*Publ. RIMS Kyoto Univ.* **52** (2016), 309–333
DOI 10.4171/PRIMS/183

# Extended Affine Lie Superalgebras

by

Malihe Yousofzadeh

**Abstract**

We introduce the notion of extended affine Lie superalgebras and investigate the properties of their root systems. Extended affine Lie algebras, invariant affine reflection algebras, finite-dimensional basic classical simple Lie superalgebras and affine Lie superalgebras are examples of extended affine Lie superalgebras.

*2010 Mathematics Subject Classification:* Primary 17B67.
*Keywords:* extended affine Lie superalgebra, extended affine root supersystem.

## §1. Introduction

Given an arbitrary $n \times n$-matrix $A$ and a subset $\tau \subseteq \{1, \dots, n\}$, one can define the contragredient Lie superalgebra $\mathcal{G}(A, \tau)$ which is presented by a finite set of generators subject to specific relations. Contragredient Lie superalgebras associated with so-called generalized Cartan matrices are known as Kac–Moody Lie superalgebras. These Lie superalgebras are of great importance among contragredient Lie superalgebras; in particular, affine Lie superalgebras, i.e., those Kac–Moody Lie superalgebras which are of finite growth, but not of finite dimension and are equipped with a nondegenerate invariant even supersymmetric bilinear form, play a significant role in the theory of Lie superalgebras. In the past 40 years, researchers in many areas of mathematics and mathematical physics have been attracted to Kac–Moody Lie superalgebras $\mathcal{G}(A, \emptyset)$ known as Kac–Moody Lie algebras. These Lie algebras are a natural generalization of finite-dimensional simple Lie algebras. One of the differences between affine Lie superalgebras and affine Lie algebras is the existence of nonsingular roots, i.e., roots which are orthogonal to themselves but not to any other roots. In 1990, R. Høegh-Krohn and B. Torresani [5] introduced irreducible quasi simple Lie algebras as a generalization of both affine

Communicated by H. Nakajima. Received May 30, 2015. Revised February 7, 2016.

M. Yousofzadeh: Department of Mathematics, University of Isfahan, Isfahan, Iran, and School of Mathematics, Institute for Research in Fundamental Sciences (IPM), Tehran, Iran;
e-mail: ma.yousofzadeh@sci.ui.ac.ir & ma.yousofzadeh@ipm.ir

Lie algebras and finite-dimensional simple Lie algebras over complex numbers. In 1997, the authors of [1] systematically studied irreducible quasi simple Lie algebras under the name of extended affine Lie algebras. The existence of isotropic roots, i.e., roots which are orthogonal to all other roots, is one of the phenomena which occur in extended affine Lie algebras but not in finite-dimensional simple Lie algebras. Since 1997, different generalizations of extended affine Lie algebras have been studied; toral type extended affine Lie algebras [4], locally extended affine Lie algebras [9] and invariant affine reflection algebras [10], as a generalization of the last two stated classes, are examples of such generalizations.

Basic classical simple Lie superalgebras, orthosymplectic Lie superalgebras of arbitrary dimension as well as specific extensions of particular root graded Lie superalgebras satisfy certain properties which are in fact the super version of the axioms defining invariant affine reflection algebras.

In the present work, we study the class of Lie superalgebras satisfying these properties; we introduce the notion of extended affine Lie superalgebras. Roughly speaking, an extended affine Lie superalgebra is a Lie superalgebra having a weight space decomposition with respect to a nontrivial abelian subalgebra of the even part and equipped with a nondegenerate invariant even supersymmetric bilinear form such that the weight vectors associated with real roots are ad-nilpotent.

We prove that the even part of an extended affine Lie superalgebra is an invariant affine reflection algebra. We show that corresponding to each nonisotropic root $\alpha$ of an extended affine Lie superalgebra $\mathcal{L}$, there exists a triple of elements of $\mathcal{L}$ generating a subsuperalgebra $\mathcal{G}(\alpha)$ isomorphic to either $\mathfrak{sl}_2$ or $\mathfrak{osp}(1,2)$, depending on whether $\alpha$ is even or not. Considering $\mathcal{L}$ as a $\mathcal{G}(\alpha)$-module, we can derive some properties of the corresponding root system of $\mathcal{L}$ which are in fact the features defining extended affine root supersystems [12].

As $\mathfrak{osp}(1,2)$-modules are important in the theory of extended affine Lie superalgebras, we devote a section to the module theory of $\mathfrak{osp}(1,2)$. Although it is well-known that finite-dimensional $\mathfrak{osp}(1,2n)$-modules are completely reducible, we reprove this using the generic features of $\mathfrak{osp}(1,2)$, in a different approach from the one in the literature.

We conclude the paper with some examples showing that starting from an extended affine Lie superalgebra, one can get a new one using an affinization process.

## §2. Finite-dimensional modules of $\mathfrak{osp}(1,2)$

Throughout this work, $\mathbb{F}$ is a field of characteristic zero and $\mathbb{Z}_2 := \{\bar{0}, \bar{1}\}$ is the unique abelian group of order 2. Unless otherwise mentioned, all vector spaces considered are over $\mathbb{F}$. We denote the dual space of a vector space $V$ by $V^*$. If $V$ is a vector space graded by an abelian group, we denote the degree of a homogeneous element $x \in V$ by $|x|$; we also make the convention that if $|x|$ appears in an expression, for an element $x$ of $V$, then by default, we assume that $x$ is homogeneous.

If $X$ is a subset of a group $A$, we denote by $\langle X \rangle$ the subgroup of $A$ generated by $X$. The cardinality of a set $S$ is denoted by $|S|$, and $\delta_{i,j}$ is the Kronecker delta. For a map $f : A \to B$ and $C \subseteq A$, we denote by $f|_C$ the restriction of $f$ to $C$. Also we use $\uplus$ to indicate disjoint union.

In the present paper, by a *module* of a Lie superalgebra $\mathfrak{g}$, we mean a superspace $\mathcal{V} = \mathcal{V}_{\bar{0}} \oplus \mathcal{V}_{\bar{1}}$ and a bilinear map $\cdot : \mathfrak{g} \times \mathcal{V} \to \mathcal{V}$ satisfying $\mathfrak{g}_{\bar{i}} \cdot \mathcal{V}_{\bar{j}} \subseteq \mathcal{V}_{\bar{i}+\bar{j}}$ for $i, j \in \{0, 1\}$ and $[x, y] \cdot v = x \cdot (y \cdot v) - (-1)^{|x|\,|y|} y \cdot (x \cdot v)$ for all $x, y \in \mathfrak{g}$, $v \in \mathcal{V}$. Also by a $\mathfrak{g}$-*module homomorphism* from a $\mathfrak{g}$-module $\mathcal{V}$ to a $\mathfrak{g}$-module $\mathcal{W}$, we mean a linear homomorphism $\phi$ of parity $\bar{i}$ ($i \in \{0, 1\}$) with $\phi(x \cdot v) = (-1)^{|x|\,|\phi|} x \cdot \phi(v)$ for $x \in \mathfrak{g}, v \in \mathcal{V}$.

Also by a *symmetric form* on an additive abelian group $A$, we mean a map $(\cdot, \cdot) : A \times A \to \mathbb{F}$ satisfying

- $(a, b) = (b, a)$ for all $a, b \in A$,
- $(a + b, c) = (a, c) + (b, c)$ and $(a, b + c) = (a, b) + (a, c)$ for all $a, b, c \in A$.

In this case, we set $A^0 := \{a \in A \mid (a, A) = \{0\}\}$ and call it the *radical* of the form $(\cdot, \cdot)$. The form is called *nondegenerate* if $A^0 = \{0\}$. We note that if the form is nondegenerate, then $A$ is torsion free and we can identify $A$ as a subset of $\mathbb{Q} \otimes_{\mathbb{Z}} A$. If $A$ is a vector space over $\mathbb{F}$, bilinear forms are used in the usual sense.

We recall that $\mathfrak{osp}(1, 2)$ is a subsuperalgebra of $\mathfrak{sl}(1, 2)$ for which

$$F_+ := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad F_- := \begin{pmatrix} 0 & 0 & 2 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

of parity one, together with

$$H := \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad E_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}, \quad E_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -8 \\ 0 & 0 & 0 \end{pmatrix},$$

of parity zero, form a basis. The triple $(F_+, F_-, H)$ is an $\mathfrak{osp}$-triple for $\mathfrak{osp}(1, 2)$ in the following sense:

**Definition 2.1.** Suppose that $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a Lie superalgebra. We call a triple $(x, y, h)$ of nonzero elements of $\mathfrak{g}$ an $\mathfrak{sl}_2$-*super triple* for $\mathfrak{g}$ if

- $\{x, y, h\}$ generates the Lie superalgebra $\mathfrak{g}$,
- $x, y$ are homogeneous of the same degree,
- $[h, x] = 2x,\ [h, y] = -2y,\ [x, y] = h.$

If $x, y \in \mathfrak{g}_{\bar{1}}$, we refer to $(x, y, h)$ as an $\mathfrak{osp}$-*triple* and note that if $x, y \in \mathfrak{g}_{\bar{0}}$, then $(x, y, h)$ is an $\mathfrak{sl}_2$-triple.

**Lemma 2.2.** *Suppose that $(x, y, h)$ is an $\mathfrak{osp}$-triple for a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$. Then $\left(\frac{1}{4}[x, x], -\frac{1}{4}[y, y], \frac{1}{2}h\right)$ is an $\mathfrak{sl}_2$-triple for $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g} \simeq \mathfrak{osp}(1, 2)$.*

*Proof.* We have

$$[[x, x], [y, y]] = -8h, \qquad [h, [x, x]] = 4[x, x], \qquad [h, [y, y]] = -4[y, y],$$
$$[[x, x], x] = 0, \qquad [[y, y], y] = 0.$$

Therefore, $\left(\frac{1}{4}[x, x], -\frac{1}{4}[y, y], \frac{1}{2}h\right)$ is an $\mathfrak{sl}_2$-triple; in particular $[x, x] \neq 0$ as well as $[y, y] \neq 0$ and we have $\mathfrak{g} = \mathbb{F}[x, x] \oplus \mathbb{F}[y, y] \oplus \mathbb{F}h \oplus \mathbb{F}y \oplus \mathbb{F}x$. Now it follows that $\mathfrak{g}$ is isomorphic to $\mathfrak{osp}(1, 2)$. □

**Lemma 2.3.** *Suppose that $(e, f, h)$ is an $\mathfrak{osp}$-triple for a Lie superalgebra $\mathfrak{g}$. Assume $(\mathcal{V}, \cdot)$ is a $\mathfrak{g}$-module with corresponding representation $\pi$. If $\lambda \in \mathbb{F} \setminus \{-2\}$ and $u \in \mathcal{V}_{\bar{i}}$ $(i \in \{0, 1\})$ are such that $h \cdot u = \lambda u$ and $[e, e] \cdot u = 0$, then the $\mathfrak{g}_{\bar{0}}$-submodule of $\mathcal{V}$ generated by $f \cdot u$ equals*

$$T := \sum_{k \in \mathbb{Z}^{\geq 0}} \mathbb{F}f^{2k} \cdot (e \cdot u) + \sum_{k \in \mathbb{Z}^{\geq 0}} \mathbb{F}f^{2k+1} \cdot (f \cdot (e \cdot u) - (\lambda + 2)u)$$

*in which the action of $f^k$ is by means of $\pi(f)^k$ for all $k \in \mathbb{Z}^{\geq 0}$.*

*Proof.* Since $h \cdot u = \lambda u$ and $e \cdot (e \cdot u) = \frac{1}{2}[e, e] \cdot u = 0$, we have $h \cdot (e \cdot u) = (\lambda + 2) e \cdot u$, $h \cdot (f \cdot (e \cdot u)) = \lambda f \cdot (e \cdot u)$ and

$$e \cdot (f \cdot (e \cdot u)) = -f \cdot (e \cdot (e \cdot u)) + h \cdot (e \cdot u) = (\lambda + 2) e \cdot u.$$

Now for $x \in \{f \cdot (e \cdot u) - (\lambda + 2)u, e \cdot u\}$, we have $e \cdot x = 0$ and for

$$\lambda_x := \begin{cases} \lambda & \text{if } x = f \cdot (e \cdot u) - (\lambda + 2)u, \\ \lambda + 2 & \text{if } x = e \cdot u, \end{cases}$$

we have

$$\begin{aligned} & h \cdot (f^k \cdot x) = (\lambda_x - 2k) f^k \cdot x, \\ & f \cdot (f^k \cdot x) = f^{k+1} \cdot x, \\ & e \cdot (f^k \cdot x) = \begin{cases} -k f^{k-1} \cdot x & \text{if } k \text{ is even}, \\ (\lambda_x - (k-1)) f^{k-1} \cdot x & \text{if } k \text{ is odd}, \end{cases} \end{aligned}$$

for $k \in \mathbb{Z}^{\geq 0}$, where $f^{-1} \cdot x$ is defined to be zero. Now it follows that $T$ is invariant under the action of $[f,f]$, $[e,e]$ and $h$, i.e. it is a $\mathfrak{g}_{\bar{0}}$-submodule of $\mathcal{V}$. On the other hand,

$$f \cdot u = \frac{1}{\lambda+2}\big(f^2 \cdot (e \cdot u) - f \cdot (f \cdot (e \cdot u - (\lambda+2)u))\big) \in T.$$

Also if $S$ is a $\mathfrak{g}_{\bar{0}}$-submodule of $\mathcal{V}$ containing $f \cdot u$, then

$$\begin{aligned} -2e \cdot u = -[h,e] \cdot u &= -h \cdot (e \cdot u) + e \cdot (h \cdot u) \\ &= -f \cdot (e \cdot (e \cdot u)) - e \cdot (f \cdot (e \cdot u)) + e \cdot (h \cdot u) \\ &= -e \cdot (f \cdot (e \cdot u)) + e \cdot (h \cdot u) \\ &= e \cdot (e \cdot (f \cdot u)) - e \cdot (h \cdot u) + e \cdot (h \cdot u) = \tfrac{1}{2}[e,e] \cdot (f \cdot u) \in S \end{aligned}$$

So $T \subseteq S$. This completes the proof. □

**Lemma 2.4.** *Suppose that $(\mathcal{V}, \cdot)$ is a finite-dimensional module for a Lie superalgebra $\mathfrak{g} \simeq \mathfrak{osp}(1,2)$ with the corresponding representation $\pi$. Let $(e,f,h)$ be an $\mathfrak{osp}$-triple for $\mathfrak{g}$. Then:*

(i) *$\pi(h)$ is a diagonalizable endomorphism of $\mathcal{V}$ with even integer eigenvalues each of which occurs together with its negative.*

(ii) *Suppose that $\mathcal{V}$ is irreducible and $\Lambda$ is the set of eigenvalues of $\pi(h)$. Then the corresponding eigenspaces are one-dimensional and there is a nonnegative even integer $\lambda$ with $\Lambda = \{-\lambda, -\lambda+2, \ldots, \lambda-2, \lambda\}$. Moreover, if $\lambda \neq 0$, then $\mathcal{V}_{\bar{0}}$ and $\mathcal{V}_{\bar{1}}$ are irreducible $\mathfrak{g}_{\bar{0}}$-submodules of $\mathcal{V}$ and there is $i \in \{0,1\}$ such that $\{-\lambda/2, -\lambda/2+2, \ldots, \lambda/2-2, \lambda/2\}$ and $\{-\lambda/2+1, -\lambda/2+3, \ldots, \lambda/2-3, \lambda/2-1\}$ are the sets of eigenvalues of $\frac{1}{2}\pi(h)|_{\mathcal{V}_{\bar{i}}}$ and $\frac{1}{2}\pi(h)|_{\mathcal{V}_{\bar{i}+\bar{1}}}$ respectively.*

*Proof.* (i) We know that $\mathfrak{g}_{\bar{0}} \simeq \mathfrak{sl}_2(\mathbb{F})$ and that $\big(\frac{1}{4}[e,e], -\frac{1}{4}[f,f], \frac{1}{2}h\big)$ is an $\mathfrak{sl}_2$-triple for $\mathfrak{g}_{\bar{0}}$. Considering $\mathcal{V}$ as a $\mathfrak{g}_{\bar{0}}$-module and using $\mathfrak{sl}_2$-module theory [6, §III.8], we find that $\pi(\frac{1}{2}h)$ acts diagonally on $\mathcal{V}$ with integer eigenvalues each of which occurs together with its negative.

