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The Homotopy Type of Spaces of Polynomials with Bounded Multiplicity

by

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Abstract

We study the homotopy type of the space of all monic polynomials of degree d in $\mathbb{C}[z]$ without roots of multiplicity $\geq n$. In particular, for $n \geq 3$ we improve the homotopy stability dimension obtained in [5].

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§1. Introduction

Throughout, X and Y are pointed connected topological spaces. Let $\mathrm{Map}(X,Y)$ (resp. $\mathrm{Map}^*(X,Y)$) denote the space consisting of all continuous maps (resp. base-point preserving maps) from X to Y with the compact-open topology. When X and Y are complex manifolds, we denote by $\mathrm{Hol}(X,Y)$ (resp. $\mathrm{Hol}^*(X,Y)$) the subspace of $\mathrm{Map}(X,Y)$ (resp. $\mathrm{Map}^*(X,Y)$) consisting of all holomorphic maps (resp. base-point preserving holomorphic maps).

From now on, we identify $S^2 = \mathbb{C} \cup \{\infty\}$. For each integer $d \geq 1$, let $\mathrm{Map}^*_d(S^2, \mathbb{C}\mathrm{P}^{n-1}) = \Omega^2_d \mathbb{C}\mathrm{P}^{n-1}$ denote the space of all base-point preserving continuous maps $f : S^2 \to \mathbb{C}\mathrm{P}^{n-1}$ such that $[f] = d \in \mathbb{Z} = \pi_2(\mathbb{C}\mathrm{P}^{n-1})$, where we choose $\infty \in S^2$ and $[1 : \cdots : 1] \in \mathbb{C}\mathrm{P}^{n-1}$ as the base points of S^2 and $\mathbb{C}\mathrm{P}^{n-1}$, respectively. Let $\mathrm{Hol}^*_d(S^2, \mathbb{C}\mathrm{P}^{n-1})$ denote the subspace of $\Omega^2_d \mathbb{C}\mathrm{P}^{n-1}$ of holomorphic maps.

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Let z be the complex variable and let $\mathrm{P}^d(\mathbb{C})$ be the space of all monic polynomials $f(z) = z^d + a_1 z^{d-1} + \cdots + a_d \in \mathbb{C}[z]$ of degree d, topologized by identifying f with $(a_1, \ldots, a_d) \in \mathbb{C}^d$. Let $\mathrm{SP}^d_n \subset \mathrm{P}^d(\mathbb{C})$ denote the subspace of polynomials without roots of multiplicity $\geq n$. Then the *jet map* $j^d_n : \mathrm{SP}^d_n \to \Omega^2_d \mathbb{C}\mathrm{P}^{n-1} \simeq \Omega^2 S^{2n-1}$ is defined by

$$j^d_n(f)(x) = \begin{cases} [f(x) : f(x) + f'(x) : \cdots : f(x) + f^{(n-1)}(x)] & \text{if } x \in \mathbb{C}, \\ [1 : \cdots : 1] & \text{if } x = \infty, \end{cases}$$

for $(f, x) \in \mathrm{SP}^d_n \times S^2$.

Remark 1.1. Note that $\mathrm{Hol}^*_d(S^2, \mathbb{C}\mathrm{P}^{n-1})$ can be identified with the space of all n-tuples $(f_1, \ldots, f_n) \in \mathrm{P}^d(\mathbb{C})^n$ of monic polynomials of the same degree d that have no common root. With this identification, the image of the map j^d_n is contained in $\mathrm{Hol}^*_d(S^2, \mathbb{C}\mathrm{P}^{n-1})$.

Remark 1.2. A map $f : X \to Y$ will be called a *homotopy equivalence* (resp. *homology equivalence*) *up to dimension* D if the induced homomorphism $f_* : \pi_k(X) \to \pi_k(Y)$ (resp. $f_* : H_k(X, \mathbb{Z}) \to H_k(Y, \mathbb{Z})$) is an isomorphism for any $k < D$ and an epimorphism if $k = D$; and f will be called a *homotopy equivalence* (resp. *homology equivalence*) *through dimension* D if $f_* : \pi_k(X) \to \pi_k(Y)$ (resp. $f_* : H_k(X, \mathbb{Z}) \to H_k(Y, \mathbb{Z})$) is an isomorphism for any $k \leq D$.

First, recall the following two results given in [5], [6] and [8].

Theorem 1.3 ([5], [6]). (i) *The jet map* $j^d_n : \mathrm{SP}^d_n \to \Omega^2_d \mathbb{C}\mathrm{P}^{n-1} \simeq \Omega^2 S^{2n-1}$ *is a homotopy equivalence up to dimension* $(2n-3)\lfloor d/n \rfloor$ *if* $n \geq 3$ *and a homology equivalence up to dimension* $\lfloor d/2 \rfloor$ *if* $n = 2$, *where* $\lfloor x \rfloor$ *denotes the integer part of a real number* x.

