Publ. RIMS Kyoto Univ. **52** (2016), 497–555
DOI 10.4171/PRIMS/188

New Realization of Cyclotomic q-Schur Algebras

by

Kentaro WADA

Abstract

We introduce a Lie algebra $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$ and an associative algebra $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ associated with the Cartan data of $\mathfrak{gl}_m$ which is separated into r parts with respect to $\mathbf{m} = (m_1, \dots, m_r)$ such that $m_1 + \cdots + m_r = m$. We show that the Lie algebra $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$ is a filtered deformation of the current Lie algebra of $\mathfrak{gl}_m$, and we can regard the algebra $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ as a "q-analogue" of $U(\mathfrak{g}_{\mathbf{Q}}(\mathbf{m}))$. Then, we realize a cyclotomic q-Schur algebra as a quotient algebra of $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ under a certain mild condition. We also study the representation theory for $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$ and $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$, and we apply it to representations of the cyclotomic q-Schur algebras.

2010 Mathematics Subject Classification: Primary 20G43; Secondary 17B10, 17B37
Keywords: cyclotomic q-Schur algebras, Lie algebras, quantum groups.

Contents

Communicated by H. Nakajima. Received November 18, 2015. Revised June 16, 2016.

K. Wada: Department of Mathematics, Faculty of Science, Shinshu University,
Asahi 3-1-1, Matsumoto 390-8621, Japan;
e-mail: wada@math.shinshu-u.ac.jp

§0. Introduction

0.1. Let $\mathscr{H}_{n,r}$ be the Ariki–Koike algebra associated with the complex reflection group of type $G(r,1,n)$ over a commutative ring R with parameters $q, Q_0, \dots, Q_{r-1} \in R$, where q is invertible in R. Let $\mathscr{S}_{n,r}(\mathbf{m})$ be the cyclotomic q-Schur algebra associated with $\mathscr{H}_{n,r}$ introduced in [DJM], where $\mathbf{m} = (m_1, \dots, m_r)$ is an r-tuple of positive integers. By [DJM], it is known that $\mathscr{S}_{n,r}(\mathbf{m})$-mod is a highest weight cover of $\mathscr{H}_{n,r}$-mod in the sense of [R] if R is a field and $\mathbf{m}$ is large enough.

In [RSVV] and independently in [L], it is proven that $\mathscr{S}_{n,r}(\mathbf{m})$-mod is equivalent to a certain highest weight subcategory of an affine parabolic category $\mathbf{O}$ in a dominant case of an affine general linear Lie algebra as a highest weight cover of $\mathscr{H}_{n,r}$-mod. It is also equivalent to the category $\mathcal{O}$ of a rational Cherednik algebra with the corresponding parameters. In the argument of [RSVV], the monoidal structure on the affine parabolic category $\mathbf{O}$ (more precisely, the structure of $\mathbf{O}$ as a bimodule category over the Kazhdan–Lusztig category) has an important role.

For $r = 1$, it is known that the q-Schur algebra $\mathscr{S}_{n,1}(m)$ is a quotient algebra of the quantum group $U_q(\mathfrak{gl}_m)$ associated with the general linear Lie algebra $\mathfrak{gl}_m$, and $\bigoplus_{n \geq 0} \mathscr{S}_{n,1}(m)$-mod is equivalent to the category $\mathcal{C}^{\geq 0}_{U_q(\mathfrak{gl}_m)}$ consisting of finite-dimensional polynomial representations of $U_q(\mathfrak{gl}_m)$ ([BLM], [D] and [J]). The category $\mathcal{C}^{\geq 0}_{U_q(\mathfrak{gl}_m)}$ has a (braided) monoidal structure which comes from the structure of $U_q(\mathfrak{gl}_m)$ as a Hopf algebra. The monoidal structure on $\mathcal{C}^{\geq 0}_{U_q(\mathfrak{gl}_m)}$ is compatible with the monoidal structure on the Kazhdan–Lusztig category by [KL]. However, for cyclotomic q-Schur algebras and $r > 1$ such structures are not known, although we may expect they exist through the equivalence in [RSVV]. This is a motivation of this paper.

In [W1], we obtained a presentation of cyclotomic q-Schur algebras by generators and defining relations. The argument in [W1] is based on the existence of the upper (resp. lower) Borel subalgebra of the cyclotomic q-Schur algebra $\mathscr{S}_{n,r}(\mathbf{m})$ which is introduced in [DR]. In [DR], it is proven that the upper (resp. lower) Borel subalgebra of $\mathscr{S}_{n,r}(\mathbf{m})$ is isomorphic to the upper (resp. lower) Borel subalgebra of $\mathscr{S}_{n,1}(m)$ (i.e. the case where $r = 1$) which is a quotient of the upper (resp. lower) Borel subalgebra of the quantum group $U_q(\mathfrak{gl}_m)$ ($m := \sum_{k=1}^{r} m_k$) if $\mathbf{m}$ is large enough. The presentation of $\mathscr{S}_{n,r}(\mathbf{m})$ in [W1] is applied to the representation theory of cyclotomic q-Schur algebras in [W2] and [W3]. However, this presentation is not so useful in general since, in the presentation, we need some non-commutative polynomials which are computable, but we cannot describe them explicitly (see [W1, Lemma 7.2]). Hence, we can hope there is a more useful realization of cyclotomic q-Schur algebras, like the fact that the q-Schur algebra $\mathscr{S}_{n,1}(m)$ is a quotient of the quantum group $U_q(\mathfrak{gl}_m)$ in the case where $r=1$. In this paper, by extending the argument in [W1], we give such a realization.

0.2. Let $\mathbf{Q}=(Q_1,\dots,Q_{r-1})$ be an $r-1$-tuple of indeterminates over $\mathbb{Z}$, and $\mathbb{Q}(\mathbf{Q})$ be the field of rational functions in variables $\mathbf{Q}$. In §2, we introduce a Lie algebra $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$ with parameters $\mathbf{Q}$ associated with the Cartan data of $\mathfrak{gl}_m$ ($m=\sum_{k=1}^{r} m_k$) which is separated into r parts with respect to $\mathbf{m}$ (see paragraph 1.3). Then, in Proposition 2.13, we prove that $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$ is a filtered deformation of the current Lie algebra $\mathfrak{gl}_m[x]=\mathbb{Q}(\mathbf{Q})[x]\otimes\mathfrak{gl}_m$ of the general linear Lie algebra $\mathfrak{gl}_m$.

In Corollary 2.8, we see that $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$ has a triangular decomposition

$$\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})=\mathfrak{n}^-\oplus\mathfrak{n}^0\oplus\mathfrak{n}^+.$$

Thus we can develop a weight theory to study representations of $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$ in the usual manner (see §3). Let $\mathcal{C}_{\mathbf{Q}}(\mathbf{m})$ be the category of finite-dimensional $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$-modules which have weight space decompositions, and all eigenvalues of the action of $\mathfrak{n}^0$ belong to $\mathbb{Q}(\mathbf{Q})$. Then we see that a simple $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$-module in $\mathcal{C}_{\mathbf{Q}}(\mathbf{m})$ is a highest weight module.

There exists a surjective homomorphism of Lie algebras $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})\to\mathfrak{gl}_m$ (see (2.16.1)) which can be regarded as a special case of evaluation homomorphisms (see Remark 2.17). Let $\mathcal{C}_{\mathfrak{gl}_m}$ be the category of finite-dimensional $\mathfrak{gl}_m$-modules which have weight space decompositions. Then $\mathcal{C}_{\mathfrak{gl}_m}$ is a full subcategory of $\mathcal{C}_{\mathbf{Q}}(\mathbf{m})$ through the above surjection (see Proposition 3.7).

Let $\widetilde{\mathbf{Q}}=(Q_0,Q_1,\dots,Q_{r-1})$ be an r-tuple of indeterminates over $\mathbb{Z}$, and $\mathbb{Q}(\widetilde{\mathbf{Q}})$ be the field of rational functions in variables $\widetilde{\mathbf{Q}}$. Set $\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m})=\mathbb{Q}(\widetilde{\mathbf{Q}})\otimes_{\mathbb{Q}(\mathbf{Q})}\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$, and define the category $\mathcal{C}_{\widetilde{\mathbf{Q}}}(\mathbf{m})$ in a similar way. Let $\mathscr{S}^{\mathbf{1}}_{n,r}(\mathbf{m})$ be the cyclotomic q-Schur algebra over $\mathbb{Q}(\widetilde{\mathbf{Q}})$ with parameters $q=1$ and $\widetilde{\mathbf{Q}}$. In Theorem 8.4, we prove that there exists a homomorphism of algebras

$$\Psi_{\mathbf{1}}:U(\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m}))\to\mathscr{S}^{\mathbf{1}}_{n,r}(\mathbf{m}),$$

where $U(\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m}))$ is the universal enveloping algebra of $\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m})$. Assume that $m_k\geq n$ for all $k=1,\dots,r-1$. Then $\Psi_{\mathbf{1}}$ is surjective, and $\mathscr{S}^{\mathbf{1}}_{n,r}(\mathbf{m})$-mod is a full subcategory of $\mathcal{C}_{\widetilde{\mathbf{Q}}}(\mathbf{m})$ through the surjection $\Psi_{\mathbf{1}}$ (see Theorem 8.4(ii)). We expect that the surjectivity of $\Psi_{\mathbf{1}}$ also holds without the condition on $\mathbf{m}$. (We need this condition for a technical reason—see Remark 8.2.)

It is known that $\mathscr{S}^{\mathbf{1}}_{n,r}(\mathbf{m})$ is semisimple, and the set $\{\Delta(\lambda)\mid\lambda\in\widetilde{\Lambda}^+_{n,r}(\mathbf{m})\}$ of Weyl (cell) modules gives a complete set of (representatives of) isomorphism classes of simple $\mathscr{S}^{\mathbf{1}}_{n,r}(\mathbf{m})$-modules (see §6 and [DJM] for definitions). The characters of the Weyl modules, denoted by $\operatorname{ch}\Delta(\lambda)$ ($\lambda\in\widetilde{\Lambda}^+_{n,r}(\mathbf{m})$), are studied in [W2]. We see that $\mathrm{ch}\,\Delta(\lambda)$ ($\lambda \in \widetilde{\Lambda}^+_{n,r}(\mathbf{m})$) is a symmetric polynomial in variables $\mathbf{x_m}$. Set $\widetilde{\Lambda}^+_{\geq 0,r}(\mathbf{m}) = \bigcup_{n\geq 0} \widetilde{\Lambda}^+_{n,r}(\mathbf{m})$. Then, for $\lambda, \mu \in \widetilde{\Lambda}^+_{\geq 0,r}(\mathbf{m})$, it was conjectured in [W2] that

$$\mathrm{ch}\,\Delta(\lambda)\,\mathrm{ch}\,\Delta(\mu) = \sum_{\nu\in\widetilde{\Lambda}^+_{\geq 0,r}(\mathbf{m})} \mathrm{LR}^{\nu}_{\lambda\mu}\,\mathrm{ch}\,\Delta(\nu), \tag{0.2.1}$$

where $\mathrm{LR}^{\nu}_{\lambda\mu}$ is the product of the Littlewood–Richardson coefficients with respect to λ, μ and ν (see §9 for details). We prove this conjecture in Proposition 9.4. We remark that the characters of Weyl modules of a cyclotomic q-Schur algebra do not depend on the choice of the base field and parameters.

By using the usual coproduct of the universal enveloping algebra $U(\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m}))$ of $\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m})$, we can consider the tensor product $M \otimes N$ in $U(\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m}))$-mod for $M, N \in U(\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m}))$-mod. We regard $\mathscr{S}^{\mathbf{1}}_{n,r}(\mathbf{m})$-modules ($n \geq 0$) as $U(\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m}))$-modules through the homomorphism $\Psi_{\mathbf{1}}$. Take $n, n_1, n_2 \in \mathbb{Z}_{>0}$ with $n = n_1 + n_2$. Then, in Proposition 10.1, we prove that, for $\lambda \in \widetilde{\Lambda}^+_{n_1,r}(\mathbf{m})$ and $\mu \in \widetilde{\Lambda}^+_{n_2,r}(\mathbf{m})$,

$$\Delta(\lambda) \otimes \Delta(\mu) \cong \bigoplus_{\nu\in\widetilde{\Lambda}^+_{n,r}(\mathbf{m})} \mathrm{LR}^{\nu}_{\lambda\mu}\,\Delta(\nu) \tag{0.2.2}$$

as $U(\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m}))$-modules if $m_k \geq n$ for all $k = 1, \dots, r-1$, where $\mathrm{LR}^{\nu}_{\lambda\mu}\,\Delta(\nu)$ means the direct sum of $\mathrm{LR}^{\nu}_{\lambda\mu}$ copies of $\Delta(\nu)$. In particular, $\Delta(\lambda)\otimes\Delta(\nu) \in \mathscr{S}_{n,r}(\mathbf{m})$-mod. The decomposition (0.2.2) gives an interpretation of formula (0.2.1) in the category $\mathcal{C}_{\widetilde{\mathbf{Q}}}(\mathbf{m})$. We expect that (0.2.2) also holds without the condition on $\mathbf{m}$. (Note that we prove (0.2.1) without the condition on $\mathbf{m}$ in Proposition 9.4.)

0.3. Set $\mathbb{A} = \mathbb{Z}[q, q^{-1}, Q_1, \dots, Q_{r-1}]$, where $q, Q_1, \dots, Q_{r-1}$ are indeterminates over $\mathbb{Z}$, and let $\mathbb{K} = \mathbb{Q}(q, Q_1, \dots, Q_{r-1})$ be the quotient field of $\mathbb{A}$. In §4, we introduce an associative algebra $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ with parameters q and $\mathbf{Q}$ associated with the Cartan data of $\mathfrak{gl}_m$ which is separated into r parts with respect to $\mathbf{m}$.

Let $\mathcal{U}^{\star}_{\mathbb{A},q,\mathbf{Q}}(\mathbf{m})$ be the $\mathbb{A}$-subalgebra of $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ generated by the defining generators of $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ (see paragraph 4.11). We regard $\mathbb{Q}(\mathbf{Q})$ as an $\mathbb{A}$-module through the ring homomorphism $\mathbb{A} \to \mathbb{Q}(\mathbf{Q})$ sending q to 1, and we consider the specialization $\mathbb{Q}(\mathbf{Q}) \otimes_{\mathbb{A}} \mathcal{U}^{\star}_{\mathbb{Q},q,\mathbf{Q}}(\mathbf{m})$ using this ring homomorphism. Then we have a surjective homomorphism of algebras

$$U(\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})) \to \mathbb{Q}(\mathbf{Q}) \otimes_{\mathbb{A}} \mathcal{U}^{\star}_{\mathbb{A},q,\mathbf{Q}}(\mathbf{m})/\mathfrak{J}, \tag{0.3.1}$$

where $\mathfrak{J}$ is a certain ideal of $\mathbb{Q}(\mathbf{Q}) \otimes_{\mathbb{A}} \mathcal{U}^{\star}_{\mathbb{A},q,\mathbf{Q}}(\mathbf{m})$ (see (4.11.2)). We conjecture that (0.3.1) is an isomorphism. Then we can regard $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ as a "q-analogue" of $U(\mathfrak{g}_{\mathbf{Q}}(\mathbf{m}))$. Dividing by the ideal $\mathfrak{J}$ in (0.3.1) means that the Cartan subalgebra

$U(\mathfrak{n}^0)$ of $U(\mathfrak{g}_{\mathbf{Q}}(\mathbf{m}))$ deforms to several directions in $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ (see paragraph 4.11 and Remark 4.12).

We find that $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ has a triangular decomposition

$$\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m}) = \mathcal{U}^{-}\mathcal{U}^{0}\mathcal{U}^{+} \tag{0.3.2}$$

in a weak sense (see (4.6.1)). We conjecture that the multiplication map $\mathcal{U}^{-} \otimes_{\mathbb{K}} \mathcal{U}^{0} \otimes_{\mathbb{K}} \mathcal{U}^{+} \to \mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ gives an isomorphism of vector spaces. More precisely, we expect the existence of a PBW type basis of $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ which is compatible with a PBW basis of $U(\mathfrak{g}_{\mathbf{Q}}(\mathbf{m}))$ through the homomorphism (0.3.1).

Anyway, thanks to the triangular decomposition (0.3.2), we can develop weight theory to study $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$-modules in the usual manner (see §5). Let $\mathcal{C}_{q,\mathbf{Q}}(\mathbf{m})$ be the category of finite-dimensional $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$-modules which have weight space decompositions, and all eigenvalues of the action of $\mathcal{U}^0$ belong to $\mathbb{K}$. Then a simple $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$-module in $\mathcal{C}_{q,\mathbf{Q}}(\mathbf{m})$ is a highest weight module.

There exists a surjective homomorphism of algebras $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m}) \to U_q(\mathfrak{gl}_m)$ (see (4.9.1)) which can be regarded as a special case of evaluation homomorphisms (see Remark 4.10). Let $\mathcal{C}_{U_q(\mathfrak{gl}_m)}$ be the category of finite-dimensional $U_q(\mathfrak{gl}_m)$-modules which have weight space decompositions. Then $\mathcal{C}_{U_q(\mathfrak{gl}_m)}$ is a full subcategory of $\mathcal{C}_{q,\mathbf{Q}}(\mathbf{m})$ through the above surjection (see Proposition 5.6).

Set $\widetilde{\mathbb{K}} = \mathbb{K}(Q_0)$, $\widetilde{\mathbb{A}} = \mathbb{A}[Q_0]$, and $\mathcal{U}_{q,\widetilde{\mathbf{Q}}}(\mathbf{m}) = \widetilde{\mathbb{K}} \otimes_{\mathbb{K}} \mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$. Let $\mathcal{U}_{\mathbb{A},q,\mathbf{Q}}(\mathbf{m})$ be the $\mathbb{A}$-form of $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ involving divided powers (see paragraph 4.13), and set $\mathcal{U}_{\widetilde{\mathbb{A}},q,\widetilde{\mathbf{Q}}}(\mathbf{m}) = \widetilde{\mathbb{A}} \otimes_{\mathbb{A}} \mathcal{U}_{\mathbb{A},q,\mathbf{Q}}(\mathbf{m})$. Let $\mathscr{S}_{n,r}^{\widetilde{\mathbb{K}}}(\mathbf{m})$ (resp. $\mathscr{S}_{n,r}^{\widetilde{\mathbb{A}}}(\mathbf{m})$) be the cyclotomic q-Schur algebra over $\widetilde{\mathbb{K}}$ (resp. over $\widetilde{\mathbb{A}}$) with parameters q and $\widetilde{\mathbf{Q}}$. In Theorem 8.1, we prove that there exists a homomorphism of algebras

$$\Psi : \mathcal{U}_{q,\widetilde{\mathbf{Q}}}(\mathbf{m}) \to \mathscr{S}_{n,r}^{\widetilde{\mathbb{K}}}(\mathbf{m}).$$

By restricting Ψ to $\mathcal{U}_{\widetilde{\mathbb{A}},q,\widetilde{\mathbf{Q}}}(\mathbf{m})$, we obtain a homomorphism $\Psi_{\widetilde{\mathbb{A}}} : \mathcal{U}_{\widetilde{\mathbb{A}},q,\widetilde{\mathbf{Q}}}(\mathbf{m}) \to \mathscr{S}_{n,r}^{\widetilde{\mathbb{A}}}(\mathbf{m})$. Then we can specialize $\Psi_{\widetilde{\mathbb{A}}}$ to any base ring and parameters. If $m_k \geq n$ for all $k = 1, \dots, r-1$, then Ψ (resp. $\Psi_{\widetilde{\mathbb{A}}}$) is surjective (see also Remark 8.2 for surjectivity of Ψ). In Theorem 8.3, we prove that $\mathscr{S}_{n,r}^{\widetilde{\mathbb{K}}}(\mathbf{m})$-mod is a full subcategory of $\mathcal{C}_{q,\widetilde{\mathbf{Q}}}(\mathbf{m})$ through the surjection Ψ if $\mathbf{m}$ is large enough.

We conjecture that $\mathcal{U}_{q,\widetilde{\mathbf{Q}}}(\mathbf{m})$ has the structure of a Hopf algebra, and that the decomposition (0.2.2) also holds for Weyl modules of $\mathscr{S}_{n,r}^{\widetilde{\mathbb{K}}}(\mathbf{m})$ $(n \geq 0)$ through the homomorphism Ψ and the Hopf algebra structure of $\mathcal{U}_{q,\widetilde{\mathbf{Q}}}(\mathbf{m})$. (Note that formula (0.2.1) holds for $\mathscr{S}_{n,r}^{\widetilde{\mathbb{K}}}(\mathbf{m})$ with $n \geq 0$.)

It is also an interesting problem to obtain a monoidal structure for $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ (resp. $\mathcal{U}_{\mathbb{A},q,\mathbf{Q}}(\mathbf{m})$ and its specialization) which should be related to the monoidal structure on the affine parabolic category $\mathbf{O}$.

§1. Notation

1.1. For a condition X, set

$$\delta_{(X)} = \begin{cases} 1 & \text{if } X \text{ is true,} \\ 0 & \text{if } X \text{ is false.} \end{cases}$$

We also set $\delta_{ij} = \delta_{(i=j)}$ for simplicity.

1.2. ***q*-integers.** Let $\mathbb{Q}(q)$ be the field of rational functions over $\mathbb{Q}$ in variable q. For $d \in \mathbb{Z}$, set $[d] = (q^d - q^{-d})/(q - q^{-1}) \in \mathbb{Q}(q)$. For $d \in \mathbb{Z}_{>0}$, set $[d]! = [d][d-1]\dots[1]$, and $[0]! = 1$. For $d \in \mathbb{Z}$ and $c \in \mathbb{Z}_{>0}$, set

$$\begin{bmatrix} d \\ c \end{bmatrix} = \frac{[d][d-1]\dots[d-c+1]}{[c][c-1]\dots[1]}, \qquad \begin{bmatrix} d \\ 0 \end{bmatrix} = 1.$$

It is well-known that all $[d]$, $[d]!$ and $\left[\begin{smallmatrix} d \\ c \end{smallmatrix}\right]$ belong to $\mathbb{Z}[q, q^{-1}]$. Thus we can specialize these elements to any ring R and $q \in R$ such that q is invertible in R, and we denote them by the same symbols.

1.3. Cartan data. Let $\mathbf{m} = (m_1, \dots, m_r)$ be an r-tuple of positive integers. Set $m = \sum_{k=1}^{r} m_k$. Let $P = \bigoplus_{i=1}^{m} \mathbb{Z}\varepsilon_i$ be the weight lattice of $\mathfrak{gl}_m$, and let $P^\vee = \bigoplus_{i=1}^{m} \mathbb{Z}h_i$ be its dual with the natural pairing $\langle\, , \rangle : P \times P^\vee \to \mathbb{Z}$ such that $\langle \varepsilon_i, h_j \rangle = \delta_{ij}$. Write $P_{\geq 0} = \bigoplus_{i=1}^{m} \mathbb{Z}_{\geq 0}\varepsilon_i$.

Set $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i = 1, \dots, m-1$. Then $\Pi = \{\alpha_i \mid 1 \leq i \leq m-1\}$ is the set of *simple roots*, and $Q = \bigoplus_{i=1}^{m-1} \mathbb{Z}\alpha_i$ is the root lattice of $\mathfrak{gl}_m$. Write $Q^+ = \bigoplus_{i=1}^{m-1} \mathbb{Z}_{\geq 0}\alpha_i$.

Set $\alpha_i^\vee = h_i - h_{i+1}$ for $i = 1, \dots, m-1$. Then $\Pi^\vee = \{\alpha_i^\vee \mid 1 \leq i \leq m-1\}$ is the set of *simple coroots*.

We define a partial order $\geq$ on P, called the *dominance order*, by $\lambda \geq \mu$ if $\lambda - \mu \in Q^+$.

Set $\Gamma(\mathbf{m}) = \{(i,k) \mid 1 \leq i \leq m_k,\ 1 \leq k \leq r\}$ and $\Gamma'(\mathbf{m}) = \Gamma(\mathbf{m}) \setminus \{(m_r, r)\}$. We identify $\Gamma(\mathbf{m})$ with $\{1, \dots, m\}$ by the bijection

$$\gamma : \Gamma(\mathbf{m}) \to \{1, \dots, m\}, \qquad (i,k) \mapsto \sum_{j=1}^{k-1} m_j + i. \tag{1.3.1}$$

Thus, $\Gamma'(\mathbf{m})$ gets identified with $\{1, \dots, m-1\}$. Under the identification (1.3.1), for $(i,k), (j,l) \in \Gamma(\mathbf{m})$, we define

$$(i,k) > (j,l) \quad \text{if } \gamma((i,k)) > \gamma((j,l)), \qquad (i,k) \pm (j,l) = \gamma((i,k)) \pm \gamma((j,l)).$$

We also have $(m_k + 1, k) = (1, k+1)$ for $k = 1, \dots, r-1$ (resp. $(1-1, k) = (m_{k-1}, k-1)$ for $k = 2, \dots, r$).

We may write

$$P = \bigoplus_{(i,k)\in\Gamma(\mathbf{m})} \mathbb{Z}\varepsilon_{(i,k)}, \quad P^\vee = \bigoplus_{(i,k)\in\Gamma(\mathbf{m})} \mathbb{Z}h_{(i,k)}, \quad Q = \bigoplus_{(i,k)\in\Gamma'(\mathbf{m})} \mathbb{Z}\alpha_{(i,k)}.$$

For $(i,k)\in\Gamma'(\mathbf{m})$, $(j,l)\in\Gamma(\mathbf{m})$, define $a_{(i,k)(j,l)} = \langle \alpha_{(i,k)}, h_{(j,l)}\rangle$. Then

$$a_{(i,k)(j,l)} = \begin{cases} 1 & \text{if } (j,l)=(i,k), \\ -1 & \text{if } (j,l)=(i+1,k), \\ 0 & \text{otherwise.} \end{cases}$$

Set

$$P^+ = \{\lambda\in P \mid \langle\lambda, \alpha^\vee_{(i,k)}\rangle \in \mathbb{Z}_{\geq 0} \text{ for all } (i,k)\in\Gamma'(\mathbf{m})\},$$
$$P^+_{\mathbf{m}} = \{\lambda\in P \mid \langle\lambda, \alpha^\vee_{(i,k)}\rangle \in \mathbb{Z}_{\geq 0} \text{ for all } (i,k)\in\Gamma(\mathbf{m}) \setminus \{(m_k,k) \mid 1\leq k\leq r\}\}.$$

Then P^+ is the set of dominant integral weights for $\mathfrak{gl}_m$, and $P^+_{\mathbf{m}}$ is the set of dominant integral weights for the Levi subalgebra $\mathfrak{gl}_{m_1}\oplus\cdots\oplus\mathfrak{gl}_{m_r}$ of $\mathfrak{gl}_m$ with respect to $\mathbf{m}=(m_1,\dots,m_r)$.

§2. The Lie algebra $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$

In this section, we introduce a Lie algebra $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$ with $r-1$ parameters $\mathbf{Q} = (Q_1,\dots,Q_{r-1})$ associated with the Cartan data of paragraph 1.3. Then we study some basic structures of $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$. In particular, we prove that $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$ is a filtered deformation of the current Lie algebra $\mathfrak{gl}_m[x]$ of the general linear Lie algebra $\mathfrak{gl}_m$.

2.1. Let $\mathbf{Q} = (Q_1,\dots,Q_{r-1})$ be an $r-1$-tuple of indeterminates over $\mathbb{Z}$. Let $\mathbb{Z}[\mathbf{Q}] = \mathbb{Z}[Q_1,\dots,Q_{r-1}]$ be the polynomial ring in variables $Q_1,\dots,Q_{r-1}$, and $\mathbb{Q}(\mathbf{Q}) = \mathbb{Q}(Q_1,\dots,Q_{r-1})$ be the quotient field of $\mathbb{Z}[\mathbf{Q}]$.

Definition 2.2. We define the Lie algebra $\mathfrak{g}=\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$ over $\mathbb{Q}(\mathbf{Q})$ by the following generators and relations:

Generators: $\mathcal{X}^\pm_{(i,k),t}$, $\mathcal{I}_{(j,l),t}$ $((i,k)\in\Gamma'(\mathbf{m}),\ (j,l)\in\Gamma(\mathbf{m}),\ t\geq 0)$.

Relations:

(L1) $[\mathcal{I}_{(i,k),s}, \mathcal{I}_{(j,l),t}] = 0,$

(L2) $[\mathcal{I}_{(j,l),s}, \mathcal{X}^\pm_{(i,k),t}] = \pm a_{(i,k)(j,l)} \mathcal{X}^\pm_{(i,k),s+t},$

(L3) $[\mathcal{X}^+_{(i,k),t}, \mathcal{X}^-_{(j,l),s}] = \delta_{(i,k),(j,l)} \begin{cases} \mathcal{J}_{(i,k),s+t} & \text{if } i\neq m_k, \\ -Q_k\mathcal{J}_{(m_k,k),s+t} + \mathcal{J}_{(m_k,k),s+t+1} & \text{if } i=m_k, \end{cases}$

(L4) $[\mathcal{X}^{\pm}_{(i,k),t}, \mathcal{X}^{\pm}_{(j,l),s}] = 0$ if $(j,l) \neq (i \pm 1, k)$,

(L5) $$[\mathcal{X}^{+}_{(i,k),t+1}, \mathcal{X}^{+}_{(i\pm 1,k),s}] = [\mathcal{X}^{+}_{(i,k),t}, \mathcal{X}^{+}_{(i\pm 1,k),s+1}],$$ $$[\mathcal{X}^{-}_{(i,k),t+1}, \mathcal{X}^{-}_{(i\pm 1,k),s}] = [\mathcal{X}^{-}_{(i,k),t}, \mathcal{X}^{-}_{(i\pm 1,k),s+1}],$$

(L6) $[\mathcal{X}^{+}_{(i,k),s}, [\mathcal{X}^{+}_{(i,k),t}, \mathcal{X}^{+}_{(i\pm 1,k),u}]] = [\mathcal{X}^{-}_{(i,k),s}, [\mathcal{X}^{-}_{(i,k),t}, \mathcal{X}^{-}_{(i\pm 1,k),u}]] = 0$,

where we have set $\mathcal{J}_{(i,k),t} = \mathcal{I}_{(i,k),t} - \mathcal{I}_{(i+1,k),t}$.

2.3. For $\tau \in \mathbb{Q}(\mathbf{Q})$, let $V_\tau = \bigoplus_{(j,l)\in\Gamma(\mathbf{m})} \mathbb{Q}(\mathbf{Q}) v_{(j,l)}$ be the $\mathbb{Q}(\mathbf{Q})$-vector space with a basis $\{v_{(j,l)} \mid (j,l) \in \Gamma(\mathbf{m})\}$. We can define an action of $\mathfrak{g}$ on V_τ by

$$\mathcal{X}^{+}_{(i,k),t} \cdot v_{(j,l)} = \begin{cases} \tau^t v_{(i,k)} & \text{if } (j,l) = (i+1,k) \text{ and } i \neq m_k, \\ (-Q_k + \tau)\tau^t v_{(m_k,k)} & \text{if } (j,l) = (1,k+1) \text{ and } i = m_k, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{X}^{-}_{(i,k),t} \cdot v_{(j,l)} = \begin{cases} \tau^t v_{(i+1,k)} & \text{if } (j,l) = (i,k), \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{I}_{(i,k),t} \cdot v_{(j,l)} = \begin{cases} \tau^t v_{(j,l)} & \text{if } (j,l) = (i,k), \\ 0 & \text{otherwise.} \end{cases}$$

We can check that the action is well-defined by direct calculations.

2.4. For $(i,k),(j,l) \in \Gamma(\mathbf{m})$ and $t \geq 0$, we define an element $\mathcal{E}^t_{(i,k)(j,l)} \in \mathfrak{g}$ by

$$\mathcal{E}^t_{(i,k),(j,l)} = \begin{cases} \mathcal{I}_{(i,k),t} & \text{if } (j,l) = (i,k), \\ [\mathcal{X}^{+}_{(i,k),0}, [\mathcal{X}^{+}_{(i+1,k),0}, \dots, [\mathcal{X}^{+}_{(j-2,l),0}, \mathcal{X}^{+}_{(j-1,l),t}] \dots]] & \text{if } (j,l) > (i,k), \\ [\mathcal{X}^{-}_{(i-1,k),0}, [\mathcal{X}^{-}_{(i-2,k),0}, \dots, [\mathcal{X}^{-}_{(j+1,l),0}, \mathcal{X}^{-}_{(j,l),t}] \dots]] & \text{if } (j,l) < (i,k); \end{cases}$$

in particular, $\mathcal{E}^t_{(i,k),(i+1,k)} = \mathcal{X}^{+}_{(i,k),t}$ and $\mathcal{E}^t_{(i+1,k),(i,k)} = \mathcal{X}^{-}_{(i,k),t}$.

