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Self-dual t-structure

by

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Abstract

We give a self-dual t-structure on the derived category of $\mathbb{R}$-constructible sheaves over any Noetherian regular ring by generalizing the notion of t-structure.

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Introduction

Let X be a complex manifold and let $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbf{k}_X)$ be the derived category of sheaves of $\mathbf{k}$-vector spaces on X with $\mathbb{C}$-constructible cohomology. Here $\mathbf{k}$ is a given base field. Then the t-structure $({}^{\mathrm{p}}\mathrm{D}^{\leq 0}_{\mathbb{C}\text{-c}}(\mathbf{k}_X), {}^{\mathrm{p}}\mathrm{D}^{\geq 0}_{\mathbb{C}\text{-c}}(\mathbf{k}_X))$ on $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbf{k}_X)$ with middle perversity is self-dual with respect to the Verdier dual functor $\mathsf{D}_X = \mathrm{R}\mathscr{H}om(\,\bullet\,, \omega_X)$. Namely, the Verdier dual functor exchanges ${}^{\mathrm{p}}\mathrm{D}^{\leq 0}_{\mathbb{C}\text{-c}}(\mathbf{k}_X)$ and ${}^{\mathrm{p}}\mathrm{D}^{\geq 0}_{\mathbb{C}\text{-c}}(\mathbf{k}_X)$. However, on a real analytic manifold X (of positive dimension), no perversity gives a self-dual t-structure on the derived category $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_X)$ of $\mathbb{R}$-constructible sheaves on X. In this paper, we construct such a self-dual t-structure after generalizing the notion of t-structure. This generalized notion already appeared in the paper of Bridgeland [2] on stability conditions (see also [4]). This construction can also be applied to the derived category $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(A)$ of finitely generated modules over a Noetherian regular ring A. We construct a (generalized) t-structure on $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(A)$ which is self-dual with respect to the duality functor $\mathrm{RHom}_A(\,\bullet\,, A)$.

Let us explain our results more precisely with the example of $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_X)$. Let X be a real analytic manifold. Recall that a sheaf F of $\mathbf{k}$-vector spaces is called $\mathbb{R}$-*constructible* if X is a locally finite union of locally closed subanalytic subsets $\{X_\alpha\}_\alpha$ such that all the restrictions $F|_{X_\alpha}$ are locally constant with finite-dimensional fibers. Let $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_X)$ be the bounded derived category of $\mathbb{R}$-construc-

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tible sheaves. Let $\mathsf{D}_X = \mathrm{R}\mathscr{H}om(\bullet, \omega_X)$ be the Verdier dual functor. For $c \in \mathbb{R}$, we define

$$(0.1)\quad \begin{aligned} {}^{1/2}\mathrm{D}^{\le c}_{\mathbb{R}\text{-c}}(\mathbf{k}_X) &:= \{K \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_X) \mid \dim \operatorname{Supp}(H^i K) \le 2(c-i) \text{ for any } i \in \mathbb{Z}\},\\ {}^{1/2}\mathrm{D}^{\ge c}_{\mathbb{R}\text{-c}}(\mathbf{k}_X) &:= \{K \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_X) \mid \mathsf{D}_X K \in {}^{1/2}\mathrm{D}^{\le -c}_{\mathbb{R}\text{-c}}(\mathbf{k}_X)\}. \end{aligned}$$

Then, the pair $(({}^{1/2}\mathrm{D}^{\le c}_{\mathbb{R}\text{-c}}(\mathbf{k}_X))_{c\in\mathbb{R}}, ({}^{1/2}\mathrm{D}^{\ge c}_{\mathbb{R}\text{-c}}(\mathbf{k}_X))_{c\in\mathbb{R}})$ satisfies the axioms of (generalized) t-structure (Definition 1.2). In particular, $({}^{1/2}\mathrm{D}^{\le c}_{\mathbb{R}\text{-c}}(\mathbf{k}_X), {}^{1/2}\mathrm{D}^{> c-1}_{\mathbb{R}\text{-c}}(\mathbf{k}_X))$ is a t-structure in the ordinary sense for any $c \in \mathbb{R}$. Here ${}^{1/2}\mathrm{D}^{> c}_{\mathbb{R}\text{-c}}(\mathbf{k}_X) := \bigcup_{b>c} {}^{1/2}\mathrm{D}^{\ge b}_{\mathbb{R}\text{-c}}(\mathbf{k}_X)$. Therefore, for any $K \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_X)$ and $c \in \mathbb{R}$, there exists a distinguished triangle $K' \to K \to K'' \xrightarrow{+1}$ in $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_X)$ such that $K' \in {}^{1/2}\mathrm{D}^{\le c}_{\mathbb{R}\text{-c}}(\mathbf{k}_X)$ and $K'' \in {}^{1/2}\mathrm{D}^{> c}_{\mathbb{R}\text{-c}}(\mathbf{k}_X)$.

Note that ${}^{1/2}\mathrm{D}^{\le c}_{\mathbb{R}\text{-c}}(\mathbf{k}_X) = {}^{1/2}\mathrm{D}^{\le s}_{\mathbb{R}\text{-c}}(\mathbf{k}_X)$ for $s \in \frac{1}{2}\mathbb{Z}$ such that $s \le c < s + 1/2$, and ${}^{1/2}\mathrm{D}^{\ge c}_{\mathbb{R}\text{-c}}(\mathbf{k}_X) = {}^{1/2}\mathrm{D}^{\ge s}_{\mathbb{R}\text{-c}}(\mathbf{k}_X)$ for $s \in \frac{1}{2}\mathbb{Z}$ such that $s - 1/2 < c \le s$.

This paper is organized as follows. In Section 1, we generalize the notion of a t-structure. In Section 2, we recall the result of [4] on a t-structure on the derived category of a quasi-abelian category. In Section 3, we give the t-structure associated with a torsion pair on an abelian category.

In Section 4, we define a self-dual t-structure on the derived category of coherent sheaves on a Noetherian regular scheme.

In Section 5, we give two t-structures on the derived category of the abelian category of $\mathbb{R}$-constructible sheaves of A-modules on a subanalytic space X. Here A is a Noetherian regular ring. One is purely topological and the other is self-dual with respect to the Verdier duality functor.

In Section 6, we study the self-dual t-structure on the derived category of the abelian category of sheaves of A-modules on a complex manifold X with $\mathbb{C}$-constructible cohomology. The main result is its microlocal characterization (Theorem 6.2).

Convention. In this paper, all subanalytic spaces and complex analytic spaces are assumed to be Hausdorff, locally compact, countable at infinity and with finite dimension.

§1. (Generalized) t-structure

Since the following lemma is elementary, we omit its proof.

Lemma 1.1. *Let X be a set.*

(i) *Let $(X^{\le c})_{c\in\mathbb{R}}$ be a family of subsets of X such that $X^{\le c} = \bigcap_{b>c} X^{\le b}$ for any $c \in \mathbb{R}$. Set $X^{<c} := \bigcup_{b<c} X^{\le b}$. Then*

(a) $X^{<c} = \bigcup_{b<c} X^{<b}$,
(b) $X^{\le c} = \bigcap_{b>c} X^{<b}$.

(ii) *Conversely, let* $(X^{<c})_{c\in\mathbb{R}}$ *be a family of subsets of* X *such that* $X^{<c} = \bigcup_{b<c} X^{<b}$ *for any* $c \in \mathbb{R}$. *Set* $X^{\le c} := \bigcap_{b>c} X^{<b}$. *Then*

(a) $X^{\le c} = \bigcap_{b>c} X^{\le b}$,
(b) $X^{<c} = \bigcup_{b<c} X^{\le b}$.

(iii) *Let* $(X^{\le c})_{c\in\mathbb{R}}$ *and* $(X^{<c})_{c\in\mathbb{R}}$ *be as above. Let* $a, b \in \mathbb{R}$ *be such that* $a < b$. *If* $X^{<c} = X^{\le c}$ *for any* c *such that* $a < c \le b$, *then* $X^{\le a} = X^{\le b}$.

Let us recall the notion of t-structure (see [1]). Let $\mathcal{T}$ be a triangulated category. Let $\mathcal{T}^{\le 0}$ and $\mathcal{T}^{\ge 0}$ be strictly full subcategories of $\mathcal{T}$. Here, a subcategory $\mathcal{C}'$ of a category $\mathcal{C}$ is called *strictly full* if it is full, i.e. $\mathrm{Hom}_{\mathcal{C}'}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)$ for any $X, Y \in \mathcal{C}'$, and any object of $\mathcal{C}$ isomorphic to some object of $\mathcal{C}'$ is an object of $\mathcal{C}'$.

For $n \in \mathbb{Z}$, we set $\mathcal{T}^{\le n} = \mathcal{T}^{\le 0}[-n]$ and $\mathcal{T}^{\ge n} = \mathcal{T}^{\ge 0}[-n]$. Let us recall that $(\mathcal{T}^{\le 0}, \mathcal{T}^{\ge 0})$ is a *t-structure* on $\mathcal{T}$ if:

(1.1) (a) $\mathcal{T}^{\le 0} \subset \mathcal{T}^{\le 1}$ and $\mathcal{T}^{\ge 1} \subset \mathcal{T}^{\ge 0}$,
(b) $\mathrm{Hom}_{\mathcal{T}}(X,Y) = 0$ for $X \in \mathcal{T}^{\le 0}$ and $Y \in \mathcal{T}^{\ge 1}$,
(c) for any $X \in \mathcal{T}$, there exists a distinguished triangle $X_0 \to X \to X_1 \xrightarrow{+1}$ in $\mathcal{T}$ such that $X_0 \in \mathcal{T}^{\le 0}$ and $X_1 \in \mathcal{T}^{\ge 1}$.

We shall generalize this notion.

Definition 1.2. Let $(\mathcal{T}^{\le c})_{c\in\mathbb{R}}$ and $(\mathcal{T}^{\ge c})_{c\in\mathbb{R}}$ be families of strictly full subcategories of a triangulated category $\mathcal{T}$, and set $\mathcal{T}^{<c} = \bigcup_{b<c} \mathcal{T}^{\le b}$ and $\mathcal{T}^{>c} = \bigcup_{b>c} \mathcal{T}^{\ge b}$. We say that $((\mathcal{T}^{\le c})_{c\in\mathbb{R}}, (\mathcal{T}^{\ge c})_{c\in\mathbb{R}})$ is a (*generalized*) *t-structure* (cf. [2]) if

(1.2) (a) $\mathcal{T}^{\le c} = \bigcap_{b>c} \mathcal{T}^{\le b}$ and $\mathcal{T}^{\ge c} = \bigcap_{b<c} \mathcal{T}^{\ge b}$ for any $c \in \mathbb{R}$,
(b) $\mathcal{T}^{\le c+1} = \mathcal{T}^{\le c}[-1]$ and $\mathcal{T}^{\ge c+1} = \mathcal{T}^{\ge c}[-1]$ for any $c \in \mathbb{R}$,
(c) $\mathrm{Hom}_{\mathcal{T}}(X,Y) = 0$ for any $c \in \mathbb{R}$, $X \in \mathcal{T}^{<c}$ and $Y \in \mathcal{T}^{>c}$,
(d) for any $X \in \mathcal{T}$ and $c \in \mathbb{R}$, there exist distinguished triangles $X_0 \to X \to X_1 \xrightarrow{+1}$ and $X_0' \to X \to X_1' \xrightarrow{+1}$ in $\mathcal{T}$ such that $X_0 \in \mathcal{T}^{\le c}$, $X_1 \in \mathcal{T}^{>c}$ and $X_0' \in \mathcal{T}^{<c}$, $X_1' \in \mathcal{T}^{\ge c}$.

Note that under conditions (a)–(c), the distinguished triangles in (d) are unique up to a unique isomorphism.

If $((\mathcal{T}^{\leq c})_{c\in\mathbb{R}},(\mathcal{T}^{\geq c})_{c\in\mathbb{R}})$ is a generalized t-structure, then the pairs $(\mathcal{T}^{\leq c},\mathcal{T}^{>c-1})$ and $(\mathcal{T}^{<c},\mathcal{T}^{\geq c-1})$ are t-structures in the original sense for any $c\in\mathbb{R}$. Hence, $\mathcal{T}^{\leq c}\cap\mathcal{T}^{>c-1}$ and $\mathcal{T}^{<c}\cap\mathcal{T}^{\geq c-1}$ are abelian categories.

Assume that $((\mathcal{T}^{\leq c})_{c\in\mathbb{R}},(\mathcal{T}^{\geq c})_{c\in\mathbb{R}})$ is a generalized t-structure. Then the inclusion functors $\mathcal{T}^{\leq c}\to\mathcal{T}$ and $\mathcal{T}^{<c}\to\mathcal{T}$ have respective right adjoints

$$\tau^{\leq c}\colon \mathcal{T}\to\mathcal{T}^{\leq c}\quad\text{and}\quad \tau^{<c}\colon \mathcal{T}\to\mathcal{T}^{<c}.$$

Similarly, the inclusion functors $\mathcal{T}^{\geq c}\to\mathcal{T}$ and $\mathcal{T}^{>c}\to\mathcal{T}$ have respective left adjoints

$$\tau^{\geq c}\colon \mathcal{T}\to\mathcal{T}^{\geq c}\quad\text{and}\quad \tau^{>c}\colon \mathcal{T}\to\mathcal{T}^{>c}.$$

We have distinguished triangles functorially in $X\in\mathcal{T}$:

$$\tau^{\leq c}X\to X\to\tau^{>c}X\xrightarrow{+1}\quad\text{and}\quad\tau^{<c}X\to X\to\tau^{\geq c}X\xrightarrow{+1}.$$

These four functors are called the *truncation functors* of the generalized t-structure $((\mathcal{T}^{\leq c})_{c\in\mathbb{R}},(\mathcal{T}^{\geq c})_{c\in\mathbb{R}})$.

