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A Non-trivial Ghost Kernel for Complex Projective Spaces with Symmetries

by

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Abstract

It is shown that the ghost kernel for certain equivariant stable cohomotopy groups of projective spaces is non-trivial. The proof is based on the Borel cohomology Adams spectral sequence and the calculations with the Steenrod algebra afforded by it.

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§1. Introduction

Equivariant stable homotopy theory deals with the algebraic topology of group actions. Symmetries are often useful even when studying problems that do not involve group actions in the first place. The latest successful implementation of this idea appears in the work of Hill, Hopkins, and Ravenel on the non-existence of elements of Kervaire invariant one (see [HHR] and the surveys [HHR10], [HHR11a], [HHR11b], and [Mil12]).

In equivariant stable homotopy theory there are many results which compare the equivariant stable homotopy category to the usual, non-equivariant, stable homotopy category (see [LMS86] and [May96] for background). For example, it is well-known that G-equivalences f can be detected by their H-fixed points f^H for the various subgroups $H \leq G$. In contrast, the analogous statement is known to be false for the class of G-null maps, and this will be amplified here.

For easy book-keeping, let us fix a prime p, and let G be a finite group of order p. These groups have precisely two subgroups, and it will turn out that even this simple case is interesting enough for a start. The group $[X, Y]^G$ of equivariant

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stable homotopy classes of maps between (pointed) G-CW-complexes X and Y can be studied by means of the *ghost map*

$$[X,Y]^G \to \mathrm{H}^0(G;[X,Y]) \oplus [X^G,Y^G],$$

which sends a stable G-map f to the pair (f,f^G). Clearly, the kernel of a ghost map is a place to look for genuinely equivariant phenomena. See [Chr98] for a conceptual framework for related matters. The following examples show that ghost kernels can be non-trivial, so that, in contrast to the case of equivalences, maps which are essential need not be detected by their ghosts.

Example 1.1. The target of the ghost map for $[\mathrm{S}^1,\mathrm{S}^0]^G$ is $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ if both spheres have the trivial G-action. However, the splitting theorem shows that the group $[\mathrm{S}^1,\mathrm{S}^0]^G$ has an extra summand $[\mathrm{S}^1,\mathrm{B}G] \cong G$ for the present choice of G.

Example 1.2. The 'boundary map' of the cofibration sequence $\mathrm{E}G_+ \to \mathrm{S}^0 \to \widetilde{\mathrm{E}}G$ is in the ghost kernel of $[\widetilde{\mathrm{E}}G, \Sigma\mathrm{E}G_+]^G$: the target of the ghost map is zero because both spaces $\widetilde{\mathrm{E}}G$ and $(\mathrm{E}G_+)^G$ are contractible. However, if that map were zero, this would split S^0 into $\mathrm{E}G_+ \vee \widetilde{\mathrm{E}}G$, but there is no non-trivial idempotent in the Burnside ring $[\mathrm{S}^0,\mathrm{S}^0]^G$.

The main aim of this text is to display a new family of elements in the ghost kernel for another naturally occurring situation: the equivariant stable cohomotopy of projective spaces. Apart from the intrinsic interest in these spaces [GW08] and maps like this, as explained above, this is also used in [Szy12] to show that the equivariant Bauer–Furuta invariants of Galois covers of smooth 4-manifolds are not determined by non-equivariant data (see also [Nak09a], [Nak09b], [Nak10], and [Nak14]). It has been these applications that originally led to the work presented here.

Let us now investigate in more detail the G-spaces X and Y to be considered here, and why they are of interest also from the point of view of pure equivariant homotopy theory. If V is a complex G-representation, then $\mathbb{C}\mathrm{P}(V)_+$ denotes the complex projective space of V with the induced G-action, and with a disjoint G-fixed base point added. The other fixed points of $\mathbb{C}\mathrm{P}(V)_+$ are the G-invariant lines L in V. In the language of representation theory, the fixed point sets are the projective spaces of the isotypical components of V which correspond to 1-dimensional irreducible representations of G. If G is abelian, as in our case, all irreducible representations are 1-dimensional. This shows that the fixed point sets all have the same dimension if and only if V is a multiple of the regular representation $\mathbb{C}G$. That singles out this case, and we shall therefore assume it from now on. Also, the G-spaces $\mathbb{C}\mathrm{P}(V)$ appear as skeleta of the classifying space for G-line bundles, underlining again their general importance.

At a fixed point L of the G-manifold $\mathbb{C}\mathrm{P}(V)$, the tangent G-representation is $\mathrm{Hom}(L, V/L)$. (For typographical reasons, a notation such as V/L may refer to the (orthogonal) complement of L in V in the following.) If V is a multiple of the regular representation, these tangent representations are all isomorphic, namely to $W = V/\mathbb{C}$. This is yet another fact that makes the present choice of V special, and it also shows that S^W is a natural target in this situation. The corresponding collapse maps $\mathbb{C}\mathrm{P}(V)_+ \to \mathrm{S}^W$ to the one-point compactification S^W of W generate the group

$$[\mathbb{C}\mathrm{P}(V)_+, \mathrm{S}^W]^G$$

(see [Szy07a], where it is shown that the ghost map is injective in that case). As it will turn out, we can change this if we replace W by $W/\mathbb{R}G$, and this is the situation studied here. Note that $\mathbb{R}G$ can be embedded in $a\mathbb{C}G/\mathbb{C}$ if and only if $a \geq 2$.

The preceding discussion motivates the following notation which will be used throughout. Choose an integer $a \geq 2$ and let

$$V_a = a\mathbb{C}G \quad \text{and} \quad W_a = a\mathbb{C}G/(\mathbb{C} \oplus \mathbb{R}G), \tag{1.1}$$

considered as a complex and a real G-representation, respectively. Then our main result is the following.

Theorem 1.3. *For all integers $a \geq 2$ and all primes $p \geq 5$, the ghost kernel for*

$$[\mathbb{C}\mathrm{P}(V_a)_+, \mathrm{S}^{W_a}]^G$$

is non-trivial.

The ghost map is an isomorphism away from the prime p. We are therefore dealing with a p-local phenomenon. For $p \geq 5$, it is easily seen that the target of the ghost map is zero, so it suffices to show that the group displayed in Theorem 1.3 is non-trivial, which is everything but obvious: It follows from our Theorem 8.2. For $p = 3$, the target of the ghost map will contain 3-torsion, and Theorem 8.2 is not sufficient to extend Theorem 1.3 to the prime $p = 3$.

Theorem 8.2 actually establishes more than the non-triviality of the groups in Theorem 1.3: For all $p \geq 3$ the p-power torsion of this group is an elementary abelian p-group of rank r for some integer r that satisfies $1 \leq r \leq (p + 1)/2$. This not only shows the non-triviality of the group, but also gives an upper bound on its structure. I have reasons to conjecture that the group has p-rank 1 in all cases (see Remark 8.3 at the end of Section 8). Furthermore, it is tempting to relate the groups for various $a \geq 2$, and this problem is discussed in the final Section 9 (see Remarks 9.1–9.3).

The proof of Theorem 8.2 relies on the Adams spectral sequence based on Borel cohomology [Gre88] and on the computations done with it in [Szy07b]. In Sections 2 and 3, we will recall the necessary facts about that spectral sequence and about the Borel cohomology of projective spaces, respectively. The filtration of these by projective subspaces will later (in Section 7) be used to feed in the computations of [Szy07b]. Before that, in Sections 4–6 a new method is introduced to algebraically calculate the groups of homomorphisms of modules over the Steenrod algebra in the relevant cases. The results obtained there are the other, new ingredient which is needed as an input for the computation. Section 8 combines all this to give a proof of the main theorem. As already mentioned above, there and in the concluding Section 9 we also discuss some related open problems.

§2. Borel cohomology and its Adams spectral sequence

Let p be an odd prime number, and let G be the cyclic group of order p. The notation H^* will be used for (reduced) ordinary cohomology with coefficients in the field $\mathbb{F}$ with p elements. For a finite (pointed) G-CW-complex X, the Borel cohomology is defined as

$$b^*X = \mathrm{H}^*(\mathrm{E}G_+ \wedge_G X).$$

Therefore, the coefficient ring $b^* = b^*\mathrm{S}^0 = \mathrm{H}^*(\mathrm{B}G_+)$ is the mod p cohomology ring of the group. Since p is odd, this is the tensor product of an exterior algebra on a generator σ in degree 1 and a polynomial algebra on a generator τ in degree 2.