If $(j,l) > (i,k)$, we have

$$\mathcal{E}^t_{(i,k),(j,l)} = [\mathcal{X}^{+}_{(i,k),0}, \mathcal{E}^t_{(i+1,k),(j,l)}] = [\mathcal{E}^t_{(i,k),(j-1,l)}, \mathcal{X}^{+}_{(j-1,l),0}].$$

If $(j,l) < (i,k)$, we have

$$\mathcal{E}^t_{(i,k),(j,l)} = [\mathcal{X}^{-}_{(i-1,k),0}, \mathcal{E}^t_{(i-1,k),(j,l)}] = [\mathcal{E}^t_{(i,k),(j+1,l)}, \mathcal{X}^{-}_{(j,l),0}].$$

Lemma 2.5. (i) *For $(i,k),(j,l) \in \Gamma(\mathbf{m})$ such that $(j,l) > (i,k)$, we have*

$$[\mathcal{X}^{+}_{(a,c),s}, \mathcal{E}^t_{(i,k),(j,l)}] = \begin{cases} \mathcal{E}^{t+s}_{(i-1,k),(j,l)} & \textit{if } (a,c) = (i-1,k), \\ -\mathcal{E}^{t+s}_{(i,k),(j+1,l)} & \textit{if } (a,c) = (j,l), \\ 0 & \textit{otherwise,} \end{cases} \tag{2.5.1}$$

$$(2.5.2)\qquad [\mathcal{I}_{(a,c),s}, \mathcal{E}^t_{(i,k),(j,l)}] = \begin{cases} \mathcal{E}^{t+s}_{(i,k),(j,l)} & \text{if } (a,c)=(i,k), \\ -\mathcal{E}^{t+s}_{(i,k),(j,l)} & \text{if } (a,c)=(j,l), \\ 0 & \text{otherwise}, \end{cases}$$

$$(2.5.3)\qquad [\mathcal{X}^-_{(a,c),s}, \mathcal{E}^t_{(i,k),(j,l)}]$$

$$= \begin{cases} -\mathcal{E}^{t+s}_{(i,k),(i,k)} + \mathcal{E}^{t+s}_{(i+1,k),(i+1,k)} & \text{if } \ell=1, (a,c)=(i,k) \text{ and } i\neq m_k, \\ Q_k(\mathcal{E}^{t+s}_{(m_k,k),(m_k,k)} - \mathcal{E}^{t+s}_{(1,k+1),(1,k+1)}) - \mathcal{E}^{t+s+1}_{(m_k,k),(m_k,k)} + \mathcal{E}^{t+s+1}_{(1,k+1),(1,k+1)} & \\ & \text{if } \ell=1, (a,c)=(i,k) \text{ and } i=m_k, \\ \mathcal{E}^{t+s}_{(i+1,k),(j,l)} & \text{if } \ell>1, (a,c)=(i,k) \text{ and } i\neq m_k, \\ -Q_k\mathcal{E}^{t+s}_{(1,k+1),(j,l)} + \mathcal{E}^{t+s+1}_{(1,k+1),(j,l)} & \text{if } \ell>1, (a,c)=(i,k) \text{ and } i=m_k, \\ -\mathcal{E}^{t+s}_{(i,k),(j-1,l)} & \text{if } \ell>1, (a,c)=(j-1,l) \text{ and } j-1\neq m_l, \\ Q_l\mathcal{E}^{t+s}_{(i,k),(m_l,l)} - \mathcal{E}^{t+s+1}_{(i,k),(m_l,l)} & \text{if } \ell>1, (a,c)=(j-1,l) \text{ and } j-1=m_l, \\ 0 & \text{otherwise}, \end{cases}$$

where we have set $\ell = (j,l)-(i,k)$.

(ii) *For* $(i,k),(j,l)\in\Gamma(\mathbf{m})$ *such that* $(j,l)<(i,k)$, *we have*

$$[\mathcal{X}^-_{(a,c),s}, \mathcal{E}^t_{(i,k),(j,l)}] = \begin{cases} \mathcal{E}^{t+s}_{(i+1,k),(j,l)} & \text{if } (a,c)=(i,k), \\ -\mathcal{E}^{t+s}_{(i,k),(j-1,l)} & \text{if } (a,c)=(j-1,l), \\ 0 & \text{otherwise}, \end{cases}$$

$$[\mathcal{I}_{(a,c),s}, \mathcal{E}^t_{(i,k),(j,l)}] = \begin{cases} \mathcal{E}^{t+s}_{(i,k),(j,l)} & \text{if } (a,c)=(i,k), \\ -\mathcal{E}^{t+s}_{(i,k),(j,l)} & \text{if } (a,c)=(j,l), \\ 0 & \text{otherwise}, \end{cases}$$

$$[\mathcal{X}^+_{(a,c),s}, \mathcal{E}^t_{(i,k),(j,l)}]$$

$$= \begin{cases} \mathcal{E}^{t+s}_{(i-1,k),(i-1,k)} - \mathcal{E}^{t+s}_{(i,k),(i,k)} & \text{if } \ell=1, (a,c)=(i-1,k) \text{ and } i-1\neq m_k, \\ -Q_k(\mathcal{E}^{t+s}_{(m_k,k),(m_k,k)} - \mathcal{E}^{t+s}_{(1,k+1),(1,k+1)}) + \mathcal{E}^{t+s+1}_{(m_k,k),(m_k,k)} - \mathcal{E}^{t+s+1}_{(1,k+1),(1,k+1)} & \\ & \text{if } \ell=1, (a,c)=(i-1,k) \text{ and } i-1=m_k, \\ \mathcal{E}^{t+s}_{(i-1,k),(j,l)} & \text{if } \ell>1, (a,c)=(i-1,k) \text{ and } i-1\neq m_k, \\ -Q_k\mathcal{E}^{t+s}_{(m_k,k),(j,l)} + \mathcal{E}^{t+s+1}_{(m_k,k),(j,l)} & \text{if } \ell>1, (a,c)=(i-1,k) \text{ and } i-1=m_k, \\ -\mathcal{E}^{t+s}_{(i,k),(j+1,l)} & \text{if } \ell>1, (a,c)=(j,l) \text{ and } j\neq m_l, \\ Q_l\mathcal{E}^{t+s}_{(i,k),(1,l+1)} - \mathcal{E}^{t+s+1}_{(i,k),(1,l+1)} & \text{if } \ell>1, (a,c)=(j,l) \text{ and } j=m_l, \\ 0 & \text{otherwise}, \end{cases}$$

where we have set $\ell = (i,k)-(j,l)$.

(iii) *For* $(i,k) \in \Gamma(\mathbf{m})$, *we have*

$$\begin{aligned} &[\mathcal{I}_{(a,c),s}, \mathcal{E}^t_{(i,k),(i,k)}] = 0, \\ &[\mathcal{X}^+_{(a,c),s}, \mathcal{E}^t_{(i,k),(i,k)}] = -a_{(a,c)(i,k)}\mathcal{E}^{t+s}_{(a,c),(a+1,c)}, \\ &[\mathcal{X}^-_{(a,c),s}, \mathcal{E}^t_{(i,k),(i,k)}] = a_{(a,c)(i,k)}\mathcal{E}^{t+s}_{(a+1,c),(a,c)}. \end{aligned}$$

Proof. We prove (2.5.1) by induction on $(j,l)-(i,k)$.

The case $(j,l)-(i,k) = 1$ follows from (L4) and (L5). Assume now that $(j,l)-(i,k) > 1$. We have

$$\begin{aligned} [\mathcal{X}^+_{(a,c),s}, \mathcal{E}^t_{(i,k),(j,l)}] &= [\mathcal{X}^+_{(a,c),s}, [\mathcal{X}^+_{(i,k),0}, \mathcal{E}^t_{(i+1,k),(j,l)}]] \\ &= [\mathcal{X}^+_{(i,k),0}, [\mathcal{X}^+_{(a,c),s}, \mathcal{E}^t_{(i+1,k),(j,l)}]] + [[\mathcal{X}^+_{(a,c),s}, \mathcal{X}^+_{(i,k),0}], \mathcal{E}^t_{(i+1,k),(j,l)}]. \end{aligned}$$

Applying the inductive assumption, we obtain

$$(2.5.4) \qquad [\mathcal{X}^+_{(a,c),s}, \mathcal{E}^t_{(i,k),(j,l)}] = \begin{cases} [\mathcal{X}^+_{(i,k),0}, \mathcal{E}^{t+s}_{(i,k),(j,l)}] & \text{if } (a,c) = (i,k), \\ -[\mathcal{X}^+_{(i,k),0}, \mathcal{E}^{t+s}_{(i+1,k),(j+1,l)}] & \text{if } (a,c) = (j,l), \\ [[\mathcal{X}^+_{(i-1,k),s}, \mathcal{X}^+_{(i,k),0}], \mathcal{E}^t_{(i+1,k),(j,l)}] & \text{if } (a,c) = (i-1,k), \\ [[\mathcal{X}^+_{(i+1,k),s}, \mathcal{X}^+_{(i,k),0}], \mathcal{E}^t_{(i+1,k),(j,l)}] & \text{if } (a,c) = (i+1,k), \\ 0 & \text{otherwise.} \end{cases}$$

We also have

$$\begin{aligned} [\mathcal{X}^+_{(a,c),s}, \mathcal{E}^t_{(i,k),(j,l)}] &= [\mathcal{X}^+_{(a,c),s}, [\mathcal{E}^t_{(i,k),(j-1,l)}, \mathcal{X}^+_{(j-1,l),0}]] \\ &= [[\mathcal{X}^+_{(j-1,l),0}, \mathcal{X}^+_{(a,c),s}], \mathcal{E}^t_{(i,k),(j-1,l)}] + [[\mathcal{X}^+_{(a,c),s}, \mathcal{E}^t_{(i,k),(j-1,l)}], \mathcal{X}^+_{(j-1,l),0}]. \end{aligned}$$

Applying the inductive assumption, we obtain

$$(2.5.5) \qquad [\mathcal{X}^+_{(a,c),s}, \mathcal{E}^t_{(i,k),(j,l)}] = \begin{cases} [[\mathcal{X}^+_{(j-1,l),0}, \mathcal{X}^+_{(j,l),s}], \mathcal{E}^t_{(i,k),(j-1,l)}] & \text{if } (a,c) = (j,l), \\ [[\mathcal{X}^+_{(j-1,l),0}, \mathcal{X}^+_{(j-2,l),s}], \mathcal{E}^t_{(i,k),(j-1,l)}] & \text{if } (a,c) = (j-2,l), \\ [\mathcal{E}^{t+s}_{(i-1,k),(j-1,l)}, \mathcal{X}^+_{(j-1,l),0}] & \text{if } (a,c) = (i-1,k), \\ -[\mathcal{E}^{t+s}_{(i,k),(j,l)}, \mathcal{X}^+_{(j-1,l),0}] & \text{if } (a,c) = (j-1,l), \\ 0 & \text{otherwise.} \end{cases}$$

By (2.5.4) and (2.5.5),

$$[\mathcal{X}^{+}_{(a,c),s}, \mathcal{E}^{t}_{(i,k),(j,l)}] = \begin{cases} \mathcal{E}^{t+s}_{(i-1,k),(j,l)} & \text{if } (a,c)=(i-1,k), \\ -\mathcal{E}^{t+s}_{(i,k),(j+1,l)} & \text{if } (a,c)=(j,l), \\ [\mathcal{X}^{+}_{(i,k),0}, \mathcal{E}^{t+s}_{(i,k),(i+2,k)}] & \text{if } (a,c)=(i,k)=(j-2,l), \\ [[\mathcal{X}^{+}_{(i+1,k),s}, \mathcal{X}^{+}_{(i,k),0}], \mathcal{E}^{t}_{(i+1,k),(i+3,k)}] & \\ & \text{if } (a,c)=(i+1,k)=(j-2,l), \\ [\mathcal{X}^{-}_{(i+1,k),0}, \mathcal{E}^{t+s}_{(i,k),(i+2,k)}] & \text{if } (a,c)=(i+1,k)=(j-1,l), \\ 0 & \text{otherwise.} \end{cases}$$

By direct calculations using (L4)–(L6), we also get

$$\begin{aligned} [\mathcal{X}^{+}_{(i,k),0}, \mathcal{E}^{t+s}_{(i,k),(i+2,k)}] &= [[\mathcal{X}^{+}_{(i+1,k),s}, \mathcal{X}^{+}_{(i,k),0}], \mathcal{E}^{t}_{(i+1,k),(i+3,k)}] \\ &= [\mathcal{X}^{-}_{(i+1,k),0}, \mathcal{E}^{t+s}_{(i,k),(i+2,k)}] = 0. \end{aligned}$$

Thus we have proved (2.5.1).

We prove (2.5.2) by induction on $(j,l)-(i,k)$. The case $(j,l)-(i,k)=1$ is just (L2). Assume that $(j,l)-(i,k)>1$. We have

$$\begin{aligned} [\mathcal{I}_{(a,c),s}, \mathcal{E}^{t}_{(i,k),(j,l)}] &= [\mathcal{I}_{(a,c),s}, [\mathcal{X}^{+}_{(i,k),0}, \mathcal{E}^{t}_{(i+1,k),(j,l)}]] \\ &= [\mathcal{X}^{+}_{(i,k),0}, [\mathcal{I}_{(a,c),s}, \mathcal{E}^{t}_{(i+1,k),(j,l)}]] + [[\mathcal{I}_{(a,c),s}, \mathcal{X}^{+}_{(i,k),0}], \mathcal{E}^{t}_{(i+1,k),(j,l)}]. \end{aligned}$$

By the inductive assumption,

$$\begin{aligned} &[\mathcal{I}_{(a,c),s}, \mathcal{E}^{t}_{(i,k),(j,l)}] \\ &= \begin{cases} [\mathcal{X}^{+}_{(i,k),0}, \mathcal{E}^{t+s}_{(i+1,k),(j,l)}] - [\mathcal{X}^{+}_{(i,k),s}, \mathcal{E}^{t}_{(i+1,k),(j,l)}] & \text{if } (a,c)=(i+1,k), \\ -[\mathcal{X}^{+}_{(i,k),0}, \mathcal{E}^{t+s}_{(i+1,k),(j,l)}] & \text{if } (a,c)=(j,l), \\ [\mathcal{X}^{+}_{(i,k),s}, \mathcal{E}^{t}_{(i+1,k),(j,l)}] & \text{if } (a,c)=(i,k), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus, we get (2.5.2) by applying (2.5.1).

We prove (2.5.3) by induction on $\ell=(j,l)-(i,k)$. For $\ell=1,2$, we can show (2.5.3) by direct calculations. Assume that $\ell>2$. Then

$$\begin{aligned} [\mathcal{X}^{-}_{(a,c),s}, \mathcal{E}^{t}_{(i,k),(j,l)}] &= [\mathcal{X}^{-}_{(a,c),s}, [\mathcal{X}^{+}_{(i,k),0}, \mathcal{E}^{t}_{(i+1,k),(j,l)}]] \\ &= [\mathcal{X}^{+}_{(i,k),0}, [\mathcal{X}^{-}_{(a,c),s}, \mathcal{E}^{t}_{(i+1,k),(j,l)}]] + [[\mathcal{X}^{-}_{(a,c),s}, \mathcal{X}^{+}_{(i,k),0}], \mathcal{E}^{t}_{(i+1,k),(j,l)}]. \end{aligned}$$

Applying the inductive assumption, we obtain

$$[\mathcal{X}^-_{(a,c),s}, \mathcal{E}^t_{(i,k),(j,l)}]$$
$$= \begin{cases} [\mathcal{X}^+_{(i,k),0}, \mathcal{E}^{t+s}_{(i+2,k),(j,l)}] & \text{if } (a,c)=(i+1,k) \text{ and } i+1 \neq m_k, \\ [\mathcal{X}^+_{(i,k),0}, -Q_k\mathcal{E}^{t+s}_{(1,k+1),(j,l)} + \mathcal{E}^{t+s+1}_{(1,k+1),(j,l)}] & \text{if } (a,c)=(i+1,k) \text{ and } i+1 = m_k \\ [\mathcal{X}^+_{(i,k),0}, -\mathcal{E}^{t+s}_{(i+1,k),(j-1,l)}] & \text{if } (a,c)=(j-1,l) \text{ and } j-1 \neq m_l, \\ [\mathcal{X}^+_{(i,k),0}, Q_l\mathcal{E}^{t+s}_{(i+1,k),(m_l,l)} - \mathcal{E}^{t+s+1}_{(i+1,k),(m_l,l)}] & \text{if } (a,c)=(j-1,l) \text{ and } j-1 = m_l, \\ [-\mathcal{I}_{(i,k),s} + \mathcal{I}_{(i+1,k),s}, \mathcal{E}^t_{(i+1,k),(j,l)}] & \text{if } (a,c)=(i,k) \text{ and } i \neq m_k, \\ [Q_k(\mathcal{I}_{(m_k,k),s} - \mathcal{I}_{(1,k+1),s}) - \mathcal{I}_{(m_k,k),s+1} + \mathcal{I}_{(1,k+1),s+1}, \mathcal{E}^t_{(1,k+1),(j,l)}] & \\ & \text{if } (a,c)=(i,k) \text{ and } i = m_k, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, (2.5.3) follows by applying (2.5.1) and (2.5.2).

(ii) is proven in a similar way. (iii) is just the relations (L1) and (L2). □

By Lemma 2.5, $\mathfrak{g}$ is spanned by $\{\mathcal{E}^t_{(i,k)(j,l)} \mid (i,k),(j,l) \in \varGamma(\mathbf{m}),\ t \geq 0\}$ as a $\mathbb{Q}(\mathbf{Q})$-vector space. In fact, this set is a basis of $\mathfrak{g}$:

Proposition 2.6. *$\{\mathcal{E}^t_{(i,k)(j,l)} \mid (i,k),(j,l) \in \varGamma(\mathbf{m}),\ t \geq 0\}$ is a basis of $\mathfrak{g} = \mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$.*

Proof. We have to show that the elements of $\{\mathcal{E}^t_{(i,k),(j,l)} \mid (i,k),(j,l) \in \varGamma(\mathbf{m}),\ t \geq 0\}$ are linearly independent.

For $\tau \in \mathbb{Q}(\mathbf{Q})$, let $V_\tau = \bigoplus_{(j,l)\in\varGamma(\mathbf{m})} \mathbb{Q}(\mathbf{Q})v_{(j,l)}$ be the $\mathfrak{g}$-module given in 2.3. Then

$$\mathcal{E}^t_{(i,k)(j,l)} \cdot v_{(a,c)} = \delta_{(a,c)(j,l)}\psi_{(i,k)(j,l)}\tau^t v_{(i,k)},$$

where

$$\psi_{(i,k)(j,l)} = \begin{cases} \displaystyle\prod_{p=0}^{l-k-1} (-Q_l + \tau) & \text{if } l-k > 0, \\ 1 & \text{otherwise.} \end{cases}$$

Thus, if $\sum_{(i,k),(j,l)\in\varGamma(\mathbf{m}),t\geq 0} r^t_{(i,k)(j,l)}\mathcal{E}^t_{(i,k),(j,l)} = 0$ (for some $r^t_{(i,k)(j,l)} \in \mathbb{Q}(\mathbf{Q})$), then

$$\Big(\sum_{(i,k),(j,l)\in\varGamma(\mathbf{m}),t\geq 0} r^t_{(i,k)(j,l)}\mathcal{E}^t_{(i,k),(j,l)}\Big) \cdot v_{(a,c)}$$
$$= \sum_{(i,k)\in\varGamma(\mathbf{m})} \psi_{(i,k)(j,l)}\Big(\sum_{t\geq 0} r^t_{(i,k)(a,c)}\tau^t\Big)v_{(i,k)} = 0.$$

Hence, for any $(i,k),(j,l) \in \Gamma(\mathbf{m})$ and any $\tau \in \mathbb{Q}(\mathbf{Q})$, we have

$$\psi_{(i,k)(j,l)}\Big(\sum_{t\geq 0} r^t_{(i,k)(j,l)}\tau^t\Big) = 0.$$

This implies that $r^t_{(i,k)(j,l)} = 0$ for any $(i,k),(j,l) \in \Gamma(\mathbf{m})$ and any $t \geq 0$. □

2.7. Let $\mathfrak{n}^+$, $\mathfrak{n}^-$ and $\mathfrak{n}^0$ be the Lie subalgebras of $\mathfrak{g}$ generated by

$$\{\mathcal{X}^+_{(i,k),t} \mid (i,k) \in \Gamma'(\mathbf{m}),\ t \geq 0\}, \quad \{\mathcal{X}^-_{(i,k),t} \mid (i,k) \in \Gamma'(\mathbf{m}),\ t \geq 0\} \quad \text{and}$$
$$\{\mathcal{I}_{(j,l),t} \mid (j,l) \in \Gamma(\mathbf{m}),\ t \geq 0\}$$

respectively. Then we have the following triangular decomposition as a corollary of Proposition 2.6.

Corollary 2.8. *We have the triangular decomposition*

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{n}^0 \oplus \mathfrak{n}^+ \quad (\textit{as vector spaces}).$$

2.9. A current Lie algebra. Let $\mathbb{Q}[x]$ be the polynomial ring over $\mathbb{Q}$, and let $\mathfrak{gl}_m[x] = \mathbb{Q}[x] \otimes \mathfrak{gl}_m$ be the current Lie algebra associated with the general linear Lie algebra $\mathfrak{gl}_m$ over $\mathbb{Q}$. The Lie bracket on $\mathfrak{gl}_m[x]$ is defined by

$$[a \otimes g, b \otimes h] = ab \otimes [g,h] \quad (a,b \in \mathbb{Q}[x],\ g,h \in \mathfrak{gl}_m).$$

Let $E_{i,j} \in \mathfrak{gl}_m$ $(1 \leq i,j \leq m)$ be the elementary matrix having 1 at the (i,j)-entry and 0 elsewhere. Set $e_i = E_{i,i+1}$, $f_i = E_{i+1,i}$ and $K_j = E_{j,j}$. Then $\mathbb{Q}[x] \otimes \mathfrak{gl}_m$ is generated by

$$x^t \otimes e_i,\ x^t \otimes f_i,\ x^t \otimes K_j \quad (1 \leq i \leq m-1,\ 1 \leq j \leq m,\ t \geq 0).$$

2.10. For $r = 1$ $(\mathbf{m} = m)$, the Lie algebra $\mathfrak{g}(m)$ over $\mathbb{Q}$ is generated by $\mathcal{X}^\pm_{i,t}$ and $\mathcal{I}_{j,t}$ $(1 \leq i \leq m-1,\ 1 \leq j \leq m,\ t \geq 0)$ with the defining relations (L1)–(L6) (for $(i,1) \in \Gamma(m)$, we denote $(i,1)$ simply by i). In this case, the relation (L3) is just

$$[\mathcal{X}^+_{i,t}, \mathcal{X}^-_{j,s}] = \delta_{i,j}(\mathcal{I}_{i,t} - \mathcal{I}_{i+1,t}).$$

Lemma 2.11. *There exists an isomorphism of Lie algebras*

$$\Phi : \mathfrak{g}(m) \to \mathfrak{gl}_m[x] \quad (\mathcal{X}^+_{i,t} \mapsto x^t \otimes e_i,\ \mathcal{X}^-_{i,t} \mapsto x^t \otimes f_i,\ \mathcal{I}_{j,t} \mapsto x^t \otimes K_j).$$

In particular, the relations (L1)–(L6) (*for* $r = 1$) *give defining relations of* $\mathfrak{gl}_m[x]$ *through the isomorphism* Φ.

Proof. We can show that Φ is well-defined by checking the defining relations of $\mathfrak{g}(m)$ directly.

For $i, j \in \{1, \dots, m\}$ and $t \geq 0$, we see that $\Phi(\mathcal{E}^t_{i,j}) = x^t \otimes E_{i,j}$. Clearly, $\{x^t \otimes E_{i,j} \mid 1 \leq i, j \leq m,\ t \geq 0\}$ is a basis of $\mathfrak{gl}_m[x]$. Thus, Proposition 2.6 implies that Φ is an isomorphism. □

2.12. For $r \geq 2$, we can regard $\mathfrak{g} = \mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$ as a deformation of the current Lie algebra $\mathbb{Q}(\mathbf{Q}) \otimes_{\mathbb{Q}} \mathfrak{gl}_m[x]$ as follows.

For $t \geq 0$, write

$$\mathcal{Y}_t = \{\mathcal{X}^{\pm}_{(i,k),t}, \mathcal{I}_{(j,l),t} \mid (i,k) \in \Gamma'(\mathbf{m}),\ (j,l) \in \Gamma(\mathbf{m})\}.$$

Let $\mathfrak{g}_t$ be the $\mathbb{Q}(\mathbf{Q})$-subspace of $\mathfrak{g}$ spanned by

$$\Big\{[Y_{t_1}, [Y_{t_2}, \dots, [Y_{t_{p-1}}, Y_{t_p}] \dots]] \Bigm| Y_{t_b} \in \mathcal{Y}_{t_b},\ \sum_{b=1}^{p} t_b \geq t,\ p \geq 1\Big\}.$$

Then

$$\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots .$$

By the defining relations (L1)–(L6), we see that

$$[\mathfrak{g}_s, \mathfrak{g}_t] \subset \mathfrak{g}_{s+t} \quad (s, t \geq 0). \tag{2.12.1}$$

For $t \geq 0$, let $\sigma_t : \mathfrak{g}_t \to \mathfrak{g}_t/\mathfrak{g}_{t+1}$ be the natural surjection. By (2.12.1), we can define the structure of a Lie algebra on $\mathbf{gr}\,\mathfrak{g} = \bigoplus_{t \geq 0} \mathfrak{g}_t/\mathfrak{g}_{t+1}$ by

$$[\sigma_s(g), \sigma_t(h)] = \sigma_{s+t}([g, h]) \quad (g \in \mathfrak{g}_s,\ h \in \mathfrak{g}_t).$$

Then we see that $\mathbf{gr}\,\mathfrak{g}$ is generated by

$$\sigma_t(\mathcal{X}^{\pm}_{(i,k),t}),\ \sigma_t(\mathcal{I}_{(j,l),t}) \quad ((i,k) \in \Gamma'(\mathbf{m}),\ (j,l) \in \Gamma(\mathbf{m}),\ t \geq 0),$$

and $\mathbf{gr}\,\mathfrak{g}$ has a basis $\{\sigma_t(\mathcal{E}^t_{(i,k),(j,l)}) \mid (i,k), (j,l) \in \Gamma(\mathbf{m}),\ t \geq 0\}$.

Proposition 2.13. *There exists an isomorphism of Lie algebras*

$$\Psi : \mathbb{Q}(\mathbf{Q}) \otimes_{\mathbb{Q}} \mathfrak{gl}_m[x] \to \mathbf{gr}\,\mathfrak{g} = \bigoplus_{t \geq 0} \mathfrak{g}_t/\mathfrak{g}_{t+1}$$

such that

$$\begin{aligned} x^t \otimes e_{(i,k)} &\mapsto \begin{cases} \sigma_t(\mathcal{X}^+_{(i,k),t}) & \text{if } i \neq m_k, \\ -Q_k^{-1} \sigma_t(\mathcal{X}^+_{(m_k,k),t}) & \text{if } i = m_k, \end{cases} \\ x^t \otimes f_{(i,k)} &\mapsto \sigma_t(\mathcal{X}^-_{(i,k),t}), \\ x^t \otimes K_{(j,l)} &\mapsto \sigma_t(\mathcal{I}_{(j,l),t}), \end{aligned}$$

where we use the identification (1.3.1) *for the indices of the generators of* $\mathfrak{gl}_m[x]$.

Proof. We can show that Ψ is well-defined by checking the defining relations of $\mathfrak{gl}_m[x]$ directly (see Lemma 2.11). We also see that

$$\Psi(x^t \otimes E_{(i,k),(j,l)}) = \psi_{(i,k)(j,l)}\sigma_t(\mathcal{E}^t_{(i,k),(j,l)}),$$

where

$$\psi_{(i,k)(j,l)} = \begin{cases} \prod_{p=0}^{l-k-1}(-Q_{k+p}^{-1}) & \text{if } l-k>0, \\ 1 & \text{otherwise.} \end{cases}$$

Thus, Ψ is an isomorphism. □

As a corollary, we have the following isomorphism between $\mathbb{Q}(\mathbf{Q})\otimes_{\mathbb{Q}}\mathfrak{g}(m)$ and $\mathbf{gr}\,\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$.

Corollary 2.14. *There exists an isomorphism of Lie algebras*

$$\widetilde{\Psi} : \mathbb{Q}(\mathbf{Q})\otimes_{\mathbb{Q}}\mathfrak{g}(m) \to \mathbf{gr}\,\mathfrak{g}_{\mathbf{Q}}(\mathbf{m}) = \bigoplus_{t\geq 0}\mathfrak{g}_t/\mathfrak{g}_{t+1}$$

such that

$$\mathcal{X}^+_{(i,k),t} \mapsto \begin{cases} \sigma_t(\mathcal{X}^+_{(i,k),t}) & \text{if } i\neq m_k, \\ -Q_k^{-1}\sigma_t(\mathcal{X}^+_{(m_k,k),t}) & \text{if } i = m_k, \end{cases}$$
$$\mathcal{X}^-_{(i,k),t} \mapsto \sigma_t(\mathcal{X}^-_{(i,k),t}), \qquad \mathcal{I}_{(j,l),t} \mapsto \sigma_t(\mathcal{I}_{(j,l),t}),$$

where we use the identification (1.3.1) *for the indices of the generators of* $\mathfrak{g}(m)$*.*

2.15. We also have some relations between the Lie algebra $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$ and the general linear Lie algebra $\mathfrak{gl}_m$. Let $\mathfrak{gl}_{m_1}\oplus\cdots\oplus\mathfrak{gl}_{m_r}$ be the Levi subalgebra of $\mathfrak{gl}_m$ associated with $\mathbf{m}=(m_1,\dots,m_r)$. Then generators of $\mathfrak{gl}_{m_1}\oplus\cdots\oplus\mathfrak{gl}_{m_r}$ are given by $e_{(i,k)}$, $f_{(i,k)}$ $(1\leq i\leq m_k-1, 1\leq k\leq r)$ and $K_{(j,l)}$ $((j,l)\in\Gamma(\mathbf{m}))$, where we use the identification (1.3.1) for indices.

Proposition 2.16. (i) *There exists a surjective homomorphism of Lie algebras*

$$g : \mathfrak{g}_{\mathbf{Q}}(\mathbf{m}) \to \mathfrak{gl}_m \tag{2.16.1}$$

such that

$$g(\mathcal{X}^+_{(i,k),0}) = \begin{cases} e_{(i,k)} & \text{if } i\neq m_k, \\ -Q_k e_{(m_k,k)} & \text{if } i=m_k, \end{cases} \qquad g(\mathcal{X}^-_{(i,k),0}) = f_{(i,k)},$$
$$g(\mathcal{I}_{(j,l),0}) = K_{(j,l)}, \qquad g(\mathcal{X}^{\pm}_{(i,k),t}) = g(\mathcal{I}_{(j,l),t}) = 0 \qquad \text{for } t\geq 1.$$

(ii) *There exists an injective homomorphism of Lie algebras*

$$\iota : \mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r} \to \mathfrak{g}_{\mathbf{Q}}(\mathbf{m}) \tag{2.16.2}$$

such that $\iota(e_{(i,k)}) = \mathcal{X}^+_{(i,k),0}$, $\iota(f_{(i,k)}) = \mathcal{X}^-_{(i,k),0}$ *and* $\iota(K_{(j,l)}) = \mathcal{I}_{(j,l),0}$.

Proof. We can check that g and ι are well-defined by direct calculations. Clearly g is surjective. Let $\iota' : \mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r} \to \mathfrak{gl}_m$ be the natural embedding. Then, by investigating the images of generators, we see that $\iota' = g \circ \iota$. This implies that ι is injective. □

Remark 2.17. The surjective homomorphism g in (2.16.1) can be regarded as a special case of evaluation homomorphisms. However, we cannot define evaluation homomorphisms for $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$ in general, although we can consider $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$-modules corresponding to some evaluation modules. They will be studied in a subsequent paper.

§3. Representations of $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$

Thanks to the triangular decomposition in Corollary 2.8, we can develop a weight theory to study representations of $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$ in the usual manner.

3.1. Let $U(\mathfrak{g}) = U(\mathfrak{g}_{\mathbf{Q}}(\mathbf{m}))$ be the universal enveloping algebra of the Lie algebra $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$. Then, by Corollary 2.8 together with the PBW theorem, we have the triangular decomposition

$$U(\mathfrak{g}) \cong U(\mathfrak{n}^-) \otimes U(\mathfrak{n}^0) \otimes U(\mathfrak{n}^+). \tag{3.1.1}$$

Thanks to this decomposition, we can develop a weight theory for $U(\mathfrak{g})$-modules.

3.2. Highest weight modules. For $\lambda \in P$ and a multiset $\boldsymbol{\varphi} = (\varphi_{(j,l),t} \mid (j,l) \in \Gamma(\mathbf{m}),\ t \geq 1)$ $(\varphi_{(j,l),t} \in \mathbb{Q}(\mathbf{Q}))$, we say that a $U(\mathfrak{g})$-module M is a *highest weight module of highest weight* $(\lambda, \boldsymbol{\varphi})$ if there exists $v_0 \in M$ such that:

(i) M is generated by v_0 as a $U(\mathfrak{g})$-module,

(ii) $\mathcal{X}^+_{(i,k),t} \cdot v_0 = 0$ for all $(i,k) \in \Gamma'(\mathbf{m})$ and $t \geq 0$,

(iii) $\mathcal{I}_{(j,l),0} \cdot v_0 = \langle \lambda, h_{(j,l)} \rangle v_0$ and $\mathcal{I}_{(j,l),t} \cdot v_0 = \varphi_{(j,l),t} v_0$ for $(j,l) \in \Gamma(\mathbf{m})$ and $t \geq 1$.

If $v_0 \in M$ satisfies (ii) and (iii), we say that v_0 is a *maximal vector of weight* $(\lambda, \boldsymbol{\varphi})$. In this case, the submodule $U(\mathfrak{g}) \cdot v_0$ of M is a highest weight module of highest weight $(\lambda, \boldsymbol{\varphi})$. If a maximal vector $v_0 \in M$ satisfies (i), we say that v_0 is a *highest weight vector.*

For a highest weight $U(\mathfrak{g})$-module M of highest weight $(\lambda, \boldsymbol{\varphi})$ with a highest weight vector $v_0 \in M$, we have $M = U(\mathfrak{n}^-) \cdot v_0$ by the decomposition (3.1.1).

Thus, the relation (L2) implies the weight space decomposition

$$M=\bigoplus_{\substack{\mu\in P\\ \mu\leq\lambda}} M_\mu \quad \text{such that} \quad \dim_{\mathbb{Q}(\mathbf{Q})} M_\lambda=1, \tag{3.2.1}$$

where $M_\mu=\{v\in M \mid \mathcal{I}_{(j,l),0}\cdot v=\langle \mu, h_{(j,l)}\rangle v \text{ for } (j,l)\in\Gamma(\mathbf{m})\}$.