(ii) *If* $n \geq 3$, *there is a homotopy equivalence* $\mathrm{SP}^d_n \simeq \mathrm{Hol}^*_{\lfloor d/n \rfloor}(S^2, \mathbb{C}\mathrm{P}^{n-1})$. □

Theorem 1.4 ([8]). *If* $n \geq 3$, *the inclusion map* $i_d : \mathrm{Hol}^*_d(S^2, \mathbb{C}\mathrm{P}^{n-1}) \to \Omega^2_d \mathbb{C}\mathrm{P}^{n-1}$ *is a homotopy equivalence through dimension* $(2n-3)(d+1) - 1$. □

The main result of this paper improves the stability dimension of SP^d_n for $n \geq 3$ as follows:

Main Theorem 1.5. *If* $n \geq 3$, *the jet map* $j^d_n : \mathrm{SP}^d_n \to \Omega^2_d \mathbb{C}\mathrm{P}^{n-1} \simeq \Omega^2 S^{2n-1}$ *is a homotopy equivalence through dimension* $D(d, n) = (2n-3)(\lfloor d/n \rfloor + 1) - 1$.

Remark 1.6. Theorem 1.3 suggests that it may be possible to deduce Theorem 1.5 from Theorem 1.4. However, since we do not know whether the homotopy equivalence given in (ii) of Theorem 1.3 preserves the C_2-structure, there seems to be no obvious way to do so.

This paper is organized as follows. In §2 we recall the stabilization map s_d and prove the main result (Theorem 1.5) by using the key unstable result (Theorem 2.6). In §3 we recall the basic facts concerning simplicial resolutions. Finally in §4, we give the proof of Theorem 2.6 by using the Vassiliev spectral sequence induced from the non-degenerate simplicial resolution of the discriminant Σ_n^d.

§2. Stabilization

In this section we review several definitions and basic results concerning stabilization from [5].

Definition 2.1. (i) Let S_d denote the symmetric group on d letters. For a space X, S_d acts on the space $X^d = X \times \cdots \times X$ (d times) by permuting coordinates. We denote by $\mathrm{SP}^d(X)$ the d-th symmetric product of X, that is, the orbit space X^d/S_d.

(ii) Let $F(X,d) \subset X^d$ denote the subspace of all $(x_1,\dots,x_n) \in X^d$ such that $x_i \neq x_j$ if $i \neq j$. Since $F(X,d)$ is S_d-invariant, we can define the orbit space $C_d(X) = F(X,d)/S_d$. The space $C_d(X)$ is usually called the *configuration space of unordered d distinct points in X*.

Remark 2.2. Note that each $\alpha \in \mathrm{SP}^d(X)$ can be represented as a formal sum $\alpha = \sum_{k=1}^r n_k x_k$, where $\{x_k\}_{k=1}^r$ are mutually distinct points in X and the n_k are positive integers such that $\sum_{k=1}^r n_k = d$. Let $\mathrm{SP}_n^d(X)$ denote the subspace of $\mathrm{SP}^d(X)$ consisting of all elements of the form $\alpha = \sum_{k=1}^r n_k x_k$ such that $n_k < n$ for any $1 \leq k \leq r$.

By using this identification, if $X = \mathbb{C}$ we can easily see that there is a natural homeomorphism

$$\mathrm{P}^d(\mathbb{C}) \cong \mathrm{SP}^d(\mathbb{C}) \tag{2.1}$$

given by $\mathrm{P}^d(\mathbb{C}) \ni \prod_{k=1}^r (z-\alpha_k)^{n_k} \mapsto \sum_{k=1}^r n_k \alpha_k \in \mathrm{SP}^d(\mathbb{C})$, where $(\alpha_1,\dots,\alpha_r) \in F(\mathbb{C},r)$ and $\sum_{k=1}^r n_k = d$. By using (2.1), it is also easy to see that there is a natural homeomorphism

$$\mathrm{SP}_n^d \cong \mathrm{SP}_n^d(\mathbb{C}). \tag{2.2}$$

Definition 2.3. For each integer $d \geq 1$, let U_d denote the open set $\{w \in \mathbb{C} : \mathrm{Re}(w) < d\}$. Since there is a homeomorphism $U_d \cong \mathbb{C}$, we have a natural homeomorphism $\mathrm{SP}_n^d \cong \mathrm{SP}_n^d(U_d)$. Then define *the stabilization map* $s_d : \mathrm{SP}_n^d \to \mathrm{SP}_n^{d+1}$

to be the composite

$$\mathrm{SP}^d_n \cong \mathrm{SP}^d_n(U_d) \xrightarrow{\tilde{s}_d} \mathrm{SP}^{d+1}_n(U_{d+1}) \cong \mathrm{SP}^{d+1}_n \tag{2.3}$$

where $\alpha_0 \in U_{d+1} \setminus \overline{U}_d$ is any fixed point and $\tilde{s}_d : \mathrm{SP}^d_n(U_d) \to \mathrm{SP}^{d+1}_n(U_{d+1})$ denotes the map given by $\tilde{s}_d(\sum_{k=1}^d \alpha_k) = \sum_{k=1}^d \alpha_k + \alpha_0$.

