Publ. RIMS Kyoto Univ. **52** (2016), 223–247
DOI 10.4171/PRIMS/179

Pfaffian of Appell's Hypergeometric System $\mathcal{F}_4$ in Terms of the Intersection Form of Twisted Cohomology Groups

by

Yoshiaki GOTO, Jyoichi KANEKO and Keiji MATSUMOTO

Abstract

We study a Pfaffian of Appell's hypergeometric system $\mathcal{F}_4(a,b,c)$ of differential equations by twisted cohomology groups associated with Euler type integrals representing solutions. We simplify its connection matrix by the pull-back under a double cover of the complement of the singular locus. We express the simplified connection matrix in terms of the intersection form between the twisted cohomology groups.

2010 Mathematics Subject Classification: Primary 33C65; Secondary 58A17.
Keywords: Appell's hypergeometric system of differential equations, Pfaffian system, twisted cohomology group.

§1. Introduction

There are several hypergeometric series in two variables $x = (x_1, x_2)$. One of them, Appell's hypergeometric series $F_4(a,b,c;x)$, is defined in (2.1) below, where a, b and $c = (c_1, c_2)$ are complex parameters. It satisfies Appell's hypergeometric system $\mathcal{F}_4(a,b,c)$ generated by the differential equations (2.2). This system is a holonomic system of rank 4 with singular locus

$$S = \{(x_1, x_2) \in \mathbb{C}^2 \mid x_1 x_2 R(x) = 0\} \cup L_\infty,$$

Communicated by T. Mochizuki. Received April 13, 2015. Revised December 22, 2015.

Y. Goto: Department of Mathematics, Graduate School of Science, Kobe University,
Kobe 657-8501, Japan;
e-mail: y-goto@math.kobe-u.ac.jp
J. Kaneko: Department of Mathematical Science, University of the Ryukyus,
Nishihara, Okinawa, 903-0213, Japan;
e-mail: kaneko@math.u-ryukyu.ac.jp
K. Matsumoto: Department of Mathematics, Hokkaido University,
Sapporo 060-0810, Japan;
e-mail: matsu@math.sci.hokudai.ac.jp

where $R(x) = x_1^2 + x_2^2 - 2x_1x_2 - 2x_1 - 2x_2 + 1$ and L_∞ is the line at infinity in the projective plane $\mathbb{P}^2$.

In this paper, we study a Pfaffian system of $\mathcal{F}_4(a,b,c)$ by a twisted cohomology group $H^2(\Omega^\bullet(\mathbb{C}_x^2), \nabla)$ associated with Euler type integrals (2.3) representing solutions to this system. By regarding this Pfaffian system as a connection ∇_X of a vector bundle

$$H^2(\Omega^{\bullet,0}(\mathfrak{X}), \nabla) = \bigcup_{x \in X} H^2(\Omega^\bullet(\mathbb{C}_x^2), \nabla)$$

over $X = \mathbb{P}^2 - S$ with fiber $H^2(\Omega^\bullet(\mathbb{C}_x^2), \nabla)$ of rank 4, we find a global frame of this vector bundle and represent the connection as $\Xi = \Xi^1 dx_1 + \Xi^2 dx_2$ in Theorem 4.1. The connection matrix satisfies the integrability condition $d\,\Xi = \Xi \wedge \Xi$ and has no apparent singularity, but Ξ_1 and Ξ_2 are complicated and $d\,\Xi \neq O$. As shown in Proposition 4.2, the system $\mathcal{F}_4(a,b,c)$ does not admit an expression of the connection matrix as a sum of constant matrices times logarithmic 1-forms associated with the singular locus S, like other Appell's hypergeometric systems. To make the connection matrix simple, we consider a double cover Y of X defined by the map

$$\mathrm{pr} : \mathbb{C}^2 \ni (y_1, y_2) \mapsto (x_1, x_2) = (y_1(1-y_2), y_2(1-y_1)) \in \mathbb{C}^2.$$

It induces the pull-back bundle and the pull-back connection ∇_Y. By changing a frame of the pull-back bundle, we express the pull-back connection ∇_Y as a sum of constant matrices times logarithmic 1-forms associated with $\widetilde{S}$ in Theorem 5.1, where $\widetilde{S}$ is the preimage of S under the double cover pr. In particular, the connection matrix $\widehat{\Xi}$ of ∇_Y satisfies $d\,\widehat{\Xi} = \widehat{\Xi} \wedge \widehat{\Xi} = O$. It is shown in [Kat] that the pull-back of $F_4(a,b,c;x)$ under a similar map satisfies a Pfaffian equation equivalent to ours.

Let $H^2(\Omega^\bullet(\mathbb{C}_x^2), \nabla^\vee)$ be the twisted cohomology group defined by the dual derivative $\nabla^\vee$ of ∇. This space is regarded as the dual of $H^2(\Omega^\bullet(\mathbb{C}_x^2), \nabla)$ via the intersection form $\mathcal{I}_c$. We have the dual vector bundle over X and the dual connection $\nabla_X^\vee$ satisfying

$$d\,\mathcal{I}_c(\varphi, \varphi') = \mathcal{I}_c(\nabla_X \varphi, \varphi') + \mathcal{I}_c(\varphi, \nabla_X^\vee \varphi'),$$

where φ and φ' are sections of the vector bundle and of its dual, respectively. Proposition 3.2 states that there is no global frame of $H^2(\Omega^{\bullet,0}(\mathfrak{X}), \nabla)$ satisfying

$$d\,\mathcal{I}_c(\varphi_i, \varphi_j^\vee) = 0, \qquad 1 \le i, j \le 4, \tag{1.1}$$

where $\varphi_i \in H^2(\Omega^\bullet(\mathbb{C}_x^2), \nabla)$ and $\varphi_j^\vee$ is the image of φ_j under the natural map from $H^2(\Omega^{\bullet,0}(\mathfrak{X}), \nabla)$ to $H^2(\Omega^{\bullet,0}(\mathfrak{X}), \nabla^\vee)$. The intersection form is also defined between the pull-back bundles. The frame of the pull-back bundle used in Theorem 5.1 satisfies (1.1). In fact, we find the double cover Y of X such that such a frame exists. The existence of the frame enables us to represent the pull-back connection ∇_Y by the intersection form $\mathcal{I}_c$ in our main result, Theorem 5.2. We remark that this expression is independent of the choice of frames with (1.1), not given in terms of matrices.

The monodromy representation of $\mathcal{F}_4(a, b, c)$ was initially given in [Kan] by a twisted homology group associated with the integrals (2.3), and reformulated in [GM1] in terms of the intersection form. By results [GM1, §4], we can express the circuit matrix along a loop turning around each component of S by the intersection form and a subspace of vanishing cycles as x approaches the component. On the other hand, we express in Proposition 5.1 the coefficient matrix of the logarithmic 1-form corresponding to a component of $\widetilde{S}$ in $\widehat{\Xi}$ by the intersection form and a subspace of vanishing forms as x approaches the component. Note that these matrices admit a common characterization by the intersection form and vanishing vectors. We also point out the similarity between the normalized intersection matrices of twisted homology and cohomology groups in Remark 5.1.

Appell's hypergeometric system $\mathcal{F}_4(a, b, c)$ generalizes to Lauricella's hypergeometric system $\mathcal{F}_C(a, b, c)$ of rank 2^m with m variables. For this system, we have twisted (co)homology groups associated with integrals representing solutions. The monodromy representation of $\mathcal{F}_C(a, b, c)$ is expressed in terms of the intersection form between twisted homology groups in [G, §5]. However, we have not been able to deduce that the system $\mathcal{F}_C(a, b, c)$ admits a Pfaffian system with a simple expression.

For Pfaffians of Lauricella's hypergeometric systems $\mathcal{F}_A$ and $\mathcal{F}_D$ in m variables, it is easy to find frames satisfying (1.1) without considering covering maps. Their Pfaffians are expressed in terms of intersection forms between twisted cohomology groups associated with integrals representing solutions [M1], [M2]. Recently, a similar result was obtained in [GM2] for the hypergeometric system $E(k+1, k+n+2; \alpha)$ (see [AoKi, Chapter 3] for its definition). Its Pfaffian system admits a frame satisfying (1.1), and its singular locus contains components given by non-linear equations.

§2. Appell's hypergeometric function F_4

In [ApKa], Appell's hypergeometric series $F_4(a, b, c; x)$ of variables x_1, x_2 with parameters a, b, $c = (c_1, c_2)$ is defined by

$$F_4(a, b, c_1, c_2; x_1, x_2) = \sum_{n_1, n_2=0}^{\infty} \frac{(a, n_1+n_2)(b, n_1+n_2)}{(c_1, n_1)(c_2, n_2) n_1! n_2!} x_1^{n_1} x_2^{n_2}, \tag{2.1}$$

where $c_1, c_2 \neq 0, -1, -2, \dots$ and $(a, k) = a(a+1)\cdots(a+k-1) = \Gamma(a+k)/\Gamma(a)$. This series converges in the domain

$$\mathbb{D} = \{x = (x_1, x_2) \in \mathbb{C} \mid \sqrt{|x_1|} + \sqrt{|x_2|} < 1\},$$

and satisfies the differential equations

$$\begin{aligned} &\big[x_1(1-x_1)\partial_1^2 - x_2^2\partial_2^2 - 2x_1x_2\partial_1\partial_2 \\ &\qquad + \{c_1 - (a+b+1)x_1\}\partial_1 - (a+b+1)x_2\partial_2 - ab\big] f(x) = 0, \\ &\big[x_2(1-x_2)\partial_2^2 - x_1^2\partial_1^2 - 2x_1x_2\partial_1\partial_2 \\ &\qquad + \{c_2 - (a+b+1)x_2\}\partial_2 - (a+b+1)x_1\partial_1 - ab\big] f(x) = 0. \end{aligned} \tag{2.2}$$

The system generated by them is called *Appell's hypergeometric system* $\mathcal{F}_4(a, b, c)$ of differential equations. This system is of rank 4 with singular locus

$$S = \{(x_1, x_2) \in \mathbb{C}^2 \mid x_1x_2R(x) = 0\} \cup L_\infty \subset \mathbb{P}^2,$$

where L_∞ is the line at infinity and

$$R(x) = x_1^2 + x_2^2 - 2x_1x_2 - 2x_1 - 2x_2 + 1.$$

We use the following integral representation to solutions of $\mathcal{F}_4(a, b, c)$:

$$\int_\Delta u(s, x)\frac{ds_1 \wedge ds_2}{s_1s_2}, \tag{2.3}$$

where

$$u(s, x) = s_1^{\lambda_1}s_2^{\lambda_2}Q(s)^{\lambda_3}L(s, x)^{\lambda_4},$$
$$Q = Q(s) = s_1s_2 - s_1 - s_2, \qquad L = L(s, x) = 1 - s_1x_1 - s_2x_2,$$
$$\lambda_1 = a - c_2 + 1, \qquad \lambda_2 = a - c_1 + 1, \qquad \lambda_3 = c_1 + c_2 - a - 2, \qquad \lambda_4 = -b,$$

and a 2-chain Δ loading a branch of $u(s, x)$ is a twisted 2-cycle. Refer to [AoKi] for twisted cycles and twisted homology groups.

Remark 2.1. It is convenient for the study of a Pfaffian system of $\mathcal{F}_4(a, b, c)$ to use the multi-valued function $u(s, x)$ instead of

$$u(t, x) = t_1^{1-c_1}t_2^{1-c_2}(1 - t_1 - t_2)^{c_1+c_2-a-1}\left(1 - \frac{x_1}{t_1} - \frac{x_2}{t_2}\right)^{-b}$$

applied in [GM1]. We get $u(s, x)$ by the change of variables $(t_1, t_2) = (1/s_1, 1/s_2)$ and the replacement $a \mapsto a + 1$ in $u(t, x)$.