3.3. Verma modules. Let $U(\mathfrak{n}^{\geq 0})$ be the subalgebra of $U(\mathfrak{g})$ generated by $U(\mathfrak{n}^0)$ and $U(\mathfrak{n}^+)$. Then, by Proposition 2.6 together with the proof of Lemma 2.5, we see that $U(\mathfrak{n}^+)$ (resp. $U(\mathfrak{n}^-)$) is isomorphic to the algebra generated by $\{\mathcal{X}^+_{(i,k),t} \mid (i,k)\in\Gamma'(\mathbf{m}),\ t\geq 0\}$ (resp. $\{\mathcal{X}^-_{(i,k),t} \mid (i,k)\in\Gamma'(\mathbf{m}),\ t\geq 0\}$) with the defining relations (L4)–(L6), $U(\mathfrak{n}^0)$ is isomorphic to the algebra generated by $\{\mathcal{I}_{(j,l),t} \mid (j,l)\in\Gamma(\mathbf{m}),\ t\geq 0\}$ with the defining relations (L1), and $U(\mathfrak{n}^{\geq 0})$ is isomorphic to the algebra generated by $\{\mathcal{X}^+_{(i,k)t}, \mathcal{I}_{(j,l)t} \mid (i,k)\in\Gamma'(\mathbf{m}),\ (j,l)\in\Gamma(\mathbf{m}),\ t\geq 0\}$ with the defining relations (L1)–(L6) except (L3). Then we have a surjective homomorphism of algebras

$$U(\mathfrak{n}^{\geq 0})\to U(\mathfrak{n}^0) \quad \text{such that} \quad \mathcal{X}^+_{(i,k),t}\mapsto 0,\ \mathcal{I}_{(j,l),t}\mapsto \mathcal{I}_{(j,l),t}. \tag{3.3.1}$$

For $\lambda\in P$ and a multiset $\boldsymbol{\varphi}=(\varphi_{(j,l),t})$, we define a (1-dimensional) simple $U(\mathfrak{n}^0)$-module $\Theta_{(\lambda,\boldsymbol{\varphi})}=\mathbb{Q}(\mathbf{Q})v_0$ by

$$\mathcal{I}_{(j,l),0}\cdot v_0=\langle\lambda, h_{(j,l)}\rangle v_0, \qquad \mathcal{I}_{(j,l),t}\cdot v_0=\varphi_{(j,l),t}v_0$$

for $(j,l)\in\Gamma(\mathbf{m})$ and $t\geq 1$. Then we define the *Verma module* $M(\lambda,\boldsymbol{\varphi})$ as the induced module

$$M(\lambda,\boldsymbol{\varphi})=U(\mathfrak{g})\otimes_{U(\mathfrak{n}^{\geq 0})}\Theta_{(\lambda,\boldsymbol{\varphi})},$$

where we regard $\Theta_{(\lambda,\boldsymbol{\varphi})}$ as a left $U(\mathfrak{n}^{\geq 0})$-module through the surjection (3.3.1).

By the definitions, the Verma module $M(\lambda,\boldsymbol{\varphi})$ is a highest weight module of highest weight $(\lambda,\boldsymbol{\varphi})$ with a highest weight vector $1\otimes v_0$. Any highest weight module of highest weight $(\lambda,\boldsymbol{\varphi})$ is a quotient of $M(\lambda,\boldsymbol{\varphi})$, by the universality of tensor products. Moreover, $M(\lambda,\boldsymbol{\varphi})$ has the unique simple top $L(\lambda,\boldsymbol{\varphi})=M(\lambda,\boldsymbol{\varphi})/\operatorname{rad} M(\lambda,\boldsymbol{\varphi})$ from the weight space decomposition (3.2.1).

By using the homomorphism $\iota: U(\mathfrak{gl}_{m_1}\oplus\cdots\oplus\mathfrak{gl}_{m_r})\to U(\mathfrak{g})$ induced from (2.16.2), we have a necessary condition for $L(\lambda,\boldsymbol{\varphi})$ to be finite-dimensional:

Proposition 3.4. *For $\lambda\in P$ and a multiset $\boldsymbol{\varphi}=(\varphi_{(j,l),t})$, if $L(\lambda,\boldsymbol{\varphi})$ is finite-dimensional, then $\lambda\in P^+_{\mathbf{m}}$.*

Proof. Assume that $L(\lambda,\boldsymbol{\varphi})$ is finite-dimensional. Let $v_0\in L(\lambda,\boldsymbol{\varphi})$ be a highest weight vector. When we regard $L(\lambda,\boldsymbol{\varphi})$ as a $U(\mathfrak{gl}_{m_1}\oplus\cdots\oplus\mathfrak{gl}_{m_r})$-module through the injection $\iota : U(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}) \to U(\mathfrak{g})$, we see that the $U(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r})$-submodule of $L(\lambda, \boldsymbol{\varphi})$ generated by v_0 is a (finite-dimensional) highest weight $U(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r})$-module of highest weight λ. Thus, the proposition follows from well-known facts about $U(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r})$-modules. □

3.5. The category $\mathcal{C}_{\mathbf{Q}}(\mathbf{m})$. Let $\mathcal{C}_{\mathbf{Q}}(\mathbf{m})$ (resp. $\mathcal{C}^{\geq 0}_{\mathbf{Q}}(\mathbf{m})$) be the full subcategory of $U(\mathfrak{g})$-mod consisting of $U(\mathfrak{g})$-modules satisfying the following conditions:

(i) If $M \in \mathcal{C}_{\mathbf{Q}}(\mathbf{m})$ (resp. $M \in \mathcal{C}^{\geq 0}_{\mathbf{Q}}(\mathbf{m})$), then M is finite-dimensional.

(ii) If $M \in \mathcal{C}_{\mathbf{Q}}(\mathbf{m})$ (resp. $M \in \mathcal{C}^{\geq 0}_{\mathbf{Q}}(\mathbf{m})$), then M has the weight space decomposition

$$M = \bigoplus_{\lambda \in P} M_\lambda \quad (\text{resp. } M = \bigoplus_{\lambda \in P_{\geq 0}} M_\lambda),$$

where $M_\lambda = \{v \in M \mid \mathcal{I}_{(j,l),0} \cdot v = \langle \lambda, h_{(j,l)} \rangle v \text{ for } (j,l) \in \Gamma(\mathbf{m})\}$.

(iii) If $M \in \mathcal{C}_{\mathbf{Q}}(\mathbf{m})$ (resp. $M \in \mathcal{C}^{\geq 0}_{\mathbf{Q}}(\mathbf{m})$), then all eigenvalues of the action of $\mathcal{I}_{(j,l),t}$ ($(j,l) \in \Gamma(\mathbf{m})$, $t \geq 0$) on M belong to $\mathbb{Q}(\mathbf{Q})$.

By the usual argument, we have the following lemma.

Lemma 3.6. *Any simple object in $\mathcal{C}_{\mathbf{Q}}(\mathbf{m})$ is a highest weight module.*

By using the surjection $g : U(\mathfrak{g}) \to U(\mathfrak{gl}_m)$ induced from (2.16.1), we obtain the following proposition.

Proposition 3.7. *Let $\mathcal{C}_{\mathfrak{gl}_m}$ be the category of finite-dimensional $U(\mathfrak{gl}_m)$-modules which have a weight space decomposition. Then:*

(i) *$\mathcal{C}_{\mathfrak{gl}_m}$ is a full subcategory of $\mathcal{C}_{\mathbf{Q}}(\mathbf{m})$ through the surjection $g : U(\mathfrak{g}) \to U(\mathfrak{gl}_m)$.*

(ii) *For $\lambda \in P^+$, the simple highest weight $U(\mathfrak{gl}_m)$-module $\Delta_{\mathfrak{gl}_m}(\lambda)$ of highest weight λ is the simple highest weight $U(\mathfrak{g})$-module of highest weight $(\lambda, \mathbf{0})$ through the surjection $g : U(\mathfrak{g}) \to U(\mathfrak{gl}_m)$, where $\mathbf{0}$ means $\varphi_{(j,l),t} = 0$ for all $(j,l) \in \Gamma(\mathbf{m})$ and $t \geq 1$.*

§4. The algebra $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$

In this section, we introduce an algebra $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ with parameters q and $\mathbf{Q} = (Q_1, \dots, Q_{r-1})$ associated with the Cartan data of paragraph 1.3. Then we study some basic structures of $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$. In particular, we can regard $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ as a "q-analogue" of the universal enveloping algebra $U(\mathfrak{g}_{\mathbf{Q}}(\mathbf{m}))$ of the Lie algebra $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$ introduced in §2.

4.1. Set $\mathbb{A} = \mathbb{Z}[\mathbf{Q}][q, q^{-1}] = \mathbb{Z}[q, q^{-1}, Q_1, \dots, Q_{r-1}]$, where $q, Q_1, \dots, Q_{r-1}$ are indeterminates over $\mathbb{Z}$, and let $\mathbb{K} = \mathbb{Q}(q, Q_1, \dots, Q_{r-1})$ be the quotient field of $\mathbb{A}$.

Definition 4.2. We define the associative algebra $\mathcal{U} = \mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ over $\mathbb{K}$ by the following generators and relations:

Generators: $\mathcal{X}^{\pm}_{(i,k),t}$, $\mathcal{I}^{\pm}_{(j,l),t}$, $\mathcal{K}^{\pm}_{(j,l)}$ $((i,k) \in \Gamma'(\mathbf{m}),\ (j,l) \in \Gamma(\mathbf{m}),\ t \geq 0)$.

Relations:

(R1) $\mathcal{K}^{+}_{(j,l)}\mathcal{K}^{-}_{(j,l)} = \mathcal{K}^{-}_{(j,l)}\mathcal{K}^{+}_{(j,l)} = 1, \quad (\mathcal{K}^{\pm}_{(j,l)})^2 = 1 \pm (q-q^{-1})\mathcal{I}^{\mp}_{(j,l),0},$

(R2) $[\mathcal{K}^{+}_{(i,k)}, \mathcal{K}^{+}_{(j,l)}] = [\mathcal{K}^{+}_{(i,k)}, \mathcal{I}^{\sigma}_{(j,l),t}] = [\mathcal{I}^{\sigma}_{(i,k),s}, \mathcal{I}^{\sigma'}_{(j,l),t}] = 0\ (\sigma,\sigma' \in \{+,-\}),$

(R3) $\mathcal{K}^{+}_{(j,l)}\mathcal{X}^{\pm}_{(i,k),t}\mathcal{K}^{-}_{(j,l)} = q^{\pm a_{(i,k)(j,l)}}\mathcal{X}^{\pm}_{(i,k),t},$

(R4)
$$q^{\pm a_{(i,k)(j,l)}}\mathcal{I}^{\pm}_{(j,l),0}\mathcal{X}^{+}_{(i,k),t} - q^{\mp a_{(i,k)(j,l)}}\mathcal{X}^{+}_{(i,k),t}\mathcal{I}^{\pm}_{(j,l),0} = a_{(i,k)(j,l)}\mathcal{X}^{+}_{(i,k),t},$$
$$q^{\mp a_{(i,k)(j,l)}}\mathcal{I}^{\pm}_{(j,l),0}\mathcal{X}^{-}_{(i,k),t} - q^{\pm a_{(i,k)(j,l)}}\mathcal{X}^{-}_{(i,k),t}\mathcal{I}^{\pm}_{(j,l),0} = -a_{(i,k)(j,l)}\mathcal{X}^{-}_{(i,k),t},$$

(R5)
$$[\mathcal{I}^{\pm}_{(j,l),s+1}, \mathcal{X}^{+}_{(i,k),t}] = q^{\pm a_{(i,k)(j,l)}}\mathcal{I}^{\pm}_{(j,l),s}\mathcal{X}^{+}_{(i,k),t+1} - q^{\mp a_{(i,k)(j,l)}}\mathcal{X}^{+}_{(i,k),t+1}\mathcal{I}^{\pm}_{(j,l),s},$$
$$[\mathcal{I}^{\pm}_{(j,l),s+1}, \mathcal{X}^{-}_{(i,k),t}] = q^{\mp a_{(i,k)(j,l)}}\mathcal{I}^{\pm}_{(j,l),s}\mathcal{X}^{-}_{(i,k),t+1} - q^{\pm a_{(i,k)(j,l)}}\mathcal{X}^{-}_{(i,k),t+1}\mathcal{I}^{\pm}_{(j,l),s},$$

(R6)
$$[\mathcal{X}^{+}_{(i,k),t}, \mathcal{X}^{-}_{(j,l),s}] = \delta_{(i,k),(j,l)}\begin{cases}\widetilde{\mathcal{K}}^{+}_{(i,k)}\mathcal{J}_{(i,k),s+t} & \text{if } i \neq m_k,\\ -Q_k\widetilde{\mathcal{K}}^{+}_{(m_k,k)}\mathcal{J}_{(m_k,k),s+t} + \widetilde{\mathcal{K}}^{+}_{(m_k,k)}\mathcal{J}_{(m_k,k),s+t+1} & \text{if } i = m_k,\end{cases}$$

(R7)
$$[\mathcal{X}^{\pm}_{(i,k),t}, \mathcal{X}^{\pm}_{(j,l),s}] = 0 \qquad \text{if } (j,l) \neq (i,k), (i\pm 1,k),$$
$$\mathcal{X}^{\pm}_{(i,k),t+1}\mathcal{X}^{\pm}_{(i,k),s} - q^{\pm 2}\mathcal{X}^{\pm}_{(i,k),s}\mathcal{X}^{\pm}_{(i,k),t+1} = q^{\pm 2}\mathcal{X}^{\pm}_{(i,k),t}\mathcal{X}^{\pm}_{(i,k),s+1} - \mathcal{X}^{\pm}_{(i,k),s+1}\mathcal{X}^{\pm}_{(i,k),t},$$
$$\mathcal{X}^{+}_{(i,k),t+1}\mathcal{X}^{+}_{(i+1,k),s} - q^{-1}\mathcal{X}^{+}_{(i+1,k),s}\mathcal{X}^{+}_{(i,k),t+1} = \mathcal{X}^{+}_{(i,k),t}\mathcal{X}^{+}_{(i+1,k),s+1} - q\mathcal{X}^{+}_{(i+1,k),s+1}\mathcal{X}^{+}_{(i,k),t},$$
$$\mathcal{X}^{-}_{(i+1,k),s}\mathcal{X}^{-}_{(i,k),t+1} - q^{-1}\mathcal{X}^{-}_{(i,k),t+1}\mathcal{X}^{-}_{(i+1,k),s} = \mathcal{X}^{-}_{(i+1,k),s+1}\mathcal{X}^{-}_{(i,k),t} - q\mathcal{X}^{-}_{(i,k),t}\mathcal{X}^{-}_{(i+1,k),s+1},$$

(R8)
$$\mathcal{X}^{+}_{(i\pm1,k),u}(\mathcal{X}^{+}_{(i,k),s}\mathcal{X}^{+}_{(i,k),t} + \mathcal{X}^{+}_{(i,k),t}\mathcal{X}^{+}_{(i,k),s}) + (\mathcal{X}^{+}_{(i,k),s}\mathcal{X}^{+}_{(i,k),t} + \mathcal{X}^{+}_{(i,k),t}\mathcal{X}^{+}_{(i,k),s})\mathcal{X}^{+}_{(i\pm1,k),u} = (q+q^{-1})(\mathcal{X}^{+}_{(i,k),s}\mathcal{X}^{+}_{(i\pm1,k),u}\mathcal{X}^{+}_{(i,k),t} + \mathcal{X}^{+}_{(i,k),t}\mathcal{X}^{+}_{(i\pm1,k),u}\mathcal{X}^{+}_{(i,k),s}),$$
$$\mathcal{X}^{-}_{(i\pm1,k),u}(\mathcal{X}^{-}_{(i,k),s}\mathcal{X}^{-}_{(i,k),t} + \mathcal{X}^{-}_{(i,k),t}\mathcal{X}^{-}_{(i,k),s}) + (\mathcal{X}^{-}_{(i,k),s}\mathcal{X}^{-}_{(i,k),t} + \mathcal{X}^{-}_{(i,k),t}\mathcal{X}^{-}_{(i,k),s})\mathcal{X}^{-}_{(i\pm1,k),u} = (q+q^{-1})(\mathcal{X}^{-}_{(i,k),s}\mathcal{X}^{-}_{(i\pm1,k),u}\mathcal{X}^{-}_{(i,k),t} + \mathcal{X}^{-}_{(i,k),t}\mathcal{X}^{-}_{(i\pm1,k),u}\mathcal{X}^{-}_{(i,k),s}),$$

where we have set $\widetilde{\mathcal{K}}^{+}_{(i,k)} = \mathcal{K}^{+}_{(i,k)}\mathcal{K}^{-}_{(i+1,k)}$, $\widetilde{\mathcal{K}}^{-}_{(i,k)} = \mathcal{K}^{-}_{(i,k)}\mathcal{K}^{+}_{(i+1,k)}$ and

$$\mathcal{J}_{(i,k),t} = \begin{cases}\mathcal{I}^{+}_{(i,k),0} - \mathcal{I}^{-}_{(i+1,k),0} + (q-q^{-1})\mathcal{I}^{+}_{(i,k),0}\mathcal{I}^{-}_{(i+1,k),0} & \text{if } t = 0,\\ q^{-t}\mathcal{I}^{+}_{(i,k),t} - q^{t}\mathcal{I}^{-}_{(i+1,k),t} - (q-q^{-1})\sum_{b=1}^{t-1} q^{-t+2b}\mathcal{I}^{+}_{(i,k),t-b}\mathcal{I}^{-}_{(i+1,k),b} & \text{if } t > 0.\end{cases}$$

Remark 4.3. The relation (R4) follows from (R1) and (R3) in $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$. Thus, we do not need (R4) among the defining relations of $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$. However, (R4) does not follow from (R1) and (R3) in the integral forms $\mathcal{U}^{\star}_{\mathbb{A},q,\mathbf{Q}}(\mathbf{m})$ and $\mathcal{U}_{\mathbb{A},q,\mathbf{Q}}(\mathbf{m})$ defined below. Therefore, we keep (R4) as a defining relation of $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$.

4.4. By (R1), for $(i,k) \in \varGamma'(\mathbf{m})$, we have

$$\widetilde{\mathcal{K}}^+_{(i,k)}\mathcal{J}_{(i,k),0} = \frac{\widetilde{\mathcal{K}}^+_{(i,k)} - \widetilde{\mathcal{K}}^-_{(i,k)}}{q-q^{-1}}. \tag{4.4.1}$$

Thus, when $s=t=0$, we can replace (R6) by

$$[\mathcal{X}^+_{(i,k),0}, \mathcal{X}^-_{(j,l),0}] = \delta_{(i,k),(j,l)} \begin{cases} \dfrac{\widetilde{\mathcal{K}}^+_{(i,k)} - \widetilde{\mathcal{K}}^-_{(i,k)}}{q-q^{-1}} & \text{if } i \neq m_k, \\ -Q_k \dfrac{\widetilde{\mathcal{K}}^+_{(m_k,k)} - \widetilde{\mathcal{K}}^-_{(m_k,k)}}{q-q^{-1}} + \widetilde{\mathcal{K}}^+_{(m_k,k)}\mathcal{J}_{(m_k,k),1} & \text{if } i = m_k. \end{cases} \tag{4.4.2}$$

By (R8), if $s=t$, we have

(4.4.3)
$$\mathcal{X}^+_{(i\pm1,k),u}(\mathcal{X}^+_{(i,k),t})^2 - (q+q^{-1})\mathcal{X}^+_{(i,k),t}\mathcal{X}^+_{(i\pm1,k),u}\mathcal{X}^+_{(i,k),t} + (\mathcal{X}^+_{(i,k),t})^2\mathcal{X}^+_{(i\pm1,k),u} = 0,$$
$$\mathcal{X}^-_{(i\pm1,k),u}(\mathcal{X}^-_{(i,k),t})^2 - (q+q^{-1})\mathcal{X}^-_{(i,k),t}\mathcal{X}^-_{(i\pm1,k),u}\mathcal{X}^-_{(i,k),t} + (\mathcal{X}^-_{(i,k),t})^2\mathcal{X}^-_{(i\pm1,k),u} = 0.$$

By (R4) and (R5), we have

$$[\mathcal{I}^+_{(j,l),1}, \mathcal{X}^\pm_{(i,k),t}] = [\mathcal{I}^-_{(j,l),1}, \mathcal{X}^\pm_{(i,k),t}] = \pm a_{(i,k)(j,l)}\mathcal{X}^\pm_{(i,k),t+1}. \tag{4.4.4}$$

By induction on s using (R6), for $s \geq 1$ we can show that

$$\begin{aligned}
[\mathcal{I}^\pm_{(j,l),s}, \mathcal{X}^+_{(i,k),t}] &= a_{(i,k)(j,l)}q^{\pm a_{(i,k)(j,l)}(s-1)}\mathcal{X}^+_{(i,k),t+s} \\
&\quad \pm a_{(i,k)(j,l)}(q-q^{-1})\sum_{p=1}^{s-1} q^{\pm a_{(i,k)(j,l)}(p-1)}\mathcal{X}^+_{(i,k),t+p}\mathcal{I}^\pm_{(j,l),s-p} \\
&= a_{(i,k)(j,l)}q^{\mp a_{(i,k)(j,l)}(s-1)}\mathcal{X}^+_{(i,k),t+s} \\
&\quad \pm a_{(i,k)(j,l)}(q-q^{-1})\sum_{p=1}^{s-1} q^{\mp a_{(i,k)(j,l)}(p-1)}\mathcal{I}^\pm_{(j,l),s-p}\mathcal{X}^+_{(i,k),t+p},
\end{aligned} \tag{4.4.5}$$

and

$$(4.4.6)\qquad \begin{aligned}&[\mathcal{I}^{\pm}_{(j,l),s}, \mathcal{X}^{-}_{(i,k),t}]\\ &\quad = -a_{(i,k)(j,l)} q^{\mp a_{(i,k)(j,l)}(s-1)} \mathcal{X}^{-}_{(i,k),t+s}\\ &\qquad \mp a_{(i,k)(j,l)}(q-q^{-1}) \sum_{p=1}^{s-1} q^{\mp a_{(i,k)(j,l)}(p-1)} \mathcal{X}^{-}_{(i,k),t+p} \mathcal{I}^{\pm}_{(j,l),s-p}\\ &\quad = -a_{(i,k)(j,l)} q^{\pm a_{(i,k)(j,l)}(s-1)} \mathcal{X}^{-}_{(i,k),t+s}\\ &\qquad \mp a_{(i,k)(j,l)}(q-q^{-1}) \sum_{p=1}^{s-1} q^{\pm a_{(i,k)(j,l)}(p-1)} \mathcal{I}^{\pm}_{(j,l),s-p} \mathcal{X}^{-}_{(i,k),t+p}.\end{aligned}$$

4.5. Let $\mathcal{U}^+ = \mathcal{U}^+_{q,\mathbf{Q}}(\mathbf{m})$, $\mathcal{U}^- = \mathcal{U}^-_{q,\mathbf{Q}}(\mathbf{m})$ and $\mathcal{U}^0 = \mathcal{U}^0_{q,\mathbf{Q}}(\mathbf{m})$ be the subalgebras of $\mathcal{U}$ generated by

$$\{\mathcal{X}^{+}_{(i,k),t} \mid (i,k) \in \Gamma'(\mathbf{m}),\ t \geq 0\}, \qquad \{\mathcal{X}^{-}_{(i,k),t} \mid (i,k) \in \Gamma'(\mathbf{m}),\ t \geq 0\} \quad \text{and}$$
$$\{\mathcal{I}^{\pm}_{(j,l),t},\ \mathcal{K}^{\pm}_{(j,l)} \mid (j,l) \in \Gamma(\mathbf{m}),\ t \geq 0\}$$

respectively. Then, we have the following triangular decomposition of $\mathcal{U}$ from the relations (R1)–(R8), (4.4.5) and (4.4.6).

Proposition 4.6.

$$(4.6.1)\qquad \mathcal{U} = \mathcal{U}^- \mathcal{U}^0 \mathcal{U}^+.$$

Remark 4.7. We conjecture that the multiplication map $\mathcal{U}^- \otimes_{\mathbb{K}} \mathcal{U}^0 \otimes_{\mathbb{K}} \mathcal{U}^+ \to \mathcal{U}$ ($x \otimes y \otimes z \mapsto xyz$) is an isomorphism of vector spaces. More precisely, we expect the existence of a PBW type basis of $\mathcal{U}$ (cf. Proposition 2.6 and (4.11.2) with Remark 4.12).

4.8. We have some relations between the algebra $\mathcal{U}$ and a quantum group associated with the general linear Lie algebra.

Let $U_q(\mathfrak{gl}_m)$ be the quantum group associated with the general linear Lie algebra $\mathfrak{gl}_m$ over $\mathbb{K}$. Namely, $U_q(\mathfrak{gl}_m)$ is the associative algebra over $\mathbb{K}$ generated by e_i, f_i $(1 \leq i \leq m-1)$ and $K^{\pm}_j$ $(1 \leq j \leq m)$ with the following defining relations:

(Q1) $\quad K^+_i K^+_j = K^+_j K^+_i, \qquad K^+_i K^-_i = K^-_i K^+_i = 1,$

(Q2) $\quad K^+_j e_i K^-_j = q^{a_{ij}} e_i, \qquad K^+_j f_i K^-_j = q^{-a_{ij}} f_i, \qquad$ where $a_{ij} = \langle \alpha_i, h_j \rangle$,

(Q3) $\quad e_i f_j - f_j e_i = \delta_{i,j} \dfrac{K^+_i K^-_{i+1} - K^-_i K^+_{i+1}}{q - q^{-1}},$

(Q4) $\quad e_{i\pm1} e_i^2 - (q+q^{-1}) e_i e_{i\pm1} e_i + e_i^2 e_{i\pm1} = 0, \qquad e_i e_j = e_j e_i \quad (|i-j| \geq 2),$

(Q5) $\quad f_{i\pm1} f_i^2 - (q+q^{-1}) f_i f_{i\pm1} f_i + f_i^2 f_{i\pm1} = 0, \qquad f_i f_j = f_j f_i \quad (|i-j| \geq 2).$

Let $U_q(\mathfrak{gl}_{m_1}\oplus\cdots\oplus\mathfrak{gl}_{m_r})\cong U_q(\mathfrak{gl}_{m_1})\otimes\cdots\otimes U_q(\mathfrak{gl}_{m_r})$ be the Levi subalgebra of $U_q(\mathfrak{gl}_m)$ associated with the Levi subalgebra $\mathfrak{gl}_{m_1}\oplus\cdots\oplus\mathfrak{gl}_{m_r}$ of $\mathfrak{gl}_m$. Then generators of $U_q(\mathfrak{gl}_{m_1}\oplus\cdots\oplus\mathfrak{gl}_{m_r})$ are given by $e_{(i,k)}, f_{(i,k)}$ $(1\le i\le m_k-1, 1\le k\le r)$ and $K^{\pm}_{(j,l)}$ $((j,l)\in\Gamma(\mathbf{m}))$, where we use the identification (1.3.1) for indices.

Proposition 4.9. (i) *There exists a surjective homomorphism of algebras*

$$g:\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})\to U_q(\mathfrak{gl}_m) \tag{4.9.1}$$

such that

$$g(\mathcal{X}^+_{(i,k),0})=\begin{cases}e_{(i,k)} & \text{if } i\ne m_k,\\ -Q_k e_{(m_k,k)} & \text{if } i=m_k,\end{cases}\qquad g(\mathcal{X}^-_{(i,k),0})=f_{(i,k)},$$
$$g(\mathcal{K}^{\pm}_{(j,l)})=K^{\pm}_{(j,l)},\qquad g(\mathcal{X}^{\pm}_{(i,k),t})=g(\mathcal{I}^{\pm}_{(j,l),t})=0\quad \text{for } t\ge 1.$$

(ii) *There exists an injective homomorphism of algebras*

$$\iota:U_q(\mathfrak{gl}_{m_1}\oplus\cdots\oplus\mathfrak{gl}_{m_r})\to\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m}) \tag{4.9.2}$$

such that $\iota(e_{(i,k)})=\mathcal{X}^+_{(i,k),0}$, $\iota(f_{(i,k)})=\mathcal{X}^-_{(i,k),0}$ *and* $\iota(K^{\pm}_{(j,l)})=\mathcal{K}^{\pm}_{(j,l)}$.

Proof. We can check that g and ι are well-defined by direct calculations. Clearly g is surjective. Let $\iota':U_q(\mathfrak{gl}_{m_1}\oplus\cdots\oplus\mathfrak{gl}_{m_r})\to U_q(\mathfrak{gl}_m)$ be the natural embedding. Then, by investigating the images of generators, we see that $\iota'=g\circ\iota$. This implies that ι is injective. □

Remark 4.10. The surjective homomorphism g in (4.9.1) can be regarded as a special case of evaluation homomorphisms. However, we cannot define evaluation homomorphisms for $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ in general although we can consider $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$-modules corresponding to some evaluation modules. They will be studied in a subsequent paper.

4.11. Let $\mathcal{U}^{\star}_{\mathbb{A}}=\mathcal{U}^{\star}_{\mathbb{A},q,\mathbf{Q}}(\mathbf{m})$ be the $\mathbb{A}$-subalgebra of $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ generated by

$$\{\mathcal{X}^{\pm}_{(i,k),t},\ \mathcal{I}^{\pm}_{(j,l),t},\ \mathcal{K}^{\pm}_{(j,l)}\mid (i,k)\in\Gamma'(\mathbf{m}),\ (j,l)\in\Gamma(\mathbf{m}),\ t\ge 0\}.$$

Then $\mathcal{U}^{\star}_{\mathbb{A}}$ is an associative algebra over $\mathbb{A}$ generated by the same generators with the defining relations (R1)–(R8). We regard $\mathbb{Q}(\mathbf{Q})$ as an $\mathbb{A}$-module through the ring homomorphism $\mathbb{A}\to\mathbb{Q}(\mathbf{Q})$ $(q\mapsto 1)$, and we consider the specialization $\mathbb{Q}(\mathbf{Q})\otimes_{\mathbb{A}}\mathcal{U}^{\star}_{\mathbb{A}}$ using this ring homomorphism. Let $\mathfrak{J}$ be the ideal of $\mathbb{Q}(\mathbf{Q})\otimes_{\mathbb{A}}\mathcal{U}^{\star}_{\mathbb{A}}$ generated by

$$\{\mathcal{K}^+_{(j,l)}-1,\ \mathcal{I}^+_{(j,l),t}-\mathcal{I}^-_{(j,l),t}\mid (i,l)\in\Gamma(\mathbf{m}),\ t\ge 0\}. \tag{4.11.1}$$

Let $U(\mathfrak{g}_{\mathbf{Q}}(\mathbf{m}))$ be the universal enveloping algebra of the Lie algebra $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$ defined in Definition 2.2. Then we can check that there exists a surjective homomorphism of algebras

$$U(\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})) \to \mathbb{Q}(\mathbf{Q}) \otimes_{\mathbb{A}} \mathcal{U}^{\star}_{\mathbb{A},q,\mathbf{Q}}(\mathbf{m})/\mathfrak{J} \tag{4.11.2}$$

such that $\mathcal{X}^{\pm}_{(i,k),t} \mapsto \mathcal{X}^{\pm}_{(i,k),t}$ and $\mathcal{I}_{(j,l),t} \mapsto \mathcal{I}^{+}_{(j,l),t}$ $(= \mathcal{I}^{-}_{(j,l),t})$.

Remark 4.12. We conjecture that (4.11.2) is an isomorphism, so we may regard $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ as a q-analogue of $U(\mathfrak{g}_{\mathbf{Q}}(\mathbf{m}))$.

We also remark that $(\mathcal{K}^{+}_{(j,l)})^2 = 1$ in $\mathcal{U}^{\star}_{\mathbb{A}}$ by (R1). On the other hand, there exists an algebra automorphism of $\mathcal{U}$ such that $\mathcal{K}^{\pm}_{(j,l)} \mapsto -\mathcal{K}^{\pm}_{(j,l)}$ and the other generators map to the same generators. Thus, the choice of signs for $\mathcal{K}^{+}_{(j,l)}$ in (4.11.1) will not cause any troubles.

4.13. To end this section, we define the $\mathbb{A}$-form of $\mathcal{U}$ involving divided powers.

For $(i,k) \in \Gamma'(\mathbf{m})$ and $t, d \in \mathbb{Z}_{\geq 0}$, set

$$\mathcal{X}^{\pm(d)}_{(i,k),t} = (\mathcal{X}^{\pm}_{(i,k),t})^d/[d]! \in \mathcal{U}.$$

For $(j,l) \in \Gamma(\mathbf{m})$ and $d \in \mathbb{Z}_{\geq 0}$, write

$$\begin{bmatrix} \mathcal{K}_{(j,l)}; 0 \\ d \end{bmatrix} = \prod_{b=1}^{d} \frac{\mathcal{K}^{+}_{(j,l)} q^{-b+1} - \mathcal{K}^{-}_{(j,l)} q^{b-1}}{q^b - q^{-b}} \in \mathcal{U}.$$

Let $\mathcal{U}_{\mathbb{A}} = \mathcal{U}_{\mathbb{A},q,\mathbf{Q}}(\mathbf{m})$ be the $\mathbb{A}$-subalgebra of $\mathcal{U}$ generated by all $\mathcal{X}^{\pm(d)}_{(i,k),t}$, $\mathcal{I}^{\pm}_{(j,l),t}$, $\mathcal{K}^{\pm}_{(j,l)}$ and $\left[\begin{smallmatrix} \mathcal{K}_{(j,l)};0 \\ d \end{smallmatrix}\right]$.

§5. Representations of $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$

Thanks to the triangular decomposition (4.6.1) of $\mathcal{U} = \mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$, we can develop a weight theory to study $\mathcal{U}$-modules in the usual manner.

5.1. Highest weight modules. For $\lambda \in P$ and a multiset $\boldsymbol{\varphi} = (\varphi^{\pm}_{(j,l),t} \mid (j,l) \in \Gamma(\mathbf{m}),\ t \geq 1)$ $(\varphi^{\pm}_{(j,l),t} \in \mathbb{K})$, we say that a $\mathcal{U}$-module M is a *highest weight module of highest weight* $(\lambda, \boldsymbol{\varphi})$ if there exists $v_0 \in M$ such that:

(i) M is generated by v_0 as a $\mathcal{U}$-module,

(ii) $\mathcal{X}^{+}_{(i,k),t} \cdot v_0 = 0$ for all $(i,k) \in \Gamma'(\mathbf{m})$ and $t \geq 0$,

(iii) $\mathcal{K}^{+}_{(j,l)} \cdot v_0 = q^{\langle \lambda, h_{(j,l)} \rangle} v_0$ and $\mathcal{I}^{\pm}_{(j,l),t} \cdot v_0 = \varphi^{\pm}_{(j,l),t} v_0$ for $(j,l) \in \Gamma(\mathbf{m})$ and $t \geq 1$.

