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Finite W-Superalgebras and Dimensional Lower Bounds for the Representations of Basic Lie Superalgebras

by

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Abstract

In this paper we show that the lower bounds of dimensions in the modular representations of basic Lie superalgebras are attainable, under an assumption on the minimal dimensions of representations of the finite W-superalgebra $U(\mathfrak{g}_{\mathbb{C}}, e)$ over the field of complex numbers. The aforementioned lower bounds for modular representations, as a super version of the Kac–Weisfeiler conjecture [26], were formulated and proved by Wang–Zhao in [35] for basic Lie superalgebras over an algebraically closed field $\Bbbk$ of positive characteristic p. We further conjecture that the assumption is actually satisfied (see Conjecture 1.3). That is to say, the complex finite W-superalgebra $U(\mathfrak{g}_{\mathbb{C}}, e)$ affords either one-dimensional or two-dimensional representations, according to the parity of the discriminant number (the difference of dimensions between the odd part of $\mathfrak{g}_{\mathbb{C}}$ and its subspace centralized by e). We demonstrate the positivity of the conjecture with examples including all the cases of type A, and finally reduce the investigation of the conjecture to the case of rigid nilpotent elements as is the situation for ordinary finite W-algebras (cf. [29]).

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§1. Introduction

This paper is a sequel to [37]. On the basis of the structure theory of finite W-superalgebras developed there, we study the modular representations of basic Lie superalgebras, as a remarkable application of finite W-superalgebras.

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§1.1.

A finite W-algebra $U(\mathfrak{g}, e)$ is a certain associative algebra associated with a complex semisimple Lie algebra $\mathfrak{g}$ and a nilpotent element $e \in \mathfrak{g}$. The study of finite W-algebras can be traced back to Kostant's work in the case when e is regular [15], then a further study was done by Lynch in the case when e is an arbitrary even nilpotent element (cf. [19]). Premet developed finite W-algebras in full generality in [27]. On his way to proving the celebrated Kac–Weisfeiler conjecture for Lie algebras of reductive groups in [26], Premet first constructed the modular version of finite W-algebras in [27] (they will be called the reduced W-algebras in the present paper). By means of a complicated but natural "admissible" procedure, the finite W-algebras over the field of complex numbers were introduced in [27], which shows that they are filtered deformations of the coordinate rings of Slodowy slices.

Aside from the advances in finite W-algebras over the field of complex numbers, the modular theory of finite W-algebras has also developed excitingly. It is remarkable that in [29] Premet proved that if the $\mathbb{C}$-algebra $U(\mathfrak{g}, e)$ has a one-dimensional representation, then under the assumption $p \gg 0$ for the positive characteristic field $\Bbbk = \overline{\mathbb{F}}_p$, the reduced enveloping algebra $U_\chi(\mathfrak{g}_\Bbbk)$ of the modular counterpart $\mathfrak{g}_\Bbbk$ of $\mathfrak{g}$ possesses an irreducible module of dimension $d(e)$ (where χ is the linear function on $\mathfrak{g}_\Bbbk$ corresponding to e, and $d(e)$ is half of the dimension of the orbit $G_\Bbbk \cdot \chi$ for the simple, simply connected algebraic group $G_\Bbbk$ with $\mathfrak{g}_\Bbbk = \mathrm{Lie}(G_\Bbbk)$), which is a lower bound predicted by the Kac–Weisfeiler conjecture mentioned above.

The existence of one-dimensional representations for $U(\mathfrak{g}, e)$ associated with $\mathfrak{g} = \mathrm{Lie}(G)$ of a simple algebraic group G over $\mathbb{C}$ was conjectured by Premet, and confirmed in the classical cases by Losev in [17, Theorem 1.2.3(1)] (see also [16, §6]). Goodwin–Röhrle–Ubly [8] proved that the W-algebras associated with exceptional Lie algebras E_6, E_7, F_4, G_2, or E_8 with e not rigid, admit one-dimensional representations (see also [29]). Finally Premet solved this problem completely in [30].

§1.2.

The theory of finite W-superalgebras was developed at the same time. In the work of De Sole–Kac [33], finite W-superalgebras were defined in terms of BRST cohomology under the background of vertex algebras and quantum reduction. The topics on finite W-superalgebras attracted many researchers (cf. [2, 21, 22, 24, 25, 23, 36, 40]).

In the work of Wang–Zhao [35], they initiated the study of modular representations of basic Lie superalgebras over an algebraically closed field of positive characteristic, formulating the super Kac–Weisfeiler property for those Lie superalgebras as well as presenting the definition of modular W-superalgebras.

§1.3.

Based on Premet's and Wang–Zhao's work as mentioned above, our previous paper [37] presents the PBW structure theorem for finite W-superalgebras (along with reduced W-superalgebras), which shows that the construction of finite W-superalgebras (and also reduced W-superalgebras) can be divided into two cases by virtue of the parity of the dimension for a specific subspace of the basic Lie superalgebra $\mathfrak{g}_{\mathbb{F}}$, where $\mathbb{F}$ is the field of complex numbers or an algebraically closed field of characteristic $p \gg 0$. To some extent, the situation of finite W-superalgebras is significantly different from that of finite W-algebras.

To be explicit, let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$ be a basic Lie superalgebra over $\mathbb{C}$ excluding type $D(2,1;a)$ ($a \in \mathbb{C}$ is not an algebraic number), and $e \in \mathfrak{g}_{\bar{0}}$ be a nilpotent element. Fix an $\mathfrak{sl}_2$-triple (e,h,f), and define $\mathfrak{g}^e := \mathrm{Ker}(\mathrm{ad}\, e)$ in $\mathfrak{g}$. The linear operator $\mathrm{ad}\, h$ defines a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(i)$. Define the Kazhdan degree on $\mathfrak{g}$ by declaring that $x \in \mathfrak{g}(j)$ is of $(j+2)$. A finite W-superalgebra is defined by

$$U(\mathfrak{g},e) = (\mathrm{End}_{\mathfrak{g}} Q_\chi)^{\mathrm{op}},$$

where Q_χ is the generalized Gelfand–Graev $\mathfrak{g}$-module associated with e. In [37] we showed that

Theorem 1.1 ([37]). *Under the Kazhdan grading, we have*

(1) $\mathrm{gr}\, U(\mathfrak{g},e) \cong S(\mathfrak{g}^e)$ *as* $\mathbb{C}$*-algebras when* $\dim \mathfrak{g}(-1)_{\bar{1}}$ *is even;*

(2) $\mathrm{gr}\, U(\mathfrak{g},e) \cong S(\mathfrak{g}^e) \otimes \mathbb{C}[\Theta]$ *as vector spaces over* $\mathbb{C}$ *when* $\dim \mathfrak{g}(-1)_{\bar{1}}$ *is odd,*

where $\mathbb{C}[\Theta]$ *is the exterior algebra generated by one element* Θ*.*

§1.4.

The main purpose of this paper is to further develop the construction and representation theory of finite W-superalgebras both over the field of complex numbers and over the field in prime characteristic. The most important part is the accessibility of lower bounds in the super Kac–Weisfeiler property. Our approach is roughly generalizing the "reduction modulo p" method introduced by Premet for the finite W-algebra case in [29], with careful analysis and examination of the variation of structural features arising from the parity of $\dim \mathfrak{g}(-1)_{\bar{1}}$. Let us explain it roughly below.

In this paper we always assume that $\mathfrak{g}$ is a basic Lie superalgebra excluding type $D(2,1;a)$ (a is not an algebraic number). Let $\mathfrak{g}_{\Bbbk}$ and $Q_{\chi,\Bbbk}$ be the modular counterparts of $\mathfrak{g}$ and of the generalized Gelfand–Graev $\mathfrak{g}$-module Q_χ, respectively. To simplify notation, we will identify the nilpotent element $e \in \mathfrak{g}$ over $\mathbb{C}$ with the element $\bar{e} = e \otimes 1$ in $\mathfrak{g}_{\mathbb{k}}$ by "reduction modulo p" in the following. Define the finite W-superalgebra over $\mathbb{k}$ by

$$U(\mathfrak{g}_{\mathbb{k}}, e) := (\mathrm{End}_{\mathfrak{g}_{\mathbb{k}}} Q_{\chi,\mathbb{k}})^{\mathrm{op}}.$$

Let $T(\mathfrak{g}_{\mathbb{k}}, e)$ be the transition subalgebra of $U(\mathfrak{g}_{\mathbb{k}}, e)$ that is derived from the $\mathbb{C}$-algebra $U(\mathfrak{g}, e)$ by "reduction modulo p". In our arguments, the transition subalgebras will play some medium role between the theory of finite W-superalgebras over $\mathbb{C}$ and that of reduced enveloping algebras over $\mathbb{k}$. So in the first part of the paper, we will investigate the structure of the transition subalgebra $T(\mathfrak{g}_{\mathbb{k}}, e)$ and the finite W-superalgebra $U(\mathfrak{g}_{\mathbb{k}}, e)$ over $\mathbb{k}$. To be explicit, for any real number $a \in \mathbb{R}$, let $\lceil a \rceil$ denote the largest integer lower bound of a, and $\lfloor a \rfloor$ the least integer upper bound of a (it is notable that this notation also works for numbers in the prime field $\mathbb{F}_p$). Set $d_i := \dim \mathfrak{g}_i - \dim \mathfrak{g}_i^e$ for $i \in \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$; then we have

Theorem 1.2. *There is a subspace $\mathfrak{a}_{\mathbb{k}}$ of $\mathfrak{g}_{\mathbb{k}}$ with $\underline{\dim}\, \mathfrak{a}_{\mathbb{k}} = (\frac{d_0}{2}, \lceil \frac{d_1}{2} \rceil)$ such that $U(\mathfrak{g}_{\mathbb{k}}, e) \cong T(\mathfrak{g}_{\mathbb{k}}, e) \otimes_{\mathbb{k}} Z_p(\mathfrak{a}_{\mathbb{k}})$ as $\mathbb{k}$-algebras, where $Z_p(\mathfrak{a}_{\mathbb{k}})$ is the p-center as usually defined, with respect to the subspace $\mathfrak{a}_{\mathbb{k}}$.*

In the second part of this paper, we exploit some remarkable applications of finite W-superalgebras to the modular representations of basic Lie superalgebras. We provide a super version of Premet's work, as aforementioned, on the accessibility of lower bounds of dimensions in the modular representations of reductive Lie algebras predicted by the Kac–Weisfeiler conjecture. For this, we will formulate a conjecture about the "small representations" of $U(\mathfrak{g}, e)$ over $\mathbb{C}$ (in the paper, we call a representation of an algebra "small" if its dimension is minimal among all the representations of this algebra). By [37, Remark 2.7] we know that d_1 has the same parity as $\dim \mathfrak{g}(-1)_{\bar{1}}$. As seen before, the variation of the parity of d_1 gives rise to a change in the structure of finite W-superalgebras, and as we will see furthermore, also of the representations of finite W-superalgebras. First, we provide the following highly plausible conjecture, generalizing a conjecture proposed by Premet on the representations of finite W-algebras that has already been confirmed (cf. [29, 30]).

Conjecture 1.3. Let $\mathfrak{g}$ be a basic Lie superalgebra over $\mathbb{C}$. Then the following statements hold:

(1) When d_1 is even, the finite W-superalgebra $U(\mathfrak{g}, e)$ affords a one-dimensional representation.

(2) When d_1 is odd, the finite W-superalgebra $U(\mathfrak{g},e)$ affords a two-dimensional representation.

So we call d_1 the discriminant number as we said in the abstract. For the case that $\mathfrak{g}$ is of type A, Conjecture 1.3 is confirmed in the present paper, which is accomplished by conversion from the verification of the attainableness of lower bounds of modular dimensions for basic Lie superalgebras of the same type by some direct computation; see [38] for more details. However, our final result on the lower bounds of modular dimensions for basic Lie superalgebras is generally dependent on the above conjecture. In the final section of the paper, we introduce the notion of rigid nilpotent elements, and reduce the investigation of Conjecture 1.3 to the case of rigid nilpotent elements, i.e.,

Theorem 1.4. *Assume that the statement of Conjecture 1.3 holds for any given basic Lie superalgebra $\mathfrak{g}$ and any given rigid nilpotent $e\in\mathfrak{g}_{\bar{0}}$. Then Conjecture 1.3 is true for all cases.*

§1.5.

Assuming Conjecture 1.3, we finally accomplish a super version of Premet's work on classical Lie algebras. To be explicit, let $(\cdot,\cdot)$ be an even nondegenerate supersymmetric bilinear form on $\mathfrak{g}_{\Bbbk}$ (see §2.2.1), and $\xi\in(\mathfrak{g}_{\Bbbk})^*_{\bar{0}}$ be any p-character of $\mathfrak{g}_{\Bbbk}$ corresponding to an element $\bar{x}\in(\mathfrak{g}_{\Bbbk})_{\bar{0}}$ such that $\xi(\bar{y})=(\bar{x},\bar{y})$ for any $\bar{y}\in\mathfrak{g}_{\Bbbk}$. Now let $d_0=\dim(\mathfrak{g}_{\Bbbk})_{\bar{0}}-\dim(\mathfrak{g}_{\Bbbk}^{\bar{x}})_{\bar{0}}$ and $d_1=\dim(\mathfrak{g}_{\Bbbk})_{\bar{1}}-\dim(\mathfrak{g}_{\Bbbk}^{\bar{x}})_{\bar{1}}$, where $\mathfrak{g}_{\Bbbk}^{\bar{x}}$ denotes the centralizer of $\bar{x}$ in $\mathfrak{g}_{\Bbbk}$. Recall that the dimension of any irreducible representation of $\mathfrak{g}_{\Bbbk}$ is divisible by $p^{\frac{d_0}{2}}2^{\lfloor\frac{d_1}{2}\rfloor}$ (cf. [35, Theorem 5.6]). The main result of the present paper is the following theorem.

Theorem 1.5. *Let $\mathfrak{g}_{\Bbbk}$ be a basic Lie superalgebra over $\Bbbk=\overline{\mathbb{F}}_p$, and let $\xi\in(\mathfrak{g}_{\Bbbk})^*_{\bar{0}}$. If Conjecture 1.3 is established for all the basic Lie superalgebras over $\mathbb{C}$ excluding type $D(2,1;a)$ ($a\in\mathbb{C}$ is not an algebraic number), then for $p\gg0$ the reduced enveloping algebra $U_\xi(\mathfrak{g}_{\Bbbk})$ admits irreducible representations of dimension $p^{\frac{d_0}{2}}2^{\lfloor\frac{d_1}{2}\rfloor}$.*

The main part of the proof of the above theorem will be to deal with the situation when ξ is nilpotent. In the present work, the arguments in Premet's work will be exploited in the super case. The greatest challenge here is to deal with the structural change arising from the variation of the parity of $\dim\mathfrak{g}(-1)_{\bar{1}}$ (or equivalently, the parity of d_1). Here we sketch some main ingredients of our proof, beyond exploiting Premet's arguments. Based on the PBW structure theorems of finite W-superalgebras established in [37], we first prove that when $\dim\mathfrak{g}(-1)_{\bar{1}}$

is odd, the $\mathbb{C}$-algebra $U(\mathfrak{g}, e)$ does not admit one-dimensional representations (Proposition 3.7). The possible two-dimensional modules of $U(\mathfrak{g}, e)$ will turn out to be of type Q, with parity involution arising from some special odd element $\Theta_{l+q+1} \in U(\mathfrak{g}, e)$ appearing only in the case of $\dim \mathfrak{g}(-1)_{\bar{1}}$ being odd (Proposition 5.2), while what we need to deal with more are lots of arguments involving the generators and defining relations of $U(\mathfrak{g}, e)$ associated with Θ_{l+q+1} (see §5.2).

As for the case when the p-character $\chi \in (\mathfrak{g}_{\mathbb{k}})^*_{\bar{0}}$ is nilpotent, corresponding to a nilpotent element $e \in (\mathfrak{g}_{\mathbb{k}})_{\bar{0}}$ such that $\chi(\bar{x}) = (e, \bar{x})$ for any $\bar{x} \in \mathfrak{g}_{\mathbb{k}}$, the following theorem releases the condition in Theorem 1.5.

Theorem 1.6. *Let $\mathfrak{g}_{\mathbb{k}}$ be a basic Lie superalgebra over $\mathbb{k} = \overline{\mathbb{F}}_p$, and let $\chi \in (\mathfrak{g}_{\mathbb{k}})^*_{\bar{0}}$ be a nilpotent p-character, with respect to the element $e \in (\mathfrak{g}_{\mathbb{k}})_{\bar{0}}$ as described above. If the corresponding finite W-superalgebra $U(\mathfrak{g}, e)$ over $\mathbb{C}$ affords a one-dimensional (resp. two-dimensional) representation when d_1 is even (resp. odd), then for $p \gg 0$ the reduced enveloping algebra $U_\chi(\mathfrak{g}_{\mathbb{k}})$ admits irreducible representations of dimension $p^{\frac{d_0}{2}} 2^{\lfloor \frac{d_1}{2} \rfloor}$, where $d_0 = \dim(\mathfrak{g}_{\mathbb{k}})_{\bar{0}} - \dim(\mathfrak{g}^e_{\mathbb{k}})_{\bar{0}}$ and $d_1 = \dim(\mathfrak{g}_{\mathbb{k}})_{\bar{1}} - \dim(\mathfrak{g}^e_{\mathbb{k}})_{\bar{1}}$.*

§1.6.

The paper is organized as follows. In §2, some basics on algebraic supergroups, Lie superalgebras, and finite W-superalgebras are recalled. In §3, the transition subalgebra $T(\mathfrak{g}_{\mathbb{k}}, e)$ over $\mathbb{k}$ is introduced and studied, then follows the structure relation among the finite W-superalgebras, the transition subalgebras, and the p-central subalgebras of some subspaces of $\mathfrak{g}_{\mathbb{k}}$. In §4 and §5, the minimal dimensions for the representations of $U(\mathfrak{g}, e)$ over $\mathbb{C}$ are estimated and conjectured. We demonstrate that the conjecture is true for some cases, including the whole case of type A. Then §6 will be devoted to the proof of our main theorems. In §6.2, we first complete the proof of Theorem 1.6. In §6.3, we improve the result on the dimensional lower bounds of modular representations for a direct sum of basic Lie superalgebras with nilpotent p-characters in Proposition 6.5 and Remark 6.6 (note that this conclusion does not depend on Conjecture 1.3), which was originally discussed by Wang–Zhao in [35, Remark 4.5]. Then the accessibility of the lower bounds for the refined version is obtained under Conjecture 1.3. Then §6.4 is devoted to the proof Theorem 1.5. By virtue of the results obtained in §6.3, we further show that the lower bounds in the super Kac–Weisfeiler property with arbitrary p-characters in [35] are also reachable under Conjecture 1.3. The main tool applied in this section is the method of nilpotent orbit theory, and also the techniques for the modular representation theory of restricted Lie superalgebras. In §7, we introduce the notion of rigid nilpotent elements, and reduce the investigation of Conjecture 1.3 to the case of rigid nilpotent elements (see Theorem 1.4).

§1.7.

Throughout, we work with the field of complex numbers $\mathbb{C}$, or the algebraically closed field $\Bbbk = \overline{\mathbb{F}}_p$ of positive characteristic p as the ground field.

Let $\mathbb{Z}_+$ be the set of all the nonnegative integers in $\mathbb{Z}$, and denote by $\mathbb{Z}_2$ the residue class ring modulo 2 in $\mathbb{Z}$. A superspace is a $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$, in which we call elements in $V_{\bar{0}}$ and $V_{\bar{1}}$ even and odd, respectively. Write $|v| \in \mathbb{Z}_2$ for the parity (or degree) of $v \in V$, which is implicitly assumed to be $\mathbb{Z}_2$-homogeneous. We will use the notation

$$\underline{\dim}\, V = (\dim V_{\bar{0}}, \dim V_{\bar{1}}), \qquad \dim V = \dim V_{\bar{0}} + \dim V_{\bar{1}}.$$

All Lie superalgebras $\mathfrak{g}$ will be assumed to be finite-dimensional.

Recall that a superalgebra analogue of Schur's lemma states that the endomorphism ring of an irreducible module of a superalgebra is either one-dimensional or two-dimensional (in the latter case it is isomorphic to a Clifford algebra), cf. for example, Kleshchev [14, Chapter 12]. An irreducible module is of type M if its endomorphism ring is one-dimensional and it is of type Q otherwise.

We consider vector spaces, subalgebras, ideals, modules, and submodules etc. in the super sense throughout the paper.

§2. Basic Lie superalgebras and finite W-superalgebras

In this section, we will recall some knowledge on basic classical Lie superalgebras along with the corresponding algebraic supergroups, and finite W-(super)algebras for use in the sequel. We refer the readers to [3, 12, 20] for Lie superalgebras, [5, 31] for algebraic supergroups, and [27, 29, 34, 37] for finite W-(super)algebras.

§2.1. Basic classical Lie superalgebras and the corresponding algebraic supergroups

2.1.1. Basic classical Lie superalgebras. Following [3, §1], [12, §2.3–§2.4], [13, §1], and [35, §2], we recall the list of basic classical Lie superalgebras over $\mathbb{F}$ for $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \Bbbk$. These Lie superalgebras, with even parts being Lie algebras of reductive algebraic groups, are simple over $\mathbb{F}$ (the general linear Lie superalgebras, though not simple, are also included), and they admit an even nondegenerate supersymmetric invariant bilinear form in the following sense.

Definition 2.1. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$-graded space and $(\cdot, \cdot)$ be a bilinear form on V.

(1) If $(a, b) = 0$ for any $a \in V_{\bar{0}}, b \in V_{\bar{1}}$, then $(\cdot, \cdot)$ is called even.

(2) If $(a,b) = (-1)^{|a||b|}(b,a)$ for any homogeneous elements $a, b \in V$, then $(\cdot,\cdot)$ is called supersymmetric.
(3) If $([a,b],c) = (a,[b,c])$ for any homogeneous elements $a, b, c \in V$, then $(\cdot,\cdot)$ is called invariant.
(4) If one can conclude from $(a,V) = 0$ that $a = 0$, then $(\cdot,\cdot)$ is called nondegenerate.

Note that when $\mathbb{F} = \Bbbk$ is a field of characteristic $p > 0$, there are restrictions on p, as shown for example in [35, Table 1]. So we have the following list.

Table 1. Basic classical Lie superalgebras over $\Bbbk$

$\mathfrak{g}_{\Bbbk}$	$\mathfrak{g}_{\bar{0}}$	Restriction of p when $\mathbb{F} = \Bbbk$
$\mathfrak{gl}(m\|n)$	$\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$	$p > 2$
$\mathfrak{sl}(m\|n)$	$\mathfrak{sl}(m) \oplus \mathfrak{sl}(n) \oplus \Bbbk$	$p > 2, p \nmid (m-n)$
$\mathfrak{osp}(m\|n)$	$\mathfrak{so}(m) \oplus \mathfrak{sp}(n)$	$p > 2$
$\mathrm{D}(2,1,\bar{a})$	$\mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$	$p > 3$
$\mathrm{F}(4)$	$\mathfrak{sl}(2) \oplus \mathfrak{so}(7)$	$p > 15$
$\mathrm{G}(3)$	$\mathfrak{sl}(2) \oplus \mathrm{G}_2$	$p > 15$

Throughout the paper, we will simply call all $\mathfrak{g}_{\mathbb{F}}$ listed above *"basic Lie superalgebras"* for both $\mathbb{F} = \mathbb{C}$ and $\mathbb{F} = \Bbbk$.

2.1.2. Algebraic supergroups and restricted Lie superalgebras. For a given basic Lie superalgebra listed in §2.1.1, there is an algebraic supergroup $G_{\mathbb{F}}$ satisfying $\mathrm{Lie}(G_{\mathbb{F}}) = \mathfrak{g}_{\mathbb{F}}$ such that

(1) $G_{\mathbb{F}}$ has a subgroup scheme $(G_{\mathbb{F}})_{\mathrm{ev}}$ that is an ordinary connected reductive group with $\mathrm{Lie}((G_{\mathbb{F}})_{\mathrm{ev}}) = (\mathfrak{g}_{\mathbb{F}})_{\bar{0}}$;
(2) there is a well-defined action of $(G_{\mathbb{F}})_{\mathrm{ev}}$ on $\mathfrak{g}_{\mathbb{F}}$, reducing to the adjoint action of $(\mathfrak{g}_{\mathbb{F}})_{\bar{0}}$.

The above algebraic supergroup can be constructed as a Chevalley supergroup in [5]. The pair $((G_{\mathbb{F}})_{\mathrm{ev}}, \mathfrak{g}_{\mathbb{F}})$ constructed in this way is called a Chevalley super Harish-Chandra pair (cf. [5, Theorem 5.35] and [6, §3.3]). Partial results on $G_{\mathbb{F}}$ and $(G_{\mathbb{F}})_{\mathrm{ev}}$ can be found in [1, Ch. 2.2], [5], [6, §3.3] etc. In the present paper, we will call $(G_{\mathbb{F}})_{\mathrm{ev}}$ the purely even subgroup of $G_{\mathbb{F}}$. When the ground field $\mathbb{F} = \Bbbk$ is of odd prime characteristic p, one easily knows that $\mathfrak{g}_{\Bbbk}$ is a restricted Lie superalgebra (cf. [31, Definition 2.1] and [32]) in the following sense.

Definition 2.2. A Lie superalgebra $\mathfrak{g}_{\Bbbk}=(\mathfrak{g}_{\Bbbk})_{\bar{0}}\oplus(\mathfrak{g}_{\Bbbk})_{\bar{1}}$ over $\Bbbk$ is called a restricted Lie superalgebra, if there is a p-th power map $(\mathfrak{g}_{\Bbbk})_{\bar{0}}\to(\mathfrak{g}_{\Bbbk})_{\bar{0}}$, denoted by $(-)^{[p]}$, satisfying

(a) $(k\bar{x})^{[p]}=k^p\bar{x}^{[p]}$ for all $k\in\Bbbk$ and $\bar{x}\in(\mathfrak{g}_{\Bbbk})_{\bar{0}}$;
(b) $[\bar{x}^{[p]},\bar{y}]=(\mathrm{ad}\,\bar{x})^p(\bar{y})$ for all $\bar{x}\in(\mathfrak{g}_{\Bbbk})_{\bar{0}}$ and $\bar{y}\in\mathfrak{g}_{\Bbbk}$;
(c) $(\bar{x}+\bar{y})^{[p]}=\bar{x}^{[p]}+\bar{y}^{[p]}+\sum_{i=1}^{p-1}s_i(\bar{x},\bar{y})$ for all $\bar{x},\bar{y}\in(\mathfrak{g}_{\Bbbk})_{\bar{0}}$, where $is_i(\bar{x},\bar{y})$ is the coefficient of λ^{i-1} in $(\mathrm{ad}(\lambda\bar{x}+\bar{y}))^{p-1}(\bar{x})$.

Let $\mathfrak{g}_{\Bbbk}$ be a restricted Lie superalgebra. For each $\bar{x}\in(\mathfrak{g}_{\Bbbk})_{\bar{0}}$, the element $\bar{x}^p-\bar{x}^{[p]}\in U(\mathfrak{g}_{\Bbbk})$ is central by Definition 2.2, and all of them generate a central subalgebra of $U(\mathfrak{g}_{\Bbbk})$. Let $\{\bar{w}_1,\ldots,\bar{w}_c\}$ and $\{\bar{w}'_1,\ldots,\bar{w}'_d\}$ be the basis of $(\mathfrak{g}_{\Bbbk})_{\bar{0}}$ and $(\mathfrak{g}_{\Bbbk})_{\bar{1}}$ respectively. For a given $\chi\in(\mathfrak{g}_{\Bbbk})_{\bar{0}}^*$, let J_χ be the ideal of the universal enveloping algebra $U(\mathfrak{g}_{\Bbbk})$ generated by the even central elements $\bar{w}^p-\bar{w}^{[p]}-\chi(\bar{w})^p$ for all $\bar{w}\in(\mathfrak{g}_{\Bbbk})_{\bar{0}}$. The quotient algebra $U_\chi(\mathfrak{g}_{\Bbbk}):=U(\mathfrak{g}_{\Bbbk})/J_\chi$ is called the reduced enveloping algebra with p-character χ. We often regard $\chi\in\mathfrak{g}_{\Bbbk}^*$ by letting $\chi((\mathfrak{g}_{\Bbbk})_{\bar{1}})=0$. If $\chi=0$, then $U_0(\mathfrak{g}_{\Bbbk})$ is called the restricted enveloping algebra. It is a direct consequence of the PBW theorem that the $\Bbbk$-algebra $U_\chi(\mathfrak{g}_{\Bbbk})$ is of dimension p^c2^d, and has a basis

$$\{\bar{w}_1^{a_1}\cdots\bar{w}_c^{a_c}(\bar{w}'_1)^{b_1}\cdots(\bar{w}'_d)^{b_d}\mid 0\leqslant a_i<p,b_j\in\{0,1\}\text{ for all }1\leqslant i\leqslant c,1\leqslant j\leqslant d\}.$$

§2.2. Finite W-superalgebras over the field of complex numbers

2.2.1. Let $\mathfrak{g}$ be a basic Lie superalgebra over $\mathbb{C}$ and $\mathfrak{h}$ be a typical Cartan subalgebra of $\mathfrak{g}$. Let Φ be a root system of $\mathfrak{g}$ relative to $\mathfrak{h}$ whose simple root system $\Delta=\{\alpha_1,\ldots,\alpha_l\}$ is distinguished (cf. [13, Proposition 1.5]). Let Φ^+ be the corresponding positive system in Φ, and put $\Phi^-:=-\Phi^+$. Let $\mathfrak{g}=\mathfrak{n}^-\oplus\mathfrak{h}\oplus\mathfrak{n}^+$ be the corresponding triangular decomposition of $\mathfrak{g}$. By [5, §3.3], we can choose a Chevalley basis $B=\{e_\gamma\mid\gamma\in\Phi\}\cup\{h_\alpha\mid\alpha\in\Delta\}$ of $\mathfrak{g}$ excluding the case $D(2,1;a)$ with $a\notin\mathbb{Z}$. (In the case $D(2,1;a)$ with $a\notin\mathbb{Z}$ being an algebraic number, one needs to adjust the definition of a Chevalley basis by changing $\mathbb{Z}$ to $\mathbb{Z}[a]$ in the range of construction constants; see [7, §3.1]. Here $\mathbb{Z}[a]$ is the $\mathbb{Z}$-algebra generated by (a).) Let $\mathfrak{g}_{\mathbb{Z}}$ denote the Chevalley $\mathbb{Z}$-form in $\mathfrak{g}$ and $U_{\mathbb{Z}}$ the Kostant $\mathbb{Z}$-form of $U(\mathfrak{g})$ associated with B. Given a $\mathbb{Z}$-module V and a $\mathbb{Z}$-algebra A, we write $V_A:=V\otimes_{\mathbb{Z}}A$.

Let G be an algebraic supergroup as in §2.1.2, with $\mathrm{Lie}(G)=\mathfrak{g}$ and the super Harish-Chandra pair $(G_{\mathrm{ev}},\mathfrak{g})$. Let $e\in\mathfrak{g}_{\bar{0}}$ be a nilpotent element. By Dynkin–Kostant theory we know that $\mathrm{ad}\,G_{\mathrm{ev}}.e$ and $(\mathfrak{g}_{\mathbb{Z}})_{\bar{0}}$ have nonempty interaction; then one can assume that the nilpotent e is in $(\mathfrak{g}_{\mathbb{Z}})_{\bar{0}}$. By the same discussion as in [27, §4.2], for any given nilpotent element $e\in(\mathfrak{g}_{\mathbb{Z}})_{\bar{0}}$ we can find $f,h\in(\mathfrak{g}_{\mathbb{Q}})_{\bar{0}}$ such that (e,h,f) is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. In [37, Proposition 2.1] it is shown that there exists an even nondegenerate supersymmetric invariant bilinear form $(\cdot,\cdot)$, under which the Chevalley basis B of $\mathfrak{g}$ takes values in $\mathbb{Q}$, and $(e,f)=1$. Define $\chi\in\mathfrak{g}^*$ by letting $\chi(x)=(e,x)$ for all $x\in\mathfrak{g}$; then we have $\chi(\mathfrak{g}_{\bar{1}})=0$.

Following [37, Definition 2.4] we call a commutative ring A *admissible* if A is a finitely generated $\mathbb{Z}$-subalgebra of $\mathbb{C}$, $(e,f)\in A^{\times}(=A\backslash\{0\})$ and all bad primes of the root system of $\mathfrak{g}$ and the determinant of the Gram matrix of $(\cdot,\cdot)$ relative to a Chevalley basis of $\mathfrak{g}$ are invertible in A. It is clear by the definition that every admissible ring is a Noetherian domain. Given a finitely generated $\mathbb{Z}$-subalgebra A of $\mathbb{C}$, denote by $\operatorname{Specm}A$ the maximal spectrum of A. It is well known that for every element $\mathfrak{P}\in\operatorname{Specm}A$, the residue field $A/\mathfrak{P}$ is isomorphic to $\mathbb{F}_q$, where q is a p-power depending on $\mathfrak{P}$. We denote by $\Pi(A)$ the set of all primes $p\in\mathbb{N}$ that occur in this way. Since the choice of A does not depend on the super structure of $\mathfrak{g}$, it follows from the arguments in the proof of [29, Lemma 4.4] that the set $\Pi(A)$ contains almost all primes in $\mathbb{N}$. We denote by $\mathfrak{g}_A$ the A-submodule of $\mathfrak{g}$ generated by the Chevalley basis B.

Let $\mathfrak{g}(i)=\{x\in\mathfrak{g}\mid[h,x]=ix\}$, then $\mathfrak{g}=\bigoplus_{i\in\mathbb{Z}}\mathfrak{g}(i)$. By $\mathfrak{sl}_2$-theory, all subspaces $\mathfrak{g}(i)$ are defined over $\mathbb{Q}$. Also, $e\in\mathfrak{g}(2)_{\bar{0}}$ and $f\in\mathfrak{g}(-2)_{\bar{0}}$. There exists a symplectic (resp. symmetric) bilinear form $\langle\cdot,\cdot\rangle$ on the $\mathbb{Z}_2$-graded subspace $\mathfrak{g}(-1)_{\bar{0}}$ (resp. $\mathfrak{g}(-1)_{\bar{1}}$) given by $\langle x,y\rangle:=(e,[x,y])=\chi([x,y])$ for all $x,y\in\mathfrak{g}(-1)_{\bar{0}}$ (resp. $x,y\in\mathfrak{g}(-1)_{\bar{1}}$). There exist bases $\{u_1,\ldots,u_{2s}\}$ of $\mathfrak{g}(-1)_{\bar{0}}$ and $\{v_1,\ldots,v_r\}$ of $\mathfrak{g}(-1)_{\bar{1}}$ contained in $\mathfrak{g}_{\mathbb{Q}}:=\mathfrak{g}_A\otimes_A\mathbb{Q}$ such that $\langle u_i,u_j\rangle=i^*\delta_{i+j,2s+1}$ for $1\leqslant i,j\leqslant 2s$, where

$$i^*=\begin{cases}-1 & \text{if } 1\leqslant i\leqslant s,\\ 1 & \text{if } s+1\leqslant i\leqslant 2s,\end{cases}$$

and $\langle v_i,v_j\rangle=\delta_{i+j,r+1}$ for $1\leqslant i,j\leqslant r$.

Set $\mathfrak{m}:=\bigoplus_{i\leqslant -2}\mathfrak{g}(i)\oplus\mathfrak{g}(-1)'$ with $\mathfrak{g}(-1)'=\mathfrak{g}(-1)'_{\bar{0}}\oplus\mathfrak{g}(-1)'_{\bar{1}}$, where $\mathfrak{g}(-1)'_{\bar{0}}$ is the $\mathbb{C}$-span of $u_{s+1},\ldots,u_{2s}$ and $\mathfrak{g}(-1)'_{\bar{1}}$ is the $\mathbb{C}$-span of $v_{\frac{r}{2}+1},\ldots,v_r$ (resp. $v_{\frac{r+3}{2}},\ldots,v_r$) when $r:=\dim\mathfrak{g}(-1)_{\bar{1}}$ is even (resp. odd); then χ vanishes on the derived subalgebra of $\mathfrak{m}$. Define $\mathfrak{p}:=\bigoplus_{i\geqslant 0}\mathfrak{g}(i)$,

$$\mathfrak{m}':=\begin{cases}\mathfrak{m} & \text{if } r \text{ is even},\\ \mathfrak{m}\oplus\mathbb{C}v_{\frac{r+1}{2}} & \text{if } r \text{ is odd}.\end{cases}$$

Write $\mathfrak{g}^e$ for the centralizer of e in $\mathfrak{g}$ and define $d_i:=\dim\mathfrak{g}_i-\dim\mathfrak{g}_i^e$ for $i\in\mathbb{Z}_2$; then r and d_1 always have the same parity by [35, Theorem 4.3]. This parity is a crucial factor in deciding the structure of finite W-superalgebras (cf. [37, Theorem 4.5]), which is called the judging parity in [37]. We further have

$$\underline{\dim}\,\mathfrak{m}=\begin{cases}(\frac{d_0}{2},\frac{d_1}{2}) & \text{if } d_1 \text{ is even},\\ (\frac{d_0}{2},\frac{d_1-1}{2}) & \text{if } d_1 \text{ is odd}.\end{cases}$$

After enlarging A one can assume that $\mathfrak{g}_A=\bigoplus_{i\in\mathbb{Z}}\mathfrak{g}_A(i)$, and each $\mathfrak{g}_A(i):=\mathfrak{g}_A\cap\mathfrak{g}(i)$ is freely generated over A by a basis of the vector space $\mathfrak{g}(i)$. Then $\{u_1,\dots,u_{2s}\}$ and $\{v_1,\dots,v_r\}$ are free bases of A-modules $\mathfrak{g}_A(-1)_{\bar 0}$ and $\mathfrak{g}_A(-1)_{\bar 1}$, respectively. It is obvious that $\mathfrak{m}_A:=\mathfrak{g}_A\cap\mathfrak{m}$, $\mathfrak{m}'_A:=\mathfrak{g}_A\cap\mathfrak{m}'$, and $\mathfrak{p}_A:=\mathfrak{g}_A\cap\mathfrak{p}$ are free A-modules and direct summands of $\mathfrak{g}_A$. Moreover, one can assume $e,f\in(\mathfrak{g}_A)_{\bar 0}$ after enlarging A possibly; $[e,\mathfrak{g}_A(i)]$ and $[f,\mathfrak{g}_A(i)]$ are direct summands of $\mathfrak{g}_A(i+2)$ and $\mathfrak{g}_A(i-2)$ respectively; and $\mathfrak{g}_A(i+2)=[e,\mathfrak{g}_A(i)]$ for each $i\geqslant -1$ by $\mathfrak{sl}_2$-theory.