For any $a,b\in\mathbb{R}$, we have isomorphisms of functors

$$\tau^{\leq a}\circ\tau^{\leq b}\simeq\tau^{\leq\min(a,b)},\quad \tau^{\geq a}\circ\tau^{\geq b}\simeq\tau^{\geq\max(a,b)},\quad \tau^{\leq a}\circ\tau^{\geq b}\simeq\tau^{\geq b}\circ\tau^{\leq a}.$$

In the last formula, we can replace $\tau^{\geq a}$ with $\tau^{>a}$ or $\tau^{\leq b}$ with $\tau^{<b}$. For any $c\in\mathbb{R}$, we have

$$\begin{aligned}
\mathcal{T}^{\leq c}&=\{X\in\mathcal{T}\mid \mathrm{Hom}_{\mathcal{T}}(X,Y)\simeq 0\text{ for any }Y\in\mathcal{T}^{>c}\},\\
\mathcal{T}^{<c}&=\{X\in\mathcal{T}\mid \mathrm{Hom}_{\mathcal{T}}(X,Y)\simeq 0\text{ for any }Y\in\mathcal{T}^{\geq c}\},\\
\mathcal{T}^{\geq c}&=\{Y\in\mathcal{T}\mid \mathrm{Hom}_{\mathcal{T}}(X,Y)\simeq 0\text{ for any }X\in\mathcal{T}^{<c}\},\\
\mathcal{T}^{>c}&=\{Y\in\mathcal{T}\mid \mathrm{Hom}_{\mathcal{T}}(X,Y)\simeq 0\text{ for any }X\in\mathcal{T}^{\leq c}\}.
\end{aligned}\tag{1.3}$$

We set $\mathcal{T}^c:=\mathcal{T}^{\leq c}\cap\mathcal{T}^{\geq c}$. Then $\mathcal{T}^c$ is a quasi-abelian category (see [2] and [6]). More generally, for $a\leq b$, we set

$$\mathcal{T}^{[a,b]}:=\mathcal{T}^{\leq b}\cap\mathcal{T}^{\geq a}.$$

Then $\mathcal{T}^{[a,b]}$ is a quasi-abelian category if $a\leq b<a+1$.

A t-structure $(\mathcal{T}^{\leq 0},\mathcal{T}^{\geq 0})$ is regarded as a generalized t-structure by setting

$$\begin{aligned}
\mathcal{T}^{\leq c}&=\mathcal{T}^{\leq 0}[-n]\quad\text{for }n\in\mathbb{Z}\text{ such that }n\leq c<n+1,\\
\mathcal{T}^{\geq c}&=\mathcal{T}^{\geq 0}[-n]\quad\text{for }n\in\mathbb{Z}\text{ such that }n-1<c\leq n.
\end{aligned}\tag{1.4}$$

Hence, a t-structure is nothing but a generalized t-structure such that $\mathcal{T}^{\leq 0}=\mathcal{T}^{<1}$ and $\mathcal{T}^{\geq 1}=\mathcal{T}^{>0}$, or equivalently $\mathcal{T}^c=0$ for any $c\notin\mathbb{Z}$.

In the following, we call a generalized t-structure simply a t-structure.

Remark 1.3. In the examples we give in this paper, the t-structures also satisfy the following condition:

(e) for any $c \in \mathbb{R}$ we can find a and b such that $a < c < b$ and
  (1) $\mathcal{T}^{<c} = \mathcal{T}^{\leq a}$, $\mathcal{T}^{\leq c} = \mathcal{T}^{<b}$,
  (2) $\mathcal{T}^{>c} = \mathcal{T}^{\geq b}$, $\mathcal{T}^{\geq c} = \mathcal{T}^{>a}$.

More precisely, in the examples in this paper, we can take $a = \max\{s \in \frac{1}{2}\mathbb{Z} \mid s < c\}$ and $b = \min\{s \in \frac{1}{2}\mathbb{Z} \mid s > c\}$. Hence $\mathcal{T}^c = 0$ if $c \notin \frac{1}{2}\mathbb{Z}$.

§2. t-structure on the derived category of a quasi-abelian category

For more details, see [4, §2].

Let $\mathcal{C}$ be a quasi-abelian category (see [6]). Recall that, for a morphism $f\colon X \to Y$ in $\mathcal{C}$, $\operatorname{Im} f := \operatorname{Ker}(Y \to \operatorname{Coker} f)$ and $\operatorname{Coim} f := \operatorname{Coker}(\operatorname{Ker} f \to X)$. Hence, we have a diagram

$$\begin{tikzcd}
\operatorname{Ker} f \arrow[r] & X \arrow[r] \arrow[rrr, bend left, "f"] & \operatorname{Coim} f \arrow[r] & \operatorname{Im} f \arrow[r] & Y \arrow[r] & \operatorname{Coker} f.
\end{tikzcd}$$

Let $\mathrm{C}(\mathcal{C})$ be the category of complexes in $\mathcal{C}$, and $\mathrm{D}(\mathcal{C})$ the derived category of $\mathcal{C}$ (see [6]). Let us define various truncation functors for $X \in \mathrm{C}(\mathcal{C})$:

$$\begin{aligned}
\tau^{\leq n}X &: \ \cdots \to X^{n-1} \to \operatorname{Ker} d_X^n \to 0 \to 0 \to \cdots ,\\
\tau^{\leq n+1/2}X &: \ \cdots \to X^{n-1} \to X^n \to \operatorname{Im} d_X^n \to 0 \to \cdots ,\\
\tau^{\geq n}X &: \ \cdots \to 0 \to \operatorname{Coker} d_X^{n-1} \to X^{n+1} \to X^{n+2} \to \cdots ,\\
\tau^{\geq n+1/2}X &: \ \cdots \to 0 \to \operatorname{Coim} d_X^n \to X^{n+1} \to X^{n+2} \to \cdots ,
\end{aligned}$$

for $n \in \mathbb{Z}$. Then we have morphisms functorial in X:

$$\tau^{\leq s}X \to \tau^{\leq t}X \to X \to \tau^{\geq s}X \to \tau^{\geq t}X$$

for $s, t \in \frac{1}{2}\mathbb{Z}$ such that $s \leq t$. We can easily check that the functors $\tau^{\leq s}, \tau^{\geq s}\colon \mathrm{C}(\mathcal{C}) \to \mathrm{C}(\mathcal{C})$ send morphisms homotopic to zero to morphisms homotopic to zero and quasi-isomorphisms to quasi-isomorphisms. Hence, they induce functors

$$\tau^{\leq s}, \tau^{\geq s}\colon \mathrm{D}(\mathcal{C}) \to \mathrm{D}(\mathcal{C})$$

and morphisms $\tau^{\leq s} \to \mathrm{id} \to \tau^{\geq s}$.

For $s \in \frac{1}{2}\mathbb{Z}$, set

$$\begin{aligned}
\mathrm{D}^{\leq s}(\mathcal{C}) &= \{X \in \mathrm{D}(\mathcal{C}) \mid \tau^{\leq s}X \to X \text{ is an isomorphism}\},\\
\mathrm{D}^{\geq s}(\mathcal{C}) &= \{X \in \mathrm{D}(\mathcal{C}) \mid X \to \tau^{\geq s}X \text{ is an isomorphism}\}.
\end{aligned}$$

Then $\{\mathrm{D}^{\leq s}(\mathcal{C})\}_{s \in \frac{1}{2}\mathbb{Z}}$ is an increasing sequence of strictly full subcategories of $\mathrm{D}(\mathcal{C})$, and $\{\mathrm{D}^{\geq s}(\mathcal{C})\}_{s \in \frac{1}{2}\mathbb{Z}}$ is a decreasing sequence of strictly full subcategories of $\mathrm{D}(\mathcal{C})$.

The functor $\tau^{\le s}\colon \mathrm{D}(\mathcal{C}) \to \mathrm{D}^{\le s}(\mathcal{C})$ is a right adjoint functor of the inclusion functor $\mathrm{D}^{\le s}(\mathcal{C}) \hookrightarrow \mathrm{D}(\mathcal{C})$, and $\tau^{\ge s}\colon \mathrm{D}(\mathcal{C}) \to \mathrm{D}^{\ge s}(\mathcal{C})$ is a left adjoint functor of $\mathrm{D}^{\ge s}(\mathcal{C}) \hookrightarrow \mathrm{D}(\mathcal{C})$.

For $c \in \mathbb{R}$, we set

$$\begin{aligned} &\mathrm{D}^{\le c}(\mathcal{C}) = \mathrm{D}^{\le s}(\mathcal{C}) \quad \text{where } s \in \tfrac{1}{2}\mathbb{Z} \text{ satisfies } s \le c < s+1/2, \\ &\mathrm{D}^{\ge c}(\mathcal{C}) = \mathrm{D}^{\ge s}(\mathcal{C}) \quad \text{where } s \in \tfrac{1}{2}\mathbb{Z} \text{ satisfies } s-1/2 < c \le s. \end{aligned} \tag{2.1}$$

Proposition 2.1 ([6], see also [4]). *$((\mathrm{D}^{\le c}(\mathcal{C}))_{c\in\mathbb{R}}, (\mathrm{D}^{\ge c}(\mathcal{C}))_{c\in\mathbb{C}})$ is a t-structure.*

We call it the *standard t-structure* on $\mathrm{D}(\mathcal{C})$. The triangulated category $\mathrm{D}(\mathcal{C})$ is equivalent to the derived category of the abelian category $\mathrm{D}^{\le c}(\mathcal{C}) \cap \mathrm{D}^{>c-1}(\mathcal{C})$ for every $c \in \mathbb{R}$. The full subcategory $\mathrm{D}^0(\mathcal{C}) := \mathrm{D}^{\le 0}(\mathcal{C}) \cap \mathrm{D}^{\ge 0}(\mathcal{C})$ is equivalent to $\mathcal{C}$.

If $\mathcal{C}$ is an abelian category, then the standard t-structure is

$$\begin{aligned} \mathrm{D}^{\le c}(\mathcal{C}) &= \{X \in \mathrm{D}(\mathcal{C}) \mid H^i(X) = 0 \text{ for any } i > c\}, \\ \mathrm{D}^{\ge c}(\mathcal{C}) &= \{X \in \mathrm{D}(\mathcal{C}) \mid H^i(X) = 0 \text{ for any } i < c\}. \end{aligned}$$

§3. t-structure associated with a torsion pair

Let $\mathcal{C}$ be an abelian category. A *torsion pair* is a pair (T, F) of strictly full subcategories of $\mathcal{C}$ such that

(3.1) (a) $\mathrm{Hom}_{\mathcal{C}}(X, Y) = 0$ for any $X \in \mathsf{T}$ and $Y \in \mathsf{F}$,

(b) for any $X \in \mathcal{C}$, there exists an exact sequence $0 \to X' \to X \to X'' \to 0$ with $X' \in \mathsf{T}$ and $X'' \in \mathsf{F}$.

Let (T, F) be a torsion pair. Then

$$\begin{aligned} \mathsf{T} &\simeq \{X \in \mathcal{C} \mid \mathrm{Hom}_{\mathcal{C}}(X, Y) = 0 \text{ for any } Y \in \mathsf{F}\}, \\ \mathsf{F} &\simeq \{Y \in \mathcal{C} \mid \mathrm{Hom}_{\mathcal{C}}(X, Y) = 0 \text{ for any } X \in \mathsf{T}\}. \end{aligned}$$

Moreover, T is stable under taking quotients and extensions, while F is stable under taking subobjects and extensions.

For any integer n, we define

(3.2)

$$\begin{aligned} {}^{\mathrm{p}}\mathrm{D}^{\le n}(\mathcal{C}) &:= \{X \in \mathrm{D}(\mathcal{C}) \mid H^i(X) \simeq 0 \text{ for any } i > n\}, \\ {}^{\mathrm{p}}\mathrm{D}^{\le n-1/2}(\mathcal{C}) &:= \{X \in \mathrm{D}(\mathcal{C}) \mid H^i(X) \simeq 0 \text{ for any } i > n \text{ and } H^n(X) \in \mathsf{T}\}, \\ {}^{\mathrm{p}}\mathrm{D}^{\ge n-1/2}(\mathcal{C}) &:= \{X \in \mathrm{D}(\mathcal{C}) \mid H^i(X) \simeq 0 \text{ for any } i < n\}, \\ {}^{\mathrm{p}}\mathrm{D}^{\ge n}(\mathcal{C}) &:= \{X \in \mathrm{D}(\mathcal{C})^{\ge n-1/2} \mid H^i(X) \simeq 0 \text{ for any } i < n \text{ and } H^n(X) \in \mathsf{F}\}. \end{aligned}$$

For any $c \in \mathbb{R}$, we define ${}^{\mathrm{p}}\mathrm{D}^{\le c}(\mathcal{C})$ and ${}^{\mathrm{p}}\mathrm{D}^{\ge c}(\mathcal{C})$ by (2.1).

Since the following proposition can be easily proved, we omit the proof.

Proposition 3.1. $((^{\mathrm{p}}\mathrm{D}^{\leq c}(\mathcal{C}))_{c\in\mathbb{R}}, (^{\mathrm{p}}\mathrm{D}^{\geq c}(\mathcal{C}))_{c\in\mathbb{R}})$ *is a t-structure.*

We have

$$\mathsf{T} \simeq {}^{\mathrm{p}}\mathrm{D}^{-1/2}(\mathcal{C}), \quad \mathsf{F} \simeq {}^{\mathrm{p}}\mathrm{D}^{0}(\mathcal{C}), \quad \text{and} \quad \mathcal{C} \simeq {}^{\mathrm{p}}\mathrm{D}^{[-1/2,0]}(\mathcal{C}).$$

Moreover, $\mathrm{D}(\mathcal{C})$ is equivalent to the derived category of the abelian category ${}^{\mathrm{p}}\mathrm{D}^{[0,1/2]}(\mathcal{C})$.

