Publ. RIMS Kyoto Univ. **53** (2017), 79–117
DOI 10.4171/PRIMS/53-1-3

Kato's Inequality for Magnetic Relativistic Schrödinger Operators

by

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Abstract

Kato's inequality is shown for the magnetic relativistic Schrödinger operator $H_{A,m}$ defined as the operator-theoretical *square root* of the self-adjoint, magnetic nonrelativistic Schrödinger operator $(-i\nabla - A(x))^2 + m^2$ with an L^2_{loc} vector potential $A(x)$.

2010 Mathematics Subject Classification: 47A50; 81Q10; 47B25; 47N50; 47D06; 47D08.
Keywords: Kato's inequality, relativistic Schrödinger operator, magnetic relativistic Schrödinger operator.

§1. Introduction

Consider the magnetic relativistic Schrödinger operator

$$H_{A,m} := \sqrt{(-i\nabla - A(x))^2 + m^2} \tag{1.1}$$

in d-dimensional space $\mathbb{R}^d$ with vector potential $A(x) := (A_1(x), \dots, A_d(x))$ and rest mass $m \geq 0$, which may be thought of as a quantum Hamiltonian corresponding to the classical relativistic Hamiltonian symbol $\sqrt{(\xi - A(x))^2 + m^2}$, $(\xi, x) \in \mathbb{R}^d \times \mathbb{R}^d$. It is known that when $A(x)$ is an $\mathbb{R}^d$-valued function belonging to $[L^2_{\mathrm{loc}}(\mathbb{R}^d)]^d \equiv L^2_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^d)$, it becomes a self-adjoint operator in $L^2(\mathbb{R}^d)$, which is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$ so that $H_{A,m}$ has a domain containing $C_0^\infty(\mathbb{R}^d)$ as an operator core (see, e.g., [CFKiSi87, p. 9]). We shall assume that $d \geq 2$, since in the case $d = 1$ any magnetic vector potential can be removed by a gauge

Communicated by H. Okamoto. Received January 15, 2016. Revised June 13, 2016; July 11, 2016.

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transformation. For $A = 0$ we put $H_{0,m} = \sqrt{-\Delta + m^2}$, where $-\Delta$ is the *minus-signed* Laplacian $-\big(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}\big)$, as well as a nonnegative self-adjoint operator realized in $L^2(\mathbb{R}^d)$ having the Sobolev space $H^2(\mathbb{R}^d)$ as its domain.

The aim of this paper is to show Kato's inequality for this magnetic relativistic Schrödinger operator $H_{A,m}$ or $H_{A,m} - m$, when A is an $\mathbb{R}^d$-valued L^2_{loc}-function in $\mathbb{R}^d$.

Theorem 1.1 (Kato's inequality). *Let $m \geq 0$ and assume $A \in [L^2_{\mathrm{loc}}(\mathbb{R}^d)]^d$. If $u \in L^2(\mathbb{R}^d)$ with $H_{A,m}u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, then the following distributional inequality holds:*

$$\mathrm{Re}[(\mathrm{sgn}\, u)H_{A,m}u] \geq H_{0,m}|u|, \tag{1.2}$$

or

$$\mathrm{Re}[(\mathrm{sgn}\, u)[H_{A,m} - m]u] \geq [H_{0,m} - m]|u|. \tag{1.3}$$

Here sgn *is a bounded function in $\mathbb{R}^d$ defined by*

$$(\mathrm{sgn}\, u)(x) = \begin{cases} \overline{u(x)}/|u(x)| & \text{if } u(x) \neq 0, \\ 0 & \text{if } u(x) = 0. \end{cases}$$

Note here that $H_{A,m}u$ with $u \in L^2(\mathbb{R}^d)$ makes sense as a distribution in $\mathbb{R}^d$ (for this, see Lemma 2.2 with $\alpha = 1$ and a few lines after its proof). A characteristic feature in this situation is that $H_{A,m}$ is a nonlocal operator defined by the operator-theoretical square root of a nonnegative self-adjoint operator. It is not a differential operator, and neither an integral operator nor a pseudo-differential operator associated with a certain tractable symbol. The point that becomes crucial is how to go without knowledge on the regularity of the weak solution $u \in L^2(\mathbb{R}^d)$ of equation $H_{A,m}u = f$ for a given $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Thus the present inequality (1.2)/(1.3) differs from an abstract form of Kato's inequality such as in [Si77] by being substantially sharp.

An immediate corollary is the following theorem, which is already known (e.g., [FLSei08, HILo12]; cf. [I93]).

Theorem 1.2 (Diamagnetic inequality). *Let $m \geq 0$ and assume $A \in [L^2_{\mathrm{loc}}(\mathbb{R}^d)]^d$. Then it holds that for $f, g \in L^2(\mathbb{R}^d)$,*

$$|(f, e^{-t[H_{A,m}-m]}g)| \leq (|f|, e^{-t[H_{0,m}-m]}|g|). \tag{1.4}$$

Once Theorem 1.1 is established, we can apply it to show the following theorem on essential self-adjointness of the relativistic Schrödinger operator with both vector and scalar potentials $A(x)$ and $V(x)$:

$$H_{A,V,m} := H_{A,m} + V. \tag{1.5}$$

Theorem 1.3. *Let $m \geq 0$, assume that $A \in [L^2_{\mathrm{loc}}(\mathbb{R}^d)]^d$ and let $V \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ with $V(x) \geq 0$ a.e. Then $H_{A,V,m} = H_{A,m} + V$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$ and its unique self-adjoint extension is bounded below by m.*

We shall show inequality (1.2)/(1.3), basically following the idea and method of Kato's original proof in [K72] for the magnetic *nonrelativistic* Schrödinger operator $\frac{1}{2}(-i\nabla - A(x))^2$. As a matter of fact, we follow the method of proof modified for the existing form of Kato's inequality in [I89, ITs92] for *another* magnetic relativistic Schrödinger operator, which is defined as a Weyl pseudo-differential operator associated with the same relativistic classical symbol $\sqrt{(\xi - A(x))^2 + m^2}$. However, this is not sufficient, and we need further modifications *using operator theory*, since pseudo-differential calculus does not seem useful. Starting from the assumption of the theorem that $u \in L^2$ and $H_{A,m}u \in L^1_{\mathrm{loc}}$, it appears to be impossible to show the regularity of u such that $\partial_j u \in L^1_{\mathrm{loc}}$, $1 \leq j \leq d$, and/or $H_{0,m}u \in L^1_{\mathrm{loc}}$, which may be due to the fact that the operators $\partial_j \cdot (-\Delta + m^2)^{-1/2}$, $1 \leq j \leq d$ are *not* bounded from L^1 to L^1, though they are *bounded* from L^1 to *weak* L^1*-space*. Therefore we make a detour by going via the case of the fractional power $(H_{A,m})^\alpha$ with $\alpha < 1$. Verifying that the assumption implies that $(H_{A,m})^\alpha u \in L^1_{\mathrm{loc}}$ for $0 < \alpha < 1$, we show the asserted inequality *first for the case* $0 < \alpha < 1$, i.e., inequality (1.2)/(1.3) with the pair $H_{A,m}$, $H_{0,m}$ replaced by the pair $(H_{A,m})^\alpha$, $(H_{0,m})^\alpha$, respectively, and *then for the case* $\alpha = 1$, appealing to the fact, to be shown, that $(H_{A,m})^\alpha u$ converges to $H_{A,m}u$ in L^1_{loc} as $\alpha \uparrow 1$. The proof is presented separately according to $m > 0$ and $m = 0$, in a self-contained manner.

A comment is in order on our starting assumption for u, namely, why the theorem is formulated with the assumption that $u \in L^2$ and $H_{A,m}u \in L^1_{\mathrm{loc}}$, but not that *both* u *and* $H_{A,m}u$ *are* L^1_{loc}. For this question, recall that the original form of Kato's inequality for nonrelativistic Schrödinger operators $\frac{1}{2}(-i\nabla - A(x))^2$ is formulated under the assumption that *both* u *and* $\frac{1}{2}(-i\nabla - A(x))^2 u$ *are* L^1_{loc}. The answer is simply because of avoiding inessential complexity coming from the fact that $H_{A,m}$ is a nonlocal operator.

The relativistic Schrödinger operator $H_{0,m} = \sqrt{-\Delta + m^2}$ without vector potential was first considered in [W74, He77] for spectral problems. The magnetic relativistic Schrödinger operator $H_{A,m}$ like (1.1) is used to study problems related to "stability of matter" in relativistic quantum mechanics in [LSei10]. On the other hand, the problem of representing the relativistic Schrödinger semigroup with generator $H_{A,m}$ by a path integral has also been studied. A result is a formula of Feynman–Kac–Itô type (cf. [Si279/05]), earlier in [DeRiSe91, DeSe90] and also in [N00], which has recently been extensively studied in [HILo12, HILo13] (cf. [LoHBe11]). The problem is connected with a Lévy process obtained by subordinating Brownian motion ([Sa99, Ap04/09]). A weaker version of Kato's inequality as well as the diamagnetic inequality was given in our paper [HILo12], to which the present work adds further results.

In Section 2 we give some technical lemmas that will be used in the proof of theorems. They concern some basic inequalities in L^2 and L^p connected with the semigroups and/or inverse (resolvent) for the magnetic *nonrelativistic* (but not *relativistic*) Schrödinger operator $(-i\nabla - A)^2 + m^2$, which is the square of our magnetic relativistic Schrödinger operator $H_{A,m}$. For the sake of regularization of $H_{A,m}$, its fractional powers $(H_{A,m})^\alpha$ with $0 < \alpha < 1$ are also considered through the semigroup of the magnetic *nonrelativistic* Schrödinger operator to estimate, in a local L^1-norm, a kind of difference, being a distance in a particular sense, between $(H_{A,m})^\alpha$ and $(H_{0,m})^\alpha$, each applied to a function.

In Section 3 we prove the theorems. Section 4 is to make concluding remarks about how the issue is going with the other two magnetic relativistic Schrödinger operators associated with the same symbol. Appendix A provides an explicit expression of the integral kernel (heat kernel) of the semigroup $e^{-t[(H_{0,m})^\alpha - m^\alpha]}$ for the free fractional power $(H_{0,m})^\alpha$ together with the density (function) of the associated Lévy measure $n^{m,\alpha}(dy)$. For basic facts on the magnetic relativistic Schrödinger operator, we refer, e.g., to [LLos01, BE11].

Finally, we note that we have defined the fractional powers of $H_{A,m}$ mainly through the magnetic nonrelativistic Schrödinger semigroup. However, an alternative way is to define them through the Dunford integral via the *resolvent of the magnetic nonrelativistic Schrödinger operator*.

§2. Technical lemmas

Throughout this paper, we denote by $(\cdot,\cdot)$ the Hilbert space inner product which is sesquilinear, i.e., conjugate-linear in the first argument and linear in the second (the physicist's convention), and by $\langle\cdot,\cdot\rangle$ the bilinear inner product which is linear in both the arguments.

Our main concern is the operator $H_{A,m} := [(-i\nabla - A)^2 + m^2]^{1/2}$ in (1.1) with the assumption that $A \in [L^2_{\mathrm{loc}}(\mathbb{R}^d)]^d$, which is a self-adjoint operator in $L^2(\mathbb{R}^d)$ defined as the square root of the nonnegative self-adjoint (Schrödinger) operator $(-i\nabla - A)^2 + m^2$ in $L^2(\mathbb{R}^d)$. For $m = 0$, $H_{A,0} = |-i\nabla - A|$. Among them, the following identity holds:

$$\begin{aligned} \|H_{A,m}u\|^2_{L^2} &= \big(u, (H_{A,m})^2 u\big) = \big(u, [(-i\nabla - A)^2 + m^2]u\big) \\ &= \sum\nolimits_{j=1}^d \|(-i\partial_j - A_j)u\|^2_{L^2} + m^2\|u\|^2_{L^2} = \|H_{A,0}u\|^2_{L^2} + m^2\|u\|^2_{L^2}, \end{aligned} \tag{2.1}$$

with $u \in C_0^\infty(\mathbb{R}^d)$ for all five expressions and with u in the domain of $H_{A,m}$ for the first, fourth and fifth expressions. The nonrelativistic Schrödinger operator $(-i\nabla - A)^2 + m^2$ concerned is the self-adjoint operator associated with this quadratic form (2.1), which has $C_0^\infty(\mathbb{R}^d)$ as a form core (e.g., [CFKiSi87, 1.3, pp. 8–9]). As a result, $H_{A,m}$ has $C_0^\infty(\mathbb{R}^d)$ as an operator core; in other words, $H_{A,m}$ is a nonnegative self-adjoint operator in $L^2(\mathbb{R}^d)$ having domain $D[H_{A,m}] := \{u \in L^2(\mathbb{R}^d);\ (i\partial_j + A_j)u \in L^2(\mathbb{R}^d),\ \partial_j := \partial/\partial_{x_j},\ 1 \le j \le d\}$, being essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$. Though $i\nabla + A \equiv (i\partial_1 + A_1, \dots, i\partial_d + A_d)$ is a closed linear operator of $[L^2(\mathbb{R}^d)]^d$ into itself with domain $D[i\nabla + A] := \{(u_1, \dots, u_d) \in [L^2(\mathbb{R}^d)]^d;\ (i\partial_j + A_j)u \in L^2(\mathbb{R}^d),\ \partial_j := \partial/\partial_{x_j},\ 1 \le j \le d\}$, we will also slightly abuse notation to write *the first term of the fourth expression* of (2.1) as $\|(-i\nabla - A)u\|_{L^2}^2$.