As in [37, §2] we can choose a basis $x_1,\dots,x_l,x_{l+1},\dots,x_m\in(\mathfrak{p}_A)_{\bar 0},y_1,\dots,y_q,$ $y_{q+1},\dots,y_n\in(\mathfrak{p}_A)_{\bar 1}$ of the free A-module $\mathfrak{p}_A=\bigoplus_{i\geqslant 0}\mathfrak{g}_A(i)$ such that

(a) $x_i\in\mathfrak{g}_A(k_i)_{\bar 0},y_j\in\mathfrak{g}_A(k'_j)_{\bar 1}$, where $k_i,k'_j\in\mathbb{Z}_+$ with $1\leqslant i\leqslant m$ and $1\leqslant j\leqslant n$;

(b) $x_1,\dots,x_l$ is a basis of $(\mathfrak{g}_A)^e_{\bar 0}$ and $y_1,\dots,y_q$ is a basis of $(\mathfrak{g}_A)^e_{\bar 1}$;

(c) $x_{l+1},\dots,x_m\in[f,(\mathfrak{g}_A)_{\bar 0}]$ and $y_{q+1},\dots,y_n\in[f,(\mathfrak{g}_A)_{\bar 1}]$.

2.2.2. Recall that a Gelfand–Graev $\mathfrak{g}$-module associated with χ is defined by

$$Q_\chi:=U(\mathfrak{g})\otimes_{U(\mathfrak{m})}\mathbb{C}_\chi,$$

where $\mathbb{C}_\chi=\mathbb{C}1_\chi$ is a one-dimensional $\mathfrak{m}$-module such that $x.1_\chi=\chi(x)1_\chi$ for all $x\in\mathfrak{m}$. For $k\in\mathbb{Z}_+$, define

$$\mathbb{Z}^k_+:=\{(i_1,\dots,i_k)\mid i_j\in\mathbb{Z}_+\},\qquad \Lambda'_k:=\{(i_1,\dots,i_k)\mid i_j\in\{0,1\}\}$$

with $1\leqslant j\leqslant k$. For $\mathbf{i}=(i_1,\dots,i_k)$ in $\mathbb{Z}^k_+$ or Λ'_k, set $|\mathbf{i}|=i_1+\cdots+i_k$. For any real number $a\in\mathbb{R}$, let $\lceil a\rceil$ denote the largest integer lower bound of a, and $\lfloor a\rfloor$ the least integer upper bound of a. Given $(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})\in\mathbb{Z}^m_+\times\Lambda'_n\times\mathbb{Z}^s_+\times\Lambda'_t$ (where $t:=\lfloor\frac{r}{2}\rfloor=\lfloor\frac{\dim\mathfrak{g}(-1)_{\bar 1}}{2}\rfloor$), let $x^{\mathbf{a}}y^{\mathbf{b}}u^{\mathbf{c}}v^{\mathbf{d}}$ denote the monomial $x_1^{a_1}\cdots x_m^{a_m}y_1^{b_1}\cdots y_n^{b_n}u_1^{c_1}\cdots u_s^{c_s}v_1^{d_1}\cdots v_t^{d_t}$ in $U(\mathfrak{g})$. Set $Q_{\chi,A}:=U(\mathfrak{g}_A)\otimes_{U(\mathfrak{m}_A)}A_\chi$, where $A_\chi=A1_\chi$. By the definition, $Q_{\chi,A}$ is a $\mathfrak{g}_A$-stable A-lattice in Q_χ with $\{x^{\mathbf{a}}y^{\mathbf{b}}u^{\mathbf{c}}v^{\mathbf{d}}\otimes 1_\chi\mid(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})\in\mathbb{Z}^m_+\times\Lambda'_n\times\mathbb{Z}^s_+\times\Lambda'_t\}$ being a free basis. Given $(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})\in\mathbb{Z}^m_+\times\Lambda'_n\times\mathbb{Z}^s_+\times\Lambda'_t$, set

$$|(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})|_e:=\sum_{i=1}^m a_i(k_i+2)+\sum_{i=1}^n b_i(k'_i+2)+\sum_{i=1}^s c_i+\sum_{i=1}^t d_i.$$

Set

$$Y_i := \begin{cases} x_i & \text{if } 1 \leqslant i \leqslant l, \\ y_{i-l} & \text{if } l+1 \leqslant i \leqslant l+q, \\ v_{\frac{r+1}{2}} & \text{if } i = l+q+1, \end{cases}$$

where $Y_i \in \mathfrak{g}(m_i)$ with $m_i \in \mathbb{Z}$ and the term Y_{l+q+1} occurs only when $r = \dim \mathfrak{g}(-1)_{\bar{1}}$ is odd. Then there are Lie superalgebra operator identities in the setting of $[Y_i, Y_j] = \sum_{k=1}^{l+q} \alpha_{ij}^k Y_k$ in $\mathfrak{g}^e$ for $1 \leqslant i, j \leqslant l+q$. Set $q' = q$ if r (or equivalently, d_1) is even, and $q' = q+1$ if r is odd. By [37, Theorem 4.7], the finite W-superalgebra $U(\mathfrak{g}, e) := (\mathrm{End}_{\mathfrak{g}} Q_\chi)^{\mathrm{op}}$ is generated by $\Theta_1, \ldots, \Theta_l \in U(\mathfrak{g}, e)_{\bar{0}}$ and $\Theta_{l+1}, \ldots, \Theta_{l+q'} \in U(\mathfrak{g}, e)_{\bar{1}}$ with

$$\Theta_k(1_\chi) = \Bigg(Y_k + \sum_{\substack{|\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}|_e = m_k+2, \\ |\mathbf{a}|+|\mathbf{b}|+|\mathbf{c}|+|\mathbf{d}| \geqslant 2}} \lambda_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}^k x^{\mathbf{a}} y^{\mathbf{b}} u^{\mathbf{c}} v^{\mathbf{d}} + \sum_{|\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}|_e < m_k+2} \lambda_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}^k x^{\mathbf{a}} y^{\mathbf{b}} u^{\mathbf{c}} v^{\mathbf{d}} \Bigg) \otimes 1_\chi, \tag{2.1}$$

for $1 \leqslant k \leqslant l+q$, where $\lambda_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}^k \in \mathbb{Q}$, and $\lambda_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}^k = 0$ if $a_{l+1} = \cdots = a_m = b_{q+1} = \cdots = b_n = c_1 = \cdots = c_s = d_1 = \cdots = d_{\lceil \frac{r}{2} \rceil} = 0$. When r is odd, set $\Theta_{l+q+1}(1_\chi) = Y_{l+q+1} \otimes 1_\chi = v_{\frac{r+1}{2}} \otimes 1_\chi$.

By [37, Theorem 4.5], the monomials $\Theta_1^{a_1} \cdots \Theta_l^{a_l} \Theta_{l+1}^{b_1} \cdots \Theta_{l+q'}^{b_{q'}}$ with $a_i \in \mathbb{Z}_+, b_j \in \{0, 1\}$ for $1 \leqslant i \leqslant l$ and $1 \leqslant j \leqslant q'$ form a basis of the vector space $U(\mathfrak{g}, e)$. The monomial $\Theta_1^{a_1} \cdots \Theta_l^{a_l} \Theta_{l+1}^{b_1} \cdots \Theta_{l+q'}^{b_{q'}}$ is said to have Kazhdan degree $\sum_{i=1}^{l} a_i(m_i+2) + \sum_{i=1}^{q'} b_i(m_{l+i}+2)$. For $k \in \mathbb{Z}_+$, let $\mathrm{F}_k U(\mathfrak{g}, e)$ denote the $\mathbb{C}$-span of all monomials $\Theta_1^{a_1} \cdots \Theta_l^{a_l} \Theta_{l+1}^{b_1} \cdots \Theta_{l+q'}^{b_{q'}}$ of Kazhdan degree $\leqslant k$. The subspaces $\mathrm{F}_k U(\mathfrak{g}, e)$ with $k \geqslant 0$ form an increasing exhaustive filtration of the algebra $U(\mathfrak{g}, e)$, which is called the Kazhdan filtration. The corresponding graded algebra $\mathrm{gr}\, U(\mathfrak{g}, e)$ is a polynomial superalgebra in $\mathrm{gr}\,\Theta_1, \ldots, \mathrm{gr}\,\Theta_{l+q'}$. Recall that [37, Theorem 4.7] shows that there are superpolynomials F_{ij} with $i, j = 1, \ldots, l+q'$ such that the defining relations on those generators can be described as

$$[\Theta_i, \Theta_j] = F_{ij}(\Theta_1, \ldots, \Theta_{l+q'}), \quad i, j = 1, \ldots, l+q', \tag{2.2}$$

while

$$F_{ij}(\Theta_1, \ldots, \Theta_{l+q'}) \equiv \sum_{k=1}^{l+q} \alpha_{ij}^k \Theta_k + q_{ij}(\Theta_1, \ldots, \Theta_{l+q'}) \pmod{\mathrm{F}_{m_i+m_j+1} U(\mathfrak{g}, e)} \tag{2.3}$$

for $i, j = 1, \dots, l+q$, where q_{ij} is a superpolynomial in $l+q'$ variables in $\mathbb{Q}$, whose constant term and linear part are zero. For the case when one of the indices i, j equals $l+q+1$, it follows from [37, Remark 3.8] that there are no obvious formulas for the superpolynomials F_{ij} as shown in (2.3). However, by the same discussion as in [37, Theorem 4.5], one can still choose the superpolynomials F_{ij} properly such that the Kazhdan degree for all the monomials in the F_{ij} is less than $m_i + m_j + 2$. Moreover,

$$F_{l+q+1,l+q+1}(\Theta_1, \dots, \Theta_{l+q+1}) = 1 \otimes 1_\chi \tag{2.4}$$

when r is odd.

In fact, some of the defining relations in (2.2) are equivalent to each other. By the same discussion as in [37, Remark 3.8(4)], after deleting all the redundant commutating relations in (2.2), the remaining ones are with indices i, j satisfying $1 \leqslant i < j \leqslant l$, $l+1 \leqslant i \leqslant j \leqslant l+q'$, and $1 \leqslant i \leqslant l < j \leqslant l+q'$.

In the following arguments, when we consider the corresponding counterparts of all of the above over the algebraic closured field $\Bbbk = \overline{\mathbb{F}}_p$ of positive characteristic p, *we assume that the prime p is large enough such that the admissible ring A contains all $\lambda^k_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}$ in (2.1) and all coefficients of the F_{ij} in (2.2)*, thereby the "admissible procedure" developed by Premet for finite W-algebras can be reproduced in the super case.

For $\mathbf{a} = (a_1, \dots, a_l) \in \mathbb{Z}^l_+$ and $\mathbf{b} = (b_1, \dots, b_{q'}) \in \Lambda'_{q'}$, let $U(\mathfrak{g}_A, e)$ be the A-span of the monomials

$$\{\Theta_1^{a_1} \cdots \Theta_l^{a_l} \Theta_{l+1}^{b_1} \cdots \Theta_{l+q'}^{b_{q'}} \mid (\mathbf{a}, \mathbf{b}) \in \mathbb{Z}^l_+ \times \Lambda'_{q'}\}.$$

2.2.3. Let I_χ denote the $\mathbb{Z}_2$-graded ideal in $U(\mathfrak{g})$ generated by all $x - \chi(x)$ with $x \in \mathfrak{m}_i, i \in \mathbb{Z}_2$. By construction, I_χ is a $(U(\mathfrak{g}), U(\mathfrak{m}))$-bimodule. The fixed point space $(U(\mathfrak{g})/I_\chi)^{\mathrm{ad}\,\mathfrak{m}}$ carries a natural algebra structure given by $(x+I_\chi)\cdot(y+I_\chi) := (xy + I_\chi)$ for all $x, y \in U(\mathfrak{g})$. Then $Q_\chi \cong U(\mathfrak{g})/I_\chi$ as $\mathfrak{g}$-modules via the $\mathfrak{g}$-module map sending $1 + I_\chi$ to 1_χ, and $Q_\chi^{\mathrm{ad}\,\mathfrak{m}} \cong U(\mathfrak{g}, e)$ as $\mathbb{C}$-algebras. Any element of $U(\mathfrak{g}, e)$ is uniquely determined by its effect on the generator $1_\chi \in Q_\chi$, and it follows from [37, Theorem 2.12] that the canonical isomorphism between $U(\mathfrak{g}, e)$ and $Q_\chi^{\mathrm{ad}\,\mathfrak{m}}$ is given by $u \mapsto u(1_\chi)$ for any $u \in U(\mathfrak{g}, e)$. In what follows we will often identify Q_χ with $U(\mathfrak{g})/I_\chi$ and $U(\mathfrak{g}, e)$ with $Q_\chi^{\mathrm{ad}\,\mathfrak{m}}$.

Let $w_1, \dots, w_c$ be a basis of $\mathfrak{g}$ over $\mathbb{C}$. Let $U(\mathfrak{g}) = \bigcup_{i \in \mathbb{Z}} \mathrm{F}_i U(\mathfrak{g})$ be the Kazhdan filtration of $U(\mathfrak{g})$, where $\mathrm{F}_i U(\mathfrak{g})$ is the $\mathbb{C}$-span of all $w_1 \cdots w_c$ with $w_1 \in \mathfrak{g}(j_1), \dots, w_c \in \mathfrak{g}(j_c)$ and $(j_1 + 2) + \cdots + (j_c + 2) \leqslant i$. The Kazhdan filtration on Q_χ is defined by $\mathrm{F}_i Q_\chi := \pi(\mathrm{F}_i U(\mathfrak{g}))$ with $\pi : U(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g})/I_\chi$ being the canonical homomorphism, which makes Q_χ into a filtered $U(\mathfrak{g})$-module. The Kazhdan filtration of Q_χ has no negative components, and the Kazhdan filtration of $U(\mathfrak{g},e)$ defined in §2.2.2 is nothing but the filtration of $U(\mathfrak{g},e) = Q_\chi^{\mathrm{ad}\,\mathfrak{m}}$ induced from the Kazhdan filtration of Q_χ through the embedding $Q_\chi^{\mathrm{ad}\,\mathfrak{m}} \hookrightarrow Q_\chi$ (see [37, §2.3]).

§2.3. Finite W-superalgebras in positive characteristic

2.3.1. Pick a prime $p \in \Pi(A)$ and denote by $\mathbb{k} = \overline{\mathbb{F}}_p$ the algebraic closure of $\mathbb{F}_p$. Since the bilinear form $(\cdot,\cdot)$ is A-valued on $\mathfrak{g}_A$, it induces a bilinear form on the Lie superalgebra $\mathfrak{g}_\mathbb{k} \cong \mathfrak{g}_A \otimes_A \mathbb{k}$. In the following we still denote this bilinear form by $(\cdot,\cdot)$. If we denote by $G_\mathbb{k}$ the algebraic $\mathbb{k}$-supergroup of distribution algebra $U_\mathbb{k} = U_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{k}$, then $\mathfrak{g}_\mathbb{k} = \mathrm{Lie}(G_\mathbb{k})$ (cf. [10, §I.7.10] and [31, §2.2]). Note that the bilinear form $(\cdot,\cdot)$ is nondegenerate and $\mathrm{Ad}\,(G_\mathbb{k})_{\mathrm{ev}}$-invariant. For $x \in \mathfrak{g}_A$, set $\bar{x} := x \otimes 1$, an element of $\mathfrak{g}_\mathbb{k}$. To simplify notation we identify e, f, h with the nilpotent elements $\bar{e} = e \otimes 1$, $\bar{f} = f \otimes 1$, and $\bar{h} = h \otimes 1$ in $\mathfrak{g}_\mathbb{k}$, and χ with the linear function $(\bar{e},\cdot)$ on $\mathfrak{g}_\mathbb{k}$.

The Lie superalgebra $\mathfrak{g}_\mathbb{k} = \mathrm{Lie}\,(G_\mathbb{k})$ carries a natural p-mapping $x \mapsto x^{[p]}$ for all $x \in (\mathfrak{g}_\mathbb{k})_{\bar{0}}$, which is equivariant under the adjoint action of $(G_\mathbb{k})_{\mathrm{ev}}$. The subalgebra of $U(\mathfrak{g}_\mathbb{k})$ generated by all $\bar{x}^p - \bar{x}^{[p]}$ with $\bar{x} \in (\mathfrak{g}_\mathbb{k})_{\bar{0}}$ is called the p-center of $U(\mathfrak{g}_\mathbb{k})$ and we denote it by $Z_p(\mathfrak{g}_\mathbb{k})$ for short. It follows from the PBW theorem of $U(\mathfrak{g}_\mathbb{k})$ that $Z_p(\mathfrak{g}_\mathbb{k})$ is isomorphic to an (ordinary) polynomial algebra in $\dim(\mathfrak{g}_\mathbb{k})_{\bar{0}}$ variables. For every maximal ideal H of $Z_p(\mathfrak{g}_\mathbb{k})$ there is a unique linear function $\eta = \eta_H \in (\mathfrak{g}_\mathbb{k})_{\bar{0}}^*$ such that

$$H = \langle \bar{x}^p - \bar{x}^{[p]} - \eta(\bar{x})^p \mid \bar{x} \in (\mathfrak{g}_\mathbb{k})_{\bar{0}} \rangle.$$

Since the Frobenius map of $\mathbb{k}$ is bijective, this enables us to identify the maximal spectrum $\mathrm{Specm}(Z_p(\mathfrak{g}_\mathbb{k}))$ of $Z_p(\mathfrak{g}_\mathbb{k})$ with $(\mathfrak{g}_\mathbb{k})_{\bar{0}}^*$.

For any $\xi \in (\mathfrak{g}_\mathbb{k})_{\bar{0}}^*$ we denote by J_ξ the two-sided ideal of $U(\mathfrak{g}_\mathbb{k})$ generated by the even central elements $\{\bar{x}^p - \bar{x}^{[p]} - \xi(\bar{x})^p \mid \bar{x} \in (\mathfrak{g}_\mathbb{k})_{\bar{0}}\}$. Then the quotient algebra $U_\xi(\mathfrak{g}_\mathbb{k}) := U(\mathfrak{g}_\mathbb{k})/J_\xi$ is called the reduced enveloping algebra with p-character ξ. We have $\dim U_\xi(\mathfrak{g}_\mathbb{k}) = p^{\dim(\mathfrak{g}_\mathbb{k})_{\bar{0}}} 2^{\dim(\mathfrak{g}_\mathbb{k})_{\bar{1}}}$ by construction. It follows from the Schur lemma that any irreducible $\mathfrak{g}_\mathbb{k}$-module V is that of $U_\xi(\mathfrak{g}_\mathbb{k})$ for a unique $\xi = \xi_V \in (\mathfrak{g}_\mathbb{k})_{\bar{0}}^*$. We often regard $\xi \in \mathfrak{g}_\mathbb{k}^*$ by letting $\xi((\mathfrak{g}_\mathbb{k})_{\bar{1}}) = 0$.

2.3.2. For $i \in \mathbb{Z}$, set $\mathfrak{g}_\mathbb{k}(i) := \mathfrak{g}_A(i) \otimes_A \mathbb{k}$ and put $\mathfrak{m}_\mathbb{k} := \mathfrak{m}_A \otimes_A \mathbb{k}$, then [37, Lemma 2.18] shows that $\mathfrak{m}_\mathbb{k}$ is a restricted subalgebra of $\mathfrak{g}_\mathbb{k}$. Denote by $\mathfrak{m}'_\mathbb{k} := \mathfrak{m}'_A \otimes_A \mathbb{k}$ and $\mathfrak{p}_\mathbb{k} := \mathfrak{p}_A \otimes_A \mathbb{k}$. Due to our assumptions on A, the elements $\bar{x}_1, \ldots, \bar{x}_l$ and $\bar{y}_1, \ldots, \bar{y}_q$ form bases of the centralizer $(\mathfrak{g}_\mathbb{k}^e)_{\bar{0}}$ and $(\mathfrak{g}_\mathbb{k}^e)_{\bar{1}}$ of e in $\mathfrak{g}_\mathbb{k}$, respectively. It follows from [35, §4.1] that the subalgebra $\mathfrak{m}_\mathbb{k}$ is p-nilpotent, and the linear function χ vanishes on the p-closure of $[\mathfrak{m}_\mathbb{k}, \mathfrak{m}_\mathbb{k}]$. Set $Q_{\chi,\mathbb{k}} := U(\mathfrak{g}_\mathbb{k}) \otimes_{U(\mathfrak{m}_\mathbb{k})} \mathbb{k}_\chi$, where $\mathbb{k}_\chi = A_\chi \otimes_A \mathbb{k} = \mathbb{k}1_\chi$. Clearly, $\mathbb{k}1_\chi$ is a one-dimensional $\mathfrak{m}_{\Bbbk}$-module with the property $\bar{x}.1_\chi = \chi(\bar{x})1_\chi$ for all $\bar{x} \in \mathfrak{m}_{\Bbbk}$, and it is obvious that $Q_{\chi,\Bbbk} \cong Q_{\chi,A} \otimes_A \Bbbk$ as $\mathfrak{g}_{\Bbbk}$-modules. Define the finite W-superalgebra over $\Bbbk$ by $U(\mathfrak{g}_{\Bbbk}, e) := (\mathrm{End}_{\mathfrak{g}_{\Bbbk}} Q_{\chi,\Bbbk})^{\mathrm{op}}$.

Let $\mathfrak{g}_A^*$ be the A-module dual to $\mathfrak{g}_A$ and let $(\mathfrak{m}_A^\perp)_{\bar{0}}$ denote the set of all linear functions on $(\mathfrak{g}_A)_{\bar{0}}$ vanishing on $(\mathfrak{m}_A)_{\bar{0}}$. By the assumptions on A, $(\mathfrak{m}_A^\perp)_{\bar{0}}$ is a free A-submodule and a direct summand of $\mathfrak{g}_A^*$. Note that $(\mathfrak{m}_A^\perp \otimes_A \mathbb{C})_{\bar{0}}$ and $(\mathfrak{m}_A^\perp \otimes_A \Bbbk)_{\bar{0}}$ can be identified with the annihilators $\mathfrak{m}_{\bar{0}}^\perp := \{f \in \mathfrak{g}_{\bar{0}}^* \mid f(\mathfrak{m}_{\bar{0}}) = 0\}$ and $(\mathfrak{m}_{\Bbbk}^\perp)_{\bar{0}} := \{f \in (\mathfrak{g}_{\Bbbk})_{\bar{0}}^* \mid f((\mathfrak{m}_{\Bbbk})_{\bar{0}}) = 0\}$, respectively. Let $I_{\chi,A}$ denote the A-span of the left ideal of $U(\mathfrak{g}_A)$ generated by all $x - \chi(x)$ with $x \in \mathfrak{m}_A$.

Given a linear function $\eta \in \chi + (\mathfrak{m}_{\Bbbk}^\perp)_{\bar{0}}$, set the $\mathfrak{g}_{\Bbbk}$-module $Q_\chi^\eta := Q_{\chi,\Bbbk}/J_\eta Q_{\chi,\Bbbk}$. Each $\mathfrak{g}_{\Bbbk}$-endomorphism $\bar{\Theta}_i$ of $Q_{\chi,\Bbbk}$ preserves $J_\eta Q_{\chi,\Bbbk}$, and hence induces a $\mathfrak{g}_{\Bbbk}$-endomorphism of Q_χ^η which is denoted by θ_i. As in [37], we set $U_\eta(\mathfrak{g}_{\Bbbk}, e) = (\mathrm{End}_{\mathfrak{g}_{\Bbbk}} Q_\chi^\eta)^{\mathrm{op}}$, and call it a *reduced W-superalgebra.*

Since the restriction of η to $\mathfrak{m}_{\Bbbk}$ coincides with that of χ, the left ideal of $U(\mathfrak{g}_{\Bbbk})$ generated by all $x - \eta(x)$ with $x \in \mathfrak{m}_{\Bbbk}$ equals $I_{\chi,\Bbbk} := I_{\chi,A} \otimes_A \Bbbk$ and $\Bbbk_\chi = \Bbbk_\eta$ as $\mathfrak{m}_{\Bbbk}$-modules. We denote by $I_{\mathfrak{m}_{\Bbbk}}$ the left ideal of $U_\eta(\mathfrak{g}_{\Bbbk})$ generated by all $x - \eta(x)$ with $x \in \mathfrak{m}_{\Bbbk}$.

For a (left) $U_\eta(\mathfrak{g}_{\Bbbk})$-module M, define

$$M^{\mathfrak{m}_{\Bbbk}} := \{v \in M \mid I_{\mathfrak{m}_{\Bbbk}}.v = 0\}.$$

It follows from [37, Proposition 2.21] that $U_\eta(\mathfrak{g}_{\Bbbk}, e)$ can be identified with the $\Bbbk$-algebra $U_\eta(\mathfrak{g}_{\Bbbk})^{\mathrm{ad}\,\mathfrak{m}_{\Bbbk}}/U_\eta(\mathfrak{g}_{\Bbbk})^{\mathrm{ad}\,\mathfrak{m}_{\Bbbk}} \cap I_{\mathfrak{m}_{\Bbbk}}$. Therefore, any left $U_\eta(\mathfrak{g}_{\Bbbk})^{\mathrm{ad}\,\mathfrak{m}_{\Bbbk}}$-module can be considered a $U_\eta(\mathfrak{g}_{\Bbbk}, e)$-module with the trivial action of the ideal $U_\eta(\mathfrak{g}_{\Bbbk})^{\mathrm{ad}\,\mathfrak{m}_{\Bbbk}} \cap I_{\mathfrak{m}_{\Bbbk}}$. Recall [37, Theorem 2.24] shows that

Theorem 2.3. *The correspondence M to $M^{\mathfrak{m}_{\Bbbk}}$ gives rise to a category equivalence between the category of $U_\eta(\mathfrak{g}_{\Bbbk})$-modules and the category of $U_\eta(\mathfrak{g}_{\Bbbk}, e)$-modules:*

$$U_\eta(\mathfrak{g}_{\Bbbk})\text{-mod} \longrightarrow U_\eta(\mathfrak{g}_{\Bbbk}, e)\text{-mod}$$

with the inverse

$$U_\eta(\mathfrak{g}_{\Bbbk}, e)\text{-mod} \longrightarrow U_\eta(\mathfrak{g}_{\Bbbk})\text{-mod}, \qquad V \mapsto U_\eta(\mathfrak{g}_{\Bbbk}) \otimes_{U_\eta(\mathfrak{g}_{\Bbbk})^{\mathrm{ad}\,\mathfrak{m}_{\Bbbk}}} V.$$

Furthermore, for any $U_\eta(\mathfrak{g}_{\Bbbk})$-module M, [37, Lemma 2.22] shows that $M^{\mathfrak{m}_{\Bbbk}}$ is a free $U_\eta(\mathfrak{m}_{\Bbbk})$-module. By the same discussion as in [35, Proposition 4.2], one can conclude that there is an isomorphism of vector spaces $M \cong U_\eta(\mathfrak{m}_{\Bbbk})^* \otimes_{\Bbbk} M^{\mathfrak{m}_{\Bbbk}}$. It is immediate that

$$\dim M^{\mathfrak{m}_{\Bbbk}} = \frac{\dim M}{\dim U_\eta(\mathfrak{m}_{\Bbbk})}. \tag{2.5}$$

In particular, we have $\dim U_\eta(\mathfrak{g}_{\Bbbk}) = \dim U_\eta(\mathfrak{m}_{\Bbbk}) \cdot \dim U_\eta(\mathfrak{g}_{\Bbbk})^{\mathrm{ad}\,\mathfrak{m}_{\Bbbk}}$.

§3. Transition subalgebras of finite W-superalgebras in prime characteristic

We will maintain the notation and conventions of §1. In particular, $\mathfrak{g}$ is a given basic Lie superalgebra over $\mathbb{C}$, A is an associated admissible ring, and $\mathbb{k}$ is an algebraically closed field of characteristic $p \in \Pi(A)$. Throughout this section, fix a nilpotent element $e \in \mathfrak{g}_{\bar{0}}$; then we can define $d_i = \dim \mathfrak{g}_i - \dim \mathfrak{g}_i^e$ for $i \in \mathbb{Z}_2$. Further recall that both $\dim \mathfrak{g}(-1)_{\bar{1}}$ and $d_1 = \dim \mathfrak{g}_{\bar{1}} - \dim \mathfrak{g}_{\bar{1}}^e$ have the same parity by §2.2.1. We always set $q' = q$ if d_1 is even, and $q' = q+1$ if d_1 is odd.

In this section we will introduce a so-called transition subalgebra of the finite W-superalgebra $U(\mathfrak{g}_{\mathbb{k}}, e)$ over $\mathbb{k}$. We present some structure relations between $U(\mathfrak{g}_{\mathbb{k}}, e)$ and its transition subalgebra that enable us to connect the information of finite W-superalgebras over $\mathbb{C}$ with the modular representations of reduced enveloping algebras of basic Lie superalgebras over $\mathbb{k}$ in the following sections.

This section is somewhat a generalization of the Lie algebra case by Premet in [29, §2], with a few modifications. The emergence of odd parts in the Lie superalgebra $\mathfrak{g}_{\mathbb{k}}$ makes the situation complicated.

§3.1. Transition subalgebras

Recall that in §2.2.2 we defined

$$\mathbb{Z}_+^k := \{(i_1, \ldots, i_k) \mid i_j \in \mathbb{Z}_+\}, \qquad \Lambda'_k := \{(i_1, \ldots, i_k) \mid i_j \in \{0,1\}\}$$

for $k \in \mathbb{Z}_+$ with $1 \leqslant j \leqslant k$. For $\mathbf{a} = (a_1, \ldots, a_l) \in \mathbb{Z}_+^l$ and $\mathbf{b} = (b_1, \ldots, b_{q'}) \in \Lambda'_{q'}$, $U(\mathfrak{g}_A, e)$ denotes the A-span of the monomials

$$\{\Theta_1^{a_1} \cdots \Theta_l^{a_l} \Theta_{l+1}^{b_1} \cdots \Theta_{l+q'}^{b_{q'}} \mid (\mathbf{a}, \mathbf{b}) \in \mathbb{Z}_+^l \times \Lambda'_{q'}\}$$

in §2.2.2. Our assumption on A guarantees $U(\mathfrak{g}_A, e)$ to be an A-subalgebra of $U(\mathfrak{g}, e)$ contained in $(\mathrm{End}_{\mathfrak{g}_A} Q_{\chi,A})^{\mathrm{op}}$. By the definition of $Q_{\chi,A}$ in §2.2.2 and $I_{\chi,A}$ in §2.3.2 we know that $Q_{\chi,A}$ can be identified with the $\mathfrak{g}_A$-module $U(\mathfrak{g}_A)/I_{\chi,A}$. Hence $U(\mathfrak{g}_A, e)$ embeds into the A-algebra $(U(\mathfrak{g}_A)/I_{\chi,A})^{\mathrm{ad}\,\mathfrak{m}_A} \cong (Q_{\chi,A})^{\mathrm{ad}\,\mathfrak{m}_A}$. As $Q_{\chi,A}$ is a free A-module with basis $\{x^{\mathbf{a}} y^{\mathbf{b}} u^{\mathbf{c}} v^{\mathbf{d}} \otimes 1_\chi \mid (\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \in \mathbb{Z}_+^m \times \Lambda'_n \times \mathbb{Z}_+^s \times \Lambda'_t\}$, an easy induction on Kazhdan degree (based on [37, Lemma 4.3] and the formulas displayed in [37, Lemma 4.2, Theorem 4.5]) shows that

$$U(\mathfrak{g}_A, e) = (\mathrm{End}_{\mathfrak{g}_A} Q_{\chi,A})^{\mathrm{op}} \cong Q_{\chi,A}^{\mathrm{ad}\,\mathfrak{m}_A}.$$

Definition 3.1. Set the $\mathbb{k}$-algebra $T(\mathfrak{g}_{\mathbb{k}}, e) := U(\mathfrak{g}_A, e) \otimes_A \mathbb{k}$. In the following we will call $T(\mathfrak{g}_{\mathbb{k}}, e)$ a transition subalgebra.

It is notable that by the definition, $T(\mathfrak{g}_{\mathbb{k}}, e)$ can be naturally identified with a subalgebra of the finite W-superalgebra $U(\mathfrak{g}_{\mathbb{k}}, e) = (\mathrm{End}_{\mathfrak{g}_{\mathbb{k}}} Q_{\chi,\mathbb{k}})^{\mathrm{op}}$ over $\mathbb{k}$. Moreover, $T(\mathfrak{g}_{\Bbbk}, e)$ has a $\Bbbk$-basis consisting of all monomials $\bar{\Theta}_1^{a_1}\cdots\bar{\Theta}_{l+q'}^{b_{q'}}$, where $\bar{\Theta}_i := \Theta_i\otimes 1\in U(\mathfrak{g}_A, e)\otimes_A\Bbbk$ for $1\leqslant i\leqslant l+q'$.

Since all the coefficients of the polynomials F_{ij} for $1\leqslant i,j\leqslant l+q'$ in §2.2.2 are in $\mathbb{Q}$, one can assume the F_{ij} are over A after enlarging A if needed. Given a superpolynomial $g\in A[T_1,\ldots,T_n]$, let ${}^pg$ denote the image of g in the polynomial superalgebra $\Bbbk[T_1,\ldots,T_n] = A[T_1,\ldots,T_n]\otimes_A\Bbbk$. By the same discussion as in [37, Theorem 4.7], we know that there exist superpolynomials ${}^pF_{ij}$ of $l+q'$ indeterminants over $\Bbbk$ $(i,j=1,\ldots,l+q')$ with the first l indeterminants being even, and the others being odd, such that

$$[\bar{\Theta}_i,\bar{\Theta}_j] = {}^pF_{ij}(\bar{\Theta}_1,\ldots,\bar{\Theta}_{l+q'}),\quad i,j=1,\ldots,l+q',$$

while the ${}^pF_{ij}(\bar{\Theta}_1,\ldots,\bar{\Theta}_{l+q'})$ satisfy the same relations as (2.3) and (2.4).

Theorem 3.2. *Maintain the notation above. Then the $\bar{\Theta}_i$ and the ${}^pF_{ij}$ for $i,j = 1,\ldots,l+q'$ constitute a data of generators and defining relations of $T(\mathfrak{g}_{\Bbbk},e)$, with $\bar{\Theta}_1,\ldots,\bar{\Theta}_l\in T(\mathfrak{g}_{\Bbbk},e)_{\bar{0}}$ and $\bar{\Theta}_{l+1},\ldots,\bar{\Theta}_{l+q'}\in T(\mathfrak{g}_{\Bbbk},e)_{\bar{1}}$ as the generators of $T(\mathfrak{g}_{\Bbbk},e)$ subject to the relations*

$$[\bar{\Theta}_i,\bar{\Theta}_j] = {}^pF_{ij}(\bar{\Theta}_1,\ldots,\bar{\Theta}_{l+q'}),$$

where $1\leqslant i,j\leqslant l+q'$.

§3.2. Revisiting reduced W-superalgebras with p-characters $\eta\in\chi+(\mathfrak{m}_{\Bbbk}^{\perp})_{\bar{0}}$

Recall that in §2.3.2 we defined the $\mathfrak{g}_{\Bbbk}$-module $Q_\chi^\eta = Q_{\chi,\Bbbk}/J_\eta Q_{\chi,\Bbbk}$ and the reduced W-superalgebra $U_\eta(\mathfrak{g}_{\Bbbk},e)$=$(\mathrm{End}_{\mathfrak{g}_{\Bbbk}}Q_\chi^\eta)^{\mathrm{op}}$, and §2.2.1 shows that $\{x_1,\ldots,x_m, y_1,\ldots,y_n\}$ is an A-basis of $\mathfrak{p}_A$. Set

$$X_i := \begin{cases} x_{i+l} & \text{if } 1\leqslant i\leqslant m-l,\\ y_{l+q-m+i} & \text{if } m-l+1\leqslant i\leqslant m+n-l-q,\\ u_{l+q-m-n+i} & \text{if } m+n-l-q+1\leqslant i\leqslant m+n-l-q+s,\\ v_{l+q-m-n-s+i} & \text{if } m+n-l-q+s+1\leqslant i\leqslant m+n-l-q+s+t',\end{cases}$$

where $t' := \lceil\frac{r}{2}\rceil = \lceil\frac{\dim\mathfrak{g}_{\Bbbk}(-1)_{\bar{1}}}{2}\rceil$.