Note that

$$\mathrm{D}^{\leq c}(\mathcal{C}) \subset {}^{\mathrm{p}}\mathrm{D}^{\leq c}(\mathcal{C}) \subset \mathrm{D}^{\leq c+1/2}(\mathcal{C}) \quad \text{and} \quad \mathrm{D}^{\geq c+1/2}(\mathcal{C}) \subset {}^{\mathrm{p}}\mathrm{D}^{\geq c}(\mathcal{C}) \subset \mathrm{D}^{\geq c}(\mathcal{C}).$$

§4. Self-dual t-structure on the derived category of coherent sheaves

Let X be a Noetherian regular scheme. Consider the duality functor $\mathsf{D}_X := \mathrm{R}\mathscr{H}om_{\mathscr{O}_X}(\,\bullet\,, \mathscr{O}_X)$. Let $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$ be the bounded derived category of $\mathscr{O}_X$-modules with coherent cohomology. We denote by $((\mathrm{D}^{\leq c}_{\mathrm{coh}}(\mathscr{O}_X))_{c\in\mathbb{R}}, (\mathrm{D}^{\geq c}_{\mathrm{coh}}(\mathscr{O}_X))_{c\in\mathbb{R}})$ the standard t-structure on $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$.

Recall that, for any coherent $\mathscr{O}_X$-module $\mathscr{F}$, its *codimension* is defined by

$$\operatorname{codim} \mathscr{F} := \operatorname{codim} \operatorname{Supp}(\mathscr{F}) = \inf_{x\in\operatorname{Supp}(\mathscr{F})} \dim \mathscr{O}_{X,\,x}.$$

Here we understand $\operatorname{codim} 0 = +\infty$.

We set

$$\begin{aligned}
{}^{1/2}\mathrm{D}^{\leq c}_{\mathrm{coh}}(\mathscr{O}_X) &:= \{\mathscr{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X) \mid \operatorname{codim} H^i(\mathscr{F}) \geq 2(i-c) \text{ for any } i \in \mathbb{Z}\},\\
{}^{1/2}\mathrm{D}^{\geq c}_{\mathrm{coh}}(\mathscr{O}_X) &:= \{\mathscr{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X) \mid \mathsf{D}_X\mathscr{F} \in \mathrm{D}^{\leq -c}_{\mathrm{coh}}(\mathscr{O}_X)\}\\
&= \{\mathscr{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X) \mid \operatorname{codim} H^i(\mathsf{D}_X\mathscr{F}) \geq 2(i+c) \text{ for any } i \in \mathbb{Z}\}.
\end{aligned}$$

These satisfy condition (a) of Definition 1.2. Note that

$${}^{1/2}\mathrm{D}^{\leq c}_{\mathrm{coh}}(\mathscr{O}_X) = \{\mathscr{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X) \mid \mathscr{F}_x \in \mathrm{D}^{\leq c+\frac{1}{2}\dim \mathscr{O}_{X,x}}(\mathscr{O}_{X,x}) \text{ for any } x \in X\}.$$

We also have

$$\begin{aligned}
{}^{1/2}\mathrm{D}^{< c}_{\mathrm{coh}}(\mathscr{O}_X) &:= \bigcup_{b<c} {}^{1/2}\mathrm{D}^{\leq b}_{\mathrm{coh}}(\mathscr{O}_X)\\
&= \{\mathscr{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X) \mid \operatorname{codim} H^i(\mathscr{F}) > 2(i-c) \text{ for any } i \in \mathbb{Z}\},\\
{}^{1/2}\mathrm{D}^{> c}_{\mathrm{coh}}(\mathscr{O}_X) &:= \bigcup_{b>c} {}^{1/2}\mathrm{D}^{\geq b}_{\mathrm{coh}}(\mathscr{O}_X)\\
&= \{\mathscr{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X) \mid \operatorname{codim} H^i(\mathsf{D}_X\mathscr{F}) > 2(i+c) \text{ for any } i \in \mathbb{Z}\}.
\end{aligned}$$

Lemma 4.1. *Let* $\mathscr{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$. *Then* $\mathscr{F} \in {}^{1/2}\mathrm{D}^{\geq c}_{\mathrm{coh}}(\mathscr{O}_X)$ *if and only if we have* $H^i \mathrm{R}\Gamma_Z \mathscr{F} = 0$ *for any closed subset* Z *and* $i < c + (\operatorname{codim} Z)/2$.

Proof. We shall use the results in [3]. Let us define the systems of support

$$\begin{aligned}\Phi^n &= \{Z \mid \operatorname{codim} Z \geq 2(n+c)\},\\ \Psi^n &= \{Z \mid n < c+1+(\operatorname{codim} Z)/2\}.\end{aligned}$$

Then it is enough to show that

$$(\Phi \circ \Psi)^n := \bigcup_{i+j=n} (\Phi^i \cap \Psi^j) = \{Z \mid \operatorname{codim} Z \geq n\}. \tag{4.1}$$

Indeed,

$$\begin{aligned}{}^{1/2}\mathrm{D}^{\leq -c}_{\mathrm{coh}}(\mathscr{O}_X) &= {}^{\Phi}\mathrm{D}^{\leq 0}_{\mathrm{coh}}(\mathscr{O}_X)\\ &:= \{\mathscr{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X) \mid \operatorname{Supp}(H^k(\mathscr{F})) \in \Phi^k \text{ for any } k \in \mathbb{Z}\},\end{aligned}$$

and hence [3, Theorem 5.9] along with (4.1) implies that ${}^{1/2}\mathrm{D}^{\geq c}_{\mathrm{coh}}(\mathscr{O}_X)$ coincides with

$$\begin{aligned}{}^{\Psi}\mathrm{D}^{\geq 0}_{\mathrm{coh}}(\mathscr{O}_X) &:= \{F \mid H^i(\mathrm{R}\Gamma_Z F) = 0 \text{ for any } Z \in \Psi^{i+1}\}\\ &= \{F \mid H^i(\mathrm{R}\Gamma_Z F) = 0 \text{ for any } i < c + (\operatorname{codim} Z)/2\}.\end{aligned}$$

Let us show (4.1) Assume that $Z \in \Phi^i \cap \Psi^j$ with $i + j = n$. Then

$$2 \operatorname{codim} Z \geq 2(i+c) + (2(j-c-1)+1) = 2n-1$$

and hence $\operatorname{codim} Z \geq n$.

Conversely, assume that $\operatorname{codim} Z \geq n$. Then take an integer i such that $i \leq (\operatorname{codim} Z)/2 - c < i+1$. Then $i > (\operatorname{codim} Z)/2 - c - 1$ and

$$j := n - i < \operatorname{codim} Z - ((\operatorname{codim} Z)/2 - c - 1) = c + 1 + (\operatorname{codim} Z)/2.$$

Hence $Z \in \Phi^i \cap \Psi^j \subset (\Phi \circ \Psi)^n$. □

Proposition 4.2. $(({}^{1/2}\mathrm{D}^{\leq c}_{\mathrm{coh}}(\mathscr{O}_X))_{c\in\mathbb{R}}, ({}^{1/2}\mathrm{D}^{\geq c}_{\mathrm{coh}}(\mathscr{O}_X))_{c\in\mathbb{R}})$ *is a t-structure on* $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$.

Proof. This follows from [3]. Indeed, $({}^{1/2}\mathrm{D}^{<c+1}_{\mathrm{coh}}(\mathscr{O}_X), {}^{1/2}\mathrm{D}^{\geq c}_{\mathrm{coh}}(\mathscr{O}_X))$ coincides with $({}^{\Psi}\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)^{\leq 0}, {}^{\Psi}\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)^{\geq 0})$ by the proof of the preceding proposition. □

Corollary 4.3. *For* $\mathscr{F} \in {}^{1/2}\mathrm{D}^{\leq c}_{\mathrm{coh}}(\mathscr{O}_X)$ *and* $\mathscr{G} \in {}^{1/2}\mathrm{D}^{\geq c'}_{\mathrm{coh}}(\mathscr{O}_X)$, *we have*

$$\mathrm{R}\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) \in \mathrm{D}^{\geq c'-c}_{\mathrm{coh}}(\mathscr{O}_X).$$

Conversely, for any $c' \in \mathbb{R}$,

$$ {}^{1/2}\mathrm{D}^{\geq c'}_{\mathrm{coh}}(\mathscr{O}_X) = \{\mathscr{G} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X) \mid \mathrm{R}\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}) \in \mathrm{D}^{\geq c'-c}_{\mathrm{coh}}(\mathscr{O}_X) \\ \textit{for any } c \in \mathbb{R} \textit{ and } \mathscr{F} \in {}^{1/2}\mathrm{D}^{\leq c}_{\mathrm{coh}}(\mathscr{O}_X)\}, $$

and for any $c \in \mathbb{R}$,

$$ {}^{1/2}\mathrm{D}^{\geq c}_{\mathrm{coh}}(\mathscr{O}_X) = \{\mathscr{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X) \mid \mathrm{R}\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}) \in \mathrm{D}^{\geq c'-c}_{\mathrm{coh}}(\mathscr{O}_X) \\ \textit{for any } c' \in \mathbb{R} \textit{ and } \mathscr{G} \in {}^{1/2}\mathrm{D}^{\geq c'}_{\mathrm{coh}}(\mathscr{O}_X)\}. $$

Proposition 4.4. *For* $\mathscr{F},\mathscr{G} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$, *we have:*

(i) *if* $\mathscr{F} \in {}^{1/2}\mathrm{D}^{\leq c}_{\mathrm{coh}}(\mathscr{O}_X)$ *and* $\mathscr{G} \in \mathrm{D}^{\leq c'}_{\mathrm{coh}}(\mathscr{O}_X)$, *then* $\mathscr{F} \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \mathscr{G} \in {}^{1/2}\mathrm{D}^{\leq c+c'}_{\mathrm{coh}}(\mathscr{O}_X)$,

(ii) *if* $\mathscr{F} \in \mathrm{D}^{\leq c}_{\mathrm{coh}}(\mathscr{O}_X)$ *and* $\mathscr{G} \in {}^{1/2}\mathrm{D}^{\geq c'}_{\mathrm{coh}}(\mathscr{O}_X)$, *then*

$$ \mathrm{R}\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}) \in {}^{1/2}\mathrm{D}^{\geq c'-c}_{\mathrm{coh}}(\mathscr{O}_X), $$

(iii) *if* $\mathscr{F} \in {}^{1/2}\mathrm{D}^{\geq c}_{\mathrm{coh}}(\mathscr{O}_X)$ *and* $\mathscr{G} \in \mathrm{D}^{\leq c'}_{\mathrm{coh}}(\mathscr{O}_X)$, *then*

$$ \mathrm{R}\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}) \in {}^{1/2}\mathrm{D}^{\leq c'-c}_{\mathrm{coh}}(\mathscr{O}_X), $$

(iv) *if* $\mathscr{F} \in {}^{1/2}\mathrm{D}^{\geq c}_{\mathrm{coh}}(\mathscr{O}_X)$ *and* $\mathscr{G} \in {}^{1/2}\mathrm{D}^{\geq c'}_{\mathrm{coh}}(\mathscr{O}_X)$, *then* $\mathscr{F} \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \mathscr{G} \in \mathrm{D}^{\geq c+c'}_{\mathrm{coh}}(\mathscr{O}_X)$.

Proof. (i) For any $\mathscr{H} \in {}^{1/2}\mathrm{D}^{\geq c''}_{\mathrm{coh}}(\mathscr{O}_X)$, we have $\mathrm{R}\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{F},\mathscr{H}) \in \mathrm{D}^{\geq c''-c}_{\mathrm{coh}}(\mathscr{O}_X)$ by Corollary 4.3. Hence,

$$ \mathrm{R}\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{F} \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \mathscr{G},\mathscr{H}) \simeq \mathrm{R}\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{G},\mathrm{R}\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{F},\mathscr{H})) $$

belongs to $\mathrm{D}^{\geq c''-c-c'}_{\mathrm{coh}}(\mathscr{O}_X)$. Since this holds for an arbitrary $\mathscr{H} \in {}^{1/2}\mathrm{D}^{\geq c''}_{\mathrm{coh}}(\mathscr{O}_X)$, we conclude that $\mathscr{F} \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \mathscr{G} \in {}^{1/2}\mathrm{D}^{\leq c+c'}_{\mathrm{coh}}(\mathscr{O}_X)$ by (1.3).

(ii) Since $\mathscr{F} \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \mathsf{D}_X\mathscr{G} \in {}^{1/2}\mathrm{D}^{\leq c-c'}_{\mathrm{coh}}(\mathscr{O}_X)$ by (i), it follows that $\mathrm{R}\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}) \simeq \mathsf{D}_X(\mathscr{F} \overset{\mathrm{L}}{\otimes} \mathsf{D}_X\mathscr{G})$ belongs to ${}^{1/2}\mathrm{D}^{\geq c'-c}_{\mathrm{coh}}(\mathscr{O}_X)$.

(iii) Since $\mathrm{R}\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}) \simeq (\mathsf{D}_X\mathscr{F}) \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \mathscr{G}$, (iii) follows from (i).