For the proof of Theorem 1.1, however, we need to consider $H_{A,m}$ on L^p-spaces also, and moreover, the fractional powers $(H_{A,m})^\alpha$, $0 < \alpha < 1$ of $H_{A,m}$. The aim of this section is to derive some estimates that will be used below.

As for the constant m, unless otherwise stated, we assume in this section that $m > 0$, and keep assuming it in Section 3 also, until we come to consider the case that includes $m = 0$ at the final stage of the proof of Theorem 1.1. Therefore, in the case $m > 0$, the operator $H_{A,m}$ has bounded inverse $(H_{A,m})^{-1}$, and $[(-i\nabla - A)^2 + m^2]$ has bounded inverse $[(-i\nabla - A)^2 + m^2]^{-1}$ as well.

§2.1. Some inequalities related to magnetic nonrelativistic Schrödinger operators on L^p

The operators $H_{A,m}$ may be considered not only in L^2 but also in L^p, $1 \le p < \infty$, in particular, for $p = 1$. The square of $H_{A,m}$ becomes a magnetic nonrelativistic Schrödinger operator $(-i\nabla - A)^2 + m^2$. Some basic inequalities are given that are related to the magnetic nonrelativistic Schrödinger semigroup $e^{-t(H_{A,m})^2}$ and inverse (resolvent) $((H_{A,m})^2)^{-1}$ on L^p, though *not* with the magnetic relativistic Schrödinger semigroup $e^{-tH_{A,m}}$ and inverse (resolvent) $(H_{A,m})^{-1}$. They will be useful throughout the paper.

To begin with, we recall the notation to be used throughout:

$$
\begin{aligned}
(H_{A,m})^2 &= (-i\nabla - A)^2 + m^2, & (H_{A,0})^2 &= (-i\nabla - A)^2,\\
(H_{0,m})^2 &= -\Delta + m^2, & (H_{0,0})^2 &= -\Delta.
\end{aligned}
$$

Lemma 2.1. *Assume $A \in [L^2_{\mathrm{loc}}(\mathbb{R}^d)]^d$. Then the following inequalities hold.*

(i) Let $1 \le p \le \infty$. For $m \ge 0$,

$$
\begin{aligned}
\|e^{-t(H_{A,m})^2}\|_{L^p \to L^p} \le \|e^{-t(H_{0,m})^2}\|_{L^p \to L^p} &\equiv \|e^{-t(-\Delta + m^2)}\|_{L^p \to L^p}\\
&\le e^{-m^2 t} \le 1, \quad t > 0.
\end{aligned}
$$

For $m > 0$ and $\beta > 0$,

$$\|((H_{A,m})^2)^{-\beta}\|_{L^p\to L^p} \le \|((H_{0,m})^2)^{-\beta}\|_{L^p\to L^p}, \quad t > 0.$$

(ii) Let $1 \le p < \infty$. The operators $e^{-t(H_{0,0})^2}(-i\nabla)$ and $e^{-t(H_{0,0})^2}(-\Delta)$ can be extended to be bounded operators on $[L^p(\mathbb{R}^d)]^d$ and $L^p(\mathbb{R}^d)$:

$$\|e^{-t(H_{0,0})^2}(-i\nabla)\|_{[L^p]^d\to[L^p]^d} \le C_{1p}t^{-1/2},$$
$$\|e^{-t(H_{0,0})^2}(-\Delta)\|_{L^p\to L^p} \le C_{2p}t^{-1}, \quad t > 0,$$

with constants $C_{1p} > 0$ and C_{2p} independent of t.

(iii) Let $m \ge 0$. The operators $H_{A,m}e^{-(H_{A,m})^2}$ and $(H_{A,m})^2e^{-t(H_{A,m})^2}$ can be extended to be bounded operators on $L^2(\mathbb{R}^d)$:

$$\|H_{A,m}e^{-t(H_{A,m})^2}\|_{L^2\to L^2} \le (2et)^{-1/2},$$
$$\|(H_{A,m})^2e^{-t(H_{A,m})^2}\|_{L^2\to L^2} \le (et)^{-1}, \quad t > 0.$$

(iv) The operators $e^{-t(-i\nabla-A)^2}(i\nabla+A)$ and $(i\nabla+A)e^{-t(-i\nabla-A)^2}$ can be extended to be bounded operators on $[L^2(\mathbb{R}^d)]^d$:

$$\|e^{-t(-i\nabla-A)^2}(i\nabla + A)\|_{[L^2]^d\to[L^2]^d} \le \left(\tfrac{d}{2et}\right)^{1/2},$$
$$\|(i\nabla + A)e^{-t(-i\nabla-A)^2}\|_{[L^2]^d\to[L^2]^d} \le \left(\tfrac{d}{2et}\right)^{1/2}, \quad t > 0.$$

Assertion (ii) of Lemma 2.1 may be an L^p-version of (iii) or (iv) above, though only for the special case of the minus-signed Laplacian $-\Delta$ without vector potential $A(x)$.

Proof of Lemma 2.1. (i) This is due to the ingenious observation given for the magnetic nonrelativistic Schrödinger operator $(-i\nabla-A(x))^2$ with $A\in L^2_{\mathrm{loc}}$ in [Si79, Thm. 2.3, p. 40], [Si82, Sec. B13, p. 490], since $(H_{A,m})^2 = (-i\nabla - A(x))^2 + m^2$ is nothing but a magnetic (nonrelativistic) Schrödinger operator plus the constant m^2. Following the arguments there we have, for $1 \le p < \infty$ and for every $u \in C_0^\infty(\mathbb{R}^d)$,

$$|e^{-t(H_{A,m})^2}u| \le e^{-t(H_{0,m})^2}|u| = e^{-m^2t}e^{-t(-\Delta)}|u|, \quad \text{pointwise a.e.},$$

so that $e^{-(H_{A,m})^2}L^p(\mathbb{R}^d) \subseteq L^\infty(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$. In fact, for $u \in L^p(\mathbb{R}^d)$,

$$\|e^{-t(H_{A,m})^2}u\|_{L^p} \le e^{-m^2t}\|e^{-t(-\Delta)}|u|\|_{L^p} \le e^{-m^2t}\|u\|_{L^p} \le \|u\|_{L^p}, \quad t \ge 0.$$

Thus we can consider $e^{-t(H_{A,m})^2}$ also as a bounded linear operator mapping $L^p(\mathbb{R}^d)$ into itself. Moreover, it is seen to be a contraction semigroup. We may use the notation $(H_{A,m})^2$, $H_{A,m}$ also to mean operators $(H_{A,m})_p^2$, $(H_{A,m})_p$ in L^p when there is no risk of confusion. Furthermore, for the crucial assertion (i), we refer to [Si82, Cor. B.13.3, p. 491].

(ii) In fact, $e^{-t(-\Delta)}$ becomes a holomorphic semigroup on $L^p(\mathbb{R}^d)$, $1 \leq p < \infty$, for $\mathrm{Re}\, t > 0$. Then for any $f \in L^p(\mathbb{R}^d)$, $v(t) := e^{-t(-\Delta)} f$ gives a unique solution of the heat equation $\frac{\partial}{\partial t} v(t) = \Delta v(t)$ (see, e.g., [K76, IX.§1.8, p. 495] and [K76, IX.§1.6, Rem. 1.22, p. 492]). This implies that $e^{-t(-\Delta)}$ has range in the domain $D[(-\Delta)]$ of $(-\Delta)$, equivalently that $te^{-t(-\Delta)}(-\Delta)$ is uniformly bounded from $L^p(\mathbb{R}^d)$ into itself for real $t > 0$, and so is $t^{1/2}e^{-t(-\Delta)}(-i\partial_j)$ for each $j = 1, 2, \ldots, d$.

(iii) For functions in L^2, the assertions are evident by the spectral theorem, because $(H_{A,m})^2$ and $H_{A,m}$ are nonnegative self-adjoint operators in the Hilbert space $L^2(\mathbb{R}^d)$. Indeed, it is easy to see that for $u \in C_0^\infty(\mathbb{R}^d)$,

$$\begin{aligned}
\|e^{-t(H_{A,m})^2} H_{A,m} u\|_{L^2}^2 &= (u, (H_{A,m})^2 e^{-2t(H_{A,m})^2} u) \\
&\leq \sup_{\lambda \geq 0} \lambda e^{-2t\lambda} \|u\|_{L^2}^2 = (2et)^{-1} \|u\|_{L^2}^2, \\
\|e^{-t(H_{A,m})^2} (H_{A,m})^2 u\|_{L^2}^2 &= (u, (H_{A,m})^4 e^{-2t(H_{A,m})^2} u) \\
&\leq \sup_{\lambda \geq 0} \lambda^2 e^{-2t\lambda} \|u\|_{L^2}^2 = (et)^{-2} \|u\|_{L^2}^2.
\end{aligned}$$

This shows (iii).

(iv) These inequalities follow from (ii). Indeed, for the first one, since

$$\|e^{-t(-i\nabla - A)^2}(i\nabla + A)\varphi\|_{L^2}^2 := \sum_{j=1}^d \|e^{-t\sum_{k=1}^d (-i\partial_k - A_k)^2}(i\partial_j + A_j)\varphi_j\|_{L^2}^2$$

for $\varphi = (\varphi_1, \ldots, \varphi_d) \in [C_0^\infty(\mathbb{R}^d)]^d$, we have only to show that for each j,

$$\|e^{-t(-i\nabla - A)^2}(i\partial_j + A_j)\varphi_j\|_{L^2}^2 \leq (2et)^{-1}\|\varphi_j\|_{L^2}^2.$$

This is seen as follows: For $m > 0$, we have by (ii),

$$\begin{aligned}
\|e^{-t(-i\nabla - A)^2}(i\partial_j + A_j)\varphi_j\|_{L^2}^2 &= e^{2m^2}\big\|[e^{-tH_{A,m}^2} H_{A,m}][H_{A,m}^{-1}((i\partial_j + A_j)^2 + m^2)^{1/2}] \\
&\qquad \times [((i\partial_j + A_j)^2 + m^2)^{-1/2}(i\partial_j + A_j)]\varphi_j\big\|_{L^2}^2 \\
&\leq e^{2m^2}(2et)^{-1}\big\|[H_{A,m}^{-1}((i\partial_j + A_j)^2 + m^2)^{1/2}] \\
&\qquad \times [((i\partial_j + A_j)^2 + m^2)^{-1/2}(i\partial_j + A_j)]\varphi_j\big\|_{L^2}^2 \\
&\leq e^{2m^2}(2et)^{-1}\|\varphi_j\|_{L^2}^2.
\end{aligned}$$

Letting $m \downarrow 0$, we have the result.

The second result is shown similarly. This shows (iv), ending the proof of Lemma 2.1. □

Remark. The nontriviality of assertion (ii) of this lemma lies in that $i\nabla + A$ *does not commute* with the operator $(i\nabla + A(x))^2 = \sum_{j=1}^d (i\partial_j + A_j(x))^2$ or $(H_{A,m})^2$.

§2.2. Estimate of a kind of difference between $(H_{A,m})^\alpha$ and $(H_{0,m})^\alpha$ in a local L^1-norm

In this subsection, we consider the operators given by the fractional powers $(H_{A,m})^\alpha := [(-i\nabla - A)^2 + m^2]^{\alpha/2}$, $0 < \alpha \leq 1$, and provide several lemmas to estimate in a local L^1-norm a kind of difference between $(H_{A,m})^\alpha$ and $(H_{0,m})^\alpha$, each applied to a function u. They are needed to prove Theorem 1.1. Of course, the case for $\alpha = 1$ turns out to be our operator itself: $(H_{A,m})^1 \equiv H_{A,m} = [(-i\nabla - A)^2 + m^2]^{1/2}$.