§3. Twisted cohomology group

We regard the parameters a, b and $c = (c_1, c_2)$ as indeterminates and we set

$$a_{00} = a, \quad a_{10} = a - c_1 + 1, \quad a_{01} = a - c_2 + 1, \quad a_{11} = a - c_1 - c_2 + 2,$$
$$b_{00} = b, \quad b_{10} = b - c_1 + 1, \quad b_{01} = b - c_2 + 1, \quad b_{11} = b - c_1 - c_2 + 2.$$

We assume that

$$a_{ij}, b_{ij} \notin \mathbb{Z} \quad (i, j \in \mathbb{Z}_2 = \{0, 1\}) \tag{3.1}$$

when we assign complex values to the parameters. Recall that

$$\lambda_1 = a - c_2 + 1, \quad \lambda_2 = a - c_1 + 1, \quad \lambda_3 = c_1 + c_2 - a - 2, \quad \lambda_4 = -b.$$

In this section, we regard vector spaces as defined over the rational function field $\mathbb{C}(\lambda) = \mathbb{C}(\lambda_1, \ldots, \lambda_4) = \mathbb{C}(a, b, c_1, c_2)$. There is an involution on this field given by

$$\mathbb{C}(\lambda) \ni f(\lambda_1, \ldots, \lambda_4) \mapsto f^\vee(\lambda_1, \ldots, \lambda_4) = f(-\lambda_1, \ldots, -\lambda_4) \in \mathbb{C}(\lambda). \tag{3.2}$$

Note that

$$a^\vee = (\lambda_1 + \lambda_2 - \lambda_3)^\vee = -\lambda_1 - \lambda_2 + \lambda_3 = -a, \quad b^\vee = (-\lambda_4)^\vee = \lambda_4 = -b,$$
$$c_1^\vee = (1+\lambda_1+\lambda_3)^\vee = 1-\lambda_1-\lambda_3 = 2-c_1, \quad c_2^\vee = (1+\lambda_2+\lambda_3)^\vee = 1-\lambda_2-\lambda_3 = 2-c_2.$$

We set

$$\mathfrak{X} = \{(s, x) \in \mathbb{C}^2 \times X \mid s_1 s_2 Q(s) L(s, x) \neq 0\} \subset (\mathbb{P}^1 \times \mathbb{P}^1) \times \mathbb{P}^2,$$

where X is the complement of the singular locus S of $\mathcal{F}_4(a, b, c)$ in $\mathbb{P}^2$. There is a natural projection

$$\mathfrak{p} : \mathfrak{X} \ni (s, x) \mapsto x \in X.$$

For any fixed $x \in X$, we have

$$\mathbb{C}^2_x = \mathfrak{p}^{-1}(x) = \{s = (s_1, s_2) \in \mathbb{C}^2 \mid s_1 s_2 Q(s) L(s, x) \neq 0\}$$

and an inclusion map

$$\imath_x : \mathbb{C}^2_x \ni s \mapsto (s, x) \in \mathfrak{X}.$$

We denote the $\mathbb{C}(\lambda)$-algebra of rational functions on $\mathbb{P}^2$ with poles only along S by $\mathcal{O}(X)$. Note that x_1, x_2 and $R(x)$ are invertible in $\mathcal{O}(X)$. We denote the vector space of rational k-forms on $(\mathbb{P}^1 \times \mathbb{P}^1) \times \mathbb{P}^2$ with poles only along the complement of $\mathfrak{X}$ by $\Omega^k(\mathfrak{X})$, and the subspace of $\Omega^{i+j}(\mathfrak{X})$ consisting of those elements which are i-forms with respect to the variables s_1, s_2 by $\Omega^{i,j}(\mathfrak{X})$.

We set

$$\omega = d_s \log(u(s,x)) = \frac{\lambda_1 ds_1}{s_1} + \frac{\lambda_2 ds_2}{s_2} + \frac{\lambda_3 d_s Q(s)}{Q(s)} + \frac{\lambda_4 d_s L(s,x)}{L(s,x)},$$
$$\omega_X = d_x \log(u(s,x)) = -\frac{\lambda_4 s_1 dx_1}{L(s,x)} - \frac{\lambda_4 s_2 dx_2}{L(s,x)},$$

where d_s and d_x are the exterior derivatives with respect to s_1, s_2 and to x_1, x_2, respectively. Note that $\omega \in \Omega^{1,0}(\mathfrak{X})$ and $\omega_X \in \Omega^{0,1}(\mathfrak{X})$. By using the twisted exterior derivative $\nabla = d_s + \omega\wedge$ on $\mathfrak{X}$, we define the quotient spaces

$$H^k(\Omega^{\bullet,0}(\mathfrak{X}), \nabla) = \ker\bigl(\nabla : \Omega^{k,0}(\mathfrak{X}) \to \Omega^{k+1,0}(\mathfrak{X})\bigr)/\nabla(\Omega^{k-1,0}(\mathfrak{X}))$$

as $\mathcal{O}(X)$-modules, where $k = 0, 1, 2$ and we regard $\Omega^{-1,0}(\mathfrak{X})$ as the zero vector space.

For a fixed x, the inclusion map $\imath_x$ induces a natural map from $H^k(\Omega^{\bullet,0}(\mathfrak{X}), \nabla)$ to the rational twisted cohomology group

$$H^k(\Omega^{\bullet}(\mathbb{C}^2_x), \nabla) = \ker\bigl(\nabla : \Omega^{k}(\mathbb{C}^2_x) \to \Omega^{k+1}(\mathbb{C}^2_x)\bigr)/\nabla(\Omega^{k-1}(\mathbb{C}^2_x))$$

on $\mathbb{C}^2_x$ with respect to the twisted exterior derivative induced from ∇. Here $\Omega^k(\mathbb{C}^2_x)$ is the vector space of rational k-forms with poles only along the complement of $\mathbb{C}^2_x$ in $\mathbb{P}^1 \times \mathbb{P}^1$. The structure of $H^k(\Omega^{\bullet}(\mathbb{C}^2_x), \nabla)$ is as follows.

Fact 3.1 ([AoKi], [C]). (i) *We have*

$$\dim H^k(\Omega^{\bullet}(\mathbb{C}^2_x), \nabla) = \begin{cases} 4 & \text{if } k = 2, \\ 0 & \text{if } k = 0, 1. \end{cases}$$

(ii) *There is a canonical isomorphism*

$$\begin{aligned} \jmath_x : H^2(\Omega^{\bullet}(\mathbb{C}^2_x), \nabla) &\to H^2(\mathcal{E}^{\bullet}_c(\mathbb{C}^2_x), \nabla) \\ &= \ker\bigl(\nabla : \mathcal{E}^2_c(\mathbb{C}^2_x) \to \mathcal{E}^3_c(\mathbb{C}^2_x)\bigr)/\nabla(\mathcal{E}^1_c(\mathbb{C}^2_x)), \end{aligned}$$

where $\mathcal{E}^k_c(\mathbb{C}^2_x)$ is the vector space of smooth k-forms with compact support in $\mathbb{C}^2_x$.

We have a twisted exterior derivation $\nabla^\vee = d_s - \omega\wedge$ for $-\omega$ and

$$H^2(\Omega^{\bullet,0}(\mathfrak{X}), \nabla^\vee) = \Omega^{2,0}(\mathfrak{X})/\nabla^\vee(\Omega^{1,0}(\mathfrak{X})),$$
$$H^2(\Omega^{\bullet}(\mathbb{C}^2_x), \nabla^\vee) = \Omega^{2}(\mathbb{C}^2_x)/\nabla^\vee(\Omega^{1}(\mathbb{C}^2_x)).$$

For any fixed $x \in X$, we define the intersection form between $H^2(\Omega^{\bullet}(\mathbb{C}^2_x), \nabla)$ and $H^2(\Omega^{\bullet}(\mathbb{C}^2_x), \nabla^\vee)$ by

$$\mathcal{I}_c(\varphi_x, \varphi'_x) = \int_{\mathbb{C}^2_x} \jmath_x(\varphi_x) \wedge \varphi'_x \in \mathbb{C}(\lambda),$$

where $\varphi_x, \varphi'_x \in \Omega^2(\mathbb{C}^2_x)$, and $\jmath_x$ is given in Fact 3.1. This integral converges since $\jmath_x(\varphi_x)$ is a smooth 2-form on $\mathbb{C}^2_x$ with compact support. It is bilinear over $\mathbb{C}(\lambda)$.

We take four elements $\varphi_1, \dots, \varphi_4$ of $H^2(\Omega^{\bullet,0}(\mathfrak{X}), \nabla)$:

$$\varphi_1 = \frac{ds_{12}}{s_1 s_2}, \qquad \varphi_2 = \frac{x_1 ds_{12}}{s_2 L(s,x)},$$
$$\varphi_3 = \frac{x_2 ds_{12}}{s_1 L(s,x)}, \qquad \varphi_4 = \frac{ds_{12}}{Q(s)L(s,x)},$$

where $ds_{12} = ds_1 \wedge ds_2$.

Proposition 3.1. *For a fixed $x \in X$, the numbers $\mathcal{I}_c(\imath_x^*(\varphi_i), \imath_x^*(\varphi_j))$ $(1 \le i,j \le 4)$ are $(2\pi\sqrt{-1})^2 C_{ij}$, where*

$$\begin{aligned}
C_{11} &= \frac{1}{\lambda_{123}}\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right) + \frac{1}{\lambda_{\overline{134}}}\left(\frac{1}{\lambda_0}+\frac{1}{\lambda_2}\right) + \frac{1}{\lambda_{\overline{234}}}\left(\frac{1}{\lambda_0}+\frac{1}{\lambda_1}\right) \\
&= \frac{1}{a_{00}}\left(\frac{1}{a_{01}}+\frac{1}{a_{10}}\right) + \frac{1}{b_{10}}\left(\frac{1}{b_{11}}+\frac{1}{a_{10}}\right) + \frac{1}{b_{01}}\left(\frac{1}{b_{11}}+\frac{1}{a_{01}}\right), \\
C_{12} &= \frac{-1}{\lambda_{\overline{134}}}\left(\frac{1}{\lambda_0}+\frac{1}{\lambda_2}\right) = \frac{-1}{b_{10}}\left(\frac{1}{b_{11}}+\frac{1}{a_{10}}\right), \\
C_{13} &= \frac{-1}{\lambda_{\overline{234}}}\left(\frac{1}{\lambda_0}+\frac{1}{\lambda_1}\right) = \frac{-1}{b_{01}}\left(\frac{1}{b_{11}}+\frac{1}{a_{01}}\right), \\
C_{14} &= 0, \\
C_{22} &= \left(\frac{1}{\lambda_0}+\frac{1}{\lambda_2}\right)\left(\frac{1}{\lambda_4}+\frac{1}{\lambda_{\overline{134}}}\right) = \left(\frac{1}{b_{11}}+\frac{1}{a_{10}}\right)\left(\frac{-1}{b_{00}}+\frac{1}{b_{10}}\right), \\
C_{23} &= \frac{-1}{\lambda_0\lambda_4} = \frac{1}{b_{11}b_{00}}, \\
C_{24} &= 0, \\
C_{33} &= \left(\frac{1}{\lambda_0}+\frac{1}{\lambda_1}\right)\left(\frac{1}{\lambda_4}+\frac{1}{\lambda_{\overline{234}}}\right) = \left(\frac{1}{b_{11}}+\frac{1}{a_{01}}\right)\left(\frac{-1}{b_{00}}+\frac{1}{b_{01}}\right), \\
C_{34} &= 0, \\
C_{44} &= \frac{2}{\lambda_3\lambda_4 R(x)} = \frac{2}{a_{11}b_{00}R(x)}, \\
C_{ji} &= C_{ij} \quad \text{for} \quad i<j, \\
\lambda_0 &= -(\lambda_1+\lambda_2+2\lambda_3+\lambda_4) = b_{11}, \qquad \lambda_{123} = \lambda_1+\lambda_2+\lambda_3 = a_{00}, \\
\lambda_{\overline{134}} &= -(\lambda_1+\lambda_3+\lambda_4) = b_{10}, \qquad \lambda_{\overline{234}} = -(\lambda_2+\lambda_3+\lambda_4) = b_{01}.
\end{aligned}$$

The matrix $C = (C_{ij})_{i,j}$ is symmetric and its determinant is

$$\det(C) = \frac{4\lambda_3}{\lambda_0\lambda_1\lambda_2\lambda_4^3\lambda_{123}\lambda_{\overline{134}}\lambda_{\overline{234}}R(x)} = \left(\prod_{i,j=0,1}\frac{1}{a_{ij}b_{ij}}\right)\frac{4a_{11}^2}{b_{00}^2 R(x)}.$$

Divisor	λ's	a_{ij}, b_{ij}	a, b, c_1, c_2
E_∞	λ_0	b_{11}	$b - c_1 - c_2 + 2$
$s_1 = 0$	λ_1	a_{01}	$a - c_2 + 1$
$s_2 = 0$	λ_2	a_{10}	$a - c_1 + 1$
$Q(s) = 0$	λ_3	$-a_{11}$	$-a + c_1 + c_2 - 2$
$L(s, x) = 0$	λ_4	$-b_{00}$	$-b$
E_0	λ_{123}	a_{00}	a
$s_1 = \infty$	λ_{134}^-	b_{10}	$b - c_1 + 1$
$s_2 = \infty$	λ_{234}^-	b_{01}	$b - c_2 + 1$

Table 1. Residues for components of the pole divisor of ω.

