

On the Section Conjecture over Function Fields and Finitely Generated Fields

by

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Abstract

We investigate sections of the arithmetic fundamental groups of hyperbolic curves over function fields. As a consequence we prove that the anabelian section conjecture of Grothendieck holds over all finitely generated fields over \mathbb{Q} if it holds over all number fields, under the condition of finiteness (of the ℓ -primary parts) of certain Shafarevich–Tate groups. We also prove that if the section conjecture holds over all number fields then it holds over all finitely generated fields for curves which are defined over a number field.

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§0. Introduction

Let k be a characteristic 0 field and X a smooth, projective, and geometrically connected hyperbolic curve (i.e., $g(X) \geq 2$) over k . Let $\pi_1(X)$ be the arithmetic étale fundamental group of X which sits in the exact sequence

$$1 \rightarrow \pi_1(\bar{X}) \rightarrow \pi_1(X) \xrightarrow{\text{pr}} G_k := \text{Gal}(\bar{k}/k) \rightarrow 1,$$

where \bar{k} is an algebraic closure of k and $\bar{X} = X \times_k \bar{k}$. In this paper we investigate continuous group-theoretic sections (i.e., splittings) $s : G_k \rightarrow \pi_1(X)$ of the projection $\text{pr} : \pi_1(X) \rightarrow G_k$, which we will refer to as sections of $\pi_1(X)$.

Sections of $\pi_1(X)$ arise naturally from k -rational points of X . More precisely, a rational point $x \in X(k)$ determines a decomposition subgroup $D_x \subset \pi_1(X)$, which is defined modulo conjugation by elements of $\pi_1(\bar{X})$, and which maps isomorphically to G_k via the projection $\text{pr} : \pi_1(X) \rightarrow G_k$. (Thus, D_x determines a

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splitting of the above exact sequence.) We will refer to such a section of $\pi_1(X)$ as *point-theoretic*, and say that it arises from the rational point $x \in X(k)$. We have a set-theoretic map

$$\varphi_X : X(k) \rightarrow \overline{\text{Sec}}_{\pi_1(X)}, \quad x \mapsto \varphi_X(x) = [s_x],$$

where $\overline{\text{Sec}}_{\pi_1(X)}$ is the set of conjugacy classes of sections of $\pi_1(X)$, modulo conjugation by elements of $\pi_1(\overline{X})$, and $[s_x]$ denotes the image (i.e., conjugacy class) of a section s_x associated to $x \in X(k)$.

Definition 0.1. (i) We say that the **SC** (section conjecture) *holds for* X , over k , if the above map $\varphi_X : X(k) \rightarrow \overline{\text{Sec}}_{\pi_1(X)}$ is bijective.
(ii) We say that the **SC holds** over k if the **SC** holds for every smooth, projective, geometrically connected hyperbolic curve X over k (cf. (i)).

In his seminal letter to Faltings, Grothendieck formulated the following conjecture (cf. [G1]).

Grothendieck's Anabelian Section Conjecture (GASC). *Assume that k is finitely generated over the prime field \mathbb{Q} . Then the **SC** holds over k .*

The injectivity of the map φ_X if k is finitely generated over \mathbb{Q} , or more generally if k is a sub- p -adic field, is well-known (cf. [M1, Theorem C]). The statement of the **GASC** is thus equivalent to the surjectivity of the map φ_X , i.e., that every section of $\pi_1(X)$ is point-theoretic under the above assumptions on the field k . The **GASC**, even over number fields, is still wide open. More generally, one can ask: for which fields (of characteristic 0) does the **SC** hold? In this paper we investigate the section conjecture over function fields of curves in characteristic 0.

Given a separated, smooth, connected curve C over k with function field $K := k(C)$, an abelian scheme $\mathcal{A} \rightarrow C$ with generic fibre $A := \mathcal{A} \times_C \text{Spec } K$, define the *Shafarevich–Tate group*

$$\text{III}(\mathcal{A}) = \text{III}(A, C) := \text{Ker} \left(H^1(G_K, A) \rightarrow \prod_{c \in C^{\text{cl}}} H^1(G_{K_c}, A_c) \right),$$

where $c \in C^{\text{cl}}$ is a closed point, K_c is the completion of K at c , $A_c := A \times_{\text{Spec } K} \text{Spec } K_c$, and the product is over all closed points of C .

Definition 0.2. Let k' be a field with $\text{char}(k') = 0$. Consider the following conditions.

- (i) The ℓ' -cyclotomic character $\chi_{\ell'} : G_{k'} \rightarrow \mathbb{Z}_{\ell'}^\times$ is *non-Tate* in the sense of [CT, §2, Definition], meaning that the 1-dimensional ℓ' -adic representation $\mathbb{Z}_{\ell'}(1)$ does not appear as a subrepresentation of the representation arising from the ℓ' -adic Tate module of an abelian variety over k' , for any prime integer ℓ' .

- (ii) The **SC** holds over k' .
- (iii) Given a separated, smooth, connected curve C over k' , with function field $K := k'(C)$, and an abelian scheme $\mathcal{A} \rightarrow C$, one has $T\text{III}(\mathcal{A}) = 0$ (cf. Notation).
- (iv) Given an abelian variety A over k' , the group $A(k')$ of k' -rational points, and a quotient $A(k') \twoheadrightarrow D$, the following hold.
 - (a) The natural map $D \rightarrow D^\wedge$ (cf. Notation) is injective.
 - (b) The torsion group $D[N]$ is finite for each $N \geq 1$, and $TD = 0$ (cf. Notation).
- (v) Every function field $K = k'(C)$ as in (iii) admits a structure of Hausdorff topological field so that $X(K)$ becomes compact for any proper, smooth, and geometrically connected hyperbolic curve over K .
- (vi) Given a separated, smooth, connected curve C over k' with function field $K := k'(C)$, and a finite étale morphism $\tilde{C} \rightarrow C$, the following hold. If $\tilde{C}_c(k'(c)) \neq \emptyset$ for all $c \in C^{\text{cl}}$ with residue field $k'(c)$ and where \tilde{C}_c is the scheme-theoretic inverse image of c in \tilde{C} , then $\tilde{C}(K) \neq \emptyset$.

Given a field k with $\text{char}(k) = 0$, we say that k *strongly* satisfies one of the conditions (i)–(vi) above if this condition is satisfied by any finite extension k'/k . We say that the field k satisfies *condition* (\star) if k strongly satisfies each of the conditions (i)–(vi).

Conditions (i), (iv), (v), and (vi) above are satisfied by finitely generated fields over \mathbb{Q} . In this case, (i) follows from the theory of weights, (iv) follows from the Mordell–Weil and Lang–Néron Theorems, (v) follows (for the discrete topology) from Mordell’s conjecture: Faltings’ Theorem and Néron’s Specialisation Theorem, and (vi) follows from the Hilbertian property (cf. Lemma 4.1.5) which holds for finitely generated fields. It is not known in general if condition (iii) holds over such fields (cf. [ST] for some partial results).

Among our main results in this paper are the following.

Theorem A. *Let k be a field with $\text{char}(k) = 0$ and C a separated, smooth, connected curve over k with function field $K = k(C)$. Assume that k strongly satisfies conditions (i), (ii), (iv), (v), and (vi) in Definition 0.2. Let $\mathcal{X} \rightarrow C$ be a flat, proper, smooth relative curve, with generic fibre $X := \mathcal{X} \times_C \text{Spec } K$ which is a geometrically connected hyperbolic curve over K , and $\mathcal{J} := \text{Pic}_{\mathcal{X}/C}^0$ its relative jacobian. Assume that $X(K) \neq \emptyset$ and $T\text{III}(\mathcal{J}) = 0$. Then the **SC** holds for X (over K).*

Theorem B. *Let k be a field with $\text{char}(k) = 0$, and $K = k(C)$ the function field of a separated, smooth, connected curve C over k . Assume that k satisfies condition (\star) (cf. Definition 0.2). Then K strongly satisfies condition (ii). Equivalently, if L/K is a finite extension then the **SC** holds over L .*

In the case of finitely generated fields one obtains immediately from Theorems A and B, respectively, the following corollaries.

Corollary A. *Assume that the **SC** holds over all finitely generated fields over \mathbb{Q} of transcendence degree $i \geq 0$. Let k be a field with $\text{tr deg}_{\mathbb{Q}} k = i$ and C a separated, smooth, connected curve over k with function field $K = k(C)$. Let $\mathcal{X} \rightarrow C$ be a flat, proper, smooth relative curve, with generic fibre $X := \mathcal{X} \times_C \text{Spec } K$ which is a geometrically connected hyperbolic curve over K , and $\mathcal{J} := \text{Pic}_{\mathcal{X}/C}^0$ its relative jacobian. Assume that $X(K) \neq \emptyset$ and $T\text{III}(\mathcal{J}) = 0$. Then the **SC** holds for X (over K).*

Corollary B. *Assume that the **SC** holds over all number fields (i.e., all finite extensions of \mathbb{Q}) and that condition (iii) in Definition 0.2 holds for any field k' which is finitely generated over \mathbb{Q} . Then the **SC** holds over all finitely generated fields over \mathbb{Q} .*

Recently, together with Akio Tamagawa, we proved that the Shafarevich–Tate group $\text{III}(\mathcal{J})$ in Theorem A is indeed finite if the jacobian $J := \text{Pic}_{X/K}^0$ of X is (isogenous to) a constant abelian variety, i.e., with the notation of Definition 0.2 condition (iii) holds for abelian schemes $\mathcal{A} \rightarrow C$ such that $A := \mathcal{A} \times_C \text{Spec } K$ (is isogenous to an abelian variety which) descends (over \bar{K}) to an abelian variety defined over a finite extension of k' (cf. [ST, Theorem H]). From this result and Theorem A, one deduces (using an induction argument) the following (cf. Corollary 5.3).