Given a positive self-adjoint operator S in a Hilbert space $L^2(\mathbb{R}^d)$ with domain $D[S]$, we adopt the following definition of its fractional powers S^α, suggested from the identity for the gamma function $\Gamma(\beta)$, $s^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} e^{-st}\, dt$ with $t > 0$ and $0 < \beta \leq 1$: for $0 \leq \alpha < 1$,

$$S^\alpha u = S^{-(1-\alpha)} \cdot Su = \frac{1}{\Gamma(1-\alpha)} \int_0^\infty t^{-\alpha} e^{-tS}\, Su\, dt, \quad u \in D[S].$$

We shall use these formulas, taking for S the nonrelativistic Schrödinger operator $[(-i\nabla - A)^2 + m^2] = (H_{A,m})^2$ and/or $[-\Delta + m^2] = (H_{0,m})^2$, but not the relativistic Schrödinger operator $H_{A,m}$ and/or $H_{0,m}$. Thus for $f \in L^2(\mathbb{R}^d)$,

$$\begin{aligned}(H_{A,m})^{-\beta} f &= [(-i\nabla - A)^2 + m^2]^{-\beta/2} f \\ &= \frac{1}{\Gamma(\frac{\beta}{2})} \int_0^\infty t^{\beta/2-1} e^{-t[(-i\nabla - A)^2 + m^2]} f\, dt \quad (0 < \beta \leq 2), \end{aligned} \tag{2.2}$$

and similarly for $(H_{0,m})^{-\beta} \equiv [-\Delta + m^2]^{-\beta/2}$ in the case $A = 0$. Therefore, for $u \in C_0^\infty(\mathbb{R}^d)$, we have

$$\begin{aligned}(H_{A,m})^\alpha u &= [(-i\nabla - A)^2 + m^2]^{\alpha/2} u \\ &= \frac{1}{\Gamma(\frac{2-\alpha}{2})} \int_0^\infty t^{(2-\alpha)/2-1} e^{-t[(-i\nabla - A)^2 + m^2]} [(-i\nabla - A)^2 + m^2] u\, dt \\ &= \frac{1}{\Gamma(\frac{2-\alpha}{2})} \int_0^\infty t^{-\alpha/2} e^{-t(H_{A,m})^2} (H_{A,m})^2 u\, dt, \quad (0 \leq \alpha < 2), \end{aligned} \tag{2.3}$$

for u in the domain of $(H_{A,m})^2$, and similarly for $(H_{0,m})^\alpha \equiv [-\Delta + m^2]^{\alpha/2}$ in the case $A = 0$. Here note that $H_{A,m}/H_{0,m}$, as well as $S = (-i\nabla - A)^2 + m^2/\ (-\Delta + m^2)$, has bounded inverse, since we are assuming in this section that $m > 0$. It may be instructive to recognize that, for $0 < \alpha < 1$, the last integral of (2.3) exists not only for $u \in D[(H_{A,m})^2]$ but also for $u \in D[H_{A,m}]$, because by Lemma 2.1(iii),

$$\begin{aligned} t^{-\alpha/2}\|e^{-t(H_{A,m})^2}(H_{A,m})^2u\|_{L^2} &\leq t^{-\alpha/2}\|e^{-t(H_{A,m})^2}H_{A,m}\|\,\|H_{A,m}u\|_{L^2} \\ &= O(t^{-(1+\alpha)/2}). \end{aligned}$$

Lemma 2.2. *Let $0 < \alpha \leq 1$. Assume that $A \in [L^2_{\mathrm{loc}}(\mathbb{R}^d)]^d$. If $\varphi \in C_0^\infty(\mathbb{R}^d)$, then $(H_{A,m})^\alpha\varphi \in L^2(\mathbb{R}^d)$. In fact, it holds for every compact subset K in $\mathbb{R}^d$ that*

(2.4)

$$\|(H_{A,m})^\alpha\varphi\|_{L^2} \leq |K|^{1/2}\big[((m^2+1)^{1/2}+1)+\|\,|A|\,\|_{L^2(K)}\big]\,\big[\|\nabla\varphi\|_{L^\infty(K)}+\|\varphi\|_{L^\infty(K)}\big],$$

for all $\varphi \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp}\varphi \subseteq K$, where $|K|$ denotes the volume (Lebesgue measure) of K.

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp}\varphi \subseteq K$. Then for $0 < \alpha \leq 1$, we have

$$\begin{aligned} \|(H_{A,m})^\alpha\varphi\|_{L^2}^2 &= \big(\varphi, (H_{A,m})^{2\alpha}\varphi\big) = \big(\varphi, [(-i\nabla - A)^2 + m^2]^\alpha\varphi\big) \\ &\leq \big(\varphi, [(-i\nabla - A)^2 + m^2 + 1]^\alpha\varphi\big) \\ &\leq \big(\varphi, [(-i\nabla - A)^2 + m^2 + 1]\varphi\big) \\ &= \|(-i\nabla - A)\varphi\|_{L^2}^2 + (m^2+1)\|\varphi\|_{L^2}^2 \\ &= \|H_{A,(m^2+1)^{1/2}}\varphi\|_{L^2}^2. \end{aligned} \tag{2.5}$$

Here for the first term of the second-to-last line recall our informal notation mentioned after (2.1). Hence,

$$\begin{aligned} \|(H_{A,m})^\alpha\varphi\|_{L^2} &\leq \|\nabla\varphi\|_{L^2} + \|A\varphi\|_{L^2} + (m^2+1)^{1/2}\|\varphi\|_{L^2} \\ &\leq |K|^{1/2}\|\nabla\varphi\|_{L^\infty(K)} + \|\,|A|\,\|_{L^2(K)}\|\varphi\|_{L^\infty(K)} \\ &\quad + (m^2+1)^{1/2}|K|^{1/2}\|\varphi\|_{L^\infty(K)} < \infty, \end{aligned}$$

which is finite by assumption on A and φ. This shows the desired assertion. □

By this lemma, for $0 < \alpha \leq 1$ we can *define* a distribution $(H_{A,m})^\alpha u$ for $u \in L^2(\mathbb{R}^d)$ by

$$\langle\, (H_{A,m})^\alpha u, \phi\,\rangle = \langle\, u, (H_{-A,m})^\alpha\phi\,\rangle = \int (u(H_{-A,m})^\alpha\phi)(x)\,dx,$$

or

$$(\,(H_{A,m})^\alpha u, \phi\,) = (\,u, (H_{A,m})^\alpha\phi\,) = \int (\bar{u}(H_{A,m})^\alpha\phi)(x)\,dx,$$

for $\phi \in C_0^\infty(\mathbb{R}^d)$, because, for every compact set K in $\mathbb{R}^d$, we have

$$|(\,(H_{A,m})^\alpha u, \phi\,)| = |(\,u, (H_{A,m})^\alpha\phi\,)| \leq \|u\|_{L^2}\|(H_{A,m})^\alpha\phi\|_{L^2}$$

$$\le \|u\|_{L^2}\big[(|K|^{1/2}((m^2+1)^{1/2}+1)) + \|\,|A|\,\|_{L^2(K)}\big] \times \big[\|\nabla\phi\|_{L^\infty(K)} + \|\phi\|_{L^\infty(K)}\big],$$

for all $\phi \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp}\phi \subseteq K$. This says that $(H_{A,m})^\alpha u$ is a continuous linear functional on $C_0^\infty(\mathbb{R}^d)$, and thus a distribution on $\mathbb{R}^d$.

Next, we study some properties of $(H_{A,m})^\alpha$ in the case $A \equiv 0$, namely, $(H_{0,m})^\alpha \equiv (-\Delta+m^2)^{\alpha/2}$, $0<\alpha\le 1$. This is the $\frac{\alpha}{2}$-power of the nonnegative self-adjoint operator $H_{0,m} \equiv -\Delta+m^2$ on $L^2(\mathbb{R}^d)$ or also a pseudo-differential operator defined through Fourier transform having the symbol $(|\xi|^2+m^2)^{\alpha/2}$. The function $\xi \mapsto (|\xi|^2+m^2)^{\alpha/2}-m^\alpha$ is conditionally negative definite in $\mathbb{R}^d$ (e.g., [ReSi78, App. 2 to XIII.12, pp. 212–222], [IkW81/89, p. 65]), so that, for each fixed $t>0$, the function $e^{-t[(|\xi|^2+m^2)^{\alpha/2}-m^\alpha]}$ is positive definite. We note that this is a specific case of a Bernstein function, providing the kinetic term of more general nonlocal Schrödinger operators that we studied in [HILo12].

As a result, its Fourier transform is a nonnegative function for each $t>0$, which is nothing but the integral kernel $k_0^{m,\alpha}(t,x)$ of the semigroup $e^{-t[(H_{0,m})^\alpha-m^\alpha]}$ satisfying $\int_{\mathbb{R}^d} k_0^{m,\alpha}(t,x)dx = 1$. Furthermore, we see that the operator $(H_{0,m})^\alpha u$, say with $u \in C_0^\infty(\mathbb{R}^d)$, has an integral operator representation:

$$((H_{0,m})^\alpha u)(x) \equiv ([-\Delta+m^2]^{\alpha/2}u)(x) \equiv (\mathcal{F}^{-1}(|\xi|^2+m^2)^{\alpha/2}\mathcal{F}u)(x)$$

$$= m^\alpha u(x) - \int_{|y|>0}[u(x+y)-u(x)-I_{\{|y|<1\}}\, y\cdot\nabla_x u(x)]n^{m,\alpha}(dy), \tag{2.6}$$

where $n^{m,\alpha}(dy)$ is a σ-finite measure on $\mathbb{R}^d\setminus\{0\}$ depending on $m\ge 0$ and $0<\alpha\le 1$, called *Lévy measure*, that satisfies $\int_{|y|>0}\frac{|y|^2}{1+|y|^2}n^{m,\alpha}(dy)<\infty$. The Lévy measure is known [IkW62, Exa. 1, p. 81] to be given from $k_0^{m,\alpha}(t,x)$ through

$$\frac{1}{t}k_0^{m,\alpha}(t,dy) \;\to\; n^{m,\alpha}(dy), \quad t\downarrow 0. \tag{2.7}$$

In our case, it has density $n^{m,\alpha}(dy) = n^{m,\alpha}(y)\,dy$.

For the expressions for the integral kernel $k_0^{m,\alpha}(t,x)$ of $e^{-t[(H_{0,m})^\alpha-m^\alpha]}$ and the density function $n^{m,\alpha}(y)$, see Appendix A, (A.2). For $\alpha=1$, they are explicitly given (e.g., [I89, (2.4ab), (2.2ab), pp. 268–269], [LLos01, 7.11 (11)]) as

$$k_0^{m,1}(t,x) = \begin{cases} 2\left(\frac{m}{2\pi}\right)^{(d+1)/2}\frac{te^{mt}K_{(d+1)/2}(m(x^2+t^2)^{1/2})}{(x^2+t^2)^{(d+1)/4}}, & m>0,\\ \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{(d+1)/2}}\frac{t}{(x^2+t^2)^{(d+1)/2}}, & m=0,\end{cases} \tag{2.8}$$

$$n^{m,1}(y) = \begin{cases} 2\left(\frac{m}{2\pi}\right)^{(d+1)/2}\frac{K_{(d+1)/2}(m|y|)}{|y|^{(d+1)/2}}, & m>0,\\ \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{(d+1)/2}}\frac{1}{|y|^{d+1}}, & m=0,\end{cases} \tag{2.9}$$

where $K_\nu(\tau)$ is the modified Bessel function of the third kind of order ν, which satisfies $0 < K_\nu(\tau) \le C \max\{\tau^{-\nu}, \tau^{-1/2}\} e^{-\tau}$, $\tau > 0$ with a constant $C > 0$ when $\nu \ge \frac{1}{2}$.

For our later use, let us calculate the commutator $\big[(H_{A,m})^2, \psi\big]$ with $\psi \in C_0^\infty(\mathbb{R}^d)$. Here, for two operators U and V, their *commutator* is denoted by $[U,V] := UV - VU$. We have

$$\begin{aligned} \big[(H_{A,m})^2, \psi\big] &= (-i\nabla - A)^2\psi - \psi(-i\nabla - A)^2 \\ &= (i\nabla + A)(i\nabla\psi) + (i\nabla\psi)(i\nabla + A) \qquad (2.10a) \\ &= [(\Delta\psi) + 2(i\nabla + A)(i\nabla\psi)] \quad \text{or} \quad [(-\Delta\psi) + 2(i\nabla\psi)(i\nabla + A)], \qquad (2.10b) \end{aligned}$$

as quadratic forms, i.e., for suitable functions u, v on $\mathbb{R}^d$,

$$\begin{aligned} \big(u, \big[(H_{A,m})^2, \psi\big]v\big) &= \big((i\nabla\psi)(i\nabla + A)u, v\big) + \big(u, (i\nabla\psi)(i\nabla + A)v\big) \\ &= \big(u, (\Delta\psi)v\big) + 2\big(u, (i\nabla + A)(i\nabla\psi)v\big) \\ &\quad \text{or } \big(u, (-\Delta\psi)v\big) + 2\big(u, (i\nabla\psi)(i\nabla + A)v\big). \end{aligned}$$

Here, note that $[i\nabla + A, \psi]v = (i\nabla\psi)v$, as well as $[i\nabla + A, (i\nabla\psi)]v = (-\Delta\psi)v$. In fact, it holds more generally with two $\mathbb{R}^d$-valued functions A and B that for a function v in $\mathbb{R}^d$,

$$\begin{aligned} [(H_{A,m})^2\psi - \psi(H_{B,m})^2]v &= (i\nabla + A)\big((i\nabla\psi) + \psi A\big)v \\ &\quad + \big((i\nabla\psi) - \psi B\big)(i\nabla + B)v \qquad (2.11) \\ &\quad + \psi A(i\nabla v) - i\nabla(\psi B v). \end{aligned}$$

Indeed, the left-hand side of (2.11) can be seen to be equal to

$$\begin{aligned} &\big[(-i\nabla - A)^2\psi - \psi(-i\nabla - B)^2\big]v \\ &\qquad = \big[(i\nabla + A)(i\nabla + A)\psi - \psi(i\nabla + B)(i\nabla + B)\big]v \\ &\qquad = (i\nabla + A)\big[\big((i\nabla\psi) + \psi A\big)v + \big(i\nabla(\psi v) - (i\nabla\psi)v\big)\big] \\ &\qquad\quad + \big[\big((i\nabla\psi) - \psi B\big) - \big(\psi(i\nabla) + (i\nabla\psi)\big)\big](i\nabla + B)v \\ &\qquad = (i\nabla + A)\big((i\nabla\psi) + \psi A\big)v + (i\nabla + A)(\psi i\nabla v) \\ &\qquad\quad + \big((i\nabla\psi) - \psi B\big)(i\nabla + B)v - i\nabla\big(\psi(i\nabla + B)v\big) \\ &\qquad = (i\nabla + A)\big((i\nabla\psi) + \psi A\big)v + \big((i\nabla\psi) - \psi B\big)(i\nabla + B)v \\ &\qquad\quad + \psi A(i\nabla v) - i\nabla(\psi B v). \end{aligned}$$

This shows (2.11). Taking $B = A$ in (2.11) yields (2.10a), which implies (2.10b).