Let SP^∞_n denote the colimit $\mathrm{SP}^\infty_n = \lim_{d\to\infty} \mathrm{SP}^d_n$ taken over the stabilization maps s_d.

Remark 2.4. Note that while the definition of the map s_d depends on the choice of the point α_0, its homotopy class does not.

Now recall the following result of [5].

Theorem 2.5 ([5]). *Let $n \geq 3$ be an integer.*

(i) *The jet map induces the homotopy equivalence*

$$\lim_{d\to\infty} j^d_n : \mathrm{SP}^\infty_n = \lim_{d\to\infty} \mathrm{SP}^d_n \to \lim_{d\to\infty} \Omega^2_d \mathbb{C}\mathrm{P}^{n-1} \simeq \Omega^2 S^{2n-1}.$$

(ii) *If $\lfloor d/n \rfloor = \lfloor (d+1)/n \rfloor$, the stabilization map $s_d : \mathrm{SP}^d_n \to \mathrm{SP}^{d+1}_n$ is a homotopy equivalence.*

Proof. Assertion (i) can be easily obtained from Theorem 1.3 as $d \to \infty$, and (ii) follows from [5, Corollary A2]. □

The key auxiliary result is the following:

Theorem 2.6. *If $n \geq 3$ and $\lfloor d/n \rfloor < \lfloor (d+1)/n \rfloor$, then the stabilization map $s_d : \mathrm{SP}^d_n \to \mathrm{SP}^{d+1}_n$ is a homology equivalence through dimension $D(d,n) = (2n-3)(\lfloor d/n \rfloor + 1) - 1$.*

We postpone the proof of Theorem 2.6 until §4 and give the proof of Theorem 1.5.

Proof of Theorem 1.5. Assume $n \geq 3$. It follows from Theorems 2.5 and 2.6 that the jet map $j^d_n : \mathrm{SP}^d_n \to \Omega^2_d \mathbb{C}\mathrm{P}^{n-1} \simeq \Omega^2 S^{2n-1}$ is a homology equivalence through dimension $D(d,n)$. However, if $n \geq 3$, the spaces SP^d_n and $\Omega^2 S^{2n-1}$ are simply connected. Hence, j^d_n is a homotopy equivalence through dimension $D(d,n)$. □

§3. Simplicial resolutions

In this section, we summarize the definitions of the non-degenerate simplicial resolution and the associated truncated simplicial resolutions ([9], [12]).

Definition 3.1. (i) For a finite set $\boldsymbol{v} = \{v_1, \dots, v_l\} \subset \mathbb{R}^N$, let $\sigma(\boldsymbol{v})$ denote the convex hull of $\boldsymbol{v}$. Let $h : X \to Y$ be a surjective map such that $h^{-1}(y)$ is finite for any $y \in Y$, and let $i : X \to \mathbb{R}^N$ be an embedding. Let $\mathcal{X}^\Delta$ and $h^\Delta : \mathcal{X}^\Delta \to Y$ denote the space and the map defined by

$$\mathcal{X}^\Delta = \{(y, u) \in Y \times \mathbb{R}^N : u \in \sigma(i(h^{-1}(y)))\} \subset Y \times \mathbb{R}^N, \quad h^\Delta(y, u) = y. \tag{3.1}$$

The pair $(\mathcal{X}^\Delta, h^\Delta)$ is called the *simplicial resolution* of (h, i). In particular, it is called *non-degenerate* if for each $y \in Y$ any k points of $i(h^{-1}(y))$ span a $(k-1)$-dimensional simplex of $\mathbb{R}^N$.

(ii) For each $k \geq 0$, let

$$\mathcal{X}_k^\Delta = \{(y, u) \in \mathcal{X}^\Delta : u \in \sigma(\boldsymbol{v}),\ \boldsymbol{v} = \{v_1, \dots, v_l\} \subset i(h^{-1}(y)),\ l \leq k\}. \tag{3.2}$$

We identify X with $\mathcal{X}_1^\Delta$ by identifying $x \in X$ with $(h(x), i(x)) \in \mathcal{X}_1^\Delta$, and we note that there is an increasing filtration

$$\emptyset = \mathcal{X}_0^\Delta \subset X = \mathcal{X}_1^\Delta \subset \mathcal{X}_2^\Delta \subset \cdots \subset \bigcup_{k=0}^{\infty} \mathcal{X}_k^\Delta = \mathcal{X}^\Delta. \tag{3.3}$$

Since the map $h^\Delta : \mathcal{X}^\Delta \to Y$ is proper, it extends to a map $h_+^\Delta : \mathcal{X}_+^\Delta \to Y_+$ between the one-point compactifications, where X_+ denotes the one-point compactification of a locally compact space X.