Theorem C. *Let K be a finitely generated field over \mathbb{Q} with algebraic closure \bar{K} , and $\bar{\mathbb{Q}}$ the algebraic closure of \mathbb{Q} in \bar{K} . Assume that the **SC** holds over all number fields. Then the **SC** holds for every projective, smooth, geometrically connected hyperbolic curve X over K such that $X \times_{\text{Spec } K} \text{Spec } \bar{K}$ is defined over $\bar{\mathbb{Q}}$.*

Our method to prove Theorem A relies on a local-global approach and follows from a thorough investigation of sections of the arithmetic fundamental groups of hyperbolic curves over local fields of equal characteristic 0, and over function fields of curves in characteristic 0.

Next, we describe the content of each section. In §1 we establish some basic facts on geometrically abelian fundamental groups and their sections. In §2 we investigate (under the assumption that condition (i) holds) sections of the arith-

arithmetic fundamental groups of curves over local fields of equal characteristic 0. We observe that the section conjecture **SC** does not hold over local fields of equal characteristic 0 (cf. Lemma 2.1.3 and Proposition 2.3.1). We discuss those sections which are point-theoretic in the case of stable curves (cf. Lemmas 2.1.4 and 2.2.2). In §3 we investigate sections of the arithmetic fundamental groups of curves over function fields (of transcendence degree 1), and establish some of the basic techniques and facts in order to investigate their point-theoreticity via a local-global approach. In §4 we prove Theorem A and explain how Theorem B can be derived from Theorem A. Finally, in §5 we relate the results and techniques of §3 and §4 to the results and framework of [ST]. As a consequence, we prove some variants of Theorems A and B involving a condition of finiteness of the ℓ -primary parts of certain Shafarevich–Tate groups (cf. Corollary 5.2 and Theorem 5.4), and deduce Theorem C.

Theorems A, B, and C concern sections of the arithmetic fundamental groups of proper curves over function fields and finitely generated fields. One can prove similar results for non-cuspidal sections of the arithmetic fundamental groups of affine curves over function fields and finitely generated fields, as well as for the birational version of the section conjecture (cf. [S3]).

Notation

Given a scheme Y over a field L with algebraic closure \bar{L} we write $Y_{\bar{L}} := Y \times_{\text{Spec } L} \text{Spec } \bar{L}$ for the geometric fibre of Y . Given a scheme C , a field L , and morphisms of schemes $Y \rightarrow C$ and $\text{Spec } L \rightarrow C$, we write $Y_L := Y \times_C \text{Spec } L$. For an algebraic group G over a field L of characteristic 0, with algebraic closure \bar{L} , we write $TG := \varprojlim_{N \geq 1} G[N](\bar{L})$ for the Tate module of G , where $G[N] := \text{Ker}(G \xrightarrow{[N]} G)$ is the kernel of the homomorphism of multiplication by N . For a profinite group H we write H^{ab} for the maximal abelian quotient of H . For an abelian group D we write $D^\wedge := \varprojlim_{N \geq 1} D/ND$, where $ND := \{N.a \mid a \in D\}$. Given an integer $N \geq 1$, we write $D[N] := \{b \in D \mid N.b = 0\}$, and $TD := \varprojlim_{N \geq 1} D[N]$ for the Tate module of D .

§1. Geometrically abelian fundamental groups

Let K be a field of characteristic 0, and $X \rightarrow \text{Spec } K$ a proper, smooth, geometrically connected hyperbolic curve over K . Let ξ be a geometric point of X with value in its generic point. Thus, ξ determines an algebraic closure \bar{K} of K . Write $\bar{\xi}$ for the geometric point of $X_{\bar{K}}$ which is induced by ξ . We have an exact sequence

of étale fundamental groups

$$(1.1) \quad 1 \rightarrow \pi_1(X_{\overline{K}}, \overline{\xi}) \rightarrow \pi_1(X, \xi) \xrightarrow{\text{pr}} G_K \rightarrow 1,$$

where $G_K := \text{Gal}(\overline{K}/K)$. Write

$$\pi_1(X, \xi)^{(\text{ab})} := \pi_1(X, \xi) / \text{Ker}(\pi_1(X_{\overline{K}}, \overline{\xi}) \rightarrow \pi_1(X_{\overline{K}}, \overline{\xi})^{\text{ab}}).$$

We will refer to $\pi_1(X, \xi)^{(\text{ab})}$ as the geometrically abelian quotient of $\pi_1(X, \xi)$.

Assume that $X(K) \neq \emptyset$. Write $J := \text{Pic}_{X/K}^0$ for the jacobian variety of X , and $\iota : X \rightarrow J$ for the embedding which maps a rational point $x_0 \in X(K)$ to the zero section of J . Then ι induces a commutative diagram of exact sequences

$$(1.2) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(X_{\overline{K}}, \overline{\xi})^{\text{ab}} & \longrightarrow & \pi_1(X, \xi)^{(\text{ab})} & \longrightarrow & G_K \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \text{id} \downarrow \\ 1 & \longrightarrow & \pi_1(J_{\overline{K}}, \overline{\xi}) & \longrightarrow & \pi_1(J, \xi) & \longrightarrow & G_K \longrightarrow 1 \end{array}$$

where the vertical maps are isomorphisms, hence an identification of G_K -modules $\pi_1(X_{\overline{K}}, \overline{\xi})^{\text{ab}} \xrightarrow{\sim} \pi_1(J_{\overline{K}}, \overline{\xi}) \xrightarrow{\sim} TJ$. Let

$$s : G_K \rightarrow \pi_1(X, \xi)$$

be a section of $\pi_1(X, \xi)$. Then s induces a section

$$s^{\text{ab}} : G_K \rightarrow \pi_1(X, \xi)^{(\text{ab})}$$

of the projection $\pi_1(X, \xi)^{(\text{ab})} \twoheadrightarrow G_K$. The set of splittings of the upper sequence in diagram (1.2) is, up to conjugation by elements of $\pi_1(X_{\overline{K}}, \overline{\xi})^{\text{ab}}$, a torsor under the Galois cohomology group $H^1(G_K, \pi_1(X_{\overline{K}}, \overline{\xi})^{\text{ab}}) \xrightarrow{\sim} H^1(G_K, TJ)$. We fix a base point of the torsor of splittings of this exact sequence to be the splitting arising from the zero section of J , i.e., from the rational point $x_0 \in X(K)$ (cf. the above discussion). Then the above (conjugacy class of the) section s^{ab} corresponds to an element

$$s^{\text{ab}} \in H^1(G_K, TJ).$$

We will refer to s^{ab} as the *abelian portion* of the section s .

The Kummer exact sequences $0 \rightarrow J[N] \rightarrow J \xrightarrow{N} J \rightarrow 0$, for $N \geq 1$, induce an exact sequence, called the *Kummer exact sequence*,

$$(1.3) \quad 0 \rightarrow J(K)^\wedge \rightarrow H^1(G_K, TJ) \rightarrow TH^1(G_K, J) \rightarrow 0.$$

We will identify $J(K)^\wedge$ with its image in $H^1(G_K, TJ)$ via the above Kummer map $J(K)^\wedge \rightarrow H^1(G_K, TJ)$. Note that there exist natural maps $X(K) \xrightarrow{\iota} J(K) \rightarrow$

$J(K)^\wedge$, where for $x \in X(K)$ the image $\iota(x)$ is the class $[x - x_0]$ of the degree 0 divisor $x - x_0$.

Definition 1.1. Let $\eta \in H^1(G_K, TJ)$. We say that η is

- (i) *pro-geometric* if it lies in the subgroup $J(K)^\wedge$ of $H^1(G_K, TJ)$;
- (ii) *geometric* if it is in the image of the natural composite homomorphism $J(K) \rightarrow J(K)^\wedge \rightarrow H^1(G_K, TJ)$;
- (iii) *point-theoretic* if it is in the image of $X(K) \xrightarrow{\iota} J(K) \rightarrow J(K)^\wedge \rightarrow H^1(G_K, TJ)$.

The following lemma follows easily from the various definitions.

Lemma 1.2. *Suppose that the section $s = s_x$, $x \in X(K)$, is point-theoretic (cf. §0). Then $s^{\text{ab}} \in H^1(G_K, TJ)$ is point-theoretic (cf. Definition 1.1(iii)), and s^{ab} is the image of x via the composite map $X(K) \xrightarrow{\iota} J(K) \rightarrow J(K)^\wedge \rightarrow H^1(G_K, TJ)$. In particular, s^{ab} is pro-geometric and geometric (cf. Definition 1.1(i), (ii)).*

§2. Sections of the arithmetic fundamental groups of curves over local fields of equal characteristic 0

In this section, K is a complete discrete valuation field of equal characteristic 0, \mathcal{O}_K its valuation ring, and k its residue field. We use the notation introduced in §0 and §1. Moreover, we assume that k satisfies condition (i) in Definition 0.2, unless we specify otherwise.