(iv) follows from Corollary 4.3 and $\mathscr{F} \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \mathscr{G} \simeq \mathrm{R}\mathscr{H}\!om_{\mathscr{O}_X}(\mathsf{D}_X\mathscr{F},\mathscr{G})$. □

Let A be a Noetherian regular ring and $X = \mathrm{Spec}(A)$. We write $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(A)$, ${}^{1/2}\mathrm{D}^{\leq c}_{\mathrm{coh}}(A)$ and ${}^{1/2}\mathrm{D}^{\geq c}_{\mathrm{coh}}(A)$ for $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$, ${}^{1/2}\mathrm{D}^{\leq c}_{\mathrm{coh}}(\mathscr{O}_X)$ and ${}^{1/2}\mathrm{D}^{\geq c}_{\mathrm{coh}}(\mathscr{O}_X)$, respectively.

Remark 4.5. (i) A similar construction is possible for a complex manifold X and coherent $\mathscr{O}_X$-modules.

(ii) For any $c \in \mathbb{R}$, we have

$$\mathrm{D}^{\le c}_{\mathrm{coh}}(\mathscr{O}_X) \subset {}^{1/2}\mathrm{D}^{\le c}_{\mathrm{coh}}(\mathscr{O}_X) \subset \mathrm{D}^{\le c+\dim X/2}_{\mathrm{coh}}(\mathscr{O}_X),$$
$$\mathrm{D}^{\ge c+\dim X/2}_{\mathrm{coh}}(\mathscr{O}_X) \subset {}^{1/2}\mathrm{D}^{\ge c}_{\mathrm{coh}}(\mathscr{O}_X) \subset \mathrm{D}^{\ge c}_{\mathrm{coh}}(\mathscr{O}_X).$$

(iii) If $\mathscr{F}$ is a Cohen–Macaulay $\mathscr{O}_X$-module with $\operatorname{codim} \mathscr{F} = r$, then we have $\mathscr{F} \in {}^{1/2}\mathrm{D}^{-r/2}_{\mathrm{coh}}(\mathscr{O}_X)$.

(iv) Assume that A is a Noetherian regular integral domain of dimension 1, and K the fraction field of A. Let $\mathcal{C} = \mathrm{Mod}_{\mathrm{coh}}(A)$. We take as $\mathsf{T} \subset \mathcal{C}$ the subcategory of torsion A-modules, and as F the subcategory of torsion free A-modules. Then the t-structure $(({}^{\mathrm{p}}\mathrm{D}^{\le c}(\mathcal{C}))_{c\in\mathbb{R}}, ({}^{\mathrm{p}}\mathrm{D}^{\ge c}(\mathcal{C}))_{c\in\mathbb{R}})$ associated with the torsion pair (T, F) (see §3) coincides with the t-structure $(({}^{1/2}\mathrm{D}^{\le c}_{\mathrm{coh}}(A))_{c\in\mathbb{R}}, ({}^{1/2}\mathrm{D}^{\ge c}_{\mathrm{coh}}(A)))_{c\in\mathbb{R}})$. Hence we have

$$\begin{aligned}
&{}^{1/2}\mathrm{D}^{\le n}_{\mathrm{coh}}(A) = \mathrm{D}^{\le n}_{\mathrm{coh}}(A),\\
&{}^{1/2}\mathrm{D}^{\le n-1/2}_{\mathrm{coh}}(A) = \{X \in \mathrm{D}^{\le n}_{\mathrm{coh}}(A) \mid K \otimes_A X \in \mathrm{D}^{\le n-1}(K)\},\\
&{}^{1/2}\mathrm{D}^{\ge n-1/2}_{\mathrm{coh}}(A) = \mathrm{D}^{\ge n}_{\mathrm{coh}}(A),\\
&{}^{1/2}\mathrm{D}^{\ge n}_{\mathrm{coh}}(A) = \{X \in \mathrm{D}^{\ge n}_{\mathrm{coh}}(A) \mid H^n(X) \text{ is torsion free}\}.
\end{aligned} \tag{4.2}$$

for any $n \in \mathbb{Z}$.

Let $\mathcal{F}$ be the quasi-abelian category of finitely generated torsion free A-modules. Then $\mathrm{D}^{\mathrm{b}}(\mathcal{F}) \simeq \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(A)$, and the t-structure $(({}^{1/2}\mathrm{D}^{\le c}_{\mathrm{coh}}(A))_{c\in\mathbb{R}}, ({}^{1/2}\mathrm{D}^{\ge c}_{\mathrm{coh}}(A))_{c\in\mathbb{R}})$ coincides with the standard t-structure of $\mathrm{D}^{\mathrm{b}}(\mathcal{F})$.

§5. Self-dual t-structure: real case

§5.1. Topological perversity

Let X be a subanalytic space (cf. [5, Exercise IX.2]). A subanalytic space is called *smooth* if it is is locally isomorphic to a real analytic manifold as a subanalytic space.

A *subanalytic stratification* $X = \bigsqcup_{\alpha\in I} X_\alpha$ of X is a locally finite family of locally closed smooth subanalytic subsets $\{X_\alpha\}_{\alpha\in I}$ (called strata) such that the closure $\overline{X_\alpha}$ is a union of strata for any α. A subanalytic stratification $X = \bigsqcup_{\alpha\in I} X_\alpha$ is called *good* if it satisfies the following condition:

(5.1) for any $K \in \mathrm{D}^{\mathrm{b}}(\mathbb{Z}_X)$ such that $K|_{X_\alpha}$ has locally constant cohomology for all α, $(\mathrm{R}\Gamma_{X_\alpha} K)|_{X_\alpha}$ has locally constant cohomology for all α.

Let $X = \bigsqcup_{\alpha\in I} X_\alpha$ and $X = \bigsqcup_{\alpha\in I'} X'_\beta$ be two stratifications. We say that $X = \bigsqcup_{\alpha\in I} X_\alpha$ is *finer* than $X = \bigsqcup_{\beta\in I'} X'_\beta$ if any X_α is contained in some X'_β. The following fact guarantees that there exist enough good stratifications:

(5.2) For any locally finite family $\{Z_j\}_j$ of locally closed subsets, there exists a good stratification such that any Z_j is a union of strata.

A *regular subanalytic filtration* of X is an increasing sequence

$$\emptyset = X_{-1} \subset \cdots \subset X_N = X$$

of closed subanalytic subsets X_k of X such that $\mathring{X}_k := X_k \setminus X_{k-1}$ is smooth of dimension k. We say that it is a *good filtration* if $\{\mathring{X}_k\}$ satisfies (5.1). Note that any subanalytic stratification $X = \bigsqcup_{\alpha\in I} X_\alpha$ gives a regular subanalytic filtration defined by $X_k := \bigsqcup_{\dim X_\alpha \le k} X_\alpha$.

Let A be a Noetherian regular ring. Denote by $\mathrm{Mod}_{\mathbb{R}\text{-c}}(A_X)$ the category of $\mathbb{R}$-constructible A_X-modules, and by $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(A_X)$ the bounded derived category of $\mathbb{R}$-constructible A_X-modules. Let $((\mathrm{D}^{\le c}_{\mathbb{R}\text{-c}}(A_X))_{c\in\mathbb{R}}, (\mathrm{D}^{\ge c}_{\mathbb{R}\text{-c}}(A_X))_{c\in\mathbb{R}})$ be the standard t-structure of $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(A_X)$, that is,

$$\mathrm{D}^{\le c}_{\mathbb{R}\text{-c}}(A_X) = \{K \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(A_X) \mid H^i(K) = 0 \text{ for any } i > c\},$$
$$\mathrm{D}^{\ge c}_{\mathbb{R}\text{-c}}(A_X) = \{K \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(A_X) \mid H^i(K) = 0 \text{ for any } i < c\}.$$

We define

$$\begin{aligned} {}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\le c}_{\mathbb{R}\text{-c}}(A_X) &= \{K \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(A_X) \mid \dim \mathrm{Supp}(H^i(K)) \le -2(i-c) \\ &\qquad\qquad \text{for any } i\}, \\ {}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\ge c}_{\mathbb{R}\text{-c}}(A_X) &= \{K \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(A_X) \mid H^i\mathrm{R}\Gamma_Z(K) = 0 \text{ for any closed} \\ &\qquad\qquad \text{subanalytic subset } Z \text{ and } i < c - \tfrac{1}{2}\dim Z\}. \end{aligned} \tag{5.3}$$

Proposition 5.1. *The pair* $(({}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\le c}_{\mathbb{R}\text{-c}}(A_X))_{c\in\mathbb{R}}, ({}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\ge c}_{\mathbb{R}\text{-c}}(A_X))_{c\in\mathbb{R}})$ *is a t-structure on* $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(A_X)$.

Proof. Indeed, $({}^{1/2}_{\mathrm{KS}}\mathrm{D}^{< c+1}_{\mathbb{R}\text{-c}}(A_X), {}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\ge c}_{\mathbb{R}\text{-c}}(A_X))$ coincides with the t-structure associated with the perversity $p(n) = \lceil c - n/2 \rceil$ (see e.g. [5, Definition 10.2.1]). □

Lemma 5.2 ([5, Proposition 10.2.4]). *Let* $K \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(A_X)$ *and let* $X = \bigsqcup_\alpha X_\alpha$ *be a subanalytic stratification of* X *such that* $(\mathsf{D}_X K)|_{X_\alpha}$ *has locally constant cohomology for any* α. *Then* $K \in {}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\ge c}_{\mathbb{R}\text{-c}}(A_X)$ *if and only if*

$$(\mathrm{R}\Gamma_{X_\alpha} K)_x \in \mathrm{D}^{\ge c - \dim X_\alpha/2}_{\mathrm{coh}}(A) \quad \textit{for any } \alpha \textit{ and } x \in X_\alpha.$$

§5.2. Self-dual t-structure: $\mathbb{R}$-constructible case

As in the preceding subsection, X is a subanalytic space and A is a Noetherian regular ring. Let D_X be the duality functor

$$\mathsf{D}_X(K) = \mathrm{R}\mathscr{H}\!om_A(K, \omega_X) \quad \text{for } K \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(A_X),$$

where $\omega_X = a_X^! A_{\mathrm{pt}}$ with the canonical projection $a_X\colon X \to \mathrm{pt}$.

For $F \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(A_X)$, we set

$$\text{(5.4)} \qquad \text{mod-dim}(F) = \sup_{m \ge 0}\bigl(\dim\{x \in X \mid \operatorname{codim} F_x = m\} - m\bigr),$$

where $\operatorname{codim} F_x$ denotes the codimension of $\mathrm{Supp}(F_x) \subset \mathrm{Spec}(A)$. Hence if $X = \bigsqcup_\alpha X_\alpha$ is a subanalytic stratification with connected strata and $F|_{X_\alpha}$ is locally constant for any α, then

$$\text{mod-dim}(F) = \sup\{\dim X_\alpha - \operatorname{codim} F_{x_\alpha} \mid F|_{X_\alpha} \neq 0\},$$

where x_α is a point of X_α. We understand $\text{mod-dim}\, 0 = -\infty$.

We set

$$\text{(5.5)} \qquad \begin{aligned} {}^{1/2}\mathrm{D}^{\le c}_{\mathbb{R}\text{-c}}(A_X) &= \{K \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(A_X) \mid \text{mod-dim}(H^i(K)) \le -2(i-c) \text{ for any } i\},\\ {}^{1/2}\mathrm{D}^{\ge c}_{\mathbb{R}\text{-c}}(A_X) &= \{K \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(A_X) \mid \mathsf{D}_X K \in {}^{1/2}\mathrm{D}^{\le -c}_{\mathbb{R}\text{-c}}(A_X)\}. \end{aligned}$$

Note that, when A is a field, they coincide with ${}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\le c}_{\mathbb{R}\text{-c}}(A_X)$ and ${}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\ge c}_{\mathbb{R}\text{-c}}(A_X)$.

Lemma 5.3. *Let $K \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(A_X)$ and $c \in \mathbb{R}$. Let $X = \bigsqcup_\alpha X_\alpha$ be a subanalytic stratification such that $K|_{X_\alpha}$ has locally constant cohomology. Then the following conditions are equivalent:*

(a) $K \in {}^{1/2}\mathrm{D}^{\le c}_{\mathbb{R}\text{-c}}(A_X)$,

(b) $\dim\{x \in X \mid K_x \notin {}^{1/2}\mathrm{D}^{\le c-k/2}_{\mathrm{coh}}(A)\} < k$ *for any* $k \in \mathbb{Z}$,

(c) $K_x \in {}^{1/2}\mathrm{D}^{\le c-(\dim X_\alpha)/2}_{\mathrm{coh}}(A)$ *for any* α *and* $x \in X_\alpha$.

Proof. (a)$\Leftrightarrow$(c). It is obvious that $K \in {}^{1/2}\mathrm{D}^{\le c}_{\mathbb{R}\text{-c}}(A_X)$ if and only if

$$\dim X_\alpha - \operatorname{codim} \mathrm{Supp}(H^i(K)_x) \le -2(i-c) \quad \text{for any } \alpha,\ x \in X_\alpha \text{ and } i \in \mathbb{Z}.$$

The last condition is equivalent to

$$\operatorname{codim} \mathrm{Supp}(H^i(K_x)) \ge 2(i - c + (\dim X_\alpha)/2),$$

or equivalently $K_x \in {}^{1/2}\mathrm{D}^{\le c-(\dim X_\alpha)/2}_{\mathrm{coh}}(A)$.