If $v_0 \in M$ satisfies (ii) and (iii), we say that v_0 is a *maximal vector of weight* $(\lambda, \boldsymbol{\varphi})$. In this case, the submodule $\mathcal{U} \cdot v_0$ of M is a highest weight module of highest weight $(\lambda, \boldsymbol{\varphi})$. If a maximal vector $v_0 \in M$ also satisfies (i), we say that v_0 is a *highest weight vector*.

If $v_0 \in M$ is a maximal vector of weight $(\lambda, \boldsymbol{\varphi})$, for $(j,l) \in \Gamma(\mathbf{m})$ we have

$$\mathcal{I}^{\pm}_{(j,l),0} \cdot v = q^{\mp \lambda_{(j,l)}}[\lambda_{(j,l)}]v, \quad \text{where} \quad \lambda_{(j,l)} = \langle \lambda, h_{(j,l)} \rangle,$$

by (R1).

For a highest weight $\mathcal{U}$-module M of highest weight $(\lambda, \boldsymbol{\varphi})$ with a highest weight vector $v_0 \in M$, we have $M = \mathcal{U}^- \cdot v_0$ by the triangular decomposition (4.6.1). Thus, the relation (R3) implies the weight space decomposition

$$M = \bigoplus_{\substack{\mu \in P \\ \mu \leq \lambda}} M_\mu \quad \text{such that} \quad \dim_{\mathbb{K}} M_\lambda = 1, \tag{5.1.1}$$

where $M_\mu = \{ v \in M \mid \mathcal{K}^+_{(j,l)} \cdot v = q^{\langle \mu, h_{(j,l)} \rangle} v \text{ for } (j,l) \in \Gamma(\mathbf{m}) \}$.

5.2. Verma modules. Let $\widetilde{\mathcal{U}}^0$ be the associative algebra over $\mathbb{K}$ generated by $\mathcal{I}^{\pm}_{(j,l),t}$ and $\mathcal{K}^{\pm}_{(j,l)}$ for all $(j,l) \in \Gamma(\mathbf{m})$ and $t \geq 0$ with the defining relations (R1) and (R2). We also define the associative algebra $\widetilde{\mathcal{U}}^{\geq 0}$ generated by $\mathcal{X}^+_{(i,k),t}$, $\mathcal{I}^{\pm}_{(j,l),t}$ and $\mathcal{K}^{\pm}_{(j,l)}$ for all $(i,k) \in \Gamma'(\mathbf{m})$, $(j,l) \in \Gamma(\mathbf{m})$ and $t \geq 0$ with the defining relations (R1)–(R8) except (R6). Then we have a homomorphism of algebras

$$\widetilde{\mathcal{U}}^{\geq 0} \to \mathcal{U} \quad \text{such that} \quad \mathcal{X}^+_{(i,k),t} \mapsto \mathcal{X}^+_{(i,k),t},\ \mathcal{I}^{\pm}_{(j,l),t} \mapsto \mathcal{I}^{\pm}_{(j,l),t}, \tag{5.2.1}$$

and a surjective homomorphism of algebras

$$\widetilde{\mathcal{U}}^{\geq 0} \to \widetilde{\mathcal{U}}^0 \quad \text{such that} \quad \mathcal{X}^+_{(i,k),t} \mapsto 0,\ \mathcal{I}^{\pm}_{(j,l),t} \mapsto \mathcal{I}^{\pm}_{(j,l),t},\ \mathcal{K}^{\pm}_{(j,l)} \mapsto \mathcal{K}^{\pm}_{(j,l)}. \tag{5.2.2}$$

For $\lambda \in P$ and a multiset $\boldsymbol{\varphi} = (\varphi^{\pm}_{(j,l),t})$, we define a (1-dimensional) simple $\widetilde{\mathcal{U}}^0$-module $\Theta_{(\lambda,\boldsymbol{\varphi})} = \mathbb{K} v_0$ by

$$\mathcal{K}^+_{(j,l)} \cdot v_0 = q^{\langle \lambda, h_{(j,l)} \rangle} v_0, \qquad \mathcal{I}^{\pm}_{(j,l),t} \cdot v_0 = \varphi^{\pm}_{(j,l),t} v_0$$

for $(j,l) \in \Gamma(\mathbf{m})$ and $t \geq 1$. Then we define the *Verma module* $M(\lambda, \boldsymbol{\varphi})$ as the induced module

$$M(\lambda, \boldsymbol{\varphi}) = \mathcal{U} \otimes_{\widetilde{\mathcal{U}}^{\geq 0}} \Theta_{(\lambda,\boldsymbol{\varphi})},$$

where we regard $\Theta_{(\lambda,\boldsymbol{\varphi})}$ (resp. $\mathcal{U}$) as a left (resp. right) $\widetilde{\mathcal{U}}^{\geq 0}$-module through the homomorphism (5.2.2) (resp. (5.2.1)).

By the definitions, the Verma module $M(\lambda, \boldsymbol{\varphi})$ is a highest weight module of highest weight $(\lambda, \boldsymbol{\varphi})$ with a highest weight vector $1 \otimes v_0$. Every highest weight module of highest weight $(\lambda, \boldsymbol{\varphi})$ is a quotient of $M(\lambda, \boldsymbol{\varphi})$, by the universality of tensor products. Moreover, $M(\lambda, \boldsymbol{\varphi})$ has the unique simple top $L(\lambda, \boldsymbol{\varphi}) = M(\lambda, \boldsymbol{\varphi}) / \operatorname{rad} M(\lambda, \boldsymbol{\varphi})$ from the weight space decomposition (5.1.1).

By using the homomorphism $\iota : U_q(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}) \to \mathcal{U}$ of (4.9.2), we get the following necessary condition for $L(\lambda, \boldsymbol{\varphi})$ to be finite-dimensional, in a similar way to Proposition 3.4.

Proposition 5.3. *For $\lambda \in P$ and a multiset $\boldsymbol{\varphi} = (\varphi^{\pm}_{(j,l),t})$, if $L(\lambda, \boldsymbol{\varphi})$ is finite-dimensional, then $\lambda \in P^+_{\mathbf{m}}$.*

5.4. The category $\mathcal{C}_{q,\mathbf{Q}}(\mathbf{m})$. Let $\mathcal{C}_{q,\mathbf{Q}}(\mathbf{m})$ (resp. $\mathcal{C}^{\geq 0}_{q,\mathbf{Q}}(\mathbf{m})$) be the full subcategory of $\mathcal{U}$-mod consisting of $\mathcal{U}$-modules satisfying the following conditions:

(i) If $M \in \mathcal{C}_{q,\mathbf{Q}}(\mathbf{m})$ (resp. $M \in \mathcal{C}^{\geq 0}_{q,\mathbf{Q}}(\mathbf{m})$), then M is finite-dimensional.

(ii) If $M \in \mathcal{C}_{q,\mathbf{Q}}(\mathbf{m})$ (resp. $M \in \mathcal{C}^{\geq 0}_{q,\mathbf{Q}}(\mathbf{m})$), then M has the weight space decomposition

$$M = \bigoplus_{\lambda \in P} M_\lambda \qquad (\text{resp. } M = \bigoplus_{\lambda \in P_{\geq 0}} M_\lambda),$$

where $M_\lambda = \{v \in M \mid \mathcal{K}^+_{(j,l)} \cdot m = q^{\langle \lambda, h_{(j,l)} \rangle} v \text{ for } (j,l) \in \Gamma(\mathbf{m})\}$,

(iii) If $M \in \mathcal{C}_{q,\mathbf{Q}}(\mathbf{m})$ (resp. $M \in \mathcal{C}^{\geq 0}_{q,\mathbf{Q}}(\mathbf{m})$), then all eigenvalues of the action of $\mathcal{I}^{\pm}_{(j,l),t}$ $((j,l) \in \Gamma(\mathbf{m}),\ t \geq 0)$ on M belong to $\mathbb{K}$.

By the usual argument, we have the following lemma.

Lemma 5.5. *Any simple object in $\mathcal{C}_{q,\mathbf{Q}}(\mathbf{m})$ is a highest weight module.*

By using the surjection $g : \mathcal{U}_{q,\mathbf{Q}}(\mathbf{m}) \to U_q(\mathfrak{gl}_m)$ of (4.9.1), we obtain the following proposition.

Proposition 5.6. *Let $\mathcal{C}_{U_q(\mathfrak{gl}_m)}$ be the category of finite-dimensional $U_q(\mathfrak{gl}_m)$-modules which have a weight space decomposition. Then:*

(i) *$\mathcal{C}_{U_q(\mathfrak{gl}_m)}$ is a full subcategory of $\mathcal{C}_{q,\mathbf{Q}}(\mathbf{m})$ through the surjection (4.9.1).*

(ii) *For $\lambda \in P^+$, the simple highest weight $U_q(\mathfrak{gl}_m)$-module $\Delta_{U_q(\mathfrak{gl}_m)}(\lambda)$ of highest weight λ is the simple highest weight $\mathcal{U}$-module of highest weight $(\lambda, \mathbf{0})$ through the surjection (4.9.1), where $\mathbf{0}$ means $\varphi^{\pm}_{(j,l),t} = 0$ for all $(j,l) \in \Gamma(\mathbf{m})$ and $t \geq 1$.*

§6. Review of cyclotomic q-Schur algebras

In this section, we recall the definition and some fundamental properties of the cyclotomic q-Schur algebra $\mathscr{S}_{n,r}(\mathbf{m})$ introduced in [DJM]. See [DJM] and [M1] for details.

6.1. Let R be a commutative ring, and take parameters $q, Q_0, Q_1, \ldots, Q_{r-1} \in R$ such that q is invertible in R. The *Ariki–Koike algebra* $\mathscr{H}_{n,r}$ associated with the complex reflection group $\mathfrak{S}_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$ is the associative algebra with 1 over R generated by $T_0, T_1, \dots, T_{n-1}$ with the following defining relations:

$$\begin{aligned}&(T_0-Q_0)(T_0-Q_1)\dots(T_0-Q_{r-1})=0, \qquad (T_i-q)(T_i+q^{-1})=0 \quad (1\le i\le n-1),\\ &T_0T_1T_0T_1=T_1T_0T_1T_0, \qquad T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1} \quad (1\le i\le n-2),\\ &T_iT_j=T_jT_i \quad (|i-j|\ge 2).\end{aligned}$$

The subalgebra of $\mathscr{H}_{n,r}$ generated by $T_1, \dots, T_{n-1}$ is isomorphic to the Iwahori–Hecke algebra $\mathscr{H}_n$ associated with the symmetric group $\mathfrak{S}_n$. For $w \in \mathfrak{S}_n$, we denote by $\ell(w)$ the length of w, and by T_w the standard basis of $\mathscr{H}_n$ corresponding to w.

6.2. Set $L_1 = T_0$ and $L_i = T_{i-1}L_{i-1}T_{i-1}$ for $i = 2, \dots, n$. Then $L_1, \dots, L_n$ are called the *Jucys–Murphy elements* of $\mathscr{H}_{n,r}$ (see [M2] for their properties). The following lemma is well-known, and one can easily check it from the defining relations of $\mathscr{H}_{n,r}$.

Lemma 6.3. (i) *L_i and L_j commute with each other for any $1 \le i, j \le n$.*
(ii) *T_i and L_j commute with each other if $j \ne i, i+1$.*
(iii) *T_i commutes with both L_iL_{i+1} and $L_i + L_{i+1}$ for any $1 \le i \le n-1$.*
(iv) *$L_{i+1}^tT_i = (q-q^{-1})\sum_{s=0}^{t-1} L_{i+1}^{t-s}L_i^s + T_iL_i^t$ for any $1 \le i \le n-1$ and $t \ge 1$.*
(v) *$L_i^tT_i = -(q-q^{-1})\sum_{s=1}^{t} L_i^{t-s}L_{i+1}^s + T_iL_{i+1}^t$ for any $1 \le i \le n-1$ and $t \ge 1$.*

6.4. Let $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}_{>0}^r$ be an r-tuple of positive integers. Set

$$\Lambda_{n,r}(\mathbf{m}) = \left\{ \mu = (\mu^{(1)}, \dots, \mu^{(r)}) \;\middle|\; \begin{matrix} \mu^{(k)} = (\mu_1^{(k)}, \dots, \mu_{m_k}^{(k)}) \in \mathbb{Z}_{\ge 0}^{m_k} \\ \sum_{k=1}^r \sum_{i=1}^{m_k} \mu_i^{(k)} = n \end{matrix} \right\}.$$

$$\Lambda^+_{n,r}(\mathbf{m}) = \{\mu \in \Lambda_{n,r}(\mathbf{m}) \mid \mu_1^{(k)} \ge \dots \ge \mu_{m_k}^{(k)} \ge 0 \text{ for each } k = 1, \dots, r\}.$$

We regard $\Lambda_{n,r}(\mathbf{m})$ as a subset of the weight lattice $P = \bigoplus_{(i,k)\in\Gamma(\mathbf{m})} \mathbb{Z}\varepsilon_{(i,k)}$ via the injection $\Lambda_{n,r}(\mathbf{m}) \to P$ such that $\mu \mapsto \sum_{(i,k)\in\Gamma(\mathbf{m})} \mu_i^{(k)}\varepsilon_{(i,k)}$. Then we see that $\Lambda^+_{n,r}(\mathbf{m}) = \Lambda_{n,r}(\mathbf{m}) \cap P^+_{\mathbf{m}}$.

For $\mu \in \Lambda_{n,r}(\mathbf{m})$, set

$$m_\mu = \Big(\sum_{w\in\mathfrak{S}_\mu} q^{\ell(w)}T_w\Big)\Big(\prod_{k=1}^{r-1}\prod_{i=1}^{a_k}(L_i - Q_k)\Big), \tag{6.4.1}$$

where $\mathfrak{S}_\mu$ is the Young subgroup of $\mathfrak{S}_n$ corresponding to μ, and $a_k = \sum_{j=1}^k |\mu^{(j)}|$. The following fact is well known:

$$m_\mu T_w = q^{\ell(w)} m_\mu \quad \text{if } w \in \mathfrak{S}_\mu. \tag{6.4.2}$$

The *cyclotomic q-Schur algebra* $\mathscr{S}_{n,r}(\mathbf{m})$ associated with $\mathscr{H}_{n,r}$ is defined by

(6.4.3)
$$\mathscr{S}_{n,r}(\mathbf{m}) = \mathrm{End}_{\mathscr{H}_{n,r}}\Big(\bigoplus_{\mu\in\Lambda_{n,r}(\mathbf{m})} m_\mu\mathscr{H}_{n,r}\Big).$$

For convenience, we set $m_\mu = 0$ for $\mu \in P \setminus \Lambda_{n,r}(\mathbf{m})$.

6.5. Write $\widetilde{\Lambda}^+_{n,r}(\mathbf{m}) = \Lambda^+_{n,r}((n,\dots,n,m_r))$. It is clear that $\widetilde{\Lambda}^+_{n,r}(\mathbf{m}) = \Lambda^+_{n,r}(\mathbf{m})$ if $m_k \geq n$ for all $k = 1,\dots,r-1$. If $m_k < n$ for some $k < r$, then $\Lambda^+_{n,r}(\mathbf{m})$ is a proper subset of $\widetilde{\Lambda}^+_{n,r}(\mathbf{m})$.

In [DJM] (see also [M1] for the case where $m_k < n$ for some k), it is proven that $\mathscr{S}_{n,r}(\mathbf{m})$ is a cellular algebra with respect to the poset $(\widetilde{\Lambda}^+_{n,r}, \geq)$. For $\lambda \in \widetilde{\Lambda}^+_{n,r}(\mathbf{m})$, let $\Delta(\lambda)$ be the Weyl (cell) module corresponding to λ constructed in [DJM] (see also [M1] and [W3, Lemma 1.18]). By the general theory of cellular algebras in [GL], $\{\Delta(\lambda) \mid \lambda \in \widetilde{\Lambda}^+_{n,r}(\mathbf{m})\}$ gives a complete set of representatives of isomorphism classes of simple $\mathscr{S}_{n,r}(\mathbf{m})$-modules if $\mathscr{S}_{n,r}(\mathbf{m})$ is semisimple. It is also proven in [DJM] that $\mathscr{S}_{n,r}(\mathbf{m})$ is a quasi-hereditary algebra such that $\{\Delta(\lambda) \mid \lambda \in \Lambda^+_{n,r}(\mathbf{m})\}$ is a complete set of (representatives of isomorphism classes of) standard modules if R is a field and $m_k \geq n$ for all $k = 1,\dots,r-1$.

From the construction of $\Delta(\lambda)$ in [DJM], $\Delta(\lambda)$ has a basis indexed by the set of semistandard tableaux. Since we use them in the later argument, we recall the definition from [DJM].

For $\lambda \in \widetilde{\Lambda}^+_{n,r}(\mathbf{m})$, the *diagram* $[\lambda]$ of λ is the set

$$[\lambda] = \{(i,j,k) \in \mathbb{Z}^3 \mid 1 \leq i \leq m_k,\ 1 \leq j \leq \lambda_i^{(k)},\ 1 \leq k \leq r\}.$$

For $x = (i,j,k) \in [\lambda]$, define

$$\mathrm{res}(x) = q^{2(j-i)}Q_{k-1}.$$

For $\lambda \in \widetilde{\Lambda}^+_{n,r}(\mathbf{m})$ and $\mu \in \Lambda_{n,r}(\mathbf{m})$, a *tableau* of shape λ with weight μ is a map

$$T : [\lambda] \to \{(a,c) \in \mathbb{Z}\times\mathbb{Z} \mid a \geq 1, 1 \leq c \leq r\}$$

such that $\mu_i^{(k)} = \sharp\{x \in [\lambda] \mid T(x) = (i,k)\}$. We define the order on $\mathbb{Z}\times\mathbb{Z}$ by $(a,c) \geq (a',c')$ if either $c > c'$, or $c = c'$ and $a \geq a'$. For a tableau T of shape λ with weight μ, we say that T is *semistandard* if:

(i) Whenever $T((i,j,k)) = (a,c)$, then $k \leq c$.
(ii) $T((i,j,k)) \leq T((i,j+1,k))$ if $(i,j+1,k) \in [\lambda]$.
(iii) $T((i,j,k)) < T((i+1,j,k))$ if $(i+1,j,k) \in [\lambda]$.

For $\lambda \in \widetilde{\Lambda}^{+}_{n,r}(\mathbf{m})$, $\mu \in \Lambda_{n,r}(\mathbf{m})$, we denote by $\mathcal{T}_0(\lambda, \mu)$ the set of semistandard tableaux of shape λ with weight μ. Then, from the cellular basis of $\mathscr{S}_{n,r}(\mathbf{m})$ in [DJM], we see that

$$\{\varphi_T \mid T \in \mathcal{T}_0(\lambda, \mu) \text{ for some } \mu \in \Lambda_{n,r}(\mathbf{m})\}$$

is a basis of $\Delta(\lambda)$. (See [DJM] for the definition of φ_T.)

§7. Generators of cyclotomic q-Schur algebras

In this section, we define some generators of the cyclotomic q-Schur algebra, and we obtain some relations among them which will be used to obtain a homomorphism from $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ in the next section.

7.1. A *partition* λ is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers. For a partition $\lambda = (\lambda_1, \lambda_2, \dots)$, we denote by $\ell(\lambda)$ the *length* of λ, which is the maximal integer l such that $\lambda_l \neq 0$. If $\sum_{i=1}^{\ell(\lambda)} \lambda_i = n$, we write $\lambda \vdash n$. For an integer k and a partition $\lambda \vdash n$ such that $\ell(\lambda) \leq k$, set

$$\mathfrak{S}_k \cdot \lambda = \{(\mu_1, \dots, \mu_k) \in \mathbb{Z}^k_{\geq 0} \mid \mu_i = \lambda_{\sigma(i)},\ \sigma \in \mathfrak{S}_k\}.$$

7.2. For integers $t, k > 0$, we define symmetric polynomials $\Phi^{\pm}_t(x_1, \dots, x_k) \in R[x_1, \dots, x_k]^{\mathfrak{S}_k}$ of degree t in variables $x_1, \dots, x_k$ as

$$\Phi^{\pm}_t(x_1, \dots, x_k) = \sum_{\substack{\lambda \vdash t \\ \ell(\lambda) \leq k}} (1 - q^{\mp 2})^{\ell(\lambda)-1} \mathfrak{m}_\lambda(x_1, \dots, x_k), \tag{7.2.1}$$

where $\mathfrak{m}_\lambda(x_1, \dots, x_k) = \sum_{\mu = (\mu_1, \dots, \mu_k) \in \mathfrak{S}_k \cdot \lambda} x_1^{\mu_1} \dots x_k^{\mu_k}$ is the monomial symmetric polynomial associated with the partition λ. For convenience, we also define

$$\Phi^{\pm}_0(x_1, \dots, x_k) = q^{\mp k \pm 1}[k]. \tag{7.2.2}$$

From the definition, we have

$$\Phi^{\pm}_1(x_1, \dots, x_k) = x_1 + \dots + x_k \quad \text{and} \quad \Phi^{\pm}_t(x_1) = x_1^t. \tag{7.2.3}$$

The polynomials $\Phi^{\pm}_t(x_1, \dots, x_k)$ satisfy the following recursive relations which will be used to calculate some relations between generators of $\mathscr{S}_{n,r}(\mathbf{m})$.

Lemma 7.3. *For $t \geq 0$,*

$$\begin{aligned} \Phi^{\pm}_{t+1}(x_1, \dots, x_k) &= \sum_{s=1}^{k} \Phi^{\pm}_t(x_1, \dots, x_s) x_s - q^{\mp 2} \sum_{s=1}^{k-1} \Phi^{\pm}_t(x_1, \dots, x_s) x_{s+1} \\ &= x_1^{t+1} + \sum_{s=2}^{k} \big(\Phi^{\pm}_t(x_1, \dots, x_s) x_s - q^{\mp 2} \Phi^{\pm}_t(x_1, \dots, x_{s-1}) x_s\big) \end{aligned} \tag{7.3.1}$$

and

$$\Phi^{\pm}_{t+1}(x_1,\dots,x_k)-\Phi^{\pm}_{t+1}(x_2,\dots,x_k) = x_1\big(\Phi^{\pm}_{t}(x_1,\dots,x_k)-q^{\mp 2}\Phi^{\pm}_{t}(x_2,\dots,x_k)\big). \tag{7.3.2}$$

Proof. For $t=0$, we can check the statements by direct calculations.

Assume that $t\geq 1$. From the definition, we have

$$\begin{aligned}
\Phi^{\pm}_{t+1}(x_1,\dots,x_k) &= \sum_{\substack{\lambda\vdash t+1\\ \ell(\lambda)\leq k}}(1-q^{\mp 2})^{\ell(\lambda)-1}\sum_{\mu\in\mathfrak{S}_k\lambda}x_1^{\mu_1}\dots x_k^{\mu_k}\\
&= \sum_{s=1}^{k}\sum_{\substack{\lambda\vdash t+1\\ \ell(\lambda)\leq s}}(1-q^{\mp 2})^{\ell(\lambda)-1}\sum_{\substack{\mu\in\mathfrak{S}_s\lambda\\ \mu_s\neq 0}}x_1^{\mu_1}\dots x_s^{\mu_s}\\
&= \sum_{s=1}^{k}\sum_{\substack{\lambda\vdash t+1\\ \ell(\lambda)\leq s}}(1-q^{\mp 2})^{\ell(\lambda)-1}\sum_{\substack{\mu\in\mathfrak{S}_s\lambda\\ \mu_s=1}}x_1^{\mu_1}\dots x_s^{\mu_s}\\
&\quad+\sum_{s=1}^{k}\sum_{\substack{\lambda\vdash t+1\\ \ell(\lambda)\leq s}}(1-q^{\mp 2})^{\ell(\lambda)-1}\sum_{\substack{\mu\in\mathfrak{S}_s\lambda\\ \mu_s\geq 2}}x_1^{\mu_1}\dots x_s^{\mu_s}\\
&= \sum_{s=1}^{k}\sum_{\substack{\lambda\vdash t\\ \ell(\lambda)\leq s}}(1-q^{\mp 2})^{\ell(\lambda)}\sum_{\substack{\mu\in\mathfrak{S}_s\lambda\\ \mu_s=0}}x_1^{\mu_1}\dots x_{s-1}^{\mu_{s-1}}x_s\\
&\quad+\sum_{s=1}^{k}\sum_{\substack{\lambda\vdash t\\ \ell(\lambda)\leq s}}(1-q^{\mp 2})^{\ell(\lambda)-1}\sum_{\substack{\mu\in\mathfrak{S}_s\lambda\\ \mu_s\neq 0}}x_1^{\mu_1}\dots x_s^{\mu_s}x_s\\
&= \sum_{s=1}^{k}\Big(\sum_{\substack{\lambda\vdash t\\ \ell(\lambda)\leq s}}(1-q^{\mp 2})^{\ell(\lambda)-1}\sum_{\mu\in\mathfrak{S}_s\lambda}x_1^{\mu_1}\dots x_s^{\mu_s}\Big)x_s\\
&\quad-q^{\mp 2}\sum_{s=2}^{k}\Big(\sum_{\substack{\lambda\vdash t\\ \ell(\lambda)\leq s}}(1-q^{\mp 2})^{\ell(\lambda)-1}\sum_{\substack{\mu\in\mathfrak{S}_s\lambda\\ \mu_s=0}}x_1^{\mu_1}\dots x_{s-1}^{\mu_{s-1}}\Big)x_s\\
&= \sum_{s=1}^{k}\Phi^{\pm}_{t}(x_1,\dots,x_s)x_s-q^{\mp 2}\sum_{s=1}^{k-1}\Phi^{\pm}_{t}(x_1,\dots,x_s)x_{s+1}.
\end{aligned}$$

We can easily check the second equality of (7.3.1).

We prove (7.3.2) by induction on t. For $t=1$, we can check (7.3.2) directly by using (7.3.1) together with (7.2.3). Assume that $t>1$. By (7.3.1),

$$
\begin{aligned}
&\Phi^{\pm}_{t+1}(x_1,\dots,x_k) - \Phi^{\pm}_{t+1}(x_2,\dots,x_k) \\
&\qquad = \Big(\sum_{s=1}^{k} \Phi^{\pm}_t(x_1,\dots,x_s)x_s - q^{\mp 2}\sum_{s=1}^{k-1} \Phi^{\pm}_t(x_1,\dots,x_s)x_{s+1}\Big) \\
&\qquad\quad - \Big(\sum_{s=2}^{k} \Phi^{\pm}_t(x_2,\dots,x_s)x_s - q^{\mp 2}\sum_{s=2}^{k-1} \Phi^{\pm}_t(x_2,\dots,x_s)x_{s+1}\Big) \\
&\qquad = \Phi^{\pm}_t(x_1)x_1 - q^{\mp 2}\Phi^{\pm}_t(x_1)x_2 + \sum_{s=2}^{k}\big(\Phi^{\pm}_t(x_1,\dots,x_s) - \Phi^{\pm}_t(x_2,\dots,x_s)\big)x_s \\
&\qquad\quad - q^{\mp 2}\sum_{s=2}^{k-1}\big(\Phi^{\pm}_t(x_1,\dots,x_s) - \Phi^{\pm}_t(x_2,\dots,x_s)\big)x_{s+1}.
\end{aligned}
$$

Applying the inductive assumption, we get

$$
\begin{aligned}
&\Phi^{\pm}_{t+1}(x_1,\dots,x_k) - \Phi^{\pm}_{t+1}(x_2,\dots,x_k) \\
&\qquad = x_1\Phi^{\pm}_{t-1}(x_1)x_1 - q^{\mp 2}x_1\Phi^{\pm}_{t-1}(x_1)x_2 \\
&\qquad\quad + \sum_{s=2}^{k} x_1\big(\Phi^{\pm}_{t-1}(x_1,\dots,x_s) - q^{\mp 2}\Phi^{\pm}_{t-1}(x_2,\dots,x_s)\big)x_s \\
&\qquad\quad - q^{\mp 2}\sum_{s=2}^{k-1} x_1\big(\Phi^{\pm}_{t-1}(x_1,\dots,x_s) - q^{\mp 2}\Phi^{\pm}_{t-1}(x_2,\dots,x_s)\big)x_{s+1} \\
&\qquad = x_1\Big\{\Big(\sum_{s=1}^{k}\Phi^{\pm}_{t-1}(x_1,\dots,x_s)x_s - q^{\mp 2}\sum_{s=1}^{k-1}\Phi^{\pm}_{t-1}(x_1,x_2,\dots,x_s)x_{s+1}\Big) \\
&\qquad\qquad - q^{\mp 2}\Big(\sum_{s=2}^{k}\Phi^{\pm}_{t-1}(x_2,\dots,x_s)x_s - q^{\mp 2}\sum_{s=2}^{k-1}\Phi^{\pm}_{t-1}(x_2,\dots,x_s)x_{s+1}\Big)\Big\}.
\end{aligned}
$$

Applying (7.3.1), we obtain (7.3.2). □

Remark 7.4. At first, the author defined the polynomials $\Phi^{\pm}_t(x_1,\dots,x_k)$ by using (7.3.1) inductively. The definition (7.2.1) was suggested by Tatsuyuki Hikita.

7.5. For $\mu \in \varLambda_{n,r}(\mathbf{m})$ and $(j,l) \in \varGamma(\mathbf{m})$, set

$$N^{\mu}_{(j,l)} = \sum_{c=1}^{l-1}|\mu^{(c)}| + \sum_{p=1}^{j}\mu^{(l)}_p.$$

For $(j,l) \in \varGamma(\mathbf{m})$ and an integer $t \geq 0$, we define elements $\mathcal{K}^{\pm}_{(j,l)}$ and $\mathcal{I}^{\pm}_{(j,l),t}$ of $\mathscr{S}_{(n,r)}(\mathbf{m})$ by

$$\mathcal{K}^{\pm}_{(j,l)}(m_\mu) = q^{\pm\mu^{(l)}_j} m_\mu,$$

$$\mathcal{I}^{+}_{(j,l),t}(m_\mu) = \begin{cases} q^{t-1} m_\mu \Phi^{+}_{t}(L_{N^{\mu}_{(j,l)}}, L_{N^{\mu}_{(j,l)}-1}, \dots, L_{N^{\mu}_{(j,l)}-\mu^{(l)}_{j}+1}) & \text{if } \mu^{(l)}_{j} \neq 0, \\ 0 & \text{if } \mu^{(l)}_{j} = 0, \end{cases}$$

$$\mathcal{I}^{-}_{(j,l),t}(m_\mu) = \begin{cases} q^{-t+1} m_\mu \Phi^{-}_{t}(L_{N^{\mu}_{(j,l)}}, L_{N^{\mu}_{(j,l)}-1}, \dots, L_{N^{\mu}_{(j,l)}-\mu^{(l)}_{j}+1}) & \text{if } \mu^{(l)}_{j} \neq 0, \\ 0 & \text{if } \mu^{(l)}_{j} = 0, \end{cases}$$

for each $\mu \in \Lambda_{n,r}(\mathbf{m})$.

It is clear that the $\mathcal{K}^{\pm}_{(j,l)}$ are well-defined. For $\mu \in \Lambda_{n,r}(\mathbf{m})$ and $(j,l) \in \Gamma(\mathbf{m})$ such that $\mu^{(l)}_{j} \neq 0$, we see that $\Phi^{\pm}_{t}(L_{N^{\mu}_{(j,l)}}, L_{N^{\mu}_{(j,l)}-1}, \dots, L_{N^{\mu}_{(j,l)}-\mu^{(l)}_{j}+1})$ commutes with T_w for any $w \in \mathfrak{S}_\mu$ by Lemma 6.3 since $\Phi^{\pm}_{t}(L_{N^{\mu}_{(j,l)}}, \dots, L_{N^{\mu}_{(j,l)}-\mu^{(l)}_{j}+1})$ is a symmetric polynomial in the variables $L_{N^{\mu}_{(j,l)}}, L_{N^{\mu}_{(j,l)}-1}, \dots, L_{N^{\mu}_{(j,l)}-\mu^{(l)}_{j}+1}$. Thus, $\Phi^{\pm}_{t}(L_{N^{\mu}_{(j,l)}}, \dots, L_{N^{\mu}_{(j,l)}-\mu^{(l)}_{j}+1})$ commutes with m_μ, and the $\mathcal{I}^{\pm}_{(j,l),t}$ are well-defined.

The following lemma is immediate from the definitions.

Lemma 7.6. *For $(i,k), (j,l) \in \Gamma(\mathbf{m})$ and $s, t \geq 0$, we have:*

(i) $\mathcal{K}^{+}_{(j,l)} \mathcal{K}^{-}_{(j,l)} = \mathcal{K}^{-}_{(j,l)} \mathcal{K}^{+}_{(j,l)} = 1$.

(ii) $[\mathcal{K}^{+}_{(i,k)}, \mathcal{K}^{+}_{(j,l)}] = [\mathcal{K}^{+}_{(i,k)}, \mathcal{I}^{\sigma}_{(j,l),t}] = [\mathcal{I}^{\sigma}_{(i,k),s}, \mathcal{I}^{\sigma'}_{(j,l),t}] = 0$ $(\sigma, \sigma' \in \{+,-\})$.

We also have the following lemma by direct calculations.

Lemma 7.7. *For $(j,l) \in \Gamma(\mathbf{m})$, we have*

$$(\mathcal{K}^{\pm}_{(j,l)})^2 = 1 \pm (q - q^{-1}) \mathcal{I}^{\mp}_{(j,l),0}.$$

7.8. For $(i,k) \in \Gamma'(\mathbf{m})$ and an integer $t \geq 0$, we define elements $\widetilde{\mathcal{K}}^{\pm}_{(i,k)}$ and $\mathcal{J}_{(i,k),t}$ of $\mathscr{S}_{n,r}(\mathbf{m})$ by

$$\widetilde{\mathcal{K}}^{\pm}_{(i,k)} = \mathcal{K}^{\pm}_{(i,k)} \mathcal{K}^{\mp}_{(i+1,k)},$$

$$\mathcal{J}_{(i,k),t} = \begin{cases} \mathcal{I}^{+}_{(i,k),0} - \mathcal{I}^{-}_{(i+1,k),0} + (q-q^{-1}) \mathcal{I}^{+}_{(i,k),0} \mathcal{I}^{-}_{(i+1,k),0} & \text{if } t = 0, \\ q^{-t} \mathcal{I}^{+}_{(i,k),t} - q^{t} \mathcal{I}^{-}_{(i+1,k),t} - (q-q^{-1}) \sum_{b=1}^{t-1} q^{-t+2b} \mathcal{I}^{+}_{(i,k),t-b} \mathcal{I}^{-}_{(i+1,k),b} & \text{if } t > 0. \end{cases}$$

Lemma 7.7 has the following corollary.