Lemma 3.2 ([12], [13]). *Let $h : X \to Y$ be a surjective map such that $h^{-1}(y)$ is finite for any $y \in Y$, and let $i : X \to \mathbb{R}^N$ be an embedding.*

- (i) *If X and Y are semi-algebraic spaces and the two maps h, i are semi-algebraic, then the map $h_+^\Delta : \mathcal{X}_+^\Delta \to Y_+$ is a homotopy equivalence.*
- (ii) *There is an embedding $j : X \to \mathbb{R}^M$ such that the associated simplicial resolution $(\tilde{\mathcal{X}}^\Delta, \tilde{h}^\Delta)$ of (h, j) is non-degenerate.*
- (iii) *If there is an embedding $j : X \to \mathbb{R}^M$ such that the associated simplicial resolution $(\tilde{\mathcal{X}}^\Delta, \tilde{h}^\Delta)$ of (h, j) is non-degenerate, then the space $\tilde{\mathcal{X}}^\Delta$ is uniquely determined up to homeomorphism. Moreover, there is a filtration preserving homotopy equivalence $q^\Delta : \tilde{\mathcal{X}}^\Delta \to \mathcal{X}^\Delta$ such that $q^\Delta | X = \mathrm{id}_X$.* □

Remark 3.3. In this paper we only need the weaker assertion that h_+^Δ is a homology equivalence. One can easily prove this by the same argument as in the second edition of Vassiliev's book [12, proof of Lemma 1, p. 90].

Remark 3.4. In the first edition of [12] (published in 1992), Vassiliev asserts that h_+^Δ is a homotopy equivalence, but his proof is based on a misuse of Whitehead's theorem. This was noted by K. Houston [7, §4.2], who proved a similar result for homology and expressed the belief that the homotopy conclusion is probably true. In the second revised edition of [12] (published in 1994) Vassiliev states that the homotopy conclusion is true but the proof is more complicated and it is omitted since it is not needed in the book. Instead he proves that h^{Δ}_{+} is a homology equivalence (which is also sufficient for our purposes). The key point of the proof is to use the well known theorem of Łojasiewicz on the triangulability of semi-algebraic spaces.[1]

However, in the 1997 Russian edition of his book [13], Vassiliev sketches a proof of the stronger statement that the h^{Δ} is a homotopy equivalence [13, proof of Lemma 1, p. 156]. The key point of the proof is to use a theorem stated in Goresky and MacPherson's book [4, Theorem, p. 43] and a theorem on triangulability of stratified mappings.[2] Combining these results we see that the spaces $\mathcal{X}^{\Delta}$ and Y can be triangulated in such a way that h^{Δ} is a simplicial map over each simplex of the target space Y, and h^{Δ} is a trivializable bundle map over each simplex, whose fibre is a simplex. It can thus be expressed as a composite of maps k_i $(i \geq 0)$ (up to homotopy) where each k_i collapses to a point the fibres over the interiors of all the strata of dimension i (here we assume that the fibres over the boundaries of the strata have been collapsed in the previous inductive step). Since each k_i is a homotopy equivalence, so is h^{Δ}. It is easy to see that an almost identical argument can be applied to the map h^{Δ}_{+} between the one-point compactifications (it is, in fact, this argument that is described in [13]), leading to the conclusion that h^{Δ}_{+} is a homotopy equivalence.

Remark 3.5. Alternatively, one can use the same results to prove that h^{Δ} is a quasifibration with a contractible fibre and thus it is a homotopy equivalence.

Remark 3.6. Even for a surjective map $h : X \to Y$ which is not finite-to-one, it is still possible to construct an associated non-degenerate simplicial resolution. Recall that it is known that there exists a sequence $\{\tilde{i}_k : X \to \mathbb{R}^{N_k}\}_{k\geq 1}$ of embeddings satisfying the following two conditions for each $k \geq 1$ ([12], [13]):

(i) For any $y \in Y$, any t points of the set $\tilde{i}_k(h^{-1}(y))$ span a $(t-1)$-dimensional affine subspace of $\mathbb{R}^{N_k}$ if $t \leq 2k$.

(ii) $N_k \leq N_{k+1}$ and if we identify $\mathbb{R}^{N_k}$ with a subspace of $\mathbb{R}^{N_{k+1}}$, then $\tilde{i}_{k+1} = \hat{i}\circ\tilde{i}_k$, where $\hat{i} : \mathbb{R}^{N_k} \to \mathbb{R}^{N_{k+1}}$ denotes the inclusion.

[1] Some errors in the first edition of [12] were corrected in the revised edition and some new material was added. For this reason, in this paper [12] always means the revised 1994 edition.