For the next lemma, we briefly mention the *weak* L^1-space $L^1_{\mathrm{w}}(X)$, given a measurable subset X of $\mathbb{R}^d$. It is by definition the linear space of all measurable functions f on X such that

$$\|f\|_{L^1_{\rm w}} := \sup_{a>0} a\,|\{x \in X;\, |f(x)| > a\}| \tag{2.12}$$

is finite, where $|Y|$ denotes the volume (Lebesgue measure) of the measurable set $Y \subseteq \mathbb{R}^d$. Note that $L^1_{\rm w}(X)$ is not a Banach space, because $\|f\|_{L^1_{\rm w}}$ is not a norm but a quasi-norm, as it does not satisfy the triangle inequality. However, it holds that $\|f+g\|_{L^1_{\rm w}} \leq 2(\|f\|_{L^1_{\rm w}} + \|g\|_{L^1_{\rm w}})$. It is shown that $L^1_{\rm w}(X)$ is a *quasi-normed complete* linear space (see, e.g., [G10, Def. 1.1.5, pp. 5–6]). We have $\|f\|_{L^1_{\rm w}} \leq \|f\|_{L^1}$, so that $L^1(X) \subseteq L^1_{\rm w}(X)$. If $f_n \to f$ in $L^1_{\rm w}$, then the $\{f_n\}$ converges to f in measure (e.g., [G10, Prop. 1.1.9, p. 7]). We say "*f is locally in $L^1_{\rm w}$*", if for every compact set K in $\mathbb{R}^d$, f belongs to $L^1_{\rm w}(K)$. In some literature $L^1_{\rm w}(X)$ may also be denoted by $L^{1,\infty}(X)$ (*Lorentz space*).

Lemma 2.3. *Let $0 < \alpha \leq 1$. Let $\psi \in C_0^\infty(\mathbb{R}^d)$. Then, for the commutator $[(H_{0,m})^\alpha, \psi]$, with a constant C_α dependent on ψ and α but independent of $m \geq 0$, it holds that*

(i) for $1 < p < \infty$,

$$\|[(H_{0,m})^\alpha, \psi]u\|_{L^p} = \|(H_{0,m})^\alpha(\psi u) - \psi(H_{0,m})^\alpha u\|_{L^p} \leq C_\alpha \|u\|_{L^p}, \tag{2.13}$$

for all $u \in L^p(\mathbb{R}^d)$. Therefore if both u and $(H_{0,m})^\alpha(\psi u)$ are in L^p, then $\psi(H_{0,m})^\alpha u$ is in L^p, and

$$\|\psi(H_{0,m})^\alpha u\|_{L^p} \leq C_\alpha \|u\|_{L^p} + \|(H_{0,m})^\alpha(\psi u)\|_{L^p};$$

(ii) for $p = 1$,

$$\|[(H_{0,m})^\alpha, \psi]u\|_{L^1_{\rm w}} = \|(H_{0,m})^\alpha(\psi u) - \psi(H_{0,m})^\alpha u\|_{L^1_{\rm w}} \leq C_\alpha \|u\|_{L^1}, \tag{2.14}$$

for all $u \in L^1(\mathbb{R}^d)$.

Remark. Inequality (2.13) does not hold for $p = 1$, and instead we have (2.14) with the L^1-norm on the left-hand side replaced by the $L^1_{\rm w}$-quasi-norm. This is dependent on the Calderón–Zygmund theorem (for this see Proposition 2.4 below).

Proof of Lemma 2.3. (i) As the second-half assertion follows from the first, i.e., inequality (2.13), we have only to show (2.13), and even only for $u \in C_0^\infty(\mathbb{R}^d)$, since $C_0^\infty(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$. The proof for the case $\alpha = 1$ was given in [ITs92, p. 274, Lem. 2.3] by using the integral operator representation (2.6) of $H_{0,m} = \sqrt{-\Delta + m^2}$. The proof for the case $0 < \alpha < 1$ is similar. So we give only an outline.

Use (2.6) to rewrite $[(H_{0,m})^\alpha, \psi]$ as

$$
\begin{aligned}
&([(H_{0,m})^\alpha, \psi]u)(x) \\
&\quad = -\int_{|y|>0} [\psi(x+y) - \psi(x) - I_{\{|y|<1\}} y \cdot \nabla_x \psi(x)] u(x+y) n^{m,\alpha}(dy) \\
&\qquad - \int_{0<|y|<1} y \cdot \nabla_x \psi(x) [u(x+y) - u(x)] n^{m,\alpha}(dy) \\
&\quad =: (I_1 u)(x) + (I_2 u)(x).
\end{aligned} \tag{2.15}
$$

We estimate the L^p-norms of $I_1 u$ and $I_2 u$ in the last line.

First, rewrite $I_1 u$ as

$$
\begin{aligned}
(I_1 u)(x) = &-\int_{0<|y|<1} [\psi(x+y) - \psi(x) - I_{\{|y|<1\}} y \cdot \nabla_x \psi(x)] u(x+y) n^{m,\alpha}(dy) \\
&- \int_{|y|\geq 1} [\psi(x+y) - \psi(x)] u(x+y) n^{m,\alpha}(dy).
\end{aligned}
$$

Hence

$$
\begin{aligned}
|(I_1 u)(x)| \leq \|\nabla^2 \psi\|_{L^\infty} &\int_{0<|y|<1} |y|^2 |u(x+y)| n^{m,\alpha}(dy) \\
&+ 2\|\psi\|_{L^\infty} \int_{|y|>1} |u(x+y)| n^{m,\alpha}(dy),
\end{aligned}
$$

so that for $1 \leq p < \infty$,

$$
\|I_1 u\|_{L^p} = \Big(\int |(I_1 u)(x)|^p dx \Big)^{1/p} \leq \big(n_1^{m,\alpha} \|\nabla^2 \psi\|_{L^\infty} + 2 n_\infty^{m,\alpha} \|\psi\|_{L^\infty} \big) \|u\|_{L^p},
$$

where

$$
n_\infty^{m,\alpha} := \int_{|y|\geq 1} n^{m,\alpha}(dy), \qquad n_\kappa^{m,\alpha} := \int_{0<|y|<1} |y|^{1+\kappa} n^{m,\alpha}(dy), \tag{2.16}
$$

where the former is finite, and the latter is finite for all $0 < \kappa \leq 1$.

Next, for $I_2 u$ we use the following known fact for an operator T on $L^p(\mathbb{R}^d)$ with Calderón–Zygmund kernel $K : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ (e.g., [St70, II.3, pp. 35–42], [G10, Thm. 5.3.3, p. 359], [MSc13, Def. 7.1, Prop. 7.4, Thm. 7.5, pp. 166–172]). It is the integral kernel that satisfies, for some constant $B > 0$, the following conditions:

(i) $|K(x)| \leq B|x|^{-d}$ for all $x \in \mathbb{R}^d$;

(ii) $\int_{|x|\geq 2|y|} |K(x) - K(x-y)|\, dx \leq B$ for all $y \neq 0$;

(iii) $\int_{R_1<|x|<R_2} K(x)\ dx = 0$ for all $0 < R_1 < R_2 < \infty$.

Proposition 2.4. *Let*

$$(Tf)(x) := \lim_{\varepsilon \downarrow 0} \int_{|x-y|\geq \epsilon} K(x-y)f(y)\,dy.$$

Then

$$\begin{aligned} &\|Tf\|_{L^p} \leq C_p \|f\|_{L^p}, && 1<p<\infty, \\ &\|Tf\|_{L^1_{\mathrm{w}}} \equiv \sup_{a>0} a\left|\{x\in\mathbb{R}^d;\ |(Tf)(x)|>a\}\right| \leq C_1\|f\|_{L^1}, && p=1. \end{aligned}$$

This proposition is going to be used just in the proof of Lemma 2.3(i).

We continue the proof of Lemma 2.3(i). It still remains to deal with $I_2 u$, which is rewritten as

$$(I_2u)(x) = -\sum_{j=1}^d \lim_{\varepsilon\downarrow 0}\int_{\varepsilon\leq|y|<1} \partial_{x_j}\psi(x)\,(x_j-y_j)n^{m,\alpha}(x-y)u(y)\,dy.$$

Here each $x_j \cdot n^{m,\alpha}(x)$, $1\leq j\leq d$, is a Calderón–Zygmund kernel (see Appendix A, (A.2)), so that we have by Proposition 2.4 with $1\leq p<\infty$ that there exists a constant $C_p>0$ such that

$$\begin{aligned} &\|I_2u\|_{L^p} \leq C_p\|\nabla\psi\|_{L^\infty}\|u\|_{L^p}, && 1<p<\infty, \\ &\|I_2u\|_{L^1_{\mathrm{w}}} = \sup_{a>0} a\left|\{x\in\mathbb{R}^d;\ |(I_2u)(x)|>a\}\right| \leq C_1\|\nabla\psi\|_{L^\infty}\|u\|_{L^1}, && p=1. \end{aligned}$$

Thus we obtain

$$\begin{aligned} \|[(H_{0,m})^\alpha,\psi]u\|_{L^p} &\leq \|I_1u\|_{L^p} + \|I_2u\|_{L^p} \\ &\leq \big(n_1^{m,\alpha}\|\nabla^2\psi\|_{L^\infty} + 2n_\infty^{m,\alpha}\|\psi\|_{L^\infty} + C_p\|\nabla\psi\|_{L^\infty}\big)\|u\|_{L^p}, \end{aligned}$$

showing (i) for $1<p<\infty$.

Next, for (ii) for $p=1$, we have

$$\begin{aligned} \|[(H_{0,m})^\alpha,\psi]u\|_{L^1_{\mathrm{w}}} &\leq 2\big(\|I_1u\|_{L^1_{\mathrm{w}}} + \|I_2u\|_{L^1_{\mathrm{w}}}\big) \leq 2\|I_1u\|_{L^1} + 2\|I_2u\|_{L^1_{\mathrm{w}}} \\ &\leq 2\big(n_1^{m,\alpha}\|\nabla^2\psi\|_{L^\infty} + 2n_\infty^{m,\alpha}\|\psi\|_{L^\infty} + C_1\|\nabla\psi\|_{L^\infty}\big)\|u\|_{L^1}, \end{aligned}$$

because $\|I_1u\|_{L^1_{\mathrm{w}}} \leq \|I_1u\|_{L^1}$. This shows (ii), ending the proof of Lemma 2.3. □

When $A\in L^2_{\mathrm{loc}}$, our self-adjoint operator $S := (-i\nabla - A)^2 + m^2$ was originally defined as the self-adjoint operator in $L^2(\mathbb{R}^d)$ associated with the closed quadratic form (2.1). As already noted in the proof of Lemma 2.1(i), it also makes sense as an operator in the spaces $L^p(\mathbb{R}^d)$, $1\leq p<\infty$, referring to the results [Si79, Thm. 2.3] or [Si82, Sec. B13] that the Schrödinger semigroup $e^{-tS} = e^{-t[(-i\nabla-A)^2+m^2]}$ satisfies

$$|e^{-t[(-i\nabla-A)^2+m^2]}g| \leq e^{-t[-\Delta+m^2]}|g| \tag{2.17}$$

pointwise for any $g \in L^2(\mathbb{R}^d)$. This yields that for $1 \le p < \infty$, $e^{-t(H_{A,m})^2}$ is a bounded operator of $L^p(\mathbb{R}^d)$ into itself for all $t > 0$, which also is a contraction semigroup.

Thus, the fractional powers of S such as $S^{\frac{\alpha}{2}} = (H_{A,m})^\alpha$ in (2.3) make equal sense in $L^p(\mathbb{R}^d)$.

Now, we give two crucial Lemmas 2.5 and 2.6.

Lemma 2.5. *Let $0 < \alpha < 1$ and assume that $A \in [L^2_{\mathrm{loc}}(\mathbb{R}^d)]^d$. Then*

(i) if $u \in L^2(\mathbb{R}^d)$, one has for χ, $\psi \in C_0^\infty(\mathbb{R}^d)$,

$$\begin{aligned}\|\chi[(H_{0,m})^\alpha\psi - \psi(H_{A,m})^\alpha]u\|_{L^1}& \\ \equiv \big\|\chi\big([-\Delta+m^2]^{\alpha/2}\psi - \psi[(-i\nabla - A)^2 + m^2]^{\alpha/2}\big)u\big\|_{L^1}& \\ \le C_{\alpha,A,m,\chi,\psi}\|u\|_{L^2},& \qquad (2.18)\end{aligned}$$

where $C_{\alpha,A,\chi,\psi}$ is a constant that depends on $0 < \alpha < 1$, A, $m > 0$, χ and ψ, and that tends to ∞ as $\alpha \uparrow 1$;

(ii) in particular, when $A = 0$, (2.18) reads, if $u \in L^2(\mathbb{R}^d)$, one has

$$\|\chi[(H_{0,m})^\alpha, \psi]u\|_{L^1} \le C_{\alpha,0,m,\chi,\psi}\|u\|_{L^2}. \qquad (2.19)$$

For $A = 0$, inequality (2.19) appears more useful in comparison with (2.14).