(ii) Suppose that $\mathcal{V}$ is irreducible. Let $\lambda$ be the largest eigenvalue of $\pi(h)$ and fix a homogeneous eigenvector $v_0$ for this eigenvalue. Set $v_{-1} := 0$ and $v_i := \pi(f)^i(v_0)$ for $i \in \mathbb{Z}^{\geq 0}$. For $i \in \mathbb{Z}^{\geq 0}$, we have

$$h \cdot v_i = (\lambda - 2i)v_i, \qquad f \cdot v_i = v_{i+1},$$
$$e \cdot v_i = \begin{cases} -iv_{i-1} & \text{if } i \text{ is even,} \\ (\lambda-(i-1))v_{i-1} & \text{if } i \text{ is odd.} \end{cases}$$

This together with the fact that $\mathcal{V}$ is irreducible shows that $\mathcal{V} = \sum_{k \in \mathbb{Z}^{\geq 0}} \mathbb{F} f^k \cdot v_0$ and so each eigenspace is one-dimensional. Since $\lambda+2$ is not an eigenvalue, by (i), neither is $-\lambda-2$; in particular, $v_{\lambda+1}=0$. So $\mathcal{V}=\sum_{k=0}^{\lambda}\mathbb{F}f^k\cdot v_0$. This completes the proof if $\lambda=0$. Now suppose $\lambda\neq 0$. For $i\in\{0,\ldots,\lambda\}$, we have $v_i\in\mathcal{V}^{m_i}$ where $m_i:=\lambda-2i$ and

$$\mathcal{V}^{m_i}:=\{v\in\mathcal{V}\mid h\cdot v=m_iv\}=\big\{v\in\mathcal{V}\;\big|\;\tfrac{1}{2}h\cdot v=\tfrac{1}{2}m_iv\big\}.$$

Set

$$U:=\operatorname{span}_{\mathbb{F}}\big\{v_{2i}\mid i\in\{0,\ldots,\lambda/2\}\big\},\qquad W:=\operatorname{span}_{\mathbb{F}}\{v_{2i+1}\mid i\in\big\{0,\ldots,\lambda/2-1\}\big\}.$$

Both $U$ and $W$ are invariant under the action of $\frac{1}{4}[e,e],-\frac{1}{4}[f,f],\frac{1}{2}h$ and so they are $\mathfrak{g}_{\bar 0}$-submodules of $\mathcal{V}$. Since $\lambda$ is the largest eigenvalue for $\pi(h)$ and $h\cdot(e\cdot v_0)=(\lambda+2)v_0$, we have $e\cdot v_0=0$ and so $0\neq\lambda v_0=h\cdot v_0=e\cdot f\cdot v_0+f\cdot e\cdot v_0=e\cdot f\cdot v_0$. This implies that $f\cdot v_0\neq 0$. So $U$ and $W$ are nonzero $\mathfrak{g}_{\bar 0}$-submodules. For $i\in\{0,\ldots,\lambda\}$, $U^{m_i/2}:=\mathcal{V}^{m_i}\cap U$ and $W^{m_i/2}:=\mathcal{V}^{m_i}\cap W$. Then

$$U=\sum_{i=0}^{\lambda/2}\mathbb{F}v_{2i}=U^{-\lambda/2}\oplus U^{-\lambda/2+2}\oplus\cdots\oplus U^{\lambda/2-2}\oplus U^{\lambda/2},$$
$$W=\sum_{i=0}^{\lambda/2-1}\mathbb{F}v_{2i+1}=W^{-\lambda/2+1}\oplus W^{-\lambda/2+3}\oplus\cdots\oplus W^{\lambda/2-3}\oplus W^{\lambda/2-1}.$$

Using the standard $\mathfrak{sl}_2$-module theory, we find that $v_i\neq 0$ for $0\leq i\leq\lambda$ and that both $U$ and $W$ are irreducible $\mathfrak{g}_{\bar 0}$-modules. If the homogeneous element $v_0$ is of degree $\bar{i}$ ($i\in\{0,1\}$), we have $\mathcal{V}_{\bar i}=U$ and $\mathcal{V}_{\bar i+\bar 1}=W$. This completes the proof. □

**Corollary 2.5.** *Each (nonzero) finite-dimensional irreducible $\mathfrak{osp}(1,2)$-module is of odd dimension. Moreover, suppose that $\lambda$ is a nonnegative even integer and $\mathcal{V}$ is a superspace with a basis $\{v_i\mid 0\leq i\leq\lambda\}$ of homogeneous elements such that $\{v_{2i}\mid 0\leq i\leq\lambda/2\}$ is a basis for either of $\mathcal{V}_{\bar 0}$ or $\mathcal{V}_{\bar 1}$. Let $(e,f,h)$ be an $\mathfrak{osp}(1,2)$-triple for a superalgebra $\mathfrak{g}$. Set $v_{\lambda+1}=v_{-1}:=0$ and define $\cdot:\mathfrak{g}\times\mathcal{V}\to\mathcal{V}$ by*

$$\begin{aligned}
&f\cdot v_i:=v_{i+1},\\
&e\cdot v_i:=\begin{cases}-iv_{i-1} & \text{if } i \text{ is even},\\ (\lambda-(i-1))v_{i-1} & \text{if } i \text{ is odd},\end{cases}\qquad \begin{aligned}[f,f]\cdot v_i&:=2f\cdot(f\cdot v_i),\\ [e,e]\cdot v_i&:=2e\cdot(e\cdot v_i),\end{aligned}\\
&h\cdot v_i=(\lambda-2i)v_i,
\end{aligned}$$

*for $0\leq i\leq\lambda$. Then up to isomorphism, $(\mathcal{V},\cdot)$ is the unique finite-dimensional irreducible $\mathfrak{g}$-module of dimension $\lambda+1$.*

**Theorem 2.6.** *Each finite-dimensional $\mathfrak{osp}(1,2)$-module is completely reducible.*

*Proof.* Fix an $\mathfrak{osp}$-triple $(e,f,h)$ for $\mathfrak{g} := \mathfrak{osp}(1,2)$ and consider the $\mathfrak{sl}_2$-triple $\big(\frac{1}{4}[e,e], -\frac{1}{4}[f,f], \frac{1}{2}h\big)$ for $\mathfrak{g}_{\bar{0}}$ as in Lemma 2.2. Suppose that $\mathcal{V}$ is a finite-dimensional $\mathfrak{osp}(1,2)$-module with the corresponding representation $\pi$. We know from Lemma 2.4 that $\pi(h)$ is diagonalizable with even integer eigenvalues. Moreover, $\mathcal{V}_{\bar{0}}$ and $\mathcal{V}_{\bar{1}}$ are both finite-dimensional $\mathfrak{g}_{\bar{0}}$-submodules of $\mathcal{V}$. Suppose that $\mathcal{V}_{\bar{0}} = \bigoplus_{j=1}^{n} W^j$ is a decomposition of $\mathcal{V}$ into irreducible $\mathfrak{g}_{\bar{0}}$-modules. For each $1 \le j \le n$, let $w_j$ be an eigenvector of the largest eigenvalue $\lambda_j$ of $\pi\big(\frac{1}{2}h\big)$ on $W^j$. We have

$$W^j = \bigoplus_{k=0}^{\lambda_j} \mathbb{F}[f,f]^k \cdot w_j = \bigoplus_{k=0}^{\lambda_j} \mathbb{F}f^{2k} \cdot w_j \tag{2.1}$$

by $\mathfrak{sl}_2$-module theory. For $j \in \{1,\dots,n\}$, set

$$T^j := \sum_{k=0}^{\infty} \mathbb{F}f^{2k} \cdot (e \cdot w_j), \qquad S^j := \sum_{k=0}^{\infty} \mathbb{F}f^{2k+1} \cdot (f \cdot (e \cdot w_j) - (2\lambda_j + 2)w_j).$$

We carry out the proof in the following steps:

**Claim 1.** *If $y$ is an eigenvector of $\pi(h)$ of eigenvalue $2\lambda$ such that $e \cdot y = 0$, then $f^{2\lambda+1} \cdot y = 0$.*

As $h \cdot (f^k \cdot y) = (2\lambda - 2k) f^k \cdot y$ for $k \in \mathbb{Z}^{\ge 0}$ and $\mathcal{V}$ is finite-dimensional, there is $k \in \mathbb{Z}^{\ge 0}$ such that $f^k \cdot y \neq 0$ but $f^{k+1} \cdot y = 0$. Therefore,

$$0 = e \cdot (f^{k+1} \cdot y) = \begin{cases} -(k+1) f^k \cdot y & \text{if } k \text{ is odd}, \\ (2\lambda - k) f^k \cdot y & \text{if } k \text{ is even}. \end{cases}$$

This implies that $k = 2\lambda$, and we are done.

**Claim 2.** *For each $j \in \{1,\dots,n\}$,*

$$T^j = \sum_{k=0}^{\lambda_j+1} \mathbb{F}f^{2k} \cdot (e \cdot w_j), \qquad S^j = \sum_{k=0}^{\lambda_j-1} \mathbb{F}f^{2k+1} \cdot \big(f \cdot (e \cdot w_j) - (2\lambda_j + 2)w_j\big).$$

Since $w_j$ is an eigenvector of $\pi(h)$ with eigenvalue $2\lambda_j$, we have $h \cdot (e \cdot w_j) = (2\lambda_j + 2)(e \cdot w_j)$ and $h \cdot (f \cdot (e \cdot w_j)) = 2\lambda_j f \cdot (e \cdot w_j)$. Also since $[e,e] \cdot w_j = 0$, as before, we have

$$e \cdot (e \cdot w_j) = 0, \qquad e \cdot \big(f \cdot (e \cdot w_j) - (2\lambda_j + 2)w_j\big) = 0,$$

and so we are done using Claim 1.

**Claim 3.** *We have*

$$\mathcal{V}_{\bar{0}} = \sum_{j=1}^{n} \sum_{k=0}^{\lambda_j} \mathbb{F}f^{2k+1} \cdot (e \cdot w_j) + \sum_{j=1}^{n} \sum_{k=0}^{\lambda_j} \mathbb{F}f^{2k} \cdot \big(f \cdot (e \cdot w_j) - (2\lambda_j + 2)w_j\big).$$

Let $X$ be the right hand side. For each $j \in \{1, \dots, n\}$,

$$w_j = \frac{1}{2\lambda_j + 2}\big(f \cdot (e \cdot w_j) - (f \cdot (e \cdot w_j) - (2\lambda_j + 2)w_j)\big).$$

Therefore

$$w_j \in \sum_{k=0}^{\lambda_j} \mathbb{F} f^{2k+1} \cdot (e \cdot w_j) + \sum_{k=0}^{\lambda_j} \mathbb{F} f^{2k} \cdot \big(f \cdot (e \cdot w_j) - (2\lambda_j + 2)w_j\big) \subseteq X$$

for all $j \in \{1, \dots, n\}$. This completes the proof as $X \subseteq \mathcal{V}_{\bar{0}}$ and $\{w_j \mid 1 \leq j \leq n\}$ is a set of generators for the $\mathfrak{g}_{\bar{0}}$-module $\mathcal{V}_{\bar{0}}$.

**Claim 4.** *For each $j \in \{1, \dots, n\}$, let $U^j$ be the $\mathfrak{g}_{\bar{0}}$-submodule of $\mathcal{V}$ generated by $f \cdot w_j$. Then $\mathcal{V}_{\bar{1}} = P + \sum_{j=1}^n U^j$ where $P := \{0\}$ if $\mathcal{V}_{\bar{1}}$ has no one-dimensional irreducible $\mathfrak{g}_{\bar{0}}$-submodule, and otherwise $P$ is the sum of all one-dimensional irreducible $\mathfrak{g}_{\bar{0}}$-submodules of $\mathcal{V}_{\bar{1}}$.*

Take $U := P + \sum_{j=1}^n U^j$. Since $\mathcal{V}_{\bar{1}}$ is a completely reducible $\mathfrak{g}_{\bar{0}}$-module, there is a $\mathfrak{g}_{\bar{0}}$-submodule $K$ of $\mathcal{V}_{\bar{1}}$ such that $\mathcal{V}_{\bar{1}} = U \oplus K$. If $K \neq \{0\}$, we pick an irreducible $\mathfrak{g}_{\bar{0}}$-submodule $S$ of $K$ and suppose $u$ is an eigenvector for the largest eigenvalue $\lambda$ of the action of $\frac{1}{2}h$ on $S$. Since $S \cap U \subseteq K \cap U = \{0\}$, $S$ is not one-dimensional, so $\lambda$ is positive and $2f \cdot (f \cdot u) = [f, f] \cdot u \neq 0$. But $f \cdot u \in \mathcal{V}_{\bar{0}} = \sum_{j=1}^n \sum_{k=0}^{\lambda_j} \mathbb{F} f^{2k} \cdot w_j$, so $f \cdot (f \cdot u) \in \sum_{j=1}^n \sum_{k=0}^{\lambda_j} \mathbb{F} f^{2k} \cdot f \cdot w_j \in U$. This means that $0 \neq f \cdot (f \cdot u) \in S \cap U \subseteq K \cap U = \{0\}$, a contradiction. Therefore, $K = \{0\}$ and we are done.

