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Mod p Decomposition of H-spaces of Low Rank

by

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Abstract

Let X be a mod p H-space whose mod p cohomology is an exterior algebra generated by finitely many generators of degrees $2n_1+1,\dots,2n_k+1$ with $1 \le n_1 \le \cdots \le n_k$. It is known that if $n_k - n_1 < p-1$ then X decomposes into a product of odd spheres, and if $n_k - n_1 < 2(p-1)$ then X decomposes into a product of odd spheres and $B_n(p)$s. In this paper we consider the case of $n_k - n_1 < 3(p-1)$, and give a product decomposition of X into irreducible factors.

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§1. Introduction

Let p be a prime, and $\mathbb{F}_p$ the prime field of characteristic p. In this paper we assume that all spaces are localized at p. Let X be a simply connected H-space whose $\mathbb{F}_p$-cohomology is an exterior algebra generated by finitely many generators of odd degree:

$$H^*(X;\mathbb{F}_p) = \Lambda(x_1,\dots,x_k),$$

where $\deg x_i = 2n_i+1$ with $1 \le n_1 \le \cdots \le n_k$. We call the sequence $(2n_1+1,\dots,2n_k+1)$ the *type* of X, and k the *rank* of X.

We study decomposition of such H-spaces into irreducible factors. For compact Lie groups, Mimura, Nishida and Toda [8] gave a complete list of such decompositions. According to their results the type $(2n_1+1,\dots,2n_k+1)$ of each irreducible factor appearing in the product decomposition of a compact Lie group

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satisfies

(1) $n_i \neq n_j$ for any $i \neq j$, and
(2) $n_i \equiv n_j \bmod p-1$.

For p-compact groups, Davis [3] showed a similar result.

From those results one can guess that similar decompositions hold for general H-spaces. However, this is not the case. In fact, Zabrodsky [13] showed that if $p \geq 5$, then for any map $f\colon S^{2m} \to S^{2n+1}$, there is an H-space $X = S^{2n+1} \cup_f e^{2m+1} \cup e^{2n+2m+2}$. Thus, if f is essential with $m \not\equiv n \bmod p-1$, then X is not decomposable into factors satisfying (2).

On the other hand, for an H-space X of type $(2n_1+1, \ldots, 2n_k+1)$ if $n_k - n_1$ is not very large, then the result by Zabrodsky is not an obstruction to the decomposition into factors satisfying (2) since $\pi_{2m}(S^{2n+1}) = 0$ for $m - n \not\equiv 0 \bmod p-1$ for small m (e.g., $m \leq n + p(p-1)$ by Toda [10]).

From this point of view, the first known result is given by Kumpel [7]. He showed that if $n_k - n_1 < p-1$ then X is p*-regular*, i.e., X decomposes into a product of odd spheres. The assumption $n_k - n_1 < p-1$ is essential. In fact, Mimura and Toda [9, §2] showed that there is an irreducible space $B_n(p)$ with $H^*(B_n(p);\mathbb{F}_p) = \Lambda(x_1, x_2)$, where $\deg x_1 = 2n+1$ and $\deg x_2 = 2n + 2(p-1)+1$ with $\mathcal{P}^1 x_1 = x_2$. Then Hemmi [6] showed that if $n_k - n_1 < 2(p-1)$ then X is *quasi* p*-regular*, i.e., X decomposes into a product of odd spheres and $B_n(p)$s.

In this paper, as the next step we consider the case of $n_k - n_1 < 3(p-1)$. Unfortunately, we have to assume that $p \geq 5$. The reason is stated later.

To study our case, we first construct spaces which appear as factors in our decompositions. Those spaces are denoted by $B(n)$, $C(n)$, $E(n)$ and $F(n)$ for $n \geq 1$, and $D(n)$ for $n \geq 2$. Their cohomology algebras are as follows, where $q = 2(p-1)$:

(1) $H^*(B(n);\mathbb{F}_p) = \Lambda(b_1, b_2)$,
$\deg b_1 = 2n+1$, $\deg b_2 = 2n+1+q$;
(2) $H^*(C(n);\mathbb{F}_p) = \Lambda(c_1, c_2)$,
$\deg c_1 = 2n+1$, $\deg c_2 = 2n+1+2q$;
(3) $H^*(D(n);\mathbb{F}_p) = \Lambda(d_1, d_2, d_3)$,
$\deg d_1 = 2n+1$, $\deg d_2 = 2n+1+q$, $\deg d_3 = 2n+1+2q$;
(4) $H^*(E(n);\mathbb{F}_p) = \Lambda(e_1, e_2, e_3)$,
$\deg e_1 = 2n+1$, $\deg e_2 = 2n+1+q$, $\deg e_3 = 2n+1+2q$;
(5) $H^*(F(n);\mathbb{F}_p) = \Lambda(f_1, f_2, f_3)$,
$\deg f_1 = 2n+1$, $\deg f_2 = 2n+1+q$, $\deg f_3 = 2n+1+2q$.

Moreover, the generators are connected by cohomology operations as follows:

$$\mathcal{P}^1 b_1 = b_2, \quad \Phi c_1 = c_2, \quad \mathcal{P}^1 d_1 = d_2, \quad \mathcal{P}^1 d_2 = d_3,$$
$$\mathcal{P}^1 e_1 = e_2, \quad \Phi e_1 = e_3, \quad \Phi f_1 = \mathcal{P}^1 f_2 = f_3.$$

Here, Φ is the secondary operation detecting the Toda class α_2. Thus in particular we see that the spaces $B(n)$, $C(n)$, $D(n)$, $E(n)$ and $F(n)$ are irreducible.

Many of the above spaces are equivalent to the spaces given in [8] and [9]. In fact, $B(n)$ is equivalent to $B_n(p)$ of [9], and $C(1)$ and $E(1)$ are equivalent to B and $B_1^3(p)$ given in Propositions 8.4 and 7.4 of [8], respectively. Moreover $D(n)$ is equivalent to $B_n^3(p)$ for some n in [9, Prop. 7.2].

To construct the above spaces we use the method introduced by Cohen and Neisendorfer [1]. Since our construction is functorial, the above spaces are characterized by the type of the cohomology rings and the action of the operations $\mathcal{P}^1$ and Φ.

Our main result is stated as follows.