Proof of Lemma 2.5. (i) We have only to show (2.18) when $u \in C_0^\infty(\mathbb{R}^d)$, since $C_0^\infty(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$. Note then that $H_{0,m}u$ and $H_{A,m}u$ belong to $L^2(\mathbb{R}^d)$.

We use formula (2.3) for $(H_{0,m})^\alpha$ as well as $(H_{A,m})^\alpha$ to calculate

$$\begin{aligned}&[(H_{0,m})^\alpha\psi - \psi(H_{A,m})^\alpha]u \\ &\quad = \frac{1}{\Gamma(\frac{2-\alpha}{2})}\int_0^\infty t^{-\alpha/2}\Big[e^{-t(H_{0,m})^2}(H_{0,m})^2\psi - \psi(H_{A,m})^2 e^{-t(H_{A,m})^2}\Big]u\,dt \\ &\quad = \frac{1}{\Gamma(\frac{2-\alpha}{2})}\int_0^\infty dt\,t^{-\alpha/2}\big(-\frac{d}{dt}\big)\Big[e^{-\theta t(H_{0,m})^2}\psi e^{-(1-\theta)t(H_{A,m})^2}\Big]_{\theta=0}^{\theta=1}u \\ &\quad = -\frac{1}{\Gamma(\frac{2-\alpha}{2})}\int_0^\infty dt\,t^{-\alpha/2}\frac{d}{dt}\int_0^1 d\theta\,\frac{d}{d\theta}\Big[e^{-\theta t(H_{0,m})^2}\psi e^{-(1-\theta)t(H_{A,m})^2}\Big]u \\ &\quad = \frac{1}{\Gamma(\frac{2-\alpha}{2})}\int_0^\infty dt\,t^{-\alpha/2} \\ &\qquad \times \frac{d}{dt}\Big(t\int_0^1 d\theta\Big[e^{-\theta t(H_{0,m})^2}[(H_{0,m})^2\psi - \psi(H_{A,m})^2]e^{-(1-\theta)t(H_{A,m})^2}\Big]u\Big).\end{aligned}$$

Then by integration by parts,

$$
\begin{aligned}
&[(H_{0,m})^\alpha \psi - \psi(H_{A,m})^\alpha]u \\
&\text{(2.20a)} \\
&= \frac{1}{\Gamma(\frac{2-\alpha}{2})} \\
&\quad \times \left[t^{-(\alpha/2)+1} \int_0^1 \Big(e^{-\theta t (H_{0,m})^2} \big[(H_{0,m})^2\psi - \psi(H_{A,m})^2\big] e^{-(1-\theta)t(H_{A,m})^2} \Big) u \, d\theta \right]_{t=0}^{t=\infty} \\
&\quad + \frac{\alpha}{2\Gamma(\frac{2-\alpha}{2})} \\
&\quad \times \int_0^\infty dt\, t^{-\alpha/2} \int_0^1 d\theta \, \Big(e^{-\theta t (H_{0,m})^2} \big[(H_{0,m})^2\psi - \psi(H_{A,m})^2\big] e^{-(1-\theta)t(H_{A,m})^2} \Big) u \\
&\text{(2.20b)} \\
&= \frac{\alpha}{2\Gamma(\frac{2-\alpha}{2})} \\
&\quad \times \int_0^\infty dt\, t^{-\alpha/2} \int_0^1 d\theta \Big(e^{-\theta t (H_{0,m})^2} \big[(H_{0,m})^2\psi - \psi(H_{A,m})^2\big] e^{-(1-\theta)t(H_{A,m})^2} \Big) u.
\end{aligned}
$$

Here we make two observations related to (2.20). First for (2.20a), the boundary value at $t \to \infty$ of the first term also vanishes, because the part

$$e^{-\theta t (H_{0,m})^2} \big[\cdots \big] e^{-(1-\theta)t(H_{A,m})^2} = e^{-\theta t(-\Delta + m^2)} \big[\cdots \big] e^{-(1-\theta)t[(-i\nabla - A)^2 + m^2]}$$

contains the factor $e^{-m^2 t}$. Second for (2.20b), note that the middle factor in the integrand is, by (2.11) with $A := 0$, $B := A$, equal to

$$[(H_{0,m})^2\psi - \psi(H_{A,m})^2] = \big[i\nabla\big((i\nabla\psi) - \psi A\big) + \big((i\nabla\psi) - \psi A\big)(i\nabla + A)\big] \tag{2.21}$$

as quadratic forms.

Substituting (2.21) into (2.20), we have with $\chi \in C_0^\infty(\mathbb{R}^d)$,

$$
\begin{aligned}
&\chi[(H_{0,m})^\alpha \psi - \psi(H_{A,m})^\alpha]u \\
&= \frac{\alpha}{2\Gamma(\frac{2-\alpha}{2})} \int_0^\infty dt\, t^{-\alpha/2} \int_0^1 d\theta \, \chi \Big(e^{-\theta t (H_{0,m})^2} i\nabla\big((i\nabla\psi) - \psi A\big)\, e^{-(1-\theta)t(H_{A,m})^2} \Big) u \\
&\quad + \frac{\alpha}{2\Gamma(\frac{2-\alpha}{2})} \int_0^\infty dt\, t^{-\alpha/2} \\
&\quad \times \int_0^1 d\theta \, \chi \Big(e^{-\theta t (H_{0,m})^2} \big((i\nabla\psi) - \psi A\big)(i\nabla + A)\, e^{-(1-\theta)t(H_{A,m})^2} \Big) u \\
&\text{(2.22)} \\
&=: I_3 u + I_4 u.
\end{aligned}
$$

We estimate the L^1-norm for I_3u and I_4u in (2.22). Note that $e^{-t(-i\nabla-A)^2}$, $t \geq 0$ is a contraction on $L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$.

First, for I_3u, integrate its absolute value in x to get

$$\|I_3u\|_{L^1} \leq \frac{\alpha}{2\Gamma(\frac{2-\alpha}{2})} \int_0^\infty t^{-\alpha/2} e^{-m^2 t} dt \int_0^1 d\theta \\ \times \big\|\chi[e^{-\theta t(-\Delta)}(i\nabla)]\big((i\nabla\psi) - \psi A\big)\, e^{-(1-\theta)t(-i\nabla-A)^2} u\big\|_{L^1}. \tag{2.23}$$

Then by Lemma 2.1(ii) for $p = 1$, the Schwarz inequality and Lemma 2.1(i) we obtain

$$\begin{aligned}\|I_3u\|_{L^1} &\leq \frac{\alpha}{2\Gamma(\frac{2-\alpha}{2})} \int_0^\infty t^{-\alpha/2} e^{-m^2 t} dt \int_0^1 d\theta \|\chi\|_{L^\infty} \|e^{-\theta t(-\Delta)}(i\nabla)\|_{[L^1]^d \to [L^1]^d} \\ &\quad \times \|\big((i\nabla\psi) - \psi A\big)\, e^{-(1-\theta)t(-i\nabla-A)^2} u\big\|_{L^1 \to [L^1]^d} \\ &\leq \frac{\alpha}{2\Gamma(\frac{2-\alpha}{2})} \int_0^\infty t^{-\alpha/2} e^{-m^2 t} dt \int_0^1 d\theta \\ &\quad \times \|\chi\|_{L^\infty} C_{11} (\theta t)^{-1/2} \|(i\nabla\psi) - \psi A\|_{L^2} \, \|e^{-(1-\theta)t(-i\nabla-A)^2} u\|_{L^2} \\ &\leq \frac{C_{11}\alpha}{2\Gamma(\frac{2-\alpha}{2})} \int_0^\infty t^{-(1+\alpha)/2} e^{-m^2 t} dt \int_0^1 \frac{d\theta}{\theta^{1/2}} \|\chi\|_{L^\infty} \|(i\nabla\psi) - \psi A\|_{L^2} \|u\|_{L^2}.\end{aligned}$$

Here recall that $\|(i\nabla\psi) - \psi A\big\|_{L^2} < \infty$ by assumption on A and notice also that

$$\int_0^\infty t^{-(1+\alpha)/2} e^{-m^2 t} dt = \Gamma(\tfrac{1-\alpha}{2})\, m^{-(1-\alpha)/2},$$

which diverges as $\alpha \uparrow 1$ with $m > 0$. Thus we have

$$\|I_3u\|_{L^1} \leq \frac{C_{11}\alpha\Gamma(\frac{1-\alpha}{2})}{\Gamma(\frac{2-\alpha}{2})\, m^{(1-\alpha)/2}} \big\|(i\nabla\psi) - \psi A\big\|_{L^2} \|\chi\|_{L^\infty} \big\|u\big\|_{L^2}. \tag{2.24}$$

Next for I_4u, in a similar way, we have from (2.22)

$$\|I_4u\|_{L^1} \leq \frac{\alpha}{2\Gamma(\frac{2-\alpha}{2})} \int_0^\infty t^{-\alpha/2} e^{-m^2 t} dt \int_0^1 d\theta \big\|\chi\big(e^{-\theta t(-\Delta)}\big((i\nabla\psi) - \psi A\big) \\ \times [(i\nabla + A)\, e^{-(1-\theta)t(-i\nabla-A)^2}]\big) u\|_{L^1}. \tag{2.25}$$

Then by the Schwarz inequality and Lemma 2.1(iv),

$$\|I_4u\|_{L^1} \leq \frac{\alpha}{2\Gamma(\frac{2-\alpha}{2})} \int_0^\infty t^{-\alpha/2} e^{-m^2 t} dt \int_0^1 d\theta \|\chi\|_{L^\infty} \|e^{-\theta t(-\Delta)}\|_{L^1 \to L^1} \\ \times \|\big((i\nabla\psi) - \psi A\big)[(i\nabla + A)\, e^{-(1-\theta)t(-i\nabla-A)^2}]\big) u\|_{L^1}$$

$$
\begin{aligned}
&\le \frac{\alpha}{2\Gamma(\frac{2-\alpha}{2})}\int_0^\infty t^{-\alpha/2}e^{-m^2t}dt \\
&\quad \times \int_0^1 d\theta \|\chi\|_{L^\infty}\|e^{-\theta t(-\Delta)}\|_{L^1\to L^1}\|(i\nabla\psi)-\psi A\|_{L^2} \\
&\quad \times \|(i\nabla + A)\, e^{-(1-\theta)t(-i\nabla-A)^2}]u\|_{L^2} \\
&\le \frac{\alpha}{2\Gamma(\frac{2-\alpha}{2})}\int_0^\infty t^{-\alpha/2}e^{-m^2t}dt \\
&\quad \times \int_0^1 d\theta \|\chi\|_{L^\infty}\|(i\nabla\psi)-\psi A\|_{L^2}\Big(\frac{d}{2e(1-\theta)}\Big)^{1/2}\|u\|_{L^2} \\
&= \Big(\frac{d}{2e}\Big)^{1/2}\frac{\alpha}{2\Gamma(\frac{2-\alpha}{2})}\int_0^\infty t^{-(1+\alpha)/2}e^{-m^2t}dt \\
&\quad \times \int_0^1 \frac{d\theta}{(1-\theta)^{1/2}}\|\chi\|_{L^\infty}\|(i\nabla\psi)-\psi A\|_{L^2}\|u\|_{L^2}.
\end{aligned}
$$

Then we have

$$
\|I_4u\|_{L^1} \le \Big(\frac{d}{2e}\Big)^{1/2}\frac{\alpha\Gamma(\frac{1-\alpha}{2})}{\Gamma(\frac{2-\alpha}{2})\, m^{(1-\alpha)/2}}\big\|(i\nabla\psi)-\psi A\big\|_{L^2}\|\chi\|_{L^\infty}\|\chi\|_{L^\infty}\|u\|_{L^2}. \tag{2.26}
$$

Putting (2.24) and (2.26) together in view of (2.22), we have

$$
\begin{aligned}
&\|\chi[(H_{0,m})^\alpha\psi - \psi(H_{A,m})^\alpha]u\|_{L^1} \\
&\quad \le 2\big(\|I_3u\|_{L^1} + \|I_4u\|_{L^1}\big) \\
&\quad \le 2\frac{C_{11} + \big(\frac{d}{2e}\big)^{1/2}}{m^{(1-\alpha)/2}}\frac{\alpha\Gamma(\frac{1-\alpha}{2})}{\Gamma(\frac{2-\alpha}{2})}\big\|(i\nabla\psi)-\psi A\big\|_{L^2}\|\chi\|_{L^\infty}\big\|u\big\|_{L^2}. \qquad (2.27)
\end{aligned}
$$

This yields (2.18), showing Lemma 2.5(i).
(ii) Inequality (2.19) is immediately derived by putting $A = 0$ in (2.18).