**Claim 5.** *$\mathcal{V}$ is a sum of irreducible $\mathfrak{g}$-modules.*

Let $U^j$ and $P$ be as in Claim 4. Fix a basis $\{v_1, \dots, v_m\}$ of $P$ if $P$ is not zero and set $x_i := f \cdot (e \cdot v_i) - 2v_i$ for $i \in \{1, \dots, m\}$. Then for $i \in \{1, \dots, m\}$, $\mathbb{F}x_i$ is a trivial one-dimensional $\mathfrak{g}$-submodule of $\mathcal{V}$, and as $e \cdot v_i \in \mathcal{V}_{\bar{0}}$, (2.1) implies that $v_i = \frac{1}{2}(f \cdot (e \cdot v_i) - x_i) \in \sum_{j=1}^n U^j + \mathbb{F}x_i$. Moreover, by Lemma 2.3, $U^j = T^j + S^j$ for $j \in \{1, \dots, n\}$, and by Claims 2–4, we have

$$\mathcal{V} = \sum_{j=1}^{n} \sum_{k=0}^{2\lambda_j+2} \mathbb{F} f^k \cdot (e \cdot w_j) + \sum_{j=1}^{n} \sum_{k=0}^{2\lambda_j} \mathbb{F} f^k \cdot \big(f \cdot (e \cdot w_j) - (\lambda_j + 2)w_j\big) + \sum_{i=1}^{m} \mathbb{F}x_i$$

in which the last sum disappears if $P = \{0\}$. This together with Claim 1 and Corollary 2.5 completes the proof. □

## §3. Extended affine Lie superalgebras

We call a triple $(\mathcal{L}, \mathcal{H}, (\cdot, \cdot))$, consisting of a nonzero Lie superalgebra $\mathcal{L} = \mathcal{L}_{\bar{0}} \oplus \mathcal{L}_{\bar{1}}$, a nontrivial subalgebra $\mathcal{H}$ of $\mathcal{L}_{\bar{0}}$ and a nondegenerate invariant even supersymmetric bilinear form $(\cdot, \cdot)$ on $\mathcal{L}$, a *super-torus* if:

- $\mathcal{L}$ has a weight space decomposition $\mathcal{L} = \bigoplus_{\alpha \in \mathcal{H}^*} \mathcal{L}^\alpha$ with respect to $\mathcal{H}$ via the adjoint representation. We note that in this case $\mathcal{H}$ is abelian; also since $\mathcal{L}_{\bar{0}}$ and $\mathcal{L}_{\bar{1}}$ are $\mathcal{H}$-submodules of $\mathcal{L}$, we have $\mathcal{L}_{\bar{0}} = \bigoplus_{\alpha \in \mathcal{H}^*} \mathcal{L}_{\bar{0}}^\alpha$ and $\mathcal{L}_{\bar{1}} = \bigoplus_{\alpha \in \mathcal{H}^*} \mathcal{L}_{\bar{1}}^\alpha$ with $\mathcal{L}_{\bar{i}}^\alpha := \mathcal{L}_{\bar{i}} \cap \mathcal{L}^\alpha$, $i = 0, 1$ [8, Prop. 2.1.1].
- The restriction of the form $(\cdot, \cdot)$ to $\mathcal{H}$ is nondegenerate.

We call $R := \{\alpha \in \mathcal{H}^* \mid \mathcal{L}^\alpha \neq \{0\}\}$ the *root system* of $\mathcal{L}$ (with respect to $\mathcal{H}$). Each element of $R$ is called a *root*. We refer to elements of $R_0 := \{\alpha \in \mathcal{H}^* \mid \mathcal{L}_{\bar{0}}^\alpha \neq \{0\}\}$ (resp. $R_1 := \{\alpha \in \mathcal{H}^* \mid \mathcal{L}_{\bar{1}}^\alpha \neq \{0\}\}$) as *even roots* (resp. *odd roots*). We note that $R = R_0 \cup R_1$. Suppose that $(\mathcal{L}, \mathcal{H}, (\cdot, \cdot))$ is a super-torus and $\mathfrak{p} : \mathcal{H} \to \mathcal{H}^*$ maps $h \in \mathcal{H}$ to $(h, \cdot)$. Since the form is nondegenerate on $\mathcal{H}$, this map is injective. So for each $\alpha \in \mathcal{H}^{\mathfrak{p}} := \mathfrak{p}(\mathcal{H})$, there is a unique $t_\alpha \in \mathcal{H}$ representing $\alpha$ through $(\cdot, \cdot)$. Now we can transfer the form on $\mathcal{H}$ to a form on $\mathcal{H}^{\mathfrak{p}}$, denoted again by $(\cdot, \cdot)$, and defined by

$$(\alpha, \beta) := (t_\alpha, t_\beta) \quad (\alpha, \beta \in \mathcal{H}^{\mathfrak{p}}).$$

**Lemma 3.1.** *Suppose that $(\mathcal{L}, \mathcal{H}, (\cdot, \cdot))$ is a super-torus with root system $R = R_0 \cup R_1$. Then:*

(i) *For $\alpha, \beta \in \mathcal{H}^*$, $[\mathcal{L}^\alpha, \mathcal{L}^\beta] \subseteq \mathcal{L}^{\alpha+\beta}$. Also for $i = 0, 1$ and $\alpha, \beta \in R_i$, we have $(\mathcal{L}_{\bar{i}}^\alpha, \mathcal{L}_{\bar{i}}^\beta) = \{0\}$ unless $\alpha + \beta = 0$; in particular, $R_0 = -R_0$ and $R_1 = -R_1$.*

(ii) *If $\alpha \in \mathcal{H}^{\mathfrak{p}}$ and $x_{\pm\alpha} \in \mathcal{L}^{\pm\alpha}$ with $[x_\alpha, x_{-\alpha}] \in \mathcal{H}$, then $[x_\alpha, x_{-\alpha}] = (x_\alpha, x_{-\alpha}) t_\alpha$.*

(iii) *If $\alpha \in R_i \setminus \{0\}$ ($i \in \{0, 1\}$), $x_\alpha \in \mathcal{L}_{\bar{i}}^\alpha$ and $x_{-\alpha} \in \mathcal{L}_{\bar{i}}^{-\alpha}$ with $[x_\alpha, x_{-\alpha}] \in \mathcal{H} \setminus \{0\}$, then $(x_\alpha, x_{-\alpha}) \neq 0$ and $\alpha \in \mathcal{H}^{\mathfrak{p}}$.*

*Proof.* (i) This is easy to see.

(ii) For $h \in \mathcal{H}$, we have

$$(h, [x_\alpha, x_{-\alpha}]) = ([h, x_\alpha], x_{-\alpha}) = \alpha(h)(x_\alpha, x_{-\alpha}). \tag{3.1}$$

Therefore

$$(h, [x_\alpha, x_{-\alpha}]) = \alpha(h)(x_\alpha, x_{-\alpha}) = (t_\alpha(x_\alpha, x_{-\alpha}), h).$$

This together with the fact that the form on $\mathcal{H}$ is symmetric and nondegenerate completes the proof.

(iii) Suppose to the contrary that $(x_\alpha, x_{-\alpha}) = 0$. Then (3.1) implies that for all $h \in \mathcal{H}$, $(h, [x_\alpha, x_{-\alpha}]) = 0$; but the form on $\mathcal{H}$ is nondegenerate, so $[x_\alpha, x_{-\alpha}] = 0$, a contradiction. Again using (3.1), we see that $\big(h, \frac{1}{(x_\alpha, x_{-\alpha})}[x_\alpha, x_{-\alpha}]\big) = \alpha(h)$ for all $h \in \mathcal{H}$ and so $\alpha = \mathfrak{p}\big(\frac{1}{(x_\alpha, x_{-\alpha})}[x_\alpha, x_{-\alpha}]\big) \in \mathcal{H}^{\mathfrak{p}}$. □

**Definition 3.2.** A super-torus $(\mathcal{L} = \mathcal{L}_{\bar{0}} \oplus \mathcal{L}_{\bar{1}}, \mathcal{H}, (\cdot, \cdot))$ (or $\mathcal{L}$ if there is no confusion), with root system $R = R_0 \cup R_1$, is called an *extended affine Lie superalgebra* if

(1) for $\alpha \in R_i \setminus \{0\}$ ($i \in \{0,1\}$), there are $x_\alpha \in \mathcal{L}_{\bar{i}}^{\alpha}$ and $x_{-\alpha} \in \mathcal{L}_{\bar{i}}^{-\alpha}$ such that $0 \neq [x_\alpha, x_{-\alpha}] \in \mathcal{H}$,

(2) for $\alpha \in R$ with $(\alpha,\alpha) \neq 0$ and $x \in \mathcal{L}^\alpha$, the map $\mathrm{ad}_x : \mathcal{L} \to \mathcal{L}$, sending $y \in \mathcal{L}$ to $[x,y]$, is a locally nilpotent linear transformation.

The extended affine Lie superalgebra $(\mathcal{L}, \mathcal{H}, (\cdot,\cdot))$ is called an *invariant affine reflection algebra* [10] if $\mathcal{L}_{\bar{1}} = \{0\}$, and a *locally extended affine Lie algebra* [9] if $\mathcal{L}_{\bar{1}} = \{0\}$ and $\mathcal{L}^0 = \mathcal{H}$. Finally, a locally extended affine Lie algebra $(\mathcal{L}, \mathcal{H}, (\cdot,\cdot))$ is called an *extended affine Lie algebra* [1] if $\mathcal{L}^0 = \mathcal{H}$ is a finite-dimensional subalgebra of $\mathcal{L}$.

We immediately have the following:

**Proposition 3.3.** *If $(\mathcal{L}, \mathcal{H}, (\cdot,\cdot))$ is an extended affine Lie superalgebra, then the triple $(\mathcal{L}_{\bar{0}}, \mathcal{H}, (\cdot,\cdot)|_{\mathcal{L}_{\bar{0}} \times \mathcal{L}_{\bar{0}}})$ is an invariant affine reflection algebra.*

**Example 3.4.** Finite-dimensional basic classical simple Lie superalgebras [7] and affine Lie superalgebras [11] are examples of extended affine Lie superalgebras.

Although in contrast with extended affine Lie algebras, for an extended affine Lie superalgebra $(\mathcal{L}, \mathcal{H}, (\cdot,\cdot))$, the root space corresponding to zero does not necessarily coincide with $\mathcal{H}$, we call such Lie superalgebras extended affine because in view of Example 3.4, they are a wide extension of affine Lie (super)algebras. Also under some natural conditions on $\mathcal{L}$ (see Proposition 3.10(iii)), non-self-orthogonal roots of the root system $R$ of $\mathcal{L}$ fall into two disjoint parts corresponding to the even and odd parts of $\mathcal{L}$, so one may refer to $R$ as a "root supersystem". We shall prove that the root system of an extended affine Lie superalgebra $(\mathcal{L}, \mathcal{H}, (\cdot,\cdot))$ is an extended affine root supersystem in the following sense:

**Definition 3.5.** Suppose that $A$ is a nontrivial additive abelian group, $(\cdot,\cdot) : A \times A \to \mathbb{F}$ is a symmetric form with radical $A^0$, and $R$ is a subset of $A$. Set

$$\begin{aligned} R^0 &:= R \cap A^0, & R^\times &:= R \setminus R^0, \\ R_{\mathrm{re}}^\times &:= \{\alpha \in R \mid (\alpha,\alpha) \neq 0\}, & R_{\mathrm{re}} &:= R_{\mathrm{re}}^\times \cup \{0\}, \\ R_{\mathrm{ns}}^\times &:= \{\alpha \in R \setminus R^0 \mid (\alpha,\alpha) = 0\}, & R_{\mathrm{ns}} &:= R_{\mathrm{ns}}^\times \cup \{0\}. \end{aligned}$$

We say $(A, (\cdot,\cdot), R)$ is an *extended affine root supersystem* if:

(S1) $0 \in R$, and $\langle R \rangle = A$,

(S2) $R = -R$,

(S3) $2(\alpha,\beta)/(\alpha,\alpha) \in \mathbb{Z}$ for $\alpha \in R_{\mathrm{re}}^\times$ and $\beta \in R$, (S4) the *root string property* holds in $R$ in the sense that for $\alpha \in R_{\rm re}^{\times}$ and $\beta \in R$, there are nonnegative integers $p, q$ with $2(\beta,\alpha)/(\alpha,\alpha) = p - q$ such that

$$\{\beta + k\alpha \mid k \in \mathbb{Z}\} \cap R = \{\beta - p\alpha, \dots, \beta + q\alpha\},$$

(S5) $\{\beta - \alpha, \beta + \alpha\} \cap R \neq \emptyset$ for $\alpha \in R_{\rm ns}$ and $\beta \in R$ with $(\alpha, \beta) \neq 0$.

If there is no confusion, for simplicity we say $R$ is an *extended affine root supersystem in $A$.*

Extended affine root supersystems have been systematically studied in [12].