(b)$\Leftrightarrow$(c). (b) is equivalent to

$$\text{for any } x \in X_\alpha, \quad K_x \notin {}^{1/2}\mathrm{D}^{\le c-k/2}_{\mathrm{coh}}(A) \text{ implies } \dim X_\alpha < k,$$

which is equivalent to

$$\text{for any } x \in X_\alpha, \quad \dim X_\alpha \geq k \text{ implies } K_x \in {}^{1/2}\mathrm{D}^{\leq c-k/2}_{\mathrm{coh}}(A).$$

This is obviously equivalent to (c). □

Lemma 5.4. *Let $K \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(A_X)$ and $c \in \mathbb{R}$. Let $X = \bigsqcup_\alpha X_\alpha$ be a subanalytic stratification such that $(\mathsf{D}_X K)|_{X_\alpha}$ has locally constant cohomology. Then the following conditions are equivalent:*

(a) $K \in {}^{1/2}\mathrm{D}^{\geq c}_{\mathbb{R}\text{-c}}(A_X)$,

(b) *for any $c' \in \mathbb{R}$ and $M \in {}^{1/2}\mathrm{D}^{\leq c'}_{\mathrm{coh}}(A)$, we have*

$$\mathrm{R}\mathscr{H}om_A(M_X, K) \in {}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\geq c-c'}_{\mathbb{R}\text{-c}}(A_X),$$

(c) $\mathrm{R}\Gamma_Z(K)_x \in {}^{1/2}\mathrm{D}^{\geq c-\dim Z/2}_{\mathrm{coh}}(A)$ *for any closed subanalytic set Z and $x \in Z$,*

(d) $(\mathrm{R}\Gamma_{X_\alpha}K)_x \in {}^{1/2}\mathrm{D}^{\geq c-\dim X_\alpha/2}_{\mathrm{coh}}(A)$ *for any α and $x \in X_\alpha$,*

(e) $\dim\{x \in X \mid (\mathrm{R}\Gamma_{\{x\}}K)_x \notin {}^{1/2}\mathrm{D}^{\geq c+k/2}_{\mathrm{coh}}(A)\} < k$ *for any $k \in \mathbb{Z}_{\geq 0}$.*

Proof. Let $i_\alpha \colon X_\alpha \to X$ be the inclusion.

(a)⇔(d). By (a)⇔(c) in the preceding lemma, condition (a) is equivalent to

$$(\mathsf{D}_X K)_x \in {}^{1/2}\mathrm{D}^{\leq -c-(\dim X_\alpha)/2}_{\mathrm{coh}}(A) \quad \text{for any } \alpha \text{ and } x \in X_\alpha.$$

On the other hand, we have $i_\alpha^{-1}\mathsf{D}_X K \simeq \mathsf{D}_{X_\alpha} i_\alpha^! K$. Hence $i_\alpha^! K$ has locally constant cohomology. Since

$$(\mathsf{D}_X K)_x \simeq (\mathsf{D}_{X_\alpha} i_\alpha^! K)_x \simeq \mathrm{RHom}_A((i_\alpha^! K)_x, A)[\dim X_\alpha],$$

the above condition is equivalent to

$$\mathrm{RHom}_A((i_\alpha^! K)_x, A) \in {}^{1/2}\mathrm{D}^{\leq -c+(\dim X_\alpha)/2}_{\mathrm{coh}}(A),$$

which is again equivalent to $(i_\alpha^! K)_x \in {}^{1/2}\mathrm{D}^{\geq c-(\dim X_\alpha)/2}_{\mathrm{coh}}(A)$.

(a)⇔(e). (a) is equivalent to $\mathsf{D}_X K \in {}^{1/2}\mathrm{D}^{\leq -c}_{\mathbb{R}\text{-c}}(\mathbf{k}_X)$. By the preceding lemma, this is equivalent to

$$\dim\{x \in X \mid (\mathsf{D}_X K)_x \notin {}^{1/2}\mathrm{D}^{\leq -c-k/2}_{\mathrm{coh}}(A)\} < k \quad \text{for any } k \in \mathbb{Z}_{\geq 0}.$$

Since $(\mathsf{D}_X K)_x \simeq \mathsf{D}_A((\mathrm{R}\Gamma_{\{x\}}K)_x)$, the condition $(\mathsf{D}_X K)_x \notin {}^{1/2}\mathrm{D}^{\leq -c-k/2}_{\mathrm{coh}}(A)$ is equivalent to $(\mathrm{R}\Gamma_{\{x\}}K)_x \notin {}^{1/2}\mathrm{D}^{\geq c+k/2}_{\mathrm{coh}}(A)$.

(d)⇔(b). Condition (d) is equivalent to

(5.6) $\mathrm{RHom}_A(M, (\mathrm{R}\Gamma_{X_\alpha}K)_x) \in \mathrm{D}^{\geq c-(\dim X_\alpha)/2-c'}_{\mathrm{coh}}(A)$ for any $M \in {}^{1/2}\mathrm{D}^{\leq c'}_{\mathrm{coh}}(A)$, α and $x \in X_\alpha$.

Since $\mathrm{RHom}_A(M,(\mathrm{R}\Gamma_{X_\alpha}K)_x) \simeq (\mathrm{R}\Gamma_{X_\alpha}\mathrm{R}\mathscr{H}om_A(M_X,K))_x$, the last condition (5.6) is equivalent to (b) by Lemma 5.2.

(c)⇒(d) is obvious.

(b)⇒(c). For any $c' \in \mathbb{R}$ and $M \in {}^{1/2}\mathrm{D}^{\le c'}_{\mathrm{coh}}(A)$, we have

$$(\mathrm{R}\Gamma_Z\mathrm{R}\mathscr{H}om_A(M_X,K))_x \in \mathrm{D}^{\ge c-c'-(\dim Z)/2}_{\mathrm{coh}}(A).$$

Since $\mathrm{RHom}_A(M,(\mathrm{R}\Gamma_Z K)_x) \simeq (\mathrm{R}\Gamma_Z\mathrm{R}\mathscr{H}om_A(M_X,K))_x$, we obtain (c). □

We shall prove the following theorem in several steps.

Theorem 5.5. $(({}^{1/2}\mathrm{D}^{\le c}_{\mathbb{R}\text{-c}}(A_X))_{c\in\mathbb{R}}, ({}^{1/2}\mathrm{D}^{\ge c}_{\mathbb{R}\text{-c}}(A_X))_{c\in\mathbb{R}})$ *is a t-structure on* $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(A_X)$.

It is obvious that conditions (a) and (b) in Definition 1.2 are satisfied. Let us show (c).

Lemma 5.6. *For* $c \in \mathbb{R}$, $K \in {}^{1/2}\mathrm{D}^{\le c}_{\mathbb{R}\text{-c}}(A_X)$ *and* $L \in {}^{1/2}\mathrm{D}^{\ge c'}_{\mathbb{R}\text{-c}}(A_X)$, *we have*

$$\mathrm{R}\mathscr{H}om(K,L) \in \mathrm{D}^{\ge c'-c}_{\mathbb{R}\text{-c}}(A_X).$$

Proof. Let us take a good regular subanalytic filtration $\emptyset = X_{-1} \subset \cdots \subset X_N = X$ such that K and L have locally constant cohomology on each $\mathring{X}_k := X_k \setminus X_{k-1}$. We may assume that $\mathring{X}_k$ is smooth of dimension k.

Let $i_k \colon \mathring{X}_k \to X$ be the inclusion.

Let us first show that

$$i_k^!\mathrm{R}\mathscr{H}om(K,L) \simeq \mathrm{R}\mathscr{H}om(i_k^{-1}K, i_k^!L) \text{ belongs to } \mathrm{D}^{\ge c'-c}_{\mathbb{R}\text{-c}}(A_{\mathring{X}_k}). \tag{5.7}$$

Since $i_k^{-1}K$, $i_k^!L$ have locally constant cohomology,

$$(i_k^!\mathrm{R}\mathscr{H}om(K,L))_x \simeq \mathrm{RHom}_A(((i_k)^{-1}K)_x, (i_k^!L)_x)$$

for any $x \in \mathring{X}_k$. Hence it is enough to show that

$$\mathrm{RHom}_A((i_k^{-1}K)_x, (i_k^!L)_x) \in \mathrm{D}^{\ge c'-c}_{\mathbb{R}\text{-c}}(A). \tag{5.8}$$

This follows from Corollary 4.3 and

$$(i_k^{-1}K)_x \in {}^{1/2}\mathrm{D}^{\le c-k/2}_{\mathrm{coh}}(A) \quad \text{and} \quad (i_k^!L)_x \in {}^{1/2}\mathrm{D}^{\ge c'-k/2}_{\mathrm{coh}}(A).$$

Now we shall show by induction on k that

$$\mathrm{R}\Gamma_{X_k}\mathrm{R}\mathscr{H}om(K,L) \in \mathrm{D}^{\ge c'-c}_{\mathbb{R}\text{-c}}(A_X).$$

By the induction hypothesis $\mathrm{R}\Gamma_{X_{k-1}}\mathrm{R}\mathscr{H}om(K,L) \in \mathrm{D}^{\geq c'-c}_{\mathbb{R}\text{-c}}(A_X)$. We have the distinguished triangle

$$\mathrm{R}\Gamma_{X_{k-1}}\mathrm{R}\mathscr{H}om(K,L) \to \mathrm{R}\Gamma_{X_k}\mathrm{R}\mathscr{H}om(K,L) \to \mathrm{R}\Gamma_{\mathring{X}_k}\mathrm{R}\mathscr{H}om(K,L) \xrightarrow{+1} .$$

Since $\mathrm{R}\Gamma_{\mathring{X}_k}\mathrm{R}\mathscr{H}om(K,L) \simeq \mathrm{R}(i_k)_* i_k^!\mathrm{R}\mathscr{H}om(K,L)$ belongs to $\mathrm{D}^{\geq c'-c}_{\mathbb{R}\text{-c}}(A_X)$, we obtain $\mathrm{R}\Gamma_{X_k}\mathrm{R}\mathscr{H}om(K,L) \in \mathrm{D}^{\geq c'-c}_{\mathbb{R}\text{-c}}(A_X)$. ☐

Now we shall show condition (d) of Definition 1.2 in a special case.

Lemma 5.7. *Let X be a smooth subanalytic space, and $c \in \mathbb{R}$. Let $K \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(A_X)$ and assume that K has locally constant cohomology. Then there exists a distinguished triangle*

$$K' \to K \to K'' \xrightarrow{+1}$$

with $K' \in {}^{1/2}\mathrm{D}^{\leq c}_{\mathbb{R}\text{-c}}(A_X)$ and $K'' \in {}^{1/2}\mathrm{D}^{> c}_{\mathbb{R}\text{-c}}(A_X)$. Moreover K' and K'' have locally constant cohomology.

Proof. We argue in three steps.

(i) Such a distinguished triangle exists locally. Indeed, for any $x \in X$, there exist an open neighborhood U of x and $M \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(A)$ such that $K|_U \simeq M_U$. Take a distinguished triangle $M' \to M \to M'' \xrightarrow{+1}$ such that $M' \in {}^{1/2}\mathrm{D}^{\leq c-(\dim X)/2}_{\mathrm{coh}}(A)$ and $M'' \in {}^{1/2}\mathrm{D}^{> c-(\dim X)/2}_{\mathrm{coh}}(A)$. Then $M'_U \to M_U \to M''_U \xrightarrow{+1}$ gives the desired distinguished triangle.

(ii) If U_i is an open subset of X and $K'_i \to K|_{U_i} \to K''_i \xrightarrow{+1}$ is a distinguished triangle with $K'_i \in {}^{1/2}\mathrm{D}^{\leq c}_{\mathbb{R}\text{-c}}(A_{U_i})$ and $K''_i \in {}^{1/2}\mathrm{D}^{> c}_{\mathbb{R}\text{-c}}(A_{U_i})$ $(i = 1, 2)$, then there exists a distinguished triangle $K' \to K|_{U_1\cup U_2} \to K'' \xrightarrow{+1}$ with $K' \in {}^{1/2}\mathrm{D}^{\leq c}_{\mathbb{R}\text{-c}}(A_{U_1\cup U_2})$ and $K'' \in {}^{1/2}\mathrm{D}^{> c}_{\mathbb{R}\text{-c}}(A_{U_1\cup U_2})$.

Indeed, by the uniqueness of such a distinguished triangle, we have $K'_1|_{U_1\cap U_2} \simeq K'_2|_{U_1\cap U_2}$. Denote both by $K_0 \in \mathrm{D}^{\mathrm{b}}(A_{U_1\cap U_2})$. Let $i_0 \colon U_1 \cap U_2 \to U_1 \cup U_2$ and $i_k \colon U_k \to U_1 \cup U_2 (k = 1, 2)$ be the open inclusions. Then embed a morphism $(i_0)_!K_0 \to (i_1)_!K'_1 \oplus (i_2)_!K'_2$ into a distinguished triangle

$$(i_0)_!K_0 \to (i_1)_!K'_1 \oplus (i_2)_!K'_2 \to K' \xrightarrow{+1} .$$

Then $K'|_{U_k} \simeq K'_k$. Since the composition $(i_0)_!K_0 \to (i_1)_!K'_1 \oplus (i_2)_!K'_2 \to K|_{U_1\cup U_2}$ vanishes, the morphism $(i_1)_!K'_1 \oplus (i_2)_!K'_2 \to K|_{U_1\cup U_2}$ factors through K'. Hence, there exists a morphism $K' \to K|_{U_1\cup U_2}$ which extends $K'_i \to K|_{U_i}$ $(i = 1, 2)$. Embedding this morphism into a distinguished triangle $K' \to K|_{U_1\cup U_2} \to K'' \xrightarrow{+1}$, we obtain the desired distinguished triangle.