Proof. For a fixed x, we blow up $\mathbb{P}^1 \times \mathbb{P}^1$ ($\supset \mathbb{C}_x^2$) at two points $(0, 0)$ and (∞, ∞). We tabulate the residue of the pull-back of ω to this space in Table 1, where E_0 and E_∞ are exceptional divisors coming from the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the points $(0, 0)$ and (∞, ∞), respectively. Now using these data, follow the proof of [GM1, Theorem 5.1]. □

Note that the matrix C is well-defined and $\det(C) \neq 0$ for any $x \in X$ under our assumption. The natural map $\imath_x^* : H^2(\Omega^{\bullet,0}(\mathfrak{X}), \nabla) \to H^2(\Omega^\bullet(\mathbb{C}_x^2), \nabla)$ is surjective by Fact 3.1.

Corollary 3.1. *The $\mathcal{O}(X)$-modules $H^2(\Omega^{\bullet,0}(\mathfrak{X}), \nabla)$ and $H^2(\Omega^{\bullet,0}(\mathfrak{X}), \nabla^\vee)$ can be regarded as vector bundles*

$$\bigcup_{x \in X} H^2(\Omega^\bullet(\mathbb{C}_x^2), \nabla), \qquad \bigcup_{x \in X} H^2(\Omega^\bullet(\mathbb{C}_x^2), \nabla^\vee),$$

over X with the natural projection p. The classes of $\varphi_1, \ldots, \varphi_4$ form a frame of $H^2(\Omega^{\bullet,0}(\mathfrak{X}), \nabla)$ and that of $H^2(\Omega^{\bullet,0}(\mathfrak{X}), \nabla^\vee)$.

Proof. We have only to prove that the natural map from $H^2(\Omega^{\bullet,0}(\mathfrak{X}), \nabla)$ to $H^2(\Omega^\bullet(\mathbb{C}_x^2), \nabla)$ is injective. We show that if $\varphi \in \Omega^{2,0}(\mathfrak{X})$ satisfies $\imath_x^*(\varphi) = 0$ as an element of $H^2(\Omega^\bullet(\mathbb{C}_x^2), \nabla)$ for any fixed $x \in X$ then $\varphi \in \nabla(\Omega^{1,0}(\mathfrak{X}))$. There exists $\psi_x \in \Omega^1(\mathbb{C}_x^2)$ such that $\nabla\psi_x = \imath_x^*(\varphi)$ for any x. Since this is a differential equation in variables s_1, s_2 with parameters x_1, x_2, ψ_x can be globally extended to ψ. Hence we obtain $\psi \in \Omega^{1,0}(\mathfrak{X})$ such that $\nabla(\psi) = \varphi$. □

By Proposition 3.1 and Corollary 3.1, the intersection form $\mathcal{I}_c$ can be regarded as a map from $H^2(\Omega^{\bullet,0}(\mathfrak{X}), \nabla) \times H^2(\Omega^{\bullet,0}(\mathfrak{X}), \nabla^\vee)$ to $\mathcal{O}(X)$. We show that it is impossible to eliminate $R(x)$ in the matrix C by any frame change of ${}^t(\varphi_1, \ldots, \varphi_4)$.

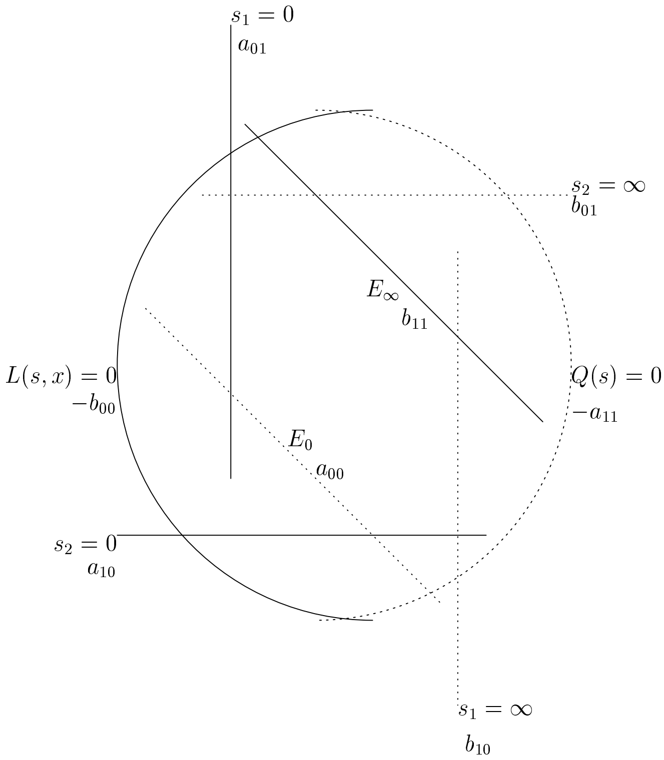

Figure 1. Pole divisor of ω.

Proposition 3.2. *There is no frame change G in*

$$\{G \in M(4,4;\mathcal{O}(X)) \mid \det(G) \neq 0\}$$

of ${}^t(\varphi_1,\dots,\varphi_4)$ such that all entries of the intersection matrix of $G^t(\varphi_1,\dots,\varphi_4)$ are independent of $x \in X$.

Proof. Suppose that such a frame change G exists. Then the intersection matrix of the frame $G\,{}^t(\varphi_1,\dots,\varphi_4)$ is given by $GC\,{}^tG^\vee$, whose entries are independent of $x \in X$. Here the matrix $G^\vee$ is obtained by applying the involution (3.2) to each entry of G. By considering its determinant, we have

$$\det(G)\det(G)^\vee = \frac{r(\lambda)}{q(\lambda)}R(x), \tag{3.3}$$

where $q(\lambda)$ and $r(\lambda)$ are in $\mathbb{C}[\lambda_1,\dots,\lambda_4]$. We set

$$\det(G) = \frac{g(\lambda,x)}{h(\lambda,x)}, \qquad g(\lambda,x), h(\lambda,x) \in \mathbb{C}[\lambda_1,\dots,\lambda_4][x_1,x_2].$$

By clearing the denominators in (3.3), we have

$$q(\lambda)g(\lambda,x)g(\lambda,x)^\vee = r(\lambda)h(\lambda,x)h(\lambda,x)^\vee R(x). \tag{3.4}$$

We decompose $g(\lambda,x)$, $h(\lambda,x)$, $q(\lambda)$ and $r(\lambda)$ into homogeneous parts of $\lambda_1,\dots,\lambda_4$:

$$g(\lambda,x)=\sum_{i=0}^{m_g} g_i(\lambda,x),\quad h(\lambda,x)=\sum_{i=0}^{m_h} h_i(\lambda,x),\quad q(\lambda)=\sum_{i=0}^{m_q} q_i(\lambda),\quad r(\lambda)=\sum_{i=0}^{m_r} r_i(\lambda),$$

where each component satisfies

$$g_i(t\lambda,x)=t^i g_i(\lambda,x),\quad h_i(t\lambda,x)=t^i h_i(\lambda,x),\quad q_i(t\lambda)=t^i q_i(\lambda),\quad r_i(t\lambda)=t^i r_i(\lambda).$$

Since

$$g(\lambda,x)^\vee=\sum_{i=0}^{m_g}(-1)^i g_i(\lambda,x),$$

we have

$$g(\lambda,x)g(\lambda,x)^\vee=\Big(\sum_{\substack{0\le i\le m_g\\ i\in 2\mathbb{N}}} g_i(\lambda,x)\Big)^2-\Big(\sum_{\substack{0\le i\le m_g\\ i\notin 2\mathbb{N}}} g_i(\lambda,x)\Big)^2.$$

The equality (3.4) is equivalent to

$$\begin{aligned}\Big(\sum_{i=0}^{m_q} q_i(\lambda)\Big)&\Big[\Big(\sum_{\substack{0\le i\le m_g\\ i\in 2\mathbb{N}}} g_i(\lambda,x)\Big)^2-\Big(\sum_{\substack{0\le i\le m_g\\ i\notin 2\mathbb{N}}} g_i(\lambda,x)\Big)^2\Big]\\ &=\Big(\sum_{i=0}^{m_r} r_i(\lambda)\Big)\Big[\Big(\sum_{\substack{0\le i\le m_h\\ i\in 2\mathbb{N}}} h_i(\lambda,x)\Big)^2-\Big(\sum_{\substack{0\le i\le m_h\\ i\notin 2\mathbb{N}}} h_i(\lambda,x)\Big)^2\Big]R(x).\end{aligned}$$

Comparing the terms with the highest degree of λ in this equality, we have

$$\pm q_{m_q}(\lambda)g_{m_g}(\lambda,x)^2=r_{m_r}(\lambda)h_{m_h}(\lambda,x)^2R(x).$$

Since $R(x)$ is irreducible in $\mathbb{C}[\lambda_1,\dots,\lambda_4][x_1,x_2]$, we have a contradiction. □

§4. Pfaffian system of $\mathcal{F}_4(a,b,c)$

For any $\varphi\in H^2(\Omega^{\bullet,0}(\mathfrak{X}),\nabla)$, we have

$$d_x\int_\Delta u(s,x)\varphi=\int_\Delta (d_x u(s,x)\wedge\varphi+u(s,x)d_x\varphi)=\int_\Delta u(s,x)(\nabla_X\varphi), \tag{4.1}$$

where $\nabla_X=d_x+\omega_X\wedge$. Thus the exterior derivative d_x on X induces the connection $\nabla_X=d_x+\omega_X\wedge$,

$$\nabla_X: H^2(\Omega^{\bullet,0}(\mathfrak{X}),\nabla)\to H^2(\Omega^{\bullet,1}(\mathfrak{X}),\nabla)=\Omega^{2,1}(\mathfrak{X})/\nabla(\Omega^{1,1}(\mathfrak{X})).$$

By considering $1/u(s,x)$ instead of $u(s,x)$, we also have the connection $\nabla_X^\vee = d_x - \omega_X \wedge$,

$$\nabla_X^\vee : H^2(\Omega^{\bullet,0}(\mathfrak{X}), \nabla^\vee) \to H^2(\Omega^{\bullet,1}(\mathfrak{X}), \nabla^\vee) = \Omega^{2,1}(\mathfrak{X})/\nabla^\vee(\Omega^{1,1}(\mathfrak{X})).$$