**Lemma 3.6.** *Suppose that $(\mathcal{L}, \mathcal{H}, (\cdot,\cdot))$ is an extended affine Lie superalgebra with root system $R = R_0 \cup R_1$. Suppose that $\alpha \in R_i$ ($i \in \{0,1\}$) with $(\alpha,\alpha) \neq 0$. Recall that $t_\alpha$ is the unique element of $\mathcal{H}$ representing $\alpha$ through the form $(\cdot,\cdot)$ restricted to $\mathcal{H}$ and set $h_\alpha := 2t_\alpha/(\alpha,\alpha)$. Then there are $y_{\pm\alpha} \in \mathcal{L}_{\bar{i}}^{\pm\alpha}$ such that $(y_\alpha, y_{-\alpha}, h_\alpha)$ is an $\mathfrak{sl}_2$-super triple for the subsuperalgebra $\mathcal{G}(\alpha)$ generated by $\{y_\alpha, y_{-\alpha}, h_\alpha\}$; in particular, if $\alpha \in R_1 \cap R_{\rm re}^{\times}$, then $2\alpha \in R_0$.*

*Proof.* Suppose that $i \in \{0,1\}$ and $\alpha \in R_i$ with $(\alpha,\alpha) \neq 0$. Fix $x_{\pm\alpha} \in \mathcal{L}_{\bar{i}}^{\pm\alpha}$ with $0 \neq [x_\alpha, x_{-\alpha}] \in \mathcal{H}$. Considering Lemma 3.1 and setting $e_\alpha := x_\alpha$ and $e_{-\alpha} := x_{-\alpha}/(x_\alpha, x_{-\alpha})$, we have $[e_\alpha, e_{-\alpha}] = t_\alpha$. Now we see that $(y_\alpha := 2e_\alpha/(t_\alpha,t_\alpha), y_{-\alpha} := e_{-\alpha}, h_\alpha)$ is an $\mathfrak{sl}_2$-super triple for $\mathcal{G}(\alpha)$. Next suppose $\alpha \in R_1 \cap R_{\rm re}^{\times}$. Using Lemma 2.2, we find that $0 \neq [y_\alpha, y_\alpha] \subseteq \mathcal{L}_{\bar{0}}^{2\alpha}$ and so $2\alpha \in R_0$. □

**Lemma 3.7.** *If $i, j \in \{0,1\}$, $\alpha \in R_i$ and $\beta \in R_j$ with $(\alpha,\beta) \neq 0$, then either $\beta - \alpha \in R$ or $\beta + \alpha \in R$.*

*Proof.* Fix $0 \neq z \in \mathcal{L}_{\bar{j}}^{\beta}$ and $x \in \mathcal{L}_{\bar{i}}^{\alpha}$, $y \in \mathcal{L}_{\bar{i}}^{-\alpha}$ with $[x,y] \in \mathcal{H} \setminus \{0\}$. Using Lemma 3.1, we obtain $[x,y] = (x,y)t_\alpha$. Therefore

$$0 \neq (x,y)(\alpha,\beta)z = (x,y)[t_\alpha, z] = [[x,y],z] = [x,[y,z]] - (-1)^{|x|\,|y|}[y,[x,z]].$$

This in turn implies that either $[y,z] \neq 0$ or $[x,z] \neq 0$. Therefore either $\beta - \alpha \in R$ or $\beta + \alpha \in R$. □

**Proposition 3.8.** *Suppose that $(\mathcal{L}, \mathcal{H}, (\cdot,\cdot))$ is an extended affine Lie superalgebra with root system $R = R_0 \cup R_1$. For $\alpha, \beta \in R$ with $(\alpha,\alpha) \neq 0$, we have:*

(i) $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$, *in particular if $k \in \mathbb{F}$ and $k\alpha \in R$, then $k \in \{0, \pm1, \pm2, \pm1/2\}$.*

(ii) $r_\alpha(\beta) := \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha \in R$.

(iii) *There are nonnegative integers $p, q$ such that $p - q = 2(\beta,\alpha)/(\alpha,\alpha)$ and $\{k \in \mathbb{Z} \mid \beta + k\alpha \in R\} = \{-p, \dots, q\}$.*

*Proof.* Suppose that $\alpha, \beta \in R$ with $(\alpha, \alpha) \neq 0$. Assume $\alpha \in R_i$ for some $i \in \{0,1\}$. By Lemma 3.6, there are $y_\alpha \in \mathcal{L}_{\bar{i}}^{\alpha}, y_{-\alpha} \in \mathcal{L}_{\bar{i}}^{-\alpha}$ such that $[y_\alpha, y_{-\alpha}] = h_\alpha = 2t_\alpha/(\alpha,\alpha)$ and $(y_\alpha, y_{-\alpha}, h_\alpha)$ is an $\mathfrak{sl}_2$-super triple for the subsuperalgebra $\mathcal{G}(\alpha)$ of $\mathcal{L}$ generated by $\{y_\alpha, y_{-\alpha}, h_\alpha\}$. Consider $\mathcal{L}$ as a $\mathcal{G}(\alpha)$-module via the adjoint representation. Then $\mathcal{M} := \sum_{k\in\mathbb{Z}} \mathcal{L}^{\beta+k\alpha}$ is a $\mathcal{G}(\alpha)$-submodule of $\mathcal{L}$. For $k \in \mathbb{Z}$ and $x \in \mathcal{L}^{\beta+k\alpha}$, set

$$\mathcal{M}(x) := \mathrm{span}_{\mathbb{F}}\{\mathrm{ad}_{y_{-\alpha}}^n \mathrm{ad}_{y_\alpha}^m x \mid m,n \in \mathbb{Z}^{\geq 0}\}.$$

We claim that $\mathcal{M}(x)$ is the $\mathfrak{g}(\alpha)$-submodule of $\mathcal{M}$ generated by $x$. Indeed, as $\mathcal{G}(\alpha)$ is generated by $\{y_\alpha, y_{-\alpha}, h_\alpha\}$, it is enough to show that $\mathcal{M}(x)$ is invariant under $\mathrm{ad}_{h_\alpha}, \mathrm{ad}_{y_\alpha}, \mathrm{ad}_{y_{-\alpha}}$. By Lemma 3.1(i), for $m,n \in \mathbb{Z}^{\geq 0}$, we have $\mathrm{ad}_{y_{-\alpha}}^n \mathrm{ad}_{y_\alpha}^m x \in \mathcal{L}^{\beta+k\alpha+m\alpha-n\alpha}$, so $\mathcal{M}(x)$ is invariant under the action of $h_\alpha$. Also it is trivial that $\mathcal{M}(x)$ is invariant under the action of $y_{-\alpha}$. We finally show that it is invariant under $\mathrm{ad}_{y_\alpha}$. We use induction on $n$ to prove that $[y_\alpha, \mathrm{ad}_{y_{-\alpha}}^n \mathrm{ad}_{y_\alpha}^m x] \in \mathcal{M}(x)$ for $n,m \in \mathbb{Z}^{\geq 0}$. If $n=0$, there is nothing to prove, so we assume that $n \in \mathbb{Z}^{\geq 1}$ and $[y_\alpha, \mathrm{ad}_{y_{-\alpha}}^{n-1} \mathrm{ad}_{y_\alpha}^m x] \in \mathcal{M}(x)$ for all $m \in \mathbb{Z}^{\geq 0}$. Now for $m \in \mathbb{Z}^{\geq 0}$, we have

$$\begin{aligned}[y_\alpha, \mathrm{ad}_{y_{-\alpha}}^n \mathrm{ad}_{y_\alpha}^m x] &= [y_\alpha, [y_{-\alpha}, \mathrm{ad}_{y_{-\alpha}}^{n-1} \mathrm{ad}_{y_\alpha}^m x]] \\ &= (-1)^{|y_\alpha|}[y_{-\alpha}, [y_\alpha, \mathrm{ad}_{y_{-\alpha}}^{n-1} \mathrm{ad}_{y_\alpha}^m x]] + [h_\alpha, \mathrm{ad}_{y_{-\alpha}}^{n-1} \mathrm{ad}_{y_\alpha}^m x].\end{aligned}$$

This together with the induction hypothesis and the fact that $\mathcal{M}(x)$ is invariant under $\mathrm{ad}_{h_\alpha}$ and $\mathrm{ad}_{y_{-\alpha}}$ completes the induction process.

Now we are ready to prove the proposition. We keep the notation above.

(i) Since $\mathrm{ad}_{y_{-\alpha}}$ and $\mathrm{ad}_{y_\alpha}$ are locally nilpotent linear transformations, for $x \in \mathcal{L}^{\beta+k\alpha}$ $(k \in \mathbb{Z})$, $\mathcal{M}(x)$ is finite-dimensional, so $\mathcal{M}$ is a sum of finite-dimensional $\mathcal{G}(\alpha)$-submodules $\mathcal{M}(x)$ $(x \in \mathcal{L}^{\beta+k\alpha}, k \in \mathbb{Z})$. We know that

(3.2) $h_\alpha$ acts diagonally on $\mathcal{M}$ with eigenvalues $\{\beta(h_\alpha)+2k \mid k \in \mathbb{Z}, \beta+k\alpha \in R\}$. Moreover, this set of eigenvalues is the union of the set of eigenvalues of the action of $h_\alpha$ on the finite-dimensional $\mathcal{G}(\alpha)$-submodules $\mathcal{M}(x)$ $(x \in \mathcal{L}^{\beta+k\alpha}, k \in \mathbb{Z})$ of $\mathcal{M}$.

Since $\mathcal{L}^\beta \neq \{0\}$, each nonzero element of $\mathcal{L}^\beta$ is an eigenvector of $\mathrm{ad}_{h_\alpha}$ restricted to $\mathcal{M}$ corresponding to the eigenvalue $\beta(h_\alpha)$. Therefore $\beta(h_\alpha)$ is an eigenvalue of $\mathrm{ad}_{h_\alpha}$ restricted to a finite-dimensional $\mathcal{G}(\alpha)$-submodule $\mathcal{M}(x)$ for some $x \in \mathcal{L}^{\beta+k\alpha}$ $(k \in \mathbb{Z})$, and so using $\mathfrak{sl}_2$-module theory together with Lemma 2.4, we conclude that $2(\beta,\alpha)/(\alpha,\alpha) = \beta(h_\alpha) \in \mathbb{Z}$.

(ii) As in the previous case, $\beta(h_\alpha)$ is an eigenvalue of $\mathrm{ad}_{h_\alpha}$ restricted to a finite-dimensional $\mathcal{G}(\alpha)$-submodule $\mathcal{M}(x)$ of $\mathcal{M}$ for some $x \in \mathcal{L}^{\beta+k\alpha}$ $(k \in \mathbb{Z})$. By Lemma 2.4 and $\mathfrak{sl}_2$-module theory, $-\beta(h_\alpha)$ is also an eigenvalue for $\mathrm{ad}_{h_\alpha}$ on $\mathcal{M}(x)$. So there is an integer $k$ such that $\beta+k\alpha \in R$ and $-\beta(h_\alpha) = \beta(h_\alpha)+2k$. This implies $k = -\beta(h_\alpha)$. In particular, $\beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha = \beta - \beta(h_\alpha)\alpha = \beta + k\alpha \in R$.

(iii) We first prove that $\{k \in \mathbb{Z} \mid \beta + k\alpha \in R\}$ is an interval. To this end, we take $r, s \in \mathbb{Z}$ with $\beta+r\alpha, \beta+s\alpha \in R$ and show that $\beta+t\alpha \in R$ for all $t$ between $r, s$. Without loss of generality, we may assume $|\beta(h_\alpha) + 2r| \geq |\beta(h_\alpha) + 2s|$. Suppose that $t$ is an integer between $r, s$. Since $\beta + t\alpha \in R$ if and only if $-\beta - t\alpha \in R$, we simultaneously replace $\beta$ with $-\beta$ and $(r, s)$ with $(-r, -s)$ if necessary and assume $\beta(h_\alpha) + 2r \geq 0$. So $-\beta(h_\alpha) - 2r \leq \beta(h_\alpha) + 2s \leq \beta(h_\alpha) + 2r$. But using (3.2), we find that $\beta(h_\alpha) + 2r$ is an eigenvalue of the action of $h_\alpha$ on $\mathcal{M}(x)$ for some $x \in \mathcal{L}^{\beta+k\alpha}$ $(k \in \mathbb{Z})$. So by Lemma 2.4 and Theorem 2.6, $\beta(h_\alpha)+2t$ is an eigenvalue for the action of $h_\alpha$ on $\mathcal{M}(x)$. Therefore, $\beta + t\alpha \in R$ by (3.2).

We next show that this interval is bounded. For $k \in \mathbb{Z}$, we find that $(\beta + k\alpha, \beta + k\alpha) = (\beta, \beta) + 2k(\beta, \alpha) + k^2(\alpha, \alpha)$. So there are at most two integers $k$ such that $\beta + k\alpha \notin R^\times_{\text{re}}$. Now to the contrary assume that the relevanr interval is not bounded. Without loss of generality, we may assume there is a positive integer $k_0$ such that for $k \in \mathbb{Z}^{\geq k_0}$, we have $\beta + k\alpha \in R^\times_{\text{re}}$ and $(\alpha, \beta + k\alpha) \neq 0$. For $k \in \mathbb{Z}^{\geq k_0}$,

$$\frac{\frac{2(\alpha,\beta)}{(\alpha,\alpha)} + 2k}{\frac{(\beta,\beta)}{(\alpha,\alpha)} + \frac{2k(\alpha,\beta)}{(\alpha,\alpha)} + k^2} = \frac{2(\alpha, \beta) + 2k(\alpha, \alpha)}{(\beta, \beta) + 2k(\alpha, \beta) + k^2(\alpha, \alpha)} = \frac{2(\alpha, \beta + k\alpha)}{(\beta + k\alpha, \beta + k\alpha)} \in \mathbb{Z}.$$

Now as $k, 2(\alpha, \beta)/(\alpha, \alpha) \in \mathbb{Z}$, we see that $(\beta, \beta)/(\alpha, \alpha) \in \mathbb{Q}$, so

$$\lim_{k\to\infty} \frac{\frac{2(\alpha,\beta)}{(\alpha,\alpha)} + 2k}{\frac{(\beta,\beta)}{(\alpha,\alpha)} + \frac{2k(\alpha,\beta)}{(\alpha,\alpha)} + k^2} = 0.$$

This is a contradiction as it is a sequence of nonzero integers. Therefore we have a bounded interval. Suppose $p, q$ are the largest nonnegative integers with $\beta - p\alpha, \beta + q\alpha \in R$. Since $r_\alpha(\beta - p\alpha) = \beta + q\alpha$, we find that $p - q = 2(\beta, \alpha)/(\alpha, \alpha)$. □

**Corollary 3.9.** *If $(\mathcal{L}, (\cdot, \cdot), \mathcal{H})$ is an extended affine Lie superalgebra with root system $R$, then $R$ is an extended affine root supersystem in its $\mathbb{Z}$-span.*

*Proof.* This is immediate using Lemma 3.7 together with Proposition 3.8. □

**Proposition 3.10.** *Suppose that $(\mathcal{L}, (\cdot, \cdot), \mathcal{H})$ is an extended affine Lie superalgebra with root system $R$.*

(i) *For $\alpha \in R^\times_{\text{re}}$, we have $2\alpha \notin R_1$; also if $\alpha \in R_{\text{re}}$ and $2\alpha \notin R$, then $\alpha \in R_0$.*
(ii) *If $\alpha \in R_0$ with $(\alpha, \alpha) = 0$, then $(\alpha, R_0) = \{0\}$; moreover, $R_0 \cap R_{\text{ns}} = \{0\}$.*
(iii) *If $\mathcal{L}^0 \subseteq \mathcal{L}_{\bar{0}}$, then $R^\times \cap R_0 \cap R_1 = \emptyset$.*

*Proof.* (i) We know from Proposition 3.8(i) that $4\alpha \notin R$. Now the result is immediate from Lemma 3.6(i).