Corollary 7.9. *For $(i,k) \in \Gamma'(\mathbf{m})$, we have*

$$\mathcal{J}_{(i,k),0} = \mathcal{I}^{+}_{(i,k),0} - (\mathcal{K}^{-}_{(i,k)})^2 \mathcal{I}^{-}_{(i+1,k),0}.$$

7.10. For $N \in \mathbb{Z}_{\geq 0}$ and $\mu \in \mathbb{Z}_{>0}$, write

$$[T;N,\mu]^+ = \begin{cases} 1 + \sum_{h=1}^{\mu-1} q^h T_{N+1}T_{N+2}\dots T_{N+h} & \text{if } N+\mu \leq n, \\ 0 & \text{otherwise,} \end{cases}$$

$$[T;N,\mu]^- = \begin{cases} 1 + \sum_{h=1}^{\mu-1} q^h T_{N-1}T_{N-2}\dots T_{N-h} & \text{if } n \geq N \geq \mu, \\ 0 & \text{otherwise.} \end{cases}$$

For convenience, we set $[T;N,0]^{\pm} = 0$ for any $N \in \mathbb{Z}_{\geq 0}$.

For $N, \mu \in \mathbb{Z}_{\geq 0}$ and $d \in \mathbb{Z}_{>0}$, write

$$\left[\begin{smallmatrix} T;N,\mu \\ d \end{smallmatrix}\right]^+ = [T;N+(d-1),\mu-(d-1)]^+ \dots [T;N+1,\mu-1]^+[T;N,\mu]^+,$$
$$\left[\begin{smallmatrix} T;N,\mu \\ d \end{smallmatrix}\right]^- = [T;N-(d-1),\mu-(d-1)]^- \dots [T;N-1,\mu-1]^-[T;N,\mu]^-.$$

We also set $\left[\begin{smallmatrix} T;N,\mu \\ 0 \end{smallmatrix}\right]^+ = \left[\begin{smallmatrix} T;N,\mu \\ 0 \end{smallmatrix}\right]^- = 1$ for any $N, \mu \in \mathbb{Z}_{\geq 0}$.

For $N \in \mathbb{Z}_{\geq 0}$ and $d \in \mathbb{Z}_{>0}$, write

$$(T;N,d)^+ = \begin{cases} 1 + \sum_{h=1}^{d-1} q^h T_{N+d-h}T_{N+d-(h-1)}\dots T_{N+d-2}T_{N+d-1} & \text{if } N+d \leq n, \\ 0 & \text{otherwise,} \end{cases}$$

$$(T;N,d)^- = \begin{cases} 1 + \sum_{h=1}^{d-1} q^h T_{N-d+h}T_{N-d+(h-1)}\dots T_{N-d+2}T_{N-d+1} & \text{if } n \geq N \geq d, \\ 0 & \text{otherwise.} \end{cases}$$

We also define

$$(T;N,d)^{\pm}! = (T;N,d)^{\pm}(T;N,d-1)^{\pm}\dots(T;N,1)^{\pm}.$$

The following lemma follows from Lemma 6.3 immediately.

Lemma 7.11. *For $N, \mu \in \mathbb{Z}_{\geq 0}$:*

(i) *L_i commutes with $[T;N,\mu]^+$ unless $N+\mu \geq i \geq N+1$.*

(ii) *L_i commutes with $[T;N,\mu]^-$ unless $N \geq i \geq N-\mu+1$.*

Lemma 7.12. (i) *For $N, \mu \in \mathbb{Z}_{\geq 0}$ such that $N+\mu \leq n$ and $\mu \geq 3$, we have*

$$(q^{\mu-2}T_{N+2}T_{N+3}\dots T_{N+\mu-1})(q^{\mu-1}T_{N+1}T_{N+2}\dots T_{N+\mu-1})$$
$$= (q^{\mu-1}T_{N+1}T_{N+2}\dots T_{N+\mu-1})(q^{\mu-2}T_{N+1}T_{N+2}\dots T_{N+\mu-2}).$$

(ii) *For $N, \mu \in \mathbb{Z}_{\geq 0}$ such that $N \geq \mu \geq 3$, we have*

$$(q^{\mu-2}T_{N-2}T_{N-3}\dots T_{N-\mu+1})(q^{\mu-1}T_{N-1}T_{N-2}\dots T_{N-\mu+1}) \\ = (q^{\mu-1}T_{N-1}T_{N-2}\dots T_{N-\mu+1})(q^{\mu-2}T_{N-1}T_{N-2}\dots T_{N-\mu+2}).$$

(iii) *For $N, \mu, c \in \mathbb{Z}_{\geq 0}$ such that $\mu \geq c \geq 1$, we have*

$$[T; N+1, c]^+(q^{\mu}T_{N+1}T_{N+2}\dots T_{N+\mu}) = (q^{\mu}T_{N+1}T_{N+2}\dots T_{N+\mu})[T; N, c]^+,$$
$$[T; N-1, c]^-(q^{\mu}T_{N-1}T_{N-2}\dots T_{N-\mu}) = (q^{\mu}T_{N-1}T_{N-2}\dots T_{N-\mu})[T; N, c]^-.$$

Proof. (i) and (ii) follow from the defining relations of $\mathscr{H}_{n,r}$. We can prove (iii) by induction on c. □

Lemma 7.13. *For $N, \mu \in \mathbb{Z}_{\geq 0}$ and $d \in \mathbb{Z}_{>0}$, we have*

$$\left[\begin{smallmatrix} T;N,\mu \\ d \end{smallmatrix}\right]^+ = \begin{cases} (T;N,d)^+\Big(\left[\begin{smallmatrix} T;N,d-1 \\ d-1 \end{smallmatrix}\right]^+ \\ \quad + \sum_{h=1}^{\mu-d}(q^h T_{N+d}T_{N+d+1}\dots T_{N+d+h-1})\left[\begin{smallmatrix} T;N,d+h-1 \\ d-1 \end{smallmatrix}\right]^+\Big) & \text{if } \mu \geq d, \\ 0 & \text{if } \mu < d, \end{cases}$$

$$\left[\begin{smallmatrix} T;N,\mu \\ d \end{smallmatrix}\right]^- = \begin{cases} (T;N,d)^-\Big(\left[\begin{smallmatrix} T;N,d-1 \\ d-1 \end{smallmatrix}\right]^- \\ \quad + \sum_{h=1}^{\mu-d}(q^h T_{N-d}T_{N-d-1}\dots T_{N-d-h+1})\left[\begin{smallmatrix} T;N,d+h-1 \\ d-1 \end{smallmatrix}\right]^-\Big) & \text{if } \mu \geq d, \\ 0 & \text{if } \mu < d. \end{cases}$$

Proof. In the case where $\mu < d$, we see that $\left[\begin{smallmatrix} T;N,\mu \\ d \end{smallmatrix}\right]^{\pm} = 0$ from the definitions.

First, we prove that if $\mu > d$,

$$\text{(7.13.1)} \qquad \left[\begin{smallmatrix} T;N,\mu \\ d \end{smallmatrix}\right]^+ = \left[\begin{smallmatrix} T;N,\mu-1 \\ d \end{smallmatrix}\right]^+ \\ + (T;N,d)^+(q^{\mu-d}T_{N+d}T_{N+d+1}\dots T_{N+\mu-1})\left[\begin{smallmatrix} T;N,\mu-1 \\ d-1 \end{smallmatrix}\right]^+$$

by induction on d. The case $d = 1$ is clear by definitions. Assume that $d > 1$. Then

$$\left[\begin{smallmatrix} T;N,\mu \\ d \end{smallmatrix}\right]^+ = [T; N+(d-1), \mu-(d-1)]^+\left[\begin{smallmatrix} T;N,\mu \\ d-1 \end{smallmatrix}\right]^+.$$

Applying the inductive assumption, we obtain

$$\left[\begin{smallmatrix} T;N,\mu \\ d \end{smallmatrix}\right]^+ = \big\{[T; N+(d-1), \mu-d]^+ + (q^{\mu-d}T_{N+d}T_{N+d+1}\dots T_{N+\mu-1})\big\} \\ \times \big\{\left[\begin{smallmatrix} T;N,\mu-1 \\ d-1 \end{smallmatrix}\right]^+ + (T;N,d-1)^+(q^{\mu-d+1}T_{N+d-1}T_{N+d}\dots T_{N+\mu-1})\left[\begin{smallmatrix} T;N,\mu-1 \\ d-2 \end{smallmatrix}\right]^+\big\}.$$

Then, by using Lemmas 7.11 and 7.12, we see that

$$\begin{aligned}
&\left[\begin{smallmatrix} T;N,\mu \\ d \end{smallmatrix}\right]^+ \\
&= [T; N+d-1, \mu-d]^+ \left[\begin{smallmatrix} T;N,\mu-1 \\ d-1 \end{smallmatrix}\right]^+ + (q^{\mu-d}T_{N+d}T_{N+d+1}\dots T_{N+\mu-1})\left[\begin{smallmatrix} T;N,\mu-1 \\ d-1 \end{smallmatrix}\right]^+ \\
&+ (T; N, d-1)^+(q^{\mu-d+1}T_{N+d-1}T_{N+d}\dots T_{N+\mu-1})[T; N+d-2, \mu-d]^+\left[\begin{smallmatrix} T;N,\mu-1 \\ d-2 \end{smallmatrix}\right]^+ \\
&+ (T; N, d-1)^+(q^{\mu-d+1}T_{N+d-1}T_{N+d}\dots T_{N+\mu-1})(q^{\mu-d}T_{N+d-1}T_{N+d}\dots T_{N+\mu-2}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \left[\begin{smallmatrix} T;N,\mu-1 \\ d-2 \end{smallmatrix}\right]^+.
\end{aligned}$$

Noting that

$$[T; N+d-2, \mu-d]^+ + q^{\mu-d}T_{N+d-1}T_{N+d}\dots T_{N+\mu-2} = [T; N+d-2, \mu-d+1]^+$$

and $[T; N+d-2, \mu-d+1]^+\left[\begin{smallmatrix} T;N,\mu-1 \\ d-2 \end{smallmatrix}\right]^+ = \left[\begin{smallmatrix} T;N,\mu-1 \\ d-1 \end{smallmatrix}\right]^+$, we have

$$\begin{aligned}
\left[\begin{smallmatrix} T;N,\mu \\ d \end{smallmatrix}\right]^+ &= \left[\begin{smallmatrix} T;N,\mu-1 \\ d \end{smallmatrix}\right]^+ \\
&\quad + (1 + (T; N, d-1)^+(qT_{N+d-1}))(q^{\mu-d}T_{N+d}T_{N+d+1}\dots T_{N+\mu-1})\left[\begin{smallmatrix} T;N,\mu-1 \\ d-1 \end{smallmatrix}\right]^+.
\end{aligned}$$

By definition, we see that $1 + (T; N, d-1)^+(qT_{N+d-1}) = (T; N, d)^+$. This yields (7.13.1).

Next, we prove that

$$\left[\begin{smallmatrix} T;N,d \\ d \end{smallmatrix}\right]^+ = (T; N, d)^+\left[\begin{smallmatrix} T;N,d-1 \\ d-1 \end{smallmatrix}\right]^+ \tag{7.13.2}$$

by induction on d. For $d = 1$, this is clear from the definitions. Assume that $d > 1$. Noting that $\left[\begin{smallmatrix} T;N,d \\ d \end{smallmatrix}\right]^+ = \left[\begin{smallmatrix} T;N,d \\ d-1 \end{smallmatrix}\right]^+$, by (7.13.1), we have

$$\begin{aligned}
\left[\begin{smallmatrix} T;N,d \\ d \end{smallmatrix}\right]^+ &= \left[\begin{smallmatrix} T;N,d-1 \\ d-1 \end{smallmatrix}\right]^+ + (T; N, d-1)^+(qT_{N+d-1})\left[\begin{smallmatrix} T;N,d-1 \\ d-2 \end{smallmatrix}\right]^+ \\
&= \big(1 + (T; N, d-1)^+(qT_{N+d-1})\big)\left[\begin{smallmatrix} T;N,d-1 \\ d-1 \end{smallmatrix}\right]^+ \\
&= (T; N, d)^+\left[\begin{smallmatrix} T;N,d-1 \\ d-1 \end{smallmatrix}\right]^+.
\end{aligned}$$

Next we prove that if $\mu \geq d$, then

$$\begin{aligned}
&\left[\begin{smallmatrix} T;N,\mu \\ d \end{smallmatrix}\right]^+ \\
&= (T; N, d)^+\Big(\left[\begin{smallmatrix} T;N,d-1 \\ d-1 \end{smallmatrix}\right]^+ + \sum_{h=1}^{\mu-d}(q^hT_{N+d}T_{N+d+1}\dots T_{N+d+h-1})\left[\begin{smallmatrix} T;N,d+h-1 \\ d-1 \end{smallmatrix}\right]^+\Big)
\end{aligned} \tag{7.13.3}$$

by induction on $\mu - d$. The case $\mu = d$ is just (7.13.2). Assume that $\mu > d$. By applying the inductive assumption to the right-hand side of (7.13.1), we get (7.13.3).

The case of $\left[\begin{smallmatrix} T;N,\mu \\ d \end{smallmatrix}\right]^-$ is similar. □

We have the following corollary which will be used in Theorem 8.1 to consider divided powers in cyclotomic q-Schur algebras.

Corollary 7.14. *For $N, \mu, d \in \mathbb{Z}_{\geq 0}$, there exists $\mathfrak{H}^{\pm}(N, \mu, d) \in \mathscr{H}_{n,r}$ such that*

$$\left[\begin{smallmatrix} T; N, \mu \\ d \end{smallmatrix}\right]^{\pm} = (T; N, d)^{\pm}! \mathfrak{H}^{\pm}(N, \mu, d).$$

Proof. Noting that $T_{N+d}T_{N+d+1} \dots T_{N+d+h-1}$ (resp. $T_{N-d}T_{N-d-1} \dots T_{N-d-h+1}$) commutes with $(T; N, d-1)^{+}!$ (resp. $(T; N, d-1)^{-}!$), we argue by induction on d using Lemma 7.13. □

7.15. For $(i,k) \in \Gamma'(\mathbf{m})$, we define elements $\mathcal{X}^{+}_{(i,k),0}$ and $\mathcal{X}^{-}_{(i,k),0}$ of $\mathscr{S}_{n,r}(\mathbf{m})$ by

$$\mathcal{X}^{+}_{(i,k),0}(m_\mu) = q^{-\mu^{(k)}_{i+1}+1} m_{\mu+\alpha_{(i,k)}} [T; N^{\mu}_{(i,k)}, \mu^{(k)}_{i+1}]^{+},$$
$$\mathcal{X}^{-}_{(i,k),0}(m_\mu) = q^{-\mu^{(k)}_{i}+1} m_{\mu-\alpha_{(i,k)}} h^{\mu}_{-(i,k)} [T; N^{\mu}_{(i,k)}, \mu^{(k)}_{i}]^{-}$$

for each $\mu \in \Lambda_{n,r}(\mathbf{m})$, where we have written $\mu^{(k)}_{m_k+1} = \mu^{(k+1)}_1$ if $i = m_k$, and

$$h^{\mu}_{-(i,k)} = \begin{cases} 1 & \text{if } i \neq m_k, \\ L_{N^{\mu}_{(m_k,k)}} - Q_k & \text{if } i = m_k. \end{cases}$$

Note that $m_{\mu \pm \alpha_{(i,k)}} = 0$ if $\mu \pm \alpha_{(i,k)} \notin \Lambda_{n,r}(\mathbf{m})$.

By [W1, Lemma 6.10], $\mathcal{X}^{\pm}_{(i,k),0}$ is well-defined (it is denoted by $\varphi^{\pm}_{(i,k)}$ in [W1]).

For $(i,k) \in \Gamma'(\mathbf{m})$ and $t \in \mathbb{Z}_{>0}$, we define $\mathcal{X}^{\pm}_{(i,k),t} \in \mathscr{S}_{n,r}(\mathbf{m})$ inductively by

(7.15.1)
$$\mathcal{X}^{+}_{(i,k),t} = \mathcal{I}^{+}_{(i,k),1} \mathcal{X}^{+}_{(i,k),t-1} - \mathcal{X}^{+}_{(i,k),t-1} \mathcal{I}^{+}_{(i,k),1},$$
$$\mathcal{X}^{-}_{(i,k),t} = -(\mathcal{I}^{-}_{(i,k),1} \mathcal{X}^{-}_{(i,k),t-1} - \mathcal{X}^{-}_{(i,k),t-1} \mathcal{I}^{-}_{(i,k),1}).$$

Lemma 7.16. *For $(i,k) \in \Gamma'(\mathbf{m})$, $(j,l) \in \Gamma(\mathbf{m})$ and $t \geq 0$, we have*

$$\mathcal{K}^{+}_{(j,l)} \mathcal{X}^{\pm}_{(i,k),t} \mathcal{K}^{-}_{(j,l)} = q^{\pm a_{(i,k)(j,l)}} \mathcal{X}^{\pm}_{(i,k),t}.$$

Proof. The case $t = 0$ comes directly from the definitions. Then we argue by induction on t using (7.15.1) together with Lemma 7.6. □

We can describe the elements $\mathcal{X}^{\pm}_{(i,k),t}$ of $\mathscr{S}_{n,r}(\mathbf{m})$ precisely as follows.

Lemma 7.17. *For $(i,k) \in \Gamma'(\mathbf{m})$, $t \geq 0$ and $\mu \in \Lambda_{n,r}(\mathbf{m})$, we have:*

(i) $\mathcal{X}^{+}_{(i,k),t}(m_\mu) = q^{-\mu^{(k)}_{i+1}+1} m_{\mu+\alpha_{(i,k)}} L^{t}_{N^{\mu}_{(i,k)}+1} [T; N^{\mu}_{(i,k)}, \mu^{(k)}_{i+1}]^{+}$.

(ii) $\mathcal{X}^{-}_{(i,k),t}(m_\mu) = q^{-\mu^{(k)}_{i}+1} m_{\mu-\alpha_{(i,k)}} L^{t}_{N^{\mu}_{(i,k)}} h^{\mu}_{-(i,k)} [T; N^{\mu}_{(i,k)}, \mu^{(k)}_{i}]^{-}$.

Proof. We prove (i). We can easily show that $\mathcal{X}^+_{(i,k),t}(m_\mu) = 0$ if $\mu^{(k)}_{i+1} = 0$ by induction on t using (7.15.1). Assume that $\mu^{(k)}_{i+1} \neq 0$. If $t = 0$, then (i) is just the definition of $\mathcal{X}^+_{(i,k),0}$. We now use induction on t. Noting that $(\mu + \alpha_{(i,k)})^{(k)}_i = \mu^{(k)}_i + 1$ and $N^{\mu+\alpha_{(i,k)}}_{(i,k)} = N^\mu_{(i,k)} + 1$, by the inductive assumption we have

$$\begin{aligned}\mathcal{I}^+_{(i,k),1}\mathcal{X}^+_{(i,k),t-1}(m_\mu) &= q^{-\mu^{(k)}_{i+1}+1} m_{\mu+\alpha_{(i,k)}} \\ &\quad\times (L_{N^\mu_{(i,k)}+1} + L_{N^\mu_{(i,k)}} + L_{N^\mu_{(i,k)}-1} + \cdots + L_{N^\mu_{(i,k)}-\mu^{(k)}_i+1}) \\ &\quad\times L^{t-1}_{N^\mu_{(i,k)}+1}[T; N^\mu_{(i,k)}, \mu^{(k)}_{i+1}]^+.\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathcal{X}^+_{(i,k),t-1}\mathcal{I}^+_{(i,k),1}(m_\mu) &= \delta_{(\mu^{(k)}_i \neq 0)} q^{-\mu^{(k)}_{i+1}+1} m_{\mu+\alpha_{(i,k)}} L^{t-1}_{N^\mu_{(i,k)}+1}[T; N^\mu_{(i,k)}, \mu^{(k)}_{i+1}]^+ \\ &\quad\times (L_{N^\mu_{(i,k)}} + L_{N^\mu_{(i,k)}-1} + \cdots + L_{N^\mu_{(i,k)}-\mu^{(k)}_i+1}).\end{aligned}$$

Thus, by (7.15.1) and Lemma 7.11, we obtain (i). The proof of (ii) is similar. □

Proposition 7.18. *For $(i,k),(j,l) \in \Gamma'(\mathbf{m})$ and $s,t \geq 0$, we have:*

(i) $[\mathcal{X}^\pm_{(i,k),t}, \mathcal{X}^\pm_{(j,l),s}] = 0$ *if* $(j,l) \neq (i,k),(i\pm1,k)$.

(ii) $$\mathcal{X}^\pm_{(i,k),t+1}\mathcal{X}^\pm_{(i,k),s} - q^{\pm2}\mathcal{X}^\pm_{(i,k),s}\mathcal{X}^\pm_{(i,k),t+1} = q^{\pm2}\mathcal{X}^\pm_{(i,k),t}\mathcal{X}^\pm_{(i,k),s+1} - \mathcal{X}^\pm_{(i,k),s+1}\mathcal{X}^\pm_{(i,k),t}.$$

(iii) $$\mathcal{X}^+_{(i,k),t+1}\mathcal{X}^+_{(i+1,k),s} - q^{-1}\mathcal{X}^+_{(i+1,k),s}\mathcal{X}^+_{(i,k),t+1} = \mathcal{X}^+_{(i,k),t}\mathcal{X}^+_{(i+1,k),s+1} - q\mathcal{X}^+_{(i+1,k),s+1}\mathcal{X}^+_{(i,k),t},$$
$$\mathcal{X}^-_{(i+1,k),s}\mathcal{X}^-_{(i,k),t+1} - q^{-1}\mathcal{X}^-_{(i,k),t+1}\mathcal{X}^-_{(i+1,k),s} = \mathcal{X}^-_{(i+1,k),s+1}\mathcal{X}^-_{(i,k),t} - q\mathcal{X}^-_{(i,k),t}\mathcal{X}^-_{(i+1,k),s+1}.$$

Proof. (i) follows from Lemma 7.17 using Lemma 6.3.

We prove (ii). We may assume that $t \geq s$ by multiplying by -1 on both sides if necessary. We prove

(7.18.1) $$\mathcal{X}^+_{(i,k),t+1}\mathcal{X}^+_{(i,k),s} - q^2\mathcal{X}^+_{(i,k),s}\mathcal{X}^+_{(i,k),t+1} = q^2\mathcal{X}^+_{(i,k),t}\mathcal{X}^+_{(i,k),s+1} - \mathcal{X}^+_{(i,k),s+1}\mathcal{X}^+_{(i,k),t}.$$

Write $N = N^\mu_{(i,k)}$. By Lemma 7.17 together with Lemma 7.11, for $\mu \in \Lambda_{n,r}(\mathbf{m})$,

(7.18.2) $$\begin{aligned}&\mathcal{X}^+_{(i,k),t+1}\mathcal{X}^+_{(i,k),s}(m_\mu) \\ &\quad= q^{-2\mu^{(k)}_{i+1}+3} m_{\mu+2\alpha_{(i,k)}} L^s_{N+1}L^{t+1}_{N+2}[T; N+1, \mu^{(k)}_{i+1}-1]^+[T; N, \mu^{(k)}_{i+1}]^+.\end{aligned}$$

Thus, we may assume that $\mu_{i+1}^{(k)} \geq 2$ since $m_{\mu+2\alpha_{(i,k)}} = 0$ if $\mu_{i+1}^{(k)} < 2$. By induction on $\mu_{i+1}^{(k)}$, we can show that

$$T_{N+1}[T; N+1, \mu_{i+1}^{(k)} - 1]^{+}[T; N, \mu_{i+1}^{(k)}]^{+} = q[T; N+1, \mu_{i+1}^{(k)} - 1]^{+}[T; N, \mu_{i+1}^{(k)}]^{+}. \tag{7.18.3}$$

We also have, by Lemma 6.3,

$$\begin{aligned} L_{N+1}^{s} L_{N+2}^{t+1} &= (L_{N+1}L_{N+2})^{s}(T_{N+1}L_{N+1}T_{N+1})L_{N+2}^{t-s} \\ &= T_{N+1}(L_{N+1}L_{N+2})^{s}L_{N+1}\Big\{L_{N+1}^{t-s}T_{N+1} + (q-q^{-1})\sum_{p=1}^{t-s} L_{N+1}^{t-s-p}L_{N+2}^{p}\Big\} \\ &= T_{N+1}L_{N+1}^{t+1}L_{N+2}^{s}T_{N+1} + (q-q^{-1})T_{N+1}\sum_{p=1}^{t-s} L_{N+1}^{t-p+1}L_{N+2}^{s+p}. \end{aligned} \tag{7.18.4}$$

Then (7.18.2) follows by using (6.4.2), (7.18.3) and (7.18.4). Moreover

$$\begin{aligned} &\mathcal{X}_{(i,k),t+1}^{+}\mathcal{X}_{(i,k),s}^{+}(m_\mu) \\ &= q^2 q^{-2\mu_{i+1}^{(k)}+3} m_{\mu+2\alpha_{(i,k)}} L_{N+1}^{t+1}L_{N+2}^{s}[T; N+1, \mu_{i+1}^{(k)}-1]^{+}[T; N, \mu_{i+1}^{(k)}]^{+} \\ &+ q(q-q^{-1})q^{-2\mu_{i+1}^{(k)}+3} m_{\mu+2\alpha_{(i,k)}} \sum_{p=1}^{t-s} L_{N+1}^{t-p+1}L_{N+2}^{s+p}[T; N+1, \mu_{i+1}^{(k)}-1]^{+}[T; N, \mu_{i+1}^{(k)}]^{+} \\ &= q^2 \mathcal{X}_{(i,k),s}^{+}\mathcal{X}_{(i,k),t+1}^{+}(m_\mu) \\ &+ q(q-q^{-1})q^{-2\mu_{i+1}^{(k)}+3} m_{\mu+2\alpha_{(i,k)}} \sum_{p=1}^{t-s} L_{N+1}^{t-p+1}L_{N+2}^{s+p}[T; N+1, \mu_{i+1}^{(k)}-1]^{+}[T; N, \mu_{i+1}^{(k)}]^{+}. \end{aligned}$$

Similarly,

$$\begin{aligned} &q^2\mathcal{X}_{(i,k),t}^{+}\mathcal{X}_{(i,k),s+1}^{+}(m_\mu) \\ &= q^{-2\mu_{i+1}^{(k)}+3} m_{\mu+2\alpha_{(i,k)}} T_{N+1}L_{N+1}^{s+1}L_{N+2}^{t}T_{N+1}[T; N+1, \mu_{i+1}^{(k)}-1]^{+}[T; N, \mu_{i+1}^{(k)}]^{+} \\ &= q^{-2\mu_{i+1}^{(k)}+3} m_{\mu+2\alpha_{(i,k)}} L_{N+1}^{t}L_{N+2}^{s+1}[T; N+1, \mu_{i+1}^{(k)}-1]^{+}[T; N, \mu_{i+1}^{(k)}]^{+} \\ &+ q(q-q^{-1})q^{-2\mu_{i+1}^{(k)}+3} m_{\mu+2\alpha_{(i,k)}} \sum_{p=1}^{t-s} L_{N+1}^{t-p+1}L_{N+2}^{s+p}[T; N+1, \mu_{i+1}^{(k)}-1]^{+}[T; N, \mu_{i+1}^{(k)}]^{+} \\ &= \mathcal{X}_{(i,k),s+1}^{+}\mathcal{X}_{(i,k),t}^{+}(m_\mu) \\ &+ q(q-q^{-1})q^{-2\mu_{i+1}^{(k)}+3} m_{\mu+2\alpha_{(i,k)}} \sum_{p=1}^{t-s} L_{N+1}^{t-p+1}L_{N+2}^{s+p}[T; N+1, \mu_{i+1}^{(k)}-1]^{+}[T; N, \mu_{i+1}^{(k)}]^{+}. \end{aligned}$$

Thus, we obtain (7.18.1). The other case of (ii) is proven in a similar way.

We now prove (iii). Write $N = N^{\mu}_{(i,k)}$. When $\mu^{(k)}_{i+1} = 0$, by Lemma 7.17 together with Lemma 7.11 we see that

$$(7.18.5)\qquad \begin{aligned}(\mathcal{X}^+_{(i,k),t+1}\mathcal{X}^+_{(i+1,k),s} - q^{-1}\mathcal{X}^+_{(i+1,k),s}\mathcal{X}^+_{(i,k),t+1})(m_\mu)\\ = (\mathcal{X}^+_{(i,k),t}\mathcal{X}^+_{(i+1,k),s+1} - q\mathcal{X}^+_{(i+1,k),s+1}\mathcal{X}^+_{(i,k),t})(m_\mu)\\ = q^{-\mu^{(k)}_{i+2}+1} m_{\mu+\alpha_{(i,k)}+\alpha_{(i+1,k)}} L^{s+t+1}_{N+1}[T; N, \mu^{(k)}_{i+2}]^+.\end{aligned}$$

Assume now that $\mu^{(k)}_{i+1} \neq 0$. By Lemmas 7.17 and 7.11, we have

$$(7.18.6)\qquad \begin{aligned}&(\mathcal{X}^+_{(i,k),t+1}\mathcal{X}^+_{(i+1,k),s} - q^{-1}\mathcal{X}^+_{(i+1,k),s}\mathcal{X}^+_{(i,k),t+1})(m_\mu)\\ &= q^{-\mu^{(k)}_{i+1}-\mu^{(k)}_{i+2}+1} m_{\mu+\alpha_{(i,k)}+\alpha_{(i+1,k)}} L^{t+1}_{N+1}(q^{\mu^{(k)}_{i+1}} T_{N+1}T_{N+2}\dots T_{N+\mu^{(k)}_{i+1}})L^s_{N+\mu^{(k)}_{i+1}+1}\\ &\qquad \times [T; N+\mu^{(k)}_{i+1}, \mu^{(k)}_{i+2}]^+\end{aligned}$$

and

$$(7.18.7)\qquad \begin{aligned}&(\mathcal{X}^+_{(i,k),t}\mathcal{X}^+_{(i+1,k),s+1} - q\mathcal{X}^+_{(i+1,k),s+1}\mathcal{X}^+_{(i,k),t})(m_\mu)\\ &= -(q-q^{-1})q^{-\mu^{(k)}_{i+1}-\mu^{(k)}_{i+2}+2} m_{\mu+\alpha_{(i,k)}+\alpha_{(i+1,k)}} L^t_{N+1}L^{s+1}_{N+\mu^{(k)}_{i+1}+1}\\ &\qquad \times [T; N, \mu^{(k)}_{i+1}]^+[T; N+\mu^{(k)}_{i+1}, \mu^{(k)}_{i+2}]^+\\ &\quad + q^{-\mu^{(k)}_{i+1}-\mu^{(k)}_{i+2}+1} m_{\mu+\alpha_{(i,k)}+\alpha_{(i+1,k)}} L^t_{N+1}(q^{\mu^{(k)}_{i+1}} T_{N+1}T_{N+2}\dots T_{N+\mu^{(k)}_{i+1}})L^{s+1}_{N+\mu^{(k)}_{i+1}+1}\\ &\qquad \times [T; N+\mu^{(k)}_{i+1}, \mu^{(k)}_{i+2}]^+.\end{aligned}$$

By induction on $\mu^{(k)}_{i+1}$ using Lemma 6.3, we can prove that

$$\begin{aligned}&(T_{N+1}T_{N+2}\dots T_{N+\mu^{(k)}_{i+1}})L_{N+\mu^{(k)}_{i+1}+1}\\ &= L_{N+1}(T_{N+1}T_{N+2}\dots T_{N+\mu^{(k)}_{i+1}}) + \delta_{(\mu^{(k)}_{i+1}\geq 2)}(q-q^{-1})L_{N+2}(T_{N+2}T_{N+3}\dots T_{N+\mu^{(k)}_{i+1}})\\ &\quad + (q-q^{-1})\sum_{p=1}^{\mu^{(k)}_{i+1}-2}(T_{N+1}T_{N+2}\dots T_{N+p})L_{N+p+2}(T_{N+p+2}T_{N+p+3}\dots T_{N+\mu^{(k)}_{i+1}})\\ &\quad + (q-q^{-1})(T_{N+1}T_{N+2}\dots T_{N+\mu^{(k)}_{i+1}-1})L_{N+\mu^{(k)}_{i+1}+1}.\end{aligned}$$

By using Lemma 6.3 and (6.4.2), this equality implies

$$(7.18.8)\qquad \begin{aligned}&m_{\mu+\alpha_{(i,k)}+\alpha_{(i+1,k)}} L^t_{N+1}(q^{\mu^{(k)}_{i+1}} T_{N+1}T_{N+2}\dots T_{N+\mu^{(k)}_{i+1}})L_{N+\mu^{(k)}_{i+1}+1}\\ &= m_{\mu+\alpha_{(i,k)}+\alpha_{(i+1,k)}} L^{t+1}_{N+1}(q^{\mu^{(k)}_{i+1}} T_{N+1}T_{N+2}\dots T_{N+\mu^{(k)}_{i+1}})\\ &\quad + q(q-q^{-1}) m_{\mu+\alpha_{(i,k)}+\alpha_{(i+1,k)}} L^t_{N+1}[T; N, \mu^{(k)}_{i+1}]^+ L_{N+\mu^{(k)}_{i+1}+1}.\end{aligned}$$

Thus, (7.18.7) and (7.18.8) imply

$$(7.18.9)\qquad (\mathcal{X}^+_{(i,k),t}\mathcal{X}^+_{(i+1,k),s+1} - q\mathcal{X}^+_{(i+1,k),s+1}\mathcal{X}^+_{(i,k),t})(m_\mu)$$
$$= q^{-\mu^{(k)}_{i+1}-\mu^{(k)}_{i+2}+1} m_{\mu+\alpha_{(i,k)}+\alpha_{(i+1,k)}} L^{t+1}_{N+1}(q^{\mu^{(k)}_{i+1}} T_{N+1}T_{N+2}\dots T_{N+\mu^{(k)}_{i+1}}) L^s_{N+\mu^{(k)}_{i+1}+1} \times [T; N+\mu^{(k)}_{i+1}, \mu^{(k)}_{i+2}]^+.$$