Theorem 1.1. *Let p be a prime with $p \geq 5$. Let X be an H-space with exterior $\mathbb{F}_p$-cohomology algebra of type $(2n_1+1, \dots, 2n_k+1)$ with $1 \leq n_1 \leq \cdots \leq n_k$. If $n_k - n_1 < 3(p-1)$, then X is homotopy equivalent to a product of the following spaces.*

(1) S^{2n+1} *with* $n_1 \leq n \leq n_k$*;*
(2) $B(n)$ *with* $n_1 \leq n \leq n_k - (p-1)$*;*
(3) $C(n)$ *with* $n_1 \leq n \leq n_k - 2(p-1)$*;*
(4) $D(n)$ *with* $n_1 \leq n \leq n_k - 2(p-1)$ $(n \neq 1)$*;*
(5) $E(n)$ *with* $n_1 \leq n \leq n_k - 2(p-1)$*;*
(6) $F(n)$ *with* $n_1 \leq n \leq n_k - 2(p-1)$*.*

The above theorem states that the type $(2n_1+1, \dots, 2n_t+1)$ of each irreducible factor satisfies (1) $n_i \neq n_j$ for any $i \neq j$, and (2) $n_i \equiv n_j \bmod p-1$. The condition $n_k - n_1 < 3(p-1)$ is essential for this fact. In fact, if $n_k - n_1 = 3(p-1)$ then there is an irreducible H-space of type

$$(2n+1, 2n+q+1, 2n+2q+1, 2n+2q+1, 2n+3q+1)$$

for which condition (1) is not satisfied.

The paper is organized as follows. In Section 2, we review the method of Cohen and Neisendorfer [1] for constructing H-spaces of low rank. Then in Section 3 we construct the spaces which appear as product factors in our decompositions of H-spaces. The main theorem is proved in Section 4.

§2. Cohen–Neisendorfer construction

In the rest of paper, we do not distinguish a continuous map and its homotopy class if there is no confusion.

There are two known methods to construct H-spaces of low rank: one by Cooke, Harper and Zabrodsky [2], and the other by Cohen and Neisendorfer [1]. Here we review the Cohen–Neisendorfer method.

Let L be a cell complex consisting of odd cells:

$$L = S^{2n_1+1} \cup e^{2n_2+1} \cup \cdots \cup e^{2n_k+1},$$

where $1 \leq n_1 \leq \cdots \leq n_k$. We call such a space an *odd cell complex* of rank k. It is proved in [1] that if $k < p-1$ then there is an H-space $M(L)$ and a map $\iota^L \colon L \to M(L)$ such that $(\iota^L)_* \colon H_*(L; \mathbb{F}_p) \to H_*(M(L); \mathbb{F}_p)$ is a monomorphism and $H_*(M(L); \mathbb{F}_p)$ is an exterior algebra generated by $(\iota^L)_*(\tilde{H}_*(L; \mathbb{F}_p))$. Thus, in particular, there are cohomology classes $x_i \in H^*(M(L); \mathbb{F}_p)$ $(1 \leq i \leq k)$ with $\deg x_i = 2n_i + 1$ such that $\{(\iota^L)^*(x_1), \dots, (\iota^L)^*(x_k)\}$ is a basis for $\tilde{H}^*(L; \mathbb{F}_p)$ and

$$H^*(M(L); \mathbb{F}_p) \cong \Lambda(x_1, \dots, x_k).$$

This construction is functorial in the sense that for any map $f \colon K \to L$ from another odd cell complex K of rank less than $p-1$, there is a map $M(f) \colon M(K) \to M(L)$ such that the following diagram is homotopy commutative:

$$\begin{array}{ccc} K & \xrightarrow{f} & L \\ \downarrow{\scriptstyle \iota^K} & & \downarrow{\scriptstyle \iota^L} \\ M(K) & \xrightarrow{M(f)} & M(L) \end{array}$$

In particular, if K is a subcomplex of L of the form $S^{2n_1+1} \cup e^{2n_2+1} \cup \cdots \cup e^{2n_t+1}$ with $t < k$, then the cofibre sequence $K \to L \to L/K$ induces a homotopy fibre sequence

$$M(K) \xrightarrow{\varepsilon} M(L) \xrightarrow{\pi} M(L/K)$$

with

$$\begin{aligned} \pi^*(z_i) &= x_i \quad (t+1 \leq i \leq k), \\ \varepsilon^*(x_i) &= y_i \quad (1 \leq i \leq t). \end{aligned}$$

where

$$H^*(M(K); \mathbb{F}_p) \cong \Lambda(y_1, \dots, y_t), \qquad H^*(M(L/K); \mathbb{F}_p) \cong \Lambda(z_{t+1}, \dots, z_k).$$

As a special case, we have

$$M(L_1 \vee L_2) \simeq M(L_1) \times M(L_2).$$

Now, to define the space $M(L)$ and the map $\iota^L \colon L \to M(L)$, a space $\lambda(L)$ and a map $\lambda(L) \to \Sigma L$ are constructed in [1] such that there is a natural fibre sequence

$$\Omega\lambda(L) \to \Omega\Sigma L \xrightarrow{\rho} M(L) \to \lambda(L) \to \Sigma L.$$

Then $\iota^L \colon L \to M(L)$ is given by the composition $\iota^L = \rho \circ E_L$, where $E_X \colon X \to \Omega\Sigma X$ denotes the adjoint of $\mathrm{id}_{\Sigma X} \colon \Sigma X \to \Sigma X$ for a space X. It is also shown that there is a section $s \colon M(L) \to \Omega\Sigma L$ so that $\rho \circ s \simeq \mathrm{id}$, and multiplication on $M(L)$ is defined by the composition

$$\rho \circ \mu \circ (s \times s) \colon M(L) \times M(L) \to M(L),$$

where $\mu \colon \Omega\Sigma L \times \Omega\Sigma L \to \Omega\Sigma L$ is the loop multiplication.

Now we show the following

Lemma 2.1. $s \circ \iota^L \simeq E_L \colon L \to \Omega\Sigma L.$

Proof. We first recall the definition of the section s from [1]. It is shown that the map $\Sigma\iota^L \colon \Sigma L \to \Sigma M(L)$ has a retraction $r \colon \Sigma M(L) \to \Sigma L$ so that $r \circ \Sigma\iota^L \simeq \mathrm{id}$. Set $s' = \Omega r \circ E_{M(L)} \colon M(L) \to \Omega\Sigma M(L) \to \Omega\Sigma L$. It is also proved in [1] that the composition $\rho \circ s' \colon M(L) \to M(L)$ is a homotopy equivalence. Then the section $s \colon M(L) \to \Omega\Sigma L$ is defined by $s = s' \circ (\rho \circ s')^{-1}$.