(ii) Although the first assertion can be obtained by modifying the argument in [9, Prop. 3.4] and [1, Prop. I.2.1], for the convenience of the readers, we give the proof. To the contrary, suppose $\alpha, \beta \in R_0$ with $(\alpha, \alpha) = 0$ and $(\alpha, \beta) \neq 0$. If there are infinitely many consecutive integers $n$ with $\beta + n\alpha \in R_0 \cap R_{\rm re}^\times$, then for such $n$, we have

$$\frac{2(\alpha, \beta + n\alpha)}{(\beta + n\alpha, \beta + n\alpha)} = \frac{2(\alpha, \beta)}{(\beta, \beta) + 2n(\alpha, \beta)} \in \mathbb{Z}.$$

Therefore, $(\beta, \beta) \neq 0$ and

$$\frac{\frac{2(\alpha,\beta)}{(\beta,\beta)}}{1 + 2n\frac{(\alpha,\beta)}{(\beta,\beta)}} \in \mathbb{Z},$$

which is absurd as it is a sequence of nonzero integers converging to 0. Hence, there is $p \in \mathbb{Z}$ with $\gamma := \beta + p\alpha \in R_0$ and $\gamma - \alpha \notin R_0$. Fix $0 \neq x \in \mathcal{L}_{\bar{0}}^{\gamma}$ and $y_{\pm\alpha} \in \mathcal{L}_{\bar{0}}^{\pm\alpha}$ with $[y_{-\alpha}, y_\alpha] = t_\alpha$. Setting $x_0 := x, x_n := (\mathrm{ad}_{y_\alpha})^n x$, for $n \in \mathbb{Z}^{\geq 1}$, we have $\mathrm{ad}_{y_{-\alpha}}(x_n) = n(\alpha, \gamma)x_{n-1}$. So $x_n \neq 0$ for all $n \in \mathbb{Z}^{\geq 0}$. This means that for consecutive integers $k_n := p + n$ $(n \in \mathbb{Z}^{\geq 0})$, we have $\beta + k_n\alpha \in R_0$, a contradiction.

For the last assertion, suppose $0 \neq \alpha \in R_0 \cap R_{\rm ns}$. Since $\alpha \notin R^0$, there is $\beta \in R$ with $(\alpha, \beta) \neq 0$. If $\beta \in R_{\rm ns}$, there is $r \in \{\pm 1\}$ with $\alpha + r\beta \in R_{\rm re}$ (see Lemma 3.7). Since $(\alpha + r\beta, \alpha) \neq 0$, we may always assume that there is a real root $\gamma$ with $(\alpha, \gamma) \neq 0$. But by the first assertion, $\gamma \in R_1 \cap R_{\rm re}^\times$ and so $2\gamma \in R_0$ by Lemma 3.6. This implies that $(\alpha, 2\gamma) = 0$, a contradiction.

(iii) To the contrary, suppose that $\alpha \in R^\times \cap R_0 \cap R_1$. By (ii), we get $\alpha \in R_{\rm re}^\times$. Consider Lemma 3.1 and fix $x_\alpha \in \mathcal{L}_{\bar{0}}^{\alpha}, y_\alpha \in \mathcal{L}_{\bar{0}}^{-\alpha}$ such that $[x_\alpha, y_\alpha] = t_\alpha$; also fix $e_\alpha \in \mathcal{L}_{\bar{1}}^{\alpha}$, $f_\alpha \in \mathcal{L}_{\bar{1}}^{-\alpha}$ with $[e_\alpha, f_\alpha] = t_\alpha$. We have $[y_\alpha, e_\alpha] \in \mathcal{L}_{\bar{1}}^{0} = \{0\}$ and by (i), $[x_\alpha, e_\alpha] \in \mathcal{L}_{\bar{1}}^{2\alpha} = \{0\}$, so we get

$$0 \neq (\alpha, \alpha)e_\alpha = [t_\alpha, e_\alpha] = [[x_\alpha, y_\alpha], e_\alpha] = [x_\alpha, [y_\alpha, e_\alpha]] - [y_\alpha, [x_\alpha, e_\alpha]] = 0,$$

a contradiction. □

**Example 3.11.** Suppose that $\Lambda$ is a torsion free additive abelian group and $\mathcal{G}$ is a locally finite basic classical simple Lie superalgebra, i.e., a direct union of finite-dimensional basic classical simple Lie superalgebras (see [13] for details) with a Cartan subalgebra $\mathcal{H}$ and a fixed nondegenerate invariant even bilinear form $f(\cdot, \cdot)$. One knows $(\mathcal{G}, \mathcal{H}, f(\cdot, \cdot))$ is an extended affine Lie superalgebra with $\mathcal{G}^0 = \mathcal{H}$. Suppose that $\theta : \Lambda \times \Lambda \to \mathbb{F} \setminus \{0\}$ is a *commutative* 2-*cocycle*, that is,

$$\theta(\zeta, \xi) = \theta(\xi, \zeta) \quad \text{and} \quad \theta(\zeta, \xi)\theta(\zeta + \xi, \eta) = \theta(\xi, \eta)\theta(\zeta, \xi + \eta)$$

for all $\zeta, \xi, \eta \in \Lambda$. Suppose that $\theta(0,0) = 1$ and note that this in turn implies that $\theta(0,\lambda) = 1$ for all $\lambda \in \Lambda$. Consider the $\mathbb{F}$-vector space $\mathcal{A} := \sum_{\lambda\in\Lambda} \mathbb{F}t^\lambda$ with a basis $\{t^\lambda \mid \lambda \in \Lambda\}$. Now $\mathcal{A}$ together with the product defined by

$$t^\zeta \cdot t^\xi := \theta(\zeta,\xi)t^{\zeta+\xi} \qquad (\xi, \zeta \in \Lambda)$$

is a $\Lambda$-graded unital commutative associative algebra with $\mathcal{A}^\lambda := \mathbb{F}t^\lambda$ (for $\lambda \in \Lambda$). We refer to $\mathcal{A}$ as the *commutative associative torus* corresponding to $(\Lambda, \theta)$. Set

$$\hat{\mathcal{G}} := \mathcal{G} \otimes \mathcal{A}$$

and define

$$|x \otimes a| := |x|, \qquad x \in \mathcal{G},\ a \in \mathcal{A}.$$

Then $\hat{\mathcal{G}}$ together with

$$[x \otimes a, y \otimes b]_{\hat{\mathcal{G}}} := [x,y] \otimes ab$$

for $x, y \in \mathcal{G}$ and $a, b \in \mathcal{A}$ is a Lie superalgebra. Now define

$$(x \otimes t^\lambda, y \otimes t^\mu) := \theta(\lambda,\mu)\delta_{\lambda+\mu,0} f(x,y)$$

for $x, y \in \mathcal{G}$ and $\lambda, \mu \in \Lambda$. This defines a nondegenerate invariant even supersymmetric bilinear form on $\hat{\mathcal{G}}$. Next take $\mathcal{V} := \mathbb{F} \otimes_{\mathbb{Z}} \Lambda$. Identify $\Lambda$ with a subset of $\mathcal{V}$ and fix a basis $B := \{\lambda_i \mid i \in I\} \subseteq \Lambda$ of $\mathcal{V}$. Suppose that $\{d_i \mid i \in I\}$ is the dual basis of $B$, and $\mathcal{V}^\dagger$ is the restricted dual of $\mathcal{V}$ with respect to this basis. Each $d \in \mathcal{V}^\dagger$ can also be considered as a derivation on $\hat{\mathcal{G}}$ mapping $x \otimes t^\lambda$ to $d(\lambda)x \otimes t^\lambda$ for $x \in \mathcal{G}$ and $\lambda \in \Lambda$; indeed, for $a, b \in \mathcal{G}$ and $\lambda, \mu \in \Lambda$, we have

$$\begin{aligned} d([a \otimes t^\lambda, b \otimes t^\mu]_{\hat{\mathcal{G}}}) &= d(\lambda+\mu)[a,b] \otimes t^\lambda t^\mu \\ &= d(\lambda)[a,b] \otimes t^\lambda t^\mu + d(\mu)[a,b] \otimes t^\lambda t^\mu \\ &= [d(a \otimes t^\lambda), b \otimes t^\mu]_{\hat{\mathcal{G}}} + [a \otimes t^\lambda, d(b \otimes t^\mu)]_{\hat{\mathcal{G}}}. \end{aligned}$$

Also for $a, b \in \mathcal{G}$, $d, d' \in \mathcal{V}^\dagger$ and $\lambda, \mu \in \Lambda$, we have

$$\begin{aligned} (d(a \otimes t^\lambda), b \otimes t^\mu) &= d(\lambda)(a \otimes t^\lambda, b \otimes t^\mu) = d(\lambda)\delta_{\lambda,-\mu}\theta(\lambda,\mu)f(a,b) \\ &= -d(\mu)\delta_{\lambda,-\mu}\theta(\lambda,\mu)f(a,b) = -d(\mu)(a \otimes t^\lambda, b \otimes t^\mu) \\ &= -(a \otimes t^\lambda, d(b \otimes t^\mu)) \end{aligned}$$

and

$$\begin{aligned} (dd'(a \otimes t^\lambda), b \otimes t^\mu) &= d(\lambda)d'(\lambda)(a \otimes t^\lambda, b \otimes t^\mu) = d(\lambda)d'(\lambda)\delta_{\lambda+\mu,0}\theta(\lambda,\mu)f(a,b) \\ &= -d(\lambda)d'(\mu)\delta_{\lambda+\mu,0}\theta(\lambda,\mu)f(a,b) = -d(\lambda)d'(\mu)(a \otimes t^\lambda, b \otimes t^\mu) \\ &= -(d(\lambda)a \otimes t^\lambda, d'(\mu)b \otimes t^\mu) = -(d(a \otimes t^\lambda), d'(b \otimes t^\mu)). \end{aligned}$$

Therefore,

$$(d(x),y)=-(x,d(y)),\qquad (dd'(x),y)=-(d(x),d'(y))\qquad (x,y\in\hat{\mathcal{G}},d,d'\in\mathcal{V}^\dagger). \tag{3.3}$$

Set

$$\mathfrak{L}:=\hat{\mathcal{G}}\oplus\mathcal{V}\oplus\mathcal{V}^\dagger=(\mathcal{G}\otimes\mathcal{A})\oplus\mathcal{V}\oplus\mathcal{V}^\dagger$$

and define

$$\begin{aligned}
&[d,x]=-[x,d]=d(x),\qquad d\in\mathcal{V}^\dagger,\ x\in\hat{\mathcal{G}},\\
&[\mathcal{V},\mathfrak{L}]=[\mathfrak{L},\mathcal{V}]=\{0\},\\
&[\mathcal{V}^\dagger,\mathcal{V}^\dagger]=\{0\},\\
&[x,y]=[x,y]_{\hat{\mathcal{G}}}+\sum_{i\in I}(d_i(x),y)\lambda_i,\qquad x,y\in\hat{\mathcal{G}}.
\end{aligned}\tag{3.4}$$

**Lemma 3.12.** *Set $\mathfrak{L}_{\bar{0}}:=\hat{\mathcal{G}}_{\bar{0}}\oplus\mathcal{V}\oplus\mathcal{V}^\dagger$ and $\mathfrak{L}_{\bar{1}}:=\hat{\mathcal{G}}_{\bar{1}}$. Then $\mathfrak{L}=\mathfrak{L}_{\bar{0}}\oplus\mathfrak{L}_{\bar{1}}$ together with the Lie bracket as in (3.4) is a Lie superalgebra.*

*Proof.* Since the form on $\mathcal{G}$ is supersymmetric, (3.3) implies that the Lie bracket defined in (3.4) is anti-supercommutative. So we just need to verify the Jacobi superidentity. We recall that the form on $\hat{\mathcal{G}}$ is invariant and supersymmetric, and $d\in\mathcal{V}^\dagger$ acts as a derivation on $\hat{\mathcal{G}}$. Take $x,y,z\in\hat{\mathcal{G}}$ and $d,d'\in\mathcal{V}^\dagger$. Then

$$\begin{aligned}
(d([x,y]_{\hat{\mathcal{G}}}),z)&\overset{(3.3)}{=}-([x,y]_{\hat{\mathcal{G}}},d(z))=-(x,[y,d(z)]_{\hat{\mathcal{G}}})\\
&=(-1)^{|y|\,|z|}(x,[d(z),y]_{\hat{\mathcal{G}}})\\
&=(-1)^{|y|\,|z|}(x,d([z,y]_{\hat{\mathcal{G}}}))-(-1)^{|y|\,|z|}(x,[z,d(y)]_{\hat{\mathcal{G}}})\\
&=-(x,d([y,z]_{\hat{\mathcal{G}}}))-(-1)^{|y|\,|z|}([x,z]_{\hat{\mathcal{G}}},d(y))\\
&=-(x,d([y,z]_{\hat{\mathcal{G}}}))+(-1)^{|y|\,|z|+|z|\,|x|}([z,x]_{\hat{\mathcal{G}}},d(y))\\
&\overset{(3.3)}{=}-(-1)^{|x|\,|z|}\big((-1)^{|x|\,|y|}(d([y,z]_{\hat{\mathcal{G}}}),x)+(-1)^{|y|\,|z|}(d([z,x]_{\hat{\mathcal{G}}}),y)\big).
\end{aligned}$$

Therefore

$$\begin{aligned}
&(-1)^{|x|\,|z|}[[x,y],z]+(-1)^{|z|\,|y|}[[z,x],y]+(-1)^{|y|\,|x|}[[y,z],x]\\
&\quad=(-1)^{|x|\,|z|}[[x,y]_{\hat{\mathcal{G}}},z]_{\hat{\mathcal{G}}}+(-1)^{|x|\,|z|}\sum_{i\in I}(d_i([x,y]_{\hat{\mathcal{G}}}),z)\lambda_i\\
&\qquad+(-1)^{|z|\,|y|}[[z,x]_{\hat{\mathcal{G}}},y]_{\hat{\mathcal{G}}}+(-1)^{|y|\,|z|}\sum_{i\in I}(d_i([z,x]_{\hat{\mathcal{G}}}),y)\lambda_i\\
&\qquad+(-1)^{|y|\,|x|}[[y,z]_{\hat{\mathcal{G}}},x]_{\hat{\mathcal{G}}}+(-1)^{|x|\,|y|}\sum_{i\in I}(d_i([y,z]_{\hat{\mathcal{G}}}),x)\lambda_i\\
&\quad=0.
\end{aligned}$$