If X and Y are finite G-CW-complexes, the Borel cohomology Adams spectral sequence takes the form

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{b^*b}(b^*Y, b^*X) \Longrightarrow ([X,Y]^G_{t-s})^\wedge_p.$$

The convergence to the indicated target has been established by Greenlees [Gre88].

As a vector space, the algebra b^*b is the tensor product $A^* \otimes b^*$, where A^* is the mod p Steenrod algebra. The algebra A^* is generated by the Bockstein element β in degree 1, and the Steenrod powers P^i for $i \geq 1$ in degree $2i(p-1)$. By convention, P^0 is the unit of the Steenrod algebra. Often the total power operation

$$P = \sum_{i=0}^{\infty} P^i$$

will be used, which acts multiplicatively on cohomology algebras. As an example, the A^*-action on the coefficient ring $b^* = b^*\mathrm{S}^0$ is given by

$$\beta(\sigma) = \tau, \quad \beta(\tau) = 0, \quad P(\sigma) = \sigma, \quad \text{and} \quad P(\tau) = \tau + \tau^p.$$

The multiplication in $b^*b = A^* \otimes b^*$ is the twisted product, the twist being given by the A^*-action on b^*.

§3. Projective spaces and their Borel cohomology

Let V by a complex G-representation. In order to describe the Borel cohomology of the projective spaces $\mathbb{C}\mathrm{P}(V)_+$, a few basic facts about Chern classes of representations [Ati61] will have to be recalled.

By definition, the Borel cohomology of $\mathbb{C}\mathrm{P}(V)_+$ is the ordinary cohomology of the space $\mathrm{E}G_+ \wedge_G \mathbb{C}\mathrm{P}(V)_+ = (\mathrm{E}G \times_G \mathbb{C}\mathrm{P}(V))_+$. The space $\mathrm{E}G \times_G \mathbb{C}\mathrm{P}(V)$ is nothing but the projective bundle associated to the vector bundle $\mathrm{E}G \times_G V$ over $\mathrm{B}G$. There is a tautological line bundle over the space $\mathrm{E}G \times_G \mathbb{C}\mathrm{P}(V)$. Write ξ_V for its first Chern class. The same symbol will be used for the reduction modulo p. Let n be the dimension of V. It follows from the Leray–Hirsch theorem that the b^*-module $b^*\mathbb{C}\mathrm{P}(V)_+$ is free of rank n with basis $1, \xi_V, \xi_V^2, \dots, \xi_V^{n-1}$. The relation

$$\sum_{j=0}^{n} (-1)^j c_j(V) \xi_V^{n-j} = 0 \tag{3.1}$$

can be used as the definition of the Chern classes $c_j(V)$ of the vector bundle $\mathrm{E}G \times_G V$ over $\mathrm{B}G$. If $V = V_1 \oplus V_2$ is a direct sum, the total Chern class $c(V) = \sum_{j=0}^{n} c_j(V)$ equals the product of the total Chern classes of the summands: $c(V_1 \oplus V_2) = c(V_1) \cdot c(V_2)$.

Since G is cyclic, the representations are easy to describe. Given an integer α, let $\mathbb{C}(\alpha)$ be the G-representation where a chosen generator of G acts as multiplication by $\exp(2\pi\mathrm{i}\alpha/p)$. We may define τ to be the first Chern class of the G-representation $\mathbb{C}(1)$. Since multiplication of irreducible 1-dimensional representations corresponds to addition in cohomology, the first Chern class of the representation $\mathbb{C}(\alpha)$ of G is $\alpha\tau$. If

$$V = \bigoplus_{j=1}^{n} \mathbb{C}(\alpha_j),$$

consider the polynomial

$$f(V) = \prod_{j=1}^{n} (x - \alpha_j \tau)$$

in $b^*[x]$. For example, if V is the complex regular representation $\mathbb{C}G$,

$$f(\mathbb{C}G) = \prod_{\alpha=1}^{p} (x - \alpha\tau) = x^p - \tau^{p-1}x = x(x^{p-1} - \tau^{p-1}). \tag{3.2}$$

This polynomial will become prominent in the next section. By (3.1), the map from $b^*[x]$ to $b^*\mathbb{C}\mathrm{P}(V)_+$ which sends x to ξ_V induces an isomorphism

$$b^*[x]/f(V) \cong b^*\mathbb{C}\mathrm{P}(V)_+. \tag{3.3}$$

The structure of $b^*\mathbb{C}\mathrm{P}(V)_+$ as a b^*-module is clear by the Leray–Hirsch theorem. As for the action of the Steenrod algebra A^*, it suffices to study it on the generators of $b^*\mathbb{C}\mathrm{P}(V)_+$ as a b^*-module and, by multiplicativity, on ξ_V. But this element has degree 2, so the action of the total Steenrod power P on it is clear: $P(\xi_V) = \xi_V + \xi_V^p$. Since ξ_V is an integral class, we have $\beta(\xi_V) = 0$, so β acts trivially on all elements of even degree.

If $U \subseteq V$ is a subrepresentation, the inclusion of $\mathbb{C}\mathrm{P}(U)_+$ into $\mathbb{C}\mathrm{P}(V)_+$ induces a surjection in Borel cohomology. In the following, the notation V/U will often denote an (orthogonal) complement of U in V, for typographical reasons. If V contains a G-line L, the cofibre sequence

$$\mathbb{C}\mathrm{P}(V/L)_+ \to \mathbb{C}\mathrm{P}(V)_+ \to \mathrm{S}^{\mathrm{Hom}(L,V/L)} \tag{3.4}$$

induces a short exact sequence

$$0 \leftarrow b^*\mathbb{C}\mathrm{P}(V/L)_+ \leftarrow b^*\mathbb{C}\mathrm{P}(V/L \oplus L)_+ \leftarrow b^*\mathrm{S}^{\mathrm{Hom}(L,V/L)} \leftarrow 0$$

of b^*b-modules, which in turn induces long exact sequences

$$\cdots \leftarrow \mathrm{Ext}_{b^*b}^{s+1,t}(b^*Y, b^*\mathrm{S}^{\mathrm{Hom}(L,V/L)}) \leftarrow \mathrm{Ext}_{b^*b}^{s,t}(b^*Y, b^*\mathbb{C}\mathrm{P}(V/L)_+) \leftarrow \cdots \tag{3.5}$$

for any Y. These will be used frequently later on. The long exact sequences (3.5) converge to the long exact sequences

$$\cdots \leftarrow [\Sigma^{-1}\mathrm{S}^{\mathrm{Hom}(L,V/L)}, Y]^G_{t-s} \leftarrow [\mathbb{C}\mathrm{P}(V/L)_+, Y]^G_{t-s} \leftarrow \cdots$$

induced by the cofibre sequence (3.4).

§4. From topology to algebra

One of our main ingredients for the later calculations with the Adams spectral sequence is the vector space $\mathrm{Hom}^1_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_a)_+)$. The purpose of this section is to prove Proposition 4.5, which establishes an isomorphism between that vector space and another one which is defined purely in terms of polynomial algebra.

Definition 4.1. Let $\overline{M}_a$ be the vector space of all elements μ in $b^*\mathbb{C}\mathrm{P}(V_a)_+$ of degree $(2a-1)p-3$ for which the equation

$$P(\mu) = (1+\tau^{p-1})^{(2a-1)(p-1)/2}\mu \tag{4.1}$$

describes the action of the total Steenrod operation.

Lemma 4.2. *Evaluation on a generator of* $b^*\mathrm{S}^{W_a}$ *gives an isomorphism between the vector space* $\mathrm{Hom}^1_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_a)_+)$ *and* $\overline{M}_a$.