Conventions 3.3. We will denote $\lceil\frac{r}{2}\rceil$ by t' once and for all. It follows from [35, Theorem 4.3] that $\dim U_\chi(\mathfrak{m}_{\Bbbk}) = p^{\frac{d_0}{2}}2^{\lceil\frac{d_1}{2}\rceil}$ and we denote it by δ afterwards. By the assumption of the notation we have $\frac{d_0}{2} = m-l+s$ and $\lceil\frac{d_1}{2}\rceil = n-q+t'$.

For $(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})\in\mathbb{Z}_+^{m-l}\times\Lambda'_{n-q}\times\mathbb{Z}_+^s\times\Lambda'_{t'}$, define

$$\begin{aligned}X^{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}:=&X_1^{a_1}\cdots X_{m-l}^{a_{m-l}}X_{m-l+1}^{b_1}\cdots X_{m+n-l-q}^{b_{n-q}}X_{m+n-l-q+1}^{c_1}\cdots X_{m+n-l-q+s}^{c_s}\\&\cdot X_{m+n-l-q+s+1}^{d_1}\cdots X_{m+n-l-q+s+t'}^{d_{t'}}\end{aligned}$$

and

$$\begin{aligned}\bar{X}^{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}:=&\bar{X}_1^{a_1}\cdots \bar{X}_{m-l}^{a_{m-l}}\bar{X}_{m-l+1}^{b_1}\cdots \bar{X}_{m+n-l-q}^{b_{n-q}}\bar{X}_{m+n-l-q+1}^{c_1}\cdots \bar{X}_{m+n-l-q+s}^{c_s}\\&\cdot \bar{X}_{m+n-l-q+s+1}^{d_1}\cdots \bar{X}_{m+n-l-q+s+t'}^{d_{t'}},\end{aligned}$$

elements of $U(\mathfrak{g}_A)$ and $U(\mathfrak{g}_{\Bbbk})$, respectively. Denote by $\bar{1}_\chi$ the image of $1_\chi\in Q_{\chi,\Bbbk}$ in Q_χ^η. For $k\in\mathbb{Z}_+$, define

$$\Lambda_k:=\{(i_1,\ldots,i_k)\mid i_j\in\mathbb{Z}_+,\ 0\leqslant i_j\leqslant p-1\}$$

with $1\leqslant j\leqslant k$.

Lemma 3.4. *The right modules $Q_{\chi,A}$ and Q_χ^η with $\eta\in\chi+(\mathfrak{m}_{\Bbbk}^\perp)_{\bar{0}}$ are free over $U(\mathfrak{g}_A,e)$ and $U_\eta(\mathfrak{g}_{\Bbbk},e)$ respectively. More precisely,*

(1) the set $\{X^{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}\otimes 1_\chi\mid(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})\in\mathbb{Z}_+^{m-l}\times\Lambda'_{n-q}\times\mathbb{Z}_+^s\times\Lambda'_{t'}\}$ is a free basis of the $U(\mathfrak{g}_A,e)$-module $Q_{\chi,A}$;

(2) the set $\{\bar{X}^{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}\otimes \bar{1}_\chi\mid(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})\in\Lambda_{m-l}\times\Lambda'_{n-q}\times\Lambda_s\times\Lambda'_{t'}\}$ is a free basis of the $U_\eta(\mathfrak{g}_{\Bbbk},e)$-module Q_χ^η.

Proof. The proof is the same as in the finite W-algebra case, thus will be omitted here (see [28, Lemma 4.2] and [29, Lemma 2.3]). □

§3.3. Transition for finite W-superalgebras

Let $\rho_{\Bbbk}$ denote the representation of $U(\mathfrak{g}_{\Bbbk})$ in $\mathrm{End}_{\Bbbk}Q_{\chi,\Bbbk}$. Given a subspace V in $\mathfrak{g}_{\Bbbk}$ we denote by $Z_p(V)$ the subalgebra of p-center $Z_p(\mathfrak{g}_{\Bbbk})$ generated by all $\bar{x}^p-\bar{x}^{[p]}$ with $\bar{x}\in V_{\bar{0}}$. Clearly, $Z_p(V)$ is isomorphic to an (ordinary) polynomial algebra in $\dim V_{\bar{0}}$ variables. We will denote $Z_p(\mathfrak{g}_{\Bbbk})$ by Z_p for short.

Let $\mathfrak{a}_{\Bbbk}$ be the $\Bbbk$-span of $\bar{X}_1,\ldots,\bar{X}_{m+n-l-q+s+t'}$ in $\mathfrak{g}_{\Bbbk}$ and put $\widetilde{\mathfrak{p}}_{\Bbbk}=\mathfrak{a}_{\Bbbk}\oplus\mathfrak{g}_{\Bbbk}^e$ (resp. $\widetilde{\mathfrak{p}}_{\Bbbk}=\mathfrak{a}_{\Bbbk}\oplus\mathfrak{g}_{\Bbbk}^e\oplus\Bbbk v_{\frac{r+1}{2}}$) when d_1 is even (resp. odd), then $\mathfrak{g}_{\Bbbk}=\mathfrak{m}_{\Bbbk}\oplus\widetilde{\mathfrak{p}}_{\Bbbk}$.

By our assumptions on $x_{l+1},\ldots,x_m,y_{q+1},\ldots,y_n$ and the inclusion $\mathfrak{g}_{\Bbbk}^f\subseteq\bigoplus_{i\leqslant0}\mathfrak{g}_{\Bbbk}(i)$, we have that

$$\mathfrak{a}_{\Bbbk}=\{\bar{x}\in\widetilde{\mathfrak{p}}_{\Bbbk}\mid(\bar{x},\mathfrak{g}_{\Bbbk}^f)=0\}\qquad(\text{resp. }\mathfrak{a}_{\Bbbk}\oplus\Bbbk v_{\frac{r+1}{2}}=\{\bar{x}\in\widetilde{\mathfrak{p}}_{\Bbbk}\mid(\bar{x},\mathfrak{g}_{\Bbbk}^f)=0\})$$

when d_1 is even (resp. odd).

Theorem 3.5. *Let $e \in (\mathfrak{g}_{\Bbbk})_{\bar{0}}$ be any even nilpotent element:*

(1) Then $\rho_{\Bbbk}(Z_p) \cong Z_p(\widetilde{\mathfrak{p}}_{\Bbbk})$ as $\Bbbk$-algebras.
(2) We have $U(\mathfrak{g}_{\Bbbk}, e)$ is a free $\rho_{\Bbbk}(Z_p)$-module of rank $p^l 2^{q'}$. In particular, $U(\mathfrak{g}_{\Bbbk}, e)$ is generated by its subalgebras $T(\mathfrak{g}_{\Bbbk}, e)$ and $\rho_{\Bbbk}(Z_p)$.
(3) Furthermore, $U(\mathfrak{g}_{\Bbbk}, e) \cong T(\mathfrak{g}_{\Bbbk}, e) \otimes_{\Bbbk} Z_p(\mathfrak{a}_{\Bbbk})$ as $\Bbbk$-algebras.

This theorem is a generalization of the finite W-algebra case in [29, Theorem 2.1]. Compared with finite W-algebras, the construction of finite W-superalgebras is much more complicated. In particular, some new phenomenon occurs when d_1 is odd. Now we will prove the theorem in detail.

Proof. (1) It follows from $\mathfrak{g}_{\Bbbk} = \mathfrak{m}_{\Bbbk} \oplus \widetilde{\mathfrak{p}}_{\Bbbk}$ that $Z_p(\mathfrak{g}_{\Bbbk}) \cong Z_p(\mathfrak{m}_{\Bbbk}) \otimes_{\Bbbk} Z_p(\widetilde{\mathfrak{p}}_{\Bbbk})$ as $\Bbbk$-algebras. As $Z_p(\mathfrak{m}_{\Bbbk}) \cap \operatorname{Ker} \rho_{\Bbbk}$ is an ideal of codimension 1 in $Z_p(\mathfrak{m}_{\Bbbk})$, one can conclude that $\rho_{\Bbbk}(Z_p) = \rho_{\Bbbk}(Z_p(\widetilde{\mathfrak{p}}_{\Bbbk}))$. As the monomials $\bar{x}^{\mathbf{a}}\bar{y}^{\mathbf{b}}\bar{u}^{\mathbf{c}}\bar{v}^{\mathbf{d}} \otimes 1_\chi$ with $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \in \mathbb{Z}_+^m \times \Lambda'_n \times \mathbb{Z}_+^s \times \Lambda'_t$ (recall that $t = \lfloor \frac{\dim \mathfrak{g}_{\Bbbk}(-1)_{\bar{1}}}{2} \rfloor$) form a basis of $Q_{\chi,\Bbbk}$, and $Z_p(\widetilde{\mathfrak{p}}_{\Bbbk})$ is a polynomial algebra in $\{\bar{x}_i^p - \bar{x}_i^{[p]} \mid 1 \leqslant i \leqslant m\} \cup \{\bar{u}_j^p - \bar{u}_j^{[p]} \mid 1 \leqslant j \leqslant s\}$, we have $Z_p(\widetilde{\mathfrak{p}}_{\Bbbk}) \cap \operatorname{Ker} \rho_{\Bbbk} = \{0\}$. It follows that $\rho_{\Bbbk}(Z_p) \cong Z_p(\widetilde{\mathfrak{p}}_{\Bbbk})$ as $\Bbbk$-algebras. This completes the proof of statement (1).

(2) As the proofs of the remaining statements are the same whether d_1 is even or odd, it is sufficient for us to make arguments under the assumption that d_1 is odd.

First recall that $S((\widetilde{\mathfrak{p}}_{\Bbbk})_{\bar{0}}) \cong \Bbbk[\chi + (\mathfrak{m}_{\Bbbk}^{\perp})_{\bar{0}}]$ by the discussion of [37, §2.3], hence $Z_p(\widetilde{\mathfrak{p}}_{\Bbbk}) \cong \Bbbk[(\chi + (\mathfrak{m}_{\Bbbk}^{\perp})_{\bar{0}})^{(1)}]$, where $(\chi + (\mathfrak{m}_{\Bbbk}^{\perp})_{\bar{0}})^{(1)} \subseteq (\mathfrak{g}_{\Bbbk}^*)_{\bar{0}}^{(1)}$ is the Frobenius twist of $\chi + (\mathfrak{m}_{\Bbbk}^{\perp})_{\bar{0}}$. Then we will manage to construct a set of free bases of $U(\mathfrak{g}_{\Bbbk}, e)$ as a $\rho_{\Bbbk}(Z_p)$-module, via the reduced W-superalgebra $U_\eta(\mathfrak{g}_{\Bbbk}, e)$ with $\eta \in \chi + (\mathfrak{m}_{\Bbbk}^{\perp})_{\bar{0}}$. Now we proceed by steps.

(2-i) Let us begin by understanding the $Z_p(\widetilde{\mathfrak{p}}_{\Bbbk})$-module $Q_{\chi,\Bbbk}$. As an immediate consequence of [37, Theorem 4.5(1)], we have

$$\bar{\Theta}_k^p(1_\chi) - \left(\bar{x}_k^p + \sum_{|(\mathbf{a},\mathbf{0},\mathbf{c},\mathbf{0})|_e = m_k+2} \mu^k_{\mathbf{a},\mathbf{0},\mathbf{c},\mathbf{0}} \bar{x}^{p\mathbf{a}} \bar{u}^{p\mathbf{c}}\right) \otimes 1_\chi \in (Q_{\chi,\Bbbk})_{p(m_k+2)-1} \quad (3.1)$$

for $1 \leqslant k \leqslant l$, where $\mu^k_{\mathbf{a},\mathbf{0},\mathbf{c},\mathbf{0}} \in \mathbb{F}_p$. Since $\bar{x}^{[p]} \in \mathfrak{g}_{\Bbbk}(pi)$ whenever $\bar{x} \in \mathfrak{g}_{\Bbbk}(i)$ for all $i \in \mathbb{Z}$ (see the proof of [37, Lemma 2.18]), within the context of the graded algebra $\operatorname{gr}(U(\mathfrak{g}_{\Bbbk}))$ under the Kazhdan filtration, we can obtain that

$$\operatorname{gr}(\bar{x}_i^p - \bar{x}_i^{[p]}) = \operatorname{gr}(\bar{x}_i)^p, \quad \operatorname{gr}(\bar{u}_j^p - \bar{u}_j^{[p]}) = \operatorname{gr}(\bar{u}_j)^p \quad (1 \leqslant i \leqslant m;\ \ 1 \leqslant j \leqslant s). \quad (3.2)$$

On the other hand, Lemma 3.4(1) implies that the vectors $\bar{X}^{(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})}\otimes 1_\chi$ with

$$(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})=(a_1,\ldots,a_{m-l};b_1,\ldots,b_{n-q};c_1,\ldots,c_s;d_1,\ldots,d_{t'})\\ \in\mathbb{Z}_+^{m-l}\times\Lambda'_{n-q}\times\mathbb{Z}_+^s\times\Lambda'_{t'}$$

form a free basis of the right $T(\mathfrak{g}_\Bbbk,e)$-module $Q_{\chi,\Bbbk}$. As $Q_{\chi,\Bbbk}$ is a Kazhdan-filtered $T(\mathfrak{g}_\Bbbk,e)$-module, straightforward induction on filtration degree based on (3.1) and (3.2) shows that $Q_{\chi,\Bbbk}$ is generated as a $Z_p(\widetilde{\mathfrak{p}}_\Bbbk)$-module by the set

$$\{\bar{X}^{(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})}\bar{\Theta}^{(\mathbf{i},\mathbf{j})}\otimes 1_\chi\mid(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{i},\mathbf{j})\in\Lambda_{m-l}\times\Lambda'_{n-q}\times\Lambda_s\times\Lambda'_{t'}\times\Lambda_l\times\Lambda'_{q+1}\}.$$

(2-ii) Let h be an arbitrary element of $U(\mathfrak{g}_\Bbbk,e)$. By the above discussion we can assume that

$$\begin{aligned}h(1_\chi)=\sum f_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{i},\mathbf{j}}&\bar{X}_1^{a_1}\cdots\bar{X}_{m-l}^{a_{m-l}}\bar{X}_{m-l+1}^{b_1}\cdots\bar{X}_{m+n-l-q}^{b_{n-q}}\\&\cdot\bar{X}_{m+n-l-q+1}^{c_1}\cdots\bar{X}_{m+n-l-q+s}^{c_s}\bar{X}_{m+n-l-q+s+1}^{d_1}\cdots\bar{X}_{m+n-l-q+s+t'}^{d_{t'}}\\&\cdot\bar{\Theta}_1^{i_1}\cdots\bar{\Theta}_l^{i_l}\bar{\Theta}_{l+1}^{j_1}\cdots\bar{\Theta}_{l+q}^{j_q}\bar{\Theta}_{l+q+1}^{j_{q+1}}(1_\chi),\end{aligned}$$

where $f_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{i},\mathbf{j}}\in Z_p(\widetilde{\mathfrak{p}}_\Bbbk)$ with $(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{i},\mathbf{j})$ in the set $\Lambda_{m-l}\times\Lambda'_{n-q}\times\Lambda_s\times\Lambda'_{t'}\times\Lambda_l\times\Lambda'_{q+1}$. For any $\eta\in\chi+(\mathfrak{m}_\Bbbk^\perp)_{\bar{0}}$, the image of $f_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{i},\mathbf{j}}$ in $U_\eta(\mathfrak{g}_\Bbbk)$ is a scalar in $\Bbbk$, which shall be denoted by $\eta(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{i},\mathbf{j})$.

Suppose $f_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{i},\mathbf{j}}\neq 0$ for a nonzero $(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})\in\Lambda_{m-l}\times\Lambda'_{n-q}\times\Lambda_s\times\Lambda'_{t'}$ and some $(\mathbf{i},\mathbf{j})\in\Lambda_l\times\Lambda'_{q+1}$. Then there exists $\eta\in\chi+(\mathfrak{m}_\Bbbk^\perp)_{\bar{0}}$ such that $\eta(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{i},\mathbf{j})\neq 0$. Let h_η be the image of $h\in U(\mathfrak{g}_\Bbbk,e)$ in $U_\eta(\mathfrak{g}_\Bbbk,e)=(\mathrm{End}_{\mathfrak{g}_\Bbbk}Q_\chi^\eta)^{\mathrm{op}}$. In [37, Remark 3.9] it is shown that there exist even elements $\theta_1,\ldots,\theta_l\in U_\eta(\mathfrak{g}_\Bbbk,e)_{\bar{0}}$ and odd elements $\theta_{l+1},\ldots,\theta_{l+q+1}\in U_\eta(\mathfrak{g}_\Bbbk,e)_{\bar{1}}$ in the same sense as in [37, Corollary 3.6], such that the monomials

$$\theta_1^{a_1}\cdots\theta_l^{a_l}\theta_{l+1}^{b_1}\cdots\theta_{l+q+1}^{b_{q+1}}$$

with $0\leqslant a_k\leqslant p-1$ for $1\leqslant k\leqslant l$ and $0\leqslant b_k\leqslant 1$ for $1\leqslant k\leqslant q+1$ form a $\Bbbk$-basis of $U_\eta(\mathfrak{g}_\Bbbk,e)$. Therefore, $h_\eta(\bar{1}_\chi)$ is a $\Bbbk$-linear combination of $\theta_1^{i_1}\cdots\theta_l^{i_l}\theta_{l+1}^{j_1}\cdots\theta_{l+q}^{j_q}\cdot\theta_{l+q+1}^{j_{q+1}}(\bar{1}_\chi)$ with

$$(i_1,\ldots,i_l;j_1,\ldots,j_q;j_{q+1})\in\Lambda_l\times\Lambda'_q\times\Lambda'_1.$$

By Lemma 3.4(2), the set

$$\{\bar{X}^{(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})}\otimes\bar{1}_\chi\mid(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})\in\Lambda_{m-l}\times\Lambda'_{n-q}\times\Lambda_s\times\Lambda'_{t'}\}$$

is a free basis of the right $U_\eta(\mathfrak{g}_\Bbbk,e)$-module Q_χ^η. Since $\eta(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{i},\mathbf{j})\neq 0$ and $\theta_1^{i_1}\cdots\theta_l^{i_l}\theta_{l+1}^{j_1}\cdots\theta_{l+q}^{j_q}\theta_{l+q+1}^{j_{q+1}}$ is the image of $\bar{\Theta}_1^{i_1}\cdots\bar{\Theta}_l^{i_l}\bar{\Theta}_{l+1}^{j_1}\cdots\bar{\Theta}_{l+q}^{j_q}\bar{\Theta}_{l+q+1}^{j_{q+1}}$ in

$U_\eta(\mathfrak{g}_\Bbbk, e)$, it is now evident that $h_\eta(\bar{1}_\chi)$ cannot be a $\Bbbk$-linear combination of $\theta_1^{i_1}\cdots\theta_l^{i_l}\theta_{l+1}^{j_1}\cdots\theta_{l+q}^{j_q}\theta_{l+q+1}^{j_{q+1}}(\bar{1}_\chi)$ with

$$(i_1,\ldots,i_l;j_1,\ldots,j_q;j_{q+1})\in\Lambda_l\times\Lambda'_q\times\Lambda'_1.$$

This contradiction shows that $f_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{i},\mathbf{j}}=0$ unless $(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})=\mathbf{0}$. As a consequence,

$$\{\bar{\Theta}_1^{i_1}\cdots\bar{\Theta}_l^{i_l}\bar{\Theta}_{l+1}^{j_1}\cdots\bar{\Theta}_{l+q}^{j_q}\bar{\Theta}_{l+q+1}^{j_{q+1}}\mid(\mathbf{i},\mathbf{j})\in\Lambda_l\times\Lambda'_{q+1}\}\tag{3.3}$$

generates $U(\mathfrak{g}_\Bbbk, e)$ as a $Z_p(\widetilde{\mathfrak{p}}_\Bbbk)$-module. Specializing at a suitable $\eta\in\chi+(\mathfrak{m}_\Bbbk^\perp)_{\bar{0}}$ and applying [37, Remark 3.9] we further deduce that the set

$$\{\bar{\Theta}_1^{i_1}\cdots\bar{\Theta}_l^{i_l}\bar{\Theta}_{l+1}^{j_1}\cdots\bar{\Theta}_{l+q}^{j_q}\bar{\Theta}_{l+q+1}^{j_{q+1}}\mid(\mathbf{i},\mathbf{j})\in\Lambda_l\times\Lambda'_{q+1}\}\tag{3.4}$$

is a free basis of the $Z_p(\widetilde{\mathfrak{p}}_\Bbbk)$-module $U(\mathfrak{g}_\Bbbk, e)$. Then $U(\mathfrak{g}_\Bbbk, e)$ is a free $\rho_\Bbbk(Z_p)$-module of rank p^l2^{q+1}. Note that the elements of (3.4) are in the $\Bbbk$-algebra $T(\mathfrak{g}_\Bbbk, e)$; then the second part of statement (2) follows. We complete the proof of statement (2).

(3) We first claim that

$$u\in T(\mathfrak{g}_\Bbbk, e)\cdot Z_p(\mathfrak{a}_\Bbbk)\text{ for each }u\in U(\mathfrak{g}_\Bbbk, e).\tag{3.5}$$

We proceed with the proof of claim (3.5) by steps, starting with some necessary preparation.

(3-i) Note that every $\mathfrak{g}_\Bbbk$-endomorphism of $Q_{\chi,\Bbbk}$ is uniquely determined by its value at 1_χ. For a nonzero $u\in U(\mathfrak{g}_\Bbbk, e)$ with highest Kazhdan degree $n(u)$, we can write

$$u(1_\chi)=\sum_{|(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})|_e\leqslant n(u)}\lambda_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}\bar{x}^{\mathbf{a}}\bar{y}^{\mathbf{b}}\bar{u}^{\mathbf{c}}\bar{v}^{\mathbf{d}}\otimes 1_\chi,$$

where $\lambda_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}\neq 0$ for at least one $(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})$ with $|(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})|_e=n(u)$. For $k\in\mathbb{Z}_+$ put

$$\Lambda^k(u):=\{(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})\in\mathbb{Z}_+^m\times\Lambda'_n\times\mathbb{Z}_+^s\times\Lambda'_t\mid\lambda_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}\neq 0\ \&\ |(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})|_e=k\},$$

and denote by $\Lambda^{\max}(u)$ the set of all $(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})\in\Lambda^{n(u)}(u)$ for which the quantity $n(u)-|\mathbf{a}|-|\mathbf{b}|-|\mathbf{c}|-|\mathbf{d}|$ assumes its maximum value. This maximum value will be denoted by $N(u)$. For each $(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})\in\Lambda^{\max}$, let $\bar{x}_i\in\mathfrak{g}_\Bbbk(k_i)_{\bar{0}}$, $\bar{y}_j\in\mathfrak{g}_\Bbbk(k'_j)_{\bar{1}}$ for $1\leqslant i\leqslant m$ and $1\leqslant j\leqslant n$ with $k_i,k'_j\in\mathbb{Z}_+$; then we have

$$\begin{aligned}|(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})|_e-|\mathbf{a}|-|\mathbf{b}|-|\mathbf{c}|-|\mathbf{d}|=&\sum_{i=1}^m(k_i+2)a_i+\sum_{i=1}^n(k'_i+2)b_i+\sum_{i=1}^s c_i+\sum_{i=1}^t d_i\\&-|\mathbf{a}|-|\mathbf{b}|-|\mathbf{c}|-|\mathbf{d}|\\\geqslant&\ 0.\end{aligned}$$

Consequently, $n(u),\ N(u)\in\mathbb{Z}_+$, and $n(u)\geqslant N(u)$.

(3-ii) The previous arguments in step (2) along with [37, Theorem 4.5(1)] show that

$$\begin{aligned}
&\Lambda^{\max}(\bar{\Theta}_i)=\{(\mathbf{e}_i,\mathbf{0},\mathbf{0},\mathbf{0})\} && \text{for } 1\leqslant i\leqslant l,\\
&\Lambda^{\max}(\rho_{\Bbbk}(\bar{x}_i^p-\bar{x}_i^{[p]}))=\{(p\mathbf{e}_i,\mathbf{0},\mathbf{0},\mathbf{0})\} && \text{for } 1\leqslant i\leqslant m,\\
&\Lambda^{\max}(\rho_{\Bbbk}(\bar{u}_j^p-\bar{u}_j^{[p]}))=\{(\mathbf{0},\mathbf{0},p\mathbf{e}_j,\mathbf{0})\} && \text{for } 1\leqslant j\leqslant s,\\
&\Lambda^{\max}(\bar{\Theta}_k)=\{(\mathbf{0},\mathbf{e}_{k-l},\mathbf{0},\mathbf{0})\} && \text{for } l+1\leqslant k\leqslant l+q,\\
&\Lambda^{\max}(\bar{\Theta}_{l+q+1})=\{(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{e}_t)\}.
\end{aligned}$$

Here $\mathbf{e}_i=(\delta_{i1},\delta_{i2},\ldots,\delta_{ij})$ is a tuple with j entries for $j\in\mathbb{Z}_+$, and δ_{ik} is the Kronecker function for $k=1,\ldots,j$.

Since $Q_{\chi,\Bbbk}$ is a Kazhdan-filtered $U(\mathfrak{g}_{\Bbbk})$-module, this implies that

$$\begin{aligned}
&\Lambda^{\max}\left(\prod_{i=1}^m\rho_{\Bbbk}\left(\bar{x}_i^p-\bar{x}_i^{[p]}\right)^{a_i}\cdot\prod_{i=1}^s\rho_{\Bbbk}\left(\bar{u}_i^p-\bar{u}_i^{[p]}\right)^{b_i}\cdot\bar{\Theta}_1^{c_1}\cdots\bar{\Theta}_l^{c_l}\bar{\Theta}_{l+1}^{d_1}\cdots\bar{\Theta}_{l+q}^{d_q}\bar{\Theta}_{l+q+1}^{d_{q+1}}\right)\\
&\quad=\left\{\left(\sum_{i=1}^m pa_i\mathbf{e}_i+\sum_{j=1}^l c_j\mathbf{e}_j,\sum_{i=1}^q d_i\mathbf{e}_i,\sum_{i=1}^s pb_i\mathbf{e}_i,d_{q+1}\mathbf{e}_t\right)\right\}
\end{aligned}$$

for all $(a_1,\ldots,a_m;b_1,\ldots,b_s;c_1,\ldots,c_l;d_1,\ldots,d_{q+1})\in\mathbb{Z}_+^m\times\mathbb{Z}_+^s\times\Lambda_l\times\Lambda'_{q+1}$.

Thanks to statement (2), $U(\mathfrak{g}_{\Bbbk},e)$ is generated as a $Z_p(\widetilde{\mathfrak{p}}_{\Bbbk})$-module by the set

$$\{\bar{\Theta}_1^{i_1}\cdots\bar{\Theta}_l^{i_l}\bar{\Theta}_{l+1}^{j_1}\cdots\bar{\Theta}_{l+q}^{j_q}\bar{\Theta}_{l+q+1}^{j_{q+1}}\mid(\mathbf{i},\mathbf{j})\in\Lambda_l\times\Lambda'_{q+1}\}.$$

It follows that for every $u\in U(\mathfrak{g}_{\Bbbk},e)$ with $(n(u),N(u))=(d,d')$ there exists a $\Bbbk$-linear combination u' of the endomorphisms

$$u(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}):=\prod_{i=1}^m\rho_{\Bbbk}(\bar{x}_i^p-\bar{x}_i^{[p]})^{a_i}\cdot\prod_{i=1}^s\rho_{\Bbbk}(\bar{u}_i^p-\bar{u}_i^{[p]})^{b_i}\cdot\bar{\Theta}_1^{c_1}\cdots\bar{\Theta}_l^{c_l}\bar{\Theta}_{l+1}^{d_1}\cdots\bar{\Theta}_{l+q}^{d_q}\bar{\Theta}_{l+q+1}^{d_{q+1}}$$

for all $(a_1,\ldots,a_m;b_1,\ldots,b_s;c_1,\ldots,c_l;d_1,\ldots,d_{q+1})\in\mathbb{Z}_+^m\times\mathbb{Z}_+^s\times\Lambda_l\times\Lambda'_{q+1}$ with $\Lambda^{\max}(u(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}))\subseteq\Lambda^{\max}(u)$ such that either $n(u-u')<d$, or $n(u-u')=d$ and $N(u-u')<d'$.

(3-iii) Let $\Omega:=\{(n_1,n_2)\in\mathbb{Z}_+^2\mid n_1\geq n_2\}$ be a totally ordered set with tuples ordered lexicographically. Due to the arguments in (3-i), $(n(u),N(u))\in\Omega$ for all $u\in U(\mathfrak{g}_{\Bbbk},e)$. Now we prove claim (3.5) by induction on $(n(u),N(u))$ in the totally ordered set Ω. The claim is clearly valid when $(n(u),N(u))=(0,0)$. Assume that $u\in T(\mathfrak{g}_{\Bbbk},e)\cdot Z_p(\mathfrak{a}_{\Bbbk})$ for all nonzero $u\in U(\mathfrak{g}_{\Bbbk},e)$ with $(n(u),N(u))\prec(d,d')$. Now let $u\in U(\mathfrak{g}_{\Bbbk},e)$ be such that $(n(u),N(u))=(d,d')$. By the preceding remark we know that there exists $u'=\sum_{(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})}\lambda_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}u(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})$ with

$\Lambda^{\max}(u(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})) \subseteq \Lambda^{\max}(u)$ for all $(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})$ with $\lambda_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}} \neq 0$ such that $(n(u-u'), N(u-u')) \prec (d,d')$. Set

$$v(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}) := u((0,\dots,0,a_{l+1},\dots,a_m),\mathbf{b},\mathbf{0},\mathbf{0}) \cdot \prod_{i=1}^{l} \bar{\Theta}_i^{pa_i} \cdot (\bar{\Theta}_1^{c_1}\cdots\bar{\Theta}_l^{c_l}\bar{\Theta}_{l+1}^{d_1}\cdots\bar{\Theta}_{l+q}^{d_q}\bar{\Theta}_{l+q+1}^{d_{q+1}}).$$

Using (3.1) it is easy to observe that $\Lambda^{\max}(u(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})) = \Lambda^{\max}(v(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}))$ and

$$(n(u(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}) - v(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})), N(u(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}) - v(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}))) \prec (n(u(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})), N(u(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}))).$$

We now put $u'' := \sum_{(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})} \lambda_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}} v(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})$, an element of $T(\mathfrak{g}_{\Bbbk},e)\cdot Z_p(\mathfrak{a}_{\Bbbk})$. Because $(n(u-u''), N(u-u'')) \prec (n(u), N(u))$, the equality $U(\mathfrak{g}_{\Bbbk},e) = T(\mathfrak{g}_{\Bbbk},e)\cdot Z_p(\mathfrak{a}_{\Bbbk})$ follows by induction on the length of (d,d') in the linearly ordered set $(\Omega,\prec)$. We complete the proof of claim (3.5).

Next we will finish the proof of statement (3). By Lemma 3.4(1) and the procedure of "modular p reduction", we know that the vectors $\bar{X}^{(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})}\otimes 1_\chi$ with

$$(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}) = (a_1,\dots,a_{m-l}; b_1,\dots,b_{n-q}; c_1,\dots,c_s; d_1,\dots,d_{t'}) \in \mathbb{Z}_+^{m-l}\times\Lambda'_{n-q}\times\mathbb{Z}_+^s\times\Lambda'_{t'}$$

form a free basis of the right $T(\mathfrak{g}_{\Bbbk},e)$-module $Q_{\chi,\Bbbk}$. Since (3.2) shows that $\bar{X}_i^p$ and $\bar{X}_i^p - \bar{X}_i^{[p]}$ have the same Kazhdan degree in $U(\mathfrak{g}_{\Bbbk})$ for $1\leqslant i\leqslant m-l$ and $m+n-l-q+1\leqslant i\leqslant m+n-l-q+s$ respectively, and $Q_{\chi,\Bbbk}$ is a Kazhdan-filtered $U(\mathfrak{g}_{\Bbbk})$-module, it follows that the vectors

$$\prod_{i=1}^{m-l}\prod_{j=m+n-l-q+1}^{m+n-l-q+s} \rho_{\Bbbk}(\bar{X}_i^p - \bar{X}_i^{[p]})^{a_i}\rho_{\Bbbk}(\bar{X}_j^p - \bar{X}_j^{[p]})^{b_j}\cdot\bar{\Theta}_1^{c_1}\cdots\bar{\Theta}_l^{c_l}\bar{\Theta}_{l+1}^{d_1}\cdots\bar{\Theta}_{l+q}^{d_q}\bar{\Theta}_{l+q+1}^{d_{q+1}}$$

are linearly independent, where $(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}) \in \mathbb{Z}_+^{m-l}\times\mathbb{Z}_+^s\times\mathbb{Z}_+^l\times\Lambda'_{q+1}$.

Combining all the above, we have an isomorphism between $\Bbbk$-algebras

$$U(\mathfrak{g}_{\Bbbk},e) \cong T(\mathfrak{g}_{\Bbbk},e)\otimes_{\Bbbk} Z_p(\mathfrak{a}_{\Bbbk}).$$

By the analysis at the beginning of step (2), we complete the proof. □

With the above theorem, we can define a "reduced" quotient algebra of $U(\mathfrak{g}_{\Bbbk},e)$, analogous to the reduced enveloping algebra of a restricted Lie (super)algebra, by

$$\widetilde{U}_\eta(\mathfrak{g}_{\Bbbk},e) := U(\mathfrak{g}_{\Bbbk},e)\otimes_{Z_p(\widetilde{\mathfrak{p}}_{\Bbbk})}\Bbbk_\eta.$$

Lemma 3.6. *For any given $\eta \in \chi + (\mathfrak{m}_{\Bbbk}^{\perp})_{\bar{0}}$, the above "reduced" quotient algebras are isomorphic to the reduced W-superalgebras, i.e., $\widetilde{U}_\eta(\mathfrak{g}_{\Bbbk}, e) \cong U_\eta(\mathfrak{g}_{\Bbbk}, e)$.*

Proof. The canonical projection $Q_{\chi,\Bbbk} \twoheadrightarrow Q_{\chi,\Bbbk}/J_\eta Q_{\chi,\Bbbk} = Q_\chi^\eta$ gives rise to an algebra homomorphism $\rho_\eta : \widetilde{U}_\eta(\mathfrak{g}_{\Bbbk}, e) \to (\mathrm{End}_{\mathfrak{g}_{\Bbbk}} Q_\chi^\eta)^{\mathrm{op}} = U_\eta(\mathfrak{g}_{\Bbbk}, e)$. As $\dim \widetilde{U}_\eta(\mathfrak{g}_{\Bbbk}, e) \leqslant p^l 2^{q'}$ by Theorem 3.5(2), [37, Remark 3.9] yields that ρ_η is an algebra isomorphism. We complete the proof. □

§3.4. Transition for minimal-dimensional representations

First notice the following important fact:

Proposition 3.7. *Assume that d_1 is odd. Then the finite W-superalgebra $U(\mathfrak{g}, e)$ over $\mathbb{C}$ cannot afford a one-dimensional representation.*

Proof. Recall that when d_1 is odd, §2.2.2 shows that there is an element $\Theta_{l+q+1} = v_{\frac{r+1}{2}} \otimes 1_\chi \in U(\mathfrak{g}, e)_{\bar{1}}$; then

$$\Theta_{l+q+1}^2(1_\chi) = v_{\frac{r+1}{2}}^2 \otimes 1_\chi = \frac{1}{2}[v_{\frac{r+1}{2}}, v_{\frac{r+1}{2}}] \otimes 1_\chi = \frac{1}{2}\chi([v_{\frac{r+1}{2}}, v_{\frac{r+1}{2}}]) \otimes 1_\chi = \frac{1}{2} \otimes 1_\chi.$$

Thus $\Theta_{l+q+1}^2 = \frac{1}{2}\mathrm{id}$.