Proposition 4.1. *The connection ∇_X is compatible with the intersection form $\mathcal{I}_c$, i.e., they satisfy*

$$d_x\mathcal{I}_c(\varphi, \varphi') = \mathcal{I}_c(\nabla_X\varphi, \varphi') + \mathcal{I}_c(\varphi, \nabla_X^\vee\varphi').$$

Proof. It is enough to show that this equality holds in any small simply connected domain U in X. We have

$$\begin{aligned} d_x\mathcal{I}_c(\varphi, \varphi') &= d_x \int_{\mathbb{C}_x^2} u(s,x)\jmath_x(\varphi) \wedge \frac{\varphi'}{u(s,x)} \\ &= \int_{\mathbb{C}_x^2} d_x[u(s,x)\jmath_x(\varphi)] \wedge \frac{\varphi'}{u(s,x)} + \int_{\mathbb{C}_x^2} u(s,x)\jmath_x(\varphi) \wedge d_x\frac{\varphi'}{u(s,x)} \\ &= \int_{\mathbb{C}_x^2} \nabla_X(\jmath_x(\varphi)) \wedge \varphi' + \int_{\mathbb{C}_x^2} \jmath_x(\varphi) \wedge \nabla_X^\vee\varphi', \end{aligned}$$

since $\jmath_x(\varphi)$ is with compact support for any point $x \in U$. By following the proof of [M1, Lemma 7.2], we can show that $\nabla_X(\jmath_x(\varphi))$ is ∇-cohomologous to $\jmath_x(\nabla_X(\varphi))$. Hence

$$\int_{\mathbb{C}_x^2} \nabla_X(\jmath_x(\varphi)) \wedge \varphi' = \int_{\mathbb{C}_x^2} \jmath_x(\nabla_X(\varphi)) \wedge \varphi' = \mathcal{I}_c(\nabla_X\varphi, \varphi'),$$

which completes the proof. □

Since $\nabla_X\varphi \in \Omega^{2,1}(\mathfrak{X})$, there exist Ξ_i^1 and Ξ_i^2 in $\mathcal{O}(X)$ such that

$$\nabla_X\varphi = dx_1 \wedge \sum_{i=1}^4 \Xi_i^1\varphi_i + dx_2 \wedge \sum_{i=1}^4 \Xi_i^2\varphi_i.$$

By calculating Ξ_i^1 and Ξ_i^2 for $\varphi = \varphi_1, \dots, \varphi_4$, we represent the connection ∇_X as

$$\nabla_X {}^t(\varphi_1, \dots, \varphi_4) = \Xi \wedge {}^t(\varphi_1, \dots, \varphi_4), \qquad \Xi = dx_1\Xi^1 + dx_2\Xi^2,$$

where Ξ^1 and Ξ^2 are 4×4-matrices over the $\mathbb{C}(\lambda)$-algebra $\mathcal{O}(X)$. By (4.1), the vector-valued function

$$F(x) = {}^t\!\left(\int_\Delta u(s,x)\varphi_1, \dots, \int_\Delta u(s,x)\varphi_4\right)$$

satisfies a system of differential equations

$$d_xF(x) = \Xi\, F(x).$$

Let us determine Ξ^1 and Ξ^2. A straightforward calculation implies the following.

Lemma 4.1. *We have*

$$\begin{aligned}
\nabla_X(\varphi_1) &= dx_1 \wedge \frac{-\lambda_4 ds_{12}}{s_2 L} + dx_2 \wedge \frac{-\lambda_4 ds_{12}}{s_1 L} \\
&= \frac{dx_1}{x_1} \wedge (-\lambda_4)\varphi_2 + \frac{dx_2}{x_2} \wedge (-\lambda_4)\varphi_3, \\
\nabla_X(\varphi_2) &= dx_1 \wedge \frac{(1-\lambda_4 s_1 x_1 - s_2 x_2) ds_{12}}{s_2 L^2} + dx_2 \wedge \frac{(1-\lambda_4) x_1 ds_{12}}{L^2}, \\
\nabla_X(\varphi_3) &= dx_1 \wedge \frac{(1-\lambda_4) x_2 ds_{12}}{L^2} + dx_2 \wedge \frac{(1-s_1 x_1 - \lambda_4 s_2 x_2) ds_{12}}{s_1 L^2}, \\
\nabla_X(\varphi_4) &= dx_1 \wedge \frac{(1-\lambda_4) s_1 ds_{12}}{Q L^2} + dx_2 \wedge \frac{(1-\lambda_4) s_2 ds_{12}}{Q L^2}.
\end{aligned}$$

To obtain Ξ^1 and Ξ^2, we express

$$\frac{ds_{12}}{L^2}, \quad \frac{s_1 ds_{12}}{s_2 L^2}, \quad \frac{s_2 ds_{12}}{s_1 L^2}, \quad \frac{ds_{12}}{s_2 L^2}, \quad \frac{ds_{12}}{s_1 L^2}, \quad \frac{s_1 ds_{12}}{Q L^2}, \quad \frac{s_2 ds_{12}}{Q L^2},$$

in terms of $\varphi_1, \ldots, \varphi_4$.

Lemma 4.2. *As elements of $H^2(\Omega^{\bullet,0}(\mathfrak{X}), \nabla)$, we have*

$$\begin{aligned}
\frac{2(\lambda_4 - 1) x_1 x_2}{L^2} ds_{12} &= (\lambda_1 + \lambda_2 + \lambda_3)\varphi_1 + (\lambda_2 - \lambda_4)\varphi_2 \\
&\quad + (\lambda_1 - \lambda_4)\varphi_3 + \lambda_3(1 - x_1 - x_2)\varphi_4, \\
\frac{(\lambda_4 - 1) x_1 s_1}{s_2 L^2} ds_{12} &= \frac{\lambda_1 + \lambda_3 + 1}{x_1}\varphi_2 + \lambda_3 \varphi_4, \\
\frac{(\lambda_4 - 1) x_2 s_2}{s_1 L^2} ds_{12} &= \frac{\lambda_2 + \lambda_3 + 1}{x_2}\varphi_3 + \lambda_3 \varphi_4.
\end{aligned}$$

Proof. Straightforward calculations imply

$$\begin{aligned}
&\nabla\left(\frac{x_1 ds_1 - x_2 ds_2}{L} + \frac{ds_1}{s_1} - \frac{ds_2}{s_2}\right) \\
&= \frac{2(\lambda_4 - 1) x_1 x_2 ds_{12}}{L^2} \\
&\quad - [(\lambda_1 + \lambda_2 + \lambda_3)\varphi_1 + (\lambda_2 - \lambda_4)\varphi_2 + (\lambda_1 - \lambda_4)\varphi_3 + \lambda_3(1 - x_1 - x_2)\varphi_4], \\
&\nabla\left(-\frac{s_1 ds_2}{s_2 L}\right) = \frac{(\lambda_4 - 1) x_1 s_1}{s_2 L^2} ds_{12} - \left[\frac{\lambda_1 + \lambda_3 + 1}{x_1}\varphi_2 + \lambda_3 \varphi_4\right], \\
&\nabla\left(\frac{s_2 ds_1}{s_1 L}\right) = \frac{(\lambda_4 - 1) x_2 s_2}{s_1 L^2} ds_{12} - \left[\frac{\lambda_2 + \lambda_3 + 1}{x_2}\varphi_3 + \lambda_3 \varphi_4\right],
\end{aligned}$$

which proves the lemma. □

Lemma 4.3. *As elements of $H^2(\Omega^{\bullet,0}(\mathfrak{X}),\nabla)$, we have*

$$\begin{aligned}\frac{2(\lambda_4-1)x_2}{s_1L^2}ds_{12} &= (\lambda_1+\lambda_2+\lambda_3)\varphi_1+(\lambda_2-\lambda_4)\varphi_2\\ &\quad+(\lambda_1+2\lambda_2+2\lambda_3+\lambda_4)\varphi_3+\lambda_3(1-x_1+x_2)\varphi_4,\\ \frac{2(\lambda_4-1)x_1}{s_2L^2}ds_{12} &= (\lambda_1+\lambda_2+\lambda_3)\varphi_1+(2\lambda_1+\lambda_2+2\lambda_3+\lambda_4)\varphi_2\\ &\quad+(\lambda_1-\lambda_4)\varphi_3+\lambda_3(1+x_1-x_2)\varphi_4.\end{aligned}$$

Proof. Note that

$$\begin{aligned}\left(\frac{1}{s_1L^2}-\frac{x_1}{L^2}-\frac{s_2x_2}{s_1L^2}\right)ds_{12}&=\frac{ds_{12}}{s_1L}=\frac{\varphi_3}{x_2},\\ \left(\frac{1}{s_2L^2}-\frac{x_2}{L^2}-\frac{s_1x_1}{s_2L^2}\right)ds_{12}&=\frac{ds_{12}}{s_2L}=\frac{\varphi_2}{x_1}.\end{aligned}$$

Use Lemma 4.2. □

Lemma 4.4. *As elements of $H^2(\Omega^{\bullet,0}(\mathfrak{X}),\nabla)$, we have*

$$\begin{aligned}\frac{s_1ds_{12}}{QL^2} &= \frac{1-x_1+x_2}{R(x)}\varphi_4+\frac{(\lambda_1+\lambda_2+2\lambda_3+\lambda_4)(1+x_1-x_2)x_2}{\lambda_3R(x)}\frac{ds_{12}}{L^2}\\ &\quad-\frac{2\lambda_1x_2}{\lambda_3R(x)}\frac{ds_{12}}{s_1L^2}-\frac{\lambda_2(1-x_1-x_2)}{\lambda_3R(x)}\frac{ds_{12}}{s_2L^2},\\ \frac{s_1ds_{12}}{QL^2} &= \frac{1+x_1-x_2}{R(x)}\varphi_4+\frac{(\lambda_1+\lambda_2+2\lambda_3+\lambda_4)(1-x_1+x_2)x_1}{\lambda_3R(x)}\frac{ds_{12}}{L^2}\\ &\quad-\frac{\lambda_1(1-x_1-x_2)}{\lambda_3R(x)}\frac{ds_{12}}{s_1L^2}-\frac{2\lambda_2x_1}{\lambda_3R(x)}\frac{ds_{12}}{s_2L^2}.\end{aligned}$$

Proof. Set

$$\eta_0=\frac{ds_{12}}{QL^2},\qquad \eta_1=\frac{s_1ds_{12}}{QL^2},\qquad \eta_2=\frac{s_2ds_{12}}{QL^2}.$$

There is a relation

$$\eta_0-x_1\eta_1-x_2\eta_2=\frac{(1-s_1x_1-s_2x_2)ds_{12}}{QL^2}=\varphi_4$$

among them. We have

$$\begin{aligned}&\nabla\left(\frac{x_1ds_1+x_2ds_2}{L^2}\right)\\ &\quad=\lambda_3(x_1-x_2)\eta_0-\lambda_3x_1\eta_1+\lambda_3x_2\eta_2+\left(\frac{\lambda_1x_2}{s_1L^2}-\frac{\lambda_2x_1}{s_2L^2}\right)ds_{12},\end{aligned}$$

$$\nabla\left(\frac{ds_1 - ds_2}{L^2} + \frac{(x_1+x_2)(-s_2 ds_1 + s_1 ds_2)}{L^2}\right)$$
$$= 2\lambda_3\eta_0 + \lambda_3(x_1+x_2-1)\eta_1 + \lambda_3(x_1+x_2-1)\eta_2$$
$$- \left(\frac{\lambda_1}{s_1 L^2} + \frac{\lambda_2}{s_2 L^2} - \frac{(\lambda_1+\lambda_2+2\lambda_3+\lambda_4)(x_1+x_2)}{L^2}\right) ds_{12},$$

which are zero as elements of $H^2(\Omega^{\bullet,0}(\mathfrak{X}),\nabla)$. By regarding these relations as linear equations in variables η_0, η_1, η_2, we can solve them. The solutions yield the lemma. □

Remark 4.1. The form $\eta_0 = ds_{12}/(QL^2)$ in the proof of Lemma 4.4 is expressed as

$$\eta_0 = \frac{1-x_1-x_2}{R(x)}\varphi_4 + \frac{2(\lambda_1+\lambda_2+2\lambda_3+\lambda_4)x_1x_2}{\lambda_3 R(x)}\,\frac{ds_{12}}{L^2}$$
$$- \frac{\lambda_1 x_2(1+x_1-x_2)}{\lambda_3 R(x)}\,\frac{ds_{12}}{s_1 L^2} - \frac{\lambda_2 x_1(1-x_1+x_2)}{\lambda_3 R(x)}\,\frac{ds_{12}}{s_2 L^2}.$$

Now we are ready to find an expresssion of the connection matrix Ξ.