Proof. First let us translate the grading into a suspension, so that we then deal with the group $\mathrm{Hom}_{b^*b}(\Sigma^{-1}b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_a)_+)$. A b^*b-linear map from $\Sigma^{-1}b^*\mathrm{S}^{W_a}$ to $b^*\mathbb{C}\mathrm{P}(V_a)_+$ is just a b^*-linear map which is also A^*-linear. A b^*-linear map like that is the same as an element μ in $b^*\mathbb{C}\mathrm{P}(V_a)_+$ of degree one less than the degree of the generator of $b^*\mathrm{S}^{W_a}$. That is, μ has degree $(2a-1)p-3$. In other symbols,

$$\mathrm{Hom}_{b^*}(\Sigma^{-1}b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_a)_+) \cong b^{(2a-1)p-3}\mathbb{C}\mathrm{P}(V_a)_+.$$

Such an element μ in $b^*\mathbb{C}\mathrm{P}(V_a)_+$ corresponds to an A^*-linear map if and only if the Steenrod algebra acts on μ as on the generator of $b^*\mathrm{S}^{W_a}$. In order to proceed, one has to know the A^*-action on the Borel cohomology of S^{W_a}. The (real) dimension of W_a is $(2a-1)p-2$ and the (real) dimension of its fixed point set is $(2a-1)-2$. The difference is $(2a-1)(p-1)$. Thus A^* acts on a generator of $b^*\mathrm{S}^{W_a}$ as on $\tau^{(2a-1)(p-1)/2}$ in b^*. This means that β acts trivially, and P acts by multiplication with

$$(1+\tau^{p-1})^{(2a-1)(p-1)/2}.$$

Note that this element is not homogeneous, since P is not homogeneous. This can now be compared to the action of the Steenrod algebra on μ. Since the degree of μ is even, β acts trivially. Thus, the only condition is on the power operations. One has to require that

$$P(\mu) = (1+\tau^{p-1})^{(2a-1)(p-1)/2}\mu$$

in $b^*\mathbb{C}\mathrm{P}(V_{a-1})_+$ for μ to represent a b^*b-linear map. □

Motivated by the lemma, let us write

$$\epsilon_a = (2a-1)\frac{p-1}{2} = (p-1)a - \frac{p-1}{2}, \tag{4.2}$$

$$h_a = (1+\tau^{p-1})^{\epsilon_a}. \tag{4.3}$$

Thus, equation (4.1) now reads $P(\mu) = h_a\mu$.

A comment on the degrees might be appropriate. Until now, all the degrees have come from the usual topologist's grading of cohomology groups. Since our problem will eventually be reduced to polynomial algebra, it will be convenient to use algebraic degrees. Then the elements x and τ of topological degree 2 will have algebraic degree 1. The algebraic degree of μ will be written

$$\delta_a = \frac{1}{2}((2a-1)p-3) = pa - \frac{p+3}{2}. \tag{4.4}$$

From now on, the degrees used will be algebraic unless stated otherwise.

Recall from (3.3) that one has an isomorphism $b^*\mathbb{C}\mathrm{P}(V)_+ \cong b^*[x]/f(V)$ for any G-representation V. This will be used to identify the two rings. In particular, we have $b^*\mathbb{C}\mathrm{P}(V_a)_+ = b^*[x]/f(V_a)$. If we set $r = f(\mathbb{C}G)$, then

$$r = \prod_{\lambda\in\mathbb{F}}(x-\lambda\tau) = x^p - x\tau^{p-1} = x(x^{p-1}-\tau^{p-1}) \tag{4.5}$$

as in (3.2), and $f(V_a) = f(\mathbb{C}G)^a = r^a$. Note that the algebraic degree of r^a is ap.

Definition 4.3. Let M_a be the vector space of polynomials m in the subring $\mathbb{F}[\tau, x]$ of the polynomial ring $b^*[x]$ such that the algebraic degree of m is δ_a, and r^a divides $P(m) - h_a m$.

Lemma 4.4. *There is an isomorphism between M_a and $\overline{M}_a$ which is the identity on representatives.*

Proof. With the notation already established, we see that $\overline{M}_a$ is the vector space of all elements μ in $b^{2\delta_a}\mathbb{C}\mathrm{P}(V_a)_+$ for which $P(\mu) = h_a\mu$.

First note that the algebraic degree of r^a, which is ap, is larger than the algebraic degree δ_a of the μ. This shows that there are as many elements μ of that degree in $b^*\mathbb{C}\mathrm{P}(V_a)_+$ as in $\mathbb{F}[\tau, x]$. In other words, the map

$$\mathbb{F}[\tau, x] \subset b^*[x] \to b^*[x]/f(V_a) = b^*\mathbb{C}\mathrm{P}(V_a)_+$$

which is the identity on representatives is an isomorphism in this degree. In the polynomial ring, the condition $P(m) \equiv h_a m$ modulo r^a ensures that the image μ of m satisfies $P(\mu) = h_a\mu$. □

Taken together, the previous two lemmas imply the following result, which yields the translation of our problem into polynomial algebra.

Proposition 4.5. *The vector space $\mathrm{Hom}^1_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_a)_+)$ is isomorphic to the vector space M_a of polynomials m in the polynomial ring $\mathbb{F}[\tau, x]$ such that the algebraic degree of m is δ_a, and r^a divides $P(m) - h_a m$. An isomorphism is given by associating to a map the representative of evaluation on the generator.*

Proof. By Lemma 4.2, evaluation on a generator of $b^*\mathrm{S}^{W_a}$ gives an isomorphism between the vector space $\mathrm{Hom}^1_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_a)_+)$ and $\overline{M}_a$. By Lemma 4.4, there is an isomorphism between M_a and $\overline{M}_a$ which is the identity on representatives. □

§5. The algebra in the case $a = 2$

As a special case of Definition 4.3, $M_2 \subset \mathbb{F}[\tau, x]$ is the space of polynomials of degree $3(p-1)/2$ that satisfy $P(m) \equiv h_2 m$ modulo r^2. The purpose of this section is to prove the following estimate.

Proposition 5.1. *The dimension of M_2 is at least $(p+1)/2$.*

This will be achieved by first producing enough elements in M_2, and then noting that they are linearly independent.

Lemma 5.2. *For every integer k such that $0 \leq k \leq (p-1)/2$, the polynomial*

$$\tau^{(p-1)/2-k} x^k (k x^{p-1} + (1-k)\tau^{p-1})$$

is in M_2.

Proof. The cases $k = 1$ and $k = 0$ can easily be dealt with directly. We will only deal with the case $k \geq 2$.

Let m be the polynomial displayed in the proposition. Since it has the correct degree, it remains to show that r^2 divides $P(m) - h_2 m$. Set

$$E_\tau = 1 + \tau^{p-1} \quad \text{and} \quad E_x = 1 + x^{p-1}.$$

Then $P(\tau) = \tau E_\tau$, $P(x) = x E_x$, $r = x(E_x - E_\tau)$ and $h_2 = E_\tau^{3(p-1)/2}$. Rearranging the terms, one sees that $P(m) - h_2 m$ equals $\tau^{(p-1)/2-k} x^k$ times

(5.1)
$$k x^{p-1}(E_x^{p-1+k} E_\tau^{(p-1)/2-k} - E_\tau^{3(p-1)/2}) + (1-k)\tau^{p-1}(E_x^k E_\tau^{3(p-1)/2-k} - E_\tau^{3(p-1)/2}).$$