Let $X \rightarrow \text{Spec } \mathcal{O}_K$ be a flat, proper, stable, geometrically connected (relative) curve over \mathcal{O}_K , with X_K smooth. Assume that the irreducible components $\{X_i\}_{i \in I}$ of $X_k = \sum_{i \in I} X_i$ are smooth, geometrically connected, and the singular points $\{x_j\}_{j \in J}$ of X_k are k -rational. Let ξ (resp. ξ') be a geometric point of X with value in the generic point of X (resp. with value in the generic point of some irreducible component X_{i_0} of X_k). Thus, ξ (resp. ξ') determines an algebraic closure \bar{K} (resp. \bar{k}) of K (resp. k). We have natural exact sequences of arithmetic “admissible” fundamental groups

$$1 \rightarrow \pi_1(X_{\bar{K}}, \bar{\xi}) \rightarrow \pi_1(X_K, \xi) \rightarrow G_K \rightarrow 1$$

and

$$1 \rightarrow \pi_1(X_{\bar{k}}, \bar{\xi}')^{\text{adm}} \rightarrow \pi_1(X_k, \xi')^{\text{adm}} \rightarrow G_K \rightarrow 1,$$

where $G_K := \text{Gal}(\bar{K}/K)$, and the geometric point $\bar{\xi}$ (resp. $\bar{\xi}'$) is naturally induced by ξ (resp. ξ'). Here, the superscript “adm” means the admissible fundamental group (cf. [M2, Definition 2.7] for more details on the definition of $\pi_1(X_{\bar{k}}, \bar{\xi}')^{\text{adm}}$). We also refer to [M2, discussion before Definition 2.1, and between Definitions 2.5 and 2.7] for the precise definition of the log structures on X_k , $\text{Spec } k$, $\text{Spec } \mathcal{O}_K$,

$X_k \rightarrow \text{Spec } k$, and $X \rightarrow \text{Spec } \mathcal{O}_X$ involved in the definition of $\pi_1(X_{\bar{k}}, \bar{\xi}')^{\text{adm}}$. Moreover, after a suitable choice of ξ and ξ' we have a commutative diagram of exact sequences

$$(2.1) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(X_{\bar{K}}, \bar{\xi}) & \longrightarrow & \pi_1(X_K, \xi) & \longrightarrow & G_K \longrightarrow 1 \\ & & \downarrow & & \text{Sp} \downarrow & & \text{id} \downarrow \\ 1 & \longrightarrow & \pi_1(X_{\bar{k}}, \bar{\xi}')^{\text{adm}} & \longrightarrow & \pi_1(X_k, \xi')^{\text{adm}} & \longrightarrow & G_K \longrightarrow 1 \end{array}$$

where the middle and left vertical maps are continuous homomorphisms of “specialisation”, which are isomorphisms since $\text{char}(k) = 0$ (cf. [M2, Definition 2.4, Lemma 2.6, and discussion before Definition 2.7]). We assume that X_K is hyperbolic, i.e., $g(X_K) \geq 2$. We have an exact sequence $1 \rightarrow I_K \rightarrow G_K \rightarrow G_k := \text{Gal}(\bar{k}/k) \rightarrow 1$ where I_K is the inertia group. Moreover, $I_K \xrightarrow{\sim} \hat{\mathbb{Z}}(1)$ where the “(1)” is a Tate twist.

§2.1. The good reduction case

Assume that X is smooth. In this case we have a commutative diagram of exact sequences of arithmetic fundamental groups

$$(2.2) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(X_{\bar{K}}, \bar{\xi}) & \longrightarrow & \pi_1(X_K, \xi) & \longrightarrow & G_K \longrightarrow 1 \\ & & \downarrow & & \text{Sp} \downarrow & & \downarrow \\ 1 & \longrightarrow & \pi_1(X_{\bar{k}}, \bar{\xi}') & \longrightarrow & \pi_1(X_k, \xi') & \longrightarrow & G_k \longrightarrow 1 \end{array}$$

where the middle vertical map is the surjective homomorphism of specialisation (defined up to conjugation, cf. [G2, Exposé X, §2]), the left vertical map is an isomorphism (since $\text{char}(k) = 0$), and the right vertical map is a surjection.

- Lemma 2.1.1.** (i) *The right square in diagram (2.1) is cartesian.*
 (ii) *The projection $\pi_1(X_K, \xi) \twoheadrightarrow G_K$ induces a natural isomorphism $\text{Ker}(\text{Sp}) \xrightarrow{\sim} I_K$ (i.e., maps $\text{Ker}(\text{Sp})$ isomorphically onto I_K).*

Proof. The proof of (i) is similar to the proof of [S1, Lemma 3.3.2]. Assertion (ii) is clear in light of (i). □

Let $s : G_K \rightarrow \pi_1(X_K, \xi)$ be a section of $\pi_1(X_K, \xi)$ so that it induces, by composing with the specialisation homomorphism $\text{Sp} : \pi_1(X_K, \xi) \rightarrow \pi_1(X_k, \xi')$, a continuous homomorphism $s' := \text{Sp} \circ s : G_K \rightarrow \pi_1(X_k, \xi')$.

Lemma 2.1.2. *We have $\text{Ker}(s') = I_K$. In particular, s' factorizes through G_k and induces a section $\tilde{s} : G_k \rightarrow \pi_1(X_k, \xi')$ of $\pi_1(X_k, \xi')$.*

Proof. It suffices to show that the image $s'(I_K)$ of the inertia subgroup in $\pi_1(X_k, \xi')$ is trivial. This image is contained in $\pi_1(X_{\bar{k}}, \bar{\xi}')$ by diagram (2.1). A standard (well-known) weight argument, using the fact that k satisfies condition (i) in Definition 0.2, shows that this image must be trivial (cf. [HM1, Lemma 1.6]). \square

Assume that the section s is point-theoretic, i.e., $s = s_x : G_K \rightarrow \pi_1(X_K, \xi)$ is associated to a rational point $x \in X(K)$ (cf. §0). Let $\bar{x} \in X(k)$ be the specialisation of the point x . Then one verifies easily that the section $\tilde{s} : G_k \rightarrow \pi_1(X_k, \xi')$ of $\pi_1(X_k, \xi')$ which is induced by s (cf. Lemma 2.1.2) is point-theoretic and arises from the k -rational point \bar{x} , i.e., $[\tilde{s}] = [s_{\bar{x}}]$ in $\overline{\text{Sec}}_{\pi_1(X_k, \xi')}$.

Lemma 2.1.3. *With the above notation, let $x' \in X(K)$ be a rational point which specialises in \bar{x} . Then $[s_x] = [s_{x'}]$. In particular, the map $\varphi_X : X(K) \rightarrow \overline{\text{Sec}}_{\pi_1(X_K, \xi)}$ (cf. §0) is not injective.*

Proof. Indeed, it follows immediately from Lemmas 2.1.1(i) and 2.1.2 that a section $s : G_K \rightarrow \pi_1(X_K, \xi)$ of $\pi_1(X_K, \xi)$ is uniquely determined by the continuous homomorphism $s' := \text{Sp} \circ s : G_K \twoheadrightarrow G_K/I_K \rightarrow \pi_1(X_k, \xi')$ it induces. In particular, all rational points $x' \in X(K)$ which specialise in \bar{x} (there are infinitely many such points x') give rise to the same conjugacy class of sections of $\pi_1(X_K, \xi)$, from which the second assertion follows. \square

Conversely, we have the following.

Lemma 2.1.4. *Assume that the section $\tilde{s} : G_k \rightarrow \pi_1(X_k, \xi')$ of $\pi_1(X_k, \xi')$ which is induced by s (cf. Lemma 2.1.2) is point-theoretic, i.e., $\tilde{s} = s_{\bar{x}}$ for some k -rational point $\bar{x} \in X(k)$. Then the section s is point-theoretic, i.e., $s = s_x$ for some (non-unique) $x \in X(K)$ which specialises in the point \bar{x} .*

Proof. Let $x \in X(K)$ be a rational point which specialises in \bar{x} (such a point x exists since X is smooth). Then $[s] = [s_x]$ by the same argument used in the proof of Lemma 2.1.3. \square

§2.2. The bad reduction case

In this section, and in addition to our assumptions, we will assume that k satisfies condition (iv)(a) in Definition 0.2 unless we specify otherwise. We suppose that X_k is singular. Recall that the irreducible components $\{X_i\}_{i \in I}$ of $X_k = \sum_{i \in I} X_i$ are smooth, geometrically connected, and the singular points $\{x_j\}_{j \in J}$ of X_k are all k -rational.

Let X_i be an irreducible component of X_k and $D_{X_i} \subset \pi_1(X_k, \xi')^{\text{adm}}$ a decomposition group associated to X_i . Thus, D_{X_i} is the decomposition group of an irreducible component of the special fibre of the universal admissible cover \tilde{X} of

X which lies above the component X_i , and D_{X_i} is only defined up to conjugation (cf. [M2, discussion before Proposition 4.1]). Let \bar{X}_i be the (unique, since X_i is geometrically connected) irreducible component of $X_{\bar{k}}$ above X_i . Then we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 1 & \longrightarrow & D_{\bar{X}_i} & \longrightarrow & D_{X_i} & \longrightarrow & G_K & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 (2.3) \quad 1 & \longrightarrow & \pi_1(\bar{X}_i, *)^{\text{adm}} & \longrightarrow & \pi_1(X_i, *)^{\text{adm}} & \longrightarrow & G_K & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 1 & \longrightarrow & \pi_1(\bar{U}_i, *)^{\text{tame}} & \longrightarrow & \pi_1(U_i, *)^{\text{tame}} & \longrightarrow & G_k & \longrightarrow & 1
 \end{array}$$

Here, $D_{\bar{X}_i}$ is defined so that the top horizontal sequence is exact, $*$ denote suitable base points, and $\pi_1(X_i, *)^{\text{adm}}$ denotes the admissible fundamental group of X_i which is marked by the cusps, i.e., the double points of X_k lying on X_i (cf. [M2, Proposition 4.2] and the discussion before it which explains the log structure on X_i and the choice of the base point $*$ involved in the definition of $\pi_1(X_i, *)^{\text{adm}}$), $\pi_1(\bar{X}_i, *)^{\text{adm}} := \text{Ker}(\pi_1(X_i, *)^{\text{adm}} \rightarrow G_K)$, $U_i := X_i \setminus \{\text{cusps}\}$, and $\bar{U}_i := \bar{X}_i \setminus \{\text{cusps}\}$, respectively. The superscript “tame” means the tame fundamental group. The left and middle upper vertical maps are isomorphisms defined up to conjugation (cf. loc. cit.), and the bottom right square is cartesian. Note that $\pi_1(\bar{U}_i, *)^{\text{tame}} = \pi_1(\bar{U}_i, *)$, and $\pi_1(U_i, *)^{\text{tame}} = \pi_1(U_i, *)$, since $\text{char}(k) = 0$.