By (7.18.5), (7.18.6) and (7.18.9), we obtain

$$\mathcal{X}^+_{(i,k),t+1}\mathcal{X}^+_{(i+1,k),s} - q^{-1}\mathcal{X}^+_{(i+1,k),s}\mathcal{X}^+_{(i,k),t+1} = \mathcal{X}^+_{(i,k),t}\mathcal{X}^+_{(i+1,k),s+1} - q\mathcal{X}^+_{(i+1,k),s+1}\mathcal{X}^+_{(i,k),t}.$$

The other case of (iii) is proven in a similar way. □

Proposition 7.19. *For $(i,k) \in \Gamma'(\mathbf{m})$ and $s,t,u \geq 0$, we have the followings.*

(i) $\mathcal{X}^+_{(i\pm1,k),u}(\mathcal{X}^+_{(i,k),s}\mathcal{X}^+_{(i,k),t} + \mathcal{X}^+_{(i,k),t}\mathcal{X}^+_{(i,k),s}) + (\mathcal{X}^+_{(i,k),s}\mathcal{X}^+_{(i,k),t} + \mathcal{X}^+_{(i,k),t}\mathcal{X}^+_{(i,k),s})\mathcal{X}^+_{(i\pm1,k),u} = (q+q^{-1})(\mathcal{X}^+_{(i,k),s}\mathcal{X}^+_{(i\pm1,k),u}\mathcal{X}^+_{(i,k),t} + \mathcal{X}^+_{(i,k),t}\mathcal{X}^+_{(i\pm1,k),u}\mathcal{X}^+_{(i,k),s}).$

(ii) $\mathcal{X}^-_{(i\pm1,k),u}(\mathcal{X}^-_{(i,k),s}\mathcal{X}^-_{(i,k),t} + \mathcal{X}^-_{(i,k),t}\mathcal{X}^-_{(i,k),s}) + (\mathcal{X}^-_{(i,k),s}\mathcal{X}^-_{(i,k),t} + \mathcal{X}^-_{(i,k),t}\mathcal{X}^-_{(i,k),s})\mathcal{X}^-_{(i\pm1,k),u} = (q+q^{-1})(\mathcal{X}^-_{(i,k),s}\mathcal{X}^-_{(i\pm1,k),u}\mathcal{X}^-_{(i,k),t} + \mathcal{X}^-_{(i,k),t}\mathcal{X}^-_{(i\pm1,k),u}\mathcal{X}^-_{(i,k),s}).$

Proof. By Lemmas 7.17 and 7.11,

$$(7.19.1)\qquad (\mathcal{X}^+_{(i+1,k),u}\mathcal{X}^+_{(i,k),s}\mathcal{X}^+_{(i,k),t} - q\mathcal{X}^+_{(i,k),s}\mathcal{X}^+_{(i+1,k),u}\mathcal{X}^+_{(i,k),t})(m_\mu)$$
$$= -\delta_{(\mu^{(k)}_{i+1}=1)} q^{-\mu^{(k)}_{i+2}+2} m_{\mu+2\alpha_{(i,k)}+\alpha_{(i+1,k)}} L^t_{N+1} L^{s+u}_{N+2}[T; N+1, \mu^{(k)}_{i+2}]^+$$
$$- \delta_{(\mu^{(k)}_{i+1}\geq 2)} q^{-2\mu^{(k)}_{i+1}-\mu^{(k)}_{i+2}+4} m_{\mu+2\alpha_{(i,k)}+\alpha_{(i+1,k)}} L^t_{N+1}L^s_{N+2}$$
$$\times (q^{\mu^{(k)}_{i+1}-1} T_{N+2}T_{N+3}\dots T_{N+\mu^{(k)}_{i+1}})[T; N, \mu^{(k)}_{i+1}]^+ L^u_{N+\mu^{(k)}_{i+1}+1}[T; N+\mu^{(k)}_{i+1}, \mu^{(k)}_{i+2}]^+$$

and

$$(\mathcal{X}^+_{(i,k),s}\mathcal{X}^+_{(i,k),t}\mathcal{X}^+_{(i+1,k),u} - q^{-1}\mathcal{X}^+_{(i,k),s}\mathcal{X}^+_{(i+1,k),u}\mathcal{X}^+_{(i,k),t})(m_\mu)$$
$$= q^{-2\mu^{(k)}_{i+1}-\mu^{(k)}_{i+2}+2} m_{\mu+2\alpha_{(i,k)}+\alpha_{(i+1,k)}} L^t_{N+1}L^s_{N+2}$$
$$\times [T; N+1, \mu^{(k)}_{i+1}]^+ (q^{\mu^{(k)}_{i+1}} T_{N+1}T_{N+2}\dots T_{N+\mu^{(k)}_{i+1}}) L^u_{N+\mu^{(k)}_{i+1}+1}[T; N+\mu^{(k)}_{i+1}, \mu^{(k)}_{i+2}]^+.$$

Applying Lemma 7.12(iii), we obtain

$$(7.19.2)\qquad (\mathcal{X}^+_{(i,k),s}\mathcal{X}^+_{(i,k),t}\mathcal{X}^+_{(i+1,k),u} - q^{-1}\mathcal{X}^+_{(i,k),s}\mathcal{X}^+_{(i+1,k),u}\mathcal{X}^+_{(i,k),t})(m_\mu)$$
$$= \delta_{(\mu^{(k)}_{i+1}=1)} q^{-\mu^{(k)}_{i+2}+1} m_{\mu+2\alpha_{(i,k)}+\alpha_{i+1,k)}} L^t_{N+1}L^s_{N+2}T_{N+1}L^u_{N+2}[T;N+\mu^{(k)}_{i+1},\mu^{(k)}_{i+2}]^+$$
$$+ \delta_{\mu^{(k)}_{i+1}\geq 2} q^{-2\mu^{(k)}_{i+1}-\mu^{(k)}_{i+2}+3} m_{\mu+2\alpha_{(i,k)}+\alpha_{(i+1,k)}} L^t_{N+1}L^s_{N+2}T_{N+1}$$
$$\times (q^{\mu^{(k)}_{i+1}-1}T_{N+2}T_{N+3}\dots T_{N+\mu^{(k)}_{i+1}})[T;N;\mu^{(k)}_{i+1}]^+ L^u_{N+\mu^{(k)}_{i+1}+1}[T;N+\mu^{(k)}_{i+1},\mu^{(k)}_{i+2}]^+.$$

We see that

$$\begin{aligned} m_{\mu+2\alpha_{(i,k)}+\alpha_{(i+1,k)}}&(L^t_{N+1}L^s_{N+2}+L^s_{N+1}L^t_{N+2})T_{N+1} \\ &= m_{\mu+2\alpha_{(i,k)}+\alpha_{(i+1,k)}}T_{N+1}(L^t_{N+1}L^s_{N+2}+L^s_{N+1}L^t_{N+2}) \\ &= q m_{\mu+2\alpha_{(i,k)}+\alpha_{(i+1,k)}}(L^t_{N+1}L^s_{N+2}+L^s_{N+1}L^t_{N+2}) \end{aligned}$$

by Lemma 6.3 and (6.4.2). Then (7.19.1) and (7.19.2) imply

$$\begin{aligned} \mathcal{X}^+_{(i+1,k),u}(\mathcal{X}^+_{(i,k),s}\mathcal{X}^+_{(i,k),t}+\mathcal{X}^+_{(i,k),t}\mathcal{X}^+_{(i,k),s})+(\mathcal{X}^+_{(i,k),s}\mathcal{X}^+_{(i,k),t}+\mathcal{X}^+_{(i,k),t}\mathcal{X}^+_{(i,k),s})\mathcal{X}^+_{(i+1,k),u} \\ = (q+q^{-1})(\mathcal{X}^+_{(i,k),s}\mathcal{X}^+_{(i+1,k),u}\mathcal{X}^+_{(i,k),t}+\mathcal{X}^+_{(i,k),t}\mathcal{X}^+_{(i+1,k),u}\mathcal{X}^+_{(i,k),s}). \end{aligned}$$

The other cases are proven in a similar way. □

By direct calculations, we get the following lemma.

Lemma 7.20. *For $(i,k)\in\Gamma'(\mathbf{m})$, $(j,l)\in\Gamma(\mathbf{m})$, $t\geq 0$, we have:*

(i) $q^{\pm a_{(i,k)(j,l)}}\mathcal{I}^\pm_{(j,l),0}\mathcal{X}^+_{(i,k),t} - q^{\mp a_{(i,k)(j,l)}}\mathcal{X}^+_{(i,k),t}\mathcal{I}^\pm_{(j,l),0} = a_{(i,k)(j,l)}\mathcal{X}^+_{(i,k),t}$.

(ii) $q^{\mp a_{(i,k)(j,l)}}\mathcal{I}^\pm_{(j,l),0}\mathcal{X}^-_{(i,k),t} - q^{\pm a_{(i,k)(j,l)}}\mathcal{X}^-_{(i,k),t}\mathcal{I}^\pm_{(j,l),0} = -a_{(i,k)(j,l)}\mathcal{X}^-_{(i,k),t}$.

We also have the following proposition.

Proposition 7.21. *For $(i,k)\in\Gamma'(\mathbf{m})$, $(j,l)\in\Gamma(\mathbf{m})$, $s,t\geq 0$, we have:*

(i) $[\mathcal{I}^\pm_{(j,l),s+1},\mathcal{X}^+_{(i,k),t}] = q^{\pm a_{(i,k)(j,l)}}\mathcal{I}^\pm_{(j,l),s}\mathcal{X}^+_{(i,k),t+1} - q^{\mp a_{(i,k)(j,l)}}\mathcal{X}^+_{(i,k),t+1}\mathcal{I}^\pm_{(j,l),s}$.

(ii) $[\mathcal{I}^\pm_{(j,l),s+1},\mathcal{X}^-_{(i,k),t}] = q^{\mp a_{(i,k)(j,l)}}\mathcal{I}^\pm_{(j,l),s}\mathcal{X}^-_{(i,k),t+1} - q^{\pm a_{(i,k)(j,l)}}\mathcal{X}^-_{(i,k),t+1}\mathcal{I}^\pm_{(j,l),s}$.

Proof. By Lemmas 7.17 and 6.3, we see that

$$[\mathcal{I}^\sigma_{(j,l),s},\mathcal{X}^{\sigma'}_{(i,k),t}] = 0 \quad \text{if } (j,l)\neq(i,k),(i+1,k),$$

where $\sigma,\sigma'\in\{+,-\}$. Thus, it is enough to handle the cases where $(j,l)=(i,k)$ or $(j,l)=(i+1,k)$. We will prove

$$(7.21.1)\qquad [\mathcal{I}^+_{(i,k),s+1},\mathcal{X}^+_{(i,k),t}] = q\mathcal{I}^+_{(i,k),s}\mathcal{X}^+_{(i,k),t+1} - q^{-1}\mathcal{X}^+_{(i,k),t+1}\mathcal{I}^+_{(i,k),s}.$$

For $\mu \in \Lambda_{n,r}(\mathbf{m})$, write $N = N^{\mu}_{(i,k)}$. Then, by Lemmas 7.17 and 7.11, we have

$$(\mathcal{I}^+_{(i,k),s+1}\mathcal{X}^+_{(i,k),t} - \mathcal{X}^+_{(i,k),t}\mathcal{I}^+_{(i,k),s+1})(m_\mu) = q^{s-\mu^{(k)}_{i+1}+1} m_{\mu+\alpha_{(i,k)}}$$
$$\times \big(\Phi^+_{s+1}(L_{N+1}, L_N, L_{N-1}, \ldots, L_{N-\mu^{(k)}_i+1}) - \Phi^+_{s+1}(L_N, L_{N-1}, \ldots, L_{N-\mu^{(k)}_i+1})\big)$$
$$\times L^t_{N+1}[T; N, \mu^{(k)}_{i+1}]^+.$$

By (7.3.2), we have

$$\begin{aligned}
&(\mathcal{I}^+_{(i,k),s+1}\mathcal{X}^+_{(i,k),t} - \mathcal{X}^+_{(i,k),t}\mathcal{I}^+_{(i,k),s+1})(m_\mu)\\
&= q^{s-\mu^{(k)}_{i+1}+1} m_{\mu+\alpha_{(i,k)}}\\
&\quad \times L_{N+1}\big(\Phi^+_s(L_{N+1}, L_N, \ldots, L_{N-\mu^{(k)}_i+1}) - q^{-2}\Phi^+_s(L_N, L_{N-1}, \ldots, L_{N-\mu^{(k)}_i+1})\big)\\
&\quad \times L^t_{N+1}[T; N, \mu^{(k)}_{i+1}]^+\\
&= q^{(s-1)-\mu^{(k)}_{i+1}+1} m_{\mu+\alpha_{(i,k)}}\\
&\quad \times \big\{q\Phi^+_s(L_{N+1}, L_N, \ldots, L_{N-\mu^{(k)}_i+1}) L^{t+1}_{N+1}[T; N, \mu^{(k)}_{i+1}]^+\\
&\qquad\qquad - q^{-1} L^{t+1}_{N+1}[T; N, \mu^{(k)}_{i+1}]^+ \Phi^+_s(L_N, L_{N-1}, \ldots, L_{N-\mu^{(k)}_i+1})\big\}\\
&= (q\mathcal{I}^+_{(i,k),s}\mathcal{X}^+_{(i,k),t+1} - q^{-1}\mathcal{X}^+_{(i,k),t+1}\mathcal{I}^+_{(i,k),s})(m_\mu).
\end{aligned}$$

Thus we have proved (7.21.1). The other cases are proven in a similar way. □

Proposition 7.22. *For* $(i,k), (j,l) \in \Gamma'(\mathbf{m})$ *such that* $(i,k) \neq (j,l)$ *and* $s, t \geq 0$,

$$[\mathcal{X}^+_{(i,k),t}, \mathcal{X}^-_{(j,l),s}] = 0.$$

Proof. By Lemma 7.17, for $\mu \in \Lambda_{n,r}(\mathbf{m})$,

$$\begin{aligned}
\mathcal{X}^+_{(i,k),t}\mathcal{X}^-_{(j,l),s}(m_\mu) &= q^{-\mu^{(l)}_j - (\mu-\alpha_{(j,l)})^{(k)}_{i+1}+2} m_{\mu+\alpha_{(i,k)}-\alpha_{(j,l)}}\\
&\times L^t_{N^{\mu-\alpha_{(j,l)}}_{(i,k)}+1}[T; N^{\mu-\alpha_{(j,l)}}_{(i,k)}, (\mu-\alpha_{(j,l)})^{(k)}_{i+1}]^+ L^s_{N^\mu_{(j,l)}} h^\mu_{-(j,l)}[T; N^\mu_{(j,l)}, \mu^{(l)}_j]^-
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{X}^-_{(j,l),s}\mathcal{X}^+_{(i,k),t}(m_\mu) &= q^{-\mu^{(k)}_{i+1} - (\mu+\alpha_{(i,k)})^{(l)}_j+2} m_{\mu+\alpha_{(i,k)}-\alpha_{(j,l)}}\\
&\times L^s_{N^{\mu+\alpha_{(i,k)}}_{(j,l)}} h^{\mu+\alpha_{(i,k)}}_{-(j,l)}[T; N^{\mu+\alpha_{(i,k)}}_{(j,l)}, (\mu+\alpha_{(i,k)})^{(l)}_j]^- L^t_{N^\mu_{(i,k)}+1}[T; N^\mu_{(i,k)}, \mu^{(k)}_{i+1}]^+.
\end{aligned}$$

Since $(i,k) \neq (j,l)$, we have

$$N^\mu_{(i,k)} = N^{\mu-\alpha_{(j,l)}}_{(i,k)}, \qquad N^\mu_{(j,l)} = N^{\mu+\alpha_{(i,k)}}_{(j,l)},$$
$$(\mu-\alpha_{(j,l)})^{(k)}_{i+1} = \begin{cases} \mu^{(k)}_{i+1} & \text{if } (j,l) \neq (i+1,k),\\ \mu^{(k)}_{i+1} - 1 & \text{if } (j,l) = (i+1,k),\end{cases}$$

$$(\mu+\alpha_{(i,k)})_j^{(l)} = \begin{cases} \mu_j^{(l)} & \text{if } (j,l) \neq (i+1,k), \\ \mu_j^{(l)} - 1 & \text{if } (j,l) = (i+1,k), \end{cases}$$

$$h^{\mu}_{-(j,l)} = h^{\mu+\alpha_{(i,k)}}_{-(j,l)} = \begin{cases} 1 & \text{if } j \neq m_j, \\ L_{N^{\mu}_{(m_l,l)}} - Q_l & \text{if } j = m_l. \end{cases}$$

Then, by Lemma 7.11,

$$\begin{aligned} &[T; N^{\mu-\alpha_{(j,l)}}_{(i,k)}, (\mu-\alpha_{(j,l)})^{(k)}_{i+1}]^{+} L^{s}_{N^{\mu}_{(j,l)}} h^{\mu}_{-(j,l)} \\ &\qquad\qquad = L^{s}_{N^{\mu}_{(j,l)}} h^{\mu}_{-(j,l)} [T; N^{\mu-\alpha_{(j,l)}}_{(i,k)}, (\mu-\alpha_{(j,l)})^{(k)}_{i+1}]^{+}, \\ &[T; N^{\mu+\alpha_{(i,k)}}_{(j,l)}, (\mu+\alpha_{(i,k)})^{(l)}_{j}]^{-} L^{t}_{N^{\mu}_{(i,k)}+1} = L^{t}_{N^{\mu}_{(i,k)}+1} [T; N^{\mu+\alpha_{(i,k)}}_{(j,l)}, (\mu+\alpha_{(i,k)})^{(l)}_{j}]^{-}. \end{aligned}$$

Thus, it is enough to show that

$$\begin{aligned} \text{(7.22.1)} \qquad & [T; N^{\mu-\alpha_{(j,l)}}_{(i,k)}, (\mu-\alpha_{(j,l)})^{(k)}_{i+1}]^{+} [T; N^{\mu}_{(j,l)}, \mu^{(l)}_{j}]^{-} \\ &\qquad = [T; N^{\mu+\alpha_{(i,k)}}_{(j,l)}, (\mu+\alpha_{(i,k)})^{(l)}_{j}]^{-} [T; N^{\mu}_{(i,k)}, \mu^{(k)}_{i+1}]^{+}. \end{aligned}$$

If $(j,l) \neq (i+1,k)$, we see easily that (7.22.1) holds since the product is commutative on each side. When $(j,l) = (i+1,k)$, we can prove (7.22.1) by induction on $\mu^{(k)}_{i+1}$. □

Remark 7.23. There is an error in the proof of [W1, Proposition 6.11(i)] (the case where $(j,l) = (i+1,k)$). The above argument also proves [W1, Proposition 6.11(i)] as a special case.

We prepare some technical lemmas.

Lemma 7.24. *For $\mu \in \Lambda_{n,r}(\mathbf{m})$ and $(i,k) \in \Gamma(\mathbf{m})$, we have:*

(i) *For $t \geq 0$ and $1 \leq p \leq \mu^{(k)}_i$,*

$$m_\mu L^{t}_{N^{\mu}_{(i,k)}} [T; N^{\mu}_{(i,k)}, p]^{-} = q^{2p-2} m_\mu \Phi^{+}_{t}(L_{N^{\mu}_{(i,k)}}, L_{N^{\mu}_{(i,k)}-1}, \dots, L_{N^{\mu}_{(i,k)}-p+1}).$$

(ii) *For $t \geq 0$ and $1 \leq p \leq \mu^{(k)}_{i+1}$,*

$$m_\mu L^{t}_{N^{\mu}_{(i,k)}+1} [T; N^{\mu}_{(i,k)}, p]^{+} = m_\mu \Phi^{-}_{t}(L_{N^{\mu}_{(i,k)}+1}, L_{N^{\mu}_{(i,k)}+2}, \dots, L_{N^{\mu}_{(i,k)}+p}).$$

Proof. For $t = 0$, we get (i) and (ii) from (6.4.2).

We now prove (i) for $t > 0$. Write $N = N^{\mu}_{(i,k)}$. For $1 \leq h \leq \mu^{(k)}_i - 1$, by induction on h together with Lemma 6.3 and (6.4.2), we can show that

(7.24.1) $\quad m_\mu L_N^t(T_{N-1}T_{N-2}\dots T_{N-h})$

$$= m_\mu\Big\{(q-q^{-1})q^{h-1}L_N^t + \sum_{s=2}^{h}(q-q^{-1})q^{h-s}L_N^{t-1}(T_{N-1}T_{N-2}\dots T_{N-s+1})L_{N-s+1} + L_N^{t-1}(T_{N-1}T_{N-2}\dots T_{N-h})L_{N-h}\Big\}.$$

We will prove that

(7.24.2) $\quad m_\mu L_N^t(T_{N-1}T_{N-2}\dots T_{N-h})$

$$= m_\mu\big(q^h\Phi_t^+(L_N, L_{N-1},\dots,L_{N-h}) - q^{h-2}\Phi_t^+(L_N, L_{N-1},\dots,L_{N-h+1})\big)$$

by induction on t. For $t=1$, by (7.24.1) together with (6.4.2), we have

$$\begin{aligned} m_\mu L_N(T_{N-1}T_{N-2}\dots T_{N-h}) &= m_\mu\Big\{(q-q^{-1})q^{h-1}L_N + \sum_{s=2}^{h}(q-q^{-1})q^{h-s}q^{s-1}L_{N-s+1} + q^hL_{N-h}\Big\} \\ &= m_\mu\big(q^h\Phi_1^+(L_N, L_{N-1},\dots,L_{N-h}) - q^{h-2}\Phi_1^+(L_N, L_{N-1},\dots,L_{N-h+1})\big). \end{aligned}$$

Assume that $t>1$. Applying the inductive assumption to (7.24.1), we get

$$\begin{aligned} &m_\mu L_N^t(T_{N-1}T_{N-2}\dots T_{N-h}) \\ &= m_\mu\Big\{(q-q^{-1})q^{h-1}L_N^t + \sum_{s=2}^{h}(q-q^{-1})q^{h-s}(q^{s-1}\Phi_{t-1}^+(L_N, L_{N-1},\dots,L_{N-s+1}) \\ &\qquad - q^{s-3}\Phi_{t-1}^+(L_N, L_{N-1},\dots,L_{N-s+2}))L_{N-s+1} \\ &\quad + (q^h\Phi_{t-1}^+(L_N, L_{N-1},\dots,L_{N-h}) - q^{h-2}\Phi_{t-1}^+(L_N, L_{N-1},\dots,L_{N-h+1}))L_{N-h}\Big\}. \end{aligned}$$

Setting $s'=s-1$, we have

$$\begin{aligned} &m_\mu L_N^t(T_{N-1}T_{N-2}\dots T_{N-h}) \\ &= m_\mu\Big\{q^h\Big(L_N^t + \sum_{s=1}^{h}(\Phi_{t-1}^+(L_N, L_{N-1},\dots,L_{N-s'})L_{N-s'} \\ &\qquad - q^{-2}\Phi_{t-1}(L_N, L_{N-1},\dots,L_{N-s'+1})L_{N-s'})\Big) \\ &\quad - q^{h-2}\Big(L_N^t + \sum_{s=1}^{h-1}(\Phi_{t-1}^+(L_N, L_{N-1},\dots,L_{N-s'})L_{N-s'} \\ &\qquad - q^{-2}\Phi_{t-1}(L_N, L_{N-1},\dots,L_{N-s'+1})L_{N-s'})\Big)\Big\}. \end{aligned}$$

Applying (7.3.1) to the right-hand side, we get (7.24.2). Hence

$$
\begin{aligned}
m_\mu L_N^t[T;N,p]^- &= m_\mu L_N^t\Big(1+\sum_{h=1}^{p-1} q^h T_{N-1}T_{N-2}\dots T_{N-h}\Big)\\
&= m_\mu\Big\{\Phi_t^+(L_N)+\sum_{h=1}^{p-1}(q^{2h}\Phi_t^+(L_N,L_{N-1},\dots,L_{N-h})\\
&\qquad\qquad - q^{2h-2}\Phi_t^+(L_N,L_{N-1},\dots,L_{N-h+1}))\Big\}\\
&= q^{2p-2}m_\mu\Phi_t^+(L_N,L_{N-1},\dots,L_{N-p}).
\end{aligned}
$$

Thus we have obtained (i).

For $t>0$ and $1\le h\le \mu_{i+1}^{(k)}-1$, by induction on h using Lemma 6.3 and (6.4.2) we can show that

(7.24.3)
$$
\begin{aligned}
&m_\mu L_{N+1}^t(T_{N+1}T_{N+2}\dots T_{N+h})\\
&= q^{-h}m_\mu L_{N+1}^{t-1}\Big\{(1-q^2)\Big(1+\sum_{s=1}^{h-1}q^sT_{N+1}T_{N+2}\dots T_{N+s}\Big)\\
&\qquad + q^hT_{N+1}T_{N+2}\dots T_{N+h}\Big\}L_{N+h+1}.
\end{aligned}
$$

We prove (ii) by induction on t. The case where $t=0$ has already been dealt with.

Assume that $t>0$. By (7.24.3), we have

$$
\begin{aligned}
m_\mu L_{N+1}^t[T;N,p]^+ &= m_\mu L_{N+1}^t\Big(1+\sum_{h=1}^{p-1}q^hT_{N+1}T_{N+2}\dots T_{N+h}\Big)\\
&= m_\mu L_{N+1}^{t-1}\Big\{L_{N+1}+\sum_{h=1}^{p-1}\Big\{(1-q^2)\Big(1+\sum_{s=1}^{h-1}q^sT_{N+1}T_{N+2}\dots T_{N+s}\Big)\\
&\qquad + q^hT_{N+1}T_{N+2}\dots T_{N+h}\Big\}L_{N+h+1}\Big\}\\
&= m_\mu L_{N+1}^{t-1}\Big\{\sum_{h=1}^{p}[T;N,h]^+L_{N+h} - q^2\sum_{h=1}^{p-1}[T;N,h]^+L_{N+h+1}\Big\}.
\end{aligned}
$$

Applying the inductive assumption, we get

$$
\begin{aligned}
m_\mu L_{N+1}^t[T;N,p]^+ &= m_\mu\Big\{\sum_{h=1}^{p}\Phi_{t-1}^-(L_{N+1},L_{N+2},\dots,L_{N+h})L_{N+h}\\
&\qquad - q^2\sum_{h=1}^{p-1}\Phi_{t-1}^-(L_{N+1},L_{N+2},\dots,L_{N+h})L_{N+h+1}\Big\}.
\end{aligned}
$$

In view of (7.3.1), we have

$$
m_\mu L_{N+1}^t[T;N,p]^+ = m_\mu\Phi_t^-(L_{N+1},L_{N+2},\dots,L_{N+p}). \qquad \square
$$

Lemma 7.25. *For* $\mu \in \Lambda_{n,r}(\mathbf{m})$ *and* $(i,k) \in \Gamma'(\mathbf{m})$, *set* $N = N^{\mu}_{(i,k)}$.

(i) *If* $\mu_i^{(k)} \neq 0$, *then*

$$\begin{aligned} m_\mu L_N^t[T; N-1, \mu_{i+1}^{(k)}+1]^+[T; N, \mu_i^{(k)}]^- \\ = q^{2\mu_i^{(k)}-2} m_\mu \Phi_t^+(L_N, L_{N-1}, \ldots, L_{N-\mu_i^{(k)}+1}) \\ + \delta_{(\mu_{i+1}^{(k)}\neq 0)} m_\mu L_N^t([T; N+1, \mu_i^{(k)}+1]^- - 1)[T; N, \mu_{i+1}^{(k)}]^+. \end{aligned}$$

(ii) *If* $\mu_i^{(k)} \neq 0$, *then*

$$\begin{aligned} m_\mu L_N^t[T; N-1, \mu_{i+1}^{(k)}+1]^+ L_N[T; N, \mu_i^{(k)}]^- \\ = q^{2\mu_i^{(k)}-2} m_\mu \Phi_{t+1}^+(L_N, L_{N-1}, \ldots, L_{N-\mu_i^{(k)}+1}) \\ - \delta_{(\mu_{i+1}^{(k)}\neq 0)}(q-q^{-1}) q^{2\mu_i^{(k)}-1} m_\mu \Phi_t^+(L_N, L_{N-1}, \ldots, L_{N-\mu_i^{(k)}+1}) \\ \times \Phi_1^-(L_{N+1}, L_{N+2}, \ldots, L_{N+\mu_{i+1}^{(k)}}) \\ + m_\mu L_N^t L_{N+1}([T; N-1, \mu_{i+1}^{(k)}+1]^+ - 1)[T; N, \mu_i^{(k)}]^-. \end{aligned}$$

(iii) *If* $\mu_{i+1}^{(k)} \neq 0$, *then*

$$\begin{aligned} m_\mu [T; N+1, \mu_i^{(k)}+1]^- L_{N+1}^t [T; N, \mu_{i+1}^{(k)}]^+ \\ = (1 + \delta_{(t\neq 0)}(q^{2\mu_i^{(k)}} - 1)) m_\mu \Phi_t^-(L_{N+1}, L_{N+2}, \ldots, L_{N+\mu_{i+1}^{(k)}}) \\ + \delta_{(\mu_i^{(k)}\neq 0)}(q-q^{-1}) \sum_{b=1}^{t-1} m_\mu q^{2\mu_i^{(k)}-1} \Phi_{t-b}^+(L_N, L_{N-1}, \ldots, L_{N-\mu_i^{(k)}+1}) \\ \times \Phi_b^-(L_{N+1}, L_{N+2}, \ldots, L_{N+\mu_{i+1}^{(k)}}) \\ + m_\mu L_N^t([T; N+1, \mu_i^{(k)}+1]^- - 1)[T; N, \mu_{i+1}^{(k)}]^+. \end{aligned}$$

(iv) *If* $\mu_{i+1}^{(k)} \neq 0$, *then*

$$\begin{aligned} m_\mu L_{N+1}[T; N+1, \mu_i^{(k)}+1]^- L_{N+1}^t [T; N, \mu_{i+1}^{(k)}]^+ \\ = (1 + \delta_{(t\neq 0)}(q^{2\mu_i^{(k)}} - 1)) m_\mu \Phi_{t+1}^-(L_{N+1}, L_{N+2}, \ldots, L_{N+\mu_{i+1}^{(k)}}) \\ + \delta_{(\mu_i^{(k)}\neq 0)}(q-q^{-1}) \sum_{b=1}^{t-1} m_\mu q^{2\mu_i^{(k)}-1} \Phi_{t-b}^+(L_N, L_{N-1}, \ldots, L_{N-\mu_i^{(k)}+1}) \\ \times \Phi_{b+1}^-(L_{N+1}, L_{N+2}, \ldots, L_{N+\mu_{i+1}^{(k)}}) \\ + m_\mu L_N^t L_{N+1}([T; N+1, \mu_i^{(k)}+1]^- - 1)[T; N, \mu_{i+1}^{(k)}]^+. \end{aligned}$$

Proof. By induction on $\mu_{i+1}^{(k)}$, we can prove that

$$
\begin{aligned}
(7.25.1)\qquad & [T; N-1, \mu_{i+1}^{(k)}+1]^+[T; N, \mu_i^{(k)}]^- \\
&= [T; N, \mu_i^{(k)}]^- + \delta_{(\mu_{i+1}^{(k)}\neq 0)}([T; N+1, \mu_i^{(k)}+1]^- - 1)[T; N, \mu_{i+1}^{(k)}]^+.
\end{aligned}
$$

Thus

$$
\begin{aligned}
& m_\mu L_N^t[T; N-1, \mu_{i+1}^{(k)}+1]^+[T; N, \mu_i^{(k)}]^- \\
&\quad = m_\mu L_N^t\big\{[T; N, \mu_i^{(k)}]^- + \delta_{(\mu_{i+1}^{(k)}\neq 0)}([T; N+1, \mu_i^{(k)}+1]^- - 1)[T; N, \mu_{i+1}^{(k)}]^+\big\}.
\end{aligned}
$$

Applying Lemma 7.24(i) yields (i).

We now prove (ii). By Lemma 6.3,

$$
\begin{aligned}
& [T; N-1, \mu_{i+1}^{(k)}+1]^+ L_N \\
& = L_N + L_{N+1}([T; N-1, \mu_{i+1}^{(k)}+1]^+ - 1) - \delta_{(\mu_{i+1}^{(k)}\neq 0)} q(q-q^{-1})L_{N+1}[T; N, \mu_{i+1}^{(k)}]^+.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& m_\mu L_N^t[T; N-1, \mu_{i+1}^{(k)}+1]^+ L_N[T; N, \mu_i^{(k)}]^- \\
&\quad = m_\mu L_N^{t+1}[T; N, \mu_i^{(k)}]^- + m_\mu L_N^t L_{N+1}([T; N-1, \mu_{i+1}^{(k)}+1]^+ - 1)[T; N, \mu_i^{(k)}]^- \\
&\qquad - \delta_{(\mu_{i+1}^{(k)}\neq 0)} q(q-q^{-1}) m_\mu L_N^t L_{N+1}[T; N, \mu_{i+1}^{(k)}]^+[T; N, \mu_i^{(k)}]^-.
\end{aligned}
$$

Applying (6.4.2), Lemma 7.11, Lemma 7.24 and (7.25.1), we get (ii).