Now $(\rho \circ s') \circ \iota^L = \rho \circ \Omega r \circ E_{M(L)} \circ \iota^L = \rho \circ \Omega r \circ \Omega\Sigma\iota^L \circ E_L \simeq \rho \circ E_L = \iota^L$. Thus, we have

$$s \circ \iota^L = s' \circ (\rho \circ s')^{-1} \circ \iota^L \simeq s' \circ \iota^L \simeq \Omega r \circ E_{M(L)} \circ \iota^L \simeq \Omega r \circ \Omega\Sigma\iota^L \circ E_L \simeq E_L. \qquad \square$$

Let L_i $(1 \leq i \leq t)$ be an odd cell complex with rank less than $p-1$. Set $L = L_1 \vee \cdots \vee L_t$. Consider the composition

$$\iota \colon L \subset L_1 \times \cdots \times L_t \xrightarrow{\iota^{L_1} \times \cdots \times \iota^{L_t}} M(L_1) \times \cdots \times M(L_t).$$

It is clear that $\iota_* \colon H_*(L;\mathbb{F}_p) \to H_*(M(L_1) \times \cdots \times M(L_t);\mathbb{F}_p)$ is a monomorphism and $H_*(M(L_1) \times \cdots \times M(L_t);\mathbb{F}_p)$ is an exterior algebra generated by $\iota_*(\tilde{H}_*(L;\mathbb{F}_p))$. Thus it is natural to write $M(L_1) \times \cdots \times M(L_t)$ as $M(L)$ and ι as ι^L:

$$\iota^L \colon L \to M(L) = M(L_1) \times \cdots \times M(L_t).$$

Then we show the following

Lemma 2.2. *Let $L = L_1 \vee \cdots \vee L_t$, where L_i are odd cell complexes of rank less than $p-1$. Let Y be an H-space and $f\colon L \to Y$ a map. Then there is a map $\hat{f}\colon M = M(L) \to Y$ such that the following diagram is homotopy commutative:*

$$\begin{array}{ccc} L & \xrightarrow{\ f\ } & Y \\ {\scriptstyle \iota^L}\big\downarrow & \nearrow{\scriptstyle \hat{f}} & \\ M(L) & & \end{array}$$

Proof. It is sufficient to handle the case of $L = L_i$. In fact, if there are maps $\hat{f}_i\colon M(L_i) \to Y$ with $\hat{f}_i|L_i = f|L_i$ for $1 \le i \le t$, then we can define $\hat{f}\colon M(L) \to Y$ by

$$\hat{f}(x_1, x_2, \dots, x_t) = (\cdots(\hat{f}_1(x_1)\hat{f}_2(x_2))\cdots)\hat{f}_t(x_t)$$

using multiplication of Y.

Now we define $\hat{f}\colon M(L) \to Y$ by the composition

$$r \circ \Omega\Sigma f \circ s\colon M(L) \to \Omega\Sigma L \to \Omega\Sigma Y \to Y,$$

where $s\colon M(L) \to \Omega\Sigma L$ is the section and $r\colon \Omega\Sigma Y \to Y$ is the retraction of an H-space so that $r \circ E_Y \simeq \mathrm{id}\colon Y \to Y$. Then by Lemma 2.1 we have

$$\hat{f} \circ \iota^L \simeq r \circ \Omega\Sigma f \circ s \circ \iota^L \simeq r \circ \Omega\Sigma f \circ E_L \simeq r \circ E_Y \circ f \simeq f. \qquad \square$$

§3. Construction of low rank H-spaces

Now we construct H-spaces $B(n)$, $C(n)$, $D(n)$, $E(n)$ and $F(n)$ by using the Cohen–Neisendorfer method. Since the rank should be less than $p-1$, we need to assume that $p \ge 5$. Our method to prove the main theorem is as follows: We construct odd cell complexes $L_A(n)$ for $A = B$, C, U, V and W such that $M(L_A(n)) = A(n)$. Then we show that there is an odd cell complex L which is a wedge sum of such complexes, and a map $f\colon L \to X$ to an H-space X with the properties in Theorem 1.1 such that $f^*(x_1), \dots, f^*(x_k)$ is a basis for $\tilde{H}^*(L; \mathbb{F}_p)$, where $H^*(X; \mathbb{F}_p) \cong \Lambda(x_1, \dots, x_k)$. Then by Lemma 2.2, we obtain Theorem 1.1.

First we note that $M(S^{2n+1}) = S^{2n+1}$ and $\iota^{S^{2n+1}} = \mathrm{id}$.

To construct the required odd cell complexes, we recall the homotopy group $\pi_{2m}(S^{2n+1})$ for $2m < 2n+3q$. In this range, the only non-trivial cases are

$$\pi_{2n+q}(S^{2n+1}) \cong \mathbb{Z}/p\{\alpha_1(2n+1)\}, \qquad \pi_{2n+2q}(S^{2n+1}) \cong \mathbb{Z}/p\{\alpha_2(2n+1)\}.$$

Here, $\alpha_i(m) \in \pi_{m+iq-1}(S^m)$ for $m \geq 3$ are well known generators with $\alpha_i(m) = \Sigma^{m-3}\alpha_i(3)$ for $m > 3$. Moreover , $\alpha_2(3)$ is defined by the Toda bracket as

$$\alpha_2(3) = \{\alpha_1(3), p, \alpha_1(2p)\}.$$

For the composition $\alpha_1(2n+1) \circ \alpha_1(2n+q)$ we have the following

Lemma 3.1 (Toda [11, Proposition 13.6]). *If $n = 1$, then $\alpha_1(3) \circ \alpha_1(2p)$ is a generator of $\pi_{q+1}(S^3) \cong \mathbb{Z}/p$, while for $n \geq 2$, $\alpha_1(2n+1) \circ \alpha_1(2n+q) = 0$.*

Hereafter, we simply denote $\alpha_i(m)$ by α_i.

Now, for $n \geq 1$, we set

$$L_B(n) = S^{2n+1} \cup_{\alpha_1} e^{2n+q+1}, \qquad L_C(n) = S^{2n+1} \cup_{\alpha_2} e^{2n+2q+1}.$$

It is clear that $L_B(n) = \Sigma^{2n-2} L_B(1)$ and $L_C(n) = \Sigma^{2n-2} L_C(1)$. We notice that $\Sigma L_B(n) = S^{2n+2} \cup_{\alpha_1} e^{2n+2q+2}$ and $\Sigma L_C(n) = S^{2n+2} \cup_{\alpha_2} e^{2n+2q+2}$.