(iii) By (i) and (ii), there exist an increasing sequence of open subsets $\{U_n\}_{n\in\mathbb{Z}_{\geq 0}}$ with $X = \bigcup_{n\in\mathbb{Z}_{\geq 0}} U_n$ and a distinguished triangle $K'_n \to K|_{U_n} \to$

$K''_n \xrightarrow{+1}$ with $K'_n \in {}^{1/2}\mathrm{D}^{\le c}_{\mathbb{R}\text{-c}}(A_{U_n})$ and $K''_n \in {}^{1/2}\mathrm{D}^{> c}_{\mathbb{R}\text{-c}}(A_{U_n})$. Let $i_n \colon U_n \to X$ be the inclusion. By the uniqueness of such distinguished triangles, we have $K'_{n+1}|_{U_n} \simeq K'_n$. Hence, we have a map $\beta_n \colon (i_n)_! K'_n \to (i_{n+1})_! K'_{n+1}$. Let K' be the hocolim of the inductive system $\{(i_n)_! K'_n\}_{n\in\mathbb{Z}_{\ge 0}}$, that is, the third term of a distinguished triangle

$$\bigoplus_{n\in\mathbb{Z}_{\ge 0}} (i_n)_! K'_n \xrightarrow{f} \bigoplus_{n\in\mathbb{Z}_{\ge 0}} (i_n)_! K'_n \to K' \xrightarrow{+1} .$$

Here f is such that the following diagram commutes for any $a \in \mathbb{Z}_{\ge 0}$:

$$\begin{tikzcd}
(i_a)_! K'_a \arrow[r, "\mathrm{id}_{(i_a)_! K'_a} \oplus (-\beta_a)"] \arrow[d] & (i_a)_! K'_a \oplus (i_{a+1})_! K'_{a+1} \arrow[d] \\
\bigoplus_{n\in\mathbb{Z}_{\ge 0}} (i_n)_! K'_n \arrow[r, "f"] & \bigoplus_{n\in\mathbb{Z}_{\ge 0}} (i_n)_! K'_n.
\end{tikzcd}$$

Then $K'|_{U_n} \simeq K'_n$. Since the composition

$$\bigoplus_{n\in\mathbb{Z}_{\ge 0}} (i_n)_! K'_n \xrightarrow{f} \bigoplus_{n\in\mathbb{Z}_{\ge 0}} (i_n)_! K'_n \to K$$

vanishes, the morphism $\bigoplus_{n\in\mathbb{Z}_{\ge 0}} (i_n)_! K'_n \to K$ factors through K'. Hence there is a morphism $K' \to K$ which extends $(i_n)_! K'_n \to K$. Embedding this morphism into a distinguished triangle $K' \to K \to K'' \xrightarrow{+1}$, we obtain the desired distinguished triangle. □

Finally we shall complete the proof of condition (d) of Definition 1.2.

Lemma 5.8. *Let $K \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(A_X)$ and $c \in \mathbb{R}$. Then there exists a distinguished triangle $K' \to K \to K'' \xrightarrow{+1}$ with $K' \in {}^{1/2}\mathrm{D}^{\le c}_{\mathbb{R}\text{-c}}(A_X)$ and $K'' \in {}^{1/2}\mathrm{D}^{> c}_{\mathbb{R}\text{-c}}(A_X)$.*

Proof. Let us take a good regular subanalytic filtration $\emptyset = X_{-1} \subset \cdots \subset X_N = X$ such that K has locally constant cohomology on each $\mathring{X}_k := X_k \setminus X_{k-1}$. We may assume that $\mathring{X}_k$ is a smooth subanalytic space of dimension k. We shall prove that

$(5.9)_k$ there exists a distinguished triangle $K' \to K|_{X\setminus X_k} \to K'' \xrightarrow{+1}$ with $K \in {}^{1/2}\mathrm{D}^{\le c}(A_{X\setminus X_k})$ and $K'' \in {}^{1/2}\mathrm{D}^{> c}_{\mathbb{R}\text{-c}}(A_X)$. Moreover, $K'|_{\mathring{X}_j}$ and $K''|_{\mathring{X}_j}$ have locally constant cohomology for $j > k$,

by descending induction on k.

Assuming $(5.9)_k$, we shall show $(5.9)_{k-1}$. Let $K' \to K|_{X\setminus X_k} \to K'' \xrightarrow{+1}$ be a distinguished triangle as in $(5.9)_k$. Let $j \colon X \setminus X_k \to X \setminus X_{k-1}$ be the open embedding and $i \colon \mathring{X}_k \to X \setminus X_{k-1}$ the closed embedding. The morphism $K' \to K|_{X\setminus X_k}$ induces $j_! K' \to K|_{X\setminus X_{k-1}}$. We embed it into a distinguished triangle in $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(A_{X\setminus X_{k-1}})$

$$j_!K' \to K|_{X\setminus X_{k-1}} \to L \xrightarrow{+1} .$$

By Lemma 5.7, there exists a distinguished triangle

$$L' \to i^!L \to L'' \xrightarrow{+1} \tag{5.10}$$

with $L' \in {}^{1/2}\mathrm{D}^{\le c}_{\mathbb{R}\text{-c}}(A_{\mathring{X}_k})$ and $L'' \in {}^{1/2}\mathrm{D}^{> c}_{\mathbb{R}\text{-c}}(A_{\mathring{X}_k})$. We embed the composition $i_!L' \to i_!i^!L \to L$ into a distinguished triangle

$$i_!L' \to L \to \widetilde{K}'' \xrightarrow{+1} . \tag{5.11}$$

Finally, we embed the composition $K|_{X\setminus X_{k-1}} \to L \to \widetilde{K}''$ into a distinguished triangle

$$\widetilde{K}' \to K|_{X\setminus X_{k-1}} \to \widetilde{K}'' \xrightarrow{+1} .$$

Let us show that

$$\widetilde{K}' \in {}^{1/2}\mathrm{D}^{\le c}_{\mathbb{R}\text{-c}}(A_{X\setminus X_{k-1}}) \quad \text{and} \quad \widetilde{K}'' \in {}^{1/2}\mathrm{D}^{> c}_{\mathbb{R}\text{-c}}(A_{X\setminus X_{k-1}}).$$

By the construction, we have $\widetilde{K}''|_{X\setminus X_k} \simeq L|_{X\setminus X_k} \simeq K''$ and $\widetilde{K}'|_{X\setminus X_k} \simeq K'$. Hence it is enough to show that $i^{-1}\widetilde{K}' \in {}^{1/2}\mathrm{D}^{\le c}_{\mathbb{R}\text{-c}}(A_{\mathring{X}_k})$ and $i^!\widetilde{K}'' \in {}^{1/2}\mathrm{D}^{> c}_{\mathbb{R}\text{-c}}(A_{\mathring{X}_k})$. Applying the functor $i^!$ to (5.11), we obtain a distinguished triangle

$$L' \to i^!L \to i^!\widetilde{K}'' \xrightarrow{+1} .$$

By the distinguished triangle (5.10), we have $i^!\widetilde{K}'' \simeq L'' \in {}^{1/2}\mathrm{D}^{> c}_{\mathbb{R}\text{-c}}(A_{\mathring{X}_k})$.

By the octahedral axiom of a triangulated category, we have a diagram

$$\begin{tikzcd}
 & \widetilde{K}' \arrow[ddl] \arrow[dr, dotted] & \\
j_!K' \arrow[d] \arrow[ur, dotted] & & i_!L' \arrow[ll, "+1"'] \arrow[ddl] \\
K|_{X\setminus X_{k-1}} \arrow[rr] \arrow[dr] & & \widetilde{K}'' \arrow[u, "+1"'] \arrow[uul, "+1"] \\
 & L \arrow[ur] \arrow[uul, "+1"] &
\end{tikzcd}$$

and a distinguished triangle

$$j_!K' \to \widetilde{K}' \to i_!L' \xrightarrow{+1} .$$

This implies $i^{-1}\widetilde{K}' \simeq L' \in {}^{1/2}\mathrm{D}^{\le c}_{\mathbb{R}\text{-c}}(A_{\mathring{X}_k})$. $\square$

This completes the proof of Theorem 5.5.

Recall the full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(A_X)$:

$$^{1/2}\mathrm{D}^{[a,b]}_{\mathbb{R}\text{-c}}(A_X) := {}^{1/2}\mathrm{D}^{\le b}_{\mathbb{R}\text{-c}}(A_X) \cap {}^{1/2}\mathrm{D}^{\ge a}_{\mathbb{R}\text{-c}}(A_X)$$

for $a \le b$.

Proposition 5.9. *Assume that $a, b \in \mathbb{R}$ satisfy $a \le b < a+1$. Then $X \supset U \mapsto {}^{1/2}\mathrm{D}^{[a,b]}_{\mathbb{R}\text{-c}}((A_U)$ is a stack on X.*

Proof. (i) Let $K, L \in {}^{1/2}\mathrm{D}^{[a,b]}_{\mathbb{R}\text{-c}}(A_X)$. Since $\mathrm{R}\mathscr{H}om_A(K,L) \in \mathrm{D}^{\ge a-b}_{\mathbb{R}\text{-c}}(A_X) = \mathrm{D}^{\ge 0}_{\mathbb{R}\text{-c}}(A_X)$, the presheaf

$$U \mapsto \mathrm{Hom}_{^{1/2}\mathrm{D}^{[a,b]}_{\mathbb{R}\text{-c}}(A_U)}(K|_U, L|_U) \simeq \Gamma(U; H^0(\mathrm{R}\mathscr{H}om_A(K,L)))$$

is a sheaf. Hence, $U \mapsto {}^{1/2}\mathrm{D}^{[a,b]}_{\mathbb{R}\text{-c}}(A_U)$ is a separated prestack on X.

(ii) Let us show the following statement:

- Let U_1 and U_2 be open subsets of X such that $X = U_1 \cup U_2$, and let $K_k \in {}^{1/2}\mathrm{D}^{[a,b]}_{\mathbb{R}\text{-c}}(A_{U_k})$ $(k=1,2)$. Assume that $K_1|_{U_1\cap U_2} \simeq K_2|_{U_1\cap U_2}$. Then there exists $K \in {}^{1/2}\mathrm{D}^{[a,b]}_{\mathbb{R}\text{-c}}(A_X)$ such that $K|_{U_k} \simeq K_k$ $(k=1,2)$.

Set $U_0 = U_1 \cap U_2$ and $K_0 = K_1|_{U_1\cap U_2} \simeq K_2|_{U_1\cap U_2} \in {}^{1/2}\mathrm{D}^{[a,b]}_{\mathbb{R}\text{-c}}(A_{U_0})$. Let $j_k\colon U_k \to X$ be the open inclusion $(k=0,1,2)$. Then we have $\beta_k\colon (j_0)_!(K_0) \to (j_k)_!K_k$ $(k=1,2)$. We embed the morphism $(\beta_1,\beta_2)\colon (j_0)_!(K_0) \to (j_1)_!K_1 \oplus (j_2)_!K_2$ into a distinguished triangle

$$(j_0)_!(K_0) \to (j_1)_!K_1 \oplus (j_2)_!K_2 \to K \xrightarrow{+1}.$$

Then K satisfies the desired condition.

(iii) Let us show the following statement:

- Let $\{U_n\}_{n\in\mathbb{Z}_{\ge 0}}$ be an increasing sequence of open subsets of X such that $X = \bigcup_{n\in\mathbb{Z}_{\ge 0}} U_n$. Let $K_n \in {}^{1/2}\mathrm{D}^{[a,b]}_{\mathbb{R}\text{-c}}(A_{U_n})$ $(n \in \mathbb{Z}_{\ge 0})$ and $K_{n+1}|_{U_n} \simeq K_n$. Then there exists $K \in {}^{1/2}\mathrm{D}^{[a,b]}_{\mathbb{R}\text{-c}}(A_X)$ such that $K|_{U_n} \simeq K_n$ $(n \in \mathbb{Z}_{\ge 0})$.

The proof is similar to the proof of Lemma 5.7. Let $j_n\colon U_n \to X$ be the open inclusion, and let $(j_n)_!K_n \to (j_{n+1})_!K_{n+1}$ be the morphism induced by the isomorphism $K_{n+1}|_{U_n} \simeq K_n$. Let K be the hocolim of the inductive system $\{(j_n)_!K_n\}_{n\in\mathbb{Z}_{\ge 0}}$. Then $K \in {}^{1/2}\mathrm{D}^{[a,b]}_{\mathbb{R}\text{-c}}(A_X)$ satisfies the desired condition.

(iv) By (i)–(iii), we conclude that $U \mapsto {}^{1/2}\mathrm{D}^{[a,b]}_{\mathbb{R}\text{-c}}(A_U)$ is a stack on X. □

Proposition 5.10. *Let $f\colon X \to Y$ be a morphism of subanalytic spaces, and $d \in \mathbb{Z}_{\ge 0}$. Assume that* $\dim f^{-1}(y) \le d$ *for any $y \in Y$. Then:*

(i) *If $G \in {}^{1/2}\mathrm{D}^{\le c}_{\mathbb{R}\text{-c}}(A_Y)$, then $f^{-1}G \in {}^{1/2}\mathrm{D}^{\le c+d/2}_{\mathbb{R}\text{-c}}(A_X)$.*

(ii) *If $G \in {}^{1/2}\mathrm{D}^{\ge c}_{\mathbb{R}\text{-c}}(A_Y)$, then $f^!G \in {}^{1/2}\mathrm{D}^{\ge c-d/2}_{\mathbb{R}\text{-c}}(A_X)$.*

(iii) *If* $F \in {}^{1/2}\mathrm{D}^{\geq c}_{\mathbb{R}\text{-c}}(A_X)$ *and* $\mathrm{R}f_*F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(A_Y)$*, then* $\mathrm{R}f_*F \in {}^{1/2}\mathrm{D}^{\geq c-d/2}_{\mathbb{R}\text{-c}}(A_Y)$*.*
(iv) *If* $F \in {}^{1/2}\mathrm{D}^{\leq c}_{\mathbb{R}\text{-c}}(A_X)$ *and* $\mathrm{R}f_!F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(A_Y)$*, then* $\mathrm{R}f_!F \in {}^{1/2}\mathrm{D}^{\leq c+d/2}_{\mathbb{R}\text{-c}}(A_Y)$*.*

Proof. (i) Assume $G \in {}^{1/2}\mathrm{D}^{\leq c}_{\mathbb{R}\text{-c}}(A_Y)$. Then

$$\begin{aligned}
&\dim\{x \in X \mid (f^{-1}G)_x \notin {}^{1/2}\mathrm{D}^{\leq c+d/2-k/2}_{\mathrm{coh}}(A)\} \\
&\qquad = \dim f^{-1}(\{y \in Y \mid G_y \notin {}^{1/2}\mathrm{D}^{\leq c+d/2-k/2}_{\mathrm{coh}}(A)\}) \\
&\qquad \leq \dim\{y \in Y \mid G_y \notin {}^{1/2}\mathrm{D}^{\leq c+d/2-k/2}_{\mathrm{coh}}(A)\} + d < (k-d) + d = k.
\end{aligned}$$

(ii) follows from (i) by duality.