Theorem 4.1. *With respect to the frame ${}^t(\varphi_1,\dots,\varphi_4)$, the connection ∇_X is represented as*

$$\nabla_X {}^t(\varphi_1,\dots,\varphi_4) = \Xi \wedge {}^t(\varphi_1,\dots,\varphi_4),$$

where $\Xi = \Xi^1 dx_1 + \Xi^2 dx_2$ *and*

$$\Xi^1 = \begin{pmatrix} 0 & -\dfrac{\lambda_4}{x_1} & 0 & 0 \\ 0 & -\dfrac{\lambda_1+\lambda_3}{x_1} & 0 & -\lambda_3 \\ -\dfrac{\lambda_1+\lambda_2+\lambda_3}{2x_1} & -\dfrac{\lambda_2-\lambda_4}{2x_1} & -\dfrac{\lambda_1-\lambda_4}{2x_1} & -\dfrac{\lambda_3(1-x_1-x_2)}{2x_1} \\ \Xi^1_{4,1} & \Xi^1_{4,2} & \Xi^1_{4,3} & \Xi^1_{4,4} \end{pmatrix},$$

$$\Xi^2 = \begin{pmatrix} 0 & 0 & -\dfrac{\lambda_4}{x_2} & 0 \\ -\dfrac{\lambda_1+\lambda_2+\lambda_3}{2x_2} & -\dfrac{\lambda_2-\lambda_4}{2x_2} & -\dfrac{\lambda_1-\lambda_4}{2x_2} & -\dfrac{\lambda_3(1-x_1-x_2)}{2x_2} \\ 0 & 0 & -\dfrac{\lambda_2+\lambda_3}{x_2} & -\lambda_3 \\ \Xi^2_{4,1} & \Xi^2_{4,2} & \Xi^2_{4,3} & \Xi^2_{4,4} \end{pmatrix},$$

$$\Xi^1_{4,1} = \frac{\lambda_1+\lambda_2+\lambda_3}{\lambda_3}\left(\frac{\lambda_1-\lambda_2}{R(x)} - \frac{(\lambda_1+2\lambda_3+\lambda_4)(1+x_1-x_2)}{2x_1 R(x)}\right),$$

$$\Xi^1_{4,2} = -\frac{(\lambda_2-\lambda_4)(\lambda_2+2\lambda_3+\lambda_4)}{\lambda_3 R(x)} + \frac{\lambda_1\lambda_2+\lambda_1\lambda_4+\lambda_2\lambda_4+2\lambda_3\lambda_4+\lambda_4^2}{2\lambda_3}\,\frac{1-x_1-x_2}{x_1 R(x)},$$

$$\Xi^1_{4,3} = \frac{\lambda_1\lambda_2+\lambda_1\lambda_4+\lambda_2\lambda_4+2\lambda_3\lambda_4+\lambda_4^2}{\lambda_3 R(x)} - \frac{(\lambda_1-\lambda_4)(\lambda_1+\lambda_4+2\lambda_3)}{2\lambda_3}\,\frac{1-x_1-x_2}{x_1 R(x)},$$

$$\Xi^1_{4,4} = -(\lambda_1-2\lambda_3-3\lambda_4+2)\frac{x_1}{2R(x)} + (\lambda_1-\lambda_4+1)\frac{1+x_2}{R(x)} - (\lambda_1+2\lambda_3+\lambda_4)\frac{(x_2-1)^2}{2x_1R(x)},$$

$$\Xi^2_{4,1} = \frac{\lambda_1+\lambda_2+\lambda_3}{\lambda_3}\left(\frac{\lambda_2-\lambda_1}{R(x)} - \frac{(\lambda_2+2\lambda_3+\lambda_4)(1-x_1+x_2)}{2x_2R(x)}\right),$$

$$\Xi^2_{4,2} = \frac{\lambda_1\lambda_2+\lambda_1\lambda_4+\lambda_2\lambda_4+2\lambda_3\lambda_4+\lambda_4^2}{\lambda_3 R(x)} - \frac{(\lambda_2-\lambda_4)(\lambda_2+2\lambda_3+\lambda_4)}{2\lambda_3}\,\frac{1-x_1-x_2}{x_2 R(x)},$$

$$\Xi^2_{4,3} = -\frac{(\lambda_1-\lambda_4)(\lambda_1+\lambda_4+2\lambda_3)}{\lambda_3 R(x)} + \frac{\lambda_1\lambda_2+\lambda_1\lambda_4+\lambda_2\lambda_4+2\lambda_3\lambda_4+\lambda_4^2}{2\lambda_3}\,\frac{1-x_1-x_2}{x_2 R(x)},$$

$$\Xi^2_{4,4} = -(\lambda_2-2\lambda_3-3\lambda_4+2)\frac{x_2}{2R(x)} + (\lambda_2-\lambda_4+1)\frac{x_1+1}{R(x)} - (\lambda_2+2\lambda_3+\lambda_4)\frac{(x_1-1)^2}{2x_2R(x)}.$$

This connection matrix and the intersection matrix C in Proposition 3.1 satisfy

$$\Xi C + C\,{}^t\Xi^\vee = d_x C = \mathrm{diag}\left(0,0,0,\frac{4(1-x_1+x_2)dx_1+4(1+x_1-x_2)dx_2}{\lambda_3\lambda_4R(x)^2}\right), \tag{4.2}$$

where $\Xi^\vee$ is the image of Ξ under the involution (3.2), and diag(…) *denotes the diagonal matrix with given entries.*

Proof. Lemmas 4.1–4.4 yield the representation of ∇_X with respect to the frame ${}^t(\varphi_1,\dots,\varphi_4)$. By applying Proposition 4.1 to the frame ${}^t(\varphi_1,\dots,\varphi_4)$, we obtain (4.2). □

Remark 4.2. Though the connection matrix Ξ is not closed, it satisfies the integrability condition, i.e.,

$$\begin{aligned}\Xi \wedge \Xi &= (\Xi^1\Xi^2 - \Xi^2\Xi^1)dx_1 \wedge dx_2 \\ &= d_x\Xi = \left(-\frac{\partial}{\partial x_2}\Xi^1 + \frac{\partial}{\partial x_1}\Xi^2\right)dx_1 \wedge dx_2 \neq O.\end{aligned}$$

Remark 4.3. We give some expressions of elements of $H^2(\Omega^{\bullet,0}(\mathfrak{X}), \nabla)$ in terms of $\varphi_1, \ldots, \varphi_4$:

$$\begin{aligned}\nabla\left(-\frac{ds_1}{s_1} + \frac{ds_2}{s_2}\right) &= \frac{\lambda_3 ds_{12}}{Q} - [-(\lambda_1+\lambda_2+\lambda_3)\varphi_1 + \lambda_4\varphi_2 + \lambda_4\varphi_3], \\ \nabla\left(\frac{ds_2}{s_2}\right) &= \frac{\lambda_3 ds_{12}}{s_1 Q} - [-(\lambda_1+\lambda_3)\varphi_1 + \lambda_4\varphi_2], \\ \nabla\left(-\frac{ds_1}{s_1}\right) &= \frac{\lambda_3 ds_{12}}{s_2 Q} - [-(\lambda_2+\lambda_3)\varphi_1 + \lambda_4\varphi_3].\end{aligned}$$

Remark 4.4. We define a function $f_1(x)$ by $\int_\Delta u(s,x)\varphi_1$ for a twisted cycle Δ loading a branch of $u(s,x)$. This function satisfies

$$\begin{aligned}x_1\frac{\partial f_1(x)}{\partial x_1} &= -\lambda_4\int_\Delta u(s,x)\varphi_2, \qquad x_2\frac{\partial f_1(x)}{\partial x_2} = -\lambda_4\int_\Delta u(s,x)\varphi_3, \\ x_1x_2\frac{\partial^2 f_1(x)}{\partial x_1\partial x_2} &= \frac{\lambda_4}{2}\left[\lambda_{123}\int_\Delta u(s,x)\varphi_1 + (\lambda_2-\lambda_4)\int_\Delta u(s,x)\varphi_2\right. \\ &\qquad \left. + (\lambda_1-\lambda_4)\int_\Delta u(s,x)\varphi_3 + \lambda_3(1-x_1-x_2)\int_\Delta u(s,x)\varphi_4\right].\end{aligned}$$

Thus the vector-valued function

$$F_\partial(x) = {}^t\left(f_1(x), x_1\frac{\partial f_1(x)}{\partial x_1}, x_2\frac{\partial f_1(x)}{\partial x_2}, x_1x_2\frac{\partial^2 f_1(x)}{\partial x_1\partial x_2}\right)$$

satisfies

$$d_xF_\partial(x) = (G_\partial\Xi G_\partial^{-1} + d_xG_\partial\, G_\partial^{-1})F_\partial(x),$$

where

$$G_\partial = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\lambda_4 & 0 & 0 \\ 0 & 0 & -\lambda_4 & 0 \\ \lambda_{123}\lambda_4/2 & (\lambda_2-\lambda_4)\lambda_4/2 & (\lambda_1-\lambda_4)\lambda_4/2 & \lambda_3\lambda_4(1-x_1-x_2)/2 \end{pmatrix},$$

$$G_\partial^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\lambda_4^{-1} & 0 & 0 \\ 0 & 0 & -\lambda_4^{-1} & 0 \\ \frac{-\lambda_{123}}{\lambda_3(1-x_1-x_2)} & \frac{\lambda_2-\lambda_4}{\lambda_3\lambda_4(1-x_1-x_2)} & \frac{\lambda_1-\lambda_4}{\lambda_3\lambda_4(1-x_1-x_2)} & \frac{2}{\lambda_3\lambda_4(1-x_1-x_2)} \end{pmatrix}.$$

Note that the matrix G_∂ does not belong to

$$\mathrm{GL}_4(\mathcal{O}(X)) = \{G \in M(4,4;\mathcal{O}(X)) \mid \det(G) \text{ is invertible in } \mathcal{O}(X)\}.$$

Proposition 4.2. *Suppose that the parameters a, b, c_1, c_2 satisfy the condition (3.1). There is no global frame of the vector bundle $H^2(\Omega^{\bullet,0}(\mathfrak{X}), \nabla)$ over X such that the connection ∇_X can be expressed in terms of logarithmic forms as*

$$A_1\frac{dx_1}{x_1} + A_2\frac{dx_2}{x_2} + A_3\frac{d_xR(x)}{R(x)},$$

where A_1, A_2 and A_3 are 4×4-matrices over the algebraic closure $\overline{\mathbb{C}(\lambda)}$ of $\mathbb{C}(\lambda)$.

