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The Hadamard Product and the Karcher Mean of Positive Invertible Operators

by

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Abstract

In this paper, we show several operator inequalities involving the Hadamard product and the Karcher mean of n (≥ 3) positive invertible operators on a separable Hilbert space, which are regarded as an n-variable operator version of results due to Ando and Aujla–Vasudeva. As applications, we show estimates from above for an n-variable version of the Fiedler-type theorem due to Fujii. Moreover, we show an n-variable version of the majorization relation due to Ando for the Hadamard product via the Karcher mean of n (≥ 3) positive-definite matrices.

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§1. Introduction

For n-square matrices $A = (a_{ij})$ and $B = (b_{ij})$, their Hadamard product is the n-square matrix of entrywise products

$$A \circ B = (a_{ij} b_{ij}).$$

For Hermitian matrices X and Y, we write that the order relation $X \leq Y$ if $Y - X$ is positive semidefinite, and $X < Y$ if $Y - X$ is positive definite.

Many researchers have studied various matrix inequalities involving the Hadamard product and the geometric matrix mean of positive-definite matrices: Fiedler [9] showed that if A is a positive-definite matrix, then

$$A \circ A^{-1} \geq I. \tag{1.1}$$

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As a generalization of the Fiedler inequality (1.1), Ando [1, 2] showed better estimates from below for the Hadamard product of positive-definite matrices A, B by using the geometric matrix mean: For each $\alpha \in [0,1]$,

$$A \circ B \geq (A \sharp_\alpha B) \circ (A \sharp_{1-\alpha} B). \tag{1.2}$$

In particular,

$$A \circ B \geq (A \sharp B) \circ (A \sharp B), \tag{1.3}$$

where the weighted geometric matrix mean $A \sharp_\alpha B$ for A, $B > 0$ and $\alpha \in [0,1]$ is defined by

$$A \sharp_\alpha B = A^{1/2} \left(A^{-1/2} B A^{-1/2}\right)^\alpha A^{1/2}$$

and we denote $A \sharp_{1/2} B$ simply by $A \sharp B$. In fact, put $B = A^{-1}$ in (1.3), then (1.3) implies the Fiedler inequality (1.1). Afterwards, Aujla and Vasudeva [3] extended results due to Ando and Fiedler as follows: If A, B, C and D are positive definite, then

$$(A \circ B) \sharp (C \circ D) \geq (A \sharp C) \circ (B \sharp D). \tag{1.4}$$

Results (1.1), (1.2) and (1.4) mentioned above hold for operators; also see [19, pp. 175–178]. In this paper, we try to consider an n-variable operator version of (1.1), (1.2) and (1.4) by virtue of the Karcher mean. Recall the Hadamard product for operators on a separable Hilbert space H: Let $\{e_j\}$ be an orthonormal basis of H and $A_1 \otimes A_2 \otimes \cdots \otimes A_n$ be the tensor product of operators $A_1, A_2, \ldots, A_n$ on H with regard to $\{e_j\}$. Let $U_n : H \mapsto H \otimes H \otimes \cdots \otimes H$ (n times) be the isometry such that $U_n e_j = e_j \otimes e_j \otimes \cdots \otimes e_j$ (n times). Following [18, 10], the Hadamard product $A_1 \circ A_2 \circ \cdots \circ A_n$ with regard to $\{e_j\}$ is expressed as

$$A_1 \circ A_2 \circ \cdots \circ A_n = U_n^*(A_1 \otimes A_2 \otimes \cdots \otimes A_n)U_n. \tag{1.5}$$

The most important property is commutativity of Hadamard multiplication:

$$A_1 \circ A_2 \circ \cdots \circ A_n = A_{\sigma(1)} \circ A_{\sigma(2)} \circ \cdots \circ A_{\sigma(n)}$$

for any permutation σ. Recall that $A \circ I$ is just the diagonal operator formed from A, where I stands for the identity operator, and

$$(A_1 \circ A_2 \circ \cdots \circ A_n) \circ I = (A_1 \circ I)(A_2 \circ I) \cdots (A_n \circ I).$$

Next, recall the Karcher mean of positive invertible operators: In 2014, Lawson and Lim [14] established the formulation of the geometric mean for n (≥ 3) positive invertible operators on a separable Hilbert space, which is a nice extension of the geometric operator mean in the Kubo–Ando theory. They showed that there exists a unique positive invertible solution of the Karcher equation

$$\sum_{i=1}^{n} \omega_i \log X^{-1/2} A_i X^{-1/2} = 0 \tag{1.6}$$

for given n positive invertible operators $A_1, \dots, A_n$, where $\omega = (\omega_1, \dots, \omega_n)$ is a weight vector, i.e., $\omega_1, \dots, \omega_n \geq 0$ and $\sum_{i=1}^{n} \omega_i = 1$. We say the solution X of (1.6) is the Karcher mean for n positive invertible operators $A_1, \dots, A_n$ and denote it by $G_{\mathrm{K}}(\omega; A_1, \dots, A_n)$. In particular, in the case $\omega = (1/n, \dots, 1/n)$, we denote it by $G_{\mathrm{K}}(A_1, \dots, A_n)$. In the case $n = 2$, the Karcher mean $G_{\mathrm{K}}((1 - \alpha, \alpha); A, B)$ coincides with the weighted geometric operator mean $A \sharp_\alpha B$. We list some properties of the Karcher mean that we need later; also see [5, 6, 16]:

- (P1) consistency with scalars: $G_{\mathrm{K}}(\omega; A_1, \dots, A_n) = A_1^{\omega_1} \cdots A_n^{\omega_n}$ if the A_i's commute;
- (P2) joint homogeneity: $G_{\mathrm{K}}(\omega; a_1 A_1, \dots, a_n A_n) = a_1^{\omega_1} \cdots a_n^{\omega_n} G_{\mathrm{K}}(\omega; A_1, \dots, A_n)$;
- (P3) permutation invariance: $G_{\mathrm{K}}(\omega_\sigma; A_{\sigma(1)}, \dots, A_{\sigma(n)}) = G_{\mathrm{K}}(\omega; A_1, \dots, A_n)$ where $\omega_\sigma = (\omega_{\sigma(1)}, \dots, \omega_{\sigma(n)})$ and σ is any permutation;
- (P4) transformer inequality: $T^* G_{\mathrm{K}}(\omega; A_1, \dots, A_n) T \leq G_{\mathrm{K}}(\omega; T^* A_1 T, \dots, T^* A_n T)$ for every operator T;
- (P5) self-duality: $G_{\mathrm{K}}(\omega; A_1^{-1}, \dots, A_n^{-1})^{-1} = G_{\mathrm{K}}(\omega; A_1, \dots, A_n)$;
- (P6) information monotonicity: $\Phi(G_{\mathrm{K}}(\omega; A_1, \dots, A_n)) \leq G_{\mathrm{K}}(\omega; \Phi(A_1), \dots, \Phi(A_n))$ for any unital positive linear map Φ;
- (P7) arithmetic–geometric–harmonic weighted mean inequality:

$$\left(\sum_{i=1}^{n} \omega_i A_i^{-1} \right)^{-1} \leq G_{\mathrm{K}}(\omega; A_1, \dots, A_n) \leq \sum_{i=1}^{n} \omega_i A_i.$$

We refer the reader to [7, 8, 12, 20] for more information on the Karcher mean.

In this paper, we show several operator inequalities involving the Hadamard product and the Karcher mean of n (≥ 3) positive invertible operators on the Hilbert space, which are regarded as n-variable version of results due to Ando (1.2) and Aujla–Vasudeva (1.4). As applications, we show estimates from above for an n-variable version of the Fiedler-type inequality. Moreover, we show an n-variable version of the majorization relation for the Hadamard product via the Karcher mean of n positive-definite matrices.

§2. Main results

First of all, we start with the following lemma, which is a key lemma in this paper:

Lemma 2.1. *For any integer $n \geq 2$, let $A_i, B_i, \ldots, Z_i$ be positive invertible operators for $i = 1, \ldots, n$ and $\omega = (\omega_1, \ldots, \omega_n)$ a weight vector. Then*

$$\begin{aligned} &G_{\mathrm{K}}(\omega; A_1 \otimes B_1 \otimes \cdots \otimes Z_1, A_2 \otimes B_2 \otimes \cdots \otimes Z_2, \ldots, A_n \otimes B_n \otimes \cdots \otimes Z_n) \\ &\quad = G_{\mathrm{K}}(\omega; A_1, \ldots, A_n) \otimes G_{\mathrm{K}}(\omega; B_1, \ldots, B_n) \otimes \cdots \otimes G_{\mathrm{K}}(\omega; Z_1, \ldots, Z_n). \end{aligned}$$

Proof. Note that $\log(A \otimes B) = (\log A) \otimes I + I \otimes (\log B)$ for positive invertible operators A and B. In fact, since $A \otimes I$ and $I \otimes B$ commute, we have

$$\log(A \otimes B) = \log(A \otimes I)(I \otimes B) = \log(A \otimes I) + \log(I \otimes B) = (\log A) \otimes I + I \otimes (\log B).$$