For any $U(\mathfrak{g}, e)$-module M, let $0 \neq v \in M$ be a $\mathbb{Z}_2$-homogeneous element. We claim that $\Theta_{l+q+1}.v \neq 0$. If not, i.e., $\Theta_{l+q+1}.v = 0$, then $\Theta_{l+q+1}^2.v = 0$. However, by the preceding remark we have $\Theta_{l+q+1}^2.v = \frac{1}{2}v$, a contradiction. Therefore, $\Theta_{l+q+1}.v$ is a nonzero element in M, which obviously shares a different parity from that of v. Thus the dimension of any $U(\mathfrak{g}, e)$-module (as a vector space) is at least two, and the algebra $U(\mathfrak{g}, e)$ cannot afford a one-dimensional representation in this case. □

Remark 3.8. Recall that in §2.2.3 we obtained an algebra isomorphism

$$\begin{aligned} \phi : (\mathrm{End}_{\mathfrak{g}} Q_\chi)^{\mathrm{op}} &\cong Q_\chi^{\mathrm{ad}\,\mathfrak{m}}, \\ \Theta &\mapsto \Theta(1_\chi). \end{aligned}$$

In the paper we will often identify $U(\mathfrak{g}, e) = (\mathrm{End}_{\mathfrak{g}} Q_\chi)^{\mathrm{op}}$ with $Q_\chi^{\mathrm{ad}\,\mathfrak{m}}$ as $\mathbb{C}$-algebras, which will cause no confusion. For any $\mathbb{Z}_2$-homogeneous elements $\Theta_1, \Theta_2 \in Q_\chi^{\mathrm{ad}\,\mathfrak{m}}$, since $\phi(\Theta_1 \cdot \Theta_2) = \phi(\Theta_1)\phi(\Theta_2)$, we can identify $\Theta_1 \cdot \Theta_2$ in $U(\mathfrak{g}, e)$ with $\Theta_1(1_\chi) \cdot \Theta_2(1_\chi)$ in $Q_\chi^{\mathrm{ad}\,\mathfrak{m}}$. When d_1 is odd, the element Θ_{l+q+1} in $(\mathrm{End}_{\mathfrak{g}} Q_\chi)^{\mathrm{op}}$ can be considered as the element $v_{\frac{r+1}{2}} \otimes 1_\chi$ in $Q_\chi^{\mathrm{ad}\,\mathfrak{m}}$, and for any $\mathbb{Z}_2$-homogeneous element v in a $U(\mathfrak{g}, e)$-module M, we have $\Theta_{l+q+1}^2.v = \frac{1}{2}v$ by Proposition 3.7.

Now we are in a position to talk about the transiting role of the transition subalgebras for the minimal dimensions of modular representations of basic Lie superalgebras.

Proposition 3.9. *Keep the above notation. If $p \gg 0$ for the field $\Bbbk = \overline{\mathbb{F}}_p$, the following hold.*

(1) Assume that d_1 is even. Then the following assertions are equivalent:

(1-i) The transition subalgebra $T(\mathfrak{g}_\Bbbk, e)$ admits one-dimensional representations.

(1-ii) There exists $\eta \in \chi + (\mathfrak{m}_\Bbbk^\perp)_{\bar{0}}$ such that $U_\eta(\mathfrak{g}_\Bbbk)$ admits irreducible representations of dimension $p^{\frac{d_0}{2}} 2^{\frac{d_1}{2}}$.

(2) Assume that d_1 is odd. Then the following assertions are equivalent:

(2-i) The transition subalgebra $T(\mathfrak{g}_\Bbbk, e)$ admits two-dimensional representations.

(2-ii) There exists $\eta \in \chi + (\mathfrak{m}_\Bbbk^\perp)_{\bar{0}}$ such that $U_\eta(\mathfrak{g}_\Bbbk)$ admits irreducible representations of dimension $p^{\frac{d_0}{2}} 2^{\frac{d_1+1}{2}}$.

Proof. Let us first prove statement (2).

"(2-i) $\Rightarrow$ (2-ii)": Recall that there is a $\Bbbk$-algebra isomorphism $U(\mathfrak{g}_\Bbbk, e) \cong T(\mathfrak{g}_\Bbbk, e) \otimes_\Bbbk Z_p(\mathfrak{a}_\Bbbk)$ by Theorem 3.5(3). Thus assumption (2-i) implies that the $\Bbbk$-algebra $U(\mathfrak{g}_\Bbbk, e)$ affords a two-dimensional representation too; we denote it by ν with the representation space V.

Let $v_{\bar{0}} \in V_{\bar{0}}$ be a nonzero even vector in V. The proof of Proposition 3.7 shows that $\Theta_{l+q+1}.v_{\bar{0}} \in V_{\bar{1}}$ is a nonzero odd vector, and we denote it by $v_{\bar{1}}$. Then V is $\Bbbk$-spanned by $v_{\bar{0}}$ and $v_{\bar{1}}$ as a vector space. For any $\bar{x} \in (\mathfrak{g}_\Bbbk)_{\bar{0}}$, since $\bar{x}^p - \bar{x}^{[p]} \in Z_p(\mathfrak{g}_\Bbbk)$ is central in $U(\mathfrak{g}_\Bbbk)$, we have $[\rho(\bar{x}^p - \bar{x}^{[p]}), \Theta_{l+q+1}] = 0$. Therefore, both $\Bbbk v_{\bar{0}}$ and $\Bbbk v_{\bar{1}}$ are one-dimensional representations of $\rho_\Bbbk(Z_p)$, decided by the same function on $(\mathfrak{g}_\Bbbk)_{\bar{0}}$. By Theorem 3.5, $\rho_\Bbbk(Z_p) \cap \operatorname{Ker} \nu$ is a maximal ideal of the algebra $\rho_\Bbbk(Z_p) \cong Z_p(\widetilde{\mathfrak{p}}_\Bbbk) \cong \Bbbk[(\chi + (\mathfrak{m}_\Bbbk^\perp)_{\bar{0}})^{(1)}]$. Then there exists $\eta \in \chi + (\mathfrak{m}_\Bbbk^\perp)_{\bar{0}}$ such that $\rho_\Bbbk(\bar{x}^p - \bar{x}^{[p]} - \eta(\bar{x})^p) \in \operatorname{Ker} \nu$ for all $\bar{x} \in (\mathfrak{g}_\Bbbk)_{\bar{0}}$. Our choice of η ensures that the "reduced" quotient algebra $\widetilde{U}_\eta(\mathfrak{g}_\Bbbk, e)$ affords a two-dimensional representation. It follows from Lemma 3.6 and Theorem 2.3 that the $\Bbbk$-algebra $U_\eta(\mathfrak{g}_\Bbbk)$ has an irreducible representation of dimension $p^{\frac{d_0}{2}} 2^{\frac{d_1+1}{2}}$.

"(2-ii) $\Rightarrow$ (2-i)": Conversely, under assumption (2-ii) it follows from Theorem 2.3 that the reduced W-superalgebra $U_\eta(\mathfrak{g}_\Bbbk, e)$ admits a two-dimensional representation, and so does the $\Bbbk$-algebra $\widetilde{U}_\eta(\mathfrak{g}_\Bbbk, e)$ by Lemma 3.6, the "reduced" quotient of $U(\mathfrak{g}_\Bbbk, e)$. Then the $\Bbbk$-algebra $U(\mathfrak{g}_\Bbbk, e)$ also affords a two-dimensional representation. Since the transition subalgebra $T(\mathfrak{g}_\Bbbk, e)$ is a subalgebra of $U(\mathfrak{g}_\Bbbk, e)$ by the definition, $T(\mathfrak{g}_\Bbbk, e)$ also affords a two-dimensional representation.

The same arguments also go through for the proof of statement (1), which will be omitted here. We complete the proof. □

§4. Conjectural one-dimensional representations for finite W-superalgebras when d_1 is even

In this and the next sections we proceed to investigate small representations for the finite W-superalgebra $U(\mathfrak{g}_{\mathbb{F}}, e)$ both over the field of complex numbers $\mathbb{F} = \mathbb{C}$ and over a field $\mathbb{F} = \Bbbk$ of positive characteristic. We will find that the parity of d_1 plays a key role for the dimensions of the small representations of $U(\mathfrak{g}_{\mathbb{F}}, e)$, which is significantly different from the case of finite W-algebras. We will present a plausible conjecture for such dimensions, and demonstrate the conjecture with some examples. Based on these results, we will discuss the accessibility of the lower bounds of dimensions predicted in the super Kac–Weisfeiler property [35, Theorems 4.3 and 5.6] in §6. For simplicity we always assume that the characteristic of the field $\Bbbk = \overline{\mathbb{F}}_p$ satisfies $p \gg 0$ unless otherwise specified.

For the ordinary finite W-algebra counterpart of the above issue, there are some remarkable work and exciting progress (cf. [30]). However, when we turn to the study of the finite W-superalgebra case, the tools available are very limited. The issue of minimal dimensions for the representations of finite W-superalgebras over $\mathbb{C}$ is in a position of being reasonably estimated, but not being solved, here.

§4.1. On the minimal-dimension conjecture when d_1 is even

Recall that the parity of d_1 plays a key role in the construction of finite W-superalgebra $U(\mathfrak{g}, e)$ in [37, Theorem 4.5]. Based on the different parities of d_1, we will consider each case separately. This section is devoted to the case when d_1 is even.

Conjecture 4.1. When d_1 is even, the $\mathbb{C}$-algebra $U(\mathfrak{g}, e)$ affords a one-dimensional representation.

Assuming that Conjecture 4.1 holds, we can assume that this one-dimensional representation $\mathsf{O} := \mathbb{C}v_{\mathsf{o}}$ is generated by a nonzero vector v_{o}. Now we will investigate the consequences.

§4.2. The analogue of commutative quotients for finite W-superalgebras in the even case

Recall that the $\mathbb{C}$-algebra $U(\mathfrak{g}, e)$ is generated by $\mathbb{Z}_2$-homogeneous elements $\Theta_1, \ldots, \Theta_l \in U(\mathfrak{g}, e)_{\bar{0}}$ and $\Theta_{l+1}, \ldots, \Theta_{l+q} \in U(\mathfrak{g}, e)_{\bar{1}}$ in [37, Theorem 4.5]. Let M be any $U(\mathfrak{g}, e)$-module. For a given odd element $u \in U(\mathfrak{g}, e)_{\bar{1}}$ and a homogeneous vector $m \in M$, we know that m and $u.m$ have different parity. As O is a one-dimensional superspace, then $\Theta_i.v_\circ = 0$ for $l+1 \leqslant i \leqslant l+q$ by consideration of parity. Set $\Theta_i.v_\circ = c_i v_\circ$ for $1 \leqslant i \leqslant l$ with $c_i \in \mathbb{C}$. Recall [37, Theorem 4.7] shows that the algebra $U(\mathfrak{g}, e)$ is completely determined by the commuting relations of $\Theta_1, \ldots, \Theta_{l+q}$ (see also §2.2.2). Based on the parity of these generators, we will consider each case separately.

(i) For $1 \leqslant i < j \leqslant l$, the element $[\Theta_i, \Theta_j]$ is even since $\Theta_i, \Theta_j \in U(\mathfrak{g}, e)_{\bar{0}}$. It is immediate from $[\Theta_i, \Theta_j].v_\circ = (\Theta_i \cdot \Theta_j - \Theta_j \cdot \Theta_i).v_\circ = (c_i c_j - c_j c_i).v_\circ = 0$ that the polynomial superalgebra $F_{ij}(\Theta_1, \ldots, \Theta_{l+q})$ in $l+q$ variables acts on O trivially. When we put each polynomial $F_{ij}(\Theta_1, \ldots, \Theta_{l+q})$ as a $\mathbb{C}$-linear combination of $\Theta_1^{a_1} \cdots \Theta_l^{a_l} \Theta_{l+1}^{b_1} \cdots \Theta_{l+q}^{b_q}$, deleting all the terms for which any of the odd elements $\Theta_{l+1}, \ldots, \Theta_{l+q}$ occurs, one can obtain an (ordinary) polynomial in l variables, and denote it by $F'_{ij}(\Theta_1, \ldots, \Theta_l)$. Since $\Theta_i.v_\circ = 0$ for $l+1 \leqslant i \leqslant l+q$ by the preceding remark, then $F'_{ij}(\Theta_1, \ldots, \Theta_l).v_\circ = 0$ for $1 \leqslant i < j \leqslant l$.

(ii) For $l+1 \leqslant i \leqslant j \leqslant l+q$, the element $[\Theta_i, \Theta_j]$ is still even since Θ_i, Θ_j are both odd. As $\Theta_i.v_\circ = 0$ for $l+1 \leqslant i \leqslant l+q$, we have $[\Theta_i, \Theta_j].v_\circ = (\Theta_i \cdot \Theta_j + \Theta_j \cdot \Theta_i).v_\circ = 0$. By the same discussion as (i) we can also get polynomials $F'_{ij}(\Theta_1, \ldots, \Theta_l)$ for $l+1 \leqslant i \leqslant j \leqslant l+q$, and the one-dimensional property of O entails that $F'_{ij}(\Theta_1, \ldots, \Theta_l).v_\circ = 0$.

(iii) For $1 \leqslant i \leqslant l < j \leqslant l+q$, the element $[\Theta_i, \Theta_j]$ is odd since $\Theta_i \in U(\mathfrak{g}, e)_{\bar{0}}$ and $\Theta_j \in U(\mathfrak{g}, e)_{\bar{1}}$. As $\Theta_i.v_\circ = 0$ for $l+1 \leqslant i \leqslant l+q$, we have $[\Theta_i, \Theta_j].v_\circ = (\Theta_i \cdot \Theta_j - \Theta_j \cdot \Theta_i).v_\circ = 0$, which entails that $F_{ij}(\Theta_1, \ldots, \Theta_{l+q})$ acts on O trivially. It is immediate from $[\Theta_i, \Theta_j] = F_{ij}(\Theta_1, \ldots, \Theta_{l+q})$ that all the $F_{ij}(\Theta_1, \ldots, \Theta_{l+q})$ are odd elements in $U(\mathfrak{g}, e)$. Therefore, when we put each polynomial $F_{ij}(\Theta_1, \ldots, \Theta_{l+q})$ as a $\mathbb{C}$-linear combination of monomials $\Theta_1^{a_1} \cdots \Theta_l^{a_l} \Theta_{l+1}^{b_1} \cdots \Theta_{l+q}^{b_q}$, in each given monomial some odd element Θ_k with $l+1 \leqslant k \leqslant l+q$ will occur at least once. Since $\Theta_k.v_\circ = 0$ for $l+1 \leqslant k \leqslant l+q$, the equations $F_{ij}(\Theta_1, \ldots, \Theta_{l+q}).v_\circ = 0$ are trivial for $1 \leqslant i \leqslant l < j \leqslant l+q$. In this case no new equations are obtained.

Keep in mind all the polynomials $F'_{ij}(\Theta_1, \ldots, \Theta_l)$ from the above arguments. Actually, since the polynomials $F_{ij}(\Theta_1, \ldots, \Theta_{l+q})$ give rise to the defining relations of $U(\mathfrak{g}, e)$, from all the above one can conclude that the one-dimensional modules of $U(\mathfrak{g}, e)$ are completely determined by the polynomials $F'_{ij}(\Theta_1, \ldots, \Theta_l)$. Set $U(\mathfrak{g}, e)^{\mathrm{ab}}$ to be the quotient algebra of $U(\mathfrak{g}, e)$ by R, where R is the ideal of $U(\mathfrak{g}, e)$ generated by all the odd generators $\Theta_{l+1}, \ldots, \Theta_{l+q}$ and all commutators $[a, b]$ with $a, b \in U(\mathfrak{g}, e)$. Then $U(\mathfrak{g}, e)^{\mathrm{ab}}$ is isomorphic to the algebra $\mathbb{C}[T_1, \ldots, T_l]/\Lambda$, where $\mathbb{C}[T_1, \ldots, T_l]$ is an (ordinary) polynomial algebra in l variables, and Λ is the ideal of $\mathbb{C}[T_1, \ldots, T_l]$ generated by all the $F'_{ij}(T_1, \ldots, T_l)$ for $1 \leqslant i < j \leqslant l$ and

$l+1\leqslant i\leqslant j\leqslant l+q$. Such a commutative quotient in the Lie algebra case is studied by Premet in [29], which leads to a lot of understanding on the small representations of finite W-algebras. Now we exploit this machinery in the super case. In fact, combining all the discussions above, Hilbert's Nullstellensatz shows that the maximal spectrum $\mathscr{E}:=\operatorname{Specm}U(\mathfrak{g},e)^{\mathrm{ab}}$ parameterizes the one-dimensional representations of $U(\mathfrak{g},e)$. Denoting by $\mathscr{E}(\mathbb{C})$ the set of all common zeros of the polynomials $F'_{ij}[T_1,\ldots,T_l]$ for $1\leqslant i<j\leqslant l$ and $l+1\leqslant i\leqslant j\leqslant l+q$ in the affine space $\mathbb{A}^l_{\mathbb{C}}$, we have

Lemma 4.2. *When d_1 is even, the Zariski closed set $\mathscr{E}(\mathbb{C})$ parameterizes the one-dimensional representations of finite W-superalgebra $U(\mathfrak{g},e)$.*

Note that for $1\leqslant i<j\leqslant l$, or $l+1\leqslant i\leqslant j\leqslant l+q$, all the coefficients of the $F_{ij}[T_1,\ldots,T_l]$ are over the admissible ring A. Thus all the coefficients of the $F'_{ij}[T_1,\ldots,T_l]$ are also over A by the definition. Set ${}^pF_{ij}[T_1,\ldots,T_l]:=F'_{ij}[T_1,\ldots,T_l]\otimes_A\mathbb{k}$, the polynomials over $\mathbb{k}$, and denote by $\mathscr{E}(\mathbb{k})$ the set of all common zeros of the polynomials ${}^pF_{ij}{}'[T_1,\ldots,T_l]$ in the affine space $\mathbb{A}^l_{\mathbb{k}}$ with $1\leqslant i<j\leqslant l$, or $l+1\leqslant i\leqslant j\leqslant l+q$. Since the transition subalgebra $T(\mathfrak{g}_{\mathbb{k}},e)$ over $\mathbb{k}$ is induced from the $\mathbb{C}$-algebra $U(\mathfrak{g},e)$ by "modular p reduction", Theorem 3.2 and Lemma 4.2 show that the Zariski closed set $\mathscr{E}(\mathbb{k})$ parameterizes the one-dimensional representations of the $\mathbb{k}$-algebra $T(\mathfrak{g}_{\mathbb{k}},e)$. By the same arguments as in [29, Theorem 2.2(a)], one can verify that

Lemma 4.3. *Assume that d_1 is even. If the finite W-superalgebra $U(\mathfrak{g},e)$ over $\mathbb{C}$ affords one-dimensional representations, then the transition subalgebra $T(\mathfrak{g}_{\mathbb{k}},e)$ over $\mathbb{k}=\overline{\mathbb{F}}_p$ also admits one-dimensional representations.*

Now we can talk about the small representations for the reduced enveloping algebra of a basic Lie superalgebra. The following result is an immediate consequence of Proposition 3.9(1) and Lemma 4.3.

Lemma 4.4. *When d_1 is even, if the finite W-superalgebra $U(\mathfrak{g},e)$ over $\mathbb{C}$ affords a one-dimensional representation, then for $p\gg 0$ there exists $\eta\in\chi+(\mathfrak{m}_{\mathbb{k}}^{\perp})_{\bar{0}}$ such that the reduced enveloping algebra $U_\eta(\mathfrak{g}_{\mathbb{k}})$ admits irreducible representations of dimension $p^{\frac{d_0}{2}}2^{\frac{d_1}{2}}$.*

§4.3. Confirmation of Conjecture 4.1 for $\mathfrak{gl}(M|N)$ and $\mathfrak{sl}(M|N)$

In [35] Wang–Zhao gave some explicit description for the Dynkin gradings of basic Lie superalgebras of all types. In particular, they showed that $\dim\mathfrak{gl}(M|N)^e_{\bar{1}}$ is always an even number for any nilpotent element $e\in\mathfrak{gl}(M|N)_{\bar{0}}$ (cf. [35, §3.2]). As the dimension of $\mathfrak{gl}(M|N)_{\bar{1}}$ is also even, then $d_1=\dim\mathfrak{gl}(M|N)_{\bar{1}}-\dim\mathfrak{gl}(M|N)^e_{\bar{1}}$ is always an even number. It is notable that although Wang–Zhao's origin arguments are carried over the field $\Bbbk$ in positive characteristic (see Table 1 in §2.1.1), as shown in [35, Remark 3.2], all the discussions there are still valid for the case of complex numbers. Actually, in order to confirm Conjecture 4.1 for the complex finite W-superalgebra associated with $\mathfrak{gl}(M|N)$ and $\mathfrak{sl}(M|N)$, we will first consider the transition subalgebra $T(\mathfrak{g}_\Bbbk, e)$ in positive characteristic.

Lemma 4.5. *Let $\mathfrak{g}_\Bbbk$ be a Lie superalgebra of type $\mathfrak{gl}(M|N)$ or $\mathfrak{sl}(M|N)$ with $M, N \in \mathbb{Z}_+$. For any nilpotent element $e \in (\mathfrak{g}_\Bbbk)_{\bar{0}}$, the $\Bbbk$-algebra $T(\mathfrak{g}_\Bbbk, e)$ affords a one-dimensional representation.*

Proof. For the Lie superalgebra $\mathfrak{g}_\Bbbk = \mathfrak{gl}(M|N)$ or $\mathfrak{sl}(M|N)$, in [38] the authors showed that the reduced enveloping algebra $U_\chi(\mathfrak{g}_\Bbbk)$ admits an irreducible representation of dimension $p^{\frac{d_0}{2}} 2^{\frac{d_1}{2}}$ over $\Bbbk = \overline{\mathbb{F}}_p$ under the assumption that (i) $p > 2$ when $\mathfrak{g} = \mathfrak{gl}(M|N)$; (ii) $p > 2$ and $p \nmid (M - N)$ when $\mathfrak{g} = \mathfrak{sl}(M|N)$. Since d_1 is even in this case, we have $\underline{\dim}\, \mathfrak{m} = (\frac{d_0}{2}, \frac{d_1}{2})$ by §2.2.1. Hence Proposition 3.9(1) yields that the $\Bbbk$-algebra $T(\mathfrak{g}_\Bbbk, e)$ admits a one-dimensional representation. □

The following result is an immediate consequence of field theory.

Lemma 4.6. *Let $\mathfrak{g}$ be a basic Lie superalgebra over $\mathbb{C}$. When d_1 is even, if the transition subalgebra $T(\mathfrak{g}_\Bbbk, e)$ with $\Bbbk = \overline{\mathbb{F}}_p$ affords one-dimensional representations for infinitely many $p \in \Pi(A)$, then the finite W-superalgebra $U(\mathfrak{g}, e)$ over $\mathbb{C}$ has a one-dimensional representation.*

Proof. Recall Lemma 4.2, and what follows show that the one-dimensional representations of finite W-superalgebra $U(\mathfrak{g}, e)$ over $\mathbb{C}$ can be parameterized by the Zariski closed set $\mathscr{E}(\mathbb{C})$, and the transition subalgebra $T(\mathfrak{g}_\Bbbk, e)$ by $\mathscr{E}(\Bbbk)$. The lemma follows by the same means as in the Lie algebra case [29, Corollary 2.1], using knowledge of Galois theory, and thus will be omitted here. □

Now we are in a position to introduce the main result of this subsection.

Proposition 4.7. *Let $\mathfrak{g} = \mathfrak{gl}(M|N)$ or $\mathfrak{sl}(M|N)$ $(M, N \in \mathbb{Z}_+)$ over $\mathbb{C}$. For any nilpotent element $e \in \mathfrak{g}_{\bar{0}}$, the finite W-superalgebra $U(\mathfrak{g}, e)$ affords a one-dimensional representation.*

Proof. The proposition readily follows from Lemmas 4.5 and 4.6. □

§5. Conjectural two-dimensional representations for finite W-superalgebras when d_1 is odd

In this section we will always assume that d_1 is odd.

§5.1. On the minimal-dimension conjecture when d_1 is odd

By virtue of Proposition 3.7, we can formulate the following conjecture on the minimal-dimensional representations of $U(\mathfrak{g}, e)$ when d_1 is odd.

Conjecture 5.1. When d_1 is odd, the finite W-superalgebra $U(\mathfrak{g}, e)$ over $\mathbb{C}$ affords a two-dimensional representation of minimal dimension.

In this section, we mainly investigate such two-dimensional modules under the assumption that the above conjecture holds, and also confirm the conjecture for the regular nilpotent elements of $\mathfrak{g} = \mathfrak{osp}(1|2n)$. Take such a module as in Conjecture 5.1 and denote it by V. Recall that in [37, §4.4] we introduced the refined finite W-superalgebra $W'_\chi := Q_\chi^{\mathrm{ad}\,\mathfrak{m}'}$ over $\mathbb{C}$, a proper subalgebra of $U(\mathfrak{g}, e)$. By virtue of this algebra, we can formulate a more precise description for the two-dimensional representation V of the $\mathbb{C}$-algebra $U(\mathfrak{g}, e)$.

Proposition 5.2. *Any irreducible $U(\mathfrak{g}, e)$-module in Conjecture 5.1 is a W'_χ-module associated with an odd module automorphism.*

Proof. Let V be a two-dimensional representation of the $\mathbb{C}$-algebra $U(\mathfrak{g}, e)$ in Conjecture 5.1. Define the $\mathbb{C}$-mapping

$$\begin{aligned} \tau : V &\to V, \\ v &\mapsto \sqrt{2}\Theta_{l+q+1}.v. \end{aligned}$$

It is easy to verify that the mapping τ is odd and surjective. In fact, τ is also injective since $\tau^2(v) = 2\Theta^2_{l+q+1}.v = v$. We claim that τ is a homomorphism of the W'_χ-module V.

For any $\mathbb{Z}_2$-homogeneous elements $\Theta \in W'_\chi$ and $v \in V$, we have $\tau(\Theta.v) = \sqrt{2}\Theta_{l+q+1}.\Theta.v$ by definition. Since $\Theta_{l+q+1} \in \mathfrak{m}'$ and $\Theta \in Q_\chi^{\mathrm{ad}\,\mathfrak{m}'}$, then $[\Theta_{l+q+1}, \Theta] = 0$. Moreover, $[\Theta_{l+q+1}, \Theta] = \Theta_{l+q+1} \cdot \Theta - (-1)^{|\Theta|}\Theta \cdot \Theta_{l+q+1}$. It is immediate that $\Theta_{l+q+1} \cdot \Theta = (-1)^{|\Theta|}\Theta \cdot \Theta_{l+q+1}$; then $\Theta_{l+q+1}.\Theta.v = (-1)^{|\Theta|}\Theta.\Theta_{l+q+1}.v$, i.e., $\tau(\Theta.v) = (-1)^{|\Theta|}\Theta.\tau(v)$. We complete the proof. □

§5.2. The analogue of commutative quotients for finite W-superalgebras in the odd case

Recall that in [37, Theorem 4.7] we chose the $\mathbb{Z}_2$-homogeneous elements $\Theta_1, \ldots, \Theta_{l+q+1}$ as a set of generators for the $\mathbb{C}$-algebra $U(\mathfrak{g}, e)$ subject to the relations

$$[\Theta_i, \Theta_j] = F_{ij}(\Theta_1, \ldots, \Theta_{l+q+1}), \quad 1 \leqslant i, j \leqslant l+q+1.$$

To simplify notation, we will denote $F_{ij}(\Theta_1, \ldots, \Theta_{l+q+1})$ by F_{ij} for short.

If the $\mathbb{C}$-algebra $U(\mathfrak{g},e)$ affords a two-dimensional representation V, Proposition 5.2 yields that V is $\mathbb{C}$-spanned by an even element $v\in V_{\bar{0}}$ and the odd element $\Theta_{l+q+1}.v\in V_{\bar{1}}$. Hence we can get $4(l+q+1)$ variables $k_i^0,k_i^1,K_i^0,K_i^1\in\mathbb{C}$ such that

$$\Theta_i.v=k_i^0v+k_i^1\Theta_{l+q+1}.v,\qquad \Theta_i.\Theta_{l+q+1}.v=K_i^0v+K_i^1\Theta_{l+q+1}.v, \tag{5.1}$$

where $1\leqslant i\leqslant l+q+1$.

Similarly, there exist $4(l+q+1)^2$ variables $(F_{ij})^0_{\bar{0}}$, $(F_{ij})^0_{\bar{1}}$, $(F_{ij})^1_{\bar{0}}$, $(F_{ij})^1_{\bar{1}}\in\mathbb{C}$ with $1\leqslant i,j\leqslant l+q+1$ such that

$$\begin{aligned} F_{ij}.v&=(F_{ij})^0_{\bar{0}}v+(F_{ij})^0_{\bar{1}}\Theta_{l+q+1}.v,\\ F_{ij}.\Theta_{l+q+1}.v&=(F_{ij})^1_{\bar{0}}v+(F_{ij})^1_{\bar{1}}\Theta_{l+q+1}.v. \end{aligned}\tag{5.2}$$

It is worth noting that each polynomial F_{ij} is generated by the $\mathbb{Z}_2$-homogeneous elements $\Theta_1,\dots,\Theta_{l+q+1}$ of $U(\mathfrak{g},e)$ over $\mathbb{Q}$. After enlarging the admissible ring A possibly, one can further assume that the F_{ij} for $1\leqslant i,j\leqslant l+q+1$ are defined over A. Since the actions of F_{ij} on v and $\Theta_{l+q+1}.v$ are completely determined by the constants in (5.1), then $(F_{ij})^0_{\bar{0}}$, $(F_{ij})^0_{\bar{1}}$, $(F_{ij})^1_{\bar{0}}$, $(F_{ij})^1_{\bar{1}}$ can be written as an A-linear combination of the monomials in k_i^0, k_i^1, K_i^0, K_i^1; thus there are no new variables appearing in (5.2).

Since each Θ_i is $\mathbb{Z}_2$-homogeneous for $1\leqslant i\leqslant l+q+1$, it follows that $2(l+q+1)$ variables in (5.1) equal zero. More precisely, $k_i^1=K_i^0=0$ for $1\leqslant i\leqslant l$ (in this case all the Θ_i are even) and $k_i^0=K_i^1=0$ for $l+1\leqslant i\leqslant l+q+1$ (in this case all the Θ_i are odd). By the definition it is obvious that all the F_{ij} are $\mathbb{Z}_2$-graded and have the same parity as $[\Theta_i,\Theta_j]$ for $1\leqslant i,j\leqslant l+q+1$. Therefore, $2(l+q+1)^2$ variables equal zero in (5.2). More precisely, $(F_{ij})^0_{\bar{1}}=(F_{ij})^1_{\bar{0}}=0$ if F_{ij} is even, and $(F_{ij})^0_{\bar{0}}=(F_{ij})^1_{\bar{1}}=0$ if F_{ij} is odd.

By virtue of (5.1), a simple calculation shows that

$$\begin{aligned} \Theta_i.\Theta_j.v&=(k_i^0k_j^0+K_i^0k_j^1)v+(k_i^1k_j^0+K_i^1k_j^1)\Theta_{l+q+1}.v,\\ \Theta_i.\Theta_j.\Theta_{l+q+1}.v&=(k_i^0K_j^0+K_i^0K_j^1)v+(k_i^1K_j^0+K_i^1K_j^1)\Theta_{l+q+1}.v \end{aligned}\tag{5.3}$$

for $1\leqslant i,j\leqslant l+q+1$.

Recall that the structure of a $\mathbb{C}$-algebra $U(\mathfrak{g},e)$ is completely determined by a data of generators and defining relations (cf. [37, Theorem 4.7]). Hence, V is completely decided by the following equalities:

$$\begin{aligned} (\Theta_i.\Theta_j-(-1)^{|\Theta_i||\Theta_j|}\Theta_j\Theta_i-F_{ij}).v&=0,\\ (\Theta_i.\Theta_j-(-1)^{|\Theta_i||\Theta_j|}\Theta_j\Theta_i-F_{ij}).\Theta_{l+q+1}.v&=0 \end{aligned}\tag{5.4}$$

for $1\leqslant i,j\leqslant l+q+1$. Simple calculation based on (5.3) and (5.4) shows that the variables k_i^0, k_i^1, K_i^0, K_i^1 and $(F_{ij})^0_{\bar{0}}$, $(F_{ij})^0_{\bar{1}}$, $(F_{ij})^1_{\bar{0}}$, $(F_{ij})^1_{\bar{1}}$ for $1\leqslant i,j\leqslant l+q+1$

should satisfy the following system of linear equations:

$$
\begin{aligned}
(k_i^0k_j^0 + K_i^0k_j^1) - (-1)^{|\Theta_i||\Theta_j|}(k_i^0k_j^0 + k_i^1K_j^0) - (F_{ij})_0^0 &= 0,\\
(k_i^1k_j^0 + K_i^1k_j^1) - (-1)^{|\Theta_i||\Theta_j|}(k_i^0k_j^1 + k_i^1K_j^1) - (F_{ij})_1^0 &= 0,\\
(k_i^0K_j^0 + K_i^0K_j^1) - (-1)^{|\Theta_i||\Theta_j|}(K_i^0k_j^0 + K_i^1K_j^0) - (F_{ij})_0^1 &= 0,\\
(k_i^1K_j^0 + K_i^1K_j^1) - (-1)^{|\Theta_i||\Theta_j|}(K_i^0k_j^1 + K_i^1K_j^1) - (F_{ij})_1^1 &= 0.
\end{aligned}
$$

It is notable that $4(l+q+1)$ variables are involved in above equations (recall that the F_{ij} $(1 \leqslant i,j \leqslant l+q+1)$ can be written as an A-linear combination of the monomials in k_i^0, k_i^1, K_i^0, K_i^1 with $1 \leqslant i \leqslant l+q+1$), and the remark preceding (5.3) shows that $2(l+q+1)$ variables equal zero. Set

$$\mathbb{C}[X_1^0, X_2^0, \ldots, X_l^0, X_{l+1}^1, \ldots, X_{l+q+1}^1, Y_1^1, Y_2^1, \ldots, Y_l^1, Y_{l+1}^0, \ldots, Y_{l+q+1}^0]$$

to be an (ordinary) polynomial algebra in $2(l+q+1)$ variables over $\mathbb{C}$. For $1 \leqslant i \leqslant l+q+1$, substitute the constants k_i^0, k_i^1, K_i^0, K_i^1 for the variables X_i^0, X_i^1, Y_i^0, Y_i^1 respectively, and define the polynomials

$$
\begin{aligned}
A_{ij} &:= (X_i^0X_j^0 + X_j^1Y_i^0) - (-1)^{|\Theta_i||\Theta_j|}(X_i^0X_j^0 + X_i^1Y_j^0) - S_{ij}^0,\\
B_{ij} &:= (X_i^1X_j^0 + X_j^1Y_i^1) - (-1)^{|\Theta_i||\Theta_j|}(X_i^0X_j^1 + X_i^1Y_j^1) - S_{ij}^1,\\
C_{ij} &:= (X_i^0Y_j^0 + Y_i^0Y_j^1) - (-1)^{|\Theta_i||\Theta_j|}(X_j^0Y_i^0 + Y_i^1Y_j^0) - T_{ij}^0,\\
D_{ij} &:= (X_i^1Y_j^0 + Y_i^1Y_j^1) - (-1)^{|\Theta_i||\Theta_j|}(X_j^1Y_i^0 + Y_i^1Y_j^1) - T_{ij}^1,
\end{aligned}
$$

for $1 \leqslant i,j \leqslant l+q+1$, where S_{ij}^0, S_{ij}^1, T_{ij}^0, T_{ij}^1 stand for the polynomials over A obtained by substituting the variables k_i^0, k_i^1, K_i^0, K_i^1 into the polynomials $(F_{ij})_0^0$, $(F_{ij})_1^0$, $(F_{ij})_0^1$, $(F_{ij})_1^1$ for the indeterminate X_i^0, X_i^1, Y_i^0, Y_i^1, respectively. By the preceding remark, for the terms in A_{ij}, B_{ij}, C_{ij}, D_{ij} with $1 \leqslant i,j \leqslant l+q+1$, we have

(1) $X_i^1 = Y_i^0 = 0$ for $1 \leqslant i \leqslant l$ (in this case the Θ_i are even);
(2) $X_i^0 = Y_i^1 = 0$ for $l+1 \leqslant i \leqslant l+q+1$ (in this case the Θ_i are odd);
(3) $S_{ij}^1 = T_{ij}^0 = 0$ when $1 \leqslant i,j \leqslant l$, or $l+1 \leqslant i,j \leqslant l+q+1$ (in this case the F_{ij} are even);
(4) $S_{ij}^0 = T_{ij}^1 = 0$ when $1 \leqslant i \leqslant l < j \leqslant l+q+1$, or $1 \leqslant j \leqslant l < i \leqslant l+q+1$ (in this case the F_{ij} are odd).

It follows from (5.3), (5.4), and [37, Theorem 4.7] that there is a 1–1 correspondence between the two-dimensional representations of $\mathbb{C}$-algebra $U(\mathfrak{g}, e)$ and the points of all common zeros of the polynomials A_{ij}, B_{ij}, C_{ij}, D_{ij} in $2(l+q+1)$ variables for $1 \leqslant i,j \leqslant l+q+1$ subject to conditions (1)–(4).

Given a subfield K of $\mathbb{C}$ containing A we denote by $\mathscr{E}(K)$ the set of all common zeros of the polynomials A_{ij}, B_{ij}, C_{ij}, D_{ij} for $1 \leqslant i,j \leqslant l+q+1$ in the affine space $\mathbb{A}_K^{2(l+q+1)}$ subject to conditions (1)–(4). Clearly, the A-defined Zariski closed set $\mathscr{E}(\mathbb{C})$ parameterizes the two-dimensional representations of $\mathbb{C}$-algebra $U(\mathfrak{g},e)$. More precisely,

Lemma 5.3. *Assume that d_1 is odd. Then the two-dimensional representations of $\mathbb{C}$-algebra $U(\mathfrak{g},e)$ are uniquely determined by all common zeros of the polynomials $A_{ij}, B_{ij}, C_{ij}, D_{ij}$ $(1 \leqslant i,j \leqslant l+q+1)$ subject to conditions (1)–(4) in the affine space $\mathbb{A}_{\mathbb{C}}^{2(l+q+1)}$.*

Similarly, let $\mathscr{E}(\Bbbk)$ be the set of common zeros of the polynomials ${}^pA_{ij}$, ${}^pB_{ij}$, ${}^pC_{ij}$, ${}^pD_{ij}$ subject to the "modular p" version of the conditions (1)–(4) in the affine space $\mathbb{A}_{\Bbbk}^{2(l+q+1)}$ with $1 \leqslant i,j \leqslant l+q+1$, where ${}^pA_{ij}$, ${}^pB_{ij}$, ${}^pC_{ij}$, ${}^pD_{ij}$ stand for the polynomials over $\Bbbk$ obtained from A_{ij}, B_{ij}, C_{ij}, D_{ij} by "modular p reduction", i.e.,

$$\begin{aligned}&\Bbbk[X_1^0, X_2^0, \ldots, X_l^0, X_{l+1}^1, \ldots, X_{l+q+1}^1, Y_1^1, Y_2^1, \ldots, Y_l^1, Y_{l+1}^0, \ldots, Y_{l+q+1}^0]\\ &\quad = A[X_1^0, X_2^0, \ldots, X_l^0, X_{l+1}^1, \ldots, X_{l+q+1}^1, Y_1^1, Y_2^1, \ldots, Y_l^1, Y_{l+1}^0, \ldots, Y_{l+q+1}^0] \otimes_A \Bbbk.\end{aligned}$$

It follows from Theorem 3.2 that the Zariski closed set $\mathscr{E}(\Bbbk)$ parameterizes the two-dimensional representations of the $\Bbbk$-algebra $T(\mathfrak{g}_{\Bbbk}, e)$.