Proof. Suppose that such a global frame exists. By the integrability condition, we have

$$\begin{aligned} O &= d_x\left(A_1\frac{dx_1}{x_1} + A_2\frac{dx_2}{x_2} + A_3\frac{d_xR(x)}{R(x)}\right) \\ &= \left(A_1\frac{dx_1}{x_1} + A_2\frac{dx_2}{x_2} + A_3\frac{d_xR(x)}{R(x)}\right) \wedge \left(A_1\frac{dx_1}{x_1} + A_2\frac{dx_2}{x_2} + A_3\frac{d_xR(x)}{R(x)}\right) \\ &= \left(\frac{[A_1,A_2]}{x_1x_2} + \frac{2[A_1,A_3](x_2-x_1-1)}{x_1R(x)} + \frac{2[A_3,A_2](x_1-x_2-1)}{x_2R(x)}\right)dx_1\wedge dx_2, \end{aligned}$$

where $[A_i, A_j] = A_iA_j - A_jA_i$, $R(x) = 1 - 2x_1 - 2x_2 + x_1^2 + x_2^2 - 2x_1x_2$ and $d_xR(x) = 2(x_1 - x_2 - 1)dx_1 + 2(x_2 - x_1 - 1)dx_2$. This equality reduces to

$$\begin{aligned} [A_1,A_2]\cdot 1 &- 2([A_1,A_2]+[A_3,A_2])\cdot x_1 - 2([A_1,A_2]+[A_1,A_3])\cdot x_2 \\ &+ ([A_1,A_2]+2[A_3,A_2])\cdot x_1^2 + ([A_1,A_2]+2[A_1,A_3])\cdot x_2^2 \\ &- 2([A_1,A_2]+[A_1,A_3]+[A_3,A_2])\cdot x_1x_2 = O, \end{aligned}$$

which is equivalent to

$$[A_1,A_2] = [A_1,A_3] = [A_2,A_3] = O.$$

Hence there exists $P \in \mathrm{GL}_4(\overline{\mathbb{C}(\lambda)})$ such that PA_iP^{-1} $(i = 1,2,3)$ are upper-triangular matrices. This means that the system $\mathcal{F}_4(a,b,c)$ is reducible. On the other hand, it is shown in [HT] that the system $\mathcal{F}_4(a,b,c)$ is irreducible under the condition (3.1). Therefore we have a contradiction. □

Remark 4.5. The same claim as in Proposition 4.2 holds for Lauricella's system $\mathcal{F}_C(a,b,c)$ under the irreducibility condition of [HT]. We can easily modify the proof of Proposition 4.2 so that it is valid for $\mathcal{F}_C(a,b,c)$.

§5. Connection matrix in terms of intersection form

As mentioned in Proposition 3.2, there is no global frame of the vector bundle $H^2(\Omega^{\bullet,0}(\mathfrak{X}),\nabla)$ over X such that its intersection matrix is independent of $x \in X$. The obstruction is that $R(x)$ is square free in $\mathbb{C}(x)$. Thus it is natural to consider the quadratic extension field of $\mathbb{C}(x)$ with $\sqrt{R(x)}$ added. In this section, we introduce the double cover of $\mathbb{C}^2$ branching along $R(x)=0$, and express the pull-back ∇_Y of the connection ∇_X under this covering by using the intersection form $\mathcal{I}_c$.

We define an affine variety

$$\widetilde{X} = \{(x_1,x_2,x_3) \in \mathbb{C}^3 \mid x_3^2 = R(x_1,x_2)\},$$

which is regarded as the double cover of $\mathbb{C}^2$ branching along the divisor $R(x_1,x_2) = 0$ by the projection

$$\mathrm{pr} : \widetilde{X} \ni (x_1,x_2,x_3) \mapsto (x_1,x_2) \in \mathbb{C}^2.$$

Note that

$$\left(\frac{1+x_1-x_2-x_3}{2}\right)\left(\frac{1+x_1-x_2+x_3}{2}\right) = x_1,$$
$$\left(\frac{1-x_1+x_2-x_3}{2}\right)\left(\frac{1-x_1+x_2+x_3}{2}\right) = x_2,$$

for $(x_1,x_2,x_3) \in \widetilde{X}$. Thus the preimages of the lines $x_1 = 0$ and $x_2 = 0$ in $\mathbb{C}^2$ under the projection pr are expressed by the equations

$$\frac{1+x_1-x_2-x_3}{2} = 0, \qquad \frac{1-x_1+x_2-x_3}{2} = 1,$$

and

$$\frac{1+x_1-x_2-x_3}{2} = 1, \qquad \frac{1-x_1+x_2-x_3}{2} = 0,$$

in $\widetilde{X}$ respectively. By means of the map

$$\widetilde{X} \ni (x_1,x_2,x_3) \mapsto (y_1,y_2) = \left(\frac{1+x_1-x_2-x_3}{2}, \frac{1-x_1+x_2-x_3}{2}\right) \in \mathbb{C}^2,$$

$\widetilde{X}$ is bi-holomorphic to $\mathbb{C}^2$; the inverse of this map is

$$\mathbb{C}^2 \ni (y_1,y_2) \mapsto (x_1,x_2,x_3) = (y_1(1-y_2), (1-y_1)y_2, 1-y_1-y_2) \in \widetilde{X}.$$

Though (x_1, x_2) are not valid as local coordinates on the set

$$\left\{(x_1, x_2, x_3) \in \widetilde{X} \;\middle|\; \frac{\partial}{\partial x_3}(x_3^2 - R(x_1, x_2)) = 2x_3 = 0\right\},$$

we can use

$$(y_1, y_2) = \left(\frac{1 + x_1 - x_2 - x_3}{2}, \frac{1 - x_1 + x_2 - x_3}{2}\right)$$

as a global system of coordinates on $\widetilde{X}$. The covering transformation

$$\rho : (x_1, x_2, x_3) \mapsto (x_1, x_2, -x_3)$$

of $\mathrm{pr} : \widetilde{X} \to \mathbb{C}^2$ is represented as

$$(y_1, y_2) \mapsto (1 - y_2, 1 - y_1).$$

The ramification locus of pr is

$$\{(y_1, y_2) \in \mathbb{C}^2 \mid y_1 + y_2 = 1\}$$

and the preimage of the singular locus of $\mathcal{F}_4(a, b, c)$ in $\mathbb{C}^2$ under the projection pr is

$$\{(y_1, y_2) \in \mathbb{C}^2 \mid [y_1(1 - y_2)] \cdot [y_2(1 - y_1)] \cdot (1 - y_1 - y_2) = 0\}.$$

We set

$$\begin{aligned} Y &= \{y = (y_1, y_2) \in \mathbb{C}^2 \mid y_1(1 - y_1)y_2(1 - y_2)(1 - y_1 - y_2) \neq 0\} \subset \mathbb{P}^2, \\ \widetilde{S} &= \mathbb{P}^2 - Y, \end{aligned}$$

and denote the restriction of pr to Y by the same symbol pr. Note that $\widetilde{S}$ and Y are invariant as sets under the action of ρ. Let $\mathcal{O}(Y)$ be the $\mathbb{C}(\lambda)$-algebra of rational functions on $\mathbb{P}^2$ with poles only along $\widetilde{S}$. From the vector bundle $H^2(\Omega^{\bullet,0}(\mathfrak{X}), \nabla)$ and the connection ∇_X over X, the projection $\mathrm{pr} : Y \to X$ induces the vector bundle

$$\mathrm{pr}^* H^2(\Omega^{\bullet,0}(\mathfrak{X}), \nabla) = \{(y, \varphi) \in Y \times H^2(\Omega^{\bullet,0}(\mathfrak{X}), \nabla) \mid \mathrm{pr}(y) = p(\varphi)\},$$

and the connection $\nabla_Y = \mathrm{pr}^* \nabla_X$ over Y, where $p : H^2(\Omega^{\bullet,0}(\mathfrak{X}), \nabla) \to X$ is the projection. There is a natural frame

$${}^t(\mathrm{pr}^*(\varphi_1), \ldots, \mathrm{pr}^*(\varphi_4))$$

given by the pull-backs of $\varphi_1, \ldots, \varphi_4$ in $H^2(\Omega^{\bullet,0}(\mathfrak{X}), \nabla)$ under the projection pr. Since

$$\mathrm{pr}^*(R(x)) = (1 - y_1 - y_2)^2,$$

there is a global frame on Y whose intersection matrix is independent of y.

Corollary 5.1. *Let* ${}^t(\widehat{\varphi}_1, \dots, \widehat{\varphi}_4)$ *be a global frame on* Y *given by*

$$G\,{}^t(\mathrm{pr}^*(\varphi_1), \dots, \mathrm{pr}^*(\varphi_4)), \qquad G = \mathrm{diag}(1, 1, 1, 1 - y_1 - y_2) \in \mathrm{GL}_4(\mathcal{O}(Y)).$$

Then the intersection numbers $\mathcal{I}_c(\widehat{\varphi}_i, \widehat{\varphi}_j)$ $(1 \le i, j \le 4)$ *are* $(2\pi\sqrt{-1})^2 \widehat{C}_{ij}$, *where*

$$\widehat{C}_{ij} = \begin{cases} C_{ij} & \text{if } 1 \le i, j \le 3, \\ \dfrac{2}{\lambda_3 \lambda_4} = \dfrac{2}{a_{11} b_{00}} & \text{if } (i, j) = (4, 4), \\ 0 & \text{otherwise}, \end{cases}$$

and the C_{ij} *are as in Proposition* 3.1.

Proof. Since the map $\mathrm{pr} : Y \to X$ is locally isomorphic, we have

$$\mathcal{I}_c(\mathrm{pr}^*(\varphi_i), \mathrm{pr}^*(\varphi_j)) = \mathcal{I}_c(\varphi_i, \varphi_j) \qquad (1 \le i, j \le 4).$$

The conclusion is now clear by the transformation G, the equality $\mathrm{pr}^*(R(x)) = (1 - y_1 - y_2)^2$, and Proposition 3.1. □

Theorem 5.1. *The connection matrix* $\widehat{\Xi}$ *of* ∇_Y *with respect to the frame* ${}^t(\widehat{\varphi}_1, \dots, \widehat{\varphi}_4)$ *is*

$$\widehat{\Xi}^1 \frac{dy_1}{y_1} + \widehat{\Xi}^2 \frac{dy_2}{y_2} + I_{3,1} \widehat{\Xi}^2 I_{3,1}^{-1} \frac{dy_1}{y_1 - 1} + I_{3,1} \widehat{\Xi}^1 I_{3,1}^{-1} \frac{dy_2}{y_2 - 1} + \widehat{\Xi}^3 \frac{dy_1 + dy_2}{y_1 + y_2 - 1},$$

where

$$\widehat{\Xi}^1 = \begin{pmatrix} 0 & -\lambda_4 & 0 & 0 \\ 0 & -\lambda_1 - \lambda_3 & 0 & 0 \\ \frac{-\lambda_{123}}{2} & \frac{-\lambda_2 + \lambda_4}{2} & \frac{-\lambda_1 + \lambda_4}{2} & \frac{-\lambda_3}{2} \\ \frac{(\lambda_{134}^- - \lambda_3)\lambda_{123}}{2\lambda_3} & \lambda_4 + \frac{(\lambda_2 + \lambda_4)(\lambda_1 + \lambda_4)}{2\lambda_3} & \frac{(\lambda_1 - \lambda_4)(\lambda_{134}^- - \lambda_3)}{2\lambda_3} & \frac{\lambda_{134}^- - \lambda_3}{2} \end{pmatrix},$$

$$\widehat{\Xi}^2 = \begin{pmatrix} 0 & 0 & -\lambda_4 & 0 \\ \frac{-\lambda_{123}}{2} & \frac{-\lambda_2 + \lambda_4}{2} & \frac{-\lambda_1 + \lambda_4}{2} & \frac{-\lambda_3}{2} \\ 0 & 0 & -\lambda_3 - \lambda_2 & 0 \\ \frac{(\lambda_{234}^- - \lambda_3)\lambda_{123}}{2\lambda_3} & \frac{(\lambda_2 - \lambda_4)(\lambda_{234}^- - \lambda_3)}{2\lambda_3} & \lambda_4 + \frac{(\lambda_2 + \lambda_4)(\lambda_1 + \lambda_4)}{2\lambda_3} & \frac{\lambda_{234}^- - \lambda_3}{2} \end{pmatrix},$$

$$\widehat{\Xi}^3 = \mathrm{diag}(0, 0, 0, 2(\lambda_3 + \lambda_4)), \qquad I_{3,1} = \mathrm{diag}(1, 1, 1, -1).$$