This shows Lemma 2.5(ii), completing the proof of Lemma 2.5. □

From Lemma 2.5 we have the following result that we shall need, particularly assertion (ii), in the proof of Theorem 1.1.

Lemma 2.6. *Let $0 < \alpha < 1$. Assume that $A \in [L^2_{\rm loc}(\mathbb{R}^d)]^d$.*

(i) If $u \in (C^\infty \cap L^2)(\mathbb{R}^d)$, then $(H_{A,m})^\alpha u$ is locally in $L^1(\mathbb{R}^d)$.
(ii) If $u \in L^2(\mathbb{R}^d)$ with $(H_{A,m})^\alpha u \in L^1_{\rm loc}(\mathbb{R}^d)$, then $(H_{0,m})^\alpha u$ is locally in $L^1(\mathbb{R}^d)$.

Proof. (i) Let $u \in (C^\infty \cap L^2)(\mathbb{R}^d)$. Then for $\psi \in C_0^\infty(\mathbb{R}^d)$,

$$
\psi(H_{A,m})^\alpha u = (H_{0,m})^\alpha(\psi u) + \big(\psi(H_{A,m})^\alpha - (H_{0,m})^\alpha\psi\big)u.
$$

Put $K = \operatorname{supp}\psi$. Then, since ψu is in $C_0^\infty(\mathbb{R}^d)$, the first term $(H_{0,m})^\alpha(\psi u)$ on the right-hand side belongs to $L^2(\mathbb{R}^d)$, as we can see from (2.6) (with ψu instead of u) or Lemma 2.2, (2.4) with $A = 0$ (with ψu instead of φ). For the second term restricted to K, it belongs to $L^1(K)$, as we can see by Lemma 2.5, (2.18). Therefore $\psi(H_{A,m})^\alpha u$ is in $L^1(K)$, so that $(H_{A,m})^\alpha u$ is locally in $L^1(\mathbb{R}^d)$. This proves assertion (i).

(ii) Let $u \in L^2$ with $(H_{A,m})^\alpha u \in L^1_{\mathrm{loc}}$ and let K be an arbitrary compact subset of $\mathbb{R}^d$. Take χ, $\psi \in C_0^\infty(\mathbb{R}^d)$ with $0 \le \chi(x) \le 1$ such that $\chi(x) = \psi(x) = 1$ on K. Then since

$$\psi(H_{0,m})^\alpha u - \psi(H_{A,m})^\alpha u = -[(H_{0,m})^\alpha, \psi]u + \big((H_{0,m})^\alpha\psi - \psi(H_{A,m})^\alpha\big)u,$$

we have by Lemma 2.5, (2.18) with $A = 0$ as well as with nonzero A,

$$\begin{aligned}
&\|(H_{0,m})^\alpha u - \psi(H_{A,m})^\alpha u\|_{L^1(K)} \\
&\quad = \|\chi\psi[(H_{0,m})^\alpha u - (H_{A,m})^\alpha u]\|_{L^1(K)} \\
&\quad \le \|\chi[(H_{0,m})^\alpha, \psi]u\|_{L^1} + \|\chi\big((H_{0,m})^\alpha\psi - \psi(H_{A,m})^\alpha\big)u\|_{L^1} \\
&\quad \le (C_{\alpha,0,m,\chi,\psi} + C_{\alpha,A,m,\chi,\psi})\|u\|_{L^2} < \infty.
\end{aligned}$$

Since, by assumption, $(H_{A,m})^\alpha u$ is locally in $L^1(\mathbb{R}^d)$, we have that $(H_{0,m})^\alpha u$ is locally $L^1(\mathbb{R}^d)$. This proves assertion (ii), ending the proof of Lemma 2.6. □

§3. Proof of the theorems

We show only Theorem 1.1 and Theorem 1.2. As for Theorem 1.3, the essential self-adjointness of $H_{A,V,m}$ follows from Theorem 1.1 by its standard application in Kato's original paper [K72]. In fact, it can be shown in the same way as in [I89, Thm. 5.1]. So the proof is omitted. The assertion that $H_{A,V,m} = H_{A,m} + V \ge m$ is trivial because $H_{A,m} \ge m$.

In this section, we continue to assume that $m > 0$ until we come to the final part (iii) of the proof of Theorem 1.1.

§3.1. Proof of Theorem 1.1

The proof will proceed similarly to Kato's original proof [K72] (e.g., [ReSi75, Thms. X.27 (p. 183), X.33 (p. 188)]) for the magnetic *nonrelativistic* Schrödinger operator $\frac{1}{2m}(-i\nabla - A(x))^2$ and to a modified one [I89, ITs92] for *another* magnetic relativistic Schrödinger operator. However, if one could show the assumption of the theorem that $u \in L^2$ with $H_{A,m}u \in L^1_{\mathrm{loc}}$ implies that $\partial_j u \in L^1_{\mathrm{loc}}$, $1 \le j \le d$, and/or $H_{0,m}u \in L^1_{\mathrm{loc}}$, there should be no problem. The obstruction seems to come from the fact that the operators $\partial_j \cdot (-\Delta + m^2)^{-1/2}$, $1 \le j \le d$ are *not* bounded from L^1 to L^1, though they are *bounded* from L^1 to L^1_{w}. The strategy we adopt to cope with this difficulty is, at the beginning, to make a detour by considering the case $(H_{A,m})^\alpha$ for $\alpha < 1$, leaving the case $\alpha = 1$ aside, in order to handle local convergence in L^1. In fact, in the first stage (Lemmas 3.1 and 3.2), we show first that if $(H_{A,m})^\alpha u \in L^1_{\mathrm{loc}}$, then $(H_{A,m})^\alpha u^\delta \to (H_{A,m})^\alpha u$ locally in L^1 as $\delta \downarrow 0$, and making use of Lemma 2.6 saying that $(H_{0,m})^\alpha u$ is locally in L^1. Next we show that the assumption $H_{A,m} u \in L^1_{\mathrm{loc}}$ implies that $(H_{A,m})^\alpha u \in L^1_{\mathrm{loc}}$ for $0 < \alpha < 1$, and $(H_{A,m})^\alpha u$ converges to $H_{A,m} u$ in L^1_{loc} as $\alpha \uparrow 1$. In the second and main stage, with $m > 0$, we show *first for* $0 < \alpha < 1$ that the asserted inequality, i.e.,

$$\mathrm{Re}((\mathrm{sgn}\, u)[(H_{A,m})^\alpha - m^\alpha]u) \ge [(H_{0,m})^\alpha - m^\alpha]|u|, \tag{3.1}$$

holds, and *next for* $\alpha = 1$, using the just-mentioned fact that $(H_{A,m})^\alpha u \to H_{A,m} u$ in L^1_{loc} as $\alpha \uparrow 1$. The final stage will deal with the remaining case for $m = 0$ and $\alpha = 1$.

We provide two lemmas playing a crucial role in the proof of Theorem 1.1.

For a function f locally in $L^1(\mathbb{R}^d)$, we write its *mollifier* as $f^\delta = \rho_\delta * f$, $0 < \delta \le 1$, where $\rho_\delta(x) := \delta^{-d}\rho(x/\delta)$, and $\rho(x)$ is a nonnegative C^∞ function $\mathbb{R}^d$ with compact support $\mathrm{supp}\,\rho \subseteq \{x;\, |x| \le 1\}$ and $\int \rho(x)dx = 1$.

Lemma 3.1. *Let $0 < \alpha < 1$. Let $u \in L^2(\mathbb{R}^d)$, so that $u^\delta := \rho_\delta * u \to u$ in L^2 as $\delta \downarrow 0$. If $(H_{A,m})^\alpha u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, then $(H_{A,m})^\alpha u^\delta = [(-i\nabla - A)^2 + m^2]^{\alpha/2} u^\delta \to (H_{A,m})^\alpha u = [(-i\nabla - A)^2 + m^2]^{\alpha/2} u$ locally in $L^1(\mathbb{R}^d)$ as $\delta \downarrow 0$.*

Proof. Let $u \in L^2$ and $(H_{A,m})^\alpha u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Then by Lemma 2.6(ii), $(H_{0,m})^\alpha u$ is locally in L^1 and since $u^\delta \in C^\infty \cap L^2$, we have by Lemma 2.6(i) that $(H_{A,m})^\alpha u^\delta$ is locally in L^1. The important point is that, thanks to the integral operator representation (2.6) of the operator $(H_{0,m})^\alpha$, the convolution commutes with $(H_{0,m})^\alpha$. Therefore we have $((H_{0,m})^\alpha u)^\delta = (H_{0,m})^\alpha u^\delta$, which converges to $(H_{0,m})^\alpha u$ locally in L^1 as $\delta \downarrow 0$. Then for a compact subset K in $\mathbb{R}^d$, let $\chi,\, \psi \in C_0^\infty(\mathbb{R}^d)$ with $0 \le \chi(x) \le 1$ on $\mathbb{R}^d$ and $\chi(x) = \psi(x) = 1$ on K. We have

$$\begin{aligned}
&\|(H_{A,m})^\alpha u^\delta - (H_{A,m})^\alpha u\|_{L^1(K)} \\
&\quad = \|\chi\psi(H_{A,m})^\alpha (u^\delta - u)\|_{L^1(K)} \\
&\quad = \|\chi\big[-(H_{0,m})^\alpha \psi + \big((H_{0,m})^\alpha \psi - \psi(H_{A,m})^\alpha\big)\big](u^\delta - u)\|_{L^1(K)} \\
&\quad \le \|\chi(H_{0,m})^\alpha \psi(u^\delta - u)\|_{L^1} + \|\chi[(H_{0,m})^\alpha \psi - \psi(H_{A,m})^\alpha](u^\delta - u)\|_{L^1}.
\end{aligned}$$

The second term in the last line of the above inequality is, by Lemma 2.5, (2.18), estimated from above by $C_{\alpha,A,m,\chi,\psi}\|u^\delta - u\|_{L^2}$. The first term is equal to

$$\begin{aligned}
&\|\chi\big([(H_{0,m})^\alpha,\psi] + \psi(H_{0,m})^\alpha\big)(u^\delta - u)\|_{L^1} \\
&\qquad \le \|\chi[(H_{0,m})^\alpha,\psi](u^\delta - u)\|_{L^1} + \|\chi\psi[(H_{0,m})^\alpha u^\delta - (H_{0,m})^\alpha u]\|_{L^1} \\
&\qquad \le C_{\alpha,0,m,\chi,\psi}\|u^\delta - u\|_{L^2} + \|((H_{0,m})^\alpha u)^\delta - (H_{0,m})^\alpha u\|_{L^1},
\end{aligned}$$

where, for the first term, we have used Lemma 2.5, (2.19) for $A = 0$, and for the second the fact that $(H_{0,m})^\alpha u^\delta = ((H_{0,m})^\alpha u)^\delta$ because, by assumption, $(H_{0,m})^\alpha u$ is locally in L^1 and $u \in L^2$. It follows that

$$\begin{aligned}
\|(H_{A,m})^\alpha u^\delta - (H_{A,m})^\alpha u\|_{L^1(K)} \le\ & C_{\alpha,0,m,\chi,\psi}\|u^\delta - u\|_{L^2} \\
&+ \|\psi\|_{L^\infty}\|((H_{0,m})^\alpha u)^\delta - (H_{0,m})^\alpha u\|_{L^1} \\
&+ C_{\alpha,A,m,\chi,\psi}\|u^\delta - u\|_{L^2},
\end{aligned}$$

which approaches zero as $\delta \downarrow 0$. This proves Lemma 3.1. □

Lemma 3.2. *Let $0 < \alpha \le 1$. Let $u \in L^2(\mathbb{R}^d)$ and $H_{A,m}u \in L^1_{\rm loc}(\mathbb{R}^d)$. Then $(H_{A,m})^\alpha u = [(-i\nabla - A)^2 + m^2]^{\alpha/2}u$ is also in $L^1_{\rm loc}(\mathbb{R}^d)$, and $\{(H_{A,m})^\alpha u\}$ converges to $H_{A,m}u$ in $L^1_{\rm loc}(\mathbb{R}^d)$ as $\alpha \uparrow 1$. Namely, for any $\psi \in C_0^\infty(\mathbb{R}^d)$, $\|\psi(H_{A,m})^\alpha u\|_{L^1}$ is uniformly bounded for $0 < \alpha \le 1$, and $\{\psi(H_{A,m})^\alpha u\}$ converges to $\psi H_{A,m}u$ in $L^1(\mathbb{R}^d)$ as $\alpha \uparrow 1$.*

Proof. Let $0 < \alpha < 1$. To begin with, suppose with $\psi \in C_0^\infty(\mathbb{R}^d)$ that some $u \in L^2(\mathbb{R}^d)$ satisfies the equality

$$\psi(H_{A,m})^\alpha u = (H_{A,m})^{-(1-\alpha)}\psi H_{A,m}u + [\psi, (H_{A,m})^{-(1-\alpha)}]H_{A,m}u. \tag{3.2}$$

This holds at least if $u \in D[H_{A,m}]$, and hence, in particular, if $u = \phi \in C_0^\infty(\mathbb{R}^d)$. Note here that $(H_{A,m})^\alpha$ has $D[H_{A,m}]$ as an operator core, while $H_{A,m}$ has $C_0^\infty(\mathbb{R}^d)$ as an operator core.