[2] No reference to any theorem on triangulability of mappings is given in [13], but the needed result can be found, for example, in [14].

In this situation, in fact, a non-degenerate simplicial resolution may be constructed by choosing a sequence $\{\tilde{i}_k : X \to \mathbb{R}^{N_k}\}_{k\geq 1}$ of embeddings satisfying the above two conditions for each $k \geq 1$.

Let $\mathcal{X}^{\Delta}_k = \{(y,u) \in Y \times \mathbb{R}^{N_k} : u \in \sigma(\boldsymbol{v}), \boldsymbol{v} = \{v_1,\dots,v_l\} \subset \tilde{i}_k(h^{-1}(y)),\, l \leq k\}$. Then by naturally identifying $\mathcal{X}^{\Delta}_k$ with a subspace of $\mathcal{X}^{\Delta}_{k+1}$, we define the non-degenerate simplicial resolution $\mathcal{X}^{\Delta}$ of h as the union $\mathcal{X}^{\Delta} = \bigcup_{k\geq 1} \mathcal{X}^{\Delta}_k$. □

§4. The Vassiliev spectral sequence

In this section we consider the Vassiliev spectral sequence induced from the non-degenerate simplicial resolution and give the proof of Theorem 2.6.

Definition 4.1. (i) Let Σ^d_n denote the *discriminant* of SP^d_n in $\mathrm{P}^d(\mathbb{C})$ given by the complement

$$\Sigma^d_n = \mathrm{P}^d(\mathbb{C}) \setminus \mathrm{SP}^d_n = \{f \in \mathrm{P}^d(\mathbb{C}) : f \text{ has a root of multiplicity} \geq n\}.$$

(ii) Let $Z_d \subset \Sigma^d_n \times \mathbb{C}$ denote the *tautological normalization* of Σ_d consisting of all pairs $(f,x) \in \Sigma^d_n \times \mathbb{C}$ such that $f(z)$ is divisible by $(z-x)^n$. Projection on the first factor gives a surjective map $\pi_d : Z_d \to \Sigma^d_n$.

Our goal in this section is to construct, by means of a *non-degenerate* simplicial resolution of the discriminant, a spectral sequence converging to the homology of SP^d_n.

Definition 4.2. Let $(\mathcal{X}^d, \pi^{\Delta}_d : \mathcal{X}^d \to \Sigma^d_n)$ be a non-degenerate simplicial resolution of the surjective map $\pi_d : Z_d \to \Sigma^d_n$ with the natural increasing filtration as in Definition 3.1,

$$\emptyset = \mathcal{X}^d_0 \subset \mathcal{X}^d_1 \subset \mathcal{X}^d_2 \subset \cdots \subset \mathcal{X}^d = \bigcup_{k=0}^{\infty} \mathcal{X}^d_k.$$

By Lemma 3.2, the map $\pi^{\Delta}_{d+} : \mathcal{X}^d_+ \to \Sigma^d_{n+}$ is a homology equivalence. Since $\mathcal{X}^d_{k+}/\mathcal{X}^d_{k-1+} \cong (\mathcal{X}^d_k \setminus \mathcal{X}^d_{k-1})_+$, we have a spectral sequence

$$\{E^{k,s}_{t;d}, d_t : E^{k,s}_{t;d} \to E^{k+t,s+1-t}_{t;d}\} \Rightarrow H^{k+s}_c(\Sigma^d_n, \mathbb{Z}),$$

where $E^{k,s}_{1;d} = \tilde{H}^{k+s}_c(\mathcal{X}^d_k \setminus \mathcal{X}^d_{k-1}, \mathbb{Z})$ and $H^k_c(X,\mathbb{Z})$ denotes the cohomology group with compact supports given by $H^k_c(X,\mathbb{Z}) = H^k(X_+,\mathbb{Z})$.

Since there is a homeomorphism $\mathrm{P}^d(\mathbb{C}) \cong \mathbb{C}^d$, by using Alexander duality there is a natural isomorphism

$$\tilde{H}_k(\mathrm{SP}^d_n, \mathbb{Z}) \cong \tilde{H}^{2d-k-1}_c(\Sigma^d_n, \mathbb{Z}) \quad \text{for any } k. \tag{4.1}$$

By reindexing we obtain a spectral sequence

$$\{E^{t;d}_{k,s}, d^t : E^{t;d}_{k,s} \to E^{t;d}_{k+t,s+t-1}\} \Rightarrow H_{s-k}(\mathrm{SP}^d_n, \mathbb{Z}), \tag{4.2}$$

where $E^{1;d}_{k,s} = \tilde{H}^{2d+k-s-1}_c(\mathcal{X}^d_k \setminus \mathcal{X}^d_{k-1}, \mathbb{Z})$.