The connection matrix $\widehat{\Xi}$ *satisfies*

$$d_y \widehat{\Xi} = \widehat{\Xi} \wedge \widehat{\Xi} = O.$$

Proof. By using

$$\begin{aligned} \mathrm{pr}^*(x_1) &= y_1(1 - y_2), & \mathrm{pr}^*(x_2) &= (1 - y_1)y_2, \\ \mathrm{pr}^*(dx_1) &= (1 - y_2)dy_1 - y_1 dy_2, & \mathrm{pr}^*(dx_2) &= -y_2 dy_1 + (1 - y_1)dy_2, \\ \mathrm{pr}^*(R(x)) &= (1 - y_1 - y_2)^2, \end{aligned}$$

we can calculate the connection matrix $\widetilde{\Xi}$ with respect to ${}^t(\mathrm{pr}^*(\varphi_1),\dots,\mathrm{pr}^*(\varphi_4))$ as the pull-back of Ξ under pr. The connection matrix $\widehat{\Xi}$ with respect to ${}^t(\widehat{\varphi}_1,\dots,\widehat{\varphi}_4)$ is given by the gauge transformation

$$G\widetilde{\Xi}G^{-1}+d_yGG^{-1}$$

of $\widetilde{\Xi}$. By straightforward calculations, we get the expression of $\widehat{\Xi}$, which implies $d_y\widehat{\Xi}=O$. By the integrability condition, $\widehat{\Xi}$ satisfies $\widehat{\Xi}\wedge\widehat{\Xi}=O$. Here note that the gauge transformation by G changes the non-closed connection matrix $\widetilde{\Xi}$ into the closed connection matrix $\widehat{\Xi}$. □

We now express the connection ∇_Y in terms of the intersection form $\mathcal{I}_c$. Let $\widehat{C}$ be the matrix $(\widehat{C}_{ij})_{1\le i,j\le 4}$ given by the intersection numbers in Corollary 5.1. This matrix is symmetric and satisfies $\widehat{C}^\vee=\widehat{C}$, where $\widehat{C}^\vee$ is the image of $\widehat{C}$ under the involution (3.2).

Lemma 5.1. *The connection matrix $\widehat{\Xi}$ satisfies*

$$\widehat{\Xi}^\vee=-\widehat{\Xi},\qquad \widehat{\Xi}\widehat{C}+\widehat{C}\,{}^t\widehat{\Xi}^\vee=O.$$

Proof. We can easily check this by using Theorem 5.1 and Corollary 5.1. The second equality can also be obtained from $d_y\widehat{C}=O$ by using Proposition 4.1. □

Lemma 5.2. (i) *The eigenvalues of $\widehat{\Xi}^1$ are 0 and $-(\lambda_1+\lambda_3)=1-c_1$, and each of the eigenspaces is 2-dimensional. The $(1-c_1)$-eigenspace is spanned by the row vectors*

$$e_2=(0,1,0,0),\qquad \left(\frac{\lambda_{123}}{\lambda_3},\frac{\lambda_2-\lambda_4}{\lambda_3},\frac{\lambda_1-\lambda_4}{\lambda_3},1\right).$$

(ii) *The eigenvalues of $\widehat{\Xi}^2$ are 0 and $-(\lambda_2+\lambda_3)=1-c_2$, and each of the eigenspaces is 2-dimensional. The $(1-c_2)$-eigenspace is spanned by the row vectors*

$$e_3=(0,0,1,0),\qquad \left(\frac{\lambda_{123}}{\lambda_3},\frac{\lambda_2-\lambda_4}{\lambda_3},\frac{\lambda_1-\lambda_4}{\lambda_3},1\right).$$

(iii) *The eigenvalues of $\widehat{\Xi}^3$ are 0 and $2(\lambda_3+\lambda_4)=2(c_1+c_2-a-b-2)$. The 0-eigenspace is 3-dimensional and the $2(c_1+c_2-a-b-2)$-eigenspace is 1-dimensional, spanned by $e_4=(0,0,0,1)$.*

Proof. Use Theorem 5.1. □

We set

$$e_5=\left(\frac{\lambda_{123}}{\lambda_3},\frac{\lambda_2-\lambda_4}{\lambda_3},\frac{\lambda_1-\lambda_4}{\lambda_3},1\right),\qquad e_6=\left(\frac{\lambda_{123}}{\lambda_3},\frac{\lambda_2-\lambda_4}{\lambda_3},\frac{\lambda_1-\lambda_4}{\lambda_3},-1\right),$$

and

$$\widehat{\varphi}_5 = \frac{\lambda_{123}}{\lambda_3}\widehat{\varphi}_1 + \frac{\lambda_2-\lambda_4}{\lambda_3}\widehat{\varphi}_2 + \frac{\lambda_1-\lambda_4}{\lambda_3}\widehat{\varphi}_3 + \widehat{\varphi}_4,$$
$$\widehat{\varphi}_6 = \frac{\lambda_{123}}{\lambda_3}\widehat{\varphi}_1 + \frac{\lambda_2-\lambda_4}{\lambda_3}\widehat{\varphi}_2 + \frac{\lambda_1-\lambda_4}{\lambda_3}\widehat{\varphi}_3 - \widehat{\varphi}_4,$$

corresponding to the vectors e_5 and e_6, respectively.

Lemma 5.3. (i) *The forms $\widehat{\varphi}_2$ and $\widehat{\varphi}_5$ vanish as $y_1 \to 0$. The forms $\widehat{\varphi}_2$ and $\widehat{\varphi}_6$ vanish as $y_2 \to 1$.*

(ii) *The forms $\widehat{\varphi}_3$ and $\widehat{\varphi}_5$ vanish as $y_2 \to 0$. The forms $\widehat{\varphi}_3$ and $\widehat{\varphi}_6$ vanish as $y_1 \to 1$.*

(iii) *The form $\widehat{\varphi}_4$ vanishes as $y_1 + y_2 \to 1$.*

Proof. (i) Since $\widehat{\varphi}_2$ equals $y_1(1-y_2)ds_{12}/(s_2L(s,y))$, it vanishes as $y_1 \to 0$ and as $y_2 \to 1$. By Lemma 4.2, the image of $2(\lambda_4-1)x_1x_2ds_{12}/L^2$ under pr^* is

$$\lambda_{123}\widehat{\varphi}_1 + (\lambda_2-\lambda_4)\widehat{\varphi}_2 + (\lambda_1-\lambda_4)\widehat{\varphi}_3 + \frac{\lambda_3(1-y_1-y_2+2y_1y_2)}{1-y_1-y_2}\widehat{\varphi}_4.$$

It is clear that this element vanishes and its last term converges to $\lambda_3\widehat{\varphi}_4$ (resp. $-\lambda_3\widehat{\varphi}_4$) as $y_1 \to 0$ (resp. $y_2 \to 1$). Thus $\lambda_3\widehat{\varphi}_5$ vanishes as $y_1 \to 0$ and $\lambda_3\widehat{\varphi}_6$ vanishes as $y_2 \to 1$.

(ii) Similar.

(iii) Since

$$\widehat{\varphi}_4 = \frac{(1-y_1-y_2)ds_{12}}{Q(s)L(s,y)},$$

it vanishes as $y_1 + y_2 \to 1$. □

We set

$$\widehat{C}_1 = \begin{pmatrix} \mathcal{I}_c(\widehat{\varphi}_2,\widehat{\varphi}_2) & \mathcal{I}_c(\widehat{\varphi}_2,\widehat{\varphi}_5) \\ \mathcal{I}_c(\widehat{\varphi}_5,\widehat{\varphi}_2) & \mathcal{I}_c(\widehat{\varphi}_5,\widehat{\varphi}_5) \end{pmatrix} = \begin{pmatrix} \frac{(\lambda_3-\lambda_{134}^-)(\lambda_1+\lambda_3)}{\lambda_0\lambda_2\lambda_4\lambda_{134}^-} & \frac{-2(\lambda_1+\lambda_3)}{\lambda_0\lambda_3\lambda_4} \\ \frac{-2(\lambda_1+\lambda_3)}{\lambda_0\lambda_3\lambda_4} & \frac{-4(\lambda_2+\lambda_3)(\lambda_1+\lambda_3)}{\lambda_0\lambda_3^2\lambda_4} \end{pmatrix},$$

$$\widehat{C}_2 = \begin{pmatrix} \mathcal{I}_c(\widehat{\varphi}_3,\widehat{\varphi}_3) & \mathcal{I}_c(\widehat{\varphi}_3,\widehat{\varphi}_5) \\ \mathcal{I}_c(\widehat{\varphi}_5,\widehat{\varphi}_3) & \mathcal{I}_c(\widehat{\varphi}_5,\widehat{\varphi}_5) \end{pmatrix} = \begin{pmatrix} \frac{(\lambda_3-\lambda_{234}^-)(\lambda_2+\lambda_3)}{\lambda_0\lambda_1\lambda_4\lambda_{234}^-} & \frac{-2(\lambda_2+\lambda_3)}{\lambda_0\lambda_3\lambda_4} \\ \frac{-2(\lambda_2+\lambda_3)}{\lambda_0\lambda_3\lambda_4} & \frac{-4(\lambda_2+\lambda_3)(\lambda_1+\lambda_3)}{\lambda_0\lambda_3^2\lambda_4} \end{pmatrix}.$$

We have

$$\begin{pmatrix} \mathcal{I}_c(\widehat{\varphi}_2,\widehat{\varphi}_2) & \mathcal{I}_c(\widehat{\varphi}_2,\widehat{\varphi}_6) \\ \mathcal{I}_c(\widehat{\varphi}_6,\widehat{\varphi}_2) & \mathcal{I}_c(\widehat{\varphi}_6,\widehat{\varphi}_6) \end{pmatrix} = \widehat{C}_1, \qquad \begin{pmatrix} \mathcal{I}_c(\widehat{\varphi}_3,\widehat{\varphi}_3) & \mathcal{I}_c(\widehat{\varphi}_3,\widehat{\varphi}_6) \\ \mathcal{I}_c(\widehat{\varphi}_6,\widehat{\varphi}_3) & \mathcal{I}_c(\widehat{\varphi}_6,\widehat{\varphi}_6) \end{pmatrix} = \widehat{C}_2.$$

Proposition 5.1. *The matrices* $\widehat{\Xi}^i$ $(i = 1, 2, 3)$ *can be expressed in terms of* $\widehat{C}$ *and its eigenvectors with non-zero eigenvalue as*

$$\widehat{\Xi}^1 = -(\lambda_1 + \lambda_3)\widehat{C}({}^te_2, {}^te_5)(\widehat{C}_1)^{-1}\begin{pmatrix} e_2 \\ e_5 \end{pmatrix},$$

$$\widehat{\Xi}^2 = -(\lambda_2 + \lambda_3)\widehat{C}({}^te_3, {}^te_5)(\widehat{C}_2)^{-1}\begin{pmatrix} e_3 \\ e_5 \end{pmatrix},$$

$$\widehat{\Xi}^3 = 2(\lambda_3 + \lambda_4)\widehat{C}\,{}^te_4(\widehat{C}_{44})^{-1}e_4.$$

Proof. We claim that

$$v\widehat{C}\,{}^tw^\vee = 0$$

for any eigenvector v of $\widehat{\Xi}^i$ with non-zero eigenvalue α and for any eigenvector w of $\widehat{\Xi}^i$ with eigenvalue 0. Lemma 5.1 implies

$$\widehat{\Xi}^i\widehat{C} = -\widehat{C}\,{}^t(\widehat{\Xi}^i)^\vee.$$

Since $\alpha \neq 0$ and

$$\begin{aligned}\alpha(v\widehat{C}\,{}^tw^\vee) &= (v\widehat{\Xi}^i)\widehat{C}\,{}^tw^\vee = v(\widehat{\Xi}^i\widehat{C})\,{}^tw^\vee = -v(\widehat{C}\,{}^t(\widehat{\Xi}^i)^\vee)\,{}^tw^\vee \\ &= -v\widehat{C}\,{}^t(w\widehat{\Xi}^i)^\vee = 0,\end{aligned}$$

the claim follows. Lemma 5.2 together with this claim gives the desired expressions of the matrices $\widehat{\Xi}^i$. □