Let $s : G_K \rightarrow \pi_1(X_K, \xi)$ be a section of $\pi_1(X_K, \xi)$, which induces a section $s' := \text{Sp} \circ s : G_K \rightarrow \pi_1(X_k, \xi')^{\text{adm}}$ of the projection $\pi_1(X_k, \xi')^{\text{adm}} \rightarrow G_K$ (cf. diagram (2.1)). Suppose that X is regular and s is point-theoretic, i.e., $s = s_x$ arises from a rational point $x \in X(K)$ (cf. §0). Then the K -rational point x specialises in a rational point $\bar{x} \in X(k)$ which is a smooth point of X_k and lies on a unique irreducible component X_i of X_k (cf. [L, Corollary 9.1.32]). Moreover, it follows from the various definitions that $s(G_K) \subset D_{X_i} \subset \pi_1(X_k, \xi')^{\text{adm}}$, where D_{X_i} is a decomposition group associated to X_i (cf. above discussion). In particular, the section $s = s_x$ induces a section $s_i : G_K \rightarrow \pi_1(X_i, *)^{\text{adm}}$ of the projection $\pi_1(X_i, *)^{\text{adm}} \rightarrow G_K$, and a continuous homomorphism $\tilde{s}_i : G_K \rightarrow \pi_1(U_i, *)^{\text{tame}}$ (cf. diagram (2.3)).

Lemma 2.2.1. (i) *The section s_i is unramified, i.e., $\tilde{s}_i(I_K) = \{1\}$. In particular, \tilde{s}_i induces a section $\bar{s}_i : G_k \rightarrow \pi_1(U_i, *)^{\text{tame}}$ of $\pi_1(U_i, *)^{\text{tame}}$.*
 (ii) *The section $\bar{s}_i : G_k \rightarrow \pi_1(U_i, *)^{\text{tame}}$ in (i) is point-theoretic and arises from the k -rational point $\bar{x} \in U_i(k)$. Moreover, \bar{s}_i is non-cuspidal, i.e., $\bar{s}_i(G_k)$ is not contained in a decomposition group associated to a cusp.*

Proof. Note that U_i is hyperbolic since X is stable. Assertion (i) follows from conditions (i) and (iv)(a) in Definition 0.2. First, condition (iv)(a) implies that the closed points of X_i are uniquely determined by the corresponding (conjugacy classes of) decomposition groups in $\pi_1(U_i, *)^{\text{tame}}$, such a decomposition group is self-normalising in $\pi_1(U_i, *)^{\text{tame}}$, and no non-cuspidal decomposition group is contained in a cuspidal decomposition group (cf. [M3, proof of Theorem 1.3] and [T, Proposition 2.8(i)]). Second, if $\tilde{s}_i(I_K)$ is non-trivial then one shows, using condition (i), that $\tilde{s}_i(I_K) \subseteq \pi_1(\bar{U}_i, *)^{\text{tame}}$ would be a non-trivial (necessarily torsion-free) procyclic group contained in an inertia group I_y at a cusp $y \in X_i \setminus U_i$ (cf. [HM1, Lemma 1.6]). Moreover, $\tilde{s}_i(G_K)$ (which normalises $\tilde{s}_i(I_K)$) is contained in a decomposition group associated to y and the latter would contain a decomposition group associated to \bar{x} , which is a contradiction (cf. above discussion). Assertion (ii) follows easily. \square

Conversely, suppose that the section s satisfies $s(G_K) \subset D_{X_i}$, i.e., the image of s is contained in a decomposition group associated to an irreducible component X_i of X_s . Thus, s induces a section $s_i : G_K \rightarrow \pi_1(X_i, *)^{\text{adm}}$ of the projection $\pi_1(X_i, *)^{\text{adm}} \rightarrow G_K$ which induces a homomorphism $\tilde{s}_i : G_K \rightarrow \pi_1(U_i, *)^{\text{tame}}$ (cf. diagram (2.2)). Assume further that s_i is unramified, i.e., $\tilde{s}_i(I_K) = \{1\}$. Then \tilde{s}_i naturally induces a section $\bar{s}_i : G_K \rightarrow \pi_1(U_i, *)^{\text{tame}}$ of $\pi_1(U_i, *)^{\text{tame}}$ (cf. diagram (2.2)).

Lemma 2.2.2. *With the same notation/assumptions as above suppose that the section $\bar{s}_i : G_K \rightarrow \pi_1(U_i, *)^{\text{tame}}$ is point-theoretic and arises from a rational point $\bar{x} \in U_i(k)$. Then the section s is point-theoretic and arises from a (non-unique) rational point $x \in X(K)$ which specialises in the point $\bar{x} \in U_i(k)$.*

Proof. Similar to the proof of Lemma 2.1.4. \square

§2.3

Next, we provide examples of sections of the arithmetic fundamental groups of hyperbolic curves over local fields of equal characteristic 0 which are not point-theoretic. We use the notation and assumptions of §2.2. Let X be a regular and stable \mathcal{O}_K -curve as above satisfying the conditions of §2.2. Let $x_j \in X(k)$ be a k -rational double point of X_k and write $D_{x_j} \subset \pi_1(X_k, \xi')^{\text{adm}}$ for the decomposition group of x_j . Thus, D_{x_j} is the decomposition group of a closed point of the special fibre of the universal admissible cover \tilde{X} of X which lies above the double point x_j , and D_{x_j} is only defined up to conjugation (cf. [M2, §5 and §6]). We have an exact sequence

$$1 \rightarrow \Delta_{x_j} \rightarrow D_{x_j} \rightarrow G_K \rightarrow 1,$$

where $\Delta_{x_j} := \text{Ker}(D_{x_j} \rightarrow G_K)$. Moreover, there exists a natural isomorphism $\Delta_{x_j} \xrightarrow{\sim} \hat{\mathbb{Z}}(1)$ (cf. loc. cit.). The profinite group D_{x_j} is isomorphic to the admissible fundamental group $\pi_1^{\text{adm}}(\mathcal{X})$ of $\mathcal{X} := \text{Spec } \mathcal{O}_K[[S, T]]/(ST - \pi)$, where π is a uniformiser of \mathcal{O}_K (cf. loc. cit.). The above exact sequence splits. Indeed, the (admissible) covers $\mathcal{Y}_N \rightarrow \mathcal{X}$ defined generically by extracting an N -th root of S with \mathcal{Y}_N normal, for all integers $N \geq 1$, define a splitting of this sequence. Such a splitting induces a section $s_{x_j} : G_K \rightarrow \pi_1(X_K, \xi) \xrightarrow{\sim} \pi_1(X_k, \xi')^{\text{adm}}$ of $\pi_1(X_K, \xi)$. The section s_{x_j} is *not point-theoretic*. Indeed, if s_{x_j} arises from a rational point $x \in X(K)$, then x specialises in a smooth non-cuspidal point of an irreducible component X_i (cf. the discussion before Lemma 2.2.1), which is necessarily adjacent to an irreducible component $X_{i'}$ passing through x_j (cf. [HM2, Corollary 1.15(iv)(c)(d)]). Let $s_i : G_k \rightarrow \pi_1(U_i, *)^{\text{tame}}$ be the section of $\pi_1(U_i, *)^{\text{tame}}$ which is induced by s_{x_j} (cf. Lemma 2.2.1). This section (which is non-cuspidal) would then be cuspidal, which contradicts Lemma 2.2.1(ii).

Proposition 2.3.1. *Let K be a complete discrete valuation field with residue field k of characteristic 0. Assume that k satisfies conditions (i) and (iv)(a) of Definition 0.2. Then there exists a flat, proper, smooth, geometrically connected hyperbolic curve C over K and a section $s : G_K \rightarrow \pi_1(C, *)$ of $\pi_1(C, *)$ which is not point-theoretic.*

Proof. Write \mathcal{O}_K for the valuation ring of K . Using formal patching techniques one can construct a proper, stable, regular \mathcal{O}_K -curve X satisfying the assumptions in 2.2. In particular, $C := X_K$ is smooth, hyperbolic, geometrically connected, and the double points of X_k are k -rational (cf. [S2, Lemma 6.3]). As in the above discussion (before Proposition 2.3.1), let $s_{x_j} : G_K \rightarrow \pi_1(C, *)$ be a section of $\pi_1(C, *)$ arising from a double point x_j of the special fibre X_s of X . Then s_{x_j} is not point-theoretic, as explained above. \square

§3. Sections of the arithmetic fundamental groups of curves over function fields in characteristic 0

We use the notation introduced in §1 and §2. In this section k is a field of characteristic 0 which strongly satisfies condition (i) in Definition 0.2. Let C be a separated, smooth, connected algebraic curve over k with function field $K := k(C)$. Let $\mathcal{X} \rightarrow C$ be a flat, proper, smooth relative curve with generic fibre $X := \mathcal{X} \times_C \text{Spec } K$ which is geometrically connected and hyperbolic (i.e. $g(X) \geq 2$). Let $c \in C^{\text{cl}}$ be a closed point and $\mathcal{X}_c := \mathcal{X} \times_C \text{Spec } k(c)$ the fibre of \mathcal{X} at c . Let ξ (resp. ξ_c) be a geometric point of \mathcal{X} with value in its generic point (resp.

with value in the generic point of \mathcal{X}_c . Then ξ (resp. ξ_c) determines an algebraic closure \bar{K} (resp. $\bar{k}(c)$) of K (resp. of the residue field $k(c)$ of C at c).