Following Premet's treatment of the Lie algebra case in [29, Theorem 2.2(a)], we have

Lemma 5.4. *When d_1 is odd, if the $\mathbb{C}$-algebra $U(\mathfrak{g},e)$ affords two-dimensional representations, then the transition subalgebra $T(\mathfrak{g}_{\Bbbk}, e)$ also admits two-dimensional representations.*

Proof. Taking Lemma 5.3 into account, we can prove the lemma by the same discussion as in Lemma 4.3. The detailed arguments are omitted here. □

As a corollary of the above lemma and Proposition 3.9(2), we have

Lemma 5.5. *When d_1 is odd, if the finite W-superalgebra $U(\mathfrak{g},e)$ over $\mathbb{C}$ affords a two-dimensional representation, then for $p \gg 0$ there exists $\eta \in \chi + (\mathfrak{m}_{\Bbbk}^{\perp})_{\bar{0}}$ associated with which the reduced enveloping algebra $U_\eta(\mathfrak{g}_{\Bbbk})$ admits irreducible representations of dimension $p^{\frac{d_0}{2}} 2^{\frac{d_1+1}{2}}$.*

§5.3. Confirmation of Conjecture 5.1 for $\mathfrak{g} = \mathfrak{osp}(1|2n)$ with regular nilpotent elements

We first need the following observation:

Lemma 5.6. *Let $\mathfrak{g}_{\Bbbk} = \mathfrak{osp}(1|2n)$ be a basic Lie superalgebra over $\Bbbk$. For any regular nilpotent element $e \in (\mathfrak{g}_{\Bbbk})_{\bar{0}}$, the transition subalgebra $T(\mathfrak{g}_{\Bbbk}, e)$ affords a two-dimensional representation.*

Proof. First note that d_1 is odd in this case (cf. [25, Corollary 2.10]). Let $\mathfrak{g}_{\Bbbk} = \mathfrak{n}_{\Bbbk}^+ \oplus \mathfrak{h}_{\Bbbk} \oplus \mathfrak{n}_{\Bbbk}^-$ denote the canonical triangular decomposition of Lie superalgebra $\mathfrak{osp}(1|2n)$. It follows from [35, Corollary 5.8] that

$$\underline{\dim}\, \mathfrak{n}_{\Bbbk}^- = \underline{\dim}\, \mathfrak{m}_{\Bbbk}' = (\dim(\mathfrak{m}_{\Bbbk})_{\bar{0}}, \dim(\mathfrak{m}_{\Bbbk})_{\bar{1}} + 1).$$

Moreover, the baby Verma module $Z_\chi(\lambda)$ of reduced enveloping algebra $U_\chi(\mathfrak{g}_{\Bbbk})$ associated with the regular p-character χ is irreducible (cf. [35, Corollary 5.8]), and has the same dimension as the vector space $U_\chi(\mathfrak{m}_{\Bbbk}')$. Theorem 2.3 shows that $Z_\chi(\lambda)^{\mathfrak{m}_{\Bbbk}}$ is a $U_\chi(\mathfrak{g}_{\Bbbk}, e)$-module, and there is an isomorphism of $U_\chi(\mathfrak{m}_{\Bbbk})$-modules $Z_\chi(\lambda) \cong U_\chi(\mathfrak{m}_{\Bbbk})^* \otimes_{\Bbbk} Z_\chi(\lambda)^{\mathfrak{m}_{\Bbbk}}$ by the proof of [35, Proposition 4.2]. Then we have

$$\dim Z_\chi(\lambda)^{\mathfrak{m}_{\Bbbk}} = \frac{\dim Z_\chi(\lambda)}{\dim U_\chi(\mathfrak{m}_{\Bbbk})} = \frac{\dim U_\chi(\mathfrak{m}_{\Bbbk}')}{\dim U_\chi(\mathfrak{m}_{\Bbbk})} = 2.$$

Therefore, the algebra $U_\chi(\mathfrak{g}_{\Bbbk}, e)$ admits a two-dimensional representation. By the same discussion as the proof of Proposition 3.9(2), one can conclude that the transition subalgebra $T(\mathfrak{g}_{\Bbbk}, e)$ also affords a two-dimensional representation. □

Recall that Lemma 5.3 shows that the two-dimensional representations of finite W-superalgebras over $\mathbb{C}$ can be parameterized by the Zariski closed set $\mathscr{E}(\mathbb{C})$ for the case when d_1 is odd. By the same consideration as in Lemma 4.6, one can also obtain

Lemma 5.7. *Let $\mathfrak{g}$ be a basic Lie superalgebra over $\mathbb{C}$. When d_1 is odd, if the transition subalgebra $T(\mathfrak{g}_{\Bbbk}, e)$ affords two-dimensional representations for infinitely many $p \in \Pi(A)$, then the finite W-superalgebra $U(\mathfrak{g}, e)$ over $\mathbb{C}$ has a two-dimensional representation.*

Now we are in a position to introduce the main result of this subsection.

Proposition 5.8. *Let $e \in \mathfrak{g}_{\bar{0}}$ be a regular nilpotent element in the Lie superalgebra $\mathfrak{g} = \mathfrak{osp}(1|2n)$ over $\mathbb{C}$; then the finite W-superalgebra $U(\mathfrak{g}, e)$ affords a two-dimensional representation.*

Proof. The proposition readily follows from Lemmas 5.6 and 5.7. □

§6. The realization of minimal-dimensional representations for reduced enveloping algebra $U_\xi(\mathfrak{g}_{\Bbbk})$

§6.1. Introduction

This section is devoted to the accessibility of dimensional lower bounds for the irreducible representations of $U_\xi(\mathfrak{g}_{\Bbbk})$ predicted by Wang–Zhao in [35, Theorem 5.6] as below:

Proposition 6.1 ([35]). *Let $\mathfrak{g}_{\Bbbk}$ be a basic Lie superalgebra over $\Bbbk = \overline{\mathbb{F}}_p$, assuming that the prime p satisfies the restriction imposed in §2.1 (Table 1). Let ξ be an arbitrary p-character in $(\mathfrak{g}_{\Bbbk})^*_{\bar{0}}$. Then the dimension of every $U_\xi(\mathfrak{g}_{\Bbbk})$-module M is divisible by $p^{\frac{d_0}{2}}2^{\lfloor\frac{d_1}{2}\rfloor}$.*

Therefore, the number $p^{\frac{d_0}{2}}2^{\lfloor\frac{d_1}{2}\rfloor}$ becomes a lower bound of dimensions for the irreducible modules of the $\Bbbk$-algebra $U_\xi(\mathfrak{g}_{\Bbbk})$. A natural question is the accessibility of this number, i.e., whether there is any irreducible module of $U_\xi(\mathfrak{g}_{\Bbbk})$ with dimension equaling such a lower bound.

For the Lie superalgebra $\mathfrak{g}_{\Bbbk}$ of type $\mathfrak{gl}(M|N)$ or $\mathfrak{sl}(M|N)$ with $M, N \in \mathbb{Z}_+$, in [38] the authors showed that every reduced enveloping algebra $U_\xi(\mathfrak{g}_{\Bbbk})$ has a "small" representation of dimension equaling the lower bound. The method applied there is to construct an appropriate parabolic subalgebra that has a one-dimensional module, then induce it to an irreducible representation of $\mathfrak{g}_{\Bbbk}$. However, this method cannot be easily exploited in the general case. Thus the general attainableness of such lower bounds of dimensions in Proposition 6.1 is an open problem.

Now we first formulate Conjecture 1.3, summarizing the previous two sections:

Conjecture 1.3. *Let $\mathfrak{g}$ be a basic Lie superalgebra over $\mathbb{C}$.*

(1) When d_1 is even, the finite W-superalgebra $U(\mathfrak{g}, e)$ affords a one-dimensional representation.

(2) When d_1 is odd, the finite W-superalgebra $U(\mathfrak{g}, e)$ affords a two-dimensional representation.

By virtue of Conjecture 1.3, we will prove that the lower bounds of dimensions in Proposition 6.1 are accessible for $p \gg 0$ in this section. In the first part, we will deal with the case for nilpotent p-character $\chi \in (\mathfrak{g}_{\Bbbk})^*_{\bar{0}}$, mainly following Premet's treatment for the Lie algebra case in [29, §2.8], with a few modifications. One can observe that the emergence of the odd part in the Lie superalgebra $\mathfrak{g}_{\Bbbk}$ makes the situation much more complicated. In the second part, we will deal with the case for arbitrary p-character $\xi \in (\mathfrak{g}_{\Bbbk})^*_{\bar{0}}$, which may be not nilpotent. A lot of precise analysis on the modular representations of basic Lie superalgebras has to be done for the second case.

§6.2. On the dimensional lower bounds for the representations of basic Lie superalgebras with nilpotent p-characters

6.2.1. Recall that in Lemmas 4.4 and 5.5 we discussed the representations of minimal dimensions for the reduced enveloping algebra $U_\eta(\mathfrak{g}_\Bbbk)$ associated with some p-character $\eta\in\chi+(\mathfrak{m}_\Bbbk^\perp)_{\bar 0}$ based on the parity of d_1. It is notable that the p-character η can be guaranteed only in $\chi+(\mathfrak{m}_\Bbbk^\perp)_{\bar 0}$, with no further information apparently contributing to $U_\chi(\mathfrak{g}_\Bbbk)$. Owing to Premet's treatment of finite W-algebras in [29, Theorem 2.2], one can translate the dimensional lower bounds for $U_\eta(\mathfrak{g}_\Bbbk)$ to those for $U_\chi(\mathfrak{g}_\Bbbk)$ (see Lemma 6.2). Taking such an approach, we will finally get at the result desired in Theorem 1.6.

Lemma 6.2. *Let $\mathfrak{g}_\Bbbk$ be a basic Lie superalgebra over $\Bbbk$. If the $\Bbbk$-algebra $U_\eta(\mathfrak{g}_\Bbbk)$ affords a representation of dimension $p^{\frac{d_0}{2}}2^{\lfloor\frac{d_1}{2}\rfloor}$ for some $\eta\in\chi+(\mathfrak{m}_\Bbbk^\perp)_{\bar 0}$, then $U_\chi(\mathfrak{g}_\Bbbk)$ also admits a representation of dimension $p^{\frac{d_0}{2}}2^{\lfloor\frac{d_1}{2}\rfloor}$.*

Proof. First note that (1) $\lfloor\frac{d_1}{2}\rfloor=\frac{d_1}{2}$ when d_1 is even; and (2) $\lfloor\frac{d_1}{2}\rfloor=\frac{d_1+1}{2}$ when d_1 is odd. Since the proof is similar for both cases, we will consider only the situation when d_1 is odd.

Let $(G_\Bbbk)_{\mathrm{ev}}$ be the reductive algebraic group associated with the even part $(\mathfrak{g}_\Bbbk)_{\bar 0}$ of the Lie superalgebra $\mathfrak{g}_\Bbbk$. For any $\xi\in(\mathfrak{g}_\Bbbk)^*_{\bar 0}$, it is well known that the construction of the $\Bbbk$-algebra $U_\xi(\mathfrak{g}_\Bbbk)$ depends only on the orbit of ξ under the coadjoint action of $(G_\Bbbk)_{\mathrm{ev}}$ up to an isomorphism. Therefore, if $\xi':=(\mathrm{Ad}^*g)(\xi)$ for some $g\in(G_\Bbbk)_{\mathrm{ev}}$, then $U_\xi(\mathfrak{g}_\Bbbk)\cong U_{\xi'}(\mathfrak{g}_\Bbbk)$ as $\Bbbk$-algebras.

Let Ξ denote the set of all $\xi\in(\mathfrak{g}_\Bbbk)^*_{\bar 0}$ for which the algebra $U_\xi(\mathfrak{g}_\Bbbk)$ contains a two-sided ideal of codimension $p^{d_0}2^{d_1+1}$. It follows from [39, Lemma 2.2] that the set Ξ is Zariski closed in $(\mathfrak{g}_\Bbbk)^*_{\bar 0}$. Moreover, the preceding remark shows that the set Ξ is stable under the coadjoint action of $(G_\Bbbk)_{\mathrm{ev}}$.

(i) We claim that $\bar t\cdot\xi\in\Xi$ for all $\bar t\in\Bbbk^\times(=\Bbbk\backslash\{0\})$.

For any $\xi\in(\mathfrak{g}_\Bbbk)^*_{\bar 0}$, we can regard $\xi\in\mathfrak{g}_\Bbbk^*$ by letting $\xi((\mathfrak{g}_\Bbbk)_{\bar 1})=0$. Now let $\xi=(\bar x,\cdot)$ for some $\bar x\in(\mathfrak{g}_\Bbbk)_{\bar 0}$. Let $\bar x=\bar s+\bar n$ be the Jordan–Chevalley decomposition of $\bar x$ in the restricted Lie algebra $(\mathfrak{g}_\Bbbk)_{\bar 0}$ and put $\xi_{\bar s}:=(\bar s,\cdot)$, $\xi_{\bar n}:=(\bar n,\cdot)$. Take a Cartan subalgebra $\mathfrak{h}_\Bbbk$ of $\mathfrak{g}_\Bbbk$ that contains $\bar s$, and let $\mathfrak{g}_\Bbbk^{\bar s}$ denote the centralizer of $\bar s$ in $\mathfrak{g}_\Bbbk$. Let Φ be a root system of $\mathfrak{g}_\Bbbk$ relative to $\mathfrak{h}_\Bbbk$. Then $\mathfrak{g}_\Bbbk^{\bar s}:=\mathfrak{l}_\Bbbk=(\mathfrak{l}_\Bbbk)_{\bar 0}+(\mathfrak{l}_\Bbbk)_{\bar 1}$ also has a root space decomposition $\mathfrak{l}_\Bbbk=\mathfrak{h}_\Bbbk\oplus\bigoplus_{\alpha\in\Phi(\mathfrak{l}_\Bbbk)}(\mathfrak{g}_\Bbbk)_\alpha$ with $\Phi(\mathfrak{l}_\Bbbk):=\{\alpha\in\Phi\mid\alpha(\bar s)=0\}$. From [35, Proposition 5.1] we know that there exists a system Δ of simple roots of $\mathfrak{g}_\Bbbk$ such that $\Delta\cap\Phi(\mathfrak{l}_\Bbbk)$ is a system of simple roots for $\Phi(\mathfrak{l}_\Bbbk)$.

In particular, $\mathfrak{l}_{\Bbbk}$ is always a direct sum of basic Lie superalgebras (note that a toral subalgebra of $\mathfrak{g}_{\Bbbk}$ may also appear in the summand). Let $\mathfrak{b}_{\Bbbk}=\mathfrak{h}_{\Bbbk}\oplus\mathfrak{n}_{\Bbbk}$ be the Borel subalgebra associated with Δ. Then we can define a parabolic subalgebra $\mathfrak{p}_{\Bbbk}=\mathfrak{l}_{\Bbbk}+\mathfrak{b}_{\Bbbk}=\mathfrak{l}_{\Bbbk}\oplus\mathfrak{u}_{\Bbbk}$, where $\mathfrak{u}_{\Bbbk}$ denotes the nilradical of $\mathfrak{p}_{\Bbbk}$. Recall [35, §5.1] shows that $\xi(\mathfrak{u}_{\Bbbk})=0$, and $\xi|_{\mathfrak{l}_{\Bbbk}}=\xi_n|_{\mathfrak{l}_{\Bbbk}}$ is nilpotent. It is notable that all subalgebras here are naturally restricted subalgebras of $\mathfrak{g}_{\Bbbk}$.

Since $\bar{x}=\bar{s}+\bar{n}$ is the Jordan–Chevalley decomposition of $\bar{x}$, it follows that $\bar{t}\bar{x}=\bar{t}\bar{s}+\bar{t}\bar{n}$ is the Jordan–Chevalley decomposition of $\bar{t}\bar{x}$ for $\bar{t}\in\Bbbk^{\times}$. Obviously, we have $\mathfrak{g}_{\Bbbk}^{\bar{t}\bar{s}}=\mathfrak{l}_{\Bbbk}$.

It follows from [35, Theorem 5.3] that every irreducible $U_\xi(\mathfrak{g}_{\Bbbk})$-module is $U_\xi(\mathfrak{u}_{\Bbbk})$-projective. Since $\mathfrak{u}_{\Bbbk}$ is nilpotent in $\mathfrak{g}_{\Bbbk}$ and $\xi|_{\mathfrak{u}_{\Bbbk}}=0$, it follows from [35, Proposition 2.6] that the $\Bbbk$-algebra $U_\xi(\mathfrak{u}_{\Bbbk})$ is local with trivial module as the unique irreducible module. Then every irreducible $U_\xi(\mathfrak{g}_{\Bbbk})$-module is $U_\xi(\mathfrak{u}_{\Bbbk})$-free, and the unique maximal ideal $N_{\mathfrak{u}_{\Bbbk}}$ of $U_\xi(\mathfrak{u}_{\Bbbk})$ is generated by the image of $\mathfrak{u}_{\Bbbk}$ in $U_\xi(\mathfrak{u}_{\Bbbk})$. We put $\bar{1}_\xi=1+N_{\mathfrak{u}_{\Bbbk}}$, the image of 1 in $\Bbbk_\xi:=U_\xi(\mathfrak{u}_{\Bbbk})/N_{\mathfrak{u}_{\Bbbk}}$. Consider the $\Bbbk$-algebra $U_\xi(\mathfrak{p}_{\Bbbk})$, which contains $U_\xi(\mathfrak{u}_{\Bbbk})$ as a subalgebra. Set $Q^0_{\mathfrak{p}_{\Bbbk}}$ to be a $U_\xi(\mathfrak{p}_{\Bbbk})$-module with the ground space $U_\xi(\mathfrak{p}_{\Bbbk})/U_\xi(\mathfrak{p}_{\Bbbk})N_{\mathfrak{u}_{\Bbbk}}$, then $Q^0_{\mathfrak{p}_{\Bbbk}}=U_\xi(\mathfrak{p}_{\Bbbk})\cdot\bar{1}_\xi$. For any $\bar{q}\in Q^0_{\mathfrak{p}_{\Bbbk}}$ there is $\bar{u}\in U_\xi(\mathfrak{p}_{\Bbbk})$ such that $\bar{q}=\bar{u}\cdot\bar{1}_\xi$. Since $[\mathfrak{p}_{\Bbbk},N_{\mathfrak{u}_{\Bbbk}}]\subseteq N_{\mathfrak{u}_{\Bbbk}}$, it follows from the Jacobi identity that $[\bar{n},\bar{u}]\in U_\xi(\mathfrak{p}_{\Bbbk})N_{\mathfrak{u}_{\Bbbk}}$ for any $\bar{n}\in N_{\mathfrak{u}_{\Bbbk}}$. Then

$$\bar{n}\cdot\bar{q}=([\bar{n},\bar{q}]+(-1)^{|\bar{n}||\bar{q}|}\bar{q}\cdot\bar{n})\cdot\bar{1}_\xi\subseteq U_\xi(\mathfrak{p}_{\Bbbk})N_{\mathfrak{u}_{\Bbbk}}$$

for any $\mathbb{Z}_2$-homogeneous elements $\bar{n}\in N_{\mathfrak{u}_{\Bbbk}}$ and $\bar{q}\in Q^0_{\mathfrak{u}_{\Bbbk}}$. Thus we have the following algebra isomorphism:

$$U_\xi(\mathfrak{p}_{\Bbbk})/U_\xi(\mathfrak{p}_{\Bbbk})N_{\mathfrak{u}_{\Bbbk}}\cong U_\xi(\mathfrak{p}_{\Bbbk}/\mathfrak{u}_{\Bbbk}).$$

Define

$$\widetilde{Q}^\xi_\xi:=U_\xi(\mathfrak{g}_{\Bbbk})/U_\xi(\mathfrak{g}_{\Bbbk})N_{\mathfrak{u}_{\Bbbk}},$$

which is clearly endowed with left $U_\xi(\mathfrak{g}_{\Bbbk})$-module structure. The algebra inclusion $U_\xi(\mathfrak{u}_{\Bbbk})\subseteq U_\xi(\mathfrak{g}_{\Bbbk})$ (resp. $U_\xi(\mathfrak{p}_{\Bbbk})\subseteq U_\xi(\mathfrak{g}_{\Bbbk})$) gives rise to the canonical embedding of spaces $\Bbbk_\xi=\Bbbk\cdot\bar{1}_\xi\hookrightarrow\widetilde{Q}^\xi_\xi$ (resp. $Q^0_{\mathfrak{p}_{\Bbbk}}\hookrightarrow\widetilde{Q}^\xi_\xi$). Those embeddings clearly satisfy the property that for any $\bar{q}\in Q^0_{\mathfrak{u}_{\Bbbk}}$ there is a unique $h_{\bar{q}}\in\mathrm{End}_{\mathfrak{g}_{\Bbbk}}\widetilde{Q}^\xi_\xi$ such that $h_{\bar{q}}(\bar{1}_\xi)=\bar{q}$. Put

$$\begin{aligned} d'_0&:=2\dim(\mathfrak{u}_{\Bbbk})_{\bar{0}}=\dim(\mathfrak{g}_{\Bbbk})_{\bar{0}}-\dim(\mathfrak{l}_{\Bbbk})_{\bar{0}},\\ d'_1&:=2\dim(\mathfrak{u}_{\Bbbk})_{\bar{1}}=\dim(\mathfrak{g}_{\Bbbk})_{\bar{1}}-\dim(\mathfrak{l}_{\Bbbk})_{\bar{1}}.\end{aligned}$$

Since every irreducible $U_\xi(\mathfrak{g}_{\Bbbk})$-module is $U_\xi(\mathfrak{u}_{\Bbbk})$-free, by the same discussion as in [35, Theorem 4.4] we can obtain a $\Bbbk$-algebra isomorphism:

$$U_\xi(\mathfrak{g}_{\Bbbk})\cong\mathrm{Mat}_{p^{\frac{d'_0}{2}}2^{\frac{d'_1}{2}}}((\mathrm{End}_{\mathfrak{g}_{\Bbbk}}\widetilde{Q}^\xi_\xi)^{\mathrm{op}}).\tag{6.1}$$

Therefore,

$$\dim(\mathrm{End}_{\mathfrak{g}_\Bbbk}\widetilde{Q}^{\xi}_{\xi})=p^{\dim(\mathfrak{g}_\Bbbk)_{\bar 0}-d_0'}2^{\dim(\mathfrak{g}_\Bbbk)_{\bar 1}-d_1'}=p^{\dim(\mathfrak{l}_\Bbbk)_{\bar 0}}2^{\dim(\mathfrak{l}_\Bbbk)_{\bar 1}}.$$

Since

$$\dim U_\xi(\mathfrak{p}_\Bbbk/\mathfrak{u}_\Bbbk)=p^{\dim(\mathfrak{p}_\Bbbk)_{\bar 0}-\dim(\mathfrak{u}_\Bbbk)_{\bar 0}}2^{\dim(\mathfrak{p}_\Bbbk)_{\bar 1}-\dim(\mathfrak{u}_\Bbbk)_{\bar 1}}=p^{\dim(\mathfrak{l}_\Bbbk)_{\bar 0}}2^{\dim(\mathfrak{l}_\Bbbk)_{\bar 1}},$$

it follows that $\mathrm{End}_{\mathfrak{g}_\Bbbk}\widetilde{Q}^{\xi}_{\xi}=\{h_{\bar q}\mid \bar q\in Q^0_{\mathfrak{u}_\Bbbk}\}$. Define the mapping

$$\begin{aligned}\tau:\mathrm{End}_{\mathfrak{g}_\Bbbk}\widetilde{Q}^{\xi}_{\xi}&\to U_\xi(\mathfrak{p}_\Bbbk/\mathfrak{u}_\Bbbk)^{\mathrm{op}},\\ \theta&\mapsto\theta(\bar 1_\xi).\end{aligned}$$

It is obvious that τ is a homomorphism of $\Bbbk$-algebras. As both $\Bbbk$-algebras have the same dimension (as vector spaces), one can deduce that τ is an isomorphism. Taking the opposite algebras for both sides, we have an algebra isomorphism

$$(\mathrm{End}_{\mathfrak{g}_\Bbbk}\widetilde{Q}^{\xi}_{\xi})^{\mathrm{op}}\cong U_\xi(\mathfrak{p}_\Bbbk/\mathfrak{u}_\Bbbk)\cong U_\xi(\mathfrak{l}_\Bbbk). \tag{6.2}$$

For the p-character $\bar t\xi$, repeating the same arguments as above for ξ, we can obtain that

$$(\mathrm{End}_{\mathfrak{g}_\Bbbk}\widetilde{Q}^{\bar t\xi}_{\bar t\xi})^{\mathrm{op}}\cong U_{\bar t\xi}(\mathfrak{l}_\Bbbk). \tag{6.3}$$

And also we have an algebra isomorphism

$$U_{\bar t\xi}(\mathfrak{g}_\Bbbk)\cong \mathrm{Mat}_{p^{\frac{d_0'}{2}}2^{\frac{d_1'}{2}}}((\mathrm{End}_{\mathfrak{g}_\Bbbk}\widetilde{Q}^{\bar t\xi}_{\bar t\xi})^{\mathrm{op}}). \tag{6.4}$$

Recall that $\mathfrak{l}_\Bbbk$ is a direct sum of basic Lie superalgebras. Set $\mathfrak{l}_\Bbbk=(\mathfrak{g}_\Bbbk)_1\oplus\cdots\oplus(\mathfrak{g}_\Bbbk)_r\oplus\mathfrak{t}'_\Bbbk$, where $(\mathfrak{g}_\Bbbk)_i$ is a basic Lie superalgebra for $1\leqslant i\leqslant r$, and $\mathfrak{t}'_\Bbbk$ is a toral subalgebra of $\mathfrak{g}_\Bbbk$. For each $1\leqslant i\leqslant r$, let $(G_\Bbbk)_i$ denote the algebraic supergroup associated with $(\mathfrak{g}_\Bbbk)_i$. It is well known that the even part of $(G_\Bbbk)_i$ $(1\leqslant i\leqslant r)$ is a reductive algebraic group, and we denote it by $((G_\Bbbk)_i)_{\mathrm{ev}}$. Since $\xi|_{\mathfrak{l}_\Bbbk}=\xi_n|_{\mathfrak{l}_\Bbbk}$ is nilpotent, it follows from [11, Lemma 2.10] that $\Bbbk^\times\cdot\xi|_{(\mathfrak{g}_\Bbbk)_i}\subseteq(\mathrm{Ad}^*((G_\Bbbk)_i)_{\mathrm{ev}})\xi|_{(\mathfrak{g}_\Bbbk)_i}$. For each $1\leqslant i\leqslant r$, since the reduced enveloping algebra $U_{\xi|_{(\mathfrak{g}_\Bbbk)_i}}((\mathfrak{g}_\Bbbk)_i)$ depends only on the orbit of $\xi|_{(\mathfrak{g}_\Bbbk)_i}$ under the coadjoint action of $((G_\Bbbk)_i)_{\mathrm{ev}}$, then $U_{\xi|_{(\mathfrak{g}_\Bbbk)_i}}((\mathfrak{g}_\Bbbk)_i)\cong U_{\bar t\xi|_{(\mathfrak{g}_\Bbbk)_i}}((\mathfrak{g}_\Bbbk)_i)$ as $\Bbbk$-algebras. With i being arbitrary, we have $\bigotimes_{i=1}^r U_{\xi|_{(\mathfrak{g}_\Bbbk)_i}}((\mathfrak{g}_\Bbbk)_i)\cong\bigotimes_{i=1}^r U_{\bar t\xi|_{(\mathfrak{g}_\Bbbk)_i}}((\mathfrak{g}_\Bbbk)_i)$. As $\mathfrak{t}'_\Bbbk$ is a toral subalgebra of $\mathfrak{g}_\Bbbk$, the reduced enveloping algebra $U_\psi(\mathfrak{t}'_\Bbbk)$ is commutative and semisimple for every $\psi\in(\mathfrak{t}'_\Bbbk)^*$.(Indeed, $\mathfrak{t}'_\Bbbk$ has a $\Bbbk$-basis $t_1,\ldots,t_d$ with $t_i^{[p]}=t_i$ for $1\leqslant i\leqslant d$. Therefore, $U_\psi(\mathfrak{t}'_\Bbbk)\cong A_1\otimes\cdots\otimes A_d$ where $A_i\cong\Bbbk[X]/(X^p-X-\psi(t_i)^p)$ is a p-dimensional commutative semisimple $\Bbbk$-algebra.) From this it is immediate that $U_{\xi|_{\mathfrak{t}'_\Bbbk}}(\mathfrak{t}'_\Bbbk)\cong U_{\bar t\xi|_{\mathfrak{t}'_\Bbbk}}(\mathfrak{t}'_\Bbbk)$ as $\Bbbk$-algebras. Since $\mathfrak{l}_\Bbbk=\bigoplus_{i=1}^r(\mathfrak{g}_\Bbbk)_i\oplus\mathfrak{t}'_\Bbbk$, we have $U_\xi(\bigoplus_{i=1}^r(\mathfrak{g}_\Bbbk)_i\oplus\mathfrak{t}'_\Bbbk)\cong\bigotimes_{i=1}^r U_{\xi|_{(\mathfrak{g}_\Bbbk)_i}}((\mathfrak{g}_\Bbbk)_i)\otimes U_{\xi|_{\mathfrak{t}'_\Bbbk}}(\mathfrak{t}'_\Bbbk)$ and $U_{\bar t\xi}(\bigoplus_{i=1}^r(\mathfrak{g}_\Bbbk)_i\oplus\mathfrak{t}'_\Bbbk)\cong$

$\bigotimes_{i=1}^r U_{\bar{t}\xi|_{(\mathfrak{g}_{\Bbbk})_i}}((\mathfrak{g}_{\Bbbk})_i)\otimes U_{\bar{t}\xi|_{\mathfrak{t}'_{\Bbbk}}}(\mathfrak{t}'_{\Bbbk})$. Therefore, $U_\xi(\mathfrak{l}_{\Bbbk})\cong U_{\bar{t}\xi}(\mathfrak{l}_{\Bbbk})$ as $\Bbbk$-algebras. It follows from (6.1)–(6.4) that there is a $\Bbbk$-algebras isomorphism

$$U_\xi(\mathfrak{g}_{\Bbbk})\cong U_{\bar{t}\xi}(\mathfrak{g}_{\Bbbk}) \tag{6.5}$$

for all $\bar{t}\in\Bbbk^\times$. Now claim (i) is an immediate consequence of (6.5), i.e., if $\xi\in\Xi$, then $\bar{t}\cdot\xi\in\Xi$ for all $\bar{t}\in\Bbbk^\times$. Moreover, combining with the arguments prior to paragraph (i), we know that the affine variety Ξ is conical.

(ii) We claim that $\chi\in\Xi$.

Recall that the assumptions of the lemma shows that $U_\eta(\mathfrak{g}_{\Bbbk})$ has an irreducible module of dimension $p^{\frac{d_0}{2}}2^{\frac{d_1+1}{2}}$, thus $\eta\in\Xi$. As $\eta\in\chi+(\mathfrak{m}_{\Bbbk}^{\perp})_{\bar{0}}$, we can write $\eta=(e+\bar{y},\cdot)$ for some $\bar{y}=\sum_{i\leqslant 1}\bar{y}_i$ with $\bar{y}_i\in\mathfrak{g}_{\Bbbk}(i)_{\bar{0}}$, $i\leqslant 1$. There is a cocharacter $\lambda:\Bbbk^\times\longrightarrow(G_{\Bbbk})_{\mathrm{ev}}$ such that $(\mathrm{Ad}\lambda(\bar{t}))\bar{x}=\bar{t}^j\bar{x}$ for all $\bar{x}\in\mathfrak{g}_{\Bbbk}(j)$ $(j\in\mathbb{Z})$ and $\bar{t}\in\Bbbk^\times$. For $i\leqslant 1$, set $\eta_i=(\bar{y}_i,\cdot)$; then $\eta=\chi+\sum_{i\leqslant 1}\eta_i$. For any even element $\bar{y}'\in\mathfrak{g}_{\Bbbk}(j)_{\bar{0}}$, by the coadjoint action of $\lambda(\bar{t})$ on $(\mathfrak{g}_{\Bbbk})_{\bar{0}}^*$ one has

$$\begin{aligned}(\mathrm{Ad}^*(\lambda(\bar{t}))(\eta))(\bar{y}')&=\eta(\mathrm{Ad}\lambda(\bar{t})^{-1}\bar{y}')\\&=\bar{t}^{-j}\left(e+\sum_{i\leqslant 1}\bar{y}_i,\bar{y}'\right)\\&=\begin{cases}\bar{t}^2(e,\bar{y}') & (j=-2),\\ \bar{t}^{-j}\sum_{i\leqslant 1}\delta_{i,-j}(\bar{y}_i,\bar{y}') & (j\neq -2).\end{cases}\end{aligned}$$

Note that $(\bar{t}^2\chi+\sum_{i\leqslant 1}\bar{t}^i\eta_i)(\bar{y}')=\bar{t}^2(e,\bar{y}')+\sum_{i\leqslant 1}\bar{t}^i(\bar{y}_i,\bar{y}')$. Then we have $(\mathrm{Ad}^*\lambda(\bar{t}))\eta=\bar{t}^2\chi+\sum_{i\leqslant 1}\bar{t}^i\eta_i$, and $(\mathrm{Ad}^*\lambda(\bar{t}))^{-1}\eta=\bar{t}^{-2}\chi+\sum_{i\leqslant 1}\bar{t}^{-i}\eta_i$. As Ξ is conical and $\mathrm{Ad}^*(G_{\Bbbk})_{\mathrm{ev}}$-invariant by step (i), this implies

$$\bar{t}^2\cdot(\mathrm{Ad}^*\lambda(\bar{t}))^{-1}\eta=\chi+\sum_{i\leqslant 1}\bar{t}^{2-i}(\bar{y}_i,\bar{y}')\in\Xi$$

for all $\bar{t}\in\Bbbk^\times$. Since Ξ is Zariski closed, this yields $\chi\in\Xi$. Then claim (ii) is proved.

From all above we know that $U_\chi(\mathfrak{g}_{\Bbbk})$ admits a two-sided ideal I of codimension $p^{d_0}2^{d_1+1}$, and all irreducible modules of the factor algebra $U_\chi(\mathfrak{g}_{\Bbbk})/I$ have dimensions $\leqslant p^{\frac{d_0}{2}}2^{\frac{d_1+1}{2}}$. Combining with Proposition 6.1, one can conclude that the $\Bbbk$-algebra $U_\chi(\mathfrak{g}_{\Bbbk})$ really has an irreducible module of dimension $p^{\frac{d_0}{2}}2^{\frac{d_1+1}{2}}$. We complete the proof. □

6.2.2. Proof of Theorem 1.6. Let $\mathfrak{g}$ be a basic Lie superalgebra over $\mathbb{C}$ and $\mathfrak{g}_{\Bbbk}$ be the corresponding Lie superalgebra over positive characteristic field $\Bbbk$. Let

$\chi \in (\mathfrak{g}_{\Bbbk})^*_{\bar{0}}$ be a nilpotent p-character of $\mathfrak{g}_{\Bbbk}$ such that $\chi(\bar{y}) = (e, \bar{y})$ for any $\bar{y} \in \mathfrak{g}_{\Bbbk}$. Under the assumption that the finite W-superalgebra $U(\mathfrak{g}, e)$ over $\mathbb{C}$ affords a one-dimensional (resp. two-dimensional) representation when d_1 is even (resp. odd), we want to prove that the reduced enveloping algebra $U_\chi(\mathfrak{g}_{\Bbbk})$ with nilpotent p-character χ possesses an irreducible module whose dimension is exactly the lower bound predicted by the super Kac–Weisfeiler property in Proposition 6.1. For the case of d_1 being even, the conclusion follows from Lemma 4.4 and Lemma 6.2. The odd case can be done by Lemma 5.5 and Lemma 6.2. Then Theorem 1.6 follows. We complete the proof.

As a corollary of Theorem 1.6, we have the following consequence on the "small representations" of reduced W-superalgebra $U_\chi(\mathfrak{g}_{\Bbbk}, e)$.