Also we have

$$
\begin{aligned}
[[x,y],d] &= [[x,y]_{\hat{\mathcal{G}}},d] = -d([x,y]_{\hat{\mathcal{G}}}) = -[d(x),y]_{\hat{\mathcal{G}}} - [x,d(y)]_{\hat{\mathcal{G}}} \\
&= -[x,d(y)]_{\hat{\mathcal{G}}} + \sum_{i\in I}(dd_i(x),y)\lambda_i - \sum_{i\in I}(dd_i(x),y)\lambda_i - [d(x),y]_{\hat{\mathcal{G}}} \\
&= -[x,d(y)]_{\hat{\mathcal{G}}} + \sum_{i\in I}(d_i d(x),y)\lambda_i - \sum_{i\in I}(dd_i(x),y)\lambda_i - [d(x),y]_{\hat{\mathcal{G}}} \\
&\overset{(3.3)}{=} -[x,d(y)]_{\hat{\mathcal{G}}} - \sum_{i\in I}(d_i(x),d(y))\lambda_i + \sum_{i\in I}(d(x),d_i(y))\lambda_i - [d(x),y]_{\hat{\mathcal{G}}} \\
&= -[x,d(y)]_{\hat{\mathcal{G}}} - \sum_{i\in I}(d_i(x),d(y))\lambda_i + (-1)^{|x|\,|y|}(\sum_{i\in I}(d_i(y),d(x))\lambda_i + [y,d(x)]_{\hat{\mathcal{G}}}) \\
&= -[x,d(y)] + (-1)^{|x|\,|y|}[y,d(x)] = [x,[y,d]] - (-1)^{|x|\,|y|}[y,[x,d]]
\end{aligned}
$$

and

$$[[d,d'],x] = 0 = dd'(x) - dd'(x) = dd'(x) - d'd(x) = [d,[d',x]] - [d',[d,x]].$$

Now the result follows immediately. □

**Lemma 3.13.** *Extend the form on $\hat{\mathcal{G}}$ to a supersymmetric bilinear form $(\cdot,\cdot)_{\mathfrak{L}}$ on $\mathfrak{L}$ by*

$$
\begin{aligned}
&(\mathcal{V},\mathcal{V})_{\mathfrak{L}} = (\mathcal{V}^\dagger,\mathcal{V}^\dagger)_{\mathfrak{L}} = (\mathcal{V},\mathcal{G}\otimes\mathcal{A})_{\mathfrak{L}} = (\mathcal{V}^\dagger,\mathcal{G}\otimes\mathcal{A})_{\mathfrak{L}} := \{0\}, \\
&(v,d)_{\mathfrak{L}} := d(v), \quad d\in\mathcal{V}^\dagger, v\in\mathcal{V}.
\end{aligned} \tag{3.5}
$$

*Then $(\cdot,\cdot)_{\mathfrak{L}}$ is a nondegenerate invariant even supersymmetric bilinear form.*

*Proof.* It is trivial that this form is nondegenerate, even and supersymmetric, so we just prove that it is invariant. We first consider the following easy table:

| $(x,y,z)\in$ | $([x,y],z)\in$ | $(x,[y,z])\in$ |
|---|---|---|
| $(\hat{\mathcal{G}},\hat{\mathcal{G}},\mathcal{V})$ | $(\hat{\mathcal{G}}+\mathcal{V},\mathcal{V})_{\mathfrak{L}}=\{0\}$ | $(\hat{\mathcal{G}},\{0\})_{\mathfrak{L}}=\{0\}$ |
| $(\mathcal{V},\hat{\mathcal{G}},\hat{\mathcal{G}})$ | $(\{0\},\hat{\mathcal{G}})_{\mathfrak{L}}=\{0\}$ | $(\mathcal{V},\hat{\mathcal{G}}+\mathcal{V})_{\mathfrak{L}}=\{0\}$ |
| $(\hat{\mathcal{G}},\mathcal{V},\mathcal{V}\cup\mathcal{V}^\dagger\cup\hat{\mathcal{G}})$ | $(\{0\},\mathfrak{L})_{\mathfrak{L}}=\{0\}$ | $(\hat{\mathcal{G}},\{0\})_{\mathfrak{L}}=\{0\}$ |
| $(\hat{\mathcal{G}},\mathcal{V}^\dagger,\mathcal{V}\cup\mathcal{V}^\dagger)$ | $(\hat{\mathcal{G}},\mathcal{V}\cup\mathcal{V}^\dagger)_{\mathfrak{L}}=\{0\}$ | $(\hat{\mathcal{G}},\{0\})_{\mathfrak{L}}=\{0\}$ |
| $(\mathcal{V}\cup\mathcal{V}^\dagger,\hat{\mathcal{G}},\mathcal{V}\cup\mathcal{V}^\dagger)$ | $(\hat{\mathcal{G}},\mathcal{V}\cup\mathcal{V}^\dagger)_{\mathfrak{L}}=\{0\}$ | $(\mathcal{V}\cup\mathcal{V}^\dagger,\hat{\mathcal{G}})_{\mathfrak{L}}=\{0\}$ |
| $(\mathcal{V}\cup\mathcal{V}^\dagger,\mathcal{V}\cup\mathcal{V}^\dagger,\hat{\mathcal{G}})$ | $(\{0\},\hat{\mathcal{G}})_{\mathfrak{L}}=\{0\}$ | $(\mathcal{V}\cup\mathcal{V}^\dagger,\hat{\mathcal{G}})_{\mathfrak{L}}=\{0\}$ |
| $(\mathcal{V}\cup\mathcal{V}^\dagger,\mathcal{V}\cup\mathcal{V}^\dagger,\mathcal{V}\cup\mathcal{V}^\dagger)$ | $(\{0\},\mathcal{V}\cup\mathcal{V}^\dagger)_{\mathfrak{L}}=\{0\}$ | $(\mathcal{V}\cup\mathcal{V}^\dagger,\{0\})_{\mathfrak{L}}=\{0\}$ |

Then we note that if $x, y, z \in \hat{\mathcal{G}}$, then

$$([x,y],z)_{\mathfrak{L}} = ([x,y]_{\hat{\mathcal{G}}},z)_{\mathfrak{L}} = ([x,y]_{\hat{\mathcal{G}}},z) = (x,[y,z]_{\hat{\mathcal{G}}}) = (x,[y,z]_{\hat{\mathcal{G}}})_{\mathfrak{L}} = (x,[y,z])_{\mathfrak{L}},$$

and for $x, y \in \hat{\mathcal{G}}$ and $z = d_j \in \mathcal{V}^\dagger$ $(j \in I)$, we get

$$\begin{aligned}([x,y],z)_{\mathfrak{L}} &= ([x,y]_{\hat{\mathcal{G}}} + \sum_{i\in I}(d_i(x),y)\lambda_i, d_j)_{\mathfrak{L}} = (d_j(x),y)\\ &\overset{(3.3)}{=} -(x,d_j(y)) = -(x,[d_j,y])_{\mathfrak{L}} = (x,[y,z])_{\mathfrak{L}}.\end{aligned}$$

Considering the latter equality, as the form is supersymmetric, for $y, z \in \hat{\mathcal{G}}$ and $x = d_j \in \mathcal{V}^\dagger$ $(j \in I)$ we have

$$\begin{aligned}([x,y],z)_{\mathfrak{L}} &= (-1)^{|y|\,|z|}(z,[x,y]) = -(-1)^{|y|\,|z|}(z,[y,x]) = -(-1)^{|y|\,|z|}([z,y],x)\\ &= -(-1)^{|y|\,|z|}(x,[z,y]) = (x,[y,z]).\end{aligned}$$

Finally, for $x, z \in \hat{\mathcal{G}}$ and $y = d_j \in \mathcal{V}^\dagger$ $(j \in I)$, one has

$$\begin{aligned}([x,y],z)_{\mathfrak{L}} &= -([d_j,x],z)_{\mathfrak{L}} = -(d_j(x),z) \overset{(3.3)}{=} (x,d_j(z)) = (x,[d_j,z])\\ &= (x,[y,z])_{\mathfrak{L}}.\end{aligned}$$

□

Now set $\mathfrak{h} := (\mathcal{H}\otimes\mathbb{F})\oplus\mathcal{V}\oplus\mathcal{V}^\dagger$ and let $R$ be the root system of $\mathcal{G}$ with respect to $\mathcal{H}$. We consider $\alpha \in R$ as an element of $\mathfrak{h}^*$ via

$$\alpha(\mathcal{V}\oplus\mathcal{V}^\dagger) := \{0\} \quad \text{and} \quad \alpha(h\otimes 1) := \alpha(h) \quad (h \in \mathcal{H}).$$

We also consider $\lambda \in \mathcal{V}$ as an element of $\mathfrak{h}^*$ via

$$\lambda((\mathfrak{h}\otimes\mathbb{F})\oplus\mathcal{V}) := \{0\} \quad \text{and} \quad \lambda(d) := d(\lambda) \quad (d \in \mathcal{V}^\dagger).$$

Then $\mathfrak{L}$ has a weight space decomposition with respect to $\mathfrak{h}$ with the corresponding root system $\mathfrak{R} = \{\alpha+\lambda \mid \alpha \in R,\ \lambda \in \Lambda\}$; moreover,

$$\mathfrak{L}^0 = \mathfrak{h} \quad \text{and} \quad \mathfrak{L}^{\alpha+\lambda} = \mathcal{G}^\alpha \otimes \mathbb{F}t^\lambda \quad (\alpha \in R,\ \lambda \in \Lambda \text{ with } \alpha+\lambda \neq 0).$$

Suppose $\lambda \in \Lambda$ and $\alpha \in R_i$ $(i \in \{0,1\})$ with $\alpha+\lambda \neq 0$. Use Lemma 3.1(iii) together with the fact that $f(\cdot,\cdot)$ is nondegenerate on $\mathcal{H}$ to fix $x \in \mathcal{G}^\alpha_{\bar{i}}$ and $y \in \mathcal{G}^{-\alpha}_{\bar{i}}$ with $f(x,y) = 1$ and $[x,y] \in \mathcal{H}$. Let $t_\alpha$ be the unique element of $\mathcal{H}$ representing $\alpha$ through $f(\cdot,\cdot)$. We have

$$[x\otimes t^\lambda, \theta(\lambda,-\lambda)^{-1}y\otimes t^{-\lambda}] = (t_\alpha\otimes 1) + \sum_{i\in I} d_i(\lambda)\lambda_i = (t_\alpha\otimes 1) + \lambda \in \mathfrak{h}\setminus\{0\}.$$

It follows that $(\mathfrak{L}, \mathfrak{h}, (\cdot,\cdot))$ is an extended affine Lie superalgebra with root system $\mathfrak{R}$.

For a unital associative algebra $\mathcal{A}$ and nonempty index supersets $I, J$, by an $I \times J$-*matrix* with entries in $\mathcal{A}$, we mean a map $A : I \times J \to \mathcal{A}$. For $i \in I$ and $j \in J$, we set $a_{ij} := A(i,j)$ and call it the $(i,j)$-*th entry* of $A$. By convention, we denote the matrix $A$ by $(a_{ij})$. We also denote by $\mathcal{A}^{I\times J}$ the set of all $I \times J$-matrices with entries in $\mathcal{A}$ and note that it is a vector superspace, under the componentwise summation and scalar product, with

$$\mathcal{A}^{I\times J}_{\bar{i}} := \{A \in \mathcal{A}^{I\times J} \mid A(I_{\bar{t}} \times J_{\bar{s}}) = 0,\ \bar{t}, \bar{s} \in \mathbb{Z}_2 \text{ with } \bar{t} + \bar{s} = \bar{i} + \bar{1}\}$$

for $\bar{i} \in \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$. If $A = (a_{ij}) \in \mathcal{A}^{I\times J}$ and $B = (b_{jk}) \in \mathcal{A}^{J\times K}$ are such that for all $i \in I$ and $k \in K$, for at most finitely many $j \in J$, $a_{ij}b_{jk}$ is nonzero, we define the product $AB$ of $A$ and $B$ to be the $I \times K$-matrix $C = (c_{ik})$ with $c_{ik} := \sum_{j\in J} a_{ij}b_{jk}$ for all $i \in I$, $k \in K$. We note that if $A, B, C$ are three matrices such that $AB$, $(AB)C$, $BC$ and $A(BC)$ are defined, then $A(BC) = (AB)C$. For $i \in I$, $j \in J$ and $a \in \mathcal{A}$, we define $E_{ij}(a)$ to be the matrix in $\mathcal{A}^{I\times J}$ whose $(i,j)$-th entry is $a$ and the other entries are zero, and if $\mathcal{A}$ is unital, we set

$$e_{i,j} := E_{i,j}(1).$$

Let $M_{I\times J}(\mathcal{A})$ be the subsuperspace of $\mathcal{A}^{I\times J}$ spanned by $\{E_{ij}(a) \mid i \in I,\ j \in J,\ a \in A\}$; in fact $M_{I\times J}(\mathcal{A})$ is a superspace with $M_{I\times J}(\mathcal{A})_{\bar{i}} = \mathrm{span}_{\mathbb{F}}\{E_{r,s}(a) \mid |r| + |s| = \bar{i}\}$ for $i = 0, 1$. Also with respect to the multiplication of matrices, the vector superspace $M_{I\times I}(\mathcal{A})$ is an associative $\mathbb{F}$-superalgebra and so it is a Lie superalgebra under the Lie bracket $[A, B] := AB - (-1)^{|A|\,|B|}BA$ for all $A, B \in M_{I\times I}(\mathcal{A})$. We denote this Lie superalgebra by $\mathfrak{pl}_I(\mathcal{A})$. For an element $X \in \mathfrak{pl}_I(\mathcal{A})$, we set $\mathrm{str}(X) := \sum_{i\in I}(-1)^{|i|}x_{i,i}$ and call it the *supertrace* of $X$. We finally make the convention that if $I$ is a disjoint union of nonempty subsets $I_1, \ldots, I_t$ of $I$, then for an $I \times I$-matrix $A$, we write

$$A = \begin{bmatrix} A_{1,1} & \cdots & A_{1,t} \\ A_{2,1} & \cdots & A_{2,t} \\ \vdots & \ddots & \vdots \\ A_{t,1} & \cdots & A_{t,t} \end{bmatrix}$$

in which for $1 \le r, s \le t$, $A_{r,s}$ is an $I_r \times I_s$-matrix whose $(i,j)$-th entry coincides with the $(i,j)$-th entry of $A$ for all $i \in I_r$ and $j \in I_s$. In this case, we say that $A \in \mathcal{A}^{I_1 \uplus \cdots \uplus I_t}$ and note that the above defined matrix product obeys the product rule for block matrices.