Lemma 3.1. *For $c \in C^{\text{cl}}$ there exists a commutative diagram*

$$(3.1) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(X_{\bar{K}}, \bar{\xi}) & \longrightarrow & \pi_1(X, \xi) & \longrightarrow & G_K & \longrightarrow & 1 \\ & & \text{id} \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \pi_1(X_{\bar{K}}, \bar{\xi}) & \longrightarrow & \pi_1(\mathcal{X}, \xi) & \longrightarrow & \pi_1(C, \xi) & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \pi_1(\mathcal{X}_{\bar{k}(c)}, \bar{\xi}_c) & \longrightarrow & \pi_1(\mathcal{X}_c, \xi_c) & \longrightarrow & G_{k(c)} & \longrightarrow & 1 \end{array}$$

where $G_K = \text{Gal}(\bar{K}/K)$, $G_{k(c)} = \text{Gal}(\bar{k}(c)/k(c))$, $\bar{\xi}$ (resp. $\bar{\xi}_c$) are geometric points induced by ξ (resp. ξ_c), the middle and right upper vertical maps are natural continuous surjective homomorphisms, the middle lower vertical map is defined up to conjugation, the right lower vertical map is injective, and both squares on the right are cartesian.

Proof. Follows from the functoriality of fundamental groups and the homotopy exact sequence for π_1 (cf. [G2, Exposé XIII, §4]). □

Let $s : G_K \rightarrow \pi_1(X, \xi)$ be a section of $\pi_1(X, \xi)$.

Lemma 3.2. *There exists a section $s_C : \pi_1(C, \xi) \rightarrow \pi_1(\mathcal{X}, \xi)$ of the projection $\pi_1(\mathcal{X}, \xi) \rightarrow \pi_1(C, \xi)$ which restricts for each closed point $c \in C^{\text{cl}}$ to a section $s_c : G_{k(c)} \rightarrow \pi_1(\mathcal{X}_c, \xi_c)$ of $\pi_1(\mathcal{X}_c, \xi_c)$ and we have a commutative diagram*

$$(3.2) \quad \begin{array}{ccc} G_K & \xrightarrow{s} & \pi_1(X, \xi) \\ \downarrow & & \downarrow \\ \pi_1(C, \xi) & \xrightarrow{s_C} & \pi_1(\mathcal{X}, \xi) \\ \uparrow & & \uparrow \\ G_{k(c)} & \xrightarrow{s_c} & \pi_1(\mathcal{X}_c, \xi_c) \end{array}$$

where the vertical maps are the ones in diagram (3.1).

Proof. Follows easily from Lemmas 2.1.2 and 3.1. □

From now on we assume that $X(K) \neq \emptyset$. Let $\mathcal{J} := \text{Pic}_{\mathcal{X}/C}^0 \rightarrow C$ be the (relative) jacobian of the (relative) curve $\mathcal{X} \rightarrow C$, $J := \mathcal{J}_K$ the jacobian variety of X , $\mathcal{J}_c := \mathcal{J}_{k(c)}$ the jacobian variety of \mathcal{X}_c , and $J_c := J_{K_c}$ the jacobian variety

of X_{K_c} , where $c \in C^{\text{cl}}$ and K_c is the completion of K at c . The commutative diagram (3.1) induces a commutative diagram of exact sequences of geometrically abelian fundamental groups (cf. §1)

$$\begin{array}{ccccccccc}
 1 & \longrightarrow & \pi_1(X_{\overline{K}}, \overline{\xi})^{\text{ab}} & \longrightarrow & \pi_1(X, \xi)^{(\text{ab})} & \longrightarrow & G_K & \longrightarrow & 1 \\
 & & \text{id} \downarrow & & \downarrow & & \downarrow & & \\
 (3.3) \quad 1 & \longrightarrow & \pi_1(X_{\overline{K}}, \overline{\xi})^{\text{ab}} & \longrightarrow & \pi_1(\mathcal{X}, \xi)^{(\text{ab})} & \longrightarrow & \pi_1(C, \xi) & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 1 & \longrightarrow & \pi_1(\mathcal{X}_{\overline{k(c)}}, \overline{\xi}_c)^{\text{ab}} & \longrightarrow & \pi_1(\mathcal{X}_c, \xi_c)^{(\text{ab})} & \longrightarrow & G_{k(c)} & \longrightarrow & 1
 \end{array}$$

satisfying similar properties to those in Lemma 3.1. Here $\pi_1(\mathcal{X}, \xi)^{(\text{ab})}$ is defined as the push forward of the group extension $1 \rightarrow \pi_1(X_{\overline{K}}, \overline{\xi}) \rightarrow \pi_1(\mathcal{X}, \xi) \rightarrow \pi_1(C, \xi) \rightarrow 1$ by the characteristic quotient $\pi_1(X_{\overline{K}}, \overline{\xi}) \twoheadrightarrow \pi_1(X_{\overline{K}}, \overline{\xi})^{\text{ab}}$. Fix a section $x \in \mathcal{X}(C) = X(K)$ of $\mathcal{X} \rightarrow C$, and the embedding $\mathcal{X} \rightarrow \mathcal{J}$ mapping x to the zero section. We identify $\pi_1(X_{\overline{K}}, \overline{\xi})^{\text{ab}}$ (resp. $\pi_1(\mathcal{X}_{\overline{k(c)}}, \overline{\xi}_c)^{\text{ab}}$) with the Tate module TJ (resp. $T\mathcal{J}_c$), and TJ_c with $T\mathcal{J}_c$ (K_c is a complete discrete valuation ring of equal characteristic 0).

Lemma 3.3. *Let $c \in C^{\text{cl}}$. We have a commutative diagram of Kummer exact sequences (cf. §1)*

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & J(K_c)^\wedge & \longrightarrow & H^1(G_{K_c}, TJ_c) & \longrightarrow & TH^1(G_{K_c}, J_c) & \longrightarrow & 0 \\
 (3.4) \quad & & \uparrow & & \uparrow & & \uparrow & & \\
 0 & \longrightarrow & \mathcal{J}_c(k(c))^\wedge & \longrightarrow & H^1(G_{k(c)}, T\mathcal{J}_c) & \longrightarrow & TH^1(G_{k(c)}, \mathcal{J}_c) & \longrightarrow & 0
 \end{array}$$

where the middle vertical map is the inflation map, the left vertical map is an isomorphism, and the middle and right vertical maps are injective.

Proof. Follows from the fact that there exists an isomorphism $H^1(G_{k(c)}, T\mathcal{J}_c) \xrightarrow{\sim} H^1(\text{Gal}(K_c^{\text{ur}}/K_c), TJ_c)$ where K_c^{ur}/K_c denotes the maximal unramified subextension of \overline{K}_c/K_c , and the fact that the kernel of the specialisation map $J_c(K_c) \rightarrow \mathcal{J}_c(k(c))$ is uniquely divisible (cf. [LT, Proposition 8]). (See also the commutative diagram in [LT, p. 675].) □

With the notation of Lemma 3.3, we will identify $H^1(G_{k(c)}, T\mathcal{J}_c)$ with its image in $H^1(G_{K_c}, TJ_c)$. Next, we fix the base point of the torsor of splittings of the middle horizontal sequence in diagram (3.3) arising from the section $x \in \mathcal{X}(C)$, and the corresponding base points of the torsor of splittings of the upper and lower horizontal sequences in diagram (3.3). The sections $s : G_K \rightarrow \pi_1(X, \xi)$,

$s_C : \pi_1(C, \xi) \rightarrow \pi_1(\mathcal{X}, \xi)$, and $s_c : G_{k(c)} \rightarrow \pi_1(\mathcal{X}_c, \xi_c)$ (cf. Lemma 3.2) give rise naturally to sections $s^{\text{ab}} : G_K \rightarrow \pi_1(X, \xi)^{(\text{ab})}$, $s_C^{\text{ab}} : \pi_1(C, \xi) \rightarrow \pi_1(\mathcal{X}, \xi)^{(\text{ab})}$, and $s_c^{\text{ab}} : G_{k(c)} \rightarrow \pi_1(\mathcal{X}_c, \xi_c)^{(\text{ab})}$, of the projections $\pi_1(X, \xi)^{(\text{ab})} \twoheadrightarrow G_K$, $\pi_1(\mathcal{X}, \xi)^{(\text{ab})} \twoheadrightarrow \pi_1(C, \xi)$, and $\pi_1(\mathcal{X}_c, \xi_c)^{(\text{ab})} \twoheadrightarrow G_{k(c)}$, respectively, which correspond to elements $s^{\text{ab}} \in H^1(G_K, TJ)$, $s_C^{\text{ab}} \in H^1(\pi_1(C, \xi), TJ)$, and $s_c^{\text{ab}} \in H^1(G_{k(c)}, TJ_c)$, for $c \in C^{\text{cl}}$.

Lemma 3.4. *Let $c \in C^{\text{cl}}$. We have a commutative diagram of homomorphisms*

$$\begin{CD} H^1(\pi_1(C, \xi), TJ) @>>> H^1(G_K, TJ) \\ @VVV @VVV \\ H^1(G_{k(c)}, TJ_c) @>>> H^1(G_{K_c}, TJ_c) \end{CD}$$

where the horizontal maps are injective inflation maps and the vertical maps are restriction homomorphisms in which s_C^{ab} maps to s^{ab} (resp. s_c^{ab}). In particular, the images of s^{ab} and s_c^{ab} in $H^1(G_{K_c}, TJ_c)$ are equal. Moreover, assume that the section s_c is point-theoretic. Then s_c^{ab} is point-theoretic (cf. Definition 1.1(iii)). In particular, s_c^{ab} is pro-geometric as an element of both $H^1(G_{k(c)}, TJ_c)$ and $H^1(G_{K_c}, TJ_c)$ (cf. Definition 1.1(i)).