Now, if $n \geq 2$, then $\alpha_1(2n+1) \circ \alpha_1(2n+q) = 0$ by Lemma 3.1, and so we have an extension $\hat{\alpha}_1 \colon S^{2n+q} \cup_{\alpha_1} e^{2n+2q} \to S^{2n+1}$ of $\alpha_1 \colon S^{2n+q} \to S^{2n+1}$:

$$\begin{array}{ccc} S^{2n+q} & \xrightarrow{\alpha_1} & S^{2n+1} \\ \downarrow & \nearrow_{\hat{\alpha}_1} & \\ S^{2n+q} \cup_{\alpha_1} e^{2n+2q} & & \end{array}$$

Then, for $n \geq 2$, we set

$$L_D(n) = S^{2n+1} \cup_{\hat{\alpha}_1} C(S^{2n+q} \cup_{\alpha_1} e^{2n+2q}).$$

It is clear that $L_D(n) = \Sigma^{2n-4} L_D(2)$, and

$$L_D(n)/S^{2n+1} = S^{2n+q+1} \cup_{\alpha_1} e^{2n+2q+1} \simeq L_B(n+p-1).$$

Finally, for $n \geq 1$, we set

$$\begin{aligned} L_E(n) &= S^{2n+1} \cup_{\nabla \circ (\alpha_1 \vee \alpha_2)} (e^{2n+q+1} \vee e^{2n+2q+1}), \\ L_F(n) &= (S^{2n+1} \vee S^{2n+q+1}) \cup_{(\alpha_2 \vee \alpha_1) \circ \Delta} e^{2n+2q+1}, \end{aligned}$$

where $\nabla \colon S^{2n+1} \vee S^{2n+1} \to S^{2n+1}$ is the folding map and $\Delta \colon S^{2n+2q} \to S^{2n+2q} \vee S^{2n+2q}$ is the coproduct. Then $L_E(n) = \Sigma^{2n-2} L_E(1)$ and $L_F(n) = \Sigma^{2n-2} L_F(1)$. It is clear that

$$\begin{gathered} L_B(n) \subset L_E(n), \quad L_C(n) \subset L_E(n), \\ L_F(n)/S^{2n+1} = L_B(n+p-1), \quad L_F(n)/S^{2n+q+1} = L_C(n). \end{gathered}$$

Now we define H-spaces $B(n)$, $C(n)$, $E(n)$ and $F(n)$ for $n \geq 1$, and $D(n)$ for $n \geq 2$ by

$$B(n) = M(L_B(n)), \quad C(n) = M(L_C(n)), \quad D(n) = M(L_D(n)),$$
$$E(n) = M(L_E(n)), \quad F(n) = M(L_F(n)).$$

Then the following proposition is clear from the construction.

Proposition 3.2. *The H-spaces $B(n)$, $C(n)$, $D(n)$, $E(n)$ and $F(n)$ are irreducible, and*

(1) $H^*(B(n); \mathbb{F}_p) = \Lambda(b_1, b_2)$,
$\deg b_1 = 2n+1$, $\deg b_2 = 2n+1+q$;
(2) $H^*(C(n); \mathbb{F}_p) = \Lambda(c_1, c_2)$,
$\deg c_1 = 2n+1$, $\deg c_2 = 2n+1+2q$;
(3) $H^*(D(n); \mathbb{F}_p) = \Lambda(d_1, d_2, d_3)$,
$\deg d_1 = 2n+1$, $\deg d_2 = 2n+1+q$, $\deg d_3 = 2n+1+2q$;
(4) $H^*(E(n); \mathbb{F}_p) = \Lambda(e_1, e_2, e_3)$,
$\deg e_1 = 2n+1$, $\deg e_2 = 2n+1+q$, $\deg e_3 = 2n+1+2q$;
(5) $H^*(F(n); \mathbb{F}_p) = \Lambda(f_1, f_2, f_3)$,
$\deg f_1 = 2n+1$, $\deg f_2 = 2n+1+q$, $\deg f_3 = 2n+1+2q$.

The generators are connected by the cohomology operations as follows:

$$\mathcal{P}^1 b_1 = b_2, \quad \Phi c_1 = c_2, \quad \mathcal{P}^1 d_1 = d_2, \quad \mathcal{P}^1 d_2 = d_3,$$
$$\mathcal{P}^1 e_1 = e_2, \quad \Phi e_1 = e_3, \quad \Phi f_1 = \mathcal{P}^1 f_2 = f_3.$$

Moreover, those spaces are characterized by the type of the cohomology rings and the action of the operations $\mathcal{P}^1$ and Φ since the Cohen–Neisendorfer method is functorial.

Various generalizations of the Cohen–Neisendorfer method have been considered by several authors. Among them are Wu [12] and Grbić, Harper, Mimura, Theriault and Wu [4], who studied the case of rank $p-1$. In particular, it is proved in [4, Proposition 1.1] that the Cohen–Neisendorfer method works also for the rank $p-1$ case but the resulting space need not be an H-space. This means that the above spaces $B(n)$ and $C(n)$ exist also for $p = 3$ as just topological spaces. Moreover, conditions for those spaces to be H-spaces are studied in [4, Theorem 7.1]. In particular, it is shown that $B(n)$ for $p = 3$ is an H-space if and only if $n = 1$ or $n \equiv -1 \bmod 3$, which coincides with the result of [14] and [5]. To study $D(n)$, $E(n)$ and $F(n)$ with $p = 3$, we need to consider the rank p case, and there are no known results for this case.

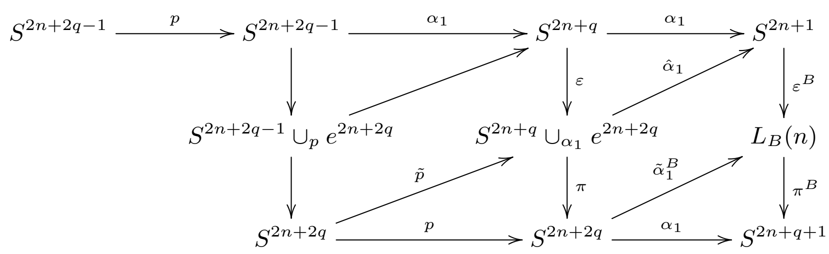
Now we study the homotopy groups of the spaces of Proposition 3.2.