Let k be the cardinal of $\{A, B, \ldots, Z\}$. Put $X_1 = G_{\mathrm{K}}(\omega; A_1, \ldots, A_n)$, $X_2 = G_{\mathrm{K}}(\omega; B_1, \ldots, B_n), \ldots, X_k = G_{\mathrm{K}}(\omega; Z_1, \ldots, Z_n)$. Then we have

$$\begin{aligned} &\sum_{i=1}^{n} \omega_i \log(X_1 \otimes X_2 \otimes \cdots \otimes X_k)^{-1/2}(A_i \otimes B_i \otimes \cdots \otimes Z_i)(X_1 \otimes X_2 \otimes \cdots \otimes X_k)^{-1/2} \\ &= \sum_{i=1}^{n} \omega_i \log \left[X_1^{-1/2} A_i X_1^{-1/2} \otimes X_2^{-1/2} B_i X_2^{-1/2} \otimes \cdots \otimes X_k^{-1/2} Z_i X_k^{-1/2}\right] \\ &= \sum_{i=1}^{n} \omega_i \log \Big[(X_1^{-1/2} A_i X_1^{-1/2} \otimes I \otimes \cdots \otimes I)(I \otimes X_2^{-1/2} B_i X_2^{-1/2} \otimes I \otimes \cdots \otimes I) \\ &\qquad\qquad \cdots (I \otimes \cdots \otimes I \otimes X_k^{-1/2} Z_i X_k^{-1/2})\Big] \\ &= \sum_{i=1}^{n} \omega_i \Big[\log(X_1^{-1/2} A_i X_1^{-1/2} \otimes I \otimes \cdots \otimes I) + \log(I \otimes X_2^{-1/2} B_i X_2^{-1/2} \otimes \cdots \otimes I) \\ &\qquad + \cdots + \log(I \otimes \cdots \otimes I \otimes X_k^{-1/2} Z_i X_k^{-1/2})\Big] \\ &= \left(\sum_{i=1}^{n} \omega_i \log X_1^{-1/2} A_i X_1^{-1/2}\right) \otimes I \otimes \cdots \otimes I \\ &\qquad + I \otimes \left(\sum_{i=1}^{n} \omega_i \log X_2^{-1/2} B_i X_2^{-1/2}\right) \otimes I \otimes \cdots \otimes I \\ &\qquad + \cdots + I \otimes \cdots \otimes I \otimes \left(\sum_{i=1}^{n} \omega_i \log X_k^{-1/2} Z_i X_k^{-1/2}\right) \\ &= 0, \end{aligned}$$

because X_j is the solution of the Karcher equation (1.6) for $j = 1, 2, \dots, k$. By the uniqueness of the Karcher mean, we have the desired equation:

$$G_{\mathrm{K}}(\omega; A_1 \otimes B_1 \otimes \cdots \otimes Z_1, A_2 \otimes B_2 \otimes \cdots \otimes Z_2, \dots, A_n \otimes B_n \otimes \cdots \otimes Z_n)$$
$$= X_1 \otimes X_2 \otimes \cdots \otimes X_k.$$

□

We shall use, for convenience, the notation

$$\prod_{i=1}^{n} \circ A_i = A_1 \circ A_2 \circ \cdots \circ A_n$$

and

$$\prod_{i=1}^{n} \circ A = A \circ A \circ \cdots \circ A \quad (n \text{ times}).$$

For a weight vector $\omega = (\omega_1, \dots, \omega_n)$,

$$\omega_{\sigma_i} = (\omega_{\sigma_i(1)}, \omega_{\sigma_i(2)}, \dots, \omega_{\sigma_i(n)}) \qquad \text{for } 1 \le i \le n, \tag{2.1}$$

where circular permutations $\sigma_i(k) = k - i + 1$ for $1 \le k \le n$.

By Lemma 2.1, we show an n-variable version of the Ando inequality (1.2) by using the Karcher mean:

Theorem 2.2. *Let $A_1, \dots, A_n$ be positive invertible operators and $\omega=(\omega_1, \dots, \omega_n)$ a weight vector. Then*

$$\prod_{i=1}^{n} \circ\ A_i \ge \prod_{i=1}^{n} \circ\ G_{\mathrm{K}}(\omega_{\sigma_i}; A_1, \dots, A_n),$$

where weight vectors ω_{σ_i} are defined as (2.1) for $1 \le i \le n$. In particular,

$$\prod_{i=1}^{n} \circ\ A_i \ge \prod_{i=1}^{n} \circ\ G_{\mathrm{K}}(A_1, \dots, A_n).$$

Proof. It follows that

$$\prod_{i=1}^{n} \circ\ A_i = G_{\mathrm{K}}(\omega; A_1 \circ \cdots \circ A_n, A_1 \circ \cdots \circ A_n, \dots, A_1 \circ \cdots \circ A_n)$$

by consistency with scalars (P1) for the Karcher mean

$$= G_{\mathrm{K}}(\omega; A_1 \circ A_2 \circ \cdots \circ A_n, A_2 \circ A_3 \circ \cdots \circ A_n \circ A_1, \dots, A_n \circ A_1 \circ \cdots \circ A_{n-1})$$

by commutativity of Hadamard multiplication

$$= G_{\mathrm{K}}\big(\omega; U_n^*(A_1 \otimes \cdots \otimes A_n)U_n, U_n^*(A_2 \otimes \cdots A_n \otimes A_1)U_n, \dots,$$
$$U_n^*(A_n \otimes A_1 \otimes \cdots \otimes A_{n-1})U_n\big)$$