Lemma 4.3. *If $1 \le k \le \lfloor d/n \rfloor$, then $\mathcal{X}^d_k \setminus \mathcal{X}^d_{k-1}$ is homeomorphic to the total space of a real affine bundle $\xi_{d,k}$ over $C_k(\mathbb{C})$ with rank $l_{d,k} = 2(d-nk)+k-1$.*

Proof. The argument is exactly analogous to the one in [1, proof of Lemma 4.4]. Namely, an element of $\mathcal{X}^d_k \setminus \mathcal{X}^d_{k-1}$ is represented by (f,u), where f is a polynomial in Σ^d_n and u is an element of the interior of the span of the images of k distinct points $\{x_1,\dots,x_k\} \in C_k(\mathbb{C})$ such that $\{x_j\}^k_{j=1}$ are the roots of $f(z)$ of multiplicity n under a suitable embedding. Since the k distinct points $\{x_j\}^k_{j=1}$ are uniquely determined by u, by the definition of the non-degenerate simplicial resolution, there are projection maps $\pi_{k,d} : \mathcal{X}^d_k \setminus \mathcal{X}^d_{k-1} \to C_k(\mathbb{C})$ defined by $((f_1,\dots,f_n),u) \mapsto \{x_1,\dots,x_k\}$.

Now suppose that $1 \le k \le \lfloor d/n \rfloor$. Fix $c = \{x_j\}^k_{j=1} \in C_k(\mathbb{C})$ and consider the fibre $\pi^{-1}_{k,d}(c)$. It is easy to see that a polynomial $f(z) \in \mathrm{P}^d(\mathbb{C})$ is divisible by $\prod^k_{j=1}(z-x_j)^n$ if and only if

$$f^{(t)}(x_j) = 0 \quad \text{for } 0 \le t < n,\ 1 \le j \le k. \tag{4.3}$$

In general, for each $0 \le t < n$ and $1 \le j < n$, the condition $f^{(t)}(x_j) = 0$ gives one linear condition on the coefficients of f_t, and it determines an affine hyperplane in $\mathrm{P}^d(\mathbb{C})$. For example, if we set $f(z) = z^d + \sum^d_{i=1} a_i z^{d-i}$, then $f(x_j) = 0$ for all $1 \le j \le k$ if and only if

$$\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^{d-1} \\ 1 & x_2 & x_2^2 & x_2^3 & \cdots & x_2^{d-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_k & x_k^2 & x_k^3 & \cdots & x_k^{d-1} \end{bmatrix} \cdot \begin{bmatrix} a_d \\ a_{d-1} \\ \vdots \\ a_1 \end{bmatrix} = - \begin{bmatrix} x_1^d \\ x_2^d \\ \vdots \\ x_k^d \end{bmatrix},$$

Similarly, $f'(x_j) = 0$ for all $1 \le j \le k$ if and only if

$$\begin{bmatrix} 0 & 1 & 2x_1 & 3x_1^2 & \cdots & (d-1)x_1^{d-2} \\ 0 & 1 & 2x_2 & 3x_2^2 & \cdots & (d-1)x_2^{d-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 2x_k & 3x_k^2 & \cdots & (d-1)x_k^{d-2} \end{bmatrix} \cdot \begin{bmatrix} a_d \\ a_{d-1} \\ \vdots \\ a_1 \end{bmatrix} = - \begin{bmatrix} dx_1^{d-1} \\ dx_2^{d-1} \\ \vdots \\ dx_k^{d-1} \end{bmatrix},$$

and $f''(x_j) = 0$ for all $1 \leq j \leq k$ if and only if

$$\begin{bmatrix} 0 & 0 & 2 & 6x_1 & \cdots & (d-1)(d-2)x_1^{d-3} \\ 0 & 0 & 2 & 6x_2 & \cdots & (d-1)(d-2)x_2^{d-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 2 & 6x_k & \cdots & (d-1)(d-2)x_k^{d-3} \end{bmatrix} \cdot \begin{bmatrix} a_d \\ a_{d-1} \\ \vdots \\ a_1 \end{bmatrix} = - \begin{bmatrix} d(d-1)x_1^{d-2} \\ d(d-1)x_2^{d-1} \\ \vdots \\ d(d-1)x_k^{d-2} \end{bmatrix},$$

and so on. Since $1 \leq k \leq \lfloor d/n \rfloor$ and $\{x_j\}_{j=1}^k \in C_k(\mathbb{C})$, it follows from the properties of Vandermonde matrices that condition (4.3) gives exactly nk independent conditions on the coefficients of $f(z)$. Hence, the space of polynomials $f \in \mathrm{P}^d(\mathbb{C})$ which satisfy (4.3) is the intersection of nk affine hyperplanes in general position, and it has codimension nk in $\mathrm{P}^d(\mathbb{C})$. Therefore, the fibre $\pi_{k,d}^{-1}(c)$ is homeomorphic to the product of an open $(k-1)$-simplex with the real affine space of dimension $2(d-nk)$. We see that $\mathcal{X}_k^d \setminus \mathcal{X}_{k-1}^d$ is a locally trivial real affine bundle over $C_k(\mathbb{C})$ of rank $l_{d,k} = 2(d-nk)+k-1$. □

Remark 4.4. If $k \geq \lfloor d/n \rfloor + 1$, then $\mathcal{X}_k^d \setminus \mathcal{X}_{k-1}^d = \emptyset$ and thus $E_{k,s}^{1;d} = 0$.