Corollary 6.3. *Let $\mathfrak{g}$ be a basic Lie superalgebra. When d_1 is even (resp. odd), if the finite W-superalgebra $U(\mathfrak{g}, e)$ over $\mathbb{C}$ affords a one-dimensional (resp. two-dimensional) representation, then for $p \gg 0$, the reduced W-superalgebra $U_\chi(\mathfrak{g}_{\Bbbk}, e)$ over $\Bbbk = \overline{\mathbb{F}}_p$ also admits a one-dimensional (resp. two-dimensional) representation.*

Proof. For any $U_\chi(\mathfrak{g}_{\Bbbk})$-module M, it follows from Theorem 2.3 that $M^{\mathfrak{m}_{\Bbbk}}$ is a $U_\chi(\mathfrak{g}_{\Bbbk}, e)$-module. For the case $\eta = \chi$, (2.5) shows that

$$\dim M^{\mathfrak{m}_{\Bbbk}} = \frac{\dim M}{\dim U_\chi(\mathfrak{m}_{\Bbbk})} = \frac{\dim M}{p^{\frac{d_0}{2}} 2^{\lceil \frac{d_1}{2} \rceil}}.$$

Then the desired result follows from Theorem 1.6. □

§6.3. The case of a direct sum of basic Lie superalgebras with nilpotent p-characters

In this section we will consider the lower bounds of the super Kac–Weisfeiler property for a direct sum of basic Lie superalgebras with nilpotent p-characters.

6.3.1. First we make digression to recall some known facts about finite-dimensional superalgebras [14, §12].

Let $\mathbb{F}$ be an algebraically closed field. Now we will recall some basics on simple superalgebras over $\mathbb{F}$ (cf. [14, §12.1]). Let V be a superspace with $\underline{\dim}\, V = (m, n)$; then $\mathcal{M}(V) := \mathrm{End}_{\mathbb{F}}(V)$ is a superalgebra with $\underline{\dim}\, \mathcal{M}(V) = (m^2 + n^2, 2mn)$. The algebra $\mathcal{M}(V)$ is defined uniquely up to an isomorphism by the superdimension (m, n) of V. So we can speak of the superalgebra $\mathcal{M}_{m,n}$. We have an isomorphism of superalgebras

$$\mathcal{M}_{m,n} \otimes \mathcal{M}_{k,l} \cong \mathcal{M}_{mk+nl, ml+nk}. \tag{6.6}$$

Moreover, [14, Example 12.1.1] shows that $\mathcal{M}_{m,n}$ is a simple superalgebra.

Let V be a superspace with $\underline{\dim}\, V = (n, n)$ and J be a degree $\bar{1}$ involution in $\mathrm{End}_{\mathbb{F}}(V)$. Consider the superalgebra $Q(V, J) := \{f \in \mathrm{End}_{\mathbb{F}}(V) \mid fJ = (-1)^{|f|} Jf\}$. Note that all degree $\bar{1}$ involutions in $\mathrm{End}_{\mathbb{F}}(V)$ are conjugate to each other by an invertible element in $\mathrm{End}_{\mathbb{F}}(V)_{\bar{0}}$. Hence another choice of J will yield an isomorphism superalgebra. So we can speak of the superalgebra $Q(V)$, defined up to an isomorphism. Pick a basis $\{v_1, \ldots, v_n\}$ of $V_{\bar{0}}$, and set $v'_i = J(v_i)$ for $1 \leqslant i \leqslant n$. Then $\{v'_1, \ldots, v'_n\}$ is a basis of $V_{\bar{1}}$. With respect to the basis $\{v_1, \ldots, v_n, v'_1, \ldots, v'_n\}$, the elements of $Q(V, J)$ have matrices of the form

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \tag{6.7}$$

where A and B are arbitrary $n \times n$ matrices, with $B = 0$ for even endomorphisms and $A = 0$ for odd ones. In particular, $\underline{\dim}\, Q(V) = (n^2, n^2)$. The superalgebra $Q(V, J)$ can be identified with the superalgebra Q_n of all matrices of the form (6.7). Moreover, [14, (12.6), (12.7)] show that

$$\mathcal{M}_{m,n} \otimes Q_k \cong Q_{(m+n)k} \tag{6.8}$$

and

$$Q_m \otimes Q_n \cong \mathcal{M}_{mn,mn} \tag{6.9}$$

as $\mathbb{F}$-algebras. Moreover, [14, Example 12.1.2] shows that Q_n is a simple superalgebra.

We say that an irreducible module of a finite-dimensional superalgebra is of type M if its endomorphism ring is one-dimensional and it is of type Q if its endomorphism ring is two-dimensional. Given a finite-dimensional superalgebra A over $\mathbb{F}$, define the parity change functor on A-*mod* (the A-module category)

$$\Upsilon : A\text{-}mod \longrightarrow A\text{-}mod.$$

For an object V, $\Upsilon(V)$ is the same underlying vector space but with the opposite $\mathbb{Z}_2$-grading. The new action of a $\mathbb{Z}_2$-homogeneous element $a \in A$ on $v \in V$ is defined in terms of the old action by $a \cdot v := (-1)^{|a|} av$.

Given left modules V and W over $\mathbb{F}$-superalgebras A and B respectively, the (outer) tensor product $V \boxtimes W$ is the space $V \otimes W$ considered as an $A \otimes B$-module via

$$(a \otimes b)(v \otimes w) = (-1)^{|b||v|} av \otimes bw \quad (a \in A,\, b \in B,\, v \in V,\, w \in W).$$

For the irreducible representations of the $\mathbb{F}$-algebra $A \otimes B$, the following result was obtained by Kleshchev in [14, Lemma 12.2.13]:

Lemma 6.4 ([14]). *Let V be an irreducible A-module and W be an irreducible B-module:*

(i) If both V and W are of type M, then $V \boxtimes W$ is an irreducible $A \otimes B$-module of type M.

(ii) If one of V and W is of type M and the other is of type Q, then $V \boxtimes W$ is an irreducible $A \otimes B$-module of type Q.

(iii) If both V and W are of type Q, then $V \boxtimes W \cong (V \circledast W) \oplus \Upsilon(V \circledast W)$ for an irreducible $A \otimes B$-module $V \circledast W$ of type M.

Moreover, all irreducible $A \otimes B$-modules arise as constituents of $V \boxtimes W$ for some choice of irreducibles V, W.

6.3.2. Now we return to the representations of reduced enveloping algebras for a direct sum of basic Lie superalgebras with nilpotent p-characters. In [35, Remark 4.6], Wang–Zhao showed that the statement in Proposition 6.1 still holds for the case when $\mathfrak{l}_{\Bbbk}$ is a direct sum of basic Lie superalgebras with nilpotent p-characters.

In fact, their result can be somewhat strengthened. Let $\mathfrak{l}_{\Bbbk} = \bigoplus_{i=1}^{r}(\mathfrak{l}_{\Bbbk})_i$ be a direct sum of basic Lie superalgebras over $\Bbbk = \overline{\mathbb{F}}_p$, where $(\mathfrak{l}_{\Bbbk})_i$ is a basic Lie superalgebra for each $1 \leqslant i \leqslant r$, and the characteristic p of the field $\Bbbk$ satisfies the restriction imposed in §2.1 (Table 1). Let $\chi = \chi_1 + \cdots + \chi_r$ be the decomposition of nilpotent p-character χ in $\mathfrak{l}_{\Bbbk}^*$ with $\chi_i \in (\mathfrak{l}_{\Bbbk})_i^*$ (each χ_i can be viewed in $\mathfrak{l}_{\Bbbk}^*$ by letting $\chi_i(\bar{y}) = 0$ for all $\bar{y} \in \bigoplus_{j\neq i}(\mathfrak{l}_{\Bbbk})_j$) for $1 \leqslant i \leqslant r$. Set $\bar{e} = \bar{e}_1 + \cdots + \bar{e}_r$ to be the corresponding decomposition of $\bar{e} \in (\mathfrak{l}_{\Bbbk})_{\bar{0}}$ with respect to the nondegenerate bilinear form $(\cdot,\cdot)$ on $\mathfrak{l}_{\Bbbk}$ such that $\chi_i(\cdot) = (\bar{e}_i, \cdot)$ for $1 \leqslant i \leqslant r$. Define

$$\begin{aligned} d_0' &:= \dim(\mathfrak{l}_{\Bbbk})_{\bar{0}} - \dim(\mathfrak{l}_{\Bbbk}^{\bar{e}})_{\bar{0}}, & d_1' &:= \dim(\mathfrak{l}_{\Bbbk})_{\bar{1}} - \dim(\mathfrak{l}_{\Bbbk}^{\bar{e}})_{\bar{1}}, \\ (d_0)_i &:= \dim((\mathfrak{l}_{\Bbbk})_i)_{\bar{0}} - \dim((\mathfrak{l}_{\Bbbk})_i^{\bar{e}_i})_{\bar{0}}, & (d_1)_i &:= \dim((\mathfrak{l}_{\Bbbk})_i)_{\bar{1}} - \dim((\mathfrak{l}_{\Bbbk})_i^{\bar{e}_i})_{\bar{1}}, \end{aligned} \tag{6.10}$$

where $\mathfrak{l}_{\Bbbk}^{\bar{e}}$ denotes the centralizer of $\bar{e}$ in $\mathfrak{l}_{\Bbbk}$, and $(\mathfrak{l}_{\Bbbk})_i^{\bar{e}_i}$ is the centralizer of $\bar{e}_i$ in $(\mathfrak{l}_{\Bbbk})_i$ for $i \in \{1,\ldots,r\}$. It is obvious that $d_0' = \sum_{i=1}^{r}(d_0)_i$ and $d_1' = \sum_{i=1}^{r}(d_1)_i$. Rearrange the summands of $\mathfrak{l}_{\Bbbk} = \bigoplus_{i=1}^{r}(\mathfrak{l}_{\Bbbk})_i$ such that each $(d_1)_i$ is odd for $1 \leqslant i \leqslant l$ (if it occurs) and each $(d_1)_i$ is even for $l+1 \leqslant i \leqslant r$ (if it occurs). In particular, d_1' and l have the same parity.

Note that all arguments in the preceding sections remain valid for a direct sum of basic Lie superalgebras. Let $\mathfrak{m}_{\Bbbk}$ and $\mathfrak{m}_{\Bbbk}'$ be the subalgebras of $\mathfrak{l}_{\Bbbk}$ as defined in §2.3.2. Write $\mathfrak{m}_{\Bbbk} = \bigoplus_{i=1}^{r}(\mathfrak{m}_{\Bbbk})_i$ and $\mathfrak{m}_{\Bbbk}' = \bigoplus_{i=1}^{r}(\mathfrak{m}_{\Bbbk}')_i$ as the decomposition of $\mathfrak{m}_{\Bbbk}$ and $\mathfrak{m}_{\Bbbk}'$ in $\mathfrak{l}_{\Bbbk}$ respectively, where $(\mathfrak{m}_{\Bbbk})_i, (\mathfrak{m}_{\Bbbk}')_i \in (\mathfrak{l}_{\Bbbk})_i$ for $1 \leqslant i \leqslant r$. As $\mathfrak{m}_{\Bbbk}$ is p-nilpotent and the linear function χ vanishes on the p-closure of $[\mathfrak{m}_{\Bbbk},\mathfrak{m}_{\Bbbk}]$, it follows from [35, Proposition 2.6] that $U_\chi(\mathfrak{m}_{\Bbbk})$ has a unique irreducible module and $U_\chi(\mathfrak{m}_{\Bbbk})/N_{\mathfrak{m}_{\Bbbk}}\cong\Bbbk$, where $N_{\mathfrak{m}_{\Bbbk}}$ is the Jacobson radical of $U_\chi(\mathfrak{m}_{\Bbbk})$ generated by all the elements $\bar{x}-\chi(\bar{x})$ with $\bar{x}\in\mathfrak{m}_{\Bbbk}$.

Recall that [35, Proposition 4.1] shows that every $\Bbbk$-algebra $U_{\chi_i}((\mathfrak{m}_{\Bbbk})_i)$ ($1\leqslant i\leqslant r$) has a unique irreducible module (there is a minor error in [35, §4.1] since this irreducible module is not necessarily a trivial module) which is one-dimensional and of type M. Let $N_{(\mathfrak{m}_{\Bbbk})_i}$ denote the Jacobson radical of $U_{\chi_i}((\mathfrak{m}_{\Bbbk})_i)$ for $1\leqslant i\leqslant r$ (which is the ideal of $U_{\chi_i}((\mathfrak{m}_{\Bbbk})_i)$ generated by all the elements $\bar{x}-\chi(\bar{x})$ with $\bar{x}\in(\mathfrak{m}_{\Bbbk})_i$). Then $U_{\chi_i}((\mathfrak{m}_{\Bbbk})_i)/N_{(\mathfrak{m}_{\Bbbk})_i}\cong\Bbbk$ for $1\leqslant i\leqslant r$.

For the case when $(d_1)_i$ is odd (i.e., $1\leqslant i\leqslant l$), the $\Bbbk$-algebra $U_{\chi_i}((\mathfrak{m}'_{\Bbbk})_i)$ also has a unique irreducible module; it is isomorphic to $V_i=U_{\chi_i}((\mathfrak{m}'_{\Bbbk})_i)\otimes_{U_{\chi_i}((\mathfrak{m}_{\Bbbk})_i)}\bar{1}_{\chi_i}$, which is two-dimensional and of type Q. Let $N_{(\mathfrak{m}'_{\Bbbk})_i}$ denote the Jacobson radical of $U_{\chi_i}((\mathfrak{m}'_{\Bbbk})_i)$ (which is the ideal of $U_{\chi_i}((\mathfrak{m}'_{\Bbbk})_i)$ generated by all the elements $\bar{x}-\chi(\bar{x})$ with $\bar{x}\in(\mathfrak{m}_{\Bbbk})_i$). Then $U_{\chi_i}((\mathfrak{m}'_{\Bbbk})_i)/N_{(\mathfrak{m}'_{\Bbbk})_i}$ is isomorphic to the simple superalgebra Q_1 for $1\leqslant i\leqslant l$. For the case when $(d_1)_i$ is even (i.e., $l+1\leqslant i\leqslant r$), one can also define $N_{(\mathfrak{m}'_{\Bbbk})_i}$ as the Jacobson radical of $U_{\chi_i}((\mathfrak{m}'_{\Bbbk})_i)$. Moreover, we have $U_{\chi_i}((\mathfrak{m}'_{\Bbbk})_i)=U_{\chi_i}((\mathfrak{m}_{\Bbbk})_i)$ and $N_{(\mathfrak{m}'_{\Bbbk})_i}=N_{(\mathfrak{m}_{\Bbbk})_i}$ for $(\mathfrak{m}'_{\Bbbk})_i=(\mathfrak{m}_{\Bbbk})_i$ by construction; thus $U_{\chi_i}((\mathfrak{m}'_{\Bbbk})_i)/N_{(\mathfrak{m}'_{\Bbbk})_i}\cong\Bbbk$.

Since $\mathfrak{l}_{\Bbbk}=\bigoplus_{i=1}^r(\mathfrak{l}_{\Bbbk})_i$, it is easy to verify that

$$U_\chi(\mathfrak{m}_{\Bbbk})\cong\bigotimes_{i=1}^r U_{\chi_i}((\mathfrak{m}_{\Bbbk})_i),\quad U_\chi(\mathfrak{m}'_{\Bbbk})\cong\bigotimes_{i=1}^r U_{\chi_i}((\mathfrak{m}'_{\Bbbk})_i)\tag{6.11}$$

as $\Bbbk$-algebras. For a $U_\chi(\mathfrak{l}_{\Bbbk})$-module M set

$$M^{\mathfrak{m}_{\Bbbk}}=\{v\in M\mid(\bar{x}-\chi(\bar{x})).v=0\text{ for all }\bar{x}\in\mathfrak{m}_{\Bbbk}\}.$$

As the finite-dimensional restricted Lie superalgebra $\mathfrak{l}_{\Bbbk}$ is a direct sum of basic Lie superalgebras, an analogous discussion of [35, Proposition 4.2] shows that every $U_\chi(\mathfrak{l}_{\Bbbk})$-module M is $U_\chi(\mathfrak{m}_{\Bbbk})$-free and $M\cong U_\chi(\mathfrak{m}_{\Bbbk})^*\otimes_{\Bbbk}M^{\mathfrak{m}_{\Bbbk}}$ as $U_\chi(\mathfrak{m}_{\Bbbk})$-modules.

Let $N_{\mathfrak{m}'_{\Bbbk}}$ denote the ideal of $U_\chi(\mathfrak{m}'_{\Bbbk})$ generated by all the elements $\bar{x}-\chi(\bar{x})$ with $\bar{x}\in\mathfrak{m}_{\Bbbk}$; then $M^{\mathfrak{m}_{\Bbbk}}$ is a $U_\chi(\mathfrak{m}'_{\Bbbk})/N_{\mathfrak{m}'_{\Bbbk}}$-module. Since $\mathfrak{m}'_{\Bbbk}=\bigoplus_{i=1}^r(\mathfrak{m}'_{\Bbbk})_i$, one can conclude from (6.11) and the remark prior to it that

$$\begin{aligned}U_\chi(\mathfrak{m}'_{\Bbbk})/N_{\mathfrak{m}'_{\Bbbk}}&\cong U_\chi(\mathfrak{m}'_{\Bbbk})\otimes_{U_\chi(\mathfrak{m}_{\Bbbk})}\bar{1}_\chi\cong U_\chi\Big(\bigoplus_{i=1}^r(\mathfrak{m}'_{\Bbbk})_i\Big)\otimes_{U_\chi(\bigoplus_{i=1}^r(\mathfrak{m}_{\Bbbk})_i)}\bar{1}_\chi\\&\cong\bigotimes_{i=1}^r(U_{\chi_i}((\mathfrak{m}'_{\Bbbk})_i)\otimes_{U_{\chi_i}((\mathfrak{m}_{\Bbbk})_i)}\bar{1}_{\chi_i})\cong\bigotimes_{i=1}^r U_{\chi_i}((\mathfrak{m}'_{\Bbbk})_i)/N_{(\mathfrak{m}'_{\Bbbk})_i}\\&\cong\overbrace{Q_1\otimes\cdots\otimes Q_1}^{l}\otimes\overbrace{\Bbbk\otimes\cdots\otimes\Bbbk}^{r-l}\cong\overbrace{Q_1\otimes\cdots\otimes Q_1}^{l}\end{aligned}$$

as $\Bbbk$-algebras.

Now we will introduce the refined super Kac–Weisfeiler property for a direct sum of basic Lie superalgebras with nilpotent p-characters.

Proposition 6.5. *Let $\mathfrak{l}_{\Bbbk}$ be a direct sum of basic Lie superalgebras over $\Bbbk=\overline{\mathbb{F}}_p$, and $\chi\in(\mathfrak{l}_{\Bbbk})^*_{\bar{0}}$ be a nilpotent p-character. Retain the notation in (6.10) and below, and assume that the prime p satisfies the restriction imposed in §2.1 (Table 1). Then the dimension of every $U_\chi(\mathfrak{l}_{\Bbbk})$-module M is divisible by $p^{\frac{d_0'}{2}}2^{\frac{d_1'+l}{2}}$ (appoint $l=0$ if the $(d_1)_i$ are all even for $1\leqslant i\leqslant r$).*

Proof. For each $U_\chi(\mathfrak{l}_{\Bbbk})$-module M, the arguments preceding the proposition show that the $U_\chi(\mathfrak{m}_{\Bbbk})$-module

$$M\cong U_\chi(\mathfrak{m}_{\Bbbk})^*\otimes_{\Bbbk}M^{\mathfrak{m}_{\Bbbk}} \tag{6.12}$$

is free. Let us first investigate the dimension of $M^{\mathfrak{m}_{\Bbbk}}$. Since $M^{\mathfrak{m}_{\Bbbk}}$ is a module over the superalgebra $U_\chi(\mathfrak{m}_{\Bbbk}')/N_{\mathfrak{m}_{\Bbbk}'}\cong\overbrace{Q_1\otimes\cdots\otimes Q_1}^{l}$, based on the parity of l we will consider each case separately.

(i) When l is odd, (6.8) and (6.9) imply that

$$U_\chi(\mathfrak{m}_{\Bbbk}')/N_{\mathfrak{m}_{\Bbbk}'}\cong\overbrace{Q_1\otimes\cdots\otimes Q_1}^{l}\cong Q_{2^{\frac{l-1}{2}}}.$$

Since $Q_{2^{\frac{l-1}{2}}}$ is a simple superalgebra whose unique irreducible module is $2\cdot2^{\frac{l-1}{2}}=2^{\frac{l+1}{2}}$-dimensional, it follows from Wedderburn's theorem [14, Theorem 12.2.9] that every $Q_{2^{\frac{l-1}{2}}}$-module has dimension divisible by $2^{\frac{l+1}{2}}$. In particular, the dimension of $M^{\mathfrak{m}_{\Bbbk}}$ is divisible by $2^{\frac{l+1}{2}}$. By the same discussion as in [35, Theorem 4.3] we can conclude that $\underline{\dim}\,\mathfrak{m}_{\Bbbk}=(\frac{d_0'}{2},\frac{d_1'-1}{2})$; then $\dim U_\chi(\mathfrak{m}_{\Bbbk})=p^{\frac{d_0'}{2}}2^{\frac{d_1'-1}{2}}$. Together with (6.12) this implies that each $U_\chi(\mathfrak{l}_{\Bbbk})$-module M has dimension divisible by $p^{\frac{d_0'}{2}}2^{\frac{d_1'-1}{2}}\cdot2^{\frac{l+1}{2}}=p^{\frac{d_0'}{2}}2^{\frac{d_1'+l}{2}}$.

(ii) When l is even, it follows from (6.8) and (6.9) that

$$U_\chi(\mathfrak{m}_{\Bbbk}')/N_{\mathfrak{m}_{\Bbbk}'}\cong\overbrace{Q_1\otimes\cdots\otimes Q_1}^{l}\cong\mathcal{M}_{2^{\frac{l}{2}-1},2^{\frac{l}{2}-1}}.$$

Since $\mathcal{M}_{2^{\frac{l}{2}-1},2^{\frac{l}{2}-1}}$ is a simple superalgebra whose unique irreducible module is $2^{\frac{l}{2}}$-dimensional, it follows from Wedderburn's theorem [14, Theorem 12.2.9] that every $\mathcal{M}_{2^{\frac{l}{2}-1},2^{\frac{l}{2}-1}}$-module has dimension divisible by $2^{\frac{l}{2}}$. In particular, the dimension of $M^{\mathfrak{m}_{\Bbbk}}$ is divisible by $2^{\frac{l}{2}}$. The same discussion as in [35, Theorem 4.3] shows that $\underline{\dim}\,\mathfrak{m}_{\Bbbk}=(\frac{d_0'}{2},\frac{d_1'}{2})$; then $\dim U_\chi(\mathfrak{m}_{\Bbbk})=p^{\frac{d_0'}{2}}2^{\frac{d_1'}{2}}$. Together with (6.12) this implies that each $U_\chi(\mathfrak{l}_{\Bbbk})$-module M has dimension divisible by $p^{\frac{d_0'}{2}}2^{\frac{d_1'}{2}}\cdot2^{\frac{l}{2}}=p^{\frac{d_0'}{2}}2^{\frac{d_1'+l}{2}}$.

The discussions in (i) and (ii) complete the proof.

□

Remark 6.6. Recall that d'_1 and l have the same parity by the preceding discussion. Thus Proposition 6.5 coincides with the consequence obtained by Wang–Zhao in [35, Remark 4.6] on the special occasion that at most one of the $(d_1)_i$ is odd for $1 \leqslant i \leqslant r$. However, when more than two of the $(d_1)_i$ are odd for $1 \leqslant i \leqslant r$, the dimensional lower bounds of the representations of $\mathfrak{l}_{\Bbbk}$ given here should be optimal (see the proof of the forthcoming Theorem 6.7), which are much larger than those mentioned in [35, Remark 4.6]. Also note that the conclusion we obtained in Proposition 6.5 does not depend on Conjecture 1.3.

6.3.3. Assuming Conjecture 1.3, the following theorem shows that the dimensional lower bounds for the representations of reduced enveloping algebra $U_\chi(\mathfrak{l}_{\Bbbk})$ with nilpotent p-character $\chi \in (\mathfrak{l}_{\Bbbk})^*_{\bar{0}}$ in Proposition 6.5 are accessible.

Theorem 6.7. *Retain the assumptions in Proposition 6.5. If Conjecture 1.3 holds and $p \gg 0$, then the reduced enveloping algebra $U_\chi(\mathfrak{l}_{\Bbbk})$ admits irreducible representations of dimension $p^{\frac{d'_0}{2}}2^{\frac{d'_1+l}{2}}$ (appoint $l = 0$ when the $(d_1)_i$ are all even for $1 \leqslant i \leqslant r$).*

Proof. For each $1 \leqslant i \leqslant r$, let $Q_{\chi_i}^{\chi_i}$ be the $(\mathfrak{l}_{\Bbbk})_i$-module as defined in §2.3.2, and denote by $U_{\chi_i}((\mathfrak{l}_{\Bbbk})_i, \bar{e}_i) = (\mathrm{End}_{(\mathfrak{l}_{\Bbbk})_i} Q_{\chi_i}^{\chi_i})^{\mathrm{op}}$ the reduced W-superalgebra of basic Lie superalgebra $(\mathfrak{l}_{\Bbbk})_i$ associated with nilpotent element $\bar{e}_i \in ((\mathfrak{l}_{\Bbbk})_i)_{\bar{0}}$. Let Q_χ^χ be the $\mathfrak{l}_{\Bbbk}$-module with the same definition as in §2.3.2, and $U_\chi(\mathfrak{l}_{\Bbbk}, \bar{e})$ be the reduced W-superalgebra of $\mathfrak{l}_{\Bbbk}$ associated with nilpotent element $\bar{e} \in (\mathfrak{l}_{\Bbbk})_{\bar{0}}$. Then we have

$$\begin{aligned} U_\chi(\mathfrak{l}_{\Bbbk}, \bar{e}) = (\mathrm{End}_{\mathfrak{l}_{\Bbbk}} Q_\chi^\chi)^{\mathrm{op}} &\cong \left(\mathrm{End}_{\bigoplus_{i=1}^r (\mathfrak{l}_{\Bbbk})_i} \bigoplus_{i=1}^r Q_{\chi_i}^{\chi_i}\right)^{\mathrm{op}} \\ &\cong \bigotimes_{i=1}^r (\mathrm{End}_{(\mathfrak{l}_{\Bbbk})_i} Q_{\chi_i}^{\chi_i})^{\mathrm{op}} = \bigotimes_{i=1}^r U_{\chi_i}((\mathfrak{l}_{\Bbbk})_i, \bar{e}_i) \end{aligned} \tag{6.13}$$

as $\Bbbk$-algebras. Now we proceed with the proof by steps.

(1) We first prove the conclusion for the $\Bbbk$-algebra $\bigotimes_{i=1}^l U_{\chi_i}((\mathfrak{l}_{\Bbbk})_i, e_i)$. We will carry out the proof by induction on l.

(1-i) For the beginning of the induction, let us first look into each single term $(\mathfrak{l}_{\Bbbk})_i$ for $1 \leqslant i \leqslant l$, and make some basic observations on the tensor product of two terms. Under the assumptions of the theorem, Corollary 6.3 shows that the $\Bbbk$-algebra $U_{\chi_i}((\mathfrak{l}_{\Bbbk})_i, \bar{e}_i)$ admits two-dimensional representations for $1 \leqslant i \leqslant l$. Denote by V_1 and V_2 the two-dimensional irreducible representations (of type

Q) of the $\Bbbk$-algebras $U_{\chi_1}((\mathfrak{l}_\Bbbk)_1, \bar{e}_1)$ and $U_{\chi_2}((\mathfrak{l}_\Bbbk)_2, \bar{e}_2)$ (if it occurs), respectively. Then it follows from Lemma 6.4(iii) that $V_1 \boxtimes V_2 \cong (V_1 \circledast V_2) \oplus \Upsilon(V_1 \circledast V_2)$ as $U_{\chi_1}((\mathfrak{l}_\Bbbk)_1, \bar{e}_1) \otimes U_{\chi_2}((\mathfrak{l}_\Bbbk)_2, \bar{e}_2)$-modules, where $V_1 \circledast V_2$ is an irreducible $U_{\chi_1}((\mathfrak{l}_\Bbbk)_1, \bar{e}_1) \otimes U_{\chi_2}((\mathfrak{l}_\Bbbk)_2, \bar{e}_2)$-module of type M. Recall that $V_1 \circledast V_2$ is the same underlying vector space as $\Upsilon(V_1 \circledast V_2)$, thereby sharing the same dimension, i.e., $\dim V_1 \circledast V_2 = \dim \Upsilon(V_1 \circledast V_2)$. Since $V_1 \boxtimes V_2$ is just $V_1 \otimes V_2$ as vector spaces, from all of the above we can conclude that $V_1 \circledast V_2$ is an irreducible $U_{\chi_1}((\mathfrak{l}_\Bbbk)_1, \bar{e}_1) \otimes U_{\chi_2}((\mathfrak{l}_\Bbbk)_2, \bar{e}_2)$-module of type M with dimension $2 = 2^{\frac{2}{2}}$.

Denote by V_3 a two-dimensional irreducible representation (of type Q) of the $\Bbbk$-algebra $U_{\chi_3}((\mathfrak{l}_\Bbbk)_3, \bar{e}_3)$ (if it occurs). It follows from Lemma 6.4(ii) that $(V_1 \circledast V_2) \boxtimes V_3$ is an irreducible $(U_{\chi_1}((\mathfrak{l}_\Bbbk)_1, \bar{e}_1) \otimes U_{\chi_2}((\mathfrak{l}_\Bbbk)_2, \bar{e}_2)) \otimes U_{\chi_3}((\mathfrak{l}_\Bbbk)_3, \bar{e}_3)$-module of type Q with dimension $2 \cdot 2 = 4 = 2^{\frac{3+1}{2}}$.

(1-ii) On the basis of (1-i), we can easily draw the conclusion by induction on the number of the terms for $\bigotimes_{i=1}^{l} U_{\chi_i}((\mathfrak{l}_\Bbbk)_i, e_i)$, summarizing it as

(1-ii-i) when l is odd, the $\Bbbk$-algebra $\bigotimes_{i=1}^{l} U_{\chi_i}((\mathfrak{l}_\Bbbk)_i, \bar{e}_i)$ admits an irreducible representation of type Q with dimension $2^{\frac{l+1}{2}}$, and we denote it by V;

(1-ii-ii) when l is even, the $\Bbbk$-algebra $\bigotimes_{i=1}^{l} U_{\chi_i}((\mathfrak{l}_\Bbbk)_i, \bar{e}_i)$ admits an irreducible representation of type M with dimension $2^{\frac{l}{2}}$ (assume that $l = 0$ when the $(d_1)_i$ are all even for $1 \leqslant i \leqslant r$), and set it as V'.

(2) Now we consider the general case with $U_\chi(\mathfrak{l}_\Bbbk, \bar{e}) \cong \bigotimes_{i=1}^{r} U_{\chi_i}((\mathfrak{l}_\Bbbk)_i, \bar{e}_i)$.

Under the assumptions of the theorem, Corollary 6.3 shows that the $\Bbbk$-algebra $U_{\chi_i}((\mathfrak{l}_\Bbbk)_i, \bar{e}_i)$ admits one-dimensional representations of type M for $l+1 \leqslant i \leqslant r$ (if it occurs). Easy induction based on Lemma 6.4(i) shows that the $\Bbbk$-algebra $\bigotimes_{i=l+1}^{r} U_{\chi_i}((\mathfrak{l}_\Bbbk)_i, \bar{e}_i)$ admits a one-dimensional representation of type M, denoted by W.

Note that $\bigotimes_{i=1}^{r} U_{\chi_i}((\mathfrak{l}_\Bbbk)_i, \bar{e}_i) \cong \bigotimes_{i=1}^{l} U_{\chi_i}((\mathfrak{l}_\Bbbk)_i, \bar{e}_i) \otimes \bigotimes_{i=l+1}^{r} U_{\chi_i}((\mathfrak{l}_\Bbbk)_i, \bar{e}_i)$. Based on the parity of l, we will consider each case separately.

(2-i) When l is odd, (1-ii-i) shows that the $\Bbbk$-algebra $\bigotimes_{i=1}^{l} U_{\chi_i}((\mathfrak{l}_\Bbbk)_i, \bar{e}_i)$ admits an irreducible representation V of type Q with dimension $2^{\frac{l+1}{2}}$, and the $\Bbbk$-algebra $\bigotimes_{i=l+1}^{r} U_{\chi_i}((\mathfrak{l}_\Bbbk)_i, \bar{e}_i)$ admits a one-dimensional representation W of type M by the preceding remark. It follows from Lemma 6.4(ii) that $V \boxtimes W$ is an irreducible module of type Q with dimension $2^{\frac{l+1}{2}}$ for the $\Bbbk$-algebra $\bigotimes_{i=1}^{l} U_{\chi_i}((\mathfrak{l}_\Bbbk)_i, \bar{e}_i) \otimes \bigotimes_{i=l+1}^{r} U_{\chi_i}((\mathfrak{l}_\Bbbk)_i, \bar{e}_i) \cong \bigotimes_{i=1}^{r} U_{\chi_i}((\mathfrak{l}_\Bbbk)_i, \bar{e}_i)$.

(2-ii) When l is even, (1-ii-ii) shows that the $\Bbbk$-algebra $\bigotimes_{i=1}^{l} U_{\chi_i}((\mathfrak{l}_\Bbbk)_i, \bar{e}_i)$ admits an irreducible representation V' of type M with dimension $2^{\frac{l}{2}}$, and the $\Bbbk$-algebra $\bigotimes_{i=l+1}^{r} U_{\chi_i}((\mathfrak{l}_\Bbbk)_i, \bar{e}_i)$ admits a one-dimensional representation W of type

M by the preceding remark. It follows from Lemma 6.4(i) that $V'\boxtimes W$ is an irreducible module of type M with dimension $2^{\frac{l}{2}}$ for the $\mathbb{k}$-algebra $\bigotimes_{i=1}^{l}U_{\chi_i}((\mathfrak{l}_{\mathbb{k}})_i,\bar{e}_i)\otimes\bigotimes_{i=l+1}^{r}U_{\chi_i}((\mathfrak{l}_{\mathbb{k}})_i,\bar{e}_i)\cong\bigotimes_{i=1}^{r}U_{\chi_i}((\mathfrak{l}_{\mathbb{k}})_i,\bar{e}_i)$.

(3) Keeping in mind the algebra isomorphism

$$U_\chi(\mathfrak{l}_{\mathbb{k}})\cong \mathrm{Mat}_{p^{\frac{d_0'}{2}}2^{\lceil\frac{d_1'}{2}\rceil}}(U_\chi(\mathfrak{l}_{\mathbb{k}},\bar{e})) \tag{6.14}$$

(cf. [35, Remark 4.6]), we can conclude

(3-i) when l is odd, it follows from (2-i) that the $\mathbb{k}$-algebra $U_\chi(\mathfrak{l}_{\mathbb{k}})$ affords irreducible representations of dimension $p^{\frac{d_0'}{2}}2^{\frac{d_1'-1}{2}}\cdot 2^{\frac{l+1}{2}}=p^{\frac{d_0'}{2}}2^{\frac{d_1'+l}{2}}$;

(3-ii) when l is even, it follows from (2-ii) that the $\mathbb{k}$-algebra $U_\chi(\mathfrak{l}_{\mathbb{k}})$ affords irreducible representations of dimension $p^{\frac{d_0'}{2}}2^{\frac{d_1'}{2}}\cdot 2^{\frac{l}{2}}=p^{\frac{d_0'}{2}}2^{\frac{d_1'+l}{2}}$.

Summing up, we complete the proof. □

Remark 6.8. Recall that the dimensional lower bounds introduced in Proposition 6.5 are much larger than the boundary obtained by Wang–Zhao in [35, Remark 4.6] when more than two of the $(d_1)_i$ are odd for $1\leqslant i\leqslant r$ (cf. Remark 6.6). In fact, careful inspection of the proof of Theorem 6.7 shows that Wang–Zhao's estimation on these lower bounds can never be reached in this case, and those introduced in Theorem 6.7 are optimal.

§6.4. Discussion for basic Lie superalgebras with arbitrary p-characters

In this subsection we will consider the accessibility of the lower bounds of the super Kac–Weisfeiler property with any p-characters in Proposition 6.1.