Theorem 5.2. *The connection* ∇_Y *of* $\mathrm{pr}^*H^2(\Omega^{\bullet,0}(\mathfrak{X}), \nabla)$ *can be expressed in terms of the intersection form* $\mathcal{I}_c$ *as*

$$\begin{aligned}\nabla_Y(\widehat{\varphi}) = &\frac{dy_1}{y_1} \wedge (1-c_1)(\mathcal{I}_c(\widehat{\varphi}, \widehat{\varphi}_2), \mathcal{I}_c(\widehat{\varphi}, \widehat{\varphi}_5))(\widehat{C}_1)^{-1}\begin{pmatrix} \widehat{\varphi}_2 \\ \widehat{\varphi}_5 \end{pmatrix} \\ &+ \frac{dy_2}{y_2} \wedge (1-c_2)(\mathcal{I}_c(\widehat{\varphi}, \widehat{\varphi}_3), \mathcal{I}_c(\widehat{\varphi}, \widehat{\varphi}_5))(\widehat{C}_2)^{-1}\begin{pmatrix} \widehat{\varphi}_3 \\ \widehat{\varphi}_5 \end{pmatrix} \\ &+ \frac{dy_1}{y_1 - 1} \wedge (1-c_2)(\mathcal{I}_c(\widehat{\varphi}, \widehat{\varphi}_3), \mathcal{I}_c(\widehat{\varphi}, \widehat{\varphi}_6))(\widehat{C}_2)^{-1}\begin{pmatrix} \widehat{\varphi}_3 \\ \widehat{\varphi}_6 \end{pmatrix} \\ &+ \frac{dy_2}{y_2 - 1} \wedge (1-c_1)(\mathcal{I}_c(\widehat{\varphi}, \widehat{\varphi}_2), \mathcal{I}_c(\widehat{\varphi}, \widehat{\varphi}_6))(\widehat{C}_1)^{-1}\begin{pmatrix} \widehat{\varphi}_2 \\ \widehat{\varphi}_6 \end{pmatrix} \\ &+ \frac{dy_1 + dy_2}{y_1 + y_2 - 1} \wedge 2(c_1 + c_2 - a - b - 2)\mathcal{I}_c(\widehat{\varphi}, \widehat{\varphi}_4)(\widehat{C}_{44})^{-1}\widehat{\varphi}_4.\end{aligned}$$

Proof. Note that the linear transformation

$$\widehat{\varphi} \mapsto (1-c_1)(\mathcal{I}_c(\widehat{\varphi}, \widehat{\varphi}_2), \mathcal{I}_c(\widehat{\varphi}, \widehat{\varphi}_5))(\widehat{C}_1)^{-1}\begin{pmatrix} \widehat{\varphi}_2 \\ \widehat{\varphi}_5 \end{pmatrix}$$

is represented by the matrix $\widehat{\Xi}^1$ with respect to the frame ${}^t(\widehat{\varphi}_1,\dots,\widehat{\varphi}_4)$. The eigenspace of $I_{3,1}\widehat{\Xi}^1 I_{3,1}^{-1}$ with non-zero eigenvalue is spanned by e_2 and e_6, which correspond to $\widehat{\varphi}_2$ and $\widehat{\varphi}_6$, respectively. Thus the linear transformation

$$\widehat{\varphi} \mapsto (1-c_1)(\mathcal{I}_c(\widehat{\varphi},\widehat{\varphi}_2),\mathcal{I}_c(\widehat{\varphi},\widehat{\varphi}_6))(\widehat{C}_1)^{-1}\begin{pmatrix}\widehat{\varphi}_2\\ \widehat{\varphi}_6\end{pmatrix}$$

is represented by $I_{3,1}\widehat{\Xi}^1 I_{3,1}^{-1}$ with respect to ${}^t(\widehat{\varphi}_1,\dots,\widehat{\varphi}_4)$. Similarly, we have representation matrices $\widehat{\Xi}^2$ and $I_{3,1}\widehat{\Xi}^2 I_{3,1}^{-1}$. The linear transformation

$$\widehat{\varphi} \mapsto 2(c_1+c_2-a-b-2)\mathcal{I}_c(\widehat{\varphi},\widehat{\varphi}_4)(\widehat{C}_{44})^{-1}\widehat{\varphi}_4$$

is represented by $\widehat{\Xi}^3$ with respect to ${}^t(\widehat{\varphi}_1,\dots,\widehat{\varphi}_4)$. Theorem 5.1 and Proposition 5.1 yield the conclusion. □

We can easily check that $\widehat{\Xi}^1$ and $\widehat{\Xi}^2$ commute. Since they are diagonalizable, there exists a frame change such that $\widehat{\Xi}^1$ and $\widehat{\Xi}^2$ are simultaneously transformed into diagonal matrices.

Proposition 5.2. *We have*

$$P\widehat{\Xi}^1P^{-1} = \mathrm{diag}(1,1-c_1,1,1-c_1), \qquad P\widehat{\Xi}^2P^{-1} = \mathrm{diag}(1,1,1-c_2,1-c_2)$$

for the matrix $P = P_1P_0$, where

$$P_1 = \frac{1}{(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)}\mathrm{diag}\big(-\lambda_4\lambda_{123},\lambda_2\lambda_{134},\lambda_1\lambda_{234},\lambda_3(\lambda_{123}+\lambda_3+\lambda_4)\big),$$

$$P_0 = \begin{pmatrix} \frac{(\lambda_1+\lambda_2+\lambda_3)\lambda_4+2(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)}{\lambda_3\lambda_4} & -\frac{\lambda_2+2\lambda_3+\lambda_4}{\lambda_3} & -\frac{\lambda_1+2\lambda_3+\lambda_4}{\lambda_3} & 1\\ \frac{\lambda_1+\lambda_2+\lambda_3}{\lambda_3} & -\frac{\lambda_2+2\lambda_3+\lambda_4}{\lambda_3} & \frac{\lambda_1-\lambda_4}{\lambda_3} & 1\\ \frac{\lambda_1+\lambda_2+\lambda_3}{\lambda_3} & \frac{\lambda_2-\lambda_4}{\lambda_3} & -\frac{\lambda_1+2\lambda_3+\lambda_4}{\lambda_3} & 1\\ \frac{\lambda_1+\lambda_2+\lambda_3}{\lambda_3} & \frac{\lambda_2-\lambda_4}{\lambda_3} & \frac{\lambda_1-\lambda_4}{\lambda_3} & 1 \end{pmatrix}.$$

The row eigenvector of $P\widehat{\Xi}^3P^{-1}$ of eigenvalue $2(c_1+c_2-a-b-2)$ is $(1,1,1,1)$, and $P\widehat{C}\,{}^tP^\vee$ is $\frac{4a}{(c_1-1)(c_2-1)(a-c_1-c_2+2)}$ times the diagonal matrix with entries

$$1, \frac{-(a-c_1+1)(b-c_1+1)}{ab}, \frac{-(a-c_2+1)(b-c_2+1)}{ab}, \frac{(a-c_1-c_2+2)(b-c_1-c_2+2)}{ab}.$$

Proof. Let $V_{i,0}$ and V_{i,c_i} be the eigenspaces of the eigenvalues 0 and $1-c_i$ of $\widehat{\Xi}^i$ for $i=1,2$. Then the intersections $V_{1,0}\cap V_{2,0}$, $V_{1,c_1}\cap V_{2,0}$, $V_{1,0}\cap V_{2,c_2}$ and $V_{1,c_1}\cap V_{2,c_2}$ are spanned by the row vectors of P_0. To normalize the eigenvector of $P_0\Xi^3P_0^{-1}$ of eigenvalue $2(c_1+c_2-a-b-2)$ into $(1,1,1,1)$, we use the matrix P_1. By straightforward calculations, we obtain the expression of $P\widehat{C}\,{}^tP^\vee$. Here note that $P^\vee = P$, since the entries of P_0 and P_1 are homogeneous of degree 0 with respect to $\lambda_1,\dots,\lambda_4$. □

Remark 5.1. The monodromy representation of $\mathcal{F}_4(a,b,c)$ is studied in [G], [GM1] and [M3]. We normalize a fundamental system of solutions satisfying:

- the circuit matrices along loops turning the divisors $x_1 = 0$ and $x_2 = 0$ are diagonal;
- the non-1 eigenvector of the circuit matrix along a loop turning the divisor $R(x) = 0$ becomes ${}^t(1,\dots,1)$.

Then the diagonal matrix

$$\operatorname{diag}\left(1, \frac{-(\alpha-\gamma_1)(\beta-\gamma_1)}{\gamma_1(\alpha-1)(\beta-1)}, \frac{-(\alpha-\gamma_2)(\beta-\gamma_2)}{\gamma_2(\alpha-1)(\beta-1)}, \frac{(\alpha-\gamma_1\gamma_2)(\beta-\gamma_1\gamma_2)}{\gamma_1\gamma_2(\alpha-1)(\beta-1)}\right)$$

appears in the intersection matrix of twisted homology groups, where $\alpha = e^{2\pi\sqrt{-1}\,a}$, $\beta = e^{2\pi\sqrt{-1}\,b}$ and $\gamma_i = e^{2\pi\sqrt{-1}\,c_i}$ $(i = 1,2)$. Notice the similarity between the normalized intersection matrices of twisted homology and cohomology groups.

References

[AoKi] K. Aomoto and M. Kita, *Theory of hypergeometric functions,* Springer, New York, 2011. Zbl 1229.33001 MR 2799182

[ApKa] P. Appell and J. Kampé de Fériet, *Fonctions hypergéométriques et hypersphériques: polynomes d'Hermite,* Gauthier-Villars, Paris, 1926. JFM 52.0361.13

[C] K. Cho, A generalization of Kita and Noumi's vanishing theorems of cohomology groups of local system, Nagoya Math. J. **147** (1997), 63–69. Zbl 0912.32023 MR 1475166

[G] Y. Goto, The monodromy representation of Lauricella's hypergeometric function F_C, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5), to appear; arXiv:1403.1654 (2014).

[GM1] Y. Goto and K. Matsumoto, The monodromy representation and twisted period relations for Appell's hypergeometric function F_4, Nagoya Math. J. **217** (2015), 61–94. Zbl 1327.32001 MR 3343839

[GM2] ———, Pfaffian equations and contiguity relations of the hypergeometric function of type $(k+1, k+n+2)$ and their applications, arXiv:1602.01637 (2016).

[HT] R. Hattori and N. Takayama, The singular locus of Lauricella's F_C, J. Math. Soc. Japan **66** (2014), 981–995. Zbl 1312.33043 MR 3238325

[Kan] J. Kaneko, Monodromy group of Appell's system (F_4), Tokyo J. Math. **4** (1981), 35–54. Zbl 0474.33010 MR 0625119

[Kat] M. Kato, A Pfaffian system of Appell's F_4, Bull. Coll. Edu. Univ. Ryukyus **33** (1988), 331–334.

[M1] K. Matsumoto, Monodromy and Pfaffian of Lauricella's F_D in terms of the intersection forms of twisted (co)homology groups, Kyushu J. Math. **67** (2013), 367–387. Zbl 1279.33023 MR 3115210

[M2] ———, Pfaffian of Lauricella's hypergeometric system F_A, arXiv:1502.00334 (2015).

[M3] ———, Monodromy representations of hypergeometric systems with respect to fundamental series solutions, Tohoku Math. J. (2), to appear; arXiv:1502.01826 (2015).