Since $k \geq 2$, the polynomial $P(m) - h_2 m$ is divisible by x^2, and so in this case it remains to show that (5.1) is divisible by $(E_x - E_\tau)^2$. Now

$$\begin{aligned} E_x^{p-1+k} E_\tau^{(p-1)/2-k} - E_\tau^{3(p-1)/2} &= E_\tau^{(p-1)/2-k}(E_x^{p-1+k} - E_\tau^{p-1+k}) \\ &= E_\tau^{(p-1)/2-k}(E_x - E_\tau) \sum_{j=0}^{p+k-2} E_x^{p+k-2-j} E_\tau^j \end{aligned}$$

and

$$E_x^k E_\tau^{3(p-1)/2-k} - E_\tau^{3(p-1)/2} = E_\tau^{3(p-1)/2-k}(E_x^k - E_\tau^k)$$
$$= E_\tau^{3(p-1)/2-k}(E_x - E_\tau)\sum_{j=0}^{k-1} E_x^{k-1-j}E_\tau^j$$

are both divisible by $E_x - E_\tau$. It remains to show that

$$kx^{p-1}E_\tau^{(p-1)/2-k}\Big(\sum_{j=0}^{p+k-2} E_x^{p+k-2-j}E_\tau^j\Big) + (1-k)\tau^{p-1}E_\tau^{3(p-1)/2-k}\Big(\sum_{j=0}^{k-1} E_x^{k-1-j}E_\tau^j\Big)$$

is divisible by $E_x - E_\tau$. But modulo $E_x - E_\tau$ we have $E_x \equiv E_\tau$, so the above is

$$kx^{p-1}E_\tau^{(p-1)/2-k}((p+k-1)E_\tau^{p+k-2}) + (1-k)\tau^{p-1}E_\tau^{3(p-1)/2-k}(kE_\tau^{k-1})$$
$$= E_\tau^{3(p-1)/2-1}(k(k-1)x^{p-1} + (1-k)k\tau^{p-1}) = E_\tau^{3(p-1)/2-1}(k^2-k)(x^{p-1}-\tau^{p-1}),$$

and $x^{p-1} - \tau^{p-1} = E_x - E_\tau \equiv 0$ modulo $E_x - E_\tau$. This finishes the proof. □

Proof of Proposition 5.1. The $(p+1)/2$ elements in the preceding lemma are linearly independent in $\mathbb{F}[\tau, x]$. This follows from an inspection of the matrix which expresses them in terms of the monomial basis: that matrix has full rank. □

§6. The algebra in the cases $a > 2$

In this section, we will see that the vector spaces M_a for $a > 2$ relate to M_2. First, some general remarks are in order.

A *linear* polynomial in $\mathbb{F}[\tau, x]$ is one of the form $\kappa\tau + \lambda x$ for some κ and λ in $\mathbb{F}$. A polynomial will be called *split* if it is a product of linear polynomials. Assume that m is split into linear factors of the form $L_j = x - \lambda_j\tau$:

$$m = \prod_j L_j.$$

Since $P(L_j) = (L_j + L_j^p) = L_j(1 + L_j^{p-1})$ and P is multiplicative, we have

$$P(m) = m\prod_j(1 + L_j^{p-1}).$$

This proves that, if m splits as above, then m divides $P(m)$. Let us write

$$Q(m) = P(m)/m$$

for the quotient in this case, and recall the definitions of the polynomial h_a and the numbers ϵ_a and δ_a from (4.3), (4.2) and (4.4). If m splits, then m divides also $P(m) - h_a m$, and the quotient is

$$Q(m) - h_a = \prod_j (1 + L_j^{p-1}) - (1 + \tau^{p-1})^{(2a-1)(p-1)/2}. \tag{6.1}$$

Let us count the number of factors in both of these terms. On the one hand, if m splits, it does so into δ_a factors. On the other hand, the polynomial h_a has ϵ_a factors $1 + \tau^{p-1}$. The difference between those two numbers is

$$\delta_a - \epsilon_a = a - 2, \tag{6.2}$$

so the numbers δ_a and ϵ_a are equal if and only if $a = 2$.

By definition (see (4.5)), the element r in $\mathbb{F}[\tau, x]$ splits. In order to give a nice formula for $Q(r)$, some more preliminaries are necessary.

Lemma 6.1. *In $\mathbb{F}[\tau, x]$, for any κ in $\mathbb{F}$, we have the equality*

$$x^p - \tau^{p-1}x = -(x - \kappa\tau)(\tau^{p-1} - (x - \kappa\tau)^{p-1}).$$

Proof. Note that

$$\prod_{\lambda \in \mathbb{F}} (x - \lambda\tau) = \prod_{\lambda \in \mathbb{F}} (x - (\kappa + \lambda)\tau) = \prod_{\lambda \in \mathbb{F}} ((x - \kappa\tau) - \lambda\tau),$$

and therefore

$$\begin{aligned} x^p - \tau^{p-1}x &= (x - \kappa\tau)^p - \tau^{p-1}(x - \kappa\tau) \\ &= -(x - \kappa\tau)(\tau^{p-1} - (x - \kappa\tau)^{p-1}), \end{aligned}$$ □

Lemma 6.2. *In $\mathbb{F}[\tau, x][K]$ we have the equality*

$$\prod_{\lambda \in \mathbb{F}} (K + (x - \lambda\tau)^{p-1}) = (x^p - \tau^{p-1}x)^{p-1} + K(K + \tau^{p-1})^{p-1}.$$

Proof. For any κ in $\mathbb{F}$ we can substitute $K = -(x - \kappa\tau)^{p-1}$ into the right hand side of the equation. We get

$$(x^p - \tau^{p-1}x)^{p-1} - (x - \kappa\tau)^{p-1}(-(x - \kappa\tau)^{p-1} + \tau^{p-1})^{p-1}.$$

But this is zero by the equality from the previous lemma, raised to the $(p-1)$-st power. This shows that the left hand side of the equation divides the right hand side. The claim follows by comparing the degrees and a coefficient. □

One gets the equality

$$\prod_{\lambda \in \mathbb{F}} (1 + (x - \lambda\tau)^{p-1}) = (x^p - \tau^{p-1}x)^{p-1} + (1 + \tau^{p-1})^{p-1}$$

by specializing to $K = 1$ in the previous lemma. Using different notation, this equation says the following.

Proposition 6.3. *For every $a \geq 2$,*

$$Q(r) = r^{p-1} + h_{a+1}/h_a. \tag{6.3}$$

Let κ be in $\mathbb{F}$. Then r is clearly divisible by $x - \kappa\tau$, but h_{a+1}/h_a is not. Therefore, (6.3) shows that $Q(r)$ is not divisible by $x - \kappa\tau$.

We can now relate the cases $a > 2$ to the case $a = 2$. Recall that M_a has been defined to be the set of polynomials m in $\mathbb{F}[\tau, x]$ of degree δ_a that satisfy the condition $P(m) \equiv h_a m$ modulo r^a.

Lemma 6.4. *If $a \geq 3$ then every element m in M_a is divisible by r.*

Proof. We can write

$$m = \sum_{i+j=\delta_a} c_{i,j}\tau^i x^j$$

with some coefficients $c_{i,j}$ in $\mathbb{F}$. Furthermore, setting $E_\tau = 1 + \tau^{p-1}$ and similarly $E_x = 1 + x^{p-1}$, we have $P(\tau) = \tau E_\tau$ as well as $P(x) = xE_x$. Consequently,

$$P(m) - h_a m = \sum_{i+j=\delta_a} c_{i,j}\tau^i x^j (E_\tau^i E_x^j - h_a).$$

By assumption on m, we have $P(m) - h_a m = r^a s$ for some s in $\mathbb{F}[\tau, x]$. Putting these together gives

$$r^a s = \sum_{i+j=\delta_a} c_{i,j}\tau^i x^j (E_\tau^i E_x^j - h_a). \tag{6.4}$$

To prove the claim, it suffices to show that $x - \kappa\tau$ divides m for each κ in $\mathbb{F}$.

First assume $\kappa \neq 0$. Modulo $x - \kappa\tau$, we have $r \equiv 0$ and $E_x \equiv E_\tau$, so that (6.4) shows that

$$0 \equiv \Big(\sum_{i+j=\delta_a} c_{i,j}\tau^i x^j\Big)(E_\tau^{\delta_a} - h_a)$$

modulo $x - \kappa\tau$. In other words, the right hand side, which is $m(E_\tau^{\delta_a} - h_a)$, is divisible by $x - \kappa\tau$. Both $E_\tau^{\delta_a}$ and h_a are powers of $1 + \tau^{p-1}$. By (6.2), the exponents differ by $a - 2$, so the assumption on a implies that $E_\tau^{\delta_a} \neq h_a$. As a consequence, $x - \kappa\tau$ does not divide $E_\tau^{\delta_a} - h_a$, so it must divide m.