$$\geq U_n^* G_{\mathrm{K}}(\omega; A_1 \otimes \cdots \otimes A_n, A_2 \otimes \cdots \otimes A_n \otimes A_1, \cdots, A_n \otimes \cdots \otimes A_{n-1}) U_n$$

by transformer inequality (P4) for the Karcher mean

$$= U_n^* \left[G_{\mathrm{K}}(\omega; A_1, \ldots, A_n) \otimes G_{\mathrm{K}}(\omega; A_2, \ldots, A_n, A_1) \otimes \cdots \otimes G_{\mathrm{K}}(\omega; A_n, A_1, \ldots, A_{n-1}) \right] U_n \quad \text{by Lemma 2.1}$$

$$= G_{\mathrm{K}}(\omega; A_1, \ldots, A_n) \circ G_{\mathrm{K}}(\omega; A_2, \ldots, A_n, A_1) \circ \cdots \circ G_{\mathrm{K}}(\omega; A_n, \ldots, A_{n-1})$$

$$= \prod_{i=1}^{n} \circ\, G_{\mathrm{K}}(\omega_{\sigma_i}; A_1, \ldots, A_n)$$

by permutation invariance (P3) for the Karcher mean.

□

Remark 2.3. The main result in Theorem 2.2 is the same as [15, Theorem 4.5] due to Lee and Kim. They showed it by using the fact that the Karcher mean is the limit of a sequence of weighted inductive means in [17], while we show it by using the fact that the Karcher mean is a unique solution of the Karcher equation (1.6).

We show an n-variable version of the Aujla–Vasudeva inequality (1.4):

Theorem 2.4. *For any integer $n \geq 2$, let $A_i, B_i, \ldots, Z_i$ be positive invertible operators for $i = 1, \ldots, n$ and $\omega = (\omega_1, \ldots, \omega_n)$ a weight vector. Then*

$$\begin{aligned} &G_{\mathrm{K}}(\omega; A_1 \circ B_1 \circ \cdots \circ Z_1, A_2 \circ B_2 \circ \cdots \circ Z_2, \ldots, A_n \circ B_n \circ \cdots \circ Z_n) \\ &\qquad \geq G_{\mathrm{K}}(\omega; A_1, \ldots, A_n) \circ G_{\mathrm{K}}(\omega; B_1, \ldots, B_n) \circ \cdots \circ G_{\mathrm{K}}(\omega; Z_1, \ldots, Z_n). \end{aligned}$$

Proof. It follows from Lemma 2.1 and the transformer inequality (P4) that

$$\begin{aligned} &G_{\mathrm{K}}(\omega; A_1, \ldots, A_n) \circ G_{\mathrm{K}}(\omega; B_1, \ldots, B_n) \circ \cdots \circ G_{\mathrm{K}}(\omega; Z_1, \ldots, Z_n) \\ &\quad = U^* \left[G_{\mathrm{K}}(\omega; A_1, \ldots, A_n) \otimes G_{\mathrm{K}}(\omega; B_1, \ldots, B_n) \otimes \cdots \otimes G_{\mathrm{K}}(\omega; Z_1, \ldots, Z_n) \right] U \\ &\quad = U^* \left[G_{\mathrm{K}}(\omega; A_1 \otimes B_1 \otimes \cdots \otimes Z_1, \ldots, A_n \otimes B_n \otimes \cdots \otimes Z_n) \right] U \\ &\quad \leq G_{\mathrm{K}}(\omega; U^*(A_1 \otimes B_1 \otimes \cdots \otimes Z_1) U, \ldots, U^*(A_n \otimes B_n \otimes \cdots \otimes Z_n) U) \\ &\quad = G_{\mathrm{K}}(\omega; A_1 \circ B_1 \circ \cdots \circ Z_1, \ldots, A_n \circ B_n \circ \cdots \circ Z_n). \end{aligned}$$

□

§3. Applications

In [10], Fujii showed the following Fiedler-type theorem for operators: If $\alpha, \beta \in \mathbb{R}$ and $s + t = 1$ for nonnegative numbers t and s, then

$$A^\alpha \circ A^\beta \geq A^{t\alpha + s\beta} \circ A^{s\alpha + t\beta} \tag{3.1}$$

for every positive operator A. In particular, we have the Fiedler inequality (1.1) as $\alpha = 1$, $\beta = -1$, $t = s = \frac{1}{2}$.