Next, we prove (iii). By Lemma 6.3,

$$
\begin{aligned}
[T; N+1, \mu_i^{(k)}+1]^- L_{N+1}^t &= L_{N+1}^t + L_N^t([T; N+1, \mu_i^{(k)}+1]^- - 1) \\
&\quad + \delta_{(\mu_i^{(k)}\neq 0)} q(q-q^{-1}) \sum_{b=1}^{t} L_N^{t-b} L_{N+1}^b [T; N, \mu_i^{(k)}]^-.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& m_\mu[T; N+1, \mu_i^{(k)}+1]^- L_{N+1}^t[T; N, \mu_{i+1}^{(k)}]^+ \\
&\qquad = m_\mu L_{N+1}^t[T; N, \mu_{i+1}^{(k)}]^+ + m_\mu L_N^t([T; N+1, \mu_i^{(k)}+1]^- - 1)[T; N, \mu_{i+1}^{(k)}]^+ \\
&\qquad\quad + \delta_{(\mu_i^{(k)}\neq 0)} q(q-q^{-1}) \sum_{b=1}^{t} m_\mu L_N^{t-b} L_{N+1}^b [T; N, \mu_i^{(k)}]^-[T; N, \mu_{i+1}^{(k)}]^+ \\
&\qquad = m_\mu L_{N+1}^t[T; N, \mu_{i+1}^{(k)}]^+ \\
&\qquad\quad + \delta_{(\mu_i^{(k)}\neq 0)} \delta_{(t\neq 0)} q(q-q^{-1}) m_\mu L_{N+1}^t [T; N, \mu_i^{(k)}]^-[T; N, \mu_{i+1}^{(k)}]^+
\end{aligned}
$$

$$+\,\delta_{(\mu_i^{(k)}\neq 0)}q(q-q^{-1})\sum_{b=1}^{t-1} m_\mu L_N^{t-b}L_{N+1}^{b}[T;N,\mu_i^{(k)}]^-[T;N,\mu_{i+1}^{(k)}]^+$$
$$+\,m_\mu L_N^t([T;N+1,\mu_i^{(k)}+1]^- - 1)[T;N,\mu_{i+1}^{(k)}]^+.$$

Applying (6.4.2), Lemma 7.11 and Lemma 7.24, we get (iii).

Finally, we prove (iv). By Lemma 6.3,

$$\begin{aligned}
&m_\mu L_{N+1}[T;N+1,\mu_i^{(k)}+1]^- L_{N+1}^t[T;N,\mu_{i+1}^{(k)}]^+\\
&\quad = m_\mu L_{N+1}^{t+1}[T;N,\mu_{i+1}^{(k)}]^+ + m_\mu L_N^t L_{N+1}([T;N+1,\mu_i^{(k)}+1]^- - 1)[T;N,\mu_{i+1}^{(k)}]^+\\
&\qquad + \delta_{(\mu_i^{(k)}\neq 0)}q(q-q^{-1})\sum_{b=1}^{t} m_\mu L_N^{t-b}L_{N+1}^{b+1}[T;N,\mu_i^{(k)}]^-[T;N,\mu_{i+1}^{(k)}]^+.
\end{aligned}$$

Applying (6.4.2) and Lemmas 7.11 and 7.24, we get (iv). □

Proposition 7.26. *For $(i,k)\in\Gamma'(\mathbf{m})$ and $s,t\geq 0$, we have*

$$[\mathcal{X}^+_{(i,k),t},\mathcal{X}^-_{(i,k),s}] = \begin{cases} \widetilde{\mathcal{K}}^+_{(i,k)}\mathcal{J}_{(i,k),s+t} & \text{if } i\neq m_k,\\ -Q_k\widetilde{\mathcal{K}}^+_{(m_k,k)}\mathcal{J}_{(m_k,k),s+t} + \widetilde{\mathcal{K}}^+_{(m_k,k)}\mathcal{J}_{(m_k,k),s+t+1} & \text{if } i = m_k.\end{cases}$$

Proof. Assume that $s=0$ and $t\geq 0$. For $\mu\in\Lambda_{n,r}(\mathbf{m})$, write $N=N^\mu_{(i,k)}$. By Lemma 7.17,

$$\begin{aligned}
\text{(7.26.1)}\qquad &\mathcal{X}^+_{(i,k),t}\mathcal{X}^-_{(i,k),0}(m_\mu)\\
&\quad = \delta_{(\mu_i^{(k)}\neq 0)}q^{-\mu_i^{(k)}-\mu_{i+1}^{(k)}+1}m_\mu L_N^t[T;N-1,\mu_{i+1}^{(k)}+1]^+ h^\mu_{-(i,k)}[T;N,\mu_i^{(k)}]^-
\end{aligned}$$

and

$$\begin{aligned}
\text{(7.26.2)}\qquad &\mathcal{X}^-_{(i,k),0}\mathcal{X}^+_{(i,k),t}(m_\mu)\\
&\quad = \delta_{(\mu_{i+1}^{(k)}\neq 0)}q^{-\mu_i^{(k)}-\mu_{i+1}^{(k)}+1}m_\mu h^{\mu+\alpha_{(i,k)}}_{-(i,k)}[T;N+1,\mu_i^{(k)}+1]^- L_{N+1}^t[T;N,\mu_{i+1}^{(k)}]^+.
\end{aligned}$$

Assume that $i\neq m_k$. By (7.26.1) and (7.26.2) together with Lemma 7.25,

$$\begin{aligned}
&(\mathcal{X}^+_{(i,k),t}\mathcal{X}^-_{(i,k),0} - \mathcal{X}^-_{(i,k),0}\mathcal{X}^+_{(i,k),t})(m_\mu)\\
&= q^{-\mu_i^{(k)}-\mu_{i+1}^{(k)}+1}m_\mu\Big\{\delta_{(\mu_i^{(k)}\neq 0)}q^{2\mu_i^{(k)}-2}\Phi_t^+(L_N,L_{N-1},\ldots,L_{N-\mu_i^{(k)}+1})\\
&\quad - \delta_{(\mu_{i+1}^{(k)}\neq 0)}(1+\delta_{(t\neq 0)}(q^{2\mu_i^{(k)}}-1))\Phi_t^-(L_{N+1},L_{N+2},\ldots,L_{N+\mu_{i+1}^{(k)}})\\
&\quad - \delta_{(\mu_i^{(k)}\neq 0)}\delta_{(\mu_{i+1}^{(k)}\neq 0)}(q-q^{-1})\sum_{b=1}^{t-1}q^{2\mu_i^{(k)}-1}\Phi^+_{t-b}(L_N,L_{N-1},\ldots,L_{N-\mu_i^{(k)}+1})\\
&\qquad\qquad \times\Phi_b^-(L_{N+1},L_{N+2},\ldots,L_{N+\mu_{i+1}^{(k)}})\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= q^{\mu_i^{(k)}-\mu_{i+1}^{(k)}} m_\mu \Big\{ \delta_{(\mu_i^{(k)}\neq 0)} q^{-t} q^{t-1} \Phi_t^+(L_N, L_{N-1}, \dots, L_{N-\mu_i^{(k)}+1}) \\
&\quad - \delta_{(\mu_{i+1}^{(k)}\neq 0)} (q^{-2\mu_i^{(k)}} + \delta_{(t\neq 0)}(1-q^{-2\mu_i^{(k)}})) q^t q^{-t+1} \Phi_t^-(L_{N+1}, L_{N+2}, \dots, L_{N+\mu_{i+1}^{(k)}}) \\
&\quad - \delta_{(\mu_i^{(k)}\neq 0)} \delta_{(\mu_{i+1}^{(k)}\neq 0)} (q-q^{-1}) \sum_{b=1}^{t-1} q^{-t+2b} q^{t-b-1} \Phi_{t-b}^+(L_N, L_{N-1}, \dots, L_{N-\mu_i^{(k)}+1}) \\
&\qquad\qquad\qquad\qquad \times q^{-b+1} \Phi_b^-(L_{N+1}, L_{N+2}, \dots, L_{N+\mu_{i+1}^{(k)}}) \Big\} \\
&= \widetilde{\mathcal{K}}_{(i,k)}^+ \mathcal{J}_{(i,k),t}(m_\mu).
\end{aligned}
$$

Thus, $[\mathcal{X}_{(i,k),t}^+, \mathcal{X}_{(i,k),0}^-] = \widetilde{\mathcal{K}}_{(i,k)}^+ \mathcal{J}_{(i,k),t}$ if $i \neq m_k$. (Note Corollary 7.9 in the case where $t = 0$.)

In a similar way, by (7.26.1) and (7.26.2) together with Lemma 7.25, we also have $[\mathcal{X}_{(m_k,k),t}^+, \mathcal{X}_{(m_k,k),0}^-] = -Q_k \widetilde{\mathcal{K}}_{(m_k,k)}^+ \mathcal{J}_{(m_k,k),s+t} + \widetilde{\mathcal{K}}_{(m_k,k)}^+ \mathcal{J}_{(m_k,k),s+t+1}$ if $i = m_k$. Thus we have proved the proposition in the case where $s = 0$ and $t \geq 0$.

Now, we use induction on s. The case $s = 0$ is already proved. Assume that $s > 0$. By (7.15.1), we have

$$
\begin{aligned}
[\mathcal{X}_{(i,k),t}^+, \mathcal{X}_{(i,k),s}^-] &= \mathcal{X}_{(i,k),t}^+ (-\mathcal{I}_{(i,k),1}^- \mathcal{X}_{(i,k),s-1}^- + \mathcal{X}_{(i,k),s-1}^- \mathcal{I}_{(i,k),1}^-) \\
&\quad - (-\mathcal{I}_{(i,k),1}^- \mathcal{X}_{(i,k),s-1}^- + \mathcal{X}_{(i,k),s-1}^- \mathcal{I}_{(i,k),1}^-) \mathcal{X}_{(i,k),t}^+.
\end{aligned}
$$

Applying Proposition 7.21 together with Lemma 7.20, we obtain

$$
\begin{aligned}
&[\mathcal{X}_{(i,k),t}^+, \mathcal{X}_{(i,k),s}^-] \\
&\quad = -\mathcal{I}_{(i,k),1}^- \mathcal{X}_{(i,k),t}^+ \mathcal{X}_{(i,k),s-1}^- + \mathcal{X}_{(i,k),t+1}^+ \mathcal{X}_{(i,k),s-1}^- + \mathcal{X}_{(i,k),t}^+ \mathcal{X}_{(i,k),s-1}^- \mathcal{I}_{(i,k),1}^- \\
&\qquad + \mathcal{I}_{(i,k),1}^- \mathcal{X}_{(i,k),s-1}^- \mathcal{X}_{(i,k),t}^+ - \mathcal{X}_{(i,k),s-1}^- \mathcal{X}_{(i,k),t}^+ \mathcal{I}_{(i,k),1}^- - \mathcal{X}_{(i,k),s-1}^- \mathcal{X}_{(i,k),t+1}^+ \\
&\quad = [\mathcal{X}_{(i,k),t+1}^+, \mathcal{X}_{(i,k),s-1}^-] \\
&\qquad - \mathcal{I}_{(i,k),1}^- [\mathcal{X}_{(i,k),t}^+, \mathcal{X}_{(i,k),s-1}^-] + [\mathcal{X}_{(i,k),t}^+, \mathcal{X}_{(i,k),s-1}^-] \mathcal{I}_{(i,k),1}^-.
\end{aligned}
$$

Then, by the inductive assumption together with Lemma 7.6, the proposition follows. □

Lemma 7.27. *For $(i,k) \in \Gamma'(\mathbf{m})$, we have:*

(i) *If $q - q^{-1}$ is invertible in R, then*

$$
\widetilde{\mathcal{K}}_{(i,k)}^+ \mathcal{J}_{(i,k),0} = \frac{\widetilde{\mathcal{K}}_{(i,k)}^+ - \widetilde{\mathcal{K}}_{(i,k)}^-}{q - q^{-1}}.
$$

(ii) *If $q = 1$, then*

$$
\widetilde{\mathcal{K}}_{(i,k)}^+ \mathcal{J}_{(i,k),0} = \mathcal{I}_{(i,k),0}^+ - \mathcal{I}_{(i+1,k),0}^-.
$$

Proof. For $\mu \in \Lambda_{n,r}(\mathbf{m})$, by the definitions together with Corollary 7.9,

$$\begin{aligned}\widetilde{\mathcal{K}}^+_{(i,k)}\mathcal{J}_{(i,k),0}(m_\mu) &= \widetilde{\mathcal{K}}^+_{(i,k)}(\mathcal{I}^+_{(i,k),0} - (\mathcal{K}^-_{(i,k)})^2\mathcal{I}^-_{(i+1,k),0})(m_\mu)\\ &= q^{\mu_i^{(k)}-\mu_{i+1}^{(k)}}(q^{-\mu_i^{(k)}}[\mu_i^{(k)}] - q^{-2\mu_i^{(k)}}q^{\mu_{i+1}^{(k)}}[\mu_{i+1}^{(k)}])m_\mu\\ &= [\mu_i^{(k)} - \mu_{i+1}^{(k)}]m_\mu.\end{aligned}$$

If $q - q^{-1}$ is invertible in R, we have

$$[\mu_i^{(k)} - \mu_{i+1}^{(k)}]m_\mu = \frac{q^{\mu_i^{(k)}-\mu_{i+1}^{(k)}} - q^{-\mu_i^{(k)}+\mu_{i+1}^{(k)}}}{q-q^{-1}}m_\mu = \frac{\widetilde{\mathcal{K}}^+_{(i,k)} - \widetilde{\mathcal{K}}^-_{(i,k)}}{q-q^{-1}}(m_\mu),$$

proving (i).

If $q = 1$, we have

$$[\mu_i^{(k)} - \mu_{i+1}^{(k)}]m_\mu = (\mu_i^{(k)} - \mu_{i+1}^{(k)})m_\mu = (\mathcal{I}^+_{(i,k),0} - \mathcal{I}^-_{(i+1),0})(m_\mu),$$

which yields (ii). □

In the case where $q = 1$, we have the following lemma.

Lemma 7.28. *Assume that $q = 1$. Then, for $(j,l) \in \Gamma(\mathbf{m})$ and $t \geq 0$, we have:*

(i) $\mathcal{K}^\pm_{(j,l)} = 1$.

(ii) $\mathcal{I}^+_{(j,l),t} = \mathcal{I}^-_{(j,l),t}$.

Proof. If $q = 1$, we see that

$$\Phi^\pm_t(x_1,\dots,x_k) = x_1^t + \cdots + x_k^t, \tag{7.28.1}$$

in particular $\Phi^+_t(x_1,\dots,x_k) = \Phi^-_t(x_1,\dots,x_k)$. Thus, the lemma follows from the definitions. □

§8. The cyclotomic q-Schur algebra as a quotient of $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$

Let $\widetilde{\mathbf{Q}} = (Q_0, Q_1, \dots, Q_{r-1})$ be an r-tuple of indeterminates over $\mathbb{Z}$, and $\mathbb{Q}(\widetilde{\mathbf{Q}}) = \mathbb{Q}(Q_0, Q_1, \dots, Q_{r-1})$ be the quotient field of $\mathbb{Z}[\widetilde{\mathbf{Q}}] = \mathbb{Z}[Q_0, Q_1, \dots, Q_{r-1}]$. Set $\widetilde{\mathbb{A}} = \mathbb{Z}[q, q^{-1}, Q_0, Q_1, \dots, Q_{r-1}]$, and let $\widetilde{\mathbb{K}} = \mathbb{Q}(q, Q_0, Q_1, \dots, Q_{r-1})$ be the quotient field of $\widetilde{\mathbb{A}}$, where q is an indeterminate over $\mathbb{Z}$. Define

$$\begin{aligned}\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m}) &= \mathbb{Q}(\widetilde{\mathbf{Q}}) \otimes_{\mathbb{Q}(\mathbf{Q})} \mathfrak{g}_{\mathbf{Q}}(\mathbf{m}),\\ \mathcal{U}_{q,\widetilde{\mathbf{Q}}}(\mathbf{m}) &= \widetilde{\mathbb{K}} \otimes_{\mathbb{K}} \mathcal{U}_{q,\mathbf{Q}}(\mathbf{m}), \quad \mathcal{U}_{\widetilde{\mathbb{A}},q,\widetilde{\mathbf{Q}}}(\mathbf{m}) = \widetilde{\mathbb{A}} \otimes_{\mathbb{A}} \mathcal{U}_{\mathbb{A},q,\mathbf{Q}}(\mathbf{m}).\end{aligned}$$

We define a full subcategory $\mathcal{C}_{\widetilde{\mathbf{Q}}}(\mathbf{m})$ and $\mathcal{C}^{\geq 0}_{\widetilde{\mathbf{Q}}}(\mathbf{m})$ (resp. $\mathcal{C}_{q,\widetilde{\mathbf{Q}}}(\mathbf{m})$ and $\mathcal{C}^{\geq 0}_{q,\widetilde{\mathbf{Q}}}(\mathbf{m})$) of $U(\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m}))$-mod (resp. $\mathcal{U}_{q,\widetilde{\mathbf{Q}}}(\mathbf{m})$-mod) in a similar manner to $\mathcal{C}_{\mathbf{Q}}(\mathbf{m})$ and $\mathcal{C}^{\geq 0}_{\mathbf{Q}}(\mathbf{m})$ (resp. $\mathcal{C}_{q,\mathbf{Q}}(\mathbf{m})$ and $\mathcal{C}^{\geq 0}_{q,\mathbf{Q}}(\mathbf{m})$).

Let $\mathscr{H}^{\widetilde{\mathbb{K}}}_{n,r}$ (resp. $\mathscr{H}^{\widetilde{\mathbb{A}}}_{n,r}$) be the Ariki–Koike algebra over $\widetilde{\mathbb{K}}$ (resp. over $\widetilde{\mathbb{A}}$) with parameters $q, Q_0, Q_1, \dots, Q_{r-1}$, and let $\mathscr{S}^{\widetilde{\mathbb{K}}}_{n,r}(\mathbf{m})$ (resp. $\mathscr{S}^{\widetilde{\mathbb{A}}}_{n,r}(\mathbf{m})$) be the cyclotomic q-Schur algebra associated with $\mathscr{H}^{\widetilde{\mathbb{K}}}_{n,r}$ (resp. $\mathscr{H}^{\widetilde{\mathbb{A}}}_{n,r}$).

Theorem 8.1. *We have a homomorphism of algebras*

$$\Psi : \mathcal{U}_{q,\widetilde{\mathbf{Q}}}(\mathbf{m}) \to \mathscr{S}^{\widetilde{\mathbb{K}}}_{n,r}(\mathbf{m}) \tag{8.1.1}$$

defined by $\Psi(\mathcal{X}^{\pm}_{(i,k),t}) = \mathcal{X}^{\pm}_{(i,k),t}$, $\Psi(\mathcal{I}^{\pm}_{(j,l),t}) = \mathcal{I}^{\pm}_{(j,l),t}$ *and* $\Psi(\mathcal{K}^{\pm}_{(j,l)}) = \mathcal{K}^{\pm}_{(j,l)}$. *The restriction of* Ψ *to* $\mathcal{U}_{\widetilde{\mathbb{A}},q,\widetilde{\mathbf{Q}}}(\mathbf{m})$ *gives a homomorphism of algebras*

$$\Psi_{\widetilde{\mathbb{A}}} : \mathcal{U}_{\widetilde{\mathbb{A}},q,\widetilde{\mathbf{Q}}}(\mathbf{m}) \to \mathscr{S}^{\widetilde{\mathbb{A}}}_{n,r}(\mathbf{m}).$$

Moreover, if $m_k \geq n$ *for all* $k = 1, \dots, r-1$, *then* Ψ (*resp.* $\Psi_{\widetilde{\mathbb{A}}}$) *is surjective.*

Proof. That Ψ is well-defined follows from Lemmas 7.6, 7.7 and 7.16, Propositions 7.18 and 7.19, Lemma 7.20, and Propositions 7.21, 7.22, and 7.26.

Note that $\mathscr{H}^{\mathbb{A}}_{n,r}$ (resp. $\mathscr{S}^{\widetilde{\mathbb{A}}}_{n,r}(\mathbf{m})$) is an $\mathbb{A}$-subalgebra of $\mathscr{H}^{\widetilde{\mathbb{K}}}_{n,r}$ (resp. $\mathscr{S}^{\widetilde{\mathbb{K}}}_{n,r}(\mathbf{m})$) by the definitions. In particular, in order to see that $\varphi \in \mathscr{S}^{\widetilde{\mathbb{K}}}_{n,r}(\mathbf{m})$ belongs to $\mathscr{S}^{\widetilde{\mathbb{A}}}_{n,r}(\mathbf{m})$, it is enough to show that $\varphi(m_\mu) \in \mathscr{H}^{\mathbb{A}}_{n,r}$ for any $\mu \in \Lambda_{n,r}(\mathbf{m})$.

For $\mu \in \Lambda_{n,r}(\mathbf{m})$ and $d \in \mathbb{Z}_{\geq 0}$, we see that

$$\begin{bmatrix} \mathcal{K}_{(j,l)};0 \\ d \end{bmatrix}(m_\mu) = \begin{cases} \left[\begin{smallmatrix} \mu^{(l)}_j \\ d \end{smallmatrix}\right] m_\mu & \text{if } d \leq \mu^{(l)}_j, \\ 0 & \text{if } d > \mu^{(l)}_j, \end{cases} \tag{8.1.2}$$

in $\mathscr{S}^{\widetilde{\mathbb{K}}}_{n,r}(\mathbf{m})$. This implies that $\Psi(\left[\begin{smallmatrix} \mathcal{K}_{(j,l)};0 \\ d \end{smallmatrix}\right]) \in \mathscr{S}^{\widetilde{\mathbb{A}}}_{n,r}(\mathbf{m})$.

For $(i,k) \in \Gamma'(\mathbf{m})$ and $t, d \in \mathbb{Z}_{\geq 0}$, we see that

$$\begin{aligned}
&(\mathcal{X}^{+}_{(i,k),t})^d(m_\mu) \\
&\quad = q^{-d\mu^{(k)}_{i+1} + d(d+1)/2} m_{\mu + d\alpha_{(i,k)}} (L_{N^\mu_{(i,k)}+1} L_{N^\mu_{(i,k)}+2} \dots L_{N^\mu_{(i,k)}+d})^t \left[\begin{smallmatrix} T; N^\mu_{(i,k)}, \mu^{(k)}_{i+1} \\ d \end{smallmatrix}\right] \\
&\quad = q^{-d\mu^{(k)}_{i+1} + d(d+1)/2} m_{\mu + d\alpha_{(i,k)}} (L_{N^\mu_{(i,k)}+1} L_{N^\mu_{(i,k)}+2} \dots L_{N^\mu_{(i,k)}+d})^t \\
&\qquad \times (T; N^\mu_{(i,k)}, d)^{+}! \mathfrak{H}^{+}(N^\mu_{(i,k)}, \mu^{(k)}_{i+1}, d)
\end{aligned}$$

by Lemmas 7.17 and 7.11 and Corollary 7.14. We also see that $(T; N^\mu_{(i,k)}, d)^{+}!$ commutes with $(L_{N^\mu_{(i,k)}+1} L_{N^\mu_{(i,k)}+2} \dots L_{N^\mu_{(i,k)}+d})^t$ by Lemma 6.3(iii), and see that $m_{\mu+d\alpha_{(i,k)}} (T; N^\mu_{(i,k)}, d)^{+}! = q^{d(d-1)/2}[d]! m_{\mu+d\alpha_{(i,k)}}$ by (6.4.2). Thus

$$(\mathcal{X}^+_{(i,k),t})^d(m_\mu)$$
$$= [d]!q^{-d\mu^{(k)}_{i+1}+d^2} m_{\mu+d\alpha_{(i,k)}}(L_{N^\mu_{(i,k)}+1}L_{N^\mu_{(i,k)}+2}\dots L_{N^\mu_{(i,k)}+d})^t\mathfrak{H}^+(N^\mu_{(i,k)},\mu^{(k)}_{i+1},d)$$

in $\mathscr{S}^{\widetilde{\mathbb{K}}}_{n,r}(\mathbf{m})$. This implies that $\Psi(\mathcal{X}^{+(d)}_{(i,k)t}) \in \mathscr{S}^{\widetilde{\mathbb{A}}}_{n,r}(\mathbf{m})$ since $\mathfrak{H}^+(N^\mu_{(i,k)},\mu^{(k)}_{i+1},d) \in \mathscr{H}^{\mathbb{A}}_{n,r}$ by the proof of Corollary 7.14. Similarly, $\Psi(\mathcal{X}^{-(d)}_{(i,k),t}) \in \mathscr{S}^{\widetilde{\mathbb{A}}}_{n,r}(\mathbf{m})$. Thus, the restriction of Ψ to $\mathcal{U}_{\widetilde{\mathbb{A}},q,\widetilde{\mathbf{Q}}}(\mathbf{m})$ gives a homomorphism $\Psi_{\widetilde{\mathbb{A}}}$.

The last assertion follows from [W1, Proposition 6.4]. □

Remark 8.2. In order to prove the surjectivity of Ψ (resp. $\Psi_{\widetilde{\mathbb{A}}}$), we use [W1, Proposition 6.4]. In fact, in [W1] we only considered the case where $m_k = n$ for all $k = 1,\dots,r$. However, we can apply the result to the case where $m_k \geq n$ for all $k = 1,\dots,r-1$ without any change since the surjectivity in [W1, Proposition 6.4] follows from [DR]. The reason we assume $m_k \geq n$ for all $k = 1,\dots,r-1$ to establish the surjectivity of Ψ is just the use of the results of [DR]. We expect that Ψ is also surjective without this condition.

Theorem 8.3. *Assume that $m_k \geq n$ for all $k = 1,\dots,r-1$. Then:*

(i) $\mathscr{S}^{\widetilde{\mathbb{K}}}_{n,r}(\mathbf{m})$-mod *is a full subcategory of $\mathcal{C}^{\geq 0}_{q,\widetilde{\mathbf{Q}}}(\mathbf{m})$ through the surjection Ψ in* (8.1.1).

(ii) *The Weyl module $\Delta(\lambda) \in \mathscr{S}^{\widetilde{\mathbb{K}}}_{n,r}(\mathbf{m})$-mod ($\lambda \in \Lambda^+_{n,r}(\mathbf{m})$) is the simple highest weight $\mathcal{U}_{q,\widetilde{\mathbf{Q}}}(\mathbf{m})$-module of highest weight $(\lambda,\boldsymbol{\varphi})$ through the surjection Ψ, where the multiset $\boldsymbol{\varphi} = (\varphi^{\pm}_{(j,l),t} \in \widetilde{\mathbb{K}} \mid (j,l) \in \varGamma(\mathbf{m}),\, t \geq 1)$ is given by*

$$\varphi^+_{(j,l),t} = Q^t_{l-1}q^{(2t-1)\lambda^{(l)}_j - t(2j-1)}[\lambda^{(l)}_j] \text{ and } \varphi^-_{(j,l),t} = Q^t_{l-1}q^{\lambda^{(l)}_j - t(2j-1)}[\lambda^{(l)}_j].$$

Proof. For $\lambda \in \Lambda_{n,r}(\mathbf{m})$, let 1_λ be the element of $\mathscr{S}^{\widetilde{\mathbb{K}}}_{n,r}(\mathbf{m})$ that is the identity on M^λ and $1_\lambda(M^\mu) = 0$ for any $\mu \neq \lambda$. Then we have $1_\lambda 1_\mu = \delta_{\lambda\mu}1_\lambda$ and $\sum_{\lambda\in\Lambda_{n,r}(\mathbf{m})} 1_\lambda = 1$. Thus, for $M \in \mathscr{S}^{\widetilde{\mathbb{K}}}_{n,r}$-mod, we have the decomposition

$$M = \bigoplus_{\mu\in\Lambda_{n,r}(\mathbf{m})} 1_\mu M. \tag{8.3.1}$$

Moreover,

$$1_\mu M = \{m \in M \mid \mathcal{K}^+_{(j,l)} \cdot m = q^{\mu^{(l)}_j} m \text{ for } (j,l) \in \varGamma(\mathbf{m})\}$$

from the definition of Ψ. Thus, any object M of $\mathscr{S}^{\widetilde{\mathbb{K}}}_{n,r}$-mod has the weight space decomposition (8.3.1) as a $\mathcal{U}_{q,\widetilde{\mathbf{Q}}}(\mathbf{m})$-module, where $\Lambda_{n,r}(\mathbf{m}) \subset P_{\geq 0}$.

For $M \in \mathscr{S}^{\widetilde{\mathbb{K}}}_{n,r}(\mathbf{m})$-mod, in order to see that all eigenvalues of the action of $\mathcal{I}^{\pm}_{(j,l),t}$ ($(j,l) \in \varGamma(\mathbf{m}), t \geq 0$) on M belong to $\widetilde{\mathbb{K}}$, it is enough to show that for

$\Delta(\lambda)$ ($\lambda \in \Lambda^+_{n,r}(\mathbf{m})$) since $\mathscr{S}^{\widetilde{\mathbb{K}}}_{n,r}(\mathbf{m})$ is semisimple and $\{\Delta(\lambda) \mid \lambda \in \Lambda^+_{n,r}(\mathbf{m})\}$ gives a complete set of representatives of isomorphism classes of simple $\mathscr{S}^{\widetilde{\mathbb{K}}}_{n,r}$-modules. Recall that $\{\varphi_T \mid T \in \mathcal{T}_0(\lambda,\mu)$ for some $\mu \in \Lambda_{n,r}(\mathbf{m})\}$ gives a basis of $\Delta(\lambda)$.

Noting that $\Phi^{\pm}_t(L_{N^{\mu}_{(j,l)}}, L_{N^{\mu}_{(j,l)}-1}, \dots, L_{N^{\mu}_{(j,l)}-\mu^{(l)}_j+1})$ commutes with T_w for any $w \in \mathfrak{S}_\mu$ by Lemma 6.3, for $T \in \mathcal{T}_0(\lambda,\mu)$, we have

(8.3.2)

$$\mathcal{I}^{\pm}_{(j,l),t} \cdot \varphi_T = \begin{cases} q^{\pm(t-1)}\Phi^{\pm}_t(\mathrm{res}_{(j,l);T})\varphi_T + \sum_{S \rhd T} r_S\varphi_S \quad (r_S \in \widetilde{\mathbb{K}}) & \text{if } \mu^{(l)}_j \neq 0, \\ 0 & \text{if } \mu^{(l)}_j = 0, \end{cases}$$

by a similar argument to the proof of [JM, Theorem 3.10], where

$$\Phi^{\pm}_t(\mathrm{res}_{(j,l);T}) = \Phi^{\pm}_t(\mathrm{res}(x_1), \dots, \mathrm{res}(x_{\mu^{(l)}_j}))$$

with $\{x_1, \dots, x_{\mu^{(l)}_j}\} = \{x \in [\lambda] \mid T(x) = (j,l)\}$, and $\rhd$ is a partial order on $\mathcal{T}_0(\lambda,\mu)$ defined in [JM, Definition 3.6]. This implies that all eigenvalues of the action of $\mathcal{I}^{\pm}_{(j,l),t}$ on $\Delta(\lambda)$ belong to $\widetilde{\mathbb{K}}$. Thus we have proved (i).

We now prove (ii). For $\lambda \in \Lambda^+_{n,r}(\mathbf{m})$, let T^λ be the unique semistandard tableau of shape λ with weight λ. Then we see easily that φ_{T^λ} is a highest weight vector of $\Delta(\lambda)$. Noting that there is no tableau such that $S \rhd T^\lambda$, we have

$$\varphi^{\pm}_{(j,l),t} = q^{\pm(t-1)}\Phi^{\pm}_t(Q_k q^{2(1-j)}, Q_k q^{2(2-j)}, \dots, Q_k q^{2(\lambda^{(l)}_j - j)}) \tag{8.3.3}$$

by (8.3.2). Then (ii) follows by induction on t using (8.3.3) and (7.3.1). □

Let $\mathscr{S}^{\mathbf{1}}_{n,r}(\mathbf{m})$ be the cyclotomic q-Schur algebra over $\mathbb{Q}(\widetilde{\mathbf{Q}})$ with parameters $q = 1, Q_0, Q_1, \dots, Q_{r-1}$.

Theorem 8.4. (i) *We have a homomorphism of algebras*

$$\Psi_{\mathbf{1}} : U(\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m})) \to \mathscr{S}^{\mathbf{1}}_{n,r}(\mathbf{m}) \tag{8.4.1}$$

defined by $\Psi_{\mathbf{1}}(\mathcal{X}^{\pm}_{(i,k),t}) = \mathcal{X}^{\pm}_{(i,k),t}$ *and* $\Psi_{\mathbf{1}}(\mathcal{I}_{(j,l),t}) = \mathcal{I}^{+}_{(j,l),t}$ $(= \mathcal{I}^{-}_{(j,l),t})$. *Moreover, if* $m_k \geq n$ *for all* $k = 1, \dots, r-1$, *then* $\Psi_{\mathbf{1}}$ *is surjective.*

(ii) *Assume that* $m_k \geq n$ *for all* $k = 1, 2, \dots, r-1$. *Then* $\mathscr{S}^{\mathbf{1}}_{n,r}(\mathbf{m})$-mod *is a full subcategory of* $\mathcal{C}^{\geq 0}_{\widetilde{\mathbf{Q}}}(\mathbf{m})$ *through the surjection* $\Psi_{\mathbf{1}}$. *Moreover, the Weyl module* $\Delta(\lambda) \in \mathscr{S}^{\mathbf{1}}_{n,r}(\mathbf{m})$-mod ($\lambda \in \Lambda^+_{n,r}$) *is the simple highest weight* $U(\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m}))$-*module of highest weight* $(\lambda, \boldsymbol{\varphi})$ *through the surjection* $\Psi_{\mathbf{1}}$, *where the multiset* $\boldsymbol{\varphi} = (\varphi_{(j,l),t} \in \mathbb{Q}(\widetilde{\mathbf{Q}}) \mid (j,l) \in \Gamma(\mathbf{m}), t \geq 1)$ *is given by*

$$\varphi_{(j,l),t} = Q^t_{l-1}\lambda^{(l)}_j.$$

Proof. Invoking Lemmas 7.27 and 7.28, we can argue in a similar way to the proof of Theorems 8.1 and 8.3. □

§9. Characters of Weyl modules of cyclotomic q-Schur algebras

In this section, we study the characters of Weyl modules of cyclotomic q-Schur algebras as symmetric polynomials. In particular, we prove the conjecture given in [W2] (formula (9.2.1) below) which will be understood as the decomposition of the tensor product of Weyl modules in the case where $q = 1$.

9.1. Characters. For $k = 1, \dots, r$, let $\mathbf{x}_{\mathbf{m}}^{(k)} = (x_{(1,k)}, \dots, x_{(m_k,k)})$ be a set of m_k independent variables, and write $\mathbf{x}_{\mathbf{m}} = \bigcup_{k=1}^{r} \mathbf{x}_{\mathbf{m}}^{(k)}$. Let $\mathbb{Z}[\mathbf{x}_{\mathbf{m}}^{\pm}]$ (resp. $\mathbb{Z}[\mathbf{x}_{\mathbf{m}}]$) be the ring of Laurent polynomials (resp. the ring of polynomials) in variables $\mathbf{x}_{\mathbf{m}}$. For $\lambda \in P$, we define the monomial $x^\lambda \in \mathbb{Z}[\mathbf{x}_{\mathbf{m}}^{\pm}]$ by $x^\lambda = \prod_{k=1}^{r} \prod_{i=1}^{m_k} x_{(i,k)}^{\langle \lambda, h_{(i,k)} \rangle}$.