Now, let $u \in L^2(\mathbb{R}^d)$ with $H_{A,m}u \in L^1_{\rm loc}(\mathbb{R}^d)$, just what is assumed by Lemma 3.2. The first term on the right-hand side of (3.2) is in $L^1(\mathbb{R}^d)$, since by Lemma 2.1(i) with $p = 1$, $(H_{A,m})^{-(1-\alpha)}$ is a bounded operator that is a contraction mapping $L^1(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$, bounded uniformly for $0 < \alpha \le 1$ and strongly continuous there, so long as $m > 0$. The term on the left-hand side of (3.2) exists as a distribution. The second term on the right-hand side lies in the dual space of the space $D[H_{A,m}]$, considered as a Hilbert space equipped with the graph norm $\|v\|_{L^2}^2 + \|H_{A,m}v\|^2$. Here recall (2.1) and note that for $\phi \in C_0^\infty(\mathbb{R}^d)$,

$$\|(H_{A,m})^\alpha\phi\|_{L^2} = \|(H_{A,m})^{-(1-\alpha)}\, H_{A,m}\phi\|_{L^2} \le \|H_{A,m}\phi\|_{L^2}.$$

Thus all the three terms on the left- and right-hand sides of (3.2) exist also as distributions.

To show the assertion of the lemma, take a C^∞ cutoff function χ with compact support, a similar one of which has already been used, such that $0 \le \chi(x) \le 1$ in $\mathbb{R}^d$ with $\chi(x) = 1$ on $\operatorname{supp}\psi$. As $\psi = \chi\psi$ holds, so does $\psi(H_{A,m})^\alpha u = \chi\,\psi(H_{A,m})^\alpha u$. Then consider (3.2) multiplied by χ, i.e.,

$$\psi(H_{A,m})^\alpha u = \chi\,(H_{A,m})^{-(1-\alpha)}\psi H_{A,m} u + \chi\,[\psi, (H_{A,m})^{-(1-\alpha)}] H_{A,m} u. \tag{3.3}$$

The first term on the right of (3.2) (and hence (3.3)) converges to $\psi H_{A,m} u$ as $\alpha \uparrow 1$, since $(H_{A,m})^{-(1-\alpha)}$ is an operator on $L^1(\mathbb{R}^d)$, bounded uniformly for $0 < \alpha \le 1$ and strongly continuous there, so long as $m > 0$. So we have only to show the second term of (3.3), i.e., $\chi[\psi, (H_{A,m})^{-(1-\alpha)}]H_{A,m}u$ lies in $L^1(\mathbb{R}^d)$, being uniformly bounded, and converges to 0 in L^1 as $\alpha \uparrow 1$.

Use formula (2.2) to rewrite this second term on the right of (3.3) as

$$\begin{aligned}
&\chi[\psi, (H_{A,m})^{-(1-\alpha)}]H_{A,m}u \\
&\quad = \frac{1}{\Gamma(\frac{1-\alpha}{2})}\int_0^\infty dt\, t^{((1-\alpha)/2)-1}\,\chi\big[\psi e^{-t(H_{A,m})^2} - e^{-t(H_{A,m})^2}\psi\big] H_{A,m}u \\
&\quad = -\frac{1}{\Gamma(\frac{1-\alpha}{2})}\int_0^\infty dt\, t^{-(1+\alpha)/2}\,\chi\int_0^1 d\theta\,\frac{d}{d\theta}\Big[e^{-\theta t(H_{A,m})^2}\psi e^{-(1-\theta)t(H_{A,m})^2}\Big] H_{A,m}u \\
&\quad = \frac{1}{\Gamma(\frac{1-\alpha}{2})}\int_0^\infty dt\, t^{(1-\alpha)/2}\big[(H_{A,m})^2, \psi\big] e^{-(1-\theta)t(H_{A,m})^2} H_{A,m}u.
\end{aligned} \tag{3.4}$$

Recall identity (2.10b) for the commutator $\big[(H_{A,m})^2, \psi\big]$, in fact, the first of the two expressions there, and substitute it into the $\big[(H_{A,m})^2, \psi\big]$ in the last line of (3.4). Then

$$\begin{aligned}
&\chi[\psi, (H_{A,m})^{-(1-\alpha)}]H_{A,m}u \\
&\quad = \frac{1}{\Gamma(\frac{1-\alpha}{2})}\int_0^\infty dt\, t^{(1-\alpha)/2}\int_0^1 d\theta\,\chi\, e^{-\theta t(H_{A,m})^2}(\Delta\psi)\, e^{-(1-\theta)t(H_{A,m})^2} H_{A,m}u \\
&\qquad + \frac{2}{\Gamma(\frac{1-\alpha}{2})}\int_0^\infty dt\, t^{(1-\alpha)/2}\int_0^1 d\theta\,\chi\, e^{-\theta t(H_{A,m})^2} \\
&\qquad\qquad \times (i\nabla + A)(i\nabla\psi) e^{-(1-\theta)t(H_{A,m})^2} H_{A,m}u \\
&\quad =: I_5 u + I_6 u.
\end{aligned} \tag{3.5}$$

We estimate the L^1-norms of $I_5 u$ and $I_6 u$ in (3.5).

First for I_5u, integrate its absolute value in x; then we have by the Schwarz inequality

$$\begin{aligned}
&\|I_5u\|_{L^1}\\
&\quad\le \frac{1}{\Gamma(\frac{1-\alpha}{2})}\int_0^\infty dt\,t^{(1-\alpha)/2}\int_0^1 d\theta\,\big\|\chi\,e^{-\theta t(H_{A,m})^2}(\Delta\psi)\,e^{-(1-\theta)t(H_{A,m})^2}H_{A,m}u\big\|_{L^1}\\
&\quad\le \frac{1}{\Gamma(\frac{1-\alpha}{2})}\int_0^\infty dt\,t^{(1-\alpha)/2}\int_0^1 d\theta\,\big\|\chi\big\|_{L^2}\\
&\qquad\times\big\|e^{-\theta t(H_{A,m})^2}\,(\Delta\psi)\,e^{-(1-\theta)t(H_{A,m})^2}H_{A,m}u\big\|_{L^2}\\
&\quad\le \frac{1}{\Gamma(\frac{1-\alpha}{2})}\int_0^\infty dt\,t^{(1-\alpha)/2}\int_0^1 d\theta\,\big\|\chi\big\|_{L^2}\big\|e^{-\theta t(H_{A,m})^2}\big\|_{L^2\to L^2}\\
&\qquad\times\big\|\Delta\psi\big\|_{L^\infty}\big\|e^{-(1-\theta)t(H_{A,m})^2}H_{A,m}\big\|_{L^2\to L^2}\big\|u\big\|_{L^2}\\
&\quad\le \frac{1}{\Gamma(\frac{1-\alpha}{2})}\int_0^\infty dt\,t^{(1-\alpha)/2}\int_0^1 d\theta\,\big\|\chi\big\|_{L^2}e^{-(m^2/2)\theta t}\big\|e^{-(\theta t/2)(H_{A,m})^2}\big\|_{L^2\to L^2}\\
&\qquad\times\big\|\Delta\psi\big\|_{L^\infty}\,e^{-(m^2/2)(1-\theta)t}\big\|e^{-((1-\theta)t/2)(H_{A,m})^2}H_{A,m}\big\|_{L^2\to L^2}\big\|u\big\|_{L^2}.
\end{aligned}$$

Then by Lemma 2.1(iii) we have the bound

$$\begin{aligned}
&\|I_5u\|_{L^1}\\
&\quad\le \frac{1}{\Gamma(\frac{1-\alpha}{2})}\int_0^\infty dt\,t^{(1-\alpha)/2}e^{-(m^2/2)t}\int_0^1 d\theta\,\|\chi\|_{L^2}\|\Delta\psi\|_{L^\infty}\Big(\frac{1}{2e(\frac{1-\theta}{2})t}\Big)^{1/2}\big\|u\big\|_{L^2}\\
&\quad\le \frac{1}{\Gamma(\frac{1-\alpha}{2})}\int_0^\infty dt\,t^{-\alpha/2}e^{-(m^2/2)t}\,(2e)^{-1/2}\int_0^1\frac{d\theta}{(\frac{1-\theta}{2})^{1/2}}\,\big\|\chi\big\|_{L^2}\big\|\Delta\psi\big\|_{L^\infty}\big\|u\big\|_{L^2}
\end{aligned}$$

(3.6)

$$\le 2^{3/2}(2e)^{-1/2}\frac{\Gamma\big(\frac{2-\alpha}{2}\big)}{\Gamma\big(\frac{1-\alpha}{2}\big)}\Big(\frac{2}{m^2}\Big)^{(2-\alpha)/2}\big\|\chi\big\|_{L^2}\big\|\Delta\psi\big\|_{L^\infty}\big\|u\big\|_{L^2}.$$

Next for I_6u, we are going to show a similar bound,

$$\|I_6u\|_{L^1}\le 4\pi(2e)^{-1/2}\frac{\Gamma\big(\frac{2-\alpha}{2}\big)}{\Gamma\big(\frac{1-\alpha}{2}\big)}\big(\tfrac{2}{m^2}\big)^{(2-\alpha)/2}C_{\chi,A}\big\|\nabla\psi\big\|_{L^\infty}\big\|u\big\|_{L^2},\tag{3.7}$$

with a constant, depending only on χ and A,

$$C_{\chi,A}:=\big[\|\nabla\chi\|_{L^2}^2+m^2\|\chi\|_{L^2}^2+\|\chi\,A\|_{L^2}^2\big]^{1/2},\tag{3.8}$$

which is bounded since $A\in L^2_{\mathrm{loc}}(\mathbb{R}^d)$. The proof is to integrate the absolute value of I_6u in x to get

$$\|I_6u\|_{L^1}\le\frac{2}{\Gamma(\frac{1-\alpha}{2})}\int_0^\infty dt\,t^{(1-\alpha)/2}\int_0^1 d\theta\,X_{A,m}(t,\theta;\chi,\psi,u),\tag{3.9}$$

where we put

(3.10)
$$X_{A,m}(t,\theta;\chi,\psi,u)=\big\|\chi\, e^{-\theta t(H_{A,m})^2}(i\nabla+A)(i\nabla\psi)e^{-(1-\theta)t(H_{A,m})^2}H_{A,m}u\big\|_{L^1}.$$

Somewhat crucial is the estimate of $X_{A,m}(t,\theta;\chi,\psi,u)$ in (3.10) which we are going to do, where the parentheses $\big(\cdot,\cdot\big)$ below stand for the L^2 inner product:

$$\begin{aligned}
&X_{A,m}(t,\theta;\chi,\psi,u)\\
&\quad=\big|\big(\chi, e^{-\theta t(H_{A,m})^2}(i\nabla+A)(i\nabla\psi)e^{-(1-\theta)t(H_{A,m})^2}H_{A,m}u\big)\big|\\
&\quad=\big|\big(\chi, e^{-\theta t(H_{A,m})^2}H_{A,m}\cdot(H_{A,m})^{-1}(i\nabla+A)(i\nabla\psi)e^{-(1-\theta)t(H_{A,m})^2}H_{A,m}u\big)\big|\\
&\quad=\big|\big(e^{-\theta t(H_{A,m})^2}H_{A,m}\,\chi,(H_{A,m})^{-1}(i\nabla+A)(i\nabla\psi)e^{-(1-\theta)t(H_{A,m})^2}H_{A,m}u\big)\big|
\end{aligned}$$

(3.11)
$$\le\big\|e^{-\theta t(H_{A,m})^2}H_{A,m}\,\chi\|_{L^2}\,\big\|(H_{A,m})^{-1}(i\nabla+A)(i\nabla\psi)e^{-(1-\theta)t(H_{A,m})^2}H_{A,m}u\big\|_{L^2}.$$

In the last line of (3.11), the first factor and the second are estimated as follows:

$$\begin{aligned}
\big\|e^{-\theta t(H_{A,m})^2}H_{A,m}\,\chi\|_{L^2}&\le e^{-m^2\theta t}\big\|H_{A,m}\chi\big\|_{L^2}\\
&=e^{-m^2\theta t}\Big[\sum_{j=1}^{d}\|(i\partial_j+A_j)\chi\|^2_{L^2}+m^2\|\chi\|^2_{L^2}\Big]^{1/2}\\
&\le e^{-m^2\theta t}\big[\|\nabla\chi\|^2_{L^2}+\|\chi A\|^2_{L^2}+m^2\|\chi\|^2_{L^2}\big]^{1/2}\\
&=e^{-m^2\theta t}\,C_{\chi,A}\,;
\end{aligned}\tag{3.12}$$

$$\begin{aligned}
&\big\|(H_{A,m})^{-1}(i\nabla+A)(i\nabla\psi)e^{-(1-\theta)t(H_{A,m})^2}H_{A,m}u\big\|_{L^2}\\
&\le\big\|(H_{A,m})^{-1}(i\nabla+A)\big\|_{L^2\to L^2}\big\|(i\nabla\psi)\big\|_{L^\infty}\big\|e^{-(1-\theta)t(H_{A,m})^2}H_{A,m}\big\|_{L^2\to L^2}\big\|u\big\|_{L^2}\\
&\le\big\|\nabla\psi\big\|_{L^\infty}e^{-\left(m^2/2\right)(1-\theta)t}\big\|e^{-((1-\theta)t/2)(H_{A,m})^2}H_{A,m}\big\|_{L^2\to L^2}\big\|u\big\|_{L^2}
\end{aligned}$$

(3.13)
$$\le\big\|\nabla\psi\big\|_{L^\infty}e^{-\left(m^2/2\right)(1-\theta)t}\Big(\frac{1}{2e\frac{(1-\theta)t}{2}}\Big)^{1/2}\big\|u\big\|_{L^2}.$$

In (3.12) and (3.13) we have used (2.1), Lemma 2.1(iii) and the estimate $\big\|(H_{A,m})^{-1}(i\nabla+A)\big\|_{L^2\to L^2}\le 1$. From (3.12) and (3.13) we obtain

$$\begin{aligned}
&\|I_6u\|_{L^1}\\
&\quad\le\frac{2}{\Gamma(\frac{1-\alpha}{2})}\int_0^\infty dt\,t^{(1-\alpha)/2}e^{-\left(m^2/2\right)t}\int_0^1 d\theta\,\Big(\frac{1}{2e\frac{(1-\theta)t}{2}}\Big)^{1/2}C_{\chi,A}\big\|\nabla\psi\big\|_{L^\infty}\big\|u\big\|_{L^2}
\end{aligned}$$

$$\leq \frac{2}{\Gamma(\frac{1-\alpha}{2})} \int_0^\infty dt\, t^{-\alpha/2} e^{-(m^2/2)t} \int_0^1 \frac{d\theta}{(1-\theta)^{1/2}} (2e)^{-1/2} C_{\chi,A} \big\|\nabla\psi\big\|_{L^\infty} \big\|u\big\|_{L^2},$$

the last line of which yields (3.7) with (3.8).