By Theorem 2.2, we show an n-variable version of Fujii's result (3.1):

Corollary 3.1. *Let A be a positive invertible operator, $\omega = (\omega_1, \dots, \omega_n)$ a weight vector and $a_1, \dots, a_n \in \mathbb{R}$. Then*

$$\prod_{i=1}^{n} \circ\ A^{a_i} \geq \prod_{i=1}^{n} \circ\ A^{\sum_{k=1}^{n} \omega_{\sigma_i(k)} a_k},$$

where weight vectors ω_{σ_i} are defined as (2.1) for $1 \leq i \leq n$. In particular, if $a_1, \dots, a_n \in \mathbb{R}$ such that $\sum_{i=1}^{n} a_i = 0$, then

$$A^{a_1} \circ \cdots \circ\ A^{a_n} \geq I.$$

Proof. It follows from Theorem 2.2 and consistency with scalars (P1) for the Karcher mean that

$$\begin{aligned}\prod_{i=1}^{n} \circ\ A^{a_i} &\geq \prod_{i=1}^{n} \circ\ G_{\mathrm{K}}(\omega_{\sigma_i}; A^{a_1}, \dots, A^{a_n}) \\ &= \prod_{i=1}^{n} \circ\ A^{\omega_{\sigma_i(1)} a_1} A^{\omega_{\sigma_i(2)} a_2} \cdots A^{\omega_{\sigma_i(n)} a_n} \\ &= \prod_{i=1}^{n} \circ\ A^{\sum_{k=1}^{n} \omega_{\sigma_i(k)} a_k}.\end{aligned}$$

In particular, if $a_1, \dots, a_n \in \mathbb{R}$ such that $\sum_{i=1}^{n} a_i = 0$ and $\omega = (1/n, 1/n, \dots, 1/n)$, then we have $A^{a_1} \circ \cdots \circ\ A^{a_n} \geq I$. □

In [1], Ando showed estimates from above for Hadamard products by diagonal matrices: If A_i is positive definite ($i = 1, 2, \dots, n, n \geq 2$) and if $p_1 \geq 1$ and $\sum_{i=1}^{n} 1/p_i \leq 1$, then

$$\prod_{i=1}^{n} \circ A_i \leq \prod_{i=1}^{n} (A_i^{p_i} \circ I)^{1/p_i}. \tag{3.2}$$

In particular,

$$\prod_{i=1}^{n} \circ A_i \leq \prod_{i=1}^{n} (A_i^{n} \circ I)^{1/n}.$$

By virtue of the Karcher mean, we present a simple proof of the operator version of (3.2):

Corollary 3.2. *Let* $A_1, A_2, \ldots, A_n$ *be positive invertible operators for* $n \geq 2$. *If* $p_i \geq 1$ *for* $i = 1, 2, \ldots, n$ *such that* $\sum_{i=1}^n \frac{1}{p_i} = 1$, *then*

$$\prod_{i=1}^n \circ A_i \leq \prod_{i=1}^n (A_i^{p_i} \circ I)^{1/p_i}.$$

In particular,

$$\prod_{i=1}^n \circ A_i \leq \prod_{i=1}^n (A_i^n \circ I)^{1/n}.$$

Proof. Put a weight vector $\omega = (\frac{1}{p_1}, \frac{1}{p_2}, \ldots, \frac{1}{p_n})$. Then it follows from Theorem 2.2 and consistency with scalars (P1) for the Karcher mean that

$$\begin{aligned} &A_1^{1/p_1} \circ A_2^{1/p_2} \circ \cdots \circ A_n^{1/p_n} \\ &\quad = G_K(\omega; A_1, I, \ldots, I) \circ G_K(\omega; I, A_2, I, \ldots, I) \circ \cdots \circ G_K(\omega; I, \ldots, I, A_n) \\ &\quad \leq G_K(\omega; A_1 \circ I \circ \cdots \circ I, I \circ A_2 \circ I \circ \cdots \circ I, \ldots, I \circ \cdots \circ I \circ A_n) \\ &\quad = G_K(\omega; A_1 \circ I, A_2 \circ I, \ldots, A_n \circ I) \\ &\quad = (A_1 \circ I)^{1/p_1} (A_2 \circ I)^{1/p_2} \cdots (A_n \circ I)^{1/p_n}. \end{aligned}$$

If we replace A_i by $A_i^{p_i}$ for $i = 1, 2, \ldots, n$, then we have the desired inequality. □

Corollary 3.3. *Let* $A_1, A_2, \ldots, A_n$ *be positive invertible operators for* $n \geq 2$. *If* $p_i \geq 1$ *for* $i = 1, 2, \ldots, n$ *such that* $\sum_{i=1}^n \frac{1}{p_i} \leq 1$, *then*

$$\prod_{i=1}^n \circ A_i \leq \prod_{i=1}^n (A_i^{p_i} \circ I)^{1/p_i}.$$

Proof. If we take $q \geq 1$ such that $\frac{1}{q} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} = 1$, we have $q \leq p_1$ and so it follows from [19, Corollary 1.22] that

$$A_1^q \circ I \leq (A_1^{p_1} \circ I)^{q/p_1}.$$

Thus we have $(A_1^q \circ I)^{1/q} \leq (A_1^{p_1} \circ I)^{1/p_1}$ and it follows from Corollary 3.2 that

$$\begin{aligned} \prod_{i=1}^n \circ A_i &\leq (A_1^q \circ I)^{1/q} (A_2^{p_2} \circ I)^{1/p_2} \cdots (A_n^{p_n} \circ I)^{1/p_n} \\ &\leq (A_1^{p_1} \circ I)^{1/p_1} (A_2^{p_2} \circ I)^{1/p_2} \cdots (A_n^{p_n} \circ I)^{1/p_n}. \end{aligned}$$