For $M \in \mathcal{C}_{\widetilde{\mathbf{Q}}}(\mathbf{m})$ (resp. $M \in \mathcal{C}_{q,\widetilde{\mathbf{Q}}}(\mathbf{m})$), we define the *character* of M by

$$\operatorname{ch} M = \sum_{\lambda \in P} \dim M_\lambda x^\lambda \in \mathbb{Z}[\mathbf{x}_{\mathbf{m}}^{\pm}]. \tag{9.1.1}$$

It is clear that $\operatorname{ch} M \in \mathbb{Z}[\mathbf{x}_{\mathbf{m}}]$ if $M \in \mathcal{C}_{\widetilde{\mathbf{Q}}}^{\geq 0}(\mathbf{m})$ (resp. $M \in \mathcal{C}_{q,\widetilde{\mathbf{Q}}}^{\geq 0}(\mathbf{m})$).

When we regard $M \in \mathcal{C}_{\widetilde{\mathbf{Q}}}(\mathbf{m})$ as a $U(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r})$-module through the injection (2.16.2), $\operatorname{ch} M$ defined by (9.1.1) coincides with the character of M as a $U(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r})$-module since M_λ is also the weight space of weight λ as a $U(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r})$-module. Thus, by the known results for $U(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r})$-modules, we see that

$$\operatorname{ch} M \in \bigotimes_{k=1}^{r} \mathbb{Z}[\mathbf{x}_{\mathbf{m}}^{(k)}]^{\mathfrak{S}_{m_k}} \qquad \text{if } M \in \mathcal{C}_{\widetilde{\mathbf{Q}}}^{\geq 0}(\mathbf{m}),$$

where $\mathbb{Z}[\mathbf{x}_{\mathbf{m}}^{(k)}]^{\mathfrak{S}_{m_k}}$ is the ring of symmetric polynomials in variables $\mathbf{x}_{\mathbf{m}}^{(k)}$, and we regard $\bigotimes_{k=1}^{r} \mathbb{Z}[\mathbf{x}_{\mathbf{m}}^{(k)}]^{\mathfrak{S}_{m_k}}$ as a subring of $\mathbb{Z}[\mathbf{x}_{\mathbf{m}}]$ through the multiplication map $\bigotimes_{k=1}^{r} \mathbb{Z}[\mathbf{x}_{\mathbf{m}}^{(k)}]^{\mathfrak{S}_{m_k}} \to \mathbb{Z}[\mathbf{x}_{\mathbf{m}}]$ $(\otimes_{k=1}^{r} f(\mathbf{x}_{\mathbf{m}}^{(k)}) \mapsto \prod_{k=1}^{r} f(\mathbf{x}_{\mathbf{m}}^{(k)}))$. The situation is similar for $M \in \mathcal{C}_{q,\widetilde{\mathbf{Q}}}(\mathbf{m})$ through the injection (4.9.2).

9.2. The character of the Weyl module $\Delta(\lambda) \in \mathscr{S}_{n,r}(\mathbf{m})$ $(\lambda \in \widetilde{\Lambda}_{n,r}^{+}(\mathbf{m}))$ is studied in [W2]. Note that $\operatorname{ch} \Delta(\lambda)$ $(\lambda \in \widetilde{\Lambda}_{n,r}^{+}(\mathbf{m}))$ does not depend on the choice of the base field and parameters. Set $\widetilde{\Lambda}_{\geq 0,r}^{+}(\mathbf{m}) = \bigcup_{n \geq 0} \widetilde{\Lambda}_{n,r}^{+}(\mathbf{m})$. For $\lambda, \mu \in \widetilde{\Lambda}_{\geq 0,r}^{+}(\mathbf{m})$, the following formula was conjectured in [W2, Conjecture 2]:

$$\operatorname{ch} \Delta(\lambda) \operatorname{ch} \Delta(\mu) = \sum_{\nu \in \widetilde{\Lambda}_{\geq 0,r}^{+}(\mathbf{m})} \mathrm{LR}_{\lambda\mu}^{\nu} \operatorname{ch} \Delta(\nu) \qquad \text{for } \lambda, \mu \in \widetilde{\Lambda}_{\geq 0,r}^{+}(\mathbf{m}), \tag{9.2.1}$$

where $\mathrm{LR}^{\nu}_{\lambda\mu} = \prod_{k=1}^{r} \mathrm{LR}^{\nu^{(k)}}_{\lambda^{(k)}\mu^{(k)}}$, and $\mathrm{LR}^{\nu^{(k)}}_{\lambda^{(k)}\mu^{(k)}}$ is the Littlewood–Richardson coefficient for the partitions $\lambda^{(k)}$, $\mu^{(k)}$ and $\nu^{(k)}$. We now prove this conjecture.

9.3. For $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)}) \in \widetilde{\Lambda}^{+}_{n,r}(\mathbf{m})$, we denote

$$(\underbrace{0, \dots, 0}_{k-1}, \lambda^{(k)}, 0, \dots, 0) \in \widetilde{\Lambda}^{+}_{n_k,r}(\mathbf{m})$$

by $(0, \dots, \lambda^{(k)}, \dots, 0)$, where $n_k = \sum_{i=1}^{m_k} \lambda_i^{(k)}$ (i.e. $\lambda^{(k)}$ appears in the k-th component in $(0, \dots, \lambda^{(k)}, \dots, 0)$). Let

$$S_{\lambda^{(k)}}(\mathbf{x}_{\mathbf{m}}^{(k)} \cup \dots \cup \mathbf{x}_{\mathbf{m}}^{(r)}) \in \mathbb{Z}[\mathbf{x}_{\mathbf{m}}^{(k)} \cup \dots \cup \mathbf{x}_{\mathbf{m}}^{(r)}]^{\mathfrak{S}(\mathbf{x}_{\mathbf{m}}^{(k)} \cup \dots \cup \mathbf{x}_{\mathbf{m}}^{(r)})}$$

be the Schur polynomial for the partition $\lambda^{(k)}$ in variables $\mathbf{x}_{\mathbf{m}}^{(k)} \cup \dots \cup \mathbf{x}_{\mathbf{m}}^{(r)}$, where we regard $\mathbb{Z}[\mathbf{x}_{\mathbf{m}}^{(k)} \cup \dots \cup \mathbf{x}_{\mathbf{m}}^{(r)}]^{\mathfrak{S}(\mathbf{x}_{\mathbf{m}}^{(k)} \cup \dots \cup \mathbf{x}_{\mathbf{m}}^{(r)})}$ as a subring of $\bigotimes_{k=1}^{r} \mathbb{Z}[\mathbf{x}_{\mathbf{m}}^{(k)}]^{\mathfrak{S}_{m_k}} \subset \mathbb{Z}[\mathbf{x}_{\mathbf{m}}]$ in the natural way. Set $\widetilde{S}_{\lambda}(\mathbf{x}_{\mathbf{m}}) = \operatorname{ch} \Delta(\lambda)$ $(\lambda \in \widetilde{\Lambda}^{+}_{\geq 0,r}(\mathbf{m}))$.

Proposition 9.4. *For $\lambda, \mu \in \widetilde{\Lambda}^{+}_{\geq 0,r}(\mathbf{m})$, we have:*

(i) $\widetilde{S}_{(0,\dots,\lambda^{(k)},\dots,0)}(\mathbf{x}_{\mathbf{m}}) = S_{\lambda^{(k)}}(\mathbf{x}_{\mathbf{m}}^{(k)} \cup \dots \cup \mathbf{x}_{\mathbf{m}}^{(r)})$.

(ii) $\widetilde{S}_{\lambda}(\mathbf{x}_{\mathbf{m}}) = \prod_{k=1}^{r} \widetilde{S}_{(0,\dots,\lambda^{(k)},\dots,0)}(\mathbf{x}_{\mathbf{m}})$.

(iii) $\widetilde{S}_{\lambda}(\mathbf{x}_{\mathbf{m}})\widetilde{S}_{\mu}(\mathbf{x}_{\mathbf{m}}) = \sum_{\nu \in \widetilde{\Lambda}^{+}_{\geq 0,r}(\mathbf{m})} \mathrm{LR}^{\nu}_{\lambda\mu}\, \widetilde{S}_{\nu}(\mathbf{x}_{\mathbf{m}})$.

Proof. (i) By the definition of the cellular basis of $\mathscr{S}_{n,r}(\mathbf{m})$ in [DJM], for $\lambda \in \widetilde{\Lambda}^{+}_{n,r}(\mathbf{m})$ we have

$$\widetilde{S}_{\lambda}(\mathbf{x}_{\mathbf{m}}) = \operatorname{ch} \Delta(\lambda) = \sum_{\mu \in \Lambda_{n,r}(\mathbf{m})} \sharp \mathcal{T}_0(\lambda, \mu) x^{\mu}. \tag{9.4.1}$$

Thus,

$$\widetilde{S}_{(0,\dots,\lambda^{(k)},\dots,0)}(\mathbf{x}_{\mathbf{m}}) = \sum_{\mu \in \Lambda_{n_k,r}(\mathbf{m})} \sharp \mathcal{T}_0((0, \dots, \lambda^{(k)}, \dots, 0), \mu) x^{\mu}, \tag{9.4.2}$$

where $n_k = \sum_{i=1}^{m_k} \lambda_i^{(k)}$. We see that

$$\mu^{(1)} = \dots = \mu^{(k-1)} = 0 \quad \text{if } \mathcal{T}_0((0, \dots, \lambda^{(k)}, \dots, 0), \mu) \neq \emptyset$$

by the definition of semistandard tableaux. Thus, $\widetilde{S}_{(0,\dots,\lambda^{(k)},\dots,0)}(\mathbf{x}_{\mathbf{m}}) \in \bigotimes_{l=k}^{r} \mathbb{Z}[\mathbf{x}_{\mathbf{m}}^{(l)}]^{\mathfrak{S}_{m_k}}$. Write

$$\Lambda^{\geq k}_{n_k,r}(\mathbf{m}) = \{\mu = (\mu^{(1)}, \dots, \mu^{(r)}) \in \Lambda_{n_k,r}(\mathbf{m}) \mid \mu^{(l)} = 0 \quad \text{for } l = 1, \dots, k-1\}.$$

Set $m' = m_k + \cdots + m_r$. We identify the set $\Lambda_{n_k,1}(m')$ with $\Lambda^{\geq k}_{n_k,r}(\mathbf{m})$ by the bijection $\theta^k : \Lambda_{n_k,1}(m') \to \Lambda^{\geq k}_{n_k,r}(\mathbf{m})$ such that

$$(\theta^k(\mu))^{(k+l)}_i = \begin{cases} \mu_i & \text{if } l = 0, \\ \mu_{m_k+m_{k+1}+\cdots+m_{k+l-1}+i} & \text{if } 1 \leq l \leq r-k, \end{cases}$$

for $\mu = (\mu_1, \ldots, \mu_{m'}) \in \Lambda_{n_k,1}(m')$. It is well-known that we can describe the Schur polynomial $S_{\lambda^{(k)}}(\mathbf{x}^{(k)}_{\mathbf{m}} \cup \cdots \cup \mathbf{x}^{(r)}_{\mathbf{m}})$ as

$$S_{\lambda^{(k)}}(\mathbf{x}^{(k)}_{\mathbf{m}} \cup \cdots \cup \mathbf{x}^{(r)}_{\mathbf{m}}) = \sum_{\mu \in \Lambda_{n_k,1}(m')} \sharp\mathcal{T}_0(\lambda^{(k)}, \mu) x^{\mu}, \tag{9.4.3}$$

where $x^{\mu} = \prod_{i=1}^{m_k} x^{\mu_i}_{(i,k)} \prod_{l=1}^{r-k} \prod_{i=1}^{m_l} x^{\mu_{m_k+m_{k+1}+\cdots+m_{k+l-1}+i}}_{(i,k+l)}$. From the definition of semistandard tableaux, we see that

$$\sharp\mathcal{T}_0(\lambda^{(k)}, \mu) = \sharp\mathcal{T}_0((0, \ldots, \lambda^{(k)}, \ldots, 0), \theta^k(\mu))$$

for $\mu \in \Lambda_{n_k,1}(m')$. Thus, by comparing the right hand sides of (9.4.2) and of (9.4.3), we obtain (i).

(ii) First we prove that

$$\widetilde{S}_{(\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(r)})}(\mathbf{x}_{\mathbf{m}}) = \widetilde{S}_{(\lambda^{(1)},0,\ldots,0)}(\mathbf{x}_{\mathbf{m}}) \widetilde{S}_{(0,\lambda^{(2)},\ldots,\lambda^{(r)})}(\mathbf{x}_{\mathbf{m}}). \tag{9.4.4}$$

By (9.4.1),

$$\widetilde{S}_{(\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(r)})}(\mathbf{x}_{\mathbf{m}}) = \sum_{\mu \in \Lambda_{n,r}(\mathbf{m})} \sharp\mathcal{T}_0(\lambda, \mu)\, x^{\mu}. \tag{9.4.5}$$

On the other hand,

$$\begin{aligned} &\widetilde{S}_{(\lambda^{(1)},0,\ldots,0)}(\mathbf{x}_{\mathbf{m}}) \widetilde{S}_{(0,\lambda^{(2)},\ldots,\lambda^{(r)})}(\mathbf{x}_{\mathbf{m}}) \\ &= \Big(\sum_{\nu \in \Lambda_{n_1,r}(\mathbf{m})} \sharp\mathcal{T}_0((\lambda^{(1)}, 0, \ldots, 0), \nu)\, x^{\nu} \Big) \Big(\sum_{\tau \in \Lambda_{n',r}(\mathbf{m})} \sharp\mathcal{T}_0((0, \lambda^{(2)}, \ldots, \lambda^{(r)}), \tau)\, x^{\tau} \Big) \\ &= \sum_{\mu \in \Lambda_{n,r}(\mathbf{m})} \Big(\sum_{\substack{\nu \in \Lambda_{n_1,r}(\mathbf{m}) \\ \tau \in \Lambda_{n',r}(\mathbf{m}) \\ \nu+\tau=\mu}} \sharp\mathcal{T}_0((\lambda^{(1)}, 0, \ldots, 0), \nu) \sharp\mathcal{T}_0((0, \lambda^{(2)}, \ldots, \lambda^{(r)}), \tau) \Big) x^{\mu} \end{aligned} \tag{9.4.6}$$

where $n_1 = \sum_{i=1}^{m_1} \lambda^{(1)}_i$ and $n' = n - n_1$. From the definition of semistandard tableaux, we can check that

$$\sharp\mathcal{T}_0(\lambda, \mu) = \sum_{\substack{\nu \in \Lambda_{n_1,r}(\mathbf{m}) \\ \tau \in \Lambda_{n',r}(\mathbf{m}) \\ \nu+\tau=\mu}} \sharp\mathcal{T}_0((\lambda^{(1)}, 0, \ldots, 0), \nu) \sharp\mathcal{T}_0((0, \lambda^{(2)}, \ldots, \lambda^{(r)}), \tau). \tag{9.4.7}$$

Thus, (9.4.5)–(9.4.7) imply (9.4.4). By applying a similar argument to $\widetilde{S}_{(0,\lambda^{(2)},\dots,\lambda^{(r)})}(\mathbf{x}_{\mathbf{m}})$ inductively, we obtain (ii).

By (i) and (ii), we have

$$\begin{aligned}\widetilde{S}_{\lambda}(\mathbf{x}_{\mathbf{m}})\widetilde{S}_{\mu}(\mathbf{x}_{\mathbf{m}}) &= \Big(\prod_{k=1}^{r} \widetilde{S}_{(0,\dots,\lambda^{(k)},\dots,0)}(\mathbf{x}_{\mathbf{m}})\Big)\Big(\prod_{k=1}^{r} \widetilde{S}_{(0,\dots,\mu^{(k)},\dots,0)}(\mathbf{x}_{\mathbf{m}})\Big)\\
&= \Big(\prod_{k=1}^{r} S_{\lambda^{(k)}}(\mathbf{x}_{\mathbf{m}}^{(k)}\cup\cdots\cup\mathbf{x}_{\mathbf{m}}^{(r)})\Big)\Big(\prod_{k=1}^{r} S_{\mu^{(k)}}(\mathbf{x}_{\mathbf{m}}^{(k)}\cup\cdots\cup\mathbf{x}_{\mathbf{m}}^{(r)})\Big)\\
&= \prod_{k=1}^{r} S_{\lambda^{(k)}}(\mathbf{x}_{\mathbf{m}}^{(k)}\cup\cdots\cup\mathbf{x}_{\mathbf{m}}^{(r)}) S_{\mu^{(k)}}(\mathbf{x}_{\mathbf{m}}^{(k)}\cup\cdots\cup\mathbf{x}_{\mathbf{m}}^{(r)})\\
&= \prod_{k=1}^{r}\Big(\sum_{\nu^{(k)}\in\Lambda^{+}_{\geq 0,1}(m_k+\cdots+m_r)} \mathrm{LR}^{\nu^{(k)}}_{\lambda^{(k)}\mu^{(k)}}\, S_{\nu^{(k)}}(\mathbf{x}_{\mathbf{m}}^{(k)}\cup\cdots\cup\mathbf{x}_{\mathbf{m}}^{(r)})\Big)\\
&= \sum_{\nu\in\widetilde{\Lambda}^{+}_{\geq 0,r}(\mathbf{m})}\Big(\prod_{k=1}^{r}\mathrm{LR}^{\nu^{(k)}}_{\lambda^{(k)}\mu^{(k)}}\Big)\prod_{k=1}^{r}\widetilde{S}_{(0,\dots,\nu^{(k)},\dots,0)}(\mathbf{x}_{\mathbf{m}})\\
&= \sum_{\nu\in\widetilde{\Lambda}^{+}_{\geq 0,r}(\mathbf{m})}\mathrm{LR}^{\nu}_{\lambda\mu}\,\widetilde{S}_{\nu}(\mathbf{x}_{\mathbf{m}}),\end{aligned}$$

where if $\ell(\lambda^{(k)}) > m_k+\cdots+m_r$ for some k, then $S_{\lambda^{(k)}}(\mathbf{x}_{\mathbf{m}}^{(k)}\cup\cdots\cup\mathbf{x}_{\mathbf{m}}^{(r)})=0$ and $\mathcal{T}_0(\lambda,\mu)=\emptyset$ for any $\mu\in\Lambda_{n,r}(\mathbf{m})$. This yields (iii). □

§10. Tensor products for Weyl modules of cyclotomic q-Schur algebras at $q=1$

By using the comultiplication $\Delta : U(\mathfrak{g}_{\widetilde{Q}}(\mathbf{m}))\to U(\mathfrak{g}_{\widetilde{Q}}(\mathbf{m}))\otimes U(\mathfrak{g}_{\widetilde{Q}}(\mathbf{m}))$ ($\Delta(x) = x\otimes 1+1\otimes x$), we define the $U(\mathfrak{g}_{\widetilde{Q}}(\mathbf{m}))$-module $M\otimes N$ for $U(\mathfrak{g}_{\widetilde{Q}}(\mathbf{m}))$-modules M and N. We regard $\mathscr{S}^{\mathbf{1}}_{n,r}(\mathbf{m})$-modules ($n\geq 0$) as $U(\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m}))$-modules through the homomorphism $\Psi_{\mathbf{1}}$ of (8.4.1). Note that $\mathscr{S}^{\mathbf{1}}_{n,r}(\mathbf{m})$ is semisimple, and $\{\Delta(\lambda)\mid \lambda\in\Lambda^{+}_{n,r}(\mathbf{m})\}$ gives a complete set of representatives of isomorphism classes of simple $\mathscr{S}^{\mathbf{1}}_{n,r}(\mathbf{m})$-modules if $m_k\geq n$ for all $k=1,\dots,r-1$.

Proposition 10.1. *Assume that $m_k\geq n$ for all $k=1,\dots,r-1$. Let $n_1,n_2\in\mathbb{Z}_{>0}$ with $n=n_1+n_2$. For $\lambda\in\Lambda^{+}_{n_1,r}(\mathbf{m})$ (resp. $\mu\in\Lambda^{+}_{n_2,r}(\mathbf{m})$), let $\Delta(\lambda)$ (resp. $\Delta(\mu)$) be the Weyl module of $\mathscr{S}^{\mathbf{1}}_{n_1,r}(\mathbf{m})$ (resp. $\mathscr{S}^{\mathbf{1}}_{n_2,r}(\mathbf{m})$) corresponding to λ (resp. μ). Then*

$$(10.1.1)\qquad \Delta(\lambda)\otimes\Delta(\mu)\cong\bigoplus_{\nu\in\Lambda^{+}_{n,r}(\mathbf{m})}\mathrm{LR}^{\nu}_{\lambda\mu}\,\Delta(\nu)\quad \textit{as } U(\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m}))\textit{-modules},$$

where $\Delta(\nu)$ is the Weyl module of $\mathscr{S}^{\mathbf{1}}_{n,r}(\mathbf{m})$ corresponding to ν, and $\mathrm{LR}^{\nu}_{\lambda\mu}\,\Delta(\nu)$ means the direct sum of $\mathrm{LR}^{\nu}_{\lambda\mu}$ copies of $\Delta(\nu)$. In particular, $\Delta(\lambda)\otimes\Delta(\mu)\in\mathscr{S}^{\mathbf{1}}_{n,r}(\mathbf{m})$-mod.

Proof. For $\tau\in P_{\geq 0}$, set

$$\pi_{\mathbf{m}}(\tau)=(|\tau^{(1)}|,\dots,|\tau^{(r)}|)\in\mathbb{Z}^{r}_{\geq 0},$$

where $|\tau^{(l)}|=\sum_{j=1}^{m_l}\langle\tau,h_{(j,l)}\rangle$ for $l=1,\dots,r$. We denote by $\geq$ the lexicographic order on $\mathbb{Z}^{r}_{\geq 0}$. Then we have the weight space decomposition

$$\Delta(\lambda)\otimes\Delta(\mu)=\bigoplus_{\substack{\tau\in\Lambda_{n,r}(\mathbf{m})\\ \pi_{\mathbf{m}}(\tau)\leq\pi_{\mathbf{m}}(\lambda+\mu)}}(\Delta(\lambda)\otimes\Delta(\mu))_{\tau}. \tag{10.1.2}$$

On the other hand, it is clear that $\Delta(\lambda)\otimes\Delta(\mu)\in\mathcal{C}^{\geq 0}_{\widetilde{\mathbf{Q}}}(\mathbf{m})$. Thus,

$$[\Delta(\lambda)\otimes\Delta(\mu)]=\sum_{\substack{\nu\in\Lambda^{+}_{n,r}(\mathbf{m})\\ \pi_{\mathbf{m}}(\nu)\leq\pi_{\mathbf{m}}(\lambda+\mu)}}\sum_{\boldsymbol{\varphi}}d_{\nu,\boldsymbol{\varphi}}[L(\nu,\boldsymbol{\varphi})]\quad\text{in }K_0(\mathcal{C}^{\geq 0}_{\widetilde{\mathbf{Q}}}(\mathbf{m})), \tag{10.1.3}$$

where $d_{\nu,\boldsymbol{\varphi}}$ is the composition multiplicity of the simple highest weight $U(\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m}))$-module $L(\nu,\boldsymbol{\varphi})$ of highest weight $(\nu,\boldsymbol{\varphi})$ in $\Delta(\lambda)\otimes\Delta(\mu)$.

Note that $L_{i+1}T_i=T_iL_i$ and $L_iT_i=T_iL_{i+1}$ since $q=1$. Then, for $(j,l)\in\Gamma(\mathbf{m})$ and $t\geq 1$, we see that

$$\mathcal{I}_{(j,l),t}\cdot v=Q^{t}_{l-1}\nu^{(l)}_{j}v\quad\text{for any }v\in(\Delta(\lambda)\otimes\Delta(\mu))_{\nu} \tag{10.1.4}$$

if $\pi_{\mathbf{m}}(\nu)=\pi_{\mathbf{m}}(\lambda+\mu)$ by the argument in [JM, proofs of Proposition 3.7 and Theorem 3.10]. This implies that

$$L(\nu,\boldsymbol{\varphi})\cong\Delta(\nu)\quad\text{if }d_{\nu,\boldsymbol{\varphi}}\neq 0\text{ and }\pi_{\mathbf{m}}(\nu)=\pi_{\mathbf{m}}(\lambda+\mu) \tag{10.1.5}$$

by Theorem 8.4(ii). By Proposition 9.4(iii) together with (10.1.3) and (10.1.5),

$$\begin{aligned}\operatorname{ch}(\Delta(\lambda)\otimes\Delta(\mu))&=\widetilde{S}_{\lambda}(\mathbf{x_m})\widetilde{S}_{\mu}(\mathbf{x_m})=\sum_{\nu\in\Lambda^{+}_{n,r}(\mathbf{m})}\mathrm{LR}^{\nu}_{\lambda\mu}\,\widetilde{S}_{\nu}(\mathbf{x_m})\\ &=\sum_{\substack{\nu\in\Lambda^{+}_{n,r}(\mathbf{m})\\ \pi_{\mathbf{m}}(\nu)=\pi_{\mathbf{m}}(\lambda+\mu)}}d_{\nu}\widetilde{S}_{\nu}(\mathbf{x_m})+\sum_{\substack{\nu\in\Lambda^{+}_{n,r}(\mathbf{m})\\ \pi_{\mathbf{m}}(\nu)<\pi_{\mathbf{m}}(\lambda+\mu)}}\sum_{\boldsymbol{\varphi}}d_{\nu,\boldsymbol{\varphi}}\operatorname{ch}L(\nu,\boldsymbol{\varphi}),\end{aligned} \tag{10.1.6}$$

where d_{ν} is the composition multiplicity of $\Delta(\nu)$ in $\Delta(\lambda)\otimes\Delta(\mu)$. Note $\mathrm{LR}^{\nu}_{\lambda\mu}=0$ unless $\pi_{\mathbf{m}}(\nu)=\pi_{\mathbf{m}}(\lambda+\mu)$. Then (10.1.6) implies $d_{\nu}=\mathrm{LR}^{\nu}_{\lambda\mu}$ if $\pi_{\mathbf{m}}(\nu)=\pi_{\mathbf{m}}(\lambda+\mu)$, and $d_{\nu,\boldsymbol{\varphi}}=0$ if $\pi_{\mathbf{m}}(\nu)<\pi_{\mathbf{m}}(\lambda+\mu)$. Thus,

$$[\Delta(\lambda)\otimes\Delta(\mu)]=\sum_{\nu\in\Lambda^{+}_{n,r}(\mathbf{m})}\mathrm{LR}^{\nu}_{\lambda\mu}[\Delta(\nu)]. \tag{10.1.7}$$

By (10.1.2), for any $k = 1, \dots, r-1$ and any $t \geq 0$, we have

$$\mathcal{X}^+_{(m_k,k),t} \cdot \Big(\bigoplus_{\substack{\nu \in \Lambda_{n,r}(\mathbf{m}) \\ \pi_{\mathbf{m}}(\nu) = \pi_{\mathbf{m}}(\lambda+\mu)}} (\Delta(\lambda) \otimes \Delta(\mu))_\nu \Big) = 0 \tag{10.1.8}$$

since $\pi_{\mathbf{m}}(\nu + \alpha_{(m_k,k)}) > \pi_{\mathbf{m}}(\nu)$. Then, by (10.1.4) and (10.1.8) together with the relation (L2), we see that

$$\begin{aligned} &\{v \in (\Delta(\lambda) \otimes \Delta(\mu))_\nu \mid \mathcal{X}^+_{(i,k),t} \cdot v \text{ for all } (i,k) \in \Gamma'(\mathbf{m}) \text{ and } t \geq 0\} \\ &= \{v \in (\Delta(\lambda) \otimes \Delta(\mu))_\nu \mid e_{(i,k)} \cdot v \text{ for all } (i,k) \in \Gamma(\mathbf{m}) \setminus \{(m_k,k) \mid 1 \leq k \leq r\}\} \end{aligned} \tag{10.1.9}$$

for $\nu \in \Lambda^+_{n,r}(\mathbf{m})$ with $\pi_{\mathbf{m}}(\nu) = \pi_{\mathbf{m}}(\lambda+\mu)$, where $e_{(i,k)} \in U(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r})$ acts on $\Delta(\lambda) \otimes \Delta(\mu)$ through the injection (2.16.2). On the other hand, $\bigoplus_{\nu \in \Lambda_{n,r}(\mathbf{m}),\, \pi_{\mathbf{m}}(\nu) = \pi_{\mathbf{m}}(\lambda+\mu)} (\Delta(\lambda) \otimes \Delta(\mu))_\nu$ is a $U(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r})$-submodule of $\Delta(\lambda) \otimes \Delta(\mu)$ and

(10.1.10)
$$\bigoplus_{\substack{\nu \in \Lambda_{n,r}(\mathbf{m}) \\ \pi_{\mathbf{m}}(\nu) = \pi_{\mathbf{m}}(\lambda+\mu)}} (\Delta(\lambda) \otimes \Delta(\mu))_\nu \cong \bigoplus_{\nu \in \Lambda^+_{n,r}(\mathbf{m})} \mathrm{LR}^\nu_{\lambda\mu}\, \Delta_{\mathfrak{gl}_{m_1}}(\nu^{(1)}) \otimes \cdots \otimes \Delta_{\mathfrak{gl}_{m_r}}(\nu^{(r)})$$

as $U(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r})$-modules by comparing the character (note [W2, Lemma 2.6]). By (10.1.7), (10.1.9) and (10.1.10), we see that

$$\Delta(\lambda) \otimes \Delta(\mu) \cong \bigoplus_{\nu \in \Lambda^+_{n,r}(\mathbf{m})} \mathrm{LR}^\nu_{\lambda\mu}\, \Delta(\nu)$$

as $U(\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m}))$-modules. □

Remarks 10.2. (i) For $M, N \in \mathcal{C}_{\widetilde{Q}}(\mathbf{m})$, we see that $\mathrm{ch}(M \otimes N) = \mathrm{ch}(M)\,\mathrm{ch}(N)$ by definition of characters. Then the decomposition (10.1.1) gives an interpretation of formula (9.2.1) (Proposition 9.4(iii)) in the category $\mathcal{C}_{\widetilde{\mathbf{Q}}}(\mathbf{m})$.

(ii) We conjecture that $\mathcal{U}_{q,\widetilde{\mathbf{Q}}}(\mathbf{m})$ has the structure of a Hopf algebra. We also conjecture that the tensor product of Weyl modules of $\mathscr{S}^{\widetilde{\mathbb{K}}}_{n,r}(\mathbf{m})$ $(n \geq 0)$ has a similar decomposition to (10.1.1).

Acknowledgements

This research was supported by JSPS KAKENHI Grant Number 24740007. The author is grateful to Tatsuyuki Hikita for his suggestion on the definition of the polynomials $\Phi^{\pm}_t(x_1, \dots, x_k)$ (see Remark 7.4).

References

[BLM] A. Beilinson, G. Lusztig and R. MacPherson, A geometric setting for the quantum deformation of GL_n, Duke Math. J. **61** (1990), 655–677. Zbl 0713.17012 MR 1074310

[DJM] R. Dipper, G. James and A. Mathas, Cyclotomic q-Schur algebras, Math. Z. **229** (1998), 385–416. Zbl 0934.20014 MR 1658581

[D] J. Du, A note on quantized Weyl reciprocity at root of unity, Algebra Colloq. **2** (1995), 363–372. Zbl 0855.17006 MR 1358684

[DR] J. Du and H. Rui, Borel type subalgebras of the q-Schur^m algebra, J. Algebra **213** (1999), 567–595. Zbl 0936.20002 MR 1673470

[GL] J. J. Graham and G. I. Lehrer, Cellular algebras, Invent. Math. **123** (1996), 1–34. Zbl 0853.20029 MR 1376244

[JM] G. James and A. Mathas, The Jantzen sum formula for cyclotomic q-Schur algebras, Trans. Amer. Math. Soc. **352** (2000), 5381–5404. Zbl 0964.16015 MR 1665333

[J] M. Jimbo, A q-analogue of $U(\mathfrak{gl}(N+1))$, Hecke algebra and the Yang–Baxter equation, Lett. Math. Phys. **11** (1986), 247–252. Zbl 0602.17005 MR 0841713

[KL] D. Kazhdan and G. Lusztig, Tensor structures arising from affine Lie algebras I–IV, J. Amer. Math. Soc. **6-7** (1993–1994), 905–947, 949–1011, 335–381, 383–453. Zbl 0802.17008 MR 1239507

[L] I. Losev, Proof of Varagnolo–Vasserot conjecture on cyclotomic categories $\mathcal{O}$, Selecta Math. (N.S.) **22** (2016), 631–668. Zbl 06568884 MR 3477332

[M1] A. Mathas, The representation theory of the Ariki–Koike and cyclotomic q-Schur algebras, in *Representation theory of algebraic groups and quantum groups*, Adv. Stud. Pure Math. 40, Math. Soc. Japan, Tokyo, 2004, 261–320. Zbl 1140.20003 MR 2074597

[M2] ———, Seminormal forms and Gram determinants for cellular algebras, J. Reine Angew. Math. **619** (2008), 141–173. Zbl 1152.20008 MR 2414949

[R] R. Rouquier, q-Schur algebras and complex reflection groups, Moscow Math. J. **8** (2008), 119–158. Zbl 1213.20007 MR 2422270

[RSVV] R. Rouquier, P. Shan, M. Varagnolo and E. Vasserot, Categorifications and cyclotomic rational double affine Hecke algebras, Invent. Math. **204** (2016), 671–786. Zbl 06592994 MR 3502064

[W1] K. Wada, Presenting cyclotomic q-Schur algebras, Nagoya Math. J. **201** (2011), 45–116. Zbl 1239.20056 MR 2772170

[W2] ———, On Weyl modules of cyclotomic q-Schur algebras, in *Algebra groups and quantum groups*, Contemp. Math. 565, Amer. Math. Soc., 2012, 261–286. Zbl 06296876 MR 2932431

[W3] ———, Induction and restriction functors for cyclotomic q-Schur algebras, Osaka J. Math. **51** (2014), 785–822. Zbl 1319.20039 MR 3272617