(iii) For any $G \in {}^{1/2}\mathrm{D}^{< c-d/2}_{\mathbb{R}\text{-c}}(A_Y)$,

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(A_Y)}(G, \mathrm{R}f_*F) \simeq \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(A_X)}(f^{-1}G, F)$$

vanishes because $f^{-1}G \in {}^{1/2}\mathrm{D}^{<c}_{\mathbb{R}\text{-c}}(A_X)$ by (i). Hence $\mathrm{R}f_*F \in {}^{1/2}\mathrm{D}^{\geq c-d/2}_{\mathbb{R}\text{-c}}(A_Y)$ by (1.3).

Similarly, (iv) follows from (ii). □

We shall give relations between the two t-structures:

$$\begin{gathered}
(({}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\leq c}_{\mathbb{R}\text{-c}}(A_X))_{c\in\mathbb{R}}, ({}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\geq c}_{\mathbb{R}\text{-c}}(A_X))_{c\in\mathbb{R}}), \\
(({}^{1/2}\mathrm{D}^{\leq c}_{\mathbb{R}\text{-c}}(A_X))_{c\in\mathbb{R}}, ({}^{1/2}\mathrm{D}^{\geq c}_{\mathbb{R}\text{-c}}(A_X))_{c\in\mathbb{R}}).
\end{gathered}$$

Lemma 5.11. *Let* $K \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(A_X)$ *and* $c \in \mathbb{R}$*.*

(i) *The following conditions are equivalent:*
  (a) $K \in {}^{1/2}\mathrm{D}^{\leq c}_{\mathbb{R}\text{-c}}(A_X)$,
  (b) *for any* $c' \in \mathbb{R}$ *and* $M \in {}^{1/2}\mathrm{D}^{\geq c'}_{\mathrm{coh}}(A)$*, we have*

$$\mathrm{R}\mathscr{H}om_A(K, M \otimes \omega_X) \in {}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\geq c'-c}_{\mathbb{R}\text{-c}}(A_X).$$

(ii) *The following conditions are equivalent:*
  (a) $K \in {}^{1/2}\mathrm{D}^{\geq c}_{\mathbb{R}\text{-c}}(A_X)$,
  (b) *for any* $c' \in \mathbb{R}$ *and* $M \in {}^{1/2}\mathrm{D}^{\leq c'}_{\mathrm{coh}}(A)$*, we have*

$$\mathrm{R}\mathscr{H}om_A(M_X, K) \in {}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\geq c-c'}_{\mathbb{R}\text{-c}}(A_X).$$

Proof. (ii) is already proved in Lemma 5.4; and (i) follows from (ii) because

$$\begin{aligned}
\mathrm{R}\mathscr{H}om_A(K, M \otimes \omega_X) &\simeq \mathrm{R}\mathscr{H}om_A(\mathsf{D}_X(M \otimes \omega_X), \mathsf{D}_X K) \\
&\simeq \mathrm{R}\mathscr{H}om_A((\mathsf{D}_A M)_X, \mathsf{D}_X K),
\end{aligned}$$

where $\mathsf{D}_A M := \mathrm{RHom}_A(M, A)$. □

Lemma 5.12. *Let X and Y be subanalytic spaces. Let $K \in {}^{1/2}\mathrm{D}^{\geq c}_{\mathbb{R}\text{-c}}(A_X)$ and $L \in {}^{1/2}\mathrm{D}^{\geq c'}_{\mathbb{R}\text{-c}}(A_Y)$. Then*

$$K \overset{\mathrm{L}}{\boxtimes} L \in {}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\geq c+c'}_{\mathbb{R}\text{-c}}(A_{X\times Y}).$$

Proof. Let $X = \bigsqcup_\alpha X_\alpha$ and $Y = \bigsqcup_\beta Y_\beta$ be good subanalytic stratifications such that $K|_{X_\alpha}$ and $L|_{Y_\beta}$ are locally constant. Then $(\mathrm{R}\Gamma_{X_\alpha} K)_x \in {}^{1/2}\mathrm{D}^{\geq c-(\dim X_\alpha)/2}_{\mathrm{coh}}(A)$ and $(\mathrm{R}\Gamma_{Y_\beta} L)_y \in {}^{1/2}\mathrm{D}^{\geq c'-(\dim Y_\beta)/2}_{\mathrm{coh}}(A)$ for $x \in X_\alpha$ and $y \in Y_\beta$. Hence by Proposition 4.4(iv),

$$(\mathrm{R}\Gamma_{X_\alpha\times Y_\beta}(K \overset{\mathrm{L}}{\boxtimes} L))_{(x,y)} \simeq (\mathrm{R}\Gamma_{X_\alpha} K)_x \overset{\mathrm{L}}{\otimes} (\mathrm{R}\Gamma_{Y_\beta} L)_y \in \mathrm{D}^{\geq c+c'-(\dim(X_\alpha\times Y_\beta))/2}_{\mathrm{coh}}(A).$$

This yields the conclusion. □

Remark 5.13. We have

$${}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\leq c}_{\mathbb{R}\text{-c}}(A_X) \subset {}^{1/2}\mathrm{D}^{\leq c}_{\mathbb{R}\text{-c}}(A_X), \qquad {}^{1/2}\mathrm{D}^{\geq c}_{\mathbb{R}\text{-c}}(A_X) \subset {}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\geq c}_{\mathbb{R}\text{-c}}(A_X).$$

§6. Self-dual t-structure: complex analytic variety case

§6.1. Middle perversity in the complex case

Let X be a complex analytic space. We denote by $\dim_{\mathbb{C}} X$ the dimension of X. Hence $\dim_{\mathbb{C}} X = (\dim X_{\mathbb{R}})/2$ where $X_{\mathbb{R}}$ is the underlying subanalytic space. For a complex submanifold Y of a complex manifold X, we denote by $\operatorname{codim}_{\mathbb{C}} Y$ the codimension of Y as complex manifolds. We sometimes write d_X for $\dim_{\mathbb{C}} X$.

Let $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(A_X)$ be the bounded derived category of the abelian category of sheaves of A-modules with $\mathbb{C}$-constructible cohomology. It is a full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(A_X)$ and it is easy to see that the self-dual t-structure on $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(A_X)$ induces a self-dual t-structure on $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(A_X)$. More precisely, if we define

$$\begin{aligned}{}^{1/2}\mathrm{D}^{\leq c}_{\mathbb{C}\text{-c}}(A_X) &:= \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(A_X) \cap {}^{1/2}\mathrm{D}^{\leq c}_{\mathbb{R}\text{-c}}(A_X),\\ {}^{1/2}\mathrm{D}^{\geq c}_{\mathbb{C}\text{-c}}(A_X) &:= \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(A_X) \cap {}^{1/2}\mathrm{D}^{\geq c}_{\mathbb{R}\text{-c}}(A_X),\end{aligned}$$

then $(({}^{1/2}\mathrm{D}^{\leq c}_{\mathbb{C}\text{-c}}(A_X))_{c\in\mathbb{C}}, ({}^{1/2}\mathrm{D}^{\geq c}_{\mathbb{C}\text{-c}}(A_X))_{c\in\mathbb{C}})$ is a t-structure on $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(A_X)$. Similarly, the t-structure $(({}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\leq c}_{\mathbb{R}\text{-c}}(A_X))_{c\in\mathbb{C}}, ({}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\geq c}_{\mathbb{R}\text{-c}}(A_X))_{c\in\mathbb{C}})$ induces the t-structure $(({}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\leq c}_{\mathbb{C}\text{-c}}(A_X))_{c\in\mathbb{C}}, ({}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\geq c}_{\mathbb{C}\text{-c}}(A_X))_{c\in\mathbb{C}})$ on $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(A_X)$.

Note that the t-structure $({}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\leq 0}_{\mathbb{C}\text{-c}}(A_X), {}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\geq 0}_{\mathbb{C}\text{-c}}(A_X))$ in the original sense is denoted by $({}^p\mathbf{D}^{\leq 0}_{\mathbb{C}\text{-}c}(X), {}^p\mathbf{D}^{\geq 0}_{\mathbb{C}\text{-}c}(X))$ in [5, §10.3].

In [5, §10.3], various properties of $({}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\leq 0}_{\mathbb{C}\text{-c}}(A_X), {}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\geq 0}_{\mathbb{C}\text{-c}}(A_X))$ are studied. By using Lemma 5.11, in the next subsection we obtain similar properties for $(({}^{1/2}\mathrm{D}^{\leq c}_{\mathbb{C}\text{-c}}(A_X))_{c\in\mathbb{C}}, ({}^{1/2}\mathrm{D}^{\geq c}_{\mathbb{C}\text{-c}}(A_X))_{c\in\mathbb{C}})$.

§6.2. Microlocal characterization

Let X be a complex manifold. Let $K \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(A_X)$. Then the microsupport $\mathrm{SS}(K)$ is a Lagrangian complex analytic subset of the cotangent bundle T^*X (see [5]).

A point p of $\mathrm{SS}(K)$ is called *good* if $\mathrm{SS}(K)$ equals the conormal bundle $T^*_Y X$ on a neighborhood of p for some locally closed complex submanifold Y of X. The complement of the set of good points of $\mathrm{SS}(K)$ is a nowhere dense closed complex analytic subset of $\mathrm{SS}(K)$. For a good point p of $\mathrm{SS}(K)$, there exists $L \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(A)$ such that K is microlocally isomorphic to $L_Y[-\operatorname{codim}_{\mathbb{C}} Y]$ on a neighborhood of p. We call L the *type* of K at p. (Note that in [5, §10.3], L is called the type of K at p with shift 0.)

The type can be calculated by the vanishing cycle functor. If f is a holomorphic function such that $f|_Y = 0$ and $df(x_0) = p$, then we have $\varphi_f(K)_{x_0} \simeq L[-\operatorname{codim}_{\mathbb{C}} Y]$. Here, $x_0 \in X$ is the image of p by the projection $T^*X \to X$, and φ_f is the vanishing cycle functor from $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(A_X)$ to $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(A_{f^{-1}(0)})$. Note that

$$\varphi_f(K) \simeq \mathrm{R}\Gamma_{\{x \mid \mathrm{Re}(f(x)) \geq 0\}}(K)|_{f^{-1}(0)}.$$

The following theorem is proved in [5, §10.3].

Theorem 6.1 ([5, Theorem 10.3.2]). *Let $K \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(A_X)$. Then the following conditions are equivalent:*

(a) $K \in {}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\leq c}_{\mathbb{C}\text{-c}}(A_X)$ (*resp.* $K \in {}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\geq c}_{\mathbb{C}\text{-c}}(A_X)$),

(b) *the type of K at any good point of* $\mathrm{SS}(K)$ *belongs to* $\mathrm{D}^{\leq c-d_X}_{\mathrm{coh}}(A)$ (*resp. belongs to* $\mathrm{D}^{\geq c-d_X}_{\mathrm{coh}}(A)$).

As a corollary, we can derive the following microlocal characterization of $(({}^{1/2}\mathrm{D}^{\leq c}_{\mathbb{C}\text{-c}}(A_X))_{c\in\mathbb{C}}, ({}^{1/2}\mathrm{D}^{\geq c}_{\mathbb{C}\text{-c}}(A_X))_{c\in\mathbb{C}})$.

Theorem 6.2. *Let $K \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(A_X)$. Then the following conditions are equivalent:*

(a) $K \in {}^{1/2}\mathrm{D}^{\leq c}_{\mathbb{C}\text{-c}}(A_X)$ (*resp.* $K \in {}^{1/2}\mathrm{D}^{\geq c}_{\mathbb{C}\text{-c}}(A_X)$),

(b) *the type of K at any good point of* $\mathrm{SS}(K)$ *belongs to* ${}^{1/2}\mathrm{D}^{\leq c-d_X}_{\mathrm{coh}}(A)$ (*resp. belongs to* ${}^{1/2}\mathrm{D}^{\geq c-d_X}_{\mathrm{coh}}(A)$).