□

§4. Reverse inequality

In this section, we consider reverse-type inequalities for Ando and Aujla–Vasudeva inequalities for the Karcher mean. The following lemma is a reverse of information monotonicity (P6), and plays an essential role in this section; also see [11, 19]. Though it is not a sharp inequality in the case of $n = 2$, we don't know any estimates better than Lemma 4.1 for the Karcher mean in the case of $n \geq 3$:

Lemma 4.1. *Let Φ be a unital positive linear map, and $A_1, \ldots, A_n$ positive invertible operators such that $0 < mI \leq A_i \leq MI$ for some scalars $0 < m \leq M$ and $i = 1, \ldots, n$. Let $\omega = (\omega_1, \ldots, \omega_n)$ be a weight vector. Then for a given $\alpha > 0$,*

$$G_{\mathrm{K}}(\omega; \Phi(A_1), \ldots, \Phi(A_n)) \leq \alpha \Phi(G_{\mathrm{K}}(\omega; A_1, \ldots, A_n)) + \beta(m, M, \alpha) I, \tag{4.1}$$

where $\beta(m, M, \alpha)$ is defined by

$$\beta(m, M, \alpha) = \begin{cases} m + M - 2\sqrt{\alpha m M} & \text{if } m \leq \sqrt{\alpha M m} \leq M, \\ (1-\alpha)M & \text{if } \sqrt{\alpha M m} \leq m, \\ (1-\alpha)m & \text{if } M \leq \sqrt{\alpha M m}. \end{cases} \tag{4.2}$$

In particular,

$$G_{\mathrm{K}}(\omega; \Phi(A_1), \ldots, \Phi(A_n)) \leq \frac{(M+m)^2}{4Mm} \Phi(G_{\mathrm{K}}(\omega; A_1, \ldots, A_n)) \tag{4.3}$$

and

$$G_{\mathrm{K}}(\omega; \Phi(A_1), \ldots, \Phi(A_n)) \leq \Phi(G_{\mathrm{K}}(\omega; A_1, \ldots, A_n)) + \left(\sqrt{M} - \sqrt{m}\right)^2 I. \tag{4.4}$$

Proof. We showed in [11] that for a given $\alpha > 0$ there exists a constant $\beta(m, M, \alpha)$ in (4.2) such that

$$\sum_{i=1}^{n} \omega_i A_i \leq \alpha \left(\sum_{i=1}^{n} \omega_i A_i^{-1} \right)^{-1} + \beta(m, M, \alpha) I.$$

Hence it follows from the arithmetic–geometric–harmonic mean inequality (P7) for the Karcher mean that

$$\begin{aligned} G_{\mathrm{K}}(\omega; \Phi(A_1), \ldots, \Phi(A_n)) &\leq \sum_{i=1}^{n} \omega_i \Phi(A_i) = \Phi\left(\sum_{i=1}^{n} \omega_i A_i \right) \\ &\leq \Phi(\alpha \left(\sum_{i=1}^{n} \omega_i A_i^{-1} \right)^{-1} + \beta(m, M, \alpha) I) \\ &\leq \alpha \Phi(G_{\mathrm{K}}(\omega; A_1, \ldots, A_n)) + \beta(m, M, \alpha) I \end{aligned}$$

and so we have (4.1).

If we put $\beta(m, M, \alpha) = 0$ in (4.1), then it follows that $\alpha = (M+m)^2/4Mm$ and $m \le \sqrt{\alpha Mm} \le M$. Hence we have (4.3). If we put $\alpha = 1$ in (4.1), then it follows that $\beta(m, M, 1) = (\sqrt{M} - \sqrt{m})^2$ and $m \le \sqrt{Mm} \le M$. Hence we have (4.4). □

By Lemma 4.1, we show the reverse inequality of Theorem 2.2:

Theorem 4.2. *Let $A_1, \dots, A_n$ be positive invertible operators such that $0 < m_i I \le A_i \le M_i I$ for some scalars $0 < m_i \le M_i$ and $i = 1, \dots, n$, and $\omega = (\omega_1, \dots, \omega_n)$ be a weight vector. Put $m = m_1 m_2 \cdots m_n$ and $M = M_1 M_2 \cdots M_n$. Then for a given $\alpha > 0$,*

$$\prod_{i=1}^{n} \circ\, A_i \le \alpha \prod_{i=1}^{n} \circ\, G_{\mathrm{K}}(\omega_{\sigma_i}; A_1, \dots, A_n) + \beta(m, M, \alpha) I,$$

where $\beta(m, M, \alpha)$ is defined as (4.2) and weight vectors ω_{σ_i} are defined as (2.1) for $1 \le i \le n$.