Lemma 4.5. *If $1 \leq k \leq \lfloor d/n \rfloor$, then there is a natural isomorphism*

$$E_{k,s}^{1;d} \cong \tilde{H}_c^{2nk-s}(C_k(\mathbb{C}), \pm\mathbb{Z}),$$

where the twisted coefficient system $\pm\mathbb{Z}$ comes from the Thom isomorphism.

Proof. As $1 \leq k \leq \lfloor d/n \rfloor$, by Lemma 4.3 there is a homeomorphism $(\mathcal{X}_k^d \setminus \mathcal{X}_{k-1}^d)_+ \cong T(\xi_{d,k})$, where $T(\xi_{d,k})$ denotes the Thom space of $\xi_{d,k}$. Since

$$(2d+k-s-1) - l_{d,k} = (2d+k-s-1) - (2d-2nk+k-1) = 2nk-s,$$

by using the Thom isomorphism there is a natural isomorphism

$$E_{k,s}^{1;d} \cong \tilde{H}^{2d+k-s-1}(T(\xi_{d,k}), \mathbb{Z}) \cong \tilde{H}_c^{2nk-s}(C_k(\mathbb{C}), \pm\mathbb{Z}),$$

and this completes the proof. □

Corollary 4.6. *For $\epsilon \in \{0,1\}$, there is an isomorphism*

$$E_{k,s}^{1;d+\epsilon} \cong \begin{cases} \mathbb{Z} & \text{if } (k,s) = (0,0), \\ \tilde{H}_c^{2nk-s}(C_k(\mathbb{C}), \pm\mathbb{Z}) & \text{if } 1 \leq k \leq \lfloor (d+\epsilon)/n \rfloor \text{ and } s \geq (2n-2)k, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Note that $\tilde{H}_c^{2nk-s}(C_k(\mathbb{C}), \pm\mathbb{Z}) = 0$ if $2nk - s > \dim C_k(\mathbb{C}) \Leftrightarrow s \leq (2n-2)k-1$. Hence, the assertion easily follows from Lemma 4.5 and Remark 4.4. □

Now recall $U_d = \{w \in \mathbb{C} : \mathrm{Re}(w) < d\}$ and the map $\tilde{s}_d : \mathrm{SP}^d_n(U_d) \to \mathrm{SP}^{d+1}_n(U_{d+1})$ given in (2.3). It naturally extends to an open embedding $\tilde{s}_d : (U_{d+1} \setminus \overline{U}_d) \times \mathrm{SP}^d(U_d) \to \mathrm{SP}^{d+1}(U_{d+1})$ by the formula

$$\tilde{s}_d\Big(\alpha, \sum_{k=1}^{d} \alpha_k\Big) = \sum_{k=1}^{d} \alpha_k + \alpha. \tag{4.4}$$

Since there is a homeomorphism $\mathbb{C} \cong U_{d+1} \setminus \overline{U}_d$, by using the identification (2.2) the stabilization map s_d also extends to an open embedding

$$\overline{s}_d : \mathbb{C} \times \mathrm{SP}^d_n \to \mathrm{SP}^{d+1}_n. \tag{4.5}$$

Since the open embedding $\tilde{s}_d : (U_{d+1} \setminus \overline{U}_d) \times \mathrm{SP}^d_n(U_d) \to \mathrm{SP}^{d+1}_n(U_{d+1})$ extends to an open embedding $(U_{d+1} \setminus \overline{U}_d) \times \mathrm{SP}^d(U_d) \to \mathrm{SP}^{d+1}(U_{d+1})$ by the same formula as in (4.4), the embedding (4.5) also naturally extends to an open embedding

$$\tilde{s}_d : \mathbb{C} \times \Sigma^d_n \to \Sigma^{d+1}_n \tag{4.6}$$

in the same way. Since one-point compactification is contravariant for open embeddings, it induces a map $\tilde{s}_{d+} : (\Sigma^{d+1}_n)_+ \to (\mathbb{C} \times \Sigma^d_n)_+ = S^2 \wedge \Sigma^d_{n+}$, and one can show that there is a commutative diagram