Proof. Assume that $K \in {}^{1/2}\mathrm{D}^{\geq c}_{\mathbb{R}\text{-c}}(A_X)$. Then for any $M \in {}^{1/2}\mathrm{D}^{\leq c'}_{\mathrm{coh}}(A)$, we have $\mathrm{R}\mathscr{H}om_A(M_X, K) \in {}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\geq c-c'}_{\mathbb{C}\text{-c}}(A_X)$. Let L be the type of K at a good point p of $\mathrm{SS}(K)$. Then $\mathrm{R}\mathscr{H}om_A(M_X, K)$ has type $\mathrm{RHom}_A(M, L)$ at p. Hence, the preceding theorem implies $\mathrm{RHom}_A(M, L) \in \mathrm{D}^{\geq c-c'-d_X}_{\mathrm{coh}}(A)$. Since this holds for any $M \in {}^{1/2}\mathrm{D}^{\leq c'}_{\mathrm{coh}}(A)$, we conclude $L \in {}^{1/2}\mathrm{D}^{\geq c-d_X}_{\mathrm{coh}}(A)$. The converse can be proved similarly.

The case of ${}^{1/2}\mathrm{D}^{\le c}_{\mathbb{C}\text{-c}}(A_X)$ can be derived from the above case by duality. The condition $K \in {}^{1/2}\mathrm{D}^{\le c}_{\mathbb{C}\text{-c}}(A_X)$ is equivalent to $\mathsf{D}_X(K) \in {}^{1/2}\mathrm{D}^{\ge -c}_{\mathbb{C}\text{-c}}(A_X)$. Let L be the type of K at a good point p of $\mathrm{SS}(K)$. Then $\mathsf{D}_X(K)$ has type $\mathsf{D}_A(L)[2d_X]$ at p, and it is enough to notice that $\mathsf{D}_A(L)[2d_X] \in {}^{1/2}\mathrm{D}^{\le -c-d_X}_{\mathrm{coh}}(A)$ if and only if $L \in {}^{1/2}\mathrm{D}^{\ge c-d_X}_{\mathrm{coh}}(A)$. □

The following proposition can be proved similarly.

Proposition 6.3. *Let Y be a closed complex submanifold of a complex manifold X. Then:*

(i) *The functor $\nu_Y\colon \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(A_X) \to \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(A_{T_YX})$ sends*

$$
{}^{1/2}\mathrm{D}^{\le c}_{\mathbb{C}\text{-c}}(A_X) \text{ to } {}^{1/2}\mathrm{D}^{\le c}_{\mathbb{C}\text{-c}}(A_{T_YX}) \quad \text{and} \quad {}^{1/2}\mathrm{D}^{\ge c}_{\mathbb{C}\text{-c}}(A_X) \text{ to } {}^{1/2}\mathrm{D}^{\ge c}_{\mathbb{C}\text{-c}}(A_{T_YX}).
$$

(ii) *The microlocalization functor $\mu_Y\colon \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(A_X) \to \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(A_{T^*_YX})$ sends*

$$
\begin{aligned}
&{}^{1/2}\mathrm{D}^{\le c}_{\mathbb{C}\text{-c}}(A_X) \text{ to } {}^{1/2}\mathrm{D}^{\le c+\operatorname{codim}_{\mathbb{C}} Y}_{\mathbb{C}\text{-c}}(A_{T^*_YX}),\\
&{}^{1/2}\mathrm{D}^{\ge c}_{\mathbb{C}\text{-c}}(A_X) \text{ to } {}^{1/2}\mathrm{D}^{\ge c+\operatorname{codim}_{\mathbb{C}} Y}_{\mathbb{C}\text{-c}}(A_{T^*_YX}).
\end{aligned}
$$

Proof. Since the proofs are similar, we show only (ii). Let $K \in {}^{1/2}\mathrm{D}^{\ge c}_{\mathbb{C}\text{-c}}(A_X)$. Then, for any $M \in {}^{1/2}\mathrm{D}^{\le c'}_{\mathrm{coh}}(A)$, we have $\mathrm{R}\mathscr{H}om_A(M_X, K) \in {}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\ge c-c'}_{\mathbb{C}\text{-c}}(A_X)$. Hence [5, Prop. 10.3.19] implies that

$$
\mu_Y(\mathrm{R}\mathscr{H}om_A(M_X, K)) \in {}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\ge c-c'+\operatorname{codim}_{\mathbb{C}} Y}_{\mathbb{C}\text{-c}}(A_{T^*_YX}).
$$

Since

$$
\mathrm{R}\mathscr{H}om_A(M_{T^*_YX}, \mu_Y K) \simeq \mu_Y(\mathrm{R}\mathscr{H}om_A(M, K)),
$$

we obtain $\mu_Y K \in {}^{1/2}\mathrm{D}^{\ge c+\operatorname{codim}_{\mathbb{C}} Y}_{\mathbb{C}\text{-c}}(A_{T^*_YX})$.

Assume now that $K \in {}^{1/2}\mathrm{D}^{\le c}_{\mathbb{C}\text{-c}}(A_X)$. Then $\mathsf{D}_X K \in {}^{1/2}\mathrm{D}^{\ge -c}_{\mathbb{C}\text{-c}}(A_X)$. Since [5, Prop. 8.4.13] implies $\mathsf{D}_{T^*_YX}(\mu_Y K) \simeq (\mu_Y \mathsf{D}_X K)^a[2\operatorname{codim}_{\mathbb{C}} Y]$, we obtain

$$
\mathsf{D}_{T^*_YX}(\mu_Y K) \in {}^{1/2}\mathrm{D}^{\ge -c-\operatorname{codim}_{\mathbb{C}} Y}_{\mathbb{C}\text{-c}}(A_{T^*_YX}).
$$

Hence $\mu_Y K \in {}^{1/2}\mathrm{D}^{\le c+\operatorname{codim}_{\mathbb{C}} Y}_{\mathbb{C}\text{-c}}(A_{T^*_YX})$. □

The following theorem is proved in [5, §10.3].

Theorem 6.4 ([5, Corollary 10.3.20]). *Let $K \in {}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\le c}_{\mathbb{C}\text{-c}}(A_X)$ and $L \in {}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\ge c'}_{\mathbb{C}\text{-c}}(A_X)$. Then $\mu hom(K, L) \in {}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\ge c'-c+d_X}_{\mathbb{C}\text{-c}}(A_{T^*X})$.*

As a corollary we obtain the following result.

Theorem 6.5. *Let* $K \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(A_X)$ *and* $L \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(A_X)$.

(i) *If* $K \in {}^{1/2}\mathrm{D}^{\leq c}_{\mathbb{C}\text{-c}}(A_X)$ *and* $L \in {}^{1/2}\mathrm{D}^{\geq c'}_{\mathbb{C}\text{-c}}(A_X)$, *then*

$$\mu hom(K,L) \in {}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\geq c'-c+d_X}_{\mathbb{C}\text{-c}}(A_{T^*X}).$$

(ii) *If* $K \in {}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\leq c}_{\mathbb{C}\text{-c}}(A_X)$ *and* $L \in {}^{1/2}\mathrm{D}^{\geq c'}_{\mathbb{C}\text{-c}}(A_X)$, *then*

$$\mu hom(K,L) \in {}^{1/2}\mathrm{D}^{\geq c'-c+d_X}_{\mathbb{C}\text{-c}}(A_{T^*X}).$$

Proof. (i) By Lemma 5.12, we have $L \overset{\mathrm{L}}{\boxtimes} \mathsf{D}_X K \in {}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\geq c'-c}_{\mathbb{R}\text{-c}}(A_X)$. Let Δ_X be the diagonal of $X \times X$. Then $\mu hom(K,L) = \mu_{\Delta_X}(L \overset{\mathrm{L}}{\boxtimes} \mathsf{D}_X K) \in {}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\geq c'-c+d_X}_{\mathbb{C}\text{-c}}(A_X)$ by [5, Proposition 10.3.19].

(ii) For any $M \in {}^{1/2}\mathrm{D}^{\leq c''}_{\mathrm{coh}}(A)$, we have $\mathrm{R}\mathscr{H}om(M_X, L) \in {}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\geq c'-c''}_{\mathbb{C}\text{-c}}(A_X)$. Hence

$$\mathrm{R}\mathscr{H}om(M_{T^*X}, \mu hom(K,L)) \simeq \mu hom(K, \mathrm{R}\mathscr{H}om(M_X, L))$$

belongs to ${}^{1/2}_{\mathrm{KS}}\mathrm{D}^{\geq c'-c''-c+d_X}_{\mathbb{C}\text{-c}}(A_{T^*X})$ by Theorem 6.4. Consequently, $\mu hom(K,L) \in {}^{1/2}\mathrm{D}^{\geq c'-c+d_X}_{\mathbb{C}\text{-c}}(A_{T^*X})$ by Lemma 5.11. $\square$

Example 6.6. Assume that 2 acts injectively on A. Let M be a finitely generated projective A-module. Let $X = \mathbb{C}^3$ and $S = \{(x,y,z) \in X \mid x^2+y^2+z^2 = 0\}$. Let $j\colon X \setminus \{0\} \to X$ be the inclusion. Since $S \setminus \{0\}$ is homeomorphic to the product of $\mathbb{R}$ and the 3-dimensional real projective space $\mathbb{P}^3(\mathbb{R})$, we have

$$(\mathrm{R}j_*j^{-1}(M_S))_0 \simeq \mathrm{R}\Gamma(S \setminus \{0\}; M_S) \simeq M \oplus (M/2M)[-2] \oplus M[-3],$$

and $\mathrm{R}\Gamma_{\{0\}}(M_S)_0 \simeq (M/2M)[-3] \oplus M[-4]$. Hence we have

$$M_S \in {}^{1/2}\mathrm{D}^{2}_{\mathbb{C}\text{-c}}(A_X),$$

and a distinguished triangle

$$M_0[-1] \to \mathrm{R}j_!j^{-1}(M_S) \to M_S \xrightarrow{+1}.$$

Consequently,

$$\begin{aligned}
\mathrm{R}j_!j^{-1}(M_S) &\in {}^{1/2}\mathrm{D}^{[1,2]}_{\mathbb{C}\text{-c}}(A_X),\\
{}^{1/2}\tau^{\geq 2}\mathrm{R}j_!j^{-1}(M_S) &\simeq M_S,\\
{}^{1/2}\tau^{<2}\mathrm{R}j_!j^{-1}(M_S) &\simeq M_0[-1] \in {}^{1/2}\mathrm{D}^{1}_{\mathbb{C}\text{-c}}(A_X).
\end{aligned}$$

Here ${}^{1/2}\tau$ denotes the truncation functor of the t-structure ${}^{1/2}\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(A_X)$.

By duality, we have

$$\begin{aligned}
\mathrm{R}j_*j^{-1}(M_S) &\in {}^{1/2}\mathrm{D}^{[2,3]}_{\mathbb{C}\text{-c}}(A_X),\\
{}^{1/2}\tau^{>2}\mathrm{R}j_*j^{-1}(M_S) &\simeq M_0[-3] \in {}^{1/2}\mathrm{D}^{3}_{\mathbb{C}\text{-c}}(A_X).
\end{aligned}$$

Hence we obtain a distinguished triangle

$$ {}^{1/2}\tau^{\leq 2}\mathrm{R}j_*j^{-1}(M_S) \to \mathrm{R}j_*j^{-1}(M_S) \to M_0[-3] \xrightarrow{+1} . $$

The canonical morphism $\mathrm{R}j_!j^{-1}(M_S) \to \mathrm{R}j_*j^{-1}(M_S)$ decomposes as

$$ \begin{tikzcd} \mathrm{R}j_!j^{-1}(M_S) \arrow[r] \arrow[d] & \mathrm{R}j_*j^{-1}(M_S) \\ M_S \arrow[r] & {}^{1/2}\tau^{\leq 2}\mathrm{R}j_*j^{-1}(M_S) \arrow[u] \end{tikzcd} $$

and the bottom arrow is embedded into a distinguished triangle

$$ M_S \to {}^{1/2}\tau^{\leq 2}\mathrm{R}j_*j^{-1}(M_S) \to (M/2M)_{\{0\}}[-2] \xrightarrow{+1} . $$

Note that $(M/2M)_{\{0\}}[-2] \in {}^{1/2}\mathrm{D}^{3/2}_{\mathbb{C}\text{-c}}(A_X)$. Hence $M_S \to {}^{1/2}\tau^{\leq 2}\mathrm{R}j_*j^{-1}(M_S)$ is a monomorphism and an epimorphism in the quasi-abelian category ${}^{1/2}\mathrm{D}^{2}_{\mathbb{C}\text{-c}}(A_X)$. Moreover, we have an exact sequence

$$ 0 \to M_S \to {}^{1/2}\tau^{\leq 2}\mathrm{R}j_*j^{-1}(M_S) \to (M/2M)_{\{0\}}[-2] \to 0 $$

in the abelian category ${}^{1/2}\mathrm{D}^{[3/2,\,2]}_{\mathbb{C}\text{-c}}(A_X)$ and an exact sequence

$$ 0 \to (M/2M)[-3]_{\{0\}} \to M_S \to {}^{1/2}\tau^{\leq 2}\mathrm{R}j_*j^{-1}(M_S) \to 0 $$

in the abelian category ${}^{1/2}\mathrm{D}^{[2,\,5/2]}_{\mathbb{C}\text{-c}}(A_X)$. Note that we have an isomorphism of distinguished triangles

$$ \begin{tikzcd} \varphi_x(M_S) \arrow[r] \arrow[d, "\wr"] & \varphi_x({}^{1/2}\tau^{\leq 2}\mathrm{R}j_*j^{-1}(M_S)) \arrow[r] \arrow[d, "\wr"] & \varphi_x((M/2M)_{\{0\}}[-2]) \arrow[r, "+1"] \arrow[d, "\wr"] & {} \\ M_{\{0\}}[-2] \arrow[r, "2"] & M_{\{0\}}[-2] \arrow[r] & (M/2M)_{\{0\}}[-2] \arrow[r, "+1"] & {} \end{tikzcd} $$

Here φ_x is the vanishing cycle functor.

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