Proof. Since $m \le A_1 \otimes A_2 \otimes \cdots \otimes A_n, \dots, A_n \otimes A_1 \otimes \cdots \otimes A_{n-1} \le M$, it follows from Lemma 4.1 that

$$\begin{aligned}
&A_1 \circ \cdots \circ A_n \\
&\quad = G_{\mathrm{K}}(\omega; A_1 \circ \cdots \circ A_n, A_1 \circ \cdots \circ A_n, \dots, A_1 \circ \cdots \circ A_n) \\
&\quad = G_{\mathrm{K}}(\omega; A_1 \circ \cdots \circ A_n, A_2 \circ \cdots \circ A_1, \dots, A_n \circ \cdots \circ A_{n-1}) \\
&\quad = G_{\mathrm{K}}(\omega; U^*(A_1 \otimes \cdots \otimes A_n)U, U^*(A_2 \otimes \cdots \otimes A_1)U, \dots, U^*(A_n \otimes \cdots \otimes A_{n-1})U) \\
&\quad \le \alpha U^* G_{\mathrm{K}}(\omega; A_1 \otimes \cdots \otimes A_n, \dots, A_n \otimes \cdots \otimes A_{n-1})U + \beta(m, M, \alpha) I \\
&\quad = \alpha U^* \left(G_{\mathrm{K}}(\omega; A_1, \dots, A_n) \otimes \cdots \otimes G_{\mathrm{K}}(\omega; A_n, \dots, A_{n-1})\right) U + \beta(m, M, \alpha) I \\
&\quad = \alpha G_{\mathrm{K}}(\omega; A_1, \dots, A_n) \circ \cdots \circ G_{\mathrm{K}}(\omega; A_n, \dots, A_{n-1}) + \beta(m, M, \alpha) I \\
&\quad = \alpha \prod_{i=1}^{n} \circ\, G_{\mathrm{K}}(\omega_{\sigma_i}; A_1, \dots, A_n) + \beta(m, M, \alpha) I.
\end{aligned}$$

□

Similarly, we have a reverse-type inequality for Theorem 2.4:

Theorem 4.3. *Let k be the cardinal of $\{A, B, \dots, Z\}$ and $\omega = (\omega_1, \dots, \omega_n)$ a weight vector. For any integer $n \ge 2$, let $A_i, B_i, \dots, Z_i$ be positive invertible operators such that $0 < m_1 I \le A_i \le M_1 I$, $0 < m_2 I \le B_i \le M_2 I, \dots, 0 < m_k I \le Z_i \le M_k I$ for some scalars $0 < m_j \le M_j$ and $i = 1, \dots, n$ and $j = 1, \dots, k$. Put*

$m' = m_1 \cdots m_k$ and $M' = M_1 \cdots M_k$. Then for a given $\alpha > 0$,

$$\begin{aligned}&G_{\mathrm{K}}(\omega; A_1 \circ B_1 \circ \cdots \circ Z_1, \ldots, A_n \circ B_n \circ \cdots \circ Z_n)\\ &\qquad \le \alpha\,[G_{\mathrm{K}}(\omega; A_1, \ldots, A_n) \circ G_{\mathrm{K}}(\omega; B_1, \ldots, B_n) \circ \cdots \circ G_{\mathrm{K}}(\omega; Z_1, \ldots, Z_n)]\\ &\qquad\quad + \beta(m', M', \alpha)I,\end{aligned}$$

where $\beta(m', M', \alpha)$ is defined as (4.2).

We consider an estimate from above for the Fiedler inequality (1.1). By the monotonicity of Hadamard multiplication, it follows that

$$A \circ A^{-1} \le \frac{M}{m} I$$

for every positive invertible A such that $0 < mI \le A \le MI$. In [13], Kitamura and the author gave an improvement of it: If A is a positive invertible operator such that $0 < mI \le A \le MI$ for some scalars $0 < m \le M$, then

$$A \circ A^{-1} \le \frac{1}{2}\left(\frac{M}{m} + \frac{m}{M}\right) I. \tag{4.5}$$

By virtue of Theorem 4.2, we show an estimate from above for an n-variable version of the Fiedler-type theorem in the case of $n \ge 3$:

Corollary 4.4. *Let A be a positive invertible operator such that $0 < mI \le A \le MI$ for some scalars $0 < m \le M$, and $a_1, \ldots, a_n \in \mathbb{R}$ such that $\sum_{i=1}^n a_i = 0$. Put $m_i = \min\{m^{a_i}, M^{a_i}\}$ and $M_i = \max\{m^{a_i}, M^{a_i}\}$ for $i = 1, \ldots, n$, and $m' = m_1 \cdots m_n$ and $M' = M_1 \cdots M_n$. Then for a given $\alpha > 0$,*

$$A^{a_1} \circ \cdots \circ A^{a_n} \le (\alpha + \beta(m', M', \alpha))I,$$

where $\beta(m', M', \alpha)$ is defined as (4.2).