It remains to show that x divides m. But modulo x, (6.4) reads

$$0 \equiv c_{\delta_a,0}\tau^{\delta_a}(E_\tau^{\delta_a} - h_a).$$

Using $E_\tau^{\delta_a} \neq h_a$ again, it follows that $c_{\delta_a,0}$ is zero. In other words, x divides m. □

Lemma 6.5. *If $3 \le a \le p$, division by r yields an injection from M_a into M_{a-1}.*

Proof. The previous result shows that every element of M_a is divisible by r. Given rm in M_a, one has to show that m is in M_{a-1}. First of all, the degree of m is correct. Since rm is in M_a, we know that $P(rm) - h_a rm$ is divisible by r^a. By assumption, r^a divides $r^p m h_{a-1}$, so that r^a divides

$$\begin{aligned} P(rm) - h_a rm - r^p m h_{a-1} &= P(rm) - (h_a + r^{p-1} h_{a-1}) rm \\ &= Q(r) r P(m) - Q(r) h_{a-1} rm \\ &= Q(r)(rP(m) - r h_{a-1} m). \end{aligned}$$

Since none of the (linear) factors of r divides $Q(r)$, we deduce that the polynomial $rP(m) - rh_{a-1}m$ must be divisible by r^a, so that r^{a-1} divides $P(m) - h_{a-1}m$, which shows that m is in M_{a-1}. Thus, the map is well-defined. Since r is not a zero-divisor, the map is injective. □

Lemma 6.6. *Let a and b be integers such that $2 \le a \le p-1$ and $a \le b$. Then multiplication with r^{b-a} is an injection from M_a into M_b.*

Proof. Injectivity is clear, if the map is well-defined at all. But, as will be shown now, it is. Equation (6.3) implies that $Q(r) \equiv h_{a+1}/h_a$ modulo r^{p-1}. Using that result $b-a$ times, we get $Q(r)^{b-a} \equiv h_b/h_a$ modulo r^{p-1}. Hence

$$h_a Q(r)^{b-a} m r^{b-a} \equiv h_b m r^{b-a}$$

modulo $r^{p-1} r^{b-a}$ and therefore, by the assumption on a, also modulo r^b. Suppose now that m is in M_a. Then mr^{b-a} has the required degree. If furthermore m satisfies $P(m) \equiv h_a m$ modulo r^a, we have

$$P(mr^{b-a}) = P(m) r^{b-a} Q(r)^{b-a} \equiv h_a m r^{b-a} Q(r)^{b-a} \equiv h_b m r^{b-a}$$

modulo r^b, as was to be shown. □

The following proposition sums up the preceding three lemmas.

Proposition 6.7. *Multiplication by r yields isomorphisms*

$$M_2 \cong M_3 \cong \cdots \cong M_{p-1} \cong M_p$$

and injections from these into M_a for every $a \ge 2$.

§7. Obstructions

As described in Section 2, the p-power torsion in $[\mathbb{C}\mathrm{P}(V_a)_+, \mathrm{S}^{W_a}]^G$ will be detected by the Borel cohomology Adams spectral sequence

$$\mathrm{Ext}^{s,t}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_a)_+) \Longrightarrow ([\mathbb{C}\mathrm{P}(V_a)_+, \mathrm{S}^{W_a}]^G_{t-s})^\wedge_p.$$

The E_2-page of that spectral sequence is hard to compute directly. Instead, we may filter the projective space $\mathbb{C}\mathrm{P}(V_a)_+$ by projective subspaces $\mathbb{C}\mathrm{P}(V)_+$ for $V \subseteq V_a$. The filtration quotients are linear G-spheres, so we may feed in results from [Szy07b] one filtration step at a time. In this section, it will be described how this can be done. For the most part, this is straightforward, and only at the end will we spot the crux of the matter. See Lemma 7.8, which uses notation introduced at the beginning of Section 7.2. The reader may wish to take this for granted for the time being and skip to the next section to see how it fits into the puzzle.

§7.1. The cases $V \subseteq V_{a-1}$

The following result is a consequence of Proposition 5 of [Szy07b] and its corollaries.

Lemma 7.1. *Let V be a complex G-representation, $L \subseteq V$ a complex line, and W any G-representation that contains $\mathrm{Hom}_{\mathbb{C}}(L, V/L)$ up to isomorphism. Then the inclusion of $\mathbb{C}\mathrm{P}(V/L)_+$ into $\mathbb{C}\mathrm{P}(V)_+$ induces an isomorphism*

$$\mathrm{Ext}^{s,t}_{b^*b}(b^*\mathrm{S}^{W}, b^*\mathbb{C}\mathrm{P}(V/L)_+) \xleftarrow{\cong} \mathrm{Ext}^{s,t}_{b^*b}(b^*\mathrm{S}^{W}, b^*\mathbb{C}\mathrm{P}(V)_+)$$

for $t - s < \dim_{\mathbb{R}} W^G - \dim_{\mathbb{R}} \mathrm{Hom}_{\mathbb{C}}(L, V/L)^G$.

An induction on the dimension of V now proves the following result.

Proposition 7.2. *Let $V \subseteq V_{a-1}$ be a complex subrepresentation. Then*

$$\mathrm{Ext}^{s,t}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V)_+) = 0$$

for $t - s \leq 0$. In particular, $\mathrm{Ext}^{s,t}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_{a-1})_+) = 0$ for $t - s \leq 0$.

As a consequence,

$$[\mathbb{C}\mathrm{P}(V_{a-1})_+, \mathrm{S}^{W_a}]^G = 0.$$

Although it will not be necessary to calculate $[\Sigma\mathbb{C}\mathrm{P}(V_{a-1})_+, \mathrm{S}^{W_a}]^G$, it will be good to know one of the groups in the relevant column of the E_2-page. These are described in the following proposition.

Proposition 7.3. *Let V be a complex subrepresentation of V_{a-2}. Then the vector space $\mathrm{Hom}^1_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V)_+)$ is zero. Let U be a complex subrepresentation of $\mathbb{C}G$, and set $V = V_{a-2} \oplus U$, so that $V_{a-2} \subseteq V \subseteq V_{a-1}$. Then*

$$\dim_{\mathbb{F}} \mathrm{Hom}^1_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V)_+) = \dim_{\mathbb{C}} U.$$

In particular, $\dim_{\mathbb{F}} \mathrm{Hom}^1_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_{a-1})_+) = p$.

Proof. The first part can also be proven by induction on the dimension of V. The result is stated in the form in which it will be used later; but the pair $(s,t) = (0,1)$ could be replaced by any pair (s,t) such that $t - s \leq 2$.

The second part can be proven by induction on $\dim_{\mathbb{C}} U$, using Propositions 5 and 9 of [Szy07b] and their corollaries. □

§7.2. The cases $V_{a-1} \subseteq V$

We may now start filtering V_a beyond V_{a-1}. Given an integer α, let $\mathbb{C}(\alpha)$ be the G-representation where a chosen generator of G acts by multiplication by $\exp(2\pi\mathrm{i}\alpha/p)$. For any integer k such that $0 \leq k \leq p$ consider the subrepresentation

$$U_k = \bigoplus_{\alpha=0}^{k-1} \mathbb{C}(\alpha)$$

of $\mathbb{C}G$. This gives a flag

$$0 = U_0 \subset U_1 \subset \cdots \subset U_p = \mathbb{C}G \tag{7.1}$$

with U_1 the trivial G-line and both $U_{(p+1)/2}/U_1$ and $U_p/U_{(p+1)/2}$ isomorphic to $\mathbb{R}G/\mathbb{R}$ as real G-representations. The flag (7.1) yields a filtration of $\mathbb{C}\mathrm{P}(V_a)_+$ by projective spaces $\mathbb{C}\mathrm{P}(V_{a-1} \oplus U_k)_+$. For $k = 0$ this is $\mathbb{C}\mathrm{P}(V_{a-1})_+$; for $k = p$ it is $\mathbb{C}\mathrm{P}(V_a)_+$. To compute the relevant part of the E_2-page of the spectral sequence for $[\mathbb{C}\mathrm{P}(V_{a-1} \oplus U_k)_+, \mathrm{S}^{W_a}]^G_*$, we will proceed inductively. The case $k = 0$ has already been settled in the previous subsection. One may therefore assume that $k \geq 1$, and that the E_2-page for $\mathbb{C}\mathrm{P}(V_{a-1} \oplus U_{k-1})_+$ has been studied. For convenience, we will write $V_{a-1,k-1}$ for $V_{a-1} \oplus U_{k-1}$ from now on, and similarly in other cases.