In the next example, we realize a certain extended affine Lie superalgebra using an "affinization" process. To this end, we need to fix some notation. Suppose that $A$ is a unital associative algebra and "$*$" is an *involution* on $\mathcal{A}$, that is, a selfinverting linear endomorphism of $A$ with $(ab)^* = b^*a^*$ for all $a, b \in A$. We next assume $\dot{I}, \dot{J}, \dot{K}$ are nonempty index sets with disjoint copies $\bar{\dot{I}} = \{\bar{i} \mid i \in \dot{I}\}$, $\bar{\dot{J}} = \{\bar{j} \mid j \in \dot{J}\}$ and $\bar{\dot{K}} = \{\bar{k} \mid k \in \dot{K}\}$ respectively. Suppose that $0, 0', 0''$ are three distinct symbols and by convention, write $\bar{0} := 0$, $\bar{0}' := 0'$ and $\bar{0}'' := 0''$. We let $I$ be either $\dot{I} \uplus \bar{\dot{I}}$ or $\{0\} \uplus \dot{I} \uplus \bar{\dot{I}}$; $J$ be either $\dot{J} \uplus \bar{\dot{J}}$ or $\{0'\} \uplus \dot{J} \uplus \bar{\dot{J}}$; and $K$ be either $\dot{K} \uplus \bar{\dot{K}}$ or $\{0''\} \uplus \dot{K} \uplus \bar{\dot{K}}$. For a matrix $X = (x_{ij}) \in M_{I\times J}(\mathcal{A})$, define $X^\diamond = (y_{ji}) \in M_{J\times I}(\mathcal{A})$ by $y_{ji} := x^*_{\bar{i}\bar{j}}$ $(i \in I,\ j \in J)$ where for an index $t \in I \cup J$, by $\bar{\bar{t}}$ we mean $t$. It is immediate that for $X = (x_{ij}) \in M_{I\times I}(\mathcal{A})$,

$$\operatorname{tr}(X^\diamond) = (\operatorname{tr}(X))^*. \tag{3.6}$$

Also if $X = (x_{ij}) \in M_{I\times J}(\mathcal{A})$ and $Y \in M_{J\times K}(\mathcal{A})$, then for $i \in I$ and $k \in K$,

$$(XY)^\diamond_{ki} = \Big(\sum_{j\in J} x_{\bar{i}j} y_{j\bar{k}}\Big)^* = \sum_{j\in J} y^*_{j\bar{k}} x^*_{\bar{i}j} = \sum_{j\in J} y^*_{\bar{j}\bar{k}} x^*_{\bar{i}\bar{j}} = \sum_{j\in J} Y^\diamond_{kj} X^\diamond_{ji} = (Y^\diamond X^\diamond)_{ki},$$

which implies that

$$(XY)^\diamond = Y^\diamond X^\diamond. \tag{3.7}$$

**Example 3.14.** In this example, we assume that the field $\mathbb{F}$ contains a fourth primitive root of unity $\zeta$. Suppose that $G$ is a torsion free additive abelian group and $\lambda$ is a commutative 2-cocycle satisfying $\lambda(0,0) = 1$. Let $\mathcal{A}$ be the commutative associative torus corresponding to $(G, \lambda)$, let $*$ be an involution of $\mathcal{A}$ mapping $\mathcal{A}^\tau$ to $\mathcal{A}^\tau$ for all $\tau \in G$ and suppose that $I$ and $J$ are as in the previous paragraph such that $I \cap J = \emptyset$, and $|I| \neq |J|$ if $I$ and $J$ are both finite. Consider $I \uplus J$ as a superset with $|i| := \bar{0}$ and $|j| := \bar{1}$ for $i \in I$ and $j \in J$ and set $\mathcal{L} := \mathfrak{pl}_{I\uplus J}(\mathcal{A})$. Then for ${}_{[\tau]}\mathcal{L} := \{(x_{ij}) \in \mathcal{L} \mid x_{ij} \in \mathcal{A}^\tau, \forall i, j \in I \uplus J\}$ $(\tau \in G)$, $\mathcal{L} = \bigoplus_{\tau\in G} {}_{[\tau]}\mathcal{L}$ is a $G$-graded Lie superalgebra. Set

$$\mathcal{G} := \mathfrak{sl}_{\mathcal{A}}(I, J) := \{A \in \mathfrak{pl}_{I\uplus J}(\mathcal{A}) \mid \operatorname{str}(A) = 0\} \simeq \mathfrak{sl}_{\mathbb{F}}(I, J) \otimes \mathcal{A}.$$

As $\mathcal{G}$ is a subsuperalgebra of $\mathcal{L}$ generated by $\{E_{i,j}(a) \mid i, j \in I \uplus J,\ i \neq j,\ a \in \mathcal{A}\}$, it follows that $\mathcal{G}$ is a $G$-graded subsuperalgebra of $\mathcal{L}$. Set $\mathcal{H} := \operatorname{span}_{\mathbb{F}}\{e_{i,i} - e_{r,r}, e_{j,j} - e_{s,s}, e_{i,i} + e_{j,j} \mid i, r \in I,\ j, s \in J\}$. Then $\mathcal{G}$ has a weight space decomposition $\mathcal{G} = \bigoplus_{\alpha\in\mathcal{H}^*} \mathcal{G}^\alpha$ with respect to $\mathcal{H}$ with root system

$$R = \{\epsilon_i - \epsilon_r, \delta_j - \delta_s, \epsilon_i - \delta_j, \delta_j - \epsilon_i \mid i, r \in I,\ j, s \in J\},$$

where for $t \in I$ and $p \in J$,

$$\begin{aligned}
\epsilon_t : \mathcal{H} \to \mathbb{F}, &\quad e_{i,i} - e_{r,r} \mapsto \delta_{i,t} - \delta_{r,t},\ e_{j,j} - e_{s,s} \mapsto 0,\ e_{i,i} + e_{s,s} \mapsto \delta_{i,t},\\
\delta_p : \mathcal{H} \to \mathbb{F}, &\quad e_{i,i} - e_{r,r} \mapsto 0,\ e_{j,j} - e_{s,s} \mapsto \delta_{j,p} - \delta_{p,s},\ e_{i,i} + e_{j,j} \mapsto \delta_{p,j},
\end{aligned}$$

$(i, r \in I,\ \ j, s \in J)$ and for $i, r \in I$ and $j, s \in J$ with $i \neq r$ and $j \neq s$, we have

$$\mathcal{G}^{\epsilon_i-\epsilon_r} = \mathcal{A}e_{i,r}, \quad \mathcal{G}^{\delta_j-\delta_s} = \mathcal{A}e_{j,s}, \quad \mathcal{G}^{\epsilon_i-\delta_j} = \mathcal{A}e_{i,j}, \quad \mathcal{G}^{\delta_j-\epsilon_i} = \mathcal{A}e_{j,i},$$
$$\mathcal{G}^0 = \Big\{A = \sum_{t\in I\uplus J} a_{tt}e_{t,t} \in \mathfrak{pl}_{I\uplus J}(\mathcal{A}) \;\Big|\; \mathrm{str}(A) = 0\Big\}.$$

For $\alpha \in R$ and $\tau \in G$, setting

$${}_{[\tau]}\mathcal{G}^{\alpha} := {}_{[\tau]}\mathcal{G} \cap \mathcal{G}^{\alpha},$$

we have

$${}_{[\tau]}\mathcal{G}^{\epsilon_i-\epsilon_r} = \mathcal{A}^{\tau}e_{i,r}, \quad {}_{[\tau]}\mathcal{G}^{\delta_j-\delta_s} = \mathcal{A}^{\tau}e_{j,s}, \quad {}_{[\tau]}\mathcal{G}^{\epsilon_i-\delta_j} = \mathcal{A}^{\tau}e_{i,j}, \quad {}_{[\tau]}\mathcal{G}^{\delta_j-\epsilon_i} = \mathcal{A}^{\tau}e_{j,i},$$
$${}_{[\tau]}\mathcal{G}^0 = \Big\{A = \sum_{t\in I\uplus J} a_{tt}e_{t,t} \in \mathfrak{pl}_{I\uplus J}(\mathcal{A}) \;\Big|\; a_{tt} \in \mathcal{A}^{\tau}\ (t \in I \uplus J),\ \mathrm{str}(A) = 0\Big\}$$

for $i, r \in I$ and $j, s \in J$ with $i \neq r$ and $j \neq s$.

Now let $\epsilon : \mathcal{A} \to \mathbb{F}$ be the linear function defined by

$$x^{\tau} \mapsto \begin{cases} 0 & \text{if } \tau \neq 0 \\ 1 & \text{if } \tau = 0 \end{cases} \qquad (\tau \in G).$$

Define

$$(\cdot,\cdot) : \mathcal{G} \times \mathcal{G} \to \mathbb{F}, \qquad (x, y) \mapsto \epsilon(\mathrm{str}(xy)).$$

This defines a nondegenerate invariant even supersymmetric bilinear form on $\mathcal{G}$.

Next set $\mathcal{V} := \mathbb{F} \otimes_{\mathbb{Z}} G$. Since $G$ is torsion free, we can identify $G$ with a subset of $\mathcal{V}$ in the usual manner. We next fix a basis $B := \{\tau_t \mid t \in T\} \subseteq G$ of $\mathcal{V}$. Suppose that $\{d_t \mid t \in T\}$ is its dual basis and let $\mathcal{V}^{\dagger}$ be the restricted dual of $\mathcal{V}$ with respect to this basis. Each $d \in \mathcal{V}^{\dagger}$ can be considered as a derivation of $\mathcal{G}$ (of degree 0) via $d(x) := d(\tau)x$ for each $x \in {}_{[\tau]}\mathcal{G}$ $(\tau \in G)$. Set

$$\mathfrak{L} := \mathcal{G} \oplus \mathcal{V} \oplus \mathcal{V}^{\dagger}.$$

We extend the form on $\mathcal{G}$ to a nondegenerate even supersymmetric bilinear form $(\cdot,\cdot)$ on $\mathfrak{L}$ by

$$(\mathcal{V}, \mathcal{V}) = (\mathcal{V}^{\dagger}, \mathcal{V}^{\dagger}) = (\mathcal{V}, \mathcal{G}) = (\mathcal{V}^{\dagger}, \mathcal{G}) := \{0\} \quad \text{and} \quad (d, v) := d(v) \quad (d \in \mathcal{V}^{\dagger}, v \in \mathcal{V}).$$

We also define

$$\begin{aligned}
&[d, x] = -[x, d] = d(x) \qquad (d \in \mathcal{V}^{\dagger},\ x \in \mathcal{G}) \\
&[\mathcal{V}, \mathfrak{L}] = \{0\}, \\
&[\mathcal{V}^{\dagger}, \mathcal{V}^{\dagger}] = \{0\}, \\
&[x, y] = [x, y]_{\mathcal{G}} + \sum_{t\in T}(d_t(x), y)\tau_t \qquad (x, y \in \mathcal{G}),
\end{aligned}$$

where $[\cdot,\cdot]_{\mathcal{G}}$ is the bracket on $\mathcal{G}$. Setting $\mathfrak{h} := \mathcal{H} \oplus \mathcal{V} \oplus \mathcal{V}^\dagger$, as in the previous example, one finds that $(\mathfrak{L}, \mathfrak{h}, (\cdot,\cdot))$ is an extended affine Lie superalgebra with root system

$$\mathfrak{R} = \{\alpha + \tau \mid \alpha \in R,\, \tau \in G\}$$

in which $\alpha \in R$ is considered as an element of $\mathfrak{h}^*$ via

$$\alpha(\mathcal{V} \oplus \mathcal{V}^\dagger) := \{0\},$$

and $\tau \in \mathcal{V}$ is considered as an element of $\mathfrak{h}^*$ via

$$\tau(\mathcal{H} \oplus \mathcal{V}) := \{0\} \quad \text{and} \quad \tau(d) := d(\tau) \quad (d \in \mathcal{V}^\dagger).$$

We also have

$$\mathfrak{L}^0 = \mathfrak{h} \quad \text{and} \quad \mathfrak{L}^{\alpha+\tau} = {}_{[\tau]}\mathcal{G}^\alpha \quad (\alpha \in R,\, \tau \in G \text{ with } \alpha + \tau \neq 0).$$

Next for $A = \left(\begin{smallmatrix} X & Y \\ Z & W \end{smallmatrix}\right) \in \mathcal{L} = \mathfrak{pl}_{I \uplus J}(\mathcal{A})$, define

$$A^\# := \begin{pmatrix} -X^\diamond & Z^\diamond \\ -Y^\diamond & -W^\diamond \end{pmatrix}.$$

We have $[A,B]^\# = [A^\#, B^\#]$ and so $\#$ is a Lie superalgebra automorphism of $\mathcal{L}$ of order 4. Since $\#$ maps $\mathcal{G}$ to $\mathcal{G}$ (see (3.6)), we consider $\#$ as a Lie superalgebra automorphism of $\mathcal{G}$ as well. Let $M = \left(\begin{smallmatrix} X & Y \\ Z & W \end{smallmatrix}\right), N = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathcal{G}$. Then

$$\begin{aligned}
(3.8) \qquad (M^\#, N^\#) &= \epsilon(\mathrm{str}(M^\# N^\#)) \\
&= \epsilon(\mathrm{tr}(X^\diamond A^\diamond - Z^\diamond B^\diamond) - \mathrm{tr}(-Y^\diamond C^\diamond + W^\diamond D^\diamond)) \\
&\overset{(3.7)}{=} \epsilon\big(\mathrm{tr}((AX)^\diamond + (CY)^\diamond - (BZ)^\diamond - (DW)^\diamond)\big) \\
&= \epsilon\big(\mathrm{tr}((AX)^\diamond) + \mathrm{tr}((CY)^\diamond) - \mathrm{tr}((BZ)^\diamond) - \mathrm{tr}((DW)^\diamond)\big) \\
&\overset{(3.6)}{=} \epsilon((\mathrm{tr}(AX))^* + (\mathrm{tr}(CY))^* - (\mathrm{tr}(BZ))^* - (\mathrm{tr}(DW))^*) \\
&= \epsilon\big(\mathrm{tr}(AX) + \mathrm{tr}(CY) - \mathrm{tr}(BZ) - \mathrm{tr}(DW)\big) \\
&= \epsilon\big(\mathrm{tr}(XA) + \mathrm{tr}(YC) - \mathrm{tr}(ZB) - \mathrm{tr}(WD)\big) \\
&= \epsilon(\mathrm{str}(MN)) = (M, N).
\end{aligned}$$

We also have

$$(3.9) \qquad d(x^\#) = (d(x))^\#, \qquad d \in \mathcal{V}^\dagger,\, x \in \mathcal{G}.$$