Proof. The first assertion is easily verified. The second assertion follows from the various definitions (cf. Lemma 1.2), and Lemma 3.3. □

Next, consider the commutative diagram

$$(3.5) \quad \begin{CD} 0 @>>> J(K)^\wedge @>>> H^1(G_K, TJ) @>>> TH^1(G_K, J) @>>> 0 \\ @. @VVV @VVV @VVV \\ 0 @>>> \prod_c J_c(K_c)^\wedge @>>> \prod_c H^1(G_{K_c}, TJ_c) @>>> \prod_c TH^1(G_{K_c}, J_c) @>>> 0 \end{CD}$$

where the horizontal sequences are the Kummer exact sequences, the vertical maps are the diagonal mappings, and the product in the bottom sequence is over all closed points $c \in C^{\text{cl}}$. The image of $s^{\text{ab}} \in H^1(G_K, TJ)$ in $H^1(G_{K_c}, TJ_c)$ via the above middle vertical map coincides with the element $s_c^{\text{ab}} \in H^1(G_{k(c)}, TJ_c) \subset H^1(G_{K_c}, TJ_c)$, for all $c \in C^{\text{cl}}$ (cf. Lemma 3.4). Recall (cf. §0)

$$\text{III}(\mathcal{J}) = \text{III}(J, C) := \text{Ker} \left(H^1(G_K, J) \rightarrow \prod_{c \in C^{\text{cl}}} H^1(G_{K_c}, J_c) \right).$$

Note that the kernel of the map $TH^1(G_K, J) \rightarrow \prod_{c \in C} TH^1(G_{K_c}, J_c)$ in diagram (3.5) is the Tate module $T\text{III}(\mathcal{J})$ of the Shafarevich–Tate group $\text{III}(\mathcal{J})$. The following is immediate from the various definitions and Lemma 3.4.

Lemma 3.5. *Suppose that $s_c^{\text{ab}} \in H^1(G_{K_c}, TJ_c)$ is pro-geometric (cf. Definition 1.1(i)) for all $c \in C^{\text{cl}}$, and $T\text{III}(\mathcal{J}) = 0$. Then $s^{\text{ab}} \in J(K)^\wedge$ is pro-geometric.*

§4. Proofs of Theorems A and B

Next, we prove Theorems A and B.

§4.1. Proof of Theorem A

Recall the notation introduced in §3 that we will use throughout. Let $k, C, K = k(C), \mathcal{X} \rightarrow C$, and $X \rightarrow \text{Spec } K$ be as in §3. We assume that $X(K) \neq \emptyset$, and k strongly satisfies conditions (i), (ii), (iv), (v), and (vi) in Definition 0.2. Let $\mathcal{J} := \text{Pic}_{\mathcal{X}/C}^0$ be the relative jacobian of $\mathcal{X} \rightarrow C$, and assume that $T\text{III}(\mathcal{J}) = 0$. We will show that the map (cf. §0)

$$\varphi_X : X(K) \rightarrow \overline{\text{Sec}}_{\pi_1(X, \xi)}, \quad x \mapsto \varphi_X(x) = [s_x],$$

is *bijective*.

First, we prove φ_X is *injective*. Let $x_1, x_2 \in X(K)$ be such that $[s_1 := s_{x_1}] = [s_2 := s_{x_2}]$ in $\overline{\text{Sec}}_{\pi_1(X, \xi)}$. Thus, $[s_{1,c}] = [s_{2,c}]$ in $\overline{\text{Sec}}_{\pi_1(\mathcal{X}_c, \xi_c)}$ for all $c \in C^{\text{cl}}$ (cf. Lemma 3.2), and $s_{i,c} = s_{x_{i,c}}$ is point-theoretic, where $x_{i,c} \in \mathcal{X}_c(k(c))$ is uniquely determined by $s_{i,c}$, for $i \in \{1, 2\}$, as we assumed that k strongly satisfies (ii). The map $\varphi_{\mathcal{X}_c} : \mathcal{X}_c(k(c)) \rightarrow \overline{\text{Sec}}_{\pi_1(\mathcal{X}_c, \xi_c)}$ is bijective by assumption, hence $x_{1,c} = x_{2,c}$ for all $c \in C^{\text{cl}}$. Moreover, $x_{i,c}$ is the specialisation of x_i in \mathcal{X}_c (cf. discussion before Lemma 2.1.3). From this it follows that $x_1 = x_2$ and φ_X is injective. Indeed, the natural specialisation map $X(K) \rightarrow \prod_{c \in C^{\text{cl}}} \mathcal{X}_c(k(c))$ is injective.

Next, we prove that φ_X is *surjective*. Let

$$s : G_K \rightarrow \pi_1(X, \xi)$$

be a section of $\pi_1(X, \xi)$. We will show that s is point-theoretic under the above assumptions. First, we have a diagram (3.2) and it follows from condition (ii) that the section s_c (cf. loc. cit.) is point-theoretic and arises from a unique rational point $x_c \in \mathcal{X}(k(c))$, for all $c \in C^{\text{cl}}$. Moreover, $s_c^{\text{ab}} \in J(K_c)^\wedge \subset H^1(G_{K_c}, TJ_c)$ is pro-geometric in the sense of Definition 1.1(i), for all $c \in C^{\text{cl}}$ (cf. Lemma 3.4). Then it follows from the assumption $T\text{III}(\mathcal{J}) = 0$ that $s^{\text{ab}} \in J(K)^\wedge \subset H^1(G_K, TJ)$ is pro-geometric (cf. Lemma 3.5).

Lemma 4.1.1. *The natural homomorphism $J(K) \rightarrow J(K)^\wedge$ is injective and $s^{\text{ab}} \in J(K) \subseteq J(K)^\wedge$ is geometric.*

Proof. There exist closed points $c_1, c_2 \in C^{\text{cl}}$ such that the natural specialisation homomorphism $J(K) \rightarrow \mathcal{J}_{c_1}(k(c_1)) \times \mathcal{J}_{c_2}(k(c_2))$ is injective (cf. [PV, Proposition 2.4]). We have a commutative diagram of exact sequences

$$\begin{array}{ccccccc}
 0 & \longrightarrow & J(K) & \longrightarrow & H := \mathcal{J}_{c_1}(k(c_1)) \times \mathcal{J}_{c_2}(k(c_2)) & \longrightarrow & H/J(K) \longrightarrow 0 \\
 & & \downarrow & & \downarrow \phi & & \downarrow \\
 0 & \longrightarrow & J(K)^\wedge & \xrightarrow{\psi} & H^\wedge = \mathcal{J}_{c_1}(k(c_1))^\wedge \times \mathcal{J}_{c_2}(k(c_2))^\wedge & \longrightarrow & (H/J(K))^\wedge \longrightarrow 0
 \end{array}$$

where the right and middle vertical maps are injective by condition (iv)(a), and the maps ψ and ϕ are the natural ones. (The exactness of the bottom sequence in the above diagram follows easily from condition (iv)(b).) In particular, the left vertical map is injective, and $J(K) = \phi(H) \cap \psi(J(K)^\wedge)$ inside H^\wedge . The image of $s^{\text{ab}} \in J(K)^\wedge$ in H^\wedge via the map ψ is the element $(s_{c_1}^{\text{ab}}, s_{c_2}^{\text{ab}}) \in H^\wedge \subset H^1(G_{k(c_1)}, T\mathcal{J}_{c_1}) \times H^1(G_{k(c_2)}, T\mathcal{J}_{c_2})$ associated to the sections $s_{c_i} : G_{k(c_i)} \rightarrow \pi_1(\mathcal{X}_{c_i}, \xi_{c_i})$ (induced by the section s), for $i \in \{1, 2\}$, which are point-theoretic by condition (ii). In particular, $s_{c_i}^{\text{ab}} \in \mathcal{J}_{c_i}(k(c_i))$ is geometric (cf. Lemma 1.2), and $s^{\text{ab}} \in J(K)$ is geometric by the above discussion. \square

We fix an embedding $\mathcal{X} \xrightarrow{\iota} \mathcal{J}$ mapping an element of $\mathcal{X}(C) = X(K)$ (which is non-empty by our assumptions) to the zero section.

Lemma 4.1.2. *The element $s^{\text{ab}} \in \iota(X(K)) \subset J(K)$ is point-theoretic.*

Proof. For each closed point $c \in C^{\text{cl}}$ the element $s_c^{\text{ab}} \in \mathcal{J}(k(c))^\wedge \subset H^1(G_{k(c)}, T\mathcal{J}_c)$ corresponding to the section $s_c : G_{k(c)} \rightarrow \pi_1(\mathcal{X}_c, \xi_c)$ lies in the subset $\iota(\mathcal{X}(k(c))) \subset \mathcal{J}(k(c)) \subset \mathcal{J}(k(c))^\wedge$, since the section s_c is point-theoretic by (ii). We view $s^{\text{ab}} \in J(K)$ as a rational section of $\mathcal{J} \rightarrow C$; in fact $s^{\text{ab}} : C \rightarrow \mathcal{J}$ is a morphism since C is a smooth curve. For each closed point $c \in C^{\text{cl}}$ the image $s^{\text{ab}}(c)$ is a closed point of $\mathcal{X}_c \subset \mathcal{J}_c$, where we view \mathcal{X}_c as a closed subscheme of \mathcal{J}_c via the closed immersion $\mathcal{X}_c \xrightarrow{\iota_c} \mathcal{J}_c$ induced by ι . From this it follows that the morphism $s^{\text{ab}} : C \rightarrow \mathcal{J}$ factorizes as $s^{\text{ab}} : C \rightarrow \mathcal{X} \xrightarrow{\iota} \mathcal{J}$ and s^{ab} belongs to the subset $\iota(X(K)) \subseteq J(K)$. \square

Let $\tilde{x} \in X(K)$ be such that $\iota(\tilde{x}) = s^{\text{ab}}$ (cf. Lemma 4.1.2). For $c \in C^{\text{cl}}$ let \tilde{x}_c be the specialisation of \tilde{x} in \mathcal{X}_c .