Let us first see what the quotient G-spheres of the filtration are. The cofibre of the inclusion of $\mathbb{C}\mathrm{P}(V_{a-1,k-1})_+$ into $\mathbb{C}\mathrm{P}(V_{a-1,k})_+$ is given by $\mathrm{S}^{\overline{H}(a-1,k)}$ with

$$\overline{H}(a-1,k) = \mathrm{Hom}_{\mathbb{C}}(U_k/U_{k-1}, V_{a-1,k-1}) \cong V_{a-1} \oplus \mathrm{Hom}_{\mathbb{C}}(U_k/U_{k-1}, U_{k-1}).$$

The cofibration sequence induces a short exact sequence in Borel cohomology. Applying the functors $\mathrm{Ext}^{s,t}_{b^*b}(b^*\mathrm{S}^{W_a}, ?)$ yields long exact sequences. In order to use them, one will have to know the groups

$$\mathrm{Ext}^{s,t}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathrm{S}^{\overline{H}(a-1,k)}) \tag{7.2}$$

for $t-s=0,1$. These groups can be simplified as follows. Since $U_k \subseteq \mathbb{C}G$, the representation $\mathrm{Hom}_{\mathbb{C}}(U_k/U_{k-1}, U_{k-1})$ has no trivial summand. If we write

$$H(k) = \mathrm{Hom}_{\mathbb{C}}(U_k/U_{k-1}, U_k) \cong \overline{H}(0,k) \oplus \mathbb{C},$$

then $H(k) \cong \mathrm{Hom}_{\mathbb{C}}(U_k/U_{k-1}, U_{k-1}) \oplus \mathbb{C}$ is a complex k-dimensional representation which has exactly one trivial summand. Because

$$H(k) \oplus V_{a-1} \cong \mathbb{C} \oplus \overline{H}(a-1,k), \qquad \mathbb{R}G \oplus V_{a-1} \cong \mathbb{C} \oplus W_a,$$

Proposition 3 of [Szy07b] implies that the group (7.2) is isomorphic to

$$\mathrm{Ext}^{s,t}_{b^*b}(b^*\mathrm{S}^{\mathbb{R}G}, b^*\mathrm{S}^{H(k)}). \tag{7.3}$$

Note that this is independent of a.

The following discussion is divided into three cases: First we deal with the cases where $k \leq (p-1)/2$, then with the case $k = (p+1)/2$, and finally with the remaining cases where $k \geq (p+3)/2$.

§7.3. The cases $k \leq (p-1)/2$

Here, $\mathrm{Ext}^{s,t}_{b^*b}(b^*\mathrm{S}^{\mathbb{R}G}, b^*\mathrm{S}^{H(k)})$ as in (7.3) is isomorphic to $\mathrm{Ext}^{s,t}_{b^*b}(b^*\mathrm{S}^{W}, b^*\mathrm{S}^{2})$ for some subrepresentation W of $\mathbb{R}G$ which properly contains the trivial subrepresentation.

Proposition 7.4. *If $k \leq (p-1)/2$ then*

$$\dim_{\mathbb{F}} \mathrm{Ext}^{s,t}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_{a-1,k})_+) = \begin{cases} 0, & t-s \leq -2, \\ k, & t-s = -1, \\ 0, & t-s = 0, \\ p, & (s,t) = (0,1). \end{cases}$$

The multiplicative structure is as expected: The extension groups belong to a Borel cohomology Adams spectral sequence that is a module over the spectral sequence $\mathrm{Ext}^{s,t}_{b^*b}(b^*, b^*) \Rightarrow ([\mathrm{S}^0, \mathrm{S}^0]^G_{t-s})^{\wedge}_p$. In the $t-s=-1$ column, multiplication with the class in $\mathrm{Ext}^{1,1}_{b^*b}(b^*, b^*)$ that represents multiplication with p is injective. In the target, this leads to a free module of rank k over the p-adic integers.

Proof of Proposition 7.4. This is again a straightforward induction on k, using the long exact sequence associated with the extension

$$0 \leftarrow b^*\mathbb{C}\mathrm{P}(V_{a-1,k-1})_+ \leftarrow b^*\mathbb{C}\mathrm{P}(V_{a-1,k})_+ \leftarrow b^*\mathrm{S}^{\overline{H}(a-1,k)} \leftarrow 0,$$

and the data from [Szy07b, Figure 8 in Section 4]. □

§7.4. The first interesting case: $k = (p+1)/2$

In this case, the sphere $\mathrm{S}^{\overline{H}(a-1,(p+1)/2)}$ is the suspension of S^{W_a}, so that there is an isomorphism $\mathrm{Ext}^{s,t}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathrm{S}^{\overline{H}(a-1,(p+1)/2)}) \cong \mathrm{Ext}^{s,t}_{b^*b}(b^*, b^*\mathrm{S}^1)$.

The long exact sequence (3.5) shows that $\mathrm{Ext}^{s,t}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_{a-1,(p+1)/2})_+)$ is zero as soon as the condition $t-s \le -2$ is fulfilled. The next case $t-s=-1$ is easy, too, since then the vector spaces $\mathrm{Ext}^{s-1,t}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_{a-1,(p-1)/2})_+)$ and $\mathrm{Ext}^{s+1,t}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathrm{S}^{\overline{H}(a-1,(p+1)/2)})$ both vanish. In this way we may therefore deduce that the vector space $\mathrm{Ext}^{s,t}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_{a-1,(p+1)/2})_+)$ is an extension of the vector space $\mathrm{Ext}^{s,t}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_{a-1,(p-1)/2})_+)$ by the vector space $\mathrm{Ext}^{s+1,t}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathrm{S}^{\overline{H}(a-1,(p+1)/2)})$. Using the data from [Szy07b, Figure 4 in Section 1], one obtains

$$(7.4)\qquad \dim_{\mathbb{F}} \mathrm{Ext}^{s,t}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_{a-1,(p+1)/2})_+) = \begin{cases} (p+1)/2, & s=0,\\ (p+3)/2, & s \ge 1.\end{cases}$$

The multiplicative structure is again as expected: In column $t-s=-1$, multiplication with the generator of $\mathrm{Ext}^{1,1}$ which represents multiplication with p is injective. Starting with $t-s=0$, the situation becomes more interesting. Then the map

$$\mathrm{Ext}^{s,t}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_{a-1,(p+1)/2})_+) \leftarrow \mathrm{Ext}^{s,t}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathrm{S}^{\overline{H}(a-1,(p+1)/2)})$$

is surjective, since $\mathrm{Ext}^{s,t}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_{a-1,(p-1)/2})_+)$ is zero. As the right hand side is non-zero only for $(s,t)=(1,1)$, so is $\mathrm{Ext}^{s,t}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_{a-1,(p+1)/2})_+)$. Thus, it remains to determine $\mathrm{Ext}^{1,1}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_{a-1,(p+1)/2})_+)$.

The vector space $\mathrm{Ext}^{1,1}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathrm{S}^{\overline{H}(a-1,(p+1)/2)})$ is 1-dimensional. The kernel of the surjection displayed right above is isomorphic—via the boundary homomorphism of the long exact sequence—to the cokernel of the map induced by the homomorphism $\mathrm{Hom}^1_{b^*b}(b^*\mathrm{S}^{W_a}, ?)$. But that induced map is injective, since the group $\mathrm{Hom}^1_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathrm{S}^{\overline{H}(a-1,(p+1)/2)})$ is zero. To summarise:

Lemma 7.5. *The vector space* $\mathrm{Ext}^{1,1}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_{a-1,(p+1)/2})_+)$ *is either* 1*-dimensional or zero, depending on whether the injection*

$$\mathrm{Hom}^1_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_{a-1,(p+1)/2})_+) \to \mathrm{Hom}^1_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_{a-1,(p-1)/2})_+)$$

is also surjective (*and therefore an isomorphism*) *or not* (*in which case the cokernel is* 1*-dimensional*)*.*

By Proposition 7.4, the vector space $\mathrm{Hom}^1_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_{a-1,(p-1)/2})_+)$ has dimension p. Thus it would be sufficient to know the dimension of the other vector space $\mathrm{Hom}^1_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_{a-1,(p+1)/2})_+)$ in order to determine the dimension of the space $\mathrm{Ext}^{1,1}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_{a-1,(p+1)/2})_+)$. For the moment, let us leave it like that and see how we can proceed.