Now extend $\#$ to $\mathfrak{L}$ by $v^\# := v$ for $v \in \mathcal{V} \oplus \mathcal{V}^\dagger$. It follows from (3.8) and (3.9) that $\#$ is an automorphism of $\mathfrak{L}$ of order 4 mapping $\mathfrak{h}$ to $\mathfrak{h}$. So $\mathfrak{L} = \bigoplus_{i=0}^{3} {}^{[i]}\mathfrak{L}$, where for $i \in \mathbb{Z}$,

$${}^{[i]}\mathfrak{L} := \{x \in \mathfrak{L} \mid x^\# = \zeta^i x\}$$

with $[i]$ indicating the congruence class of $i \in \mathbb{Z}$ modulo $4\mathbb{Z}$. Using (3.8) together with the fact that the form on $\mathfrak{L}$ is nondegenerate, for $i, j \in \mathbb{Z}$ we have

$$({}^{[i]}\mathfrak{L}, {}^{[j]}\mathfrak{L}) \neq \{0\} \quad \text{if and only if} \quad i + j \in 4\mathbb{Z}. \tag{3.10}$$

Next let $\sigma$ be the restriction of $\#$ to $\mathfrak{h}$, and let $\mathfrak{h}^\sigma$ be the set of fixed points of $\mathfrak{h}$ under $\sigma$. Consider a linear endomorphism of $\mathfrak{h}^*$ mapping $\alpha \in \mathfrak{h}^*$ to $\alpha \circ \sigma^{-1}$ and denote it again by $\sigma$. The Lie superalgebra $\mathfrak{L}$ has a weight space decomposition $\mathfrak{L} = \sum_{\{\pi(\alpha) \mid \alpha \in \mathfrak{R}\}} \mathfrak{L}^{\pi(\alpha)}$ with respect to $\mathfrak{h}^\sigma$ where

$$\pi(\alpha) := \tfrac{1}{4}(\alpha + \sigma(\alpha) + \sigma^2(\alpha) + \sigma^3(\alpha)) \qquad (\alpha \in \mathfrak{R})$$

(see [3, (2.11) & Lem. 3.7]). Moreover,

$$\begin{aligned}
\pi(\mathfrak{R}) &= \{\pi(\alpha) \mid \alpha \in \mathfrak{R}\}\\
&= \{\tau \mid \tau \in G\}\\
&\quad \cup \big\{\tfrac{1}{2}((\epsilon_i - \epsilon_r) + (\epsilon_{\bar r} - \epsilon_{\bar i})) + \tau \mid \tau \in G,\ i, r \in I,\ i \neq r\big\}\\
&\quad \cup \big\{\tfrac{1}{2}((\delta_j - \delta_s) + (\delta_{\bar s} - \delta_{\bar j})) + \tau \mid \tau \in G,\ j, s \in J,\ j \neq s\big\}\\
&\quad \cup \big\{\tfrac{1}{2}((\epsilon_i - \delta_j) + (\delta_{\bar j} - \epsilon_{\bar i})) + \tau \mid \tau \in G,\ i \in I,\ j \in J\big\}\\
&\quad \cup \big\{\tfrac{1}{2}((\delta_j - \epsilon_i) + (\epsilon_{\bar i} - \delta_{\bar j}) + \tau \mid \tau \in G,\ i \in I,\ j \in J\big\}\\
&= \{\tau \mid \tau \in G\}\\
&\quad \cup \big\{\tfrac{1}{2}((\epsilon_i - \epsilon_{\bar i}) - (\epsilon_r - \epsilon_{\bar r})) + \tau \mid \tau \in G,\ i, r \in I,\ i \neq r\big\}\\
&\quad \cup \big\{\tfrac{1}{2}((\delta_j - \delta_{\bar j}) - (\delta_s - \delta_{\bar s})) + \tau \mid \tau \in G,\ j, s \in J,\ j \neq s\big\}\\
&\quad \cup \big\{\tfrac{1}{2}((\epsilon_i - \epsilon_{\bar i}) - (\delta_j - \delta_{\bar j})) + \tau \mid \tau \in G,\ i \in I,\ j \in J\big\}\\
&\quad \cup \big\{\tfrac{1}{2}((\delta_j - \delta_{\bar j}) - (\epsilon_i - \epsilon_{\bar i})) + \tau \mid \tau \in G,\ i \in I,\ j \in J\big\}
\end{aligned}$$

which is an extended affine root supersystem of type $BC(I, J)$ if $\{0, 0'\} \cap (I \cup J) \neq \emptyset$, and of type $C(I, J)$ otherwise (see [12] for the notion of type for an extended affine root supersystem). We next note that for $i \in \{0, 1, 2, 3\}$, as ${}^{[i]}\mathfrak{L}$ is an $\mathfrak{h}^\sigma$-submodule of $\mathfrak{L}$, it inherits the weight space decomposition

$${}^{[i]}\mathfrak{L} = \sum_{\{\pi(\alpha) \mid \alpha \in \mathfrak{R}\}} {}^{[i]}\mathfrak{L}^{\pi(\alpha)}$$

from $\mathfrak{L}$ with ${}^{[i]}\mathfrak{L}^{\pi(\alpha)} := {}^{[i]}\mathfrak{L} \cap \mathfrak{L}^{\pi(\alpha)}$. We recall (3.10) together with the fact that the form $(\cdot, \cdot)$ is nondegenerate. So if $\alpha, \beta \in \mathfrak{R}$ and $i, j \in \mathbb{Z}$, we deduce that

(3.11) for $0 \neq x \in {}^{[i]}\mathfrak{L}^{\pi(\alpha)}$, $(x, {}^{[j]}\mathfrak{L}^{\pi(\beta)}) \neq \{0\}$ if and only if $i + j \in 4\mathbb{Z}$ and $\pi(\alpha) + \pi(\beta) = 0$.

Now we set

$$\tilde{\mathfrak{L}} := \sum_{i \in \mathbb{Z}} ({}^{[i]}\mathfrak{L} \otimes \mathbb{F}t^i) \oplus \mathbb{F}c \oplus \mathbb{F}d$$

where $c, d$ are two symbols. Since $\#$ preserves the $\mathbb{Z}_2$-grading on $\mathfrak{L}$, $\tilde{\mathfrak{L}}$ is a superspace with

$$\tilde{\mathfrak{L}}_{\bar{0}} := \sum_{i\in\mathbb{Z}}(({}^{[i]}\mathfrak{L}\cap\mathfrak{L}_{\bar{0}})\otimes\mathbb{F}t^i)\oplus\mathbb{F}c\oplus\mathbb{F}d \quad\text{and}\quad \tilde{\mathfrak{L}}_{\bar{1}} := \sum_{i\in\mathbb{Z}}(({}^{[i]}\mathfrak{L}\cap\mathfrak{L}_{\bar{1}})\otimes\mathbb{F}t^i).$$

Moreover, $\tilde{\mathfrak{L}}$ with the bracket

$$\begin{aligned}[x\otimes t^i+rc+sd, y\otimes t^j+r'c+s'd]^{\sim} := {}&[x,y]\otimes t^{i+j}+i\delta_{i,-j}(x,y)c\\ &+sjy\otimes t^j-s'ix\otimes t^i\end{aligned}$$

is a Lie superalgebra equipped with a weight space decomposition with respect to $(\mathfrak{h}^\sigma\otimes\mathbb{F})\oplus\mathbb{F}c\oplus\mathbb{F}d$. More precisely, if we define

$$\delta:(\mathfrak{h}^\sigma\otimes\mathbb{F})\oplus\mathbb{F}c\oplus\mathbb{F}d\to\mathbb{F},\qquad c\mapsto 0,\ d\mapsto 1,\ h\otimes 1\mapsto 0\quad (h\in\mathfrak{h}^\sigma),$$

then $\tilde{\mathfrak{R}} := \{\pi(\alpha)+i\delta \mid \alpha\in\mathfrak{R},\ i\in\mathbb{Z},\ {}^{[i]}\mathfrak{L}\cap\mathfrak{L}^{\pi(\alpha)}\neq\{0\}\}$ is the corresponding root system of $\tilde{\mathfrak{L}}$ in which $\pi(\alpha)$ is considered as an element of the dual space of $(\mathfrak{h}^\sigma\otimes\mathbb{F})\oplus\mathbb{F}c\oplus\mathbb{F}d$ mapping $h\otimes 1\in\mathfrak{h}^\sigma\otimes\mathbb{F}$ to $\alpha(h)$ and $c, d$ to 0. Furthermore, since $\pi(\alpha)\neq 0$ for each $\alpha\in\mathfrak{R}\setminus\{0\}$, as in [2, Cor. 3.26] we have ${}^{[0]}\mathfrak{L}^{\pi(0)}=\mathfrak{h}^\sigma$. So for $\alpha\in\mathfrak{R}$ and $i\in\mathbb{Z}$,

$$\tilde{\mathfrak{L}}^{\pi(\alpha)+i\delta}=\begin{cases}({}^{[i]}\mathfrak{L}\cap\mathfrak{L}^{\pi(\alpha)})\otimes t^i & \text{if } (\alpha,i)\neq(0,0),\\ (\mathfrak{h}^\sigma\otimes 1)\oplus\mathbb{F}c\oplus\mathbb{F}d & \text{if } (\alpha,i)=(0,0).\end{cases}$$

We extend the form on $\mathfrak{L}$ to a supersymmetric bilinear form $(\cdot,\cdot)^{\sim}$ on $\tilde{\mathfrak{L}}$ by

(3.12)
$$\begin{aligned}&(c,d)^{\sim}=1,\ (c,c)^{\sim}=(d,d)^{\sim}=(c,{}^{[i]}\mathfrak{L}\otimes\mathbb{F}t^i)^{\sim}=(d,{}^{[i]}\mathfrak{L}\otimes\mathbb{F}t^i)^{\sim}:=\{0\}\quad(i\in\mathbb{Z}),\\ &(x\otimes t^i,y\otimes t^j)^{\sim}=\delta_{i+j,0}(x,y)\qquad(i,j\in\mathbb{Z},\ x\in{}^{[i]}\mathfrak{L},\ y\in{}^{[j]}\mathfrak{L}).\end{aligned}$$

Since the form on $\mathfrak{L}$ is even and nondegenerate, so is $(\cdot,\cdot)^{\sim}$; moreover, by (3.11), if $j\in\{0,1\}$, $\alpha\in\mathfrak{R}$ and $i\in\mathbb{Z}$ with $(i,\alpha)\neq(0,0)$ such that ${}^{[i]}\mathfrak{L}^{\pi(\alpha)}\cap\mathfrak{L}_{\bar{j}}\neq\{0\}$, then $({}^{[i]}\mathfrak{L}^{\pi(\alpha)}\cap\mathfrak{L}_{\bar{j}},{}^{[-i]}\mathfrak{L}^{\pi(-\alpha)}\cap\mathfrak{L}_{\bar{j}})\neq\{0\}$. Using this together with the fact that ${}^{[0]}\mathfrak{L}^{\pi(0)}=\mathfrak{h}^\sigma$ and Lemma 3.1(iii), one finds $x\in{}^{[i]}\mathfrak{L}^{\pi(\alpha)}\cap\mathfrak{L}_{\bar{j}}$, $y\in{}^{[-i]}\mathfrak{L}^{\pi(-\alpha)}\cap\mathfrak{L}_{\bar{j}}$ with $0\neq[x,y]\in\mathfrak{h}^\sigma$. So

$$[x\otimes t^i,y\otimes t^{-i}]^{\sim}=[x,y]\otimes 1+i(x,y)c\in(\mathfrak{h}^\sigma\otimes 1)\oplus\mathbb{F}c\oplus\mathbb{F}d\setminus\{0\}.$$

Also as $\pi(\mathfrak{R})$ is an extended affine root supersystem, it follows that $\mathrm{ad}_x$ is locally nilpotent for all $x\in\tilde{\mathfrak{L}}^{\tilde{\alpha}}$ where $\tilde{\alpha}\in\tilde{\mathfrak{R}}^{\times}_{\mathrm{re}}$. So

$$(\tilde{\mathfrak{L}},(\mathfrak{h}^\sigma\otimes\mathbb{F})\oplus\mathbb{F}c\oplus\mathbb{F}d,(\cdot,\cdot)^{\sim})$$

is an extended affine Lie superalgebra with root system $\tilde{\mathfrak{R}}$.

## Acknowledgements

This research was in part supported by a grant from IPM (No. 92170415) and partially carried out at IPM-Isfahan branch.

## References

[1] B. Allison, S. Azam, S. Berman, Y. Gao and A. Pianzola, Extended affine Lie algebras and their root systems, Mem. Amer. Math. Soc. **126** (1997), no. 603, x+122 pp. Zbl 0879.17012 MR 1376741

[2] B. Allison, S. Berman and A. Pianzola, Covering algebras I: extended affine Lie algebras. J. Algebra **250** (2002), 485–516. Zbl 1002.17010 MR 1899862

[3] S. Azam, S. R. Hosseini and M. Yousofzadeh, Extended affinization of invariant affine reflection alegbras, Osaka J. Math. **50** (2013), 1039–1072. Zbl 06254652 MR 3161427

[4] S. Azam, V. Khalili and M. Yousofzadeh, Extended affine root systems of type BC, J. Lie Theory **15** (2005), 145–181. Zbl 1100.17003 MR 2115234

[5] R. Høegh-Krohn and B. Torresani, Classification and construction of quasi-simple Lie algebras, J. Funct. Anal. **89** (1990), 106–136. Zbl 0792.17019 MR 1040958

[6] N. Jacobson, *Lie algebras*, Dover Publ., New York. Zbl 0121.27504 MR 0143793

[7] V. Kac, Lie superalgebras, Adv. Math. **26** (1977), 8–96. Zbl 0366.17012 MR 0486011

[8] R. Moody and A. Pianzola, *Lie algebras with triangular decompositions*, Wiley-Interscience, 1995. Zbl 0874.17026 MR 1323858

[9] J. Morita and Y. Yoshii, Locally extended affine Lie algebras, J. Algebra **301** (2006), 59–81. Zbl 1147.17019 MR 2230320

[10] E. Neher, Extended affine Lie algebras and other generalizations of affine Lie algebras—a survey, in *Developments and trends in infinite-dimensional Lie theory*, Progr. Math. 228, Birkhäuser Boston, Boston, MA, 2011, 53–126. Zbl 1261.17023 MR 2743761

[11] J. W. Van de Lour, A classification of contragredient Lie superalgebras of finite growth, Comm. Algebra **17** (1989), 1815–1841.

[12] M. Yousofzadeh, Extended affine root supersystems, J. Algebra **449** (2016), 539–564. Zbl 06535196 MR 3448184

[13] ———, Locally finite basic classical simple Lie superalgebras, arXiv:1502.04586 (2015).