Let $\varepsilon^B\colon S^{2n+1}\to L_B(n)$, $\varepsilon^C\colon S^{2n+1}\to L_C(n)$ and $\varepsilon_1^F\colon S^{2n+1}\to L_F(n)$ be the natural inclusions. Then, for $n\geq 1$, we set

$$\begin{aligned}\alpha_2^B &= (\varepsilon^B)_*(\alpha_2)\in\pi_{2n+2q}(L_B(n)), &\quad \alpha_1^F &= (\varepsilon_1^F)_*(\alpha_1)\in\pi_{2n+q}(L_F(n)),\\ \alpha_1^C &= (\varepsilon^C)_*(\alpha_1)\in\pi_{2n+q}(L_C(n)), &\quad \alpha_2^F &= (\varepsilon_1^F)_*(\alpha_2)\in\pi_{2n+2q}(L_F(n)).\end{aligned}$$

Next we define $\tilde\alpha_1^B\in\pi_{2n+2q}(L_B(n))$ for $n\geq 2$. Since $\alpha_1\circ\alpha_1=0\in\pi_{2n+2q-1}(S^{2n+1})$ by Lemma 3.1 and α_1 has order p, we have the Toda bracket $\{\alpha_1,\alpha_1,p\}\subset\pi_{2n+2q}(S^{2n+1})$. It is proved in [11, Chapter XIII] that $\{\alpha_1,\alpha_1,p\}$ consists of a single element $2^{-1}\alpha_2$. In other words, we have $\hat\alpha_1\circ\tilde p=2^{-1}\alpha_2$ in the following diagram:

(3.1)



We note that the above diagram is homotopy commutative except for two central parallelograms which are homotopy anti-commutative, i.e., homotopy commutative up to sign.

Then we define

$$\tilde\alpha_1^B\in\pi_{2n+2q}(L_B(n))$$

to be the coextension of $\alpha_1\colon S^{2n+2q}\to S^{2n+q+1}$ as defined in the above diagram. By definition we have

$$(\pi^B)_*(\tilde\alpha_1^B)=\alpha_1\quad\text{and}\quad p\tilde\alpha_1^B=-2^{-1}(\varepsilon^B)_*(\alpha_2),\tag{3.2}$$

where $\pi^B\colon L_B(n)\to L_B(n)/S^{2n+1}=S^{2n+q+1}$ is the projection.

Finally, we set

$$\tilde\alpha_1^E=(\varepsilon_1^E)_*(\tilde\alpha_1^B)\in\pi_{2n+2q}(L_E(n)),$$

where $\varepsilon_1^E\colon L_B(n)\to L_E(n)$ is the inclusion.

Then we show the following

Lemma 3.3.

$$
\begin{aligned}
L_B(n) \cup_{\tilde{\alpha}_1^B} e^{2n+2q+1} &\simeq L_D(n),\\
L_B(n) \cup_{\alpha_2^B} e^{2n+2q+1} &\simeq L_E(n),\\
L_C(n) \cup_{\alpha_1^C} e^{2n+q+1} &\simeq L_E(n),\\
L_E(n) \cup_{\tilde{\alpha}_1^E} e^{2n+2q+1} &\simeq L_D(n) \vee S^{2n+2q+1},\\
L_F(n) \cup_{\alpha_2^F} e^{2n+2q+1} &\simeq L_C(n) \vee L_B(n+p-1).
\end{aligned}
$$

Proof. The first three relations are easy to show.

Let $\varepsilon^D \colon S^{2n+1} \to L_D(n)$ and $\varepsilon_1 \colon S^{2n+1} \to S^{2n+1} \vee S^{2n+q+1}$ be the inclusions. Then the last two relations are shown as follows:

$$
\begin{aligned}
L_E(n) \cup_{\tilde{\alpha}_1^E} e^{2n+2q+1} &\simeq L_D(n) \cup_{(\varepsilon^D)_*(\alpha_2)} e^{2n+2q+1}\\
&\simeq L_D(n) \vee S^{2n+2q+1},\\
L_F(n) \cup_{\alpha_2^F} e^{2n+2q+1} &\simeq ((S^{2n+1} \vee S^{2n+q+1}) \cup_{(\varepsilon_1)_*(\alpha_2)} e^{2n+2q+1}) \cup e^{2n+2q+1}\\
&\simeq (L_C(n) \vee S^{2n+q+1}) \cup_{(*,\alpha_1)} e^{2n+2q+1}\\
&\simeq L_C(n) \vee L_B(n+p-1).
\end{aligned}
$$

□

We can also prove the following relation, but we do not give the proof since we do not use it in this paper:

$$L_F(n) \cup_{\alpha_1^F} e^{2n+q+1} \simeq L_B(n) \vee L_B(n+p-1).$$

Let $\iota^A \colon L_A(n) \to M(L_A(n)) = A(n)$ be the natural map for $A = B$, C, U, V or W. Then we have the following fact, parts of which were already proved in [9, Thm. 3.2] and [8, Prop. 6.3].

Proposition 3.4. *The even-dimensional non-trivial homotopy groups of* $B(n)$, $C(n)$, $D(n)$, $E(n)$ *and* $F(n)$ *for dimension less than* $2n+3q$ *are as follows:*

$$
\begin{aligned}
\pi_{2n+2q}(B(n)) &\cong \begin{cases} \mathbb{Z}/p\{(\iota^B)_*(\alpha_2^B)\} & (n=1),\\ \mathbb{Z}/p^2\{(\iota^B)_*(\tilde{\alpha}_1^B)\} & (n \geq 2),\end{cases}\\
\pi_{2n+q}(C(n)) &\cong \mathbb{Z}/p\{(\iota^C)_*(\alpha_1^C)\},\\
\pi_{2n+2q}(E(n)) &\cong \mathbb{Z}/p\{(\iota^E)_*(\tilde{\alpha}_1^E)\} \quad (n \geq 2),\\
\pi_{2n+q}(F(n)) &\cong \mathbb{Z}/p\{(\iota^F)_*(\alpha_1^F)\},\\
\pi_{2n+2q}(F(n)) &\cong \mathbb{Z}/p\{(\iota^F)_*(\alpha_2^F)\}.
\end{aligned}
$$

We remark that $\pi_{2m}(D(n)) = 0$ for $2m < 2n+3q$.

Proof. Almost all parts are easy to show by studying homotopy exact sequences of fibre sequences. Here we give just an outline.

For the case of $B(n)$, we consider the following fibre sequence coming from the cofibre sequence $S^{2n+1} \to L_B(n) \to L_B(n)/S^{2n+1} = S^{2n+q+1}$:

$$S^{2n+1} \to B(n) \to S^{2n+q+1}.$$

It is easy to show that the even-dimensional non-trivial homotopy groups of $B(n)$ occur only in dimensions $2n + 2q$. Since the connecting homomorphism $\partial_* \colon \pi_{2n+2q}(S^{2n+q+1}) \to \pi_{2n+2q-1}(S^{2n+1})$ satisfies $\partial_*(\alpha_1) = \alpha_1 \circ \alpha_1$, if $n = 1$ then by Lemma 3.1 we have $\pi_{2q+2}(B(1)) \cong \mathbb{Z}/p\{(\iota^B)_*(\alpha_2^B)\}$. For $n \geq 2$, we have $\pi_{2n+2q}(B(n)) \cong \mathbb{Z}/p^2\{(\iota^B)_*(\tilde{\alpha}_1^B)\}$ by Lemma 3.2.