§7.5. The final cases: $k \geq (p+3)/2$

In these cases the vector spaces $\mathrm{Ext}^{s,t}_{b^*b}(b^*\mathrm{S}^{\mathbb{R}G}, b^*\mathrm{S}^{H(k)})$ as in (7.3) are isomorphic to $\mathrm{Ext}^{s,t}_{b^*b}(b^*, b^*\mathrm{S}^V)$ for some subrepresentation $V \subseteq \mathbb{R}G$ properly containing the trivial representation. The calculations summarised in [Szy07b, Figure 6 of Section 3] are relevant here.

To determine the dimension of $\mathrm{Ext}^{s,t}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_{a-1,k})_+)$, the long exact sequence (3.5) will be invoked again. Some of these groups will be non-zero for $t-s \leq -2$, since this holds for $\mathrm{Ext}^{s+1,t}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathrm{S}^{\overline{H}(a-1,k)})$. One can ignore these, since eventually only the case $t-s=0$ is of interest. For the latter, it is good to know the groups with $t-s=-1$. In this case, the groups $\mathrm{Ext}^{s-1,t}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_{a-1,k-1})_+)$ vanish except for the case $s=2$ and $t=1$ that we will discuss below, around (7.5). Also $\mathrm{Ext}^{s+1,t}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathrm{S}^{\overline{H}(a-1,k)}) = 0$. As in the previous case, we will get a splittable short exact sequence. By induction, using (7.4), this yields

$$\dim_{\mathbb{F}} \mathrm{Ext}^{s,t}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_{a-1,k})_+) = \begin{cases} (p+1)/2, & s=0, \\ k+1, & s \geq 1. \end{cases}$$

The multiplicative structure is again as expected: In column $t-s=-1$, multiplication with the generator of $\mathrm{Ext}^{1,1}$ which represents multiplication with p is injective.

Now let us turn to the most interesting situation: $t-s=0$. If in addition $s \neq 1$, the groups $\mathrm{Ext}^{s,t}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathrm{S}^{\overline{H}(a-1,k)})$ are zero. Using that and the corresponding result from the first interesting case as an input, an induction shows that the groups $\mathrm{Ext}^{s,t}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_{a-1,k})_+)$ vanish for $s \neq 1$. For $k=p$ this implies the following two results.

Proposition 7.6. *All p-power torsion in $[\mathbb{C}\mathrm{P}(V_a)_+, \mathrm{S}^{W_a}]^G$ has order p.*

Therefore, the p-adic completion of that group is elementary abelian of some rank r. Eventually, we will prove upper and lower bounds on r (see Section 8).

Proposition 7.7. *The non-trivial elements in $[\mathbb{C}\mathrm{P}(V_a)_+, \mathrm{S}^{W_a}]^G$ are detected by their Borel cohomology e-invariants.*

It remains to discuss the vector spaces $\mathrm{Ext}^{1,1}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_{a-1,k})_+)$. This is a bit more complicated than in the first interesting case since this time the vector space $\mathrm{Ext}^{1,1}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_{a-1,k-1})_+)$ may already be non-zero. Moreover, the boundary homomorphism

$$\mathrm{Ext}^{2,1}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathrm{S}^{\overline{H}(a-1,k)}) \leftarrow \mathrm{Ext}^{1,1}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_{a-1,k-1})_+) \tag{7.5}$$

maps into a non-zero group. Nevertheless, it is the zero map. This follows from the multiplicative structure and the fact that the boundary map respects it. Thus, while the argument is a little more complicated than in the first interesting case, the result is the same:

Lemma 7.8. *For every integer k such that $(p+1)/2 \le k \le p$, the map*

$$\mathrm{Ext}^{1,1}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_{a-1,k-1})_+) \leftarrow \mathrm{Ext}^{1,1}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_{a-1,k})_+)$$

is a surjection. The kernel of this homomorphism is either 1-dimensional or zero, depending on whether the injection from $\mathrm{Hom}^1_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_{a-1,k})_+)$ into $\mathrm{Hom}^1_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_{a-1,k-1})_+)$ is also surjective (and therefore an isomorphism) or not (in which case the cokernel is 1-dimensional).

§8. Upper and lower bounds

We are now able to put together the algebraic calculations from Sections 4–6 and the obstruction theory of the previous section to prove our main result, Theorem 8.2. The following result summarises Propositions 4.5, 5.1, and 6.7.

Proposition 8.1. *For any integer $a \ge 2$, the dimension of the vector space*

$$\mathrm{Hom}^1_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_a)_+)$$

is at least $(p+1)/2$.

In Section 7 we have investigated two chains of homomorphisms, displayed in Figure 8.1 below. Lemmas 7.5 and 7.8 show that the homomorphisms on the left are surjective and each kernel is at most 1-dimensional. The maps on the right are injections, and each cokernel is at most 1-dimensional. A map on the left is an isomorphism if and only if the corresponding map on the right is not.

Theorem 8.2. *The p-power torsion of the group $[\mathbb{C}\mathrm{P}(V_a)_+, \mathrm{S}^{W_a}]^G$ is non-zero elementary abelian of rank r with $1 \le r \le (p+1)/2$.*

Proof. We know from Proposition 7.6 that the group $[\mathbb{C}\mathrm{P}(V_a)_+, \mathrm{S}^{W_a}]^G$ is elementary abelian.

As for the upper bound on its rank, the length of the chain on the left of Figure 8.1, together with Lemmas 7.5 and 7.8, implies that the dimension of the vector space $\mathrm{Ext}^{1,1}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_a)_+)$ is at most $(p+1)/2$.

$$\begin{array}{cc}
\mathrm{Ext}^{1,1}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_{a-1,(p-1)/2})_+) & \mathrm{Hom}^1_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_{a-1,(p-1)/2})_+) \\
\uparrow & \uparrow \text{Lemma 7.5} \\
\mathrm{Ext}^{1,1}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_{a-1,(p+1)/2})_+) & \mathrm{Hom}^1_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_{a-1,(p+1)/2})_+) \\
\uparrow & \uparrow \\
\vdots & \vdots \\
\uparrow & \uparrow \\
\mathrm{Ext}^{1,1}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_{a-1,p})_+) & \mathrm{Hom}^1_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_{a-1,p})_+)
\end{array}$$

Figure 8.1. Two chains of homomorphisms

In order to obtain the lower bound on the p-torsion of $[\mathbb{C}\mathrm{P}(V_a)_+, \mathrm{S}^{W_a}]^G$, we can use Proposition 8.1, which states that for any integer $a \geq 2$, the dimension of the vector space

$$\mathrm{Hom}^1_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_a)_+)$$

is a least $(p+1)/2$. This implies that one of the inclusions in the right chain must be an isomorphism. Therefore, one of the surjections on the left cannot be an isomorphism. So the group $\mathrm{Ext}^{1,1}_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_a)_+)$ is non-zero. By multiplicativity, all the differentials must be zero on that group. Thus, the elements survive to the E_∞-page. □

Remark 8.3. There are indications that the upper bound given in Theorem 8.2 is far from being sharp. Using methods similar to those of Section 5, it can be improved roughly by a factor of 2. My results in this direction do not seem to justify a detailed account, in particular as I suspect that the p-torsion in $[\mathbb{C}\mathrm{P}(V_a)_+, \mathrm{S}^{W_a}]^G$ is isomorphic to $\mathbb{Z}/p$ for all primes $p \geq 3$ and all integers $a \geq 2$. While I do not know how to prove this, it is consistent with computer experiments covering all the cases with $pa \leq 50$.