Lemma 4.1.3. *The equality $\tilde{x}_c = x_c$ holds in $\mathcal{X}(k(c))$, for all $c \in C^{\text{cl}}$.*

Proof. First, $s_c^{\text{ab}} = s_{\tilde{x}_c}^{\text{ab}} = s_{x_c}^{\text{ab}}$ in $H^1(G_{k(c)}, T\mathcal{J}_c)$ (cf. Lemma 3.4). The equality $\tilde{x}_c = x_c$ then follows from the injectivity of the maps $\iota(\mathcal{X}_c) \hookrightarrow \mathcal{J}_c(k(c)) \hookrightarrow \mathcal{J}_c(k(c))^\wedge \rightarrow H^1(G_{k(c)}, T\mathcal{J}_c)$ for $c \in C^{\text{cl}}$ (see condition (iv)(a)). \square

Next, in order to show that the section s is point-theoretic, it suffices to show, by a well-known limit argument in anabelian geometry (cf. [T, Proposition 2.8(iv)]), using condition (v) of Definition 0.2, the following. Let $H \subseteq \pi_1(X, \xi)$ be an open subgroup such that $s(G_K) \subset H$, corresponding to an étale cover $Y \rightarrow X$. Then $Y(K) \neq \emptyset$.

There is a natural identification $H = \pi_1(Y, \xi)$. Moreover, the cover $Y \rightarrow X$ extends to an étale cover $\mathcal{Y} \rightarrow \mathcal{X}$ and \mathcal{Y} is a smooth model of Y over C (cf. Lemmas 3.1 and 3.2). For a closed point $c \in C^{\text{cl}}$ write $\mathcal{Y}_c := \mathcal{Y}_{k(c)}$. We will show that $Y(K) \neq \emptyset$.

Let $s' : G_K \rightarrow \pi_1(Y, \xi)$ be the section of $\pi_1(Y, \xi) = H$ induced by s , which extends to a section $s' : \pi_1(C, \xi) \rightarrow \pi_1(\mathcal{Y}, \xi)$ of the projection $\pi_1(\mathcal{Y}, \xi) \rightarrow \pi_1(C, \xi)$, and further induces a section $s'_c : G_{k(c)} \rightarrow \pi_1(\mathcal{Y}_c, \xi_c)$ of $\pi_1(\mathcal{Y}_c, \xi_c)$, for all $c \in C^{\text{cl}}$ (cf. loc. cit.). Note that s'_c is induced by s_c . The section s'_c is point-theoretic and arises from a unique rational point $y_c \in \mathcal{Y}_c(k(c))$. Moreover, x_c is the image of y_c in \mathcal{X}_c via the morphism $\mathcal{Y}_c \rightarrow \mathcal{X}_c$ (cf. condition (ii) and the fact that s_c is point-theoretic and arises from x_c , and s'_c is induced by s_c). View $\tilde{x} \in X(K) = \mathcal{X}(C)$ as a section $\tilde{x} : C \rightarrow \mathcal{X}$, and let $\mathcal{Y}_{\tilde{x}}$ be the scheme-theoretic inverse image of $\tilde{x}(C)$ in \mathcal{Y} via the above étale map $\mathcal{Y} \rightarrow \mathcal{X}$. Thus, $\mathcal{Y}_{\tilde{x}} \rightarrow \tilde{x}(C)$ is a finite étale map. We have $y_c \in \mathcal{Y}_{\tilde{x}}(k(c))$ for all $c \in C^{\text{cl}}$, as follows from the various definitions. Then $\mathcal{Y}_{\tilde{x}}(K) \neq \emptyset$ by condition (vi), and a fortiori $\mathcal{Y}_{\tilde{x}}(K) \subseteq \mathcal{Y}(K) = Y(K) \neq \emptyset$.

Thus, we have proved that $[s] = [s_x]$ in $\overline{\text{Sec}}_{\pi_1(X, \xi)}$ for a (unique) $x \in X(K)$. The following follows from Lemma 4.1.3 (cf. the above proof that φ_X is injective).

Lemma 4.1.4. *The equality $x = \tilde{x}$ holds.*

This finishes the proof of Theorem A. □

Finally, we show that Hilbertian fields satisfy condition (vi).

Lemma 4.1.5. *Let k be a Hilbertian field. Then k strongly satisfies condition (vi).*

Proof. Let k'/k be a finite extension and C a separated, smooth, connected curve over k' with function field $K := k'(C)$. Let $\tilde{C} \rightarrow C$ be a finite étale cover with $\tilde{C}_c(k'(c)) \neq \emptyset$ for all $c \in C^{\text{cl}}$ with residue field $k'(c)$. Note that k' is Hilbertian (cf. [Se, 9.5]). We show $\tilde{C}(K) \neq \emptyset$.

Assume that $\tilde{C}(K) = \emptyset$. Then for each connected component \tilde{C}_α of \tilde{C} the degree of the morphism $\tilde{C}_\alpha \rightarrow C$ is ≥ 2 . Hilbert’s irreducibility theorem (cf. [Se, 9.2]) implies that there exist points $c \in C^{\text{cl}}$ such that the fibre of c in each connected component of \tilde{C} is irreducible. This contradicts the assumption that $\tilde{C}_c(k(c)) \neq \emptyset$ for all $c \in C^{\text{cl}}$. Thus, $\tilde{C}(K) \neq \emptyset$. □

In the course of proving Theorem A we proved the following “adelic Mordell–Lang” statement.

Proposition 4.1.6. *With the above notation, assume that k only satisfies condition (iv) in Definition 0.2 (where we take $k' = k$) and $\mathcal{X}(C) \neq \emptyset$. Then the map $J(K)^\wedge \rightarrow \prod_{c \in C^{\text{cl}}} J_c(K_c)^\wedge \xrightarrow{\sim} \prod_{c \in C^{\text{cl}}} \mathcal{J}_c(k(c))^\wedge$ (cf. diagram (3.4) and*

Lemma 3.3) is injective. Further, inside $\prod_{c \in C^{\text{cl}}} \mathcal{J}_c(k(c))^\wedge$ the equality $J(K)^\wedge \cap \prod_{c \in C^{\text{cl}}} \iota(\mathcal{X}_c(k(c))) = \iota(X(K))$ holds.

Proof. See the statements and proofs of Lemmas 4.1.1 and 4.1.2. □

§4.2. Proof of Theorem B

We briefly explain how Theorem B can be derived from Theorem A. Let k be a field with $\text{char}(k) = 0$, and $K = k(C)$ the function field of a separated, smooth, connected curve C over k . Assume that k satisfies condition (\star) (cf. Definition 0.2). Let L/K be a finite extension. Then we prove the **SC** holds over L . We can, without loss of generality, assume that $L = K$. Let X be a proper, smooth, hyperbolic curve over K ; we need to prove that the **SC** holds for X , i.e., the map (cf. §0) $\varphi_X : X(K) \rightarrow \overline{\text{Sec}}_{\pi_1(X, \xi)}$ is bijective. The injectivity of φ_X follows as in the proof of Theorem A, where one only uses the fact that k strongly satisfies condition (ii) (cf. loc. cit.)

Next, we prove that φ_X is surjective. Let $s : G_K \rightarrow \pi_1(X, \xi)$ be a section of $\pi_1(X, \xi)$. We need to show that s is point-theoretic under the above assumptions. To this end we can, in the course of the proof, replace K by a finite extension L'/K . Indeed, let L'/K be a finite Galois extension, $s' : G_{L'} := \text{Gal}(\overline{K}/L') \rightarrow \pi_1(X_{L'}, \xi)$ the section of $\pi_1(X_{L'}, \xi)$ which is induced by s , and assume that $s' = s_{x'}$ is point-theoretic where $x' \in X(L')$. Then $s'(G_{L'})$ is self-normalising in $\pi_1(X_{L'}, \xi)$, and $s(G_K)$ is contained in the normaliser of $s'(G_{L'})$ in $\pi_1(X, \xi)$, which coincides with a decomposition group associated to the image x of x' in X (this follows from condition (\star) (iv)(a), cf. proof of Lemma 2.2.1 and the references therein). The point x is then necessarily K -rational. Now consider a finite (Galois) extension L'/K such that $X(L') \neq \emptyset$. Then the corresponding section $s' : G_{L'} := \text{Gal}(\overline{K}/L') \rightarrow \pi_1(X_{L'}, \xi)$ of $\pi_1(X_{L'}, \xi)$ which is induced by s is point-theoretic by Theorem A and our assumption that k satisfies condition (\star) . From this it follows that s is point theoretic (cf. above discussion). □

§5. Sections of the arithmetic fundamental groups of curves over finitely generated fields in characteristic 0

In this section we combine the techniques/results in §3 and §4 with the results in [ST] to prove variants of Theorems A and B, as well as Theorem C. We use the notation introduced in previous sections. All fields in this section are finitely generated over \mathbb{Q} . In order to deduce the validity of the **SC** over finitely generated fields from its validity over number fields (which for the time being is not known), one is reduced (by an induction argument) to proving the following.