6.4.1. For a given p-character $\xi\in(\mathfrak{g}_{\mathbb{k}})_{\bar{0}}^*$, we have its Jordan–Chevalley decomposition $\xi=\xi_{\bar{s}}+\xi_{\bar{n}}$ under the $\mathrm{Ad}\,(G_{\mathbb{k}})_{\mathrm{ev}}$-equivariant isomorphism $(\mathfrak{g}_{\mathbb{k}})_{\bar{0}}^*\cong(\mathfrak{g}_{\mathbb{k}})_{\bar{0}}$ induced by the nondegenerate bilinear form $(\cdot,\cdot)$ on $(\mathfrak{g}_{\mathbb{k}})_{\bar{0}}$. That is to say, the decomposition of ξ can be identified with the usual Jordan decomposition $\bar{x}=\bar{s}+\bar{n}$ when ξ corresponds to $\bar{x}$ under the isomorphism $(\mathfrak{g}_{\mathbb{k}})_{\bar{0}}^*\cong(\mathfrak{g}_{\mathbb{k}})_{\bar{0}}$. Let $\mathfrak{h}_{\mathbb{k}}$ be a Cartan subalgebra of $\mathfrak{g}_{\mathbb{k}}$ that contains $\bar{s}$ and denote by $\mathfrak{l}_{\mathbb{k}}=\mathfrak{g}_{\mathbb{k}}^{\bar{s}}$ the centralizer of $\bar{s}$ in $\mathfrak{g}_{\mathbb{k}}$. Let Φ be the root system of $\mathfrak{g}_{\mathbb{k}}$ and $\Phi(\mathfrak{l}_{\mathbb{k}}):=\{\alpha\in\Phi\mid\alpha(\bar{s})=0\}$. By [35, Proposition 5.1], $\mathfrak{l}_{\mathbb{k}}$ is always a direct sum of basic Lie superalgebras with a system Δ of simple roots of $\mathfrak{g}_{\mathbb{k}}$ such that $\Delta\cap\Phi(\mathfrak{l}_{\mathbb{k}})$ is a system of simple roots of $\Phi(\mathfrak{l}_{\mathbb{k}})$ (note that a toral subalgebra of $\mathfrak{g}_{\mathbb{k}}$ may also appear in the summand). This is to say,

$$\mathfrak{l}_{\mathbb{k}}=\mathfrak{g}_{\mathbb{k}}^{\bar{s}}=\bigoplus_{i=1}^{r}(\mathfrak{g}_{\mathbb{k}})_i\oplus\mathfrak{t}_{\mathbb{k}}',$$

where each $(\mathfrak{g}_{\Bbbk})_i$ is a basic Lie superalgebra for $1 \leqslant i \leqslant r$, and $\mathfrak{t}'_{\Bbbk}$ is a toral subalgebra of $\mathfrak{g}_{\Bbbk}$. Then by [35, §5.1], $\xi_{\bar{n}} = \xi_1 + \cdots + \xi_r$ is a nilpotent p-character of $\mathfrak{l}_{\Bbbk}$ with $\xi_i \in (\mathfrak{g}_{\Bbbk})_i^*$ (each ξ_i can be viewed in $\mathfrak{l}_{\Bbbk}^*$ by letting $\xi_i(\bar{y}) = 0$ for all $\bar{y} \in \bigoplus_{j\neq i}(\mathfrak{g}_{\Bbbk})_j \oplus \mathfrak{t}'_{\Bbbk}$) for $1 \leqslant i \leqslant r$. Let $\bar{n} = \bar{n}_1 + \cdots + \bar{n}_r$ be the corresponding decomposition of $\bar{n}$ in $\mathfrak{l}_{\Bbbk}$ such that $\xi_i(\cdot) = (\bar{n}_i, \cdot)$ for $1 \leqslant i \leqslant r$. Then we can obtain the reduced W-superalgebra $U_{\xi_i}((\mathfrak{g}_{\Bbbk})_i, \bar{n}_i)$ of $(\mathfrak{g}_{\Bbbk})_i$ associated with nilpotent element $\bar{n}_i \in (\mathfrak{g}_{\Bbbk})_i$. It is easily verified that

$$U_{\xi_{\bar{n}}}\left(\bigoplus_{i=1}^{r}(\mathfrak{g}_{\Bbbk})_i, \bar{n}\right) \cong \bigotimes_{i=1}^{r} U_{\xi_i}((\mathfrak{g}_{\Bbbk})_i, \bar{n}_i)$$

by the same discussion as (6.13). Define

$$\begin{aligned} d_0 &:= \dim(\mathfrak{g}_{\Bbbk})_{\bar{0}} - \dim(\mathfrak{g}_{\Bbbk}^{\bar{x}})_{\bar{0}}, \qquad d_1 := \dim(\mathfrak{g}_{\Bbbk})_{\bar{1}} - \dim(\mathfrak{g}_{\Bbbk}^{\bar{x}})_{\bar{1}}, \\ (d_0)_i &:= \dim((\mathfrak{g}_{\Bbbk})_i)_{\bar{0}} - \dim((\mathfrak{g}_{\Bbbk})_i^{\bar{n}_i})_{\bar{0}},\ (d_1)_i := \dim((\mathfrak{g}_{\Bbbk})_i)_{\bar{1}} - \dim((\mathfrak{g}_{\Bbbk})_i^{\bar{n}_i})_{\bar{1}}, \end{aligned} \tag{6.15}$$

where $\mathfrak{g}_{\Bbbk}^{\bar{x}}$ denotes the centralizer of $\bar{x}$ in $\mathfrak{g}_{\Bbbk}$, and $(\mathfrak{g}_{\Bbbk})_i^{\bar{n}_i}$ is the centralizer of $\bar{n}_i$ in $(\mathfrak{g}_{\Bbbk})_i$ for each $i \in \{1, \ldots, r\}$. Rearrange the summands of $\bigoplus_{i=1}^{r}(\mathfrak{g}_{\Bbbk})_i$ such that each $(d_1)_i$ is odd for $1 \leqslant i \leqslant l$ (if it occurs), and each $(d_1)_i$ is even for $l+1 \leqslant i \leqslant r$ (if it occurs).

Let $\mathfrak{b}_{\Bbbk} = \mathfrak{h}_{\Bbbk} \oplus \mathfrak{n}_{\Bbbk}$ be the Borel subalgebra associated with Δ. Let $\mathfrak{p}_{\Bbbk}$ be a parabolic subalgebra of $\mathfrak{g}_{\Bbbk}$ with Levi factor $\mathfrak{l}_{\Bbbk}$, i.e., $\mathfrak{p}_{\Bbbk} = \mathfrak{l}_{\Bbbk} + \mathfrak{b}_{\Bbbk} = \mathfrak{l}_{\Bbbk} \oplus \mathfrak{u}_{\Bbbk}$, where $\mathfrak{u}_{\Bbbk}$ is the nilradical of $\mathfrak{p}_{\Bbbk}$. Set $\mathfrak{u}_{\Bbbk}^-$ to be the complement nilradical of $\mathfrak{p}_{\Bbbk}$ such that $\mathfrak{g}_{\Bbbk} = \mathfrak{p}_{\Bbbk} \oplus \mathfrak{u}_{\Bbbk}^- = \mathfrak{u}_{\Bbbk} \oplus \mathfrak{l}_{\Bbbk} \oplus \mathfrak{u}_{\Bbbk}^-$ as vector spaces. Since $\xi(\mathfrak{u}_{\Bbbk}) = 0$ and $\xi|_{\mathfrak{l}_{\Bbbk}} = \xi_{\bar{n}}|_{\mathfrak{l}_{\Bbbk}}$ is nilpotent by [35, §5.1], any $U_\xi(\mathfrak{l}_{\Bbbk})$-mod can be regarded as a $U_\xi(\mathfrak{p}_{\Bbbk})$-mod with a trivial action of $\mathfrak{u}_{\Bbbk}$. Wang–Zhao proved that the $\Bbbk$-algebras $U_\xi(\mathfrak{g}_{\Bbbk})$ and $U_\xi(\mathfrak{l}_{\Bbbk})$ are Morita equivalent in [35, Theorem 5.2], and any irreducible $U_\xi(\mathfrak{g}_{\Bbbk})$-module can be induced from an irreducible $U_\xi(\mathfrak{l}_{\Bbbk})$-mod (which is also a $U_\xi(\mathfrak{p}_{\Bbbk})$-mod with a trivial action of $\mathfrak{u}_{\Bbbk}$) by

$$U_\xi(\mathfrak{g}_{\Bbbk}) \otimes_{U_\xi(\mathfrak{p}_{\Bbbk})} - : \quad U_\xi(\mathfrak{l}_{\Bbbk})\text{-mod} \to U_\xi(\mathfrak{g}_{\Bbbk})\text{-mod}. \tag{6.16}$$

6.4.2. Keep the notation and assumptions as in §6.4.1. We first recall the following result.

Proposition 6.1 ([35]). *Let $\mathfrak{g}_{\Bbbk}$ be a basic Lie superalgebra over $\Bbbk = \overline{\mathbb{F}}_p$, assuming that the prime p satisfies the restriction imposed in §2.1 (Table 1). Let ξ be an arbitrary p-character in $(\mathfrak{g}_{\Bbbk})_{\bar{0}}^*$. Then the dimension of every $U_\xi(\mathfrak{g}_{\Bbbk})$-module M is divisible by $p^{\frac{d_0}{2}}2^{\lfloor \frac{d_1}{2} \rfloor}$.*

The above proposition was first verified by Wang–Zhao in [35, Theorem 5.6]. Compared with the proof we present below, Wang–Zhao's proof in [35] is more concise since they did not consider the parity of the $(d_1)_i$ $(1 \leqslant i \leqslant r)$ for the summands of $\mathfrak{l}_{\mathbb{k}} = \mathfrak{g}_{\mathbb{k}}^{\bar{s}}$ decomposed into $\bigoplus_{i=1}^{r}(\mathfrak{g}_{\mathbb{k}})_i \oplus \mathfrak{t}'_{\mathbb{k}}$.

As our approach to the main goal of this subsection will be much dependent on the analysis of the parities of those $(d_1)_i$, along with the above result, then we have to formulate another proof of Proposition 6.1, based on Proposition 6.5. This will be important for us to get at the goal.

Proof. (1) First note that [35, Theorem 5.6] shows

$$\begin{aligned}\underline{\dim}\,\mathfrak{g}_{\mathbb{k}} - \underline{\dim}\,\mathfrak{g}_{\mathbb{k}}^{\bar{x}} &= \underline{\dim}\,\mathfrak{g}_{\mathbb{k}} - \underline{\dim}\,\mathfrak{l}_{\mathbb{k}}^{\bar{n}} \\ &= 2\underline{\dim}\,\mathfrak{u}_{\mathbb{k}}^{-} + (\underline{\dim}\,\mathfrak{l}_{\mathbb{k}} - \underline{\dim}\,\mathfrak{l}_{\mathbb{k}}^{\bar{n}}),\end{aligned} \tag{6.17}$$

where $\mathfrak{l}_{\mathbb{k}}^{\bar{n}}$ denotes the centralizer of $\bar{n}$ in $\mathfrak{l}_{\mathbb{k}}$. Since $\mathfrak{l}_{\mathbb{k}} = \bigoplus_{i=1}^{r}(\mathfrak{g}_{\mathbb{k}})_i \oplus \mathfrak{t}'_{\mathbb{k}}$ and $\bar{n} \in \bigoplus_{i=1}^{r}(\mathfrak{g}_{\mathbb{k}})_i$, it is obvious that $(\mathfrak{t}'_{\mathbb{k}})^{\bar{n}} = \mathfrak{t}'_{\mathbb{k}}$; then

$$\begin{aligned}\underline{\dim}\,\mathfrak{l}_{\mathbb{k}} - \underline{\dim}\,\mathfrak{l}_{\mathbb{k}}^{\bar{n}} &= \sum_{i=1}^{r}\left(\underline{\dim}\,(\mathfrak{g}_{\mathbb{k}})_i - \underline{\dim}\,(\mathfrak{g}_{\mathbb{k}})_i^{\bar{n}_i}\right) + \underline{\dim}\,\mathfrak{t}'_{\mathbb{k}} - \underline{\dim}\,(\mathfrak{t}'_{\mathbb{k}})^{\bar{n}} \\ &= \left(\sum_{i=1}^{r}(d_0)_i, \sum_{i=1}^{r}(d_1)_i\right).\end{aligned} \tag{6.18}$$

As $\underline{\dim}\,\mathfrak{u}_{\mathbb{k}}^{-} = \underline{\dim}\,\mathfrak{u}_{\mathbb{k}}$, (6.17) shows that

$$\begin{aligned}\dim(\mathfrak{u}_{\mathbb{k}}^{-})_{\bar{0}} = \dim(\mathfrak{u}_{\mathbb{k}})_{\bar{0}} &= \frac{d_0 - \sum_{i=1}^{r}(d_0)_i}{2}, \\ \dim(\mathfrak{u}_{\mathbb{k}}^{-})_{\bar{1}} = \dim(\mathfrak{u}_{\mathbb{k}})_{\bar{1}} &= \frac{d_1 - \sum_{i=1}^{r}(d_1)_i}{2}.\end{aligned} \tag{6.19}$$

(2) For $\mathfrak{l}_{\mathbb{k}} = \bigoplus\limits_{i=1}^{r}(\mathfrak{g}_{\mathbb{k}})_i \oplus \mathfrak{t}'_{\mathbb{k}}$, we have an algebra isomorphism

$$U_{\xi_{\bar{n}}}(\mathfrak{l}_{\mathbb{k}}) \cong U_{\xi_{\bar{n}}}\left(\bigoplus_{i=1}^{r}(\mathfrak{g}_{\mathbb{k}})_i \oplus \mathfrak{t}'_{\mathbb{k}}\right) \cong U_{\xi_{\bar{n}}}\left(\bigoplus_{i=1}^{r}(\mathfrak{g}_{\mathbb{k}})_i\right) \otimes U_0(\mathfrak{t}'_{\mathbb{k}}). \tag{6.20}$$

Let us first consider the representations of the $\mathbb{k}$-algebra $U_{\xi_{\bar{n}}}(\bigoplus_{i=1}^{r}(\mathfrak{g}_{\mathbb{k}})_i)$. Since $\xi_{\bar{n}}|_{\mathfrak{l}_{\mathbb{k}}}$ is nilpotent and each $(\mathfrak{g}_{\mathbb{k}})_i$ is a basic Lie superalgebra for $1 \leqslant i \leqslant r$, it follows from Proposition 6.5 that every $U_{\xi_{\bar{n}}}(\bigoplus_{i=1}^{r}(\mathfrak{g}_{\mathbb{k}})_i)$-module is divisible by $p^{\frac{\sum_{i=1}^{r}(d_0)_i}{2}}2^{\frac{l+\sum_{i=1}^{r}(d_1)_i}{2}}$.

Next we look at the $\mathbb{k}$-algebra $U_0(\mathfrak{t}'_{\mathbb{k}})$. As $\mathfrak{t}'_{\mathbb{k}}$ is a toral subalgebra of $\mathfrak{g}_{\mathbb{k}}$ with a basis $\{t_1, \ldots, t_d\}$ such that $t_i^{[p]} = t_i$ for all $1 \leqslant i \leqslant d$, then $U_0(\mathfrak{t}'_{\mathbb{k}}) \cong A_1^{\otimes d}$ where $A_1 \cong \mathbb{k}[X]/(X^p - X)$ is a p-dimensional commutative semisimple algebra whose irreducible representations are one-dimensional (naturally of type M). Hence we can conclude from Lemma 6.4(i) that all irreducible representations of $U_0(\mathfrak{t}'_{\mathbb{k}})$ are one-dimensional and of type M.

Recall that $U_{\xi_{\bar{n}}}(\mathfrak{l}_{\Bbbk}) \cong U_{\xi_{\bar{n}}}(\bigoplus_{i=1}^{r}(\mathfrak{g}_{\Bbbk})_i) \otimes U_0(\mathfrak{t}'_{\Bbbk})$. Summing up, we can conclude that any $U_{\xi_{\bar{n}}}(\mathfrak{l}_{\Bbbk})$-module is divisible by $p^{\frac{\sum_{i=1}^{r}(d_0)_i}{2}} 2^{\frac{l+\sum_{i=1}^{r}(d_1)_i}{2}}$.

(3) Recall that an object in the representation category of $U_{\xi}(\mathfrak{l}_{\Bbbk})$ can be regarded as one in the representation category of $U_{\xi}(\mathfrak{p}_{\Bbbk})$ with a trivial action of $\mathfrak{u}_{\Bbbk}$; then it follows from (6.16) and (6.19) that the dimension of every $U_{\xi}(\mathfrak{g}_{\Bbbk})$-mod is divisible by

$$p^{\sum_{i=1}^{r}\frac{(d_0)_i}{2}} 2^{\frac{l}{2}+\sum_{i=1}^{r}\frac{(d_1)_i}{2}} \cdot p^{\frac{d_0-\sum_{i=1}^{r}(d_0)_i}{2}} 2^{\frac{d_1-\sum_{i=1}^{r}(d_1)_i}{2}} = p^{\frac{d_0}{2}} 2^{\frac{d_1+l}{2}}. \tag{6.21}$$

(4) We now claim that in (6.21) we have $l = 0$ or 1; that is to say,

$$\text{at most one of the } (d_1)_i \text{ is odd for } 1 \leqslant i \leqslant r. \tag{6.22}$$

The proof of the claim (6.22) will be given for each case separately. Recall that for $\mathfrak{l}_{\Bbbk} = \mathfrak{g}_{\Bbbk}^{\bar{s}}$ with a direct-sum decomposition $\mathfrak{l}_{\Bbbk} = \bigoplus_{i=1}^{r}(\mathfrak{g}_{\Bbbk})_i \oplus \mathfrak{t}'_{\Bbbk}$, we have (i) d_1 and $\sum_{i=1}^{r}(d_1)_i$ share the same parity (see (6.15), (6.17), and (6.18)); (ii) the $(d_1)_i$ are odd for $1 \leqslant i \leqslant l$, and the $(d_1)_i$ are even for $l+1 \leqslant i \leqslant r$ (see §6.4.1). It follows that d_1 and l have the same parity. Combining this with the claim (6.22), Proposition 6.1 follows. □

Proof of the claim (6.22). We will complete the proof by steps.

(1) The above discussion shows that we need to consider only the summands of $\bigoplus_{i=1}^{r}(\mathfrak{g}_{\Bbbk})_i$, investigating the situation with nonzero odd parts. Recall that an explicit list of non-W-equivalent systems of positive roots was found by Kac in [12, §2.5.4] (a system of simple roots for $F(4)$ is missing; see the remark above [35, Proposition 5.1]). Note that in the examples given by Kac, the Cartan subalgebra $\mathfrak{h}_{\Bbbk}$ is a subspace of the space D of diagonal matrices; the roots are expressed in terms of the standard basis ϵ_i of D^* (more accurately, the restrictions of the ϵ_i to $\mathfrak{h}_{\Bbbk}$). In the following we assume that the semisimple element $\bar{s} \in \mathfrak{h}_{\Bbbk}$.

(2) Given any nilpotent element $e \in \mathfrak{sl}(M|N)_{\bar{0}}$, we have $\mathfrak{gl}(M|N)_{\bar{1}} = \mathfrak{sl}(M|N)_{\bar{1}}$ and $\mathfrak{gl}(M|N)_{\bar{1}}^{e} = \mathfrak{sl}(M|N)_{\bar{1}}^{e}$ for all $M, N \in \mathbb{Z}_{+}$, where $\mathfrak{gl}(M|N)_{\bar{1}}^{e}$ and $\mathfrak{sl}(M|N)_{\bar{1}}^{e}$ denote the centralizers of e in $\mathfrak{gl}(M|N)_{\bar{1}}$ and $\mathfrak{sl}(M|N)_{\bar{1}}$, respectively. It follows from [35, §3.2] that $(d_1)_i$ $(1 \leqslant i \leqslant r)$ is always even for the summand $(\mathfrak{g}_{\Bbbk})_i$ that is isomorphic to the basic Lie superalgebra of type $A(m,n)$ with $m, n \in \mathbb{Z}_{+}$. By virtue of this consequence, completely elementary yet tedious case-by-case calculations show the following:

(i) When $\mathfrak{g}_{\Bbbk}$ is of type $A(M,N)$, each summand of $\bigoplus_{i=1}^{r}(\mathfrak{g}_{\Bbbk})_i$ with nonzero odd part is always isomorphic to a basic Lie superalgebra of type $A(m,n)$ with $m,n\in\mathbb{Z}_+$, thus for $1\leqslant i\leqslant r$ the $(d_1)_i$ are all even in this case;

(ii) When $\mathfrak{g}_{\Bbbk}$ is of type $B(M,N)$, $C(M,N)$, or $D(M,N)$, each summand of $\bigoplus_{i=1}^{r}(\mathfrak{g}_{\Bbbk})_i$ with nonzero odd part is either isomorphic to a basic Lie superalgebra of type $A(m,n)$ with $m,n\in\mathbb{Z}_+$; or at most one summand is isomorphic to $B(m,n)$, $C(m,n)$, or $D(m,n)$ $(m,n\in\mathbb{Z}_+)$ respectively (which is of the same type as $\mathfrak{g}_{\Bbbk}$). Hence for $1\leqslant i\leqslant r$, at most one of the $(d_1)_i$ is odd in this case.

(iii) When $\mathfrak{g}_{\Bbbk}$ is of type $D(2,1;\bar{a})$ or $G(3)$, each summand of $\bigoplus_{i=1}^{r}(\mathfrak{g}_{\Bbbk})_i$ with nonzero odd part is either isomorphic to a basic Lie superalgebra of type $A(m,n)$ with $m,n\in\mathbb{Z}_+$; or at most one summand is isomorphic to $B(m,n)$ $(m,n\in\mathbb{Z}_+)$. At the extreme, $\mathfrak{l}_{\Bbbk}=\mathfrak{g}_{\Bbbk}^{\bar{s}}$ equals $D(2,1;\bar{a})$ or $G(3)$ respectively when $\bar{s}=0$. Hence for $1\leqslant i\leqslant r$, at most one of the $(d_1)_i$ is odd in this case.

(iv) When $\mathfrak{g}_{\Bbbk}$ is of type $F(4)$, each summand of $\bigoplus_{i=1}^{r}(\mathfrak{g}_{\Bbbk})_i$ with nonzero odd part is either isomorphic to a basic Lie superalgebra of type $A(m,n)$ with $m,n\in\mathbb{Z}_+$; or at most one summand is either isomorphic to $B(m,n)$ with $m,n\in\mathbb{Z}_+$, or to $D(2,1;\bar{a})$. At the extreme, $\mathfrak{l}_{\Bbbk}=\mathfrak{g}_{\Bbbk}^{\bar{s}}=F(4)$ when $\bar{s}=0$. Hence for $1\leqslant i\leqslant r$, at most one of the $(d_1)_i$ is odd in this case.

Summing up the above discussions, we can conclude that at most one of the $(d_1)_i$ $(1\leqslant i\leqslant r)$ is odd in the summands of $\bigoplus_{i=1}^{r}(\mathfrak{g}_{\Bbbk})_i$. Then we complete the proof of the claim (6.22). $\square$

Remark 6.9. The significance of the above new proof can be seen as below:

(1) From the precise analysis of the proof, one can conclude that the lower bounds for the dimension of $U_\xi(\mathfrak{g}_{\Bbbk})$-modules critically depend on the parities of the $(d_1)_i$ for $1\leqslant i\leqslant r$. Without careful inspection of the summands of $\mathfrak{l}_{\Bbbk}\cong\bigoplus_{i=1}^{r}(\mathfrak{g}_{\Bbbk})_i\oplus\mathfrak{t}'_{\Bbbk}$, we cannot ensure that the lower bounds introduced in [35, Theorem 5.6] are optimal.

(2) What we are concerned with is the realization of "small representations" of dimensions equaling the lower bounds in [35, Theorem 5.6], when assuming Conjecture 1.3. By the above proof, we can demonstrate how to realize the $U_\xi(\mathfrak{g}_{\Bbbk})$-modules of minimal dimensions by the representation theory of the $\Bbbk$-algebra $U_\xi(\mathfrak{l}_{\Bbbk})$; see §6.4.3 for more details.

6.4.3. The proof of Theorem 1.5. Now we are in a position to prove Theorem 1.5, attacking the problem of the accessibility of the lower bounds in the super Kac–Weisfeiler property [35, Theorem 5.6].

Proof of Theorem 1.5. Retain the notation in (6.15). For the subalgebra $\mathfrak{l}_{\mathbb{k}} = \mathfrak{g}_{\mathbb{k}}^{\bar{s}} = \bigoplus_{i=1}^{r}(\mathfrak{g}_{\mathbb{k}})_i \oplus \mathfrak{t}'_{\mathbb{k}}$ of $\mathfrak{g}_{\mathbb{k}}$, recall that Theorem 6.7 shows that, assuming Conjecture 1.3, the $\mathbb{k}$-algebra $U_{\xi_{\bar{n}}}(\bigoplus_{i=1}^{r}(\mathfrak{g}_{\mathbb{k}})_i)$ admits an irreducible representation of dimension $p^{\sum_{i=1}^{r}\frac{(d_0)_i}{2}}2^{\frac{l}{2}+\sum_{i=1}^{r}\frac{(d_1)_i}{2}}$, and we denote it by V. By the arguments of step (4) in the proof of Proposition 6.1, one can conclude that

$$\dim V = p^{\sum\limits_{i=1}^{r}\frac{(d_0)_i}{2}}2^{\frac{l}{2}+\sum\limits_{i=1}^{r}\frac{(d_1)_i}{2}} = p^{\sum\limits_{i=1}^{r}\frac{(d_0)_i}{2}}2^{\lfloor\sum\limits_{i=1}^{r}\frac{(d_1)_i}{2}\rfloor}.$$

Step (2) in the proof of Proposition 6.1 shows that the $\mathbb{k}$-algebra $U_0(\mathfrak{t}'_{\mathbb{k}})$ affords an irreducible representation of dimension one, denoted by W (note that it is of type M). Thus it follows from Lemma 6.4 that $V \boxtimes W$ is an irreducible representation of the $\mathbb{k}$-algebra $U_{\xi_{\bar{n}}}(\bigoplus_{i=1}^{r}(\mathfrak{g}_{\mathbb{k}})_i) \otimes U_0(\mathfrak{t}'_{\mathbb{k}}) \cong U_{\xi}(\mathfrak{l}_{\mathbb{k}})$ with dimension $p^{\sum_{i=1}^{r}\frac{(d_0)_i}{2}}2^{\lfloor\sum_{i=1}^{r}\frac{(d_1)_i}{2}\rfloor}$.

Recall that any $U_{\xi}(\mathfrak{l}_{\mathbb{k}})$-mod can be regarded as a $U_{\xi}(\mathfrak{p}_{\mathbb{k}})$-mod with a trivial action of $\mathfrak{u}_{\mathbb{k}}$. It follows from (6.16) that the $U_{\xi}(\mathfrak{g}_{\mathbb{k}})$-module induced from the $U_{\xi}(\mathfrak{p}_{\mathbb{k}})$-module $V \boxtimes W$ is irreducible. By (6.19) we can conclude that the dimension of this induced module is equal to

$$\begin{aligned} &p^{\sum_{i=1}^{r}\frac{(d_0)_i}{2}}2^{\lfloor\sum_{i=1}^{r}\frac{(d_1)_i}{2}\rfloor}\cdot p^{\frac{d_0-\sum_{i=1}^{r}(d_0)_i}{2}}2^{\frac{d_1-\sum_{i=1}^{r}(d_1)_i}{2}} \\ &\quad= p^{\frac{d_0}{2}}2^{\frac{d_1}{2}+(\lfloor\sum_{i=1}^{r}\frac{(d_1)_i}{2}\rfloor-\sum_{i=1}^{r}\frac{(d_1)_i}{2})}. \end{aligned} \tag{6.23}$$

Owing to (6.15), (6.17), and (6.18), $\sum_{i=1}^{r}(d_1)_i$ and d_1 have the same parity. Hence, the desired result follows from (6.23). □

Remark 6.10. For the reduced enveloping algebra $U_{\xi}(\mathfrak{g}_{\mathbb{k}})$ with arbitrary p-character $\xi \in (\mathfrak{g}_{\mathbb{k}})^*_{\bar{0}}$, the formulation of the super Kac–Weisfeiler property in Proposition 6.1 is dependent on the $\mathfrak{g}_{\mathbb{k}}^{\bar{s}}$ that equals the direct sum of some basic Lie superalgebras and a toral subalgebra, and set it to be $\bigoplus_{i=1}^{r}(\mathfrak{g}_{\mathbb{k}})_i \oplus \mathfrak{t}'_{\mathbb{k}}$ (cf. [35]). However, after the super Kac–Weisfeiler property for the $\mathbb{k}$-algebra $U_{\xi_{\bar{n}}}(\bigoplus_{i=1}^{r}(\mathfrak{g}_{\mathbb{k}})_i)$ is refined in Proposition 6.5, one may be worried whether the real minimal dimensions of the representations of $U_{\xi}(\mathfrak{g}_{\mathbb{k}})$ are much larger than those introduced in Proposition 6.1. Fortunately, Theorem 1.5 certifies that they are exactly the real minimal dimensions of those representations, when assuming Conjecture 1.3.

§7. Reducing Conjecture 1.3 to the rigid nilpotent cases

In the concluding section we will reduce the investigation of Conjecture 1.3 to the rigid nilpotent cases, generalizing the arguments of [29, §3.2] in the ordinary finite W-algebra case. We maintain the same notation as used previously.

§7.1. Rigid nilpotent elements

We first introduce the notion of induced nilpotent elements for basic Lie superalgebras, analogous to the case of reductive Lie algebras (cf. [4], or [18]). Given a basic Lie superalgebra $\mathfrak{g}_{\mathbb{C}}$ over the field of complex numbers, [5, §3.3] shows that there is a Chevalley basis B of $\mathfrak{g}_{\mathbb{C}}$ excluding the case $D(2,1;a)$ with $a \notin \mathbb{Z}$ (in the case $D(2,1;a)$ with $a \notin \mathbb{Z}$ being an algebraic number, adjust the definition of the Chevalley basis by changing $\mathbb{Z}$ to $\mathbb{Z}[a]$). Denote the Chevalley $\mathbb{Z}$-form in $\mathfrak{g}_{\mathbb{C}}$ associated with B by $\mathfrak{g}_{\mathbb{Z}}$. We continue to assume that the field $\mathbb{F}$ is either $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{k}$, and $p \gg 0$ when $\mathbb{k} = \overline{\mathbb{F}}_p$. Let $\mathfrak{g}_{\mathbb{F}} := \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F}$ be a basic Lie superalgebra over $\mathbb{F}$. Let $\mathfrak{p}_{\mathbb{F}}$ be a parabolic subalgebra of $\mathfrak{g}_{\mathbb{F}}$ with the proper Levi subalgebra $\mathfrak{l}_{\mathbb{F}}$ and nilradical $\mathfrak{u}_{\mathbb{F}}^+$. Then we have $\mathfrak{p}_{\mathbb{F}} = \mathfrak{l}_{\mathbb{F}} \oplus \mathfrak{u}_{\mathbb{F}}^+$. Denote by $\mathfrak{u}_{\mathbb{F}}^-$ the opposite nilradical of $\mathfrak{p}_{\mathbb{F}}$; then $\mathfrak{g}_{\mathbb{F}} = \mathfrak{u}_{\mathbb{F}}^- \oplus \mathfrak{l}_{\mathbb{F}} \oplus \mathfrak{u}_{\mathbb{F}}^+$.

Definition 7.1. Retain the notation above. A nilpotent element $e \in (\mathfrak{p}_{\mathbb{F}})_{\bar{0}}$ is said to be induced from a nilpotent element $e_0 \in (\mathfrak{l}_{\mathbb{F}})_{\bar{0}}$ if the following items are satisfied:

(1) $[\mathfrak{p}_{\mathbb{F}}, e] = [\mathfrak{l}_{\mathbb{F}}, e_0] \oplus \mathfrak{u}_{\mathbb{F}}^+$ as vector spaces;

(2) $\mathfrak{g}_{\mathbb{F}}^e \subseteq \mathfrak{p}_{\mathbb{F}}$, where $\mathfrak{g}_{\mathbb{F}}^e$ denotes the centralizer of e in $\mathfrak{g}_{\mathbb{F}}$.

Call e rigid nilpotent if there is no e_0 as above such that e is induced from e_0.

For a given pair of nilpotent elements (e, e_0) in the reducing relation as in Definition 7.1, one can give a further description as follows.

Proposition 7.2. *Retain the notation above, and let $e \in \mathfrak{p}_{\mathbb{F}}$ be a nilpotent element induced from a nilpotent element $e_0' \in \mathfrak{l}_{\mathbb{F}}$. Then there exist nilpotent elements $e_0 \in \mathfrak{l}_{\mathbb{F}}$ and $e_1 \in \mathfrak{u}_{\mathbb{F}}^+$ such that $e = e_0 + e_1$, and e can also be considered induced from e_0.*

Proof. Since $e \in \mathfrak{p}_{\mathbb{F}}$ and $\mathfrak{p}_{\mathbb{F}} = \mathfrak{l}_{\mathbb{F}} \oplus \mathfrak{u}_{\mathbb{F}}^+$, there exist $e_0 \in \mathfrak{l}_{\mathbb{F}}$ and $e_1 \in \mathfrak{u}_{\mathbb{F}}^+$ such that $e = e_0 + e_1$. Obviously, the elements e_0 and e_1 are nilpotent. In the following, we will verify that $[\mathfrak{p}_{\mathbb{F}}, e] = [\mathfrak{l}_{\mathbb{F}}, e_0] \oplus \mathfrak{u}_{\mathbb{F}}^+$.

First note that

$$[\mathfrak{p}_{\mathbb{F}}, e] = [\mathfrak{l}_{\mathbb{F}} + \mathfrak{u}_{\mathbb{F}}^+, e_0 + e_1] = [\mathfrak{l}_{\mathbb{F}}, e_0] + [\mathfrak{l}_{\mathbb{F}}, e_1] + [\mathfrak{u}_{\mathbb{F}}^+, e_0] + [\mathfrak{u}_{\mathbb{F}}^+, e_1]. \tag{7.1}$$

Since $e_0 \in \mathfrak{l}_{\mathbb{F}}$, $e_1 \in \mathfrak{u}_{\mathbb{F}}^+$, and $\mathfrak{u}_{\mathbb{F}}^+$ is the nilradical of $\mathfrak{p}_{\mathbb{F}}$, we have $[\mathfrak{l}_{\mathbb{F}}, e_0] \subseteq \mathfrak{l}_{\mathbb{F}}$, $[\mathfrak{l}_{\mathbb{F}}, e_1] + [\mathfrak{u}_{\mathbb{F}}^+, e_0] + [\mathfrak{u}_{\mathbb{F}}^+, e_1] \subseteq \mathfrak{u}_{\mathbb{F}}^+$.

On the other hand, as e is induced from $e_0' \in \mathfrak{l}_{\mathbb{F}}$, we have

$$[\mathfrak{p}_{\mathbb{F}}, e] = [\mathfrak{l}_{\mathbb{F}}, e_0'] \oplus \mathfrak{u}_{\mathbb{F}}^+. \tag{7.2}$$

Combining (7.1) with (7.2), we have that $[\mathfrak{l}_{\mathbb{F}}, e_0] = [\mathfrak{l}_{\mathbb{F}}, e_0']$, $\mathfrak{u}_{\mathbb{F}}^+ = [\mathfrak{l}_{\mathbb{F}}, e_1] + [\mathfrak{u}_{\mathbb{F}}^+, e_0] + [\mathfrak{u}_{\mathbb{F}}^+, e_1]$ as vector spaces. Therefore, (7.1) shows that $[\mathfrak{p}_{\mathbb{F}}, e] = [\mathfrak{l}_{\mathbb{F}}, e_0] \oplus \mathfrak{u}_{\mathbb{F}}^+$, i.e., the nilpotent element e can also be considered induced from $e_0 \in \mathfrak{l}_{\mathbb{F}}$. This completes the proof. □

In the following we will always assume that $e \in (\mathfrak{p}_{\mathbb{F}})_{\bar{0}}$ is induced from an element $e_0 \in (\mathfrak{l}_{\mathbb{F}})_{\bar{0}}$ such that $e = e_0 + e_1$ for some $e_1 \in \mathfrak{u}_{\mathbb{F}}^+$.

Remark 7.3.

(1) The notions of "induced nilpotent" and "rigid nilpotent" can also be introduced for odd nilpotent elements. Proposition 7.2 still holds for them.

(2) Hoyt introduced the notion of Richardson elements for a basic Lie superalgebra $\mathfrak{g}_{\mathbb{F}}$ in [9, §4.3]. Our setting in Definition 7.1 includes Hoyt's definition of Richardson elements as a special case when $e_0 = 0$, which is parallel to the relationship between their counterparts in the Lie algebra case.

Now continue to consider the super case. One can easily verify

Proposition 7.4. *Keep the above notation. The following are true:*

(1) $\underline{\dim}\,\mathfrak{g}_{\mathbb{F}}^e = \underline{\dim}\,\mathfrak{l}_{\mathbb{F}}^{e_0}$, *where* $\mathfrak{l}_{\mathbb{F}}^{e_0}$ *denotes the centralizer of* e_0 *in* $\mathfrak{l}_{\mathbb{F}}$*;*

(2) $\underline{\dim}\,[\mathfrak{g}_{\mathbb{F}}, e] = \underline{\dim}\,[\mathfrak{l}_{\mathbb{F}}, e_0] + 2\underline{\dim}\,\mathfrak{u}_{\mathbb{F}}^+$.