For $E(n)$, we consider the homotopy exact sequence of the fibre sequence

$$S^{2n+1} \to E(n) \to S^{2n+q+1} \times S^{2n+2q+1}.$$

Then the connecting homomorphism $\partial_* \colon \pi_{2n+2q}(S^{2n+q+1} \times S^{2n+2q+1}) \to \pi_{2n+2q-1}(S^{2n+1})$ satisfies $\partial_*(\alpha_1, *) = \alpha_1 \circ \alpha_1$. Thus, for the same reason as in the case of $B(n)$, we obtain the result.

The other cases are easy to show by considering homotopy exact sequences of the following fibrations:

$$\begin{gathered} S^{2n+1} \to C(n) \to S^{2n+2q+1}, \\ B(n) \to D(n) \to S^{2n+2q+1}, \\ S^{2n+1} \times S^{2n+q+1} \to F(n) \to S^{2n+2q+1}. \end{gathered}$$ □

For positive integers n_1 and n with $n_1 \leq n \leq n_1 + 3(p-1)$, let $S_{n_1,n}$ be the set consisting of the pairs (A, γ), where A is

(1) S^{2m+1} with $n_1 \leq m \leq n$,
(2) $B(m)$ with $n_1 \leq m \leq n - (p-1)$, or
(3) $C(m)$, $D(m)$, $E(m)$ or $F(m)$ with $n_1 \leq m \leq n - 2(p-1)$,

and $\gamma \in \pi_{2n}(A)$.

By Proposition 3.4, if $\gamma \neq 0$, then A must be S^{2n-q+1}, $S^{2n-2q+1}$, $B(n-2(p-1))$, $E(n-2(p-1))$ or $F(n-2(p-1))$, and γ is one of the classes in $\pi_{2n}(A)$ given in Proposition 3.4 up to unit. We note that A is neither $C(m)$ nor $D(m)$, and $\gamma \neq (\iota^F)_*(\alpha_1^F)$ even if $A = F(m)$ for dimensional reasons.

We define a preorder on $S_{n_1,n}$ by writing $(A_1, \gamma_1) \preceq (A_2, \gamma_2)$ for $(A_1, \gamma_1), (A_2, \gamma_2) \in S_{n_1,n}$ if there is a map $f \colon A_1 \to A_2$ with $f_*(\gamma_1) = \gamma_2$. It is clear that $(A_1, \gamma_1) \preceq (A_2, \gamma_2)$ if $\gamma_2 = *$, or $A_1 = A_2$ with $\gamma_1 = \gamma_2$ up to unit. For the other cases, we have

Lemma 3.5. *Let $n_1 \le n \le n_1 + 3(p-1)$ and $m = n - 2(p-1)$. Then in $S_{n_1,n}$ we have*

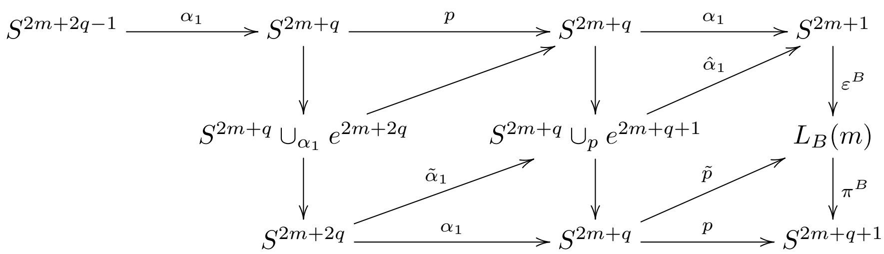
$$(S^{2m+(2-i)q+1}, \alpha_i) \preceq (F(m), (\iota^F)_*(\alpha_2^F)) \preceq (B(m), (\iota^B)_*(\alpha_2^B))$$

for $i = 1, 2$. Moreover, if $m \ge 2$, then also for $i = 1, 2$ we have

$$(B(m), (\iota^B)_*(\tilde{\alpha}_1^B)) \preceq (E(m), (\iota^E)_*(\tilde{\alpha}_1^E)) \preceq (S^{2m+(2-i)q+1}, \alpha_i).$$

Proof. First we show that there is a map $f\colon L_F(m) \to L_B(m)$ such that $f_*(\alpha_2^F) = \alpha_2^B$. Then $M(f)\colon F(m) \to B(m)$ satisfies $M(f)_*((\iota^F)_*(\alpha_2^F)) = (\iota^B)_*(\alpha_2^B)$, and we have $(F(m), (\iota^F)_*(\alpha_2^F)) \preceq (B(m), (\iota^B)_*(\alpha_2^B))$.

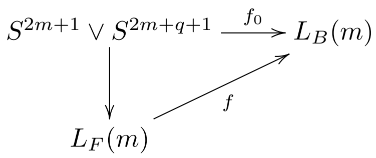
Now since $\{\alpha_1, p, \alpha_1\} = \alpha_2$, we have $\hat{\alpha}_1 \circ \tilde{\alpha}_1 = \alpha_2$ in the following diagram, which is homotopy commutative except for two central homotopy anti-commutative parallelograms:



Then the map $\tilde{p}\colon S^{2m+q} \to L_B(m)$ satisfies

$$\tilde{p}_*(\alpha_1) = -(\varepsilon^B)_*(\alpha_2) = -\alpha_2^B.$$

Consequently, for the map $f_0 = \nabla \circ (\varepsilon^B \vee \tilde{p})\colon S^{2m+1} \vee S^{2m+q+1} \to L_B(m)$ we have $f_0 \circ ((\alpha_2 \vee \alpha_1) \circ \Delta) \simeq *$, and there is an extension $f\colon L_F(m) \to L_B(m)$ of f_0:



Then

$$f_*(\alpha_2^F) = (\varepsilon_1^F)_*(\alpha_2) = (\varepsilon^B)_*(\alpha_2) = \alpha_2^B.$$

Next we show that $(S^{2m+(2-i)q+1}, \alpha_i) \preceq (F(m), (\iota^F)_*(\alpha_2^F))$ for $i = 1, 2$. Clearly $(S^{2m+1}, \alpha_2) \preceq (F(m), (\iota^F)_*(\alpha_2^F))$ since $\alpha_2^F = (\varepsilon_1^F)_*(\alpha_2)$ for $\varepsilon_1^F : S^{2m+1} \to L_F(m)$. On the other hand, for the other inclusion $\varepsilon_2^F : S^{2m+q+1} \to L_F(m)$ we have

$$(\varepsilon_2^F)_*(-\alpha_1) = (\varepsilon_1^F)_*(\alpha_2) = \alpha_2^F.$$

Thus, $(S^{2m+q+1}, \alpha_1) \preceq (F(m), (\iota^F)_*(\alpha_2^F))$.