Thus, taking (3.5) into account and putting together (3.6) and (3.7), we see the L^1-norm of the second term on the right-hand side of (3.3) is estimated as

$$\begin{aligned} &\|\chi[\psi,(H_{A,m})^{-(1-\alpha)}]H_{A,m}u\|_{L^1} \\ &\quad\leq \|I_5 u\|_{L^1} + \|I_6 u\|_{L^1} \\ &\quad\leq \frac{\Gamma\big(\frac{2-\alpha}{2}\big)}{\Gamma\big(\frac{1-\alpha}{2}\big)} \Big(\frac{2}{m^2}\Big)^{(2-\alpha)/2} \\ &\qquad\times \Big[2^{3/2}(2e)^{-1/2}\big\|\chi\big\|_{L^2}\big\|\Delta\psi\big\|_{L^2} + 4\pi(2e)^{-1}C_{\chi,A}\|\nabla\psi\|_{L^\infty}\Big]\|u\|_{L^2}. \end{aligned} \tag{3.14}$$

Since the last line of (3.14) tends to zero as $\alpha \uparrow 1$, because $\Gamma(z) \uparrow \infty$ as $z \downarrow 0$ and hence $\frac{1}{\Gamma(\frac{1-\alpha}{2})} \to 0$ as $\alpha \uparrow 1$, we see the left-hand side is uniformly bounded for $0 < \alpha < 1$, and convergent to zero as $\alpha \uparrow 1$. This shows the desired assertion of Lemma 3.2. □

Now we are in a position to prove Theorem 1.1.

Completion of proof of Theorem 1.1.

As (1.2) and (1.3) are equivalent, we have only to show (1.3). The proof is divided into three parts:

(i) the case where $m > 0$ and $0 < \alpha < 1$,
(ii) the case where $m > 0$ and $\alpha = 1$,
(iii) the case where $m = 0$ and $\alpha = 1$.

(i) *The case where* $m > 0$ *and* $0 < \alpha < 1$. We prove this in two steps: first step (i-I) for $u \in (C^\infty \cap L^2)(\mathbb{R}^d)$, and next step (i-II) for general $u \in L^2$ with $(H_{A,m})^\alpha u \in L^1_{\mathrm{loc}}$.

(i-I) *For* $u \in (C^\infty \cap L^2)(\mathbb{R}^d)$ $(0 < \alpha < 1)$.

For a function $v(x) \in C^\infty(\mathbb{R}^d)$ and $\varepsilon > 0$, put $v_\varepsilon(x) = \sqrt{|v(x)|^2 + \varepsilon^2}$. Then note that $v_\varepsilon(x) \geq \varepsilon$, and, since $v_\varepsilon(x)^2 = |v(x)|^2 + \varepsilon^2$, we have

$$-|v(x)||v(x+y)| + |v(x)|^2 \geq -v_\varepsilon(x)v_\varepsilon(x+y) + v_\varepsilon(x)^2. \tag{3.15}$$

Then we will show that $u_\varepsilon = \sqrt{|u|^2 + \varepsilon^2}$, $\varepsilon > 0$, satisfies

$$\mathrm{Re}[\overline{u(x)}([(H_{A,m})^\alpha - m^\alpha]u)(x)] \geq u_\varepsilon(x)\big([(H_{0,m})^\alpha - m^\alpha], (u_\varepsilon - \varepsilon)\big)(x), \tag{3.16}$$

pointwise a.e., which amounts to the same thing as

$$\mathrm{Re}\left[\frac{\overline{u(x)}}{u_\varepsilon(x)}([(H_{A,m})^\alpha - m^\alpha]u)(x)\right] \geq \big([(H_{0,m})^\alpha - m^\alpha](u_\varepsilon - \varepsilon)\big)(x), \tag{3.17}$$

pointwise a.e., and thus in the distribution sense. Here note that the function $u_\varepsilon - \varepsilon$ is nonnegative, C^∞, and has the same compact support as u.

We show (3.16) or (3.17) first for $u \in C_0^\infty(\mathbb{R}^d)$ and then for $u \in (C^\infty \cap L^2)(\mathbb{R}^d)$. To do so, we employ analogous arguments to those used in [I93, p. 223, Lem. 2] for the case $\alpha = 1$, i.e., for $H_{A,m} - m$. We will use the same notation S as in Section 2 for the self-adjoint operator $(-i\nabla - A(x))^2 + m^2$ in $L^2(\mathbb{R}^d)$, which may be considered as the magnetic nonrelativistic Schrödinger operator with mass $\frac{1}{2}$ with constant scalar potential m^2. Then we have $H_{A,m} = S^{1/2}$. Since the domain of $H_{A,m}$ includes $C_0^\infty(\mathbb{R}^d)$ as the operator core, the operator $[H_{A,m} - m]u$ can be written as $\text{s-lim}_{t\downarrow 0} t^{-1}\big(1 - e^{-t[H_{A,m}-m]}\big)u$. It is known from the theory of fractional powers of a linear operator (see, e.g., [Y78, IX, 11, pp. 259–261]) that the semigroup $e^{-t[(H_{A,m})^\alpha - m^\alpha]}$ with generator $(H_{A,m})^\alpha = S^{\alpha/2}$ is obtained from the semigroup e^{-tS} with generator S as

$$e^{-t[(H_{A,m})^\alpha - m^\alpha]}u = \begin{cases} e^{m^\alpha t}\int_0^\infty f_{t,\alpha/2}(\lambda)e^{-\lambda S}u\,d\lambda, & t > 0,\\ u, & t = 0,\end{cases} \tag{3.18}$$

where for $t > 0$ and $0 < \alpha \leq 1$, $f_{t,\alpha/2}(\lambda)$ is a nonnegative function of exponential growth in $\lambda \in \mathbb{R}$ given by

$$f_{t,\alpha/2}(\lambda) = \begin{cases} (2\pi i)^{-1}\int_{\sigma - i\infty}^{\sigma + i\infty} e^{z\lambda - tz^{\alpha/2}}\,dz, & \lambda \geq 0,\\ 0, & \lambda < 0,\end{cases} \tag{3.19}$$

with $\sigma > 0$, where the branch of $z^{\alpha/2}$ is so taken that $\mathrm{Re}\, z^{\alpha/2} > 0$ for $\mathrm{Re}\, z > 0$. In passing, we note that equation (3.18) is valid even for $1 < \alpha < 2$, though we do not need this case in the present paper.

We continue our preceding arguments and recall that $|e^{-tS}u| \leq e^{-t[-\Delta + m^2]}|u|$ pointwise a.e., what is referred to in (2.17). It follows with (3.18), (3.19), that

$$\begin{aligned} |e^{-t[(H_{A,m})^\alpha - m^\alpha]}u| &\leq e^{m^\alpha t}\int_0^\infty f_{t,\alpha/2}(\lambda)|e^{-\lambda S}u|\,d\lambda\\ &\leq e^{m^\alpha t}\int_0^\infty f_{t,\alpha/2}(\lambda)e^{-\lambda(-\Delta + m^2)}|u|\,d\lambda\\ &= e^{-t[(H_{0,m})^\alpha - m^\alpha]}|u|, \end{aligned} \tag{3.20}$$

pointwise a.e. Hence for $t > 0$,

(3.21)
$$\mathrm{Re}\Big[\overline{u(x)}\Big(\frac{1-e^{-t[(H_{A,m})^\alpha - m^\alpha]}}{t}u\Big)(x)\Big] \geq |u(x)|\Big(\frac{1-e^{-t[(H_{0,m})^\alpha - m^\alpha]}}{t}|u|\Big)(x),$$

pointwise a.e. Now put $n^{m,\alpha}(t,y) := \frac{1}{t}k_0^{m,\alpha}(t,y)$, taking account of the relation (2.7) between the integral kernel $k_0^{m,\alpha}(t,y)$ of $e^{-t[(H_{0,m})^\alpha - m^\alpha]}$ and the density (function) $n^{m,\alpha}(y)$ of the Lévy measure.

Then we see by (2.6), that the right-hand side of (3.21) is equal to

$$\begin{aligned}
&|u(x)|\int_{|y|>0}[|u(x)|-|u(x+y)|]\,\frac{k_0^{m,\alpha}(t,y)}{t}dy \\
&\quad = -\int_{|y|>0}\big[\,|u(x)||u(x+y)| - |u(x)|^2\,\big]\,n^{m,\alpha}(t,y)\,dy \\
&\quad \geq -\int_{|y|>0}\big[\,u_\varepsilon(x)u_\varepsilon(x+y) - u_\varepsilon(x)^2\,\big]\,n^{m,\alpha}(t,y)\,dy \\
&\quad = u_\varepsilon(x)\Big[-\int_{|y|>0}\big[\,u_\varepsilon(x+y)-u_\varepsilon(x) - I_{\{|y|<1\}}y\cdot\nabla u_\varepsilon(x)\,\big]\,n^{m,\alpha}(t,y)\,dy\Big],
\end{aligned}$$

for every $\varepsilon > 0$, where we have used (3.15) and the y-rotational invariance of $k_0^{m,\alpha}(t,y)$ or $n^{m,\alpha}(t,y)$. Notice that the integral $\big[-\int_{|y|>0}\cdots\big]$ in the last line is equal to that with $(u_\varepsilon - \varepsilon)$ in place of u_ε, i.e.,

$$\Big(\frac{1-e^{-t[(H_{0,m})^\alpha - m^\alpha]}}{t}(u_\varepsilon-\varepsilon)\Big)(x).$$

Thus we have from (3.21),

(3.22)
$$\mathrm{Re}\Big[\overline{u(x)}\Big(\frac{1-e^{-t[(H_{A,m})^\alpha - m^\alpha]}}{t}u\Big)(x)\Big] \geq u_\varepsilon(x)\Big(\frac{1-e^{-t[(H_{0,m})^\alpha - m^\alpha]}}{t}(u_\varepsilon-\varepsilon)\Big)(x).$$

Then letting $t \downarrow 0$ on both sides of (3.22), we obtain (3.16). Indeed, recalling that the function $u_\varepsilon - \varepsilon$ has compact support, the right-hand side tends to that of (3.16). For the left-hand side, since u is in the domain of $(H_{A,m})^\alpha - m^\alpha$, we have $t^{-1}[1-e^{-t[(H_{A,m})^\alpha - m^\alpha]}]u \to [(H_{A,m})^\alpha - m^\alpha]u$ in L^2, and pointwise a.e. by passing to a subsequence. This shows (3.16)/(3.17) for $u \in C_0^\infty(\mathbb{R}^d)$.

Next we show (3.16)/(3.17) when $u \in (C^\infty \cap L^2)(\mathbb{R}^d)$. Take a sequence $\{u_n\} \in C_0^\infty(\mathbb{R}^d)$ such that $u_n \to u$ in $(C^\infty \cap L^2)(\mathbb{R}^d)$, i.e., in the topology of $C^\infty(\mathbb{R}^d)$ as well as in the norm of $L^2(\mathbb{R}^d)$, as $n \to \infty$. Then from the case $u \in C_0^\infty(\mathbb{R}^d)$ above, we have for all $\varepsilon > 0$,

$$\mathrm{Re}\left[\frac{\overline{u_n(x)}}{u_{n,\varepsilon}(x)}\big([(H_{A,m})^\alpha - m^\alpha]u_n\big)(x)\right] \geq \big([(H_{0,m})^\alpha - m^\alpha](u_{n,\varepsilon}-\varepsilon)\big)(x),$$

pointwise, and hence for any $\psi \in C_0^\infty(\mathbb{R}^d)$ with