In particular,

$$A^{a_1} \circ \cdots \circ A^{a_n} \le (m' + M' - 1)I.$$

Proof. By assumption, it follows that $0 < m_i I \le A^{a_i} \le M_i I$ for $i = 1, \ldots, n$ and so $m'I \le A^{a_1} \otimes \cdots \otimes A^{a_n} \le M'I$. Hence by Theorem 4.2,

$$\begin{aligned}&A^{a_1} \circ \cdots \circ A^{a_n}\\ &\quad \le \alpha\,[G_{\mathrm{K}}(A^{a_1}, A^{a_2}, \ldots, A^{a_n}) \circ \cdots \circ G_{\mathrm{K}}(A^{a_n}, A^{a_1}, \ldots, A^{a_{n-1}})] + \beta(m', M', \alpha)I\\ &\quad = \alpha\left[(A^{a_1} \cdots A^{a_n})^{1/n} \circ \cdots \circ (A^{a_n} A^{a_1} \cdots A^{a_{n-1}})^{1/n}\right] + \beta(m', M', \alpha)I\\ &\quad = \alpha(I \circ \cdots \circ I) + \beta(m', M', \alpha)I\\ &\quad = (\alpha + \beta(m', M', \alpha))I.\end{aligned}$$

If $\alpha = 1$, then we have $1+\beta(m', M', 1) = 1 + m' + M' - 2\sqrt{m'M'} = m' + M' - 1$, because $m'M' = 1$ by construction of M' and m'. □

Remark 4.5. By the monotonicity of Hadamard multiplication, we have a simple estimate from above: $A^{a_1} \circ \cdots \circ A^{a_n} \le M'I$. By Corollary 4.4, we have the following estimate:

$$A^{a_1} \circ \cdots \circ A^{a_n} \le (M' + m' - 1)I \le M'I.$$

In fact, since $m'M' = 1$ and $m' \le 1$, we have $M' + m' - 1 \le M'$.

§5. Majorization for Hadamard products

In this section, we deal with the case of matrices. Given a $d \times d$ Hermitian matrix A, let us always arrange its eigenvalues $\lambda_i(A)$ $(i = 1, 2, \ldots, d)$ in decreasing order

$$\lambda_1(A) \ge \lambda_2(A) \ge \cdots \ge \lambda_d(A).$$

Let us write $A \succ B$ for Hermitian A, B to mean the majorization $[\lambda_i(A)] \succ [\lambda_i(B)]$, i.e.,

$$\sum_{i=1}^{k} \lambda_i(A) \ge \sum_{i=1}^{k} \lambda_i(B) \quad (k = 1, 2, \ldots, d)$$

and

$$\sum_{i=1}^{d} \lambda_i(A) = \sum_{i=1}^{d} \lambda_i(B).$$

Note that

$$A \succ B \quad \Longrightarrow \quad \sum_{i=k}^{d} \lambda_i(A) \le \sum_{i=k}^{d} \lambda_i(B) \quad (k = 1, 2, \ldots, d).$$

In [2], Ando showed that for positive-definite A, B,

$$\prod_{i=k}^{d} \lambda_i(A \circ B) \ge \prod_{i=k}^{d} \lambda_i(A \sharp B)^2 \quad (k = 1, 2, \ldots, d). \tag{5.1}$$

We show an n-variable version of (5.1) via the Karcher mean:

Theorem 5.1. *If $A_1, A_2, \ldots, A_n$ are positive definite, then*

$$\prod_{i=k}^{d} \lambda_i(A_1 \circ A_2 \circ \cdots \circ A_n) \ge \prod_{i=k}^{d} \lambda_i(G_{\mathrm{K}}(A_1, A_2, \ldots, A_n))^n \quad (k = 1, 2, \ldots, d).$$

Proof. By Theorem 2.2, we have

$$\prod_{i=1}^{n} \circ A_i \geq \prod_{i=1}^{n} \circ\, G_{\mathrm{K}}(A_1, \ldots, A_n).$$

Hence it follows that

$$\prod_{i=k}^{d} \lambda_i(A_1 \circ A_2 \circ \cdots \circ A_n) \geq \prod_{i=k}^{d} \lambda_i(G_{\mathrm{K}}(A_1, A_2, \ldots, A_n) \circ \cdots \circ G_{\mathrm{K}}(A_1, A_2, \ldots, A_n))$$

for $k = 1, 2, \ldots, d$. Bapat and Sunder [4] showed that

$$\prod_{i=k}^{d} \lambda_i(A \circ B) \geq \prod_{i=k}^{d} \lambda_i(A)\lambda_i(B) \quad (k = 1, 2, \ldots, d)$$

for positive-definite A and B. This implies

$$\prod_{i=k}^{d} \lambda_i(G_{\mathrm{K}}(A_1, A_2, \ldots, A_n) \circ \cdots \circ G_{\mathrm{K}}(A_1, A_2, \ldots, A_n)) \geq \prod_{i=k}^{d} \lambda_i(G_{\mathrm{K}}(A_1, A_2, \ldots, A_n))^n$$

for $k = 1, 2, \ldots, d$. This completes the proof of Theorem 5.1. □

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