$$\begin{array}{ccc}
\tilde{H}_k(\mathrm{SP}^d_n, \mathbb{Z}) & \xrightarrow{s_{d*}} & \tilde{H}_k(\mathrm{SP}^{d+1}_n, \mathbb{Z}) \\
{\scriptstyle Al}\downarrow \cong & & {\scriptstyle Al}\downarrow \cong \\
\tilde{H}^{2d-k-1}_c(\Sigma^d_n, \mathbb{Z}) & \xrightarrow{\tilde{s}^*_{d+}} & \tilde{H}^{2(d+1)-k-1}_c(\Sigma^{d+1}_n, \mathbb{Z})
\end{array} \tag{4.7}$$

where $\hat{s} : \tilde{H}^*_c(\Sigma^d_n, \mathbb{Z}) \xrightarrow{\cong} \tilde{H}^{*+2}_c(\Sigma^{d+1}_n, \mathbb{Z})$ is the suspension isomorphism, Al denotes the Alexander duality isomorphism, and $\tilde{s}^*_{d+}$ is the composite homomorphism

$$\tilde{H}^{2d-k-1}_c(\Sigma^d_n, \mathbb{Z}) \underset{\cong}{\xrightarrow{\hat{s}}} \tilde{H}^{2(d+1)-k-1}_c(\mathbb{C} \times \Sigma^d_n, \mathbb{Z}) \xrightarrow{(\tilde{s}_{d+})^*} \tilde{H}^{2(d+1)-k-1}_c(\Sigma^{d+1}_n, \mathbb{Z}).$$

By using the universality of non-degenerate simplicial resolutions [9, pp. 286–287], one can see that the open embedding (4.6) also naturally extends to a filtration preserving open embedding

$$\tilde{s}_d : \mathbb{C} \times \mathcal{X}^d \to \mathcal{X}^{d+1}. \tag{4.8}$$

It induces a filtration preserving map $(\tilde{s}_d)_+ : \mathcal{X}^{d+1}_+ \to (\mathbb{C} \times \mathcal{X}^d)_+ = S^2 \wedge \mathcal{X}^d_+$, and we obtain a homomorphism of spectral sequences

$$\{\theta^t_{k,s} : E^{t;d}_{k,s} \to E^{t;d+1}_{k,s}\}, \tag{4.9}$$

where $E^{1;d+\epsilon}_{k,s} = \tilde{H}^{2(d+\epsilon)+k-1-s}_c(\mathcal{X}^d_k \setminus \mathcal{X}^d_{k-1}, \mathbb{Z})$ for $\epsilon \in \{0, 1\}$.

Lemma 4.7. *If* $1 \leq k \leq \lfloor d/n \rfloor$*, then* $\theta^1_{k,s} : E^{1;d}_{k,s} \to E^{1;d+1}_{k,s}$ *is an isomorphism for any* s*.*

Proof. Suppose that $1 \leq k \leq \lfloor d/n \rfloor$. Then it follows from the proof of Lemma 4.3 that there is a homotopy commutative diagram of affine vector bundles

$$\begin{array}{ccc} \mathcal{X}^d_k \setminus \mathcal{X}^d_{k-1} & \xrightarrow{\pi_{k,d}} & C_k(\mathbb{C}) \\ \downarrow & & \| \\ \mathcal{X}^{d+1}_k \setminus \mathcal{X}^{d+1}_{k-1} & \xrightarrow{\pi_{k,d+1}} & C_k(\mathbb{C}) \end{array}$$

Hence, we have a commutative diagram

$$\begin{array}{ccc} E^{1,d}_{k,s} & \xrightarrow[\cong]{} & \tilde{H}^{2nk-s}_c(C_k(\mathbb{C}), \pm\mathbb{Z}) \\ \theta^1_{k,s} \downarrow & & \| \\ E^{1,d+1}_{k,s} & \xrightarrow[\cong]{} & \tilde{H}^{2nk-s}_c(C_k(\mathbb{C}), \pm\mathbb{Z}) \end{array}$$

and the assertion follows. □

Proof of Theorem 2.6. Assume that $\lfloor d/n \rfloor < \lfloor (d+1)/n \rfloor$. Then it is easy to see that $\lfloor (d+1)/n \rfloor = \lfloor d/n \rfloor + 1$. Consider the homomorphism of spectral sequences $\{\theta^t_{k,s} : E^{t;d}_{k,s} \to E^{t;d+1}_{k,s}\}$ given by (4.9). If we consider the differential $d^t : E^{t;d+\epsilon}_{k,s} \to E^{t;d+\epsilon}_{k+t,s+t-1}$ for $\epsilon = 0$ or 1, it follows from Corollary 4.6 and Lemma 4.7 that we can easily see that $\theta^t_{k,s}$ is always an isomorphism for any (k,s) with any $t \geq 1$ as long as the condition $s - k \leq D(d,n) = (2n-3)(\lfloor d/n \rfloor + 1) - 1$ is satisfied. Hence, $\theta^\infty_{k,s}$ is an isomorphism for any (k,s) if $s - k \leq D(d,n)$. Thus, s_d is a homology equivalence through dimension $D(d,n)$. □

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