X) \neq 0$, we have $i_0 < X_1 < i_0+n$ by Theorem 4.2(2), so such a lift X is unique and has to be in σ'. Moreover, in this case, $\mathrm{Ext}^1_{\mathrm{CM}^{\mathbb{Z}}\Lambda}(A,X) = \mathrm{Ext}^1_{\mathrm{CM}\Lambda}(A,X')$. Therefore, the unique lift T_1 of T'_1 which is in $\mathrm{add}(T'_{\sigma'})$ satisfies $\mathrm{Ext}^1_{\mathrm{CM}^{\mathbb{Z}}\Lambda}(A,T_1) = \mathrm{Ext}^1_{\mathrm{CM}\Lambda}(A,T'_1)$, so we can lift the short exact sequence

$$0 \to T'_1 \to T'_0 \to A' \to 0$$

to a short exact sequence

$$0 \to T_1 \to T_0 \to A \to 0$$

of graded Cohen–Macaulay Λ-modules. As any indecomposable summand X of T_0 is between T_1 and A in the Auslander–Reiten quiver, we get $i_0 < X_1 \leq A_1 = i_0+n$, so $X \in \sigma'$. Finally, T_0 and T_1 are in $\mathrm{add}(T'_{\sigma'})$, so $A \in \mathrm{thick}(T'_{\sigma'})$.

For any $i \in \mathbb{Z}$ such that $i_0+n < i < i_0+2n-1$, there is a short exact sequence

$$0 \to (i_0+n, i+1) \to (i, i+1) \oplus (i_0+n, *) \to (i, *) \to 0,$$

so, as $(i_0+n, i+1)$ and $(i_0+n, *)$ are in $\mathrm{thick}(T'_{\sigma'})$ and $(i, i+1)$ is projective, $(i, *)$ is in $\mathrm{thick}(T'_{\sigma'})$. As we already got the result for $(i_0+n, *)$ and $(i_0+n, \bowtie)$ and $\Omega^{-1}((i_0+n, \bowtie)) = (i_0+2n-1, *)$, all the $(i, *)$ for $i_0+n \leq i \leq i_0+2n-1$ are in $\mathrm{thick}(T'_{\sigma'})$. Up to a shift by $1-i_0-n$, we already saw that these $(i, *)$ generate $\underline{\mathrm{CM}}^{\mathbb{Z}}\,\Lambda$. Therefore, $T'_{\sigma'}$ is a tilting object in $\underline{\mathrm{CM}}^{\mathbb{Z}}\,\Lambda$.

(2) Take the triangulation σ whose set of tagged arcs is

$$\{(P_1, P_3), (P_1, P_4), \dots, (P_1, P_n), (P_1, *), (P_1, \bowtie)\}.$$

The full subquiver Q of Q_σ with the set $Q_{\sigma,0} \setminus F$ of vertices is of type D_n. Thus,

$$\Gamma_\sigma^{\mathrm{op}}/(e_F) \cong (KQ)^{\mathrm{op}}.$$

Let σ' be constructed from σ as before. Namely, $i_0 = 2$ and

$$\sigma' = \{(n+1, n+3), (n+1, n+4), \dots, (n+1, 2n), (n+1, *), (n+1, \bowtie)\}.$$

For any $A, B \in \sigma'$ and $k \in \mathbb{Z}$, we have $B(k)_1 = n+1+kn$, so

$$\mathrm{Hom}_{\underline{\mathrm{CM}}^{\mathbb{Z}}\,\Lambda}(A, B(k)) = 0$$

for $k \neq 0$. Indeed, if $k < 0$ this is immediate as morphisms go increasingly, and if $k > 0$, $\mathrm{Hom}_{\underline{\mathrm{CM}}^{\mathbb{Z}}\,\Lambda}(A, B(k)) = \mathrm{Ext}^1_{\mathrm{CM}^{\mathbb{Z}}\,\Lambda}(A, \Omega B(k))$. Moreover, $\Omega B(k)_1 = 2+kn \geq n+2$ and for the same reason as before $\mathrm{Ext}^1_{\mathrm{CM}^{\mathbb{Z}}\,\Lambda}(A, \Omega B(k)) = 0$.

Thus, by Theorem 3.16,

$$\mathrm{End}_{\underline{\mathrm{CM}}^{\mathbb{Z}}\,\Lambda}(e_F\Gamma_\sigma) \cong \mathrm{End}_{\underline{\mathrm{CM}}\,\Lambda}(e_F\Gamma_\sigma) \cong \Gamma_\sigma^{\mathrm{op}}/(e_F).$$

Because $\underline{\mathrm{CM}}^{\mathbb{Z}}\,\Lambda$ is an algebraic triangulated Krull–Schmidt category, and $e_F\Gamma_\sigma$ is a tilting object in $\underline{\mathrm{CM}}^{\mathbb{Z}}\,\Lambda$, by Theorem 4.4 there exists a triangle-equivalence

$$\underline{\mathrm{CM}}^{\mathbb{Z}}\,\Lambda \cong \mathcal{K}^{\mathrm{b}}\big(\mathrm{proj}\,\mathrm{End}^{\mathrm{op}}_{\underline{\mathrm{CM}}^{\mathbb{Z}}\,\Lambda}(e_F\Gamma_\sigma)\big).$$

Since $\mathrm{gl.dim}\,KQ < \infty$, we have a triangle-equivalence

$$\mathcal{K}^{\mathrm{b}}\big(\mathrm{proj}\,\mathrm{End}^{\mathrm{op}}_{\underline{\mathrm{CM}}^{\mathbb{Z}}\,\Lambda}(e_F\Gamma_\sigma)\big) \cong \mathcal{K}^{\mathrm{b}}(\mathrm{proj}\,KQ) \cong \mathcal{D}^{\mathrm{b}}(KQ).$$

Therefore there is a triangle-equivalence $\underline{\mathrm{CM}}^{\mathbb{Z}}\,\Lambda \cong \mathcal{D}^{\mathrm{b}}(KQ)$. □

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