Proof. (1) Since $\mathfrak{p}_{\mathbb{F}} = \mathfrak{l}_{\mathbb{F}} \oplus \mathfrak{u}_{\mathbb{F}}^+$, one can conclude from $[\mathfrak{p}_{\mathbb{F}}, e] = [\mathfrak{l}_{\mathbb{F}}, e_0] \oplus \mathfrak{u}_{\mathbb{F}}^+$ that

$$\underline{\dim}\,\mathfrak{p}_{\mathbb{F}} - \underline{\dim}\,[\mathfrak{p}_{\mathbb{F}}, e] = \underline{\dim}\,\mathfrak{l}_{\mathbb{F}} - \underline{\dim}\,[\mathfrak{l}_{\mathbb{F}}, e_0].$$

Then it follows from

$$\underline{\dim}\,\mathfrak{p}_{\mathbb{F}}^e = \underline{\dim}\,\mathfrak{p}_{\mathbb{F}} - \underline{\dim}\,[\mathfrak{p}_{\mathbb{F}}, e], \qquad \underline{\dim}\,\mathfrak{l}_{\mathbb{F}}^{e_0} = \underline{\dim}\,\mathfrak{l}_{\mathbb{F}} - \underline{\dim}\,[\mathfrak{l}_{\mathbb{F}}, e_0]$$

that

$$\underline{\dim}\,\mathfrak{p}_{\mathbb{F}}^e = \underline{\dim}\,\mathfrak{l}_{\mathbb{F}}^{e_0}. \tag{7.3}$$

On the other hand, since $\mathfrak{p}_{\mathbb{F}}$ is a subalgebra of $\mathfrak{g}_{\mathbb{F}}$, it is immediate from $\mathfrak{g}_{\mathbb{F}}^e \subseteq \mathfrak{p}_{\mathbb{F}}$ that $\mathfrak{p}_{\mathbb{F}}^e = \mathfrak{g}_{\mathbb{F}}^e$. Combining this with (7.3), we have $\underline{\dim}\,\mathfrak{g}_{\mathbb{F}}^e = \underline{\dim}\,\mathfrak{l}_{\mathbb{F}}^{e_0}$, proving (1).

(2) Since $\underline{\dim}\,\mathfrak{u}_{\mathbb{F}}^- = \underline{\dim}\,\mathfrak{u}_{\mathbb{F}}^+$, we have $\underline{\dim}\,\mathfrak{g}_{\mathbb{F}} = \underline{\dim}\,\mathfrak{l}_{\mathbb{F}} + 2\underline{\dim}\,\mathfrak{u}_{\mathbb{F}}^+$. Note that $\underline{\dim}\,\mathfrak{g}_{\mathbb{F}}^e = \underline{\dim}\,\mathfrak{g}_{\mathbb{F}} - \underline{\dim}\,[\mathfrak{g}_{\mathbb{F}}, e]$; then we have

$$\begin{aligned}\underline{\dim}\,[\mathfrak{g}_{\mathbb{F}}, e] = \underline{\dim}\,\mathfrak{g}_{\mathbb{F}} - \underline{\dim}\,\mathfrak{g}_{\mathbb{F}}^e &= \underline{\dim}\,\mathfrak{l}_{\mathbb{F}} - \underline{\dim}\,\mathfrak{g}_{\mathbb{F}}^e + 2\underline{\dim}\,\mathfrak{u}_{\mathbb{F}}^+ \\ &= \underline{\dim}\,\mathfrak{l}_{\mathbb{F}} - \underline{\dim}\,\mathfrak{l}_{\mathbb{F}}^{e_0} + 2\underline{\dim}\,\mathfrak{u}_{\mathbb{F}}^+ \\ &= \underline{\dim}\,[\mathfrak{l}_{\mathbb{F}}, e_0] + 2\underline{\dim}\,\mathfrak{u}_{\mathbb{F}}^+,\end{aligned}$$

where the third equality follows from (1). All these complete the proof of (2). □

In the following, we will give an example of the induced nilpotent elements associated with basic Lie superalgebras.

Example 7.5. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a superspace with $\underline{\dim} V = (2,4)$; then $\mathfrak{gl}(2|4) = \mathrm{End}(V)$ is a general linear Lie superalgebra associated with V. Choose ordered bases of $V_{\bar{0}}$ and $V_{\bar{1}}$ that combine to a homogeneous basis for V. Then parameterize such a basis by the set $I(2|4) = \{\bar{1}, \bar{2}; 1, 2, 3, 4\}$. The elementary matrices are accordingly denoted by $e_{i,j}$ with $i, j \in I(2|4)$. With respect to such an ordered basis, $\mathfrak{gl}(2|4)$ can be realized as 6×6 complex matrices of the block form

$$\left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right), \tag{7.4}$$

where a, b, c, d are respectively 2×2, 2×4, 4×2, 4×4 matrices. The even subalgebra $\mathfrak{gl}(2|4)_{\bar{0}}$ consists of matrices of the form (7.4) with $b = c = 0$, while the odd space $\mathfrak{gl}(2|4)_{\bar{1}}$ consists of those with $a = d = 0$.

Let $\mathfrak{h}$ be the typical Cartan subalgebra of $\mathfrak{g} := \mathfrak{gl}(2|4)$ consisting all of diagonal matrices. Let Φ be a root system of $\mathfrak{g}$ relative to $\mathfrak{h}$ whose simple root system

$$\Delta = \{\delta_1 - \delta_2, \delta_2 - \varepsilon_1, \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4\}$$

is distinguished, where $\{\delta_i, \varepsilon_j\}_{i,j}$ is the basis of $\mathfrak{h}^*$ dual to $\{e_{\bar{i},\bar{i}}, e_{j,j}\}_{i,j}$ for $i \in \{1,2\}$ and $j \in \{1,2,3,4\}$. Consider $\Delta' := \Delta \backslash \{\varepsilon_2 - \varepsilon_3\}$, a subset of Δ. Associated with Δ', there are a standard Levi subalgebra and a parabolic subalgebra of $\mathfrak{g}$, which are denoted by $\mathfrak{l}$ and $\mathfrak{p}$ respectively. Then $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}^+$ with

$$\mathfrak{u}^+ := \mathbb{C}e_{\bar{1},3} \oplus \mathbb{C}e_{\bar{1},4} \oplus \mathbb{C}e_{\bar{2},3} \oplus \mathbb{C}e_{\bar{2},4} \oplus \mathbb{C}e_{1,3} \oplus \mathbb{C}e_{1,4} \oplus \mathbb{C}e_{2,3} \oplus \mathbb{C}e_{2,4}.$$

Let $e_0 := e_{\bar{1},\bar{2}} + e_{1,2} + e_{3,4}$ be an element in $\mathfrak{l}$. Then $e = e_0 + e_1$ for $e_1 = e_{2,3} \in \mathfrak{u}^+$. One can verify by a direct computation that $e \in \mathfrak{p}_{\bar{0}}$ is induced from $e_0 \in \mathfrak{l}_{\bar{0}}$.

§7.2. Some conventions

We now make some conventions for accomplishing the "modular p reduction" in connection with the "induced nilpotent elements". First recall the notation in §2.1.2. For a given basic Lie superalgebra $\mathfrak{g}$ over the field $\mathbb{C}$, there exists an algebraic supergroup G satisfying $\mathrm{Lie}(G) = \mathfrak{g}$, and G has an ordinary connected reductive group G_{ev} with $\mathrm{Lie}(G_{\mathrm{ev}}) = \mathfrak{g}_{\bar{0}}$ such that the action of G_{ev} on $\mathfrak{g}$ can be reduced to the adjoint action of $\mathfrak{g}_{\bar{0}}$.

Let $e \in \mathfrak{p}_{\bar{0}}$ be induced from an even nilpotent element e_0 in a proper Levi subalgebra $\mathfrak{l}$. Up to an $\mathrm{Ad}\, G_{\mathrm{ev}}$-isomorphism, Dynkin–Kostant theory ensures that the nilpotent element e may be assumed in the $\mathbb{Z}$-span of the vectors of a Chevalley $\mathbb{Z}$-form $\mathfrak{g}_{\mathbb{Z}}$ associated with the distinguished simple root system (cf. [5, §3.3] for the Chevalley basis of $\mathfrak{g}$). What is more, we can further assume that e is also in the $\mathbb{C}$-span of the positive root vectors relative to a W-equivalent root system of the parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$, where W is the Weyl group of $\mathfrak{g}$. Put such a root system as Φ', with a simple root system $\Delta'=\{\alpha'_1,\ldots,\alpha'_l\}$ (in contrast to the Lie algebra case, it is a super phenomenon that all the systems of simple roots, or the systems of positive roots of $\mathfrak{g}$, are not equivalent under the Weyl group, thus Δ' is not necessarily a distinguished one). Let $(\Phi')^+$ be the corresponding positive system in Φ', and set $(\Phi')^-:=-(\Phi')^+$. Denote by $\mathfrak{h}'$ the corresponding Cartan subalgebra of $\mathfrak{g}$ such that the root system Φ' is relative to $\mathfrak{h}'$ with simple root system Δ'. Write $\mathfrak{g}_{\mathbb{Q}}:=\mathfrak{g}_{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathbb{Q}$ for the $\mathbb{Q}$-span of $\mathfrak{g}_{\mathbb{Z}}$ in $\mathfrak{g}_{\mathbb{C}}$. As $e\in(\mathfrak{g}_{\mathbb{Z}})_{\bar{0}}$, by the same discussion as in [27, §4.2] we can find $f,h\in(\mathfrak{g}_{\mathbb{Q}})_{\bar{0}}$ such that (e,h,f) is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. Then the above settings ensure that $h\in\mathfrak{h}'$. Let $\mathfrak{g}=(\mathfrak{n}')^-\oplus\mathfrak{h}'\oplus(\mathfrak{n}')^+$ be the corresponding triangular decomposition of $\mathfrak{g}$, with $\mathfrak{b}'=\mathfrak{h}'\oplus(\mathfrak{n}')^+$ being the Borel subalgebra associated with Δ'. Let $\mathfrak{g}=\mathfrak{h}'\oplus(\bigoplus_{\alpha\in\Phi'}\mathfrak{g}_\alpha)$ be the root space decomposition of $\mathfrak{g}$ associated with Φ'. By $\mathfrak{sl}_2$-representation theory, one can choose a basis of $\mathfrak{g}$ such that all simple root vectors $X_{\pm\alpha}\in\mathfrak{g}_\alpha$ $(\alpha\in\Delta')$, together with all root vectors $X_\alpha\in\mathfrak{g}_\alpha$ $(\alpha\in\Phi')$ generated by these simple root vectors, and also a basis of $\mathfrak{h}'$ with $\alpha\in\Phi'$ (denote them by $H_\alpha\in\mathfrak{h}'$), are in the $\mathbb{Q}$-span of the vectors of the Chevalley $\mathbb{Z}$-form $\mathfrak{g}_{\mathbb{Z}}$, with e also being in the $\mathbb{Q}$-span of the positive root vectors $X_\alpha\in\mathfrak{g}_\alpha$ with $\alpha\in(\Phi')^+$.

Under the above settings, we can assume that $\Delta'\cap\Phi'(\mathfrak{l})$ is a system of simple roots for the root system $\Phi'(\mathfrak{l})$ of $\mathfrak{l}$ with $\Phi'(\mathfrak{l})$ being a subsystem of Φ'; then we have $\mathfrak{l}=\bigoplus_{\alpha\in\Delta'}\mathbb{C}H_\alpha\oplus(\bigoplus_{\alpha\in\Phi'(\mathfrak{l})}\mathbb{C}X_\alpha)$. Let $(\Phi'(\mathfrak{l}))^+$ be the corresponding positive system of $\mathfrak{l}$ in $\Phi'(\mathfrak{l})$. Put $\mathfrak{p}=\mathfrak{l}+\mathfrak{b}'=\mathfrak{l}\oplus\mathfrak{u}^+$, where $\mathfrak{u}^+$ is the nilradical of $\mathfrak{p}$. It can be easily observed that $\mathfrak{p}$ is the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{h}'$ and all the root spaces $\mathfrak{g}_\alpha$ with $\alpha\in\Delta'$ or $-\alpha\in\Delta'\cap\Phi'(\mathfrak{l})$. Moreover, we have $\mathfrak{u}^+=\bigoplus_{\alpha\in(\Phi')^+\backslash(\Phi'(\mathfrak{l}))^+}\mathbb{C}X_\alpha$, and $\mathfrak{p}=\bigoplus_{\alpha\in\Delta'}\mathbb{C}H_\alpha\oplus(\bigoplus_{\alpha\in\Phi'(\mathfrak{l})\cup(\Phi')^+}\mathbb{C}X_\alpha)$. Since e is in $(\mathfrak{n}')^+$ by the preceding discussion, $\mathfrak{sl}_2$-representation theory yields that $\mathfrak{g}^e\subseteq\mathfrak{p}$.

§7.3. Reduction to the rigid nilpotents: Theorem 1.4 and its proof

Now we can go into the main results of this section. Given a Lie superalgebra $\mathfrak{g}_{\mathbb{F}}$ over $\mathbb{F}=\mathbb{C}$ or $\Bbbk=\overline{\mathbb{F}}_p$ with $p\gg0$, note that we always have $\underline{\dim}\,\mathfrak{g}_{\mathbb{C}}=\underline{\dim}\,\mathfrak{g}_{\Bbbk}$, and so do the subalgebras of $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}_{\Bbbk}$. Therefore, we will not distinguish between them in the following discussion.

Proposition 7.6. *For a basic Lie superalgebra $\mathfrak{g}$ over $\mathbb{C}$, let e be an even nilpotent element in a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$, which is induced from an even nilpotent element e_0 in a proper Levi subalgebra $\mathfrak{l}$. Define $d:=\dim([\mathfrak{l},\mathfrak{l}])_{\bar{1}}-\dim([\mathfrak{l},\mathfrak{l}]^{e_0})_{\bar{1}}$, where $[\mathfrak{l},\mathfrak{l}]^{e_0}$ denotes the centralizer of e_0 in $[\mathfrak{l},\mathfrak{l}]$. When d is even (resp. odd), if the finite W-superalgebra $U([\mathfrak{l},\mathfrak{l}],e_0)$ affords a one-dimensional (resp. two-dimensional) rep-*

resentation, then the finite W-superalgebra $U(\mathfrak{g}, e)$ also admits a one-dimensional (resp. two-dimensional) representation.

The above proposition is a generalization of [29, Theorem 3.1]. Compared with the finite W-algebra case, we have to bypass the machinery of nilpotent orbits for Lie superalgebras in the proof for the lack of related settings. Moreover, the ramifications of the possible minimal dimension for the representations of finite W-superalgebras in Conjecture 1.3 (which resulted from the parity of d) makes the situation much more complicated. So we will still give the details of the proof.

Proof. (1) Let $\mathfrak{u}^+$ be the nilradical of $\mathfrak{p}$ such that $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}^+$ as vector spaces. By Proposition 7.2 one can assume that $e = e_0 + e_1$ for an element $e_1 \in \mathfrak{u}^+$. Retaining all the settings in §7.2, we further assume that $e, e_0, e_1 \in (\mathfrak{g}_{\mathbb{Z}})_{\bar{0}}$.

Let $\mathfrak{g} = \bigoplus_{\alpha\in\Delta'} \mathbb{C}H_\alpha \oplus (\bigoplus_{\alpha\in\Phi'} \mathbb{C}X_\alpha)$ be the root space decomposition of $\mathfrak{g}$ associated with Φ' such that all simple root vectors $X_{\pm\alpha} \in \mathfrak{g}_\alpha$ ($\alpha \in \Delta'$), together with all root vectors $X_\alpha \in \mathfrak{g}_\alpha$ ($\alpha \in \Phi'$) and the semisimple elements $H_\alpha \in \mathfrak{h}'$ ($\alpha \in \Phi'$) generated by them, are in the $\mathbb{Q}$-span of the vectors in the Chevalley $\mathbb{Z}$-form $\mathfrak{g}_{\mathbb{Z}}$, with e also being in the $\mathbb{Q}$-span of the positive root vectors $X_\alpha \in \mathfrak{g}_\alpha$ for $\alpha \in (\Phi')^+$. Set $\mathfrak{g}_{\mathbb{Q}} = \bigoplus_{\alpha\in\Delta'} \mathbb{Q}H_\alpha \oplus (\bigoplus_{\alpha\in\Phi'} \mathbb{Q}X_\alpha)$. Denote by $\mathfrak{h}'_{\mathbb{Q}}$ the $\mathbb{Q}$-span of all the vectors $H_\alpha \in \mathfrak{h}'$ with $\alpha \in \Delta'$, and define $\mathfrak{l}_{\mathbb{Q}} := \mathfrak{h}'_{\mathbb{Q}} \oplus (\bigoplus_{\alpha\in\Phi'(\mathfrak{l})} \mathbb{Q}X_\alpha)$. Obviously $\mathfrak{h}'_{\mathbb{Q}}$ and $\mathfrak{l}_{\mathbb{Q}}$ are subalgebras of $\mathfrak{h}'$ and $\mathfrak{l}$, respectively. One can further define the $\mathbb{Q}$-subalgebras of $\mathfrak{u}^+$ and $\mathfrak{p}$ by

$$\mathfrak{u}^+_{\mathbb{Q}} := \bigoplus_{\alpha\in(\Phi')^+\setminus(\Phi'(\mathfrak{l}))^+} \mathbb{Q}X_\alpha, \qquad \mathfrak{p}_{\mathbb{Q}} := \mathfrak{l}_{\mathbb{Q}} \oplus \mathfrak{u}^+_{\mathbb{Q}},$$

respectively.

We claim that

$$[\mathfrak{p}_{\mathbb{Q}}, e] = [\mathfrak{l}_{\mathbb{Q}}, e_0] \oplus \mathfrak{u}^+_{\mathbb{Q}}. \tag{7.5}$$

(a) First recall that $e = e_0 + e_1$, where $e_0 \in \mathfrak{l}_{\bar{0}}$, $e_1 \in \mathfrak{u}^+_{\bar{0}}$, and $e_0, e_1 \in (\mathfrak{g}_{\mathbb{Z}})_{\bar{0}}$. By the same discussion as in (7.1), one can show that

$$[\mathfrak{p}_{\mathbb{Q}}, e] \subseteq [\mathfrak{l}_{\mathbb{Q}}, e_0] \oplus \mathfrak{u}^+_{\mathbb{Q}} \tag{7.6}$$

as vector spaces.

(b) On the other hand, since the base vectors of $\mathfrak{g}$ that we have chosen can be written as the $\mathbb{Q}$-span of the vectors in the Chevalley $\mathbb{Z}$-form $\mathfrak{g}_{\mathbb{Z}}$ of $\mathfrak{g}$, it follows from $\mathfrak{sl}_2$-representation theory that $\underline{\dim}_{\mathbb{C}}\, \mathfrak{g}^e = \underline{\dim}_{\mathbb{Q}}\, \mathfrak{g}^e_{\mathbb{Q}}$ and $\underline{\dim}_{\mathbb{C}}\, \mathfrak{l}^{e_0} = \underline{\dim}_{\mathbb{Q}}\, \mathfrak{l}^{e_0}_{\mathbb{Q}}$. As $\mathfrak{g}^e \subseteq \mathfrak{p}$ by Definition 7.1(2), combining this with Proposition 7.4(1), one can conclude that $\underline{\dim}_{\mathbb{Q}}\,\mathfrak{p}_{\mathbb{Q}}^e = \underline{\dim}_{\mathbb{Q}}\,\mathfrak{l}_{\mathbb{Q}}^{e_0}$. Thus we have

$$\begin{aligned}\underline{\dim}_{\mathbb{Q}}\,[\mathfrak{p}_{\mathbb{Q}}, e] &= \underline{\dim}_{\mathbb{Q}}\,\mathfrak{p}_{\mathbb{Q}} - \underline{\dim}_{\mathbb{Q}}\,\mathfrak{p}_{\mathbb{Q}}^e \\ &= \underline{\dim}_{\mathbb{Q}}\,\mathfrak{l}_{\mathbb{Q}} - \underline{\dim}_{\mathbb{Q}}\,\mathfrak{l}_{\mathbb{Q}}^{e_0} + \underline{\dim}_{\mathbb{Q}}\,\mathfrak{u}_{\mathbb{Q}}^+ \\ &= \underline{\dim}_{\mathbb{Q}}\,[\mathfrak{l}_{\mathbb{Q}}, e_0] + \underline{\dim}_{\mathbb{Q}}\,\mathfrak{u}_{\mathbb{Q}}^+.\end{aligned} \tag{7.7}$$

Then the claim (7.5) is an immediate corollary of (7.6) and (7.7).

(2) Recall that in §2.2.1 we introduced the admissible ring A. Enlarging A if necessary, one can assume that all the base vectors of $\mathfrak{g}$ given in step (1) (which are in the $\mathbb{Q}$-span of the vectors in the Chevalley $\mathbb{Z}$-form $\mathfrak{g}_{\mathbb{Z}}$ of $\mathfrak{g}$) are also in the A-span of the Chevalley basis of $\mathfrak{g}$. Let $\mathfrak{g}_A$, $\mathfrak{h}'_A$, $\mathfrak{l}_A$, $\mathfrak{u}_A^+$, and $\mathfrak{p}_A$ be the A-span of the corresponding base vectors that define $\mathfrak{g}_{\mathbb{Q}}$, $\mathfrak{h}'_{\mathbb{Q}}$, $\mathfrak{l}_{\mathbb{Q}}$, $\mathfrak{u}_{\mathbb{Q}}^+$ and $\mathfrak{p}_{\mathbb{Q}}$ in step (1), respectively. Enlarging A further, we may assume $[\mathfrak{l}_A, e_0]$ is a direct summand of $\mathfrak{l}_A$, and claim (7.5) shows that $[\mathfrak{p}_A, e] = [\mathfrak{l}_A, e_0] \oplus \mathfrak{u}_A^+$. As $\mathfrak{g}_{\Bbbk} \cong \mathfrak{g}_A \otimes_A \Bbbk$ (cf. §2.3.1), one can further define $\mathfrak{h}'_{\Bbbk}$, $\mathfrak{l}_{\Bbbk}$, $\mathfrak{u}_{\Bbbk}^+$, and $\mathfrak{p}_{\Bbbk}$ in $\mathfrak{g}_{\Bbbk}$ to be the modular counterparts of $\mathfrak{h}'_A$, $\mathfrak{l}_A$, $\mathfrak{u}_A^+$, and $\mathfrak{p}_A$ in $\mathfrak{g}_A$, respectively. Define $\bar{e} := e \otimes_A 1$ and $\bar{e}_0 := e_0 \otimes_A 1$; then $[\mathfrak{p}_{\Bbbk}, \bar{e}] = [\mathfrak{l}_{\Bbbk}, \bar{e}_0] \oplus \mathfrak{u}_{\Bbbk}^+$ for all $p \gg 0$.

Put $\chi_0 := (\bar{e}_0, \cdot)$ and $\chi := (\bar{e}, \cdot)$, which are the linear functions on $\mathfrak{l}_{\Bbbk}$ and $\mathfrak{g}_{\Bbbk}$, respectively. Note that χ vanishes on $\mathfrak{u}_{\Bbbk}^+$ and the restriction of χ to $\mathfrak{l}_{\Bbbk}$ equals χ_0. Denote by $\mathfrak{z}(\mathfrak{l}_{\Bbbk})$ the center of $\mathfrak{l}_{\Bbbk}$ in $\mathfrak{l}_{\Bbbk}$. Since $\mathfrak{l}_{\Bbbk}$ is a Levi subalgebra of $\mathfrak{g}_{\Bbbk}$, we have $\mathfrak{l}_{\Bbbk} = [\mathfrak{l}_{\Bbbk}, \mathfrak{l}_{\Bbbk}] \oplus \mathfrak{z}(\mathfrak{l}_{\Bbbk})$, and

$$U_{\chi_0}(\mathfrak{l}_{\Bbbk}) \cong U_{\chi_0}([\mathfrak{l}_{\Bbbk}, \mathfrak{l}_{\Bbbk}]) \otimes_{\Bbbk} U_0(\mathfrak{z}(\mathfrak{l}_{\Bbbk})) \tag{7.8}$$

as $\Bbbk$-algebras. As $\Phi'(\mathfrak{l}) \subseteq \Phi'$ is the root system of $\mathfrak{l}_{\Bbbk}$, by knowledge of linear algebras one can find an element $\bar{s}$ in the Cartan subalgebra $\mathfrak{h}'_{\Bbbk}$ of $\mathfrak{g}_{\Bbbk}$ such that $\mathfrak{g}_{\Bbbk}^{\bar{s}} = \mathfrak{l}_{\Bbbk}$, where $\mathfrak{g}_{\Bbbk}^{\bar{s}}$ denotes the centralizer of $\bar{s}$ in $\mathfrak{g}_{\Bbbk}$. Moreover, $[\mathfrak{l}_{\Bbbk}, \mathfrak{l}_{\Bbbk}]$ is a direct sum of basic Lie superalgebras, and $\mathfrak{z}(\mathfrak{l}_{\Bbbk}) \subseteq \mathfrak{h}'_{\Bbbk}$ is a toral subalgebra of $\mathfrak{g}_{\Bbbk}$.

(i) Set $\dim \mathfrak{z}(\mathfrak{l}_{\Bbbk}) = k$. As $\mathfrak{z}(\mathfrak{l}_{\Bbbk})$ is a toral subalgebra of $\mathfrak{g}_{\Bbbk}$ with a basis $\{t_1, \ldots, t_k\}$ such that $t_i^{[p]} = t_i$ for all $1 \leqslant i \leqslant k$, then $U_0(\mathfrak{z}(\mathfrak{l}_{\Bbbk})) \cong A_1^{\otimes d}$ where $A_1 \cong \Bbbk[X]/(X^p - X)$ is a p-dimensional commutative semisimple algebra whose irreducible representations are one-dimensional. Hence the $\Bbbk$-algebra $U_0(\mathfrak{z}(\mathfrak{l}_{\Bbbk}))$ has a one-dimensional representation; set it as W.

(ii) When

$$d = \dim[\mathfrak{l}, \mathfrak{l}]_{\bar{1}} - \dim[\mathfrak{l}, \mathfrak{l}]_{\bar{1}}^{e_0} \tag{7.9}$$

is even (resp. odd), the $\mathbb{C}$-algebra $U([\mathfrak{l}, \mathfrak{l}], e_0)$ admits a one-dimensional (resp. two-dimensional) representation by the assumptions in the proposition. Set

$$d' := \dim[\mathfrak{l}, \mathfrak{l}]_{\bar{0}} - \dim[\mathfrak{l}, \mathfrak{l}]_{\bar{0}}^{e_0}. \tag{7.10}$$

Since Proposition 6.1 holds for a direct sum of basic Lie superalgebras in the case that all the p-characters associated with them are nilpotent (cf. [35, Remark 4.6]), by the same discussion as the proof of Theorem 1.6 one can check that Theorem 1.6 is also true in this situation. Thus the $\mathbb{k}$-algebra $U_{\chi_0}([\mathfrak{l}_\mathbb{k},\mathfrak{l}_\mathbb{k}])$ affords a representation of dimension $p^{\frac{d'}{2}}2^{\lfloor\frac{d}{2}\rfloor}$ whether d is even or odd; set it as V.

Combining (i) and (ii), one can conclude that $V\boxtimes W$ is a representation of the $\mathbb{k}$-algebra $U_{\chi_0}(\mathfrak{l}_\mathbb{k})\cong U_{\chi_0}([\mathfrak{l}_\mathbb{k},\mathfrak{l}_\mathbb{k}])\otimes_\mathbb{k} U_0(\mathfrak{z}(\mathfrak{l}_\mathbb{k}))$ with dimension $p^{\frac{d'}{2}}2^{\lfloor\frac{d}{2}\rfloor}$ whether d is even or odd; set this module as V'. Write

$$d_0':=\dim\mathfrak{l}_{\bar{0}}-\dim\mathfrak{l}_{\bar{0}}^{\bar{e}_0},\qquad d_1':=\dim\mathfrak{l}_{\bar{1}}-\dim\mathfrak{l}_{\bar{1}}^{\bar{e}_0}. \tag{7.11}$$

Since $\mathfrak{l}_\mathbb{k}=[\mathfrak{l}_\mathbb{k},\mathfrak{l}_\mathbb{k}]\oplus\mathfrak{z}(\mathfrak{l}_\mathbb{k})$ and $\mathfrak{z}(\mathfrak{l}_\mathbb{k})$ is the center of $\mathfrak{l}_\mathbb{k}$, it can be easily observed that $d_0'=d'$ and $d_1'=d$ by direct computation.

(3) For the $U_{\chi_0}(\mathfrak{l}_\mathbb{k})$-module V', we may regard it as a $U_\chi(\mathfrak{p}_\mathbb{k})$-module with the trivial action of $\mathfrak{u}_\mathbb{k}^+$ and consider the induced $U_\chi(\mathfrak{g}_\mathbb{k})$-module $\widetilde{V}:=U_\chi(\mathfrak{g}_\mathbb{k})\otimes_{U_\chi(\mathfrak{p}_\mathbb{k})}V'$. Set

$$d_0:=\dim\mathfrak{g}_{\bar{0}}-\dim\mathfrak{g}_{\bar{0}}^{e},\, d_1:=\dim\mathfrak{g}_{\bar{1}}-\dim\mathfrak{g}_{\bar{1}}^{e}.$$

Since

$$\underline{\dim}\,\mathfrak{g}_\mathbb{k}^{\bar{e}}=\underline{\dim}\,\mathfrak{l}_\mathbb{k}^{\bar{e}_0}$$

by Proposition 7.4(1), we have

$$\underline{\dim}\,\mathfrak{g}_\mathbb{k}-\underline{\dim}\,\mathfrak{g}_\mathbb{k}^{\bar{e}}=\underline{\dim}\,\mathfrak{g}_\mathbb{k}-\underline{\dim}\,\mathfrak{l}_\mathbb{k}^{\bar{e}_0}=2\underline{\dim}\,\mathfrak{u}_\mathbb{k}^{+}+(\underline{\dim}\,\mathfrak{l}_\mathbb{k}-\underline{\dim}\,\mathfrak{l}_\mathbb{k}^{\bar{e}_0}). \tag{7.12}$$

It follows from (7.12) that d_1' and d_1 have the same parity. Applying Proposition 7.4(1) again, it follows from the PBW theorem that

$$\begin{aligned}\dim\widetilde{V}&=p^{\dim(\mathfrak{g}_\mathbb{k})_{\bar{0}}-\dim(\mathfrak{p}_\mathbb{k})_{\bar{0}}}2^{\dim(\mathfrak{g}_\mathbb{k})_{\bar{1}}-\dim(\mathfrak{p}_\mathbb{k})_{\bar{1}}}\cdot p^{\frac{d'}{2}}2^{\lfloor\frac{d}{2}\rfloor}\\&=p^{\frac{\dim(\mathfrak{g}_\mathbb{k})_{\bar{0}}-\dim(\mathfrak{l}_\mathbb{k})_{\bar{0}}+\dim(\mathfrak{l}_\mathbb{k})_{\bar{0}}-\dim(\mathfrak{l}_\mathbb{k}^{\bar{e}_0})_{\bar{0}}}{2}}2^{\lfloor\frac{\dim(\mathfrak{g}_\mathbb{k})_{\bar{1}}-\dim(\mathfrak{l}_\mathbb{k})_{\bar{1}}+\dim(\mathfrak{l}_\mathbb{k})_{\bar{1}}-\dim(\mathfrak{l}_\mathbb{k}^{\bar{e}_0})_{\bar{1}}}{2}\rfloor}\\&=p^{\frac{\dim(\mathfrak{g}_\mathbb{k})_{\bar{0}}-\dim(\mathfrak{g}_\mathbb{k}^{\bar{e}})_{\bar{0}}}{2}}2^{\lfloor\frac{\dim(\mathfrak{g}_\mathbb{k})_{\bar{1}}-(\dim\mathfrak{g}_\mathbb{k}^{\bar{e}})_{\bar{1}}}{2}\rfloor}\\&=p^{\frac{d_0}{2}}2^{\lfloor\frac{d_1}{2}\rfloor}.\end{aligned}$$

Then Proposition 3.9 entails that the reduced W-superalgebra $T(\mathfrak{g}_\mathbb{k},\bar{e})$ admits one-dimensional (resp. two-dimensional) representations when d_1 is even (resp. odd). Since this holds for all $p\gg0$, Lemma 4.6 (resp. Lemma 5.7) yields that the finite W-superalgebra $U(\mathfrak{g},e)$ affords a one-dimensional (resp. two-dimensional) representation when d_1 is even (resp. odd). As d and d_1 have the same parity, this completes the proof. □

Recall that in [29, Theorem 3.1], Premet reduced the problem of the existence of one-dimensional representations of finite W-algebras to one with rigid elements.

One may expect that Proposition 7.6 can play the same role for the representations of finite W-superalgebras. In fact, this idea is indeed feasible. We put it as follows:

Retain all the notation introduced in Proposition 7.6 and its proof. Recall that $[\mathfrak{l}_{\mathbb{k}}, \mathfrak{l}_{\mathbb{k}}]$ is a direct sum of basic Lie superalgebras, and so is the Lie superalgebra $[\mathfrak{l}, \mathfrak{l}]$ over $\mathbb{C}$. Set $[\mathfrak{l}, \mathfrak{l}] = \bigoplus_{i=1}^{r} \mathfrak{l}_i$, where $\mathfrak{l}_i$ is a basic Lie superalgebra for each $1 \leqslant i \leqslant r$, and let $e_0 = e_1 + \cdots + e_r$ be the decomposition of e_0 with each $e_i \in \mathfrak{l}_i$ for $1 \leqslant i \leqslant r$. Write

$$(d_0)_i := \dim(\mathfrak{l}_i)_{\bar{0}} - \dim(\mathfrak{l}_i^{e_i})_{\bar{0}}, \qquad (d_1)_i := \dim(\mathfrak{l}_i)_{\bar{1}} - \dim(\mathfrak{l}_i^{e_i})_{\bar{1}} \tag{7.13}$$

for $1 \leqslant i \leqslant r$.

Retain the notation of (7.9), (7.10), and (7.13). As $d'_0 = d'$ and $d'_1 = d$, we have $d'_0 = \sum_{i=1}^{r} (d_0)_i$ and $d'_1 = \sum_{i=1}^{r} (d_1)_i$. For $1 \leqslant i \leqslant r$, denote by $U_{\chi_i}(\mathfrak{l}_i, e_i)$ the finite W-superalgebras over $\mathbb{C}$ associated with each pair $(\mathfrak{l}_i, e_i)$. By the same discussion as in (6.13), one can conclude that

$$U([\mathfrak{l}, \mathfrak{l}], e_0) \cong \bigotimes_{i=1}^{r} U(\mathfrak{l}_i, e_i) \tag{7.14}$$

as $\mathbb{C}$-algebras.

In order to reduce proving Conjecture 1.3 to the case of basic Lie superalgebras with rigid nilpotent elements, we must start from the representations of the $\mathbb{C}$-algebras $U(\mathfrak{l}_i, e_i)$ for $1 \leqslant i \leqslant r$, instead of those of finite W-superalgebra $U([\mathfrak{l}, \mathfrak{l}], e_0)$ associated with the direct sum of the $\mathfrak{l}_i$.

Now we will discuss the relationship between the representations of these algebras. Based on the parity of d, we will consider each case separately.

(a) When d is even, applying (7.14) it can be easily verified that all the $\mathbb{C}$-algebras $U(\mathfrak{l}_i, e_i)$ for $1 \leqslant i \leqslant r$ afford one-dimensional representations if and only if the $\mathbb{C}$-algebra $U([\mathfrak{l}, \mathfrak{l}], e_0)$ has a one-dimensional representation. Therefore, in this case we have already achieved our goal just by applying Proposition 7.6.

Remark 7.7. Note that when $\mathfrak{l}$ is a Lie algebra, we have $d = 0$. Thus the Lie algebra case belongs to this situation.

(b) When d is odd, the situation becomes much more complicated. In fact, we cannot just restrict all our attention to the odd case. The major difficulty comes from the parities of the d_i for $1 \leqslant i \leqslant r$. What is more, even if all the $\mathbb{C}$-algebras $U(\mathfrak{l}_i, e_i)$ for $1 \leqslant i \leqslant r$ afford two-dimensional representations, we still cannot ensure that the $\mathbb{C}$-algebra $U([\mathfrak{l}, \mathfrak{l}], e_0)$ has a two-dimensional representation without careful inspection.

Now let us make further investigations of the case when d is odd. Recall that $\mathfrak{g}_{\mathbb{k}}^{\bar{s}} = \mathfrak{l}_{\mathbb{k}} = [\mathfrak{l}_{\mathbb{k}}, \mathfrak{l}_{\mathbb{k}}] \oplus \mathfrak{z}(\mathfrak{l}_{\mathbb{k}})$ by the proof of Proposition 7.6, and so do their counterparts over $\mathbb{C}$, i.e., $\mathfrak{g}^s = \mathfrak{l} = [\mathfrak{l}, \mathfrak{l}] \oplus \mathfrak{z}(\mathfrak{l})$. Then claim (6.22) shows that at most one of the $(d_1)_i$ is odd for $1 \leqslant i \leqslant r$. Without loss of generality, we can assume that $(d_1)_1$ is odd, and all the $(d_1)_i$ are even for $2 \leqslant i \leqslant r$. Since $(d_1)_1$ is odd, Proposition 3.7 yields that $U(\mathfrak{l}_1, e_1)$ cannot afford a one-dimensional representation. If we assume that the $\mathbb{C}$-algebra $U(\mathfrak{l}_1, e_1)$ has a two-dimensional representation, and each $\mathbb{C}$-algebra $U(\mathfrak{l}_i, e_i)$ for $2 \leqslant i \leqslant r$ has a one-dimensional representation, by the same discussion as step (2-ii) in the proof of Theorem 6.7 endowed with $l = 1$, one can conclude from (7.14) that the $\mathbb{C}$-algebra $U([\mathfrak{l}, \mathfrak{l}], e_0)$ has a two-dimensional representation.

Now we can complete the proof Theorem 1.4, which follows from Proposition 7.6 and all the considerations as above.

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