Conjecture D. *Assume that the **SC** holds over all finitely generated fields of transcendence degree $i \geq 0$. Then the **SC** holds over all finitely generated fields of transcendence degree $i + 1$.*

Next, we recall the discrete Selmer conjecture formulated in [ST]. Let k be a finitely generated field over \mathbb{Q} , C a separated, smooth, geometrically connected curve over k , $K := k(C)$ the function field of C , and $\mathcal{A} \rightarrow C$ an abelian scheme over C . Write $A := \mathcal{A} \times_C \text{Spec } K$ for the generic fibre of \mathcal{A} . For a closed point $c \in C^{\text{cl}}$, let $k(c)$ be the residue field at c , and $\mathcal{A}_c := \mathcal{A}_{k(c)} = \mathcal{A} \times_C \text{Spec } k(c)$. Then we have a commutative diagram of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & A(K)^\wedge & \longrightarrow & H^1(G_K, TA) & \longrightarrow & TH^1(G_K, A) \longrightarrow 0 \\ & & \downarrow & & \Pi_c \text{res}_c \downarrow & & \Pi_c \text{res}_c \downarrow \\ 0 & \longrightarrow & \prod_c \mathcal{A}_c(k(c))^\wedge & \longrightarrow & \prod_c H^1(G_{k(c)}, T\mathcal{A}_c) & \longrightarrow & \prod_c TH^1(G_{k(c)}, \mathcal{A}_c) \longrightarrow 0 \end{array}$$

where the horizontal sequences are the Kummer exact sequences, the vertical maps are natural restriction homomorphisms, and the product is taken over all closed points $c \in C^{\text{cl}}$ (cf. loc. cit., diagram (0.1)). Define the *profinite Selmer group*

$$\text{Sel}(\mathcal{A}) := \text{Sel}(A, C) := \text{Ker}\left(H^1(G_K, TA) \rightarrow \prod_c TH^1(G_{k(c)}, \mathcal{A}_c)\right),$$

and the *Shafarevich–Tate group*

$$\text{III}(\mathcal{A}) := \text{III}(A, C) := \text{Ker}\left(H^1(G_K, A) \rightarrow \prod_c H^1(G_{k(c)}, \mathcal{A}_c)\right),$$

where the product is taken over all closed points $c \in C^{\text{cl}}$. For each $c \in C^{\text{cl}}$ the group $\mathcal{A}(k(c))$ of $k(c)$ -rational points of \mathcal{A} is finitely generated as $k(c)$ is finitely generated over \mathbb{Q} (Mordell–Weil Theorem), hence injects into its profinite completion $\mathcal{A}(k(c))^\wedge$. We identify $\mathcal{A}(k(c))$ with its image in $\mathcal{A}(k(c))^\wedge$. Define the *discrete Selmer group* by

$$\mathfrak{Sel}(\mathcal{A}) := \mathfrak{Sel}(A, C) := \text{Sel}(\mathcal{A}) \cap \prod_c \mathcal{A}(k(c)) \subset \prod_c H^1(G_{k(c)}, T\mathcal{A}_c),$$

where the product is taken over all closed points $c \in C^{\text{cl}}$. Note that $A(K) \subseteq \mathfrak{Sel}(\mathcal{A})$. In [ST, Conjecture E] we conjectured the following.

Conjecture E. *The equality $\mathfrak{Sel}(\mathcal{A}) = A(K)$ holds.*

Moreover, we proved the following facts.

Facts. (i) $\mathfrak{Sel}(\mathcal{A})$ is a finitely generated abelian group (cf. loc. cit., Proposition 2.5).

- (ii) Assume that there exists a prime integer ℓ such that the ℓ -primary part $\text{III}(\mathcal{A})[\ell^\infty] := \bigcup_{n \geq 1} \text{III}(\mathcal{A})[\ell^n]$ of $\text{III}(\mathcal{A}) := \text{III}(A, C)$ is finite. Then $\mathfrak{Sel}(\mathcal{A}) = A(K)$, i.e., Conjecture E holds in this case (cf. loc. cit., Proposition 3.7).
- (iii) Assume that A is isotrivial, i.e., $A_{\bar{K}}$ is defined over \bar{k} . Then $\text{III}(\mathcal{A})$ is finite (cf. loc. cit., Theorem 4.2). In particular, $\mathfrak{Sel}(\mathcal{A}) = A(K)$, i.e., Conjecture E holds in this case (cf. (ii)).

One of our main results in this paper is a (conditional) proof of Conjecture D. More precisely, we prove the following (cf. discussion after Corollary 5.3).

Theorem 5.1. *Assume that Conjecture E holds. Then Conjecture D holds true.*

In other words, Theorem 5.1 asserts that the validity of the above discrete Selmer conjecture implies that one can reduce the section conjecture **SC** over all finitely generated fields to the case of number fields. The following corollaries follow immediately from Theorem 5.1 and the above cited Facts (ii) and (iii), respectively. (For the proof of Corollary 5.3, use an induction argument.)

Corollary 5.2. *Assume that given a finitely generated field k over \mathbb{Q} , a separated, smooth, and connected curve C over k with generic point η , and an abelian scheme $\mathcal{A} \rightarrow C$ with generic fibre $A := \mathcal{A} \times_C \eta$, there exists a prime integer ℓ such that the ℓ -primary part $\text{III}(\mathcal{A})[\ell^\infty] := \bigcup_{n \geq 1} \text{III}(\mathcal{A})[\ell^n]$ of $\text{III}(\mathcal{A}) := \text{III}(A, C)$ is finite. Then Conjecture D holds true.*

Corollary 5.3. *Let K be a finitely generated field with algebraic closure \bar{K} , and \mathbb{Q} the algebraic closure of \mathbb{Q} in \bar{K} . Assume that the **SC** holds over all number fields. Then the **SC** holds for every projective smooth and geometrically connected hyperbolic curve X over K such that $X \times_{\text{Spec } K} \text{Spec } \bar{K}$ is defined over $\bar{\mathbb{Q}}$.*

The rest of this section is devoted to the proof of Theorem 5.1. To this end it suffices to prove the following (compare with Theorem A). (Note that the assumption $X(K) \neq \emptyset$ in Theorem 5.4 is not restrictive in order to deduce Theorem 5.1 from Theorem 5.4 (cf. arguments in the proof of Theorem B).)

Theorem 5.4. *Assume that the **SC** holds over all finitely generated fields of transcendence degree $i \geq 0$. Let k be a field with $\text{tr deg}_{\mathbb{Q}} k = i$, and $K = k(C)$ the function field of a separated, smooth, connected curve C over k . Let $\mathcal{X} \rightarrow C$ be a flat, proper, smooth relative curve with generic fibre $X := \mathcal{X} \times_C \text{Spec } K$ which is hyperbolic and geometrically connected, $J := \text{Pic}_{X/K}^0$, and $\mathcal{J} := \text{Pic}_{\mathcal{X}/C}^0$. Assume that $X(K) \neq \emptyset$, and either there exists a prime integer ℓ such that the ℓ -primary part $\text{III}(\mathcal{J})[\ell^\infty]$ of $\text{III}(\mathcal{J}) := \text{III}(J, C)$ is finite, or $\mathfrak{Sel}(\mathcal{J}) := \mathfrak{Sel}(J, C) = J(K)$. Then the **SC** holds for X (over K).*

Proof. The proof of the injectivity of the map $\varphi_X : X(K) \rightarrow \overline{\text{Sec}}_{\pi_1(X, \xi)}$ is similar to the proof of injectivity in Theorem A (cf. discussion before Lemma 4.1.1). In what follows we prove φ is surjective.

Let $s : G_K \rightarrow \pi_1(X, \xi)$ be a section of $\pi_1(X, \xi)$. We will show that s is point-theoretic under the above assumptions. We use the notation of §3. Recall that s induces a section $s_c : G_{k(c)} \rightarrow \pi_1(\mathcal{X}_c, \xi_c)$ of $\pi_1(\mathcal{X}_c, \xi_c)$ for all $c \in C^{\text{cl}}$ (cf. Lemma 3.2). From our assumption that the **SC** holds over all finitely generated fields of transcendence degree i we deduce that the section s_c is point-theoretic and arises from a unique rational point $x_c \in \mathcal{X}(k(c))$, for all $c \in C^{\text{cl}}$. The element $s_c^{\text{ab}} \in H^1(G_{k(c)}, T\mathcal{J}_c)$ is then point-theoretic, hence is geometric (cf. Lemma 1.2). Thus, $s_c^{\text{ab}} \in \mathcal{J}(k(c)) \subseteq \mathcal{J}(k(c))^\wedge \subseteq H^1(G_{k(c)}, T\mathcal{J}_c)$ for all $c \in C^{\text{cl}}$. In particular, $s^{\text{ab}} \in \mathfrak{Sel}(\mathcal{J}) := \mathfrak{Sel}(J, C)$ is in the discrete Selmer group. By our second assumption, that either there exists a prime integer ℓ such that the ℓ -primary part $\text{III}(\mathcal{J})[\ell^\infty]$ of $\text{III}(\mathcal{J}) := \text{III}(J, C)$ is finite or $\mathfrak{Sel}(\mathcal{J}) := \mathfrak{Sel}(J, C) = J(K)$ (note that the former condition implies the latter one by Fact (ii) above), we deduce that $s^{\text{ab}} \in J(K)$ is geometric. Moreover, a similar proof to that of Lemma 4.1.2 shows that s^{ab} lies in the subset $X(K)$ of $J(K)$ (cf. loc. cit.). The rest of the proof is similar to that of Theorem A (cf. Lemmas 4.1.3 and 4.1.4).

This finishes the proof of Theorem 5.4, and of Theorem 5.1. \square

Remark 5.5. The condition in Theorem 5.4 involving $\text{III}(\mathcal{J})[\ell^\infty]$ is weaker than the condition in Theorem A and Corollary A involving $\text{III}(\mathcal{J})$.

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