§9. Blind alleys and dead ends

In this section, we will pursue the question of how the groups $[\mathbb{C}\mathrm{P}(V_a)_+, \mathrm{S}^{W_a}]^G$ may be related for varying $a \geq 2$. It follows from Propositions 4.5 and 6.7 that there are isomorphisms

$$\begin{aligned}\mathrm{Hom}^1_{b^*b}(b^*\mathrm{S}^{W_2}, b^*\mathbb{C}\mathrm{P}(V_2)_+) &\cong \mathrm{Hom}^1_{b^*b}(b^*\mathrm{S}^{W_3}, b^*\mathbb{C}\mathrm{P}(V_3)_+) \cong \cdots \\ &\cong \mathrm{Hom}^1_{b^*b}(b^*\mathrm{S}^{W_p}, b^*\mathbb{C}\mathrm{P}(V_p)_+),\end{aligned}$$

and injections from these into $\mathrm{Hom}^1_{b^*b}(b^*\mathrm{S}^{W_a}, b^*\mathbb{C}\mathrm{P}(V_a)_+)$ for every $a \geq 2$. As in the proof of Theorem 8.2, this implies that there are p-local isomorphisms

$$[\mathbb{C}\mathrm{P}(V_2)_+, \mathrm{S}^{W_2}]^G \cong [\mathbb{C}\mathrm{P}(V_3)_+, \mathrm{S}^{W_3}]^G \cong \cdots \cong [\mathbb{C}\mathrm{P}(V_p)_+, \mathrm{S}^{W_p}]^G,$$

and injections from these into $[\mathbb{C}\mathrm{P}(V_a)_+, \mathrm{S}^{W_a}]^G$ for every $a \geq 2$. However, the morphisms are given algebraically on the level of Adams spectral sequences by multiplication with a class originating from the regular representation. It would be enlightening to see a more geometric and less computational explanation of the phenomenon. However, there are some blind alleys and dead ends on the way towards such an interpretation, and it seems only fair to disclose three of them here.

Remark 9.1. First, note that $[\mathbb{C}\mathrm{P}(V_a)_+, \mathrm{S}^{W_a}]^G \not\cong [\mathbb{C}\mathrm{P}(V_{a+1})_+, \mathrm{S}^{W_{a+1}}]^G$ integrally: just compute the structure of these groups away from p. Thus the phenomenon is genuinely p-local.

Remark 9.2. Second, the groups $[\mathbb{C}\mathrm{P}(V_a)_+, \mathrm{S}^{W_a}]^G$ for varying a are related as shown in the following commutative diagram, in which all the arrows are induced by inclusions.

$$\begin{array}{ccc}
[\mathbb{C}\mathrm{P}(V_a)_+, \mathrm{S}^{W_a}]^G & \longrightarrow & [\mathbb{C}\mathrm{P}(V_a)_+, \mathrm{S}^{W_{a+1}}]^G \\
\uparrow & & \uparrow \\
[\mathbb{C}\mathrm{P}(V_{a+1})_+, \mathrm{S}^{W_a}]^G & \longrightarrow & [\mathbb{C}\mathrm{P}(V_{a+1})_+, \mathrm{S}^{W_{a+1}}]^G
\end{array}$$

However it is not possible to explain the phenomenon from this point of view: The horizontal maps are zero, since they are multiplication with the Euler class of $\mathbb{C}G$, which has non-zero fixed points. The group $[\mathbb{C}\mathrm{P}(V_{a+1})_+, \mathrm{S}^{W_a}]^G$ seems to be even more difficult to compute than the other three, whereas $[\mathbb{C}\mathrm{P}(V_a)_+, \mathrm{S}^{W_{a+1}}]^G$ is zero.

Remark 9.3. Third, one might wonder whether, after translating the situation into a $(\mathbb{T} \times G)$-equivariant setting, a suspension isomorphism could be used to prove Theorem 1.3 or Theorem 8.2. But this is not the case: in the $(\mathbb{T} \times G)$-equivariant setting, both sides differ by a $2p$-dimensional sphere, but $\mathbb{T}$ acts trivially on one of them and non-trivially on the other.

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References

[Ati61] M. F. Atiyah, Characters and cohomology of finite groups, Publ. Math. Inst. Hautes Études Sci. **9** (1961), 23–64. Zbl 0107.02303 MR 0148722

[Chr98] J. D. Christensen, Ideals in triangulated categories: phantoms, ghosts and skeleta, Adv. Math. **136** (1998), 284–339. Zbl 0928.55010 MR 1626856

[GS] D. R. Grayson and M. E. Stillman, Macaulay2, a software system for research in algebraic geometry, www.math.uiuc.edu/Macaulay2

[Gre88] J. P. C. Greenlees, Stable maps into free G-spaces, Trans. Amer. Math. Soc. **310** (1988), 199–215. Zbl 0706.55007 MR 0938918

[GW08] J. P. C. Greenlees and G. R. Williams, Poincaré duality for K-theory of equivariant complex projective spaces, Glasgow Math. J.**50** (2008), 111–127. Zbl 1142.57019 MR 2381737

[HHR10] M. A. Hill, M. J. Hopkins and D. C. Ravenel, The Arf–Kervaire invariant problem in algebraic topology: introduction, in *Current developments in mathematics, 2009*, Int. Press, Somerville, MA, 2010, 23–57. Zbl 1223.55009 MR 2757358

[HHR11a] ———, The Arf–Kervaire problem in algebraic topology: sketch of the proof, in *Current developments in mathematics, 2010*, Int. Press, Somerville, MA, 2011, 1–43. Zbl 1249.55005 MR 2906370

[HHR11b] ———, A solution to the Arf–Kervaire invariant problem, in *Proceedings of the Gökova Geometry-Topology Conference 2010*, Int. Press, Somerville, MA, 2011, 21–63. Zbl 1246.55014 MR 2931879

[HHR] ———, On the non-existence of elements of Kervaire invariant one, arXiv:0908.3724 (2009).

[LMS86] L. G. Lewis, J. P. May and M. Steinberger, *Equivariant stable homotopy theory*, Lecture Notes in Math. 1213, Springer, Berlin, 1986. Zbl 0611.55001 MR 0866482

[May96] J. P. May, *Equivariant homotopy and cohomology theory*, CBMS Reg. Conf. Ser. Math. 91, Amer. Math. Soc., Providence, RI, 1996. Zbl 0890.55001 MR 1413302

[Mil12] H. R. Miller, Kervaire invariant one [after M. A. Hill, M. J. Hopkins, and D. C. Ravenel], in *Séminaire Bourbaki: Vol. 2010/2011. Exposés 1027–1042*, Astérisque **348** (2012), exp. 1029, vii, 65–98. Zbl 06149577 MR 3050712

[Nak09a] N. Nakamura, Mod p vanishing theorem of Seiberg–Witten invariants for 4-manifolds with $\mathbb{Z}/p$-actions, Asian J. Math. **10** (2006), 731–748. Zbl 1129.57038 MR 2282361

[Nak09b] ———, Bauer–Furuta invariants under $\mathbb{Z}/2$-actions, Math. Z. **262** (2009), 219–233. Zbl 1196.57030 MR 2491607

[Nak10] ———, Smoothability of $\mathbb{Z} \times \mathbb{Z}$-actions on 4-manifolds, Proc. Amer. Math. Soc. **138** (2010), 2973–2978. Zbl 1205.57020 MR 2644908

[Nak14] ———, Mod p equality theorem for Seiberg–Witten invariants under $\mathbb{Z}/p$-actions, Tokyo J. Math. **37** (2014), 21–29. Zbl 1302.14037 MR 3264511

[Szy07a] M. Szymik, A stable approach to the equivariant Hopf theorem, Topology Appl. **154** (2007), 2323–2332. Zbl 1125.57016 MR 2328015

[Szy07b] ———, Equivariant stable stems for prime order groups, J. Homotopy Related Structures **2** (2007), 141–162. Zbl 1196.55019 MR 2369156

[Szy12] ———, Bauer–Furuta invariants and Galois symmetries, Quart. J. Math. **63** (2012), 1033–1054. Zbl 1275.57041 MR 2999996