Next suppose $m \geq 2$. The relation $(E(m), (\iota^E)_*(\tilde{\alpha}_1^E)) \preceq (S^{2m+(2-i)q+1}, \alpha_i)$ for $i = 1, 2$ is clear by considering the equalities $(\pi_1^E)_*(\tilde{\alpha}_1^E) = \alpha_1$ and $(\pi_2^E)_*(\tilde{\alpha}_1^E) = \alpha_2$, where $\pi_1^E\colon L_E(m) \to L_E(m)/L_B(m) = S^{2n+q+1}$ and $\pi_2^E\colon L_E(m) \to L_E(m)/L_C(m) = S^{2n+1}$ are the projections.

Moreover, the relation $(B(m), (\iota^B)_*(\tilde{\alpha}_1^B)) \preceq (E(m), (\iota^E)_*(\tilde{\alpha}_1^E))$ is clear since $(\varepsilon_1^E)_*(\tilde{\alpha}_1^B) = \tilde{\alpha}_1^E$. □

§4. Proof of Main Theorem

First we show the following

Lemma 4.1. *Let B be an H-space, and $f = (f_1, f_2)\colon S^{2m} \to A \times B$ a map, Suppose that there is a map $\eta\colon A \to B$ such that $\eta \circ f_1 \simeq f_2$. Then there is a homotopy equivalence $\varphi\colon A \times B \to A \times B$ such that $\varphi \circ f \simeq (f_1, *)$.*

Proof. Define $\psi\colon A \times B \to A \times B$ by

$$\psi(a, b) = \big(a, \mu(\eta(a), b)\big),$$

where μ is multiplication of B. Then $\psi \circ (f_1, *) \simeq f$. Since ψ is a homotopy equivalence, the homotopy inverse φ of ψ is the desired map. □

Now we prove the main theorem.

Proof of Theorem 1.1. We show that there are odd cell complexes L_i and maps $f_i\colon L_i \to X$ for $1 \leq i \leq k$ such that the following conditions are satisfied:

(1) L_i is a wedge of spaces S^{2m+1}, $L_B(m)$, $L_C(m)$, $L_D(m)$, $L_E(m)$ and $L_F(m)$ for suitable m so that $M(L_i)$ is a product of S^{2m+1}, $B(m)$, $C(m)$, $D(m)$, $E(m)$ and $F(m)$.

(2) $f_i^*(x_1), \ldots, f_i^*(x_i)$ is a basis for $\tilde{H}^*(L_i; \mathbb{F}_p)$.

Then by Lemma 2.2 there is a map $\hat{f}_i\colon M(L_i) \to X$ such that

$$H^*(M(L_i); \mathbb{F}_p) \cong \Lambda(\hat{f}_i^*(x_1), \ldots, \hat{f}_i^*(x_i)).$$

In particular, $\hat{f}_k\colon M(L_k) \to X$ is a homotopy equivalence, and so gives the desired decomposition.

For $i = 1$, we take $L_1 = S^{2n_1+1}$ and f_1 the obvious map.

Suppose inductively that we have spaces L_i and maps f_i for $i < t$. We can change the generators x_i in $H^*(X;\mathbb{F}_p)$ if necessary so that f_{t-1} satisfies $f_{t-1}^*(x_i) = 0$ for $i \geq t$.

Take a map $\beta\colon S^{2n_t} \to M(L_{t-1})$ such that $\hat{f}_{t-1} \circ \beta \simeq *$ and for an extension $g_{t-1}\colon M(L_{t-1}) \cup_\beta e^{2n_t+1} \to X$ we have

$$H^*(M(L_{t-1}) \cup_\beta e^{2n_t+1};\mathbb{F}_p) \cong \Lambda(g_{t-1}^*(x_1),\ldots,g_{t-1}^*(x_{t-1})) \oplus \mathbb{Z}/p\{g_{t-1}^*(x_t)\}.$$

If $\beta \simeq *$, then $M(L_{t-1}) \cup_\beta e^{2n_t+1} \simeq M(L_{t-1}) \vee S^{2n_t+1}$. Thus we can set $L_t = L_{t-1} \vee S^{2n_t+1}$ and define $f_t\colon L_t \to X$ by

$$f_t = \nabla \circ (f_{t-1} \vee (g_{t-1}|S^{2n_t+1}))\colon L_t = L_{t-1} \vee S^{2n_t+1} \to X \vee X \to X.$$

Then it is clear that L_t and f_t satisfy the desired conditions (1) and (2).

Suppose that $\beta \not\simeq *$. We write $L_{t-1} = K_1 \vee \cdots \vee K_s$, where each K_i is one of S^{2m+1}, $L_B(m)$, $L_C(m)$, $L_D(m)$, $L_E(m)$ or $L_F(m)$, and $\beta = (\beta_1,\ldots,\beta_s)$ with $\beta_i\colon S^{2n_t} \to M(K_i)$. Moreover, if $\beta_i \not\simeq *$ then $(M(K_i),\beta_i)$ is one of (S^{2n_t-q+1},α_1), (S^{2n_t-2q+1},α_2), $(B(n_t-2p+2),(\iota^B)_*(\alpha_2^B))$, $(B(n_t-2p+2),(\iota^B)_*(\tilde{\alpha}_1^B))$, $(E(n_t-2p+2),(\iota^E)_*(\tilde{\alpha}_1^E))$ or $(F(n_t-2p+2),(\iota^F)_*(\alpha_2^F))$ by Proposition 3.4. We remark that the pairs $(C(n_t-p+1),(\iota^C)_*(\alpha_1^C))$ and $(F(n_t-p+1),(\iota^F)_*(\alpha_1^F))$ do not occur for dimensional reasons. We assume that the $\{(M(K_i),\beta_i)\}$ are arranged so that if $(M(K_i),\beta_i) \preceq (M(K_j),\beta_j)$ then $i \leq j$.

We show there is a homotopy equivalence $\psi\colon M(L_{t-1}) \to M(L_{t-1})$ such that $\psi \circ \beta \simeq (\beta_1,*,\ldots,*)$ or $\psi \circ \beta \simeq (\alpha_2,\alpha_1,*,\ldots,*)$ with $K_1 = S^{2n_t-2q+1}$ and $K_2 = S^{2n_t-q+1}$. In fact, if $(M(K_1),\beta_1)$ is a minimum pair, then by applying Lemma 4.1 with $A = M(K_1)$ and $B = M(K_2) \times \cdots \times M(K_s)$ we get such a homotopy equivalence $\psi\colon M(L_{t-1}) \to M(L_{t-1})$. On the other hand, if there are no minimum pairs in $\{(M(K_i),\beta_i)\}$, then we can assume that $(M(K_1),\beta_1) = (S^{2n_t-2q+1},\alpha_2)$, $(M(K_2),\beta_2) = (S^{2n_t-q+1},\alpha_1)$, and $(S^{2n_t-2q+1},\alpha_2) \preceq (M(K_i),\beta_i)$ and $(S^{2n_t-q+1},\alpha_1) \preceq (M(K_i),\beta_i)$ for $i \geq 3$. Then by applying Lemma 4.1 with $A = M(K_1) \times M(K_2)$ and $B = M(K_3) \times \cdots \times M(K_s)$ we obtain a homotopy equivalence $\psi\colon M(L_{t-1}) \to M(L_{t-1})$ as desired.

Let A and B be the spaces in the above argument. We replace f_{t-1} by $\hat{f}_{t-1} \circ \psi^{-1} \circ \iota^{L_{t-1}}\colon L_{t-1} \to X$, and β by $\psi \circ \beta$. Then we can write $L_{t-1} = L_A \vee L_B$ with $M(L_A) = A$ and $M(L_B) = B$, and $\beta = (\beta_A,\beta_B)$, where $\beta_B \simeq *\colon S^{2n_t} \to B$ and (A,β_A) is one of (S^{2n_t-2q+1},α_2), (S^{2n_t-q+1},α_1), $(S^{2n_t-2q+1} \times S^{2n_t-q+1},(\alpha_2,\alpha_1))$, $(B(n_t-2p+2),(\iota^B)_*(\alpha_2^B))$, $(B(n_t-2p+2),(\iota^B)_*(\tilde{\alpha}_1^B))$, $(E(n_t-2p+2),(\iota^E)_*(\tilde{\alpha}_1^E))$ or $(F(n_t-2p+2),(\iota^F)_*(\alpha_2^F))$. Then $\beta_A \simeq \iota^A \circ \gamma$, where (L_A,γ) is one of (S^{2n_t-2q+1},α_2), (S^{2n_t-q+1},α_1), $(S^{2n_t-2q+1} \vee S^{2n_t-q+1},(\alpha_2 \vee \alpha_1) \circ \Delta)$, $(L_B(n_t-2p+2),\alpha_2^B)$, $(L_B(n_t-2p+2),\tilde{\alpha}_1^B)$, $(L_E(n_t-2p+2),\tilde{\alpha}_1^E)$ or $(L_F(n_t-2p+2),\alpha_2^F)$.

Set $h_A = f_{t-1}|L_A$ and $h_B = f_{t-1}|L_B$, and consider the extension $\hat{h}_A\colon L_A \cup_\gamma e^{2n_t+1} \to X$. We write $L_t = (L_A \cup_\gamma e^{2n_t+1}) \vee L_B$ and define $f_t\colon L_t \to X$ by

$$f_t = \nabla \circ (\hat{h}_A \vee h_B)\colon L_t \to X \vee X \to X.$$

Since $L_A \cup_\gamma e^{2n_t+1}$ is a wedge of S^{2m+1}, $L_B(m)$, $L_C(m)$, $L_D(m)$, $L_E(m)$ and $L_F(m)$ for suitable m by Lemma 3.3, L_t and f_t satisfy the desired conditions (1) and (2). This completes the proof. □

References

[1] F. R. Cohen and J. A. Neisendorfer, A construction of p-local H-spaces, in *Algebraic topology* (Aarhus, 1982), Lecture Notes in Math. 1051, Springer, 1984, 351–359. Zbl 0582.55010 MR 0764588

[2] G. Cooke, J. Harper, and A. Zabrodsky, Torsion free mod p H-spaces of low rank, Topology **18** (1979), 349–359. Zbl 0426.55009 MR 0551016

[3] D. M. Davis, Homotopy type and v_1-periodic homotopy groups of p-compact groups, Topology Appl. **156** (2008), 300–321. Zbl 1160.55010 MR 2475117

[4] J. Grbić, J. Harper, M. Mimura, S. Theriault and J. Wu, Rank $p-1$ mod-p H-spaces, Israel J. Math. **194** (2013), 641–688. Zbl 1277.55003 MR 3047086

[5] J. Harper, Rank 2 mod 3 H-spaces, in *Current trends in algebraic topology*, Part 1 (London, Ont., 1981), Canad. Math. Soc. Conf. Proc. 2, Amer. Math. Soc., 1982, 375–388. Zbl 0559.55011 MR 0686126

[6] Y. Hemmi, Mod p decompositions of mod p finite H-spaces, Mem. Fac. Sci. Kochi Univ. Ser. A Math. **22** (2001), 59–65. Zbl 0976.55007 MR 1822064

[7] P. G. Kumpel, Jr., On p-equivalences of mod p H-spaces, Quart. J. Math. Oxford (2) **23** (1972), 173–178. Zbl 0235.57016 MR 0300275

[8] M. Mimura, G. Nishida and H. Toda, Mod p decomposition of compact Lie groups, Publ. RIMS Kyoto Univ. **13** (1977), 627–680. Zbl 0383.22007 MR 0478187

[9] M. Mimura and H. Toda, Cohomology operations and the homotopy of compact Lie groups —I, Topology **9** (1970), 317–336. Zbl 0204.23803 MR 0266237

[10] H. Toda, p-primary components of homotopy groups of spheres, IV, Mem. Coll. Sci. Univ. Kyoto Ser. A Math. **32** (1959), 297–332. Zbl 0095.16802 MR 0111041

[11] ———, *Composition methods in homotopy groups of spheres*, Princeton Univ. Press, Princeton, 1962. Zbl 0101.40703 MR 0143217

[12] J. Wu, The functor $A^{\min}$ for $(p-1)$-cell complexes and EHP sequences, Israel J. Math. **178** (2010), 349–391. Zbl 1231.55006 MR 2733074

[13] A. Zabrodsky, On rank 2 mod odd H-spaces, in: *New developments in topology*, G. Segal (ed.), London Math. Soc. Lecture Note Ser. 11, Cambridge Univ. Press, 1974, 119–128. Zbl 0275.55021 MR 0336742

[14] ———, Some relations in the mod 3 cohomology of H-spaces, Israel J. Math. **33** (1979), 59–72. Zbl 0417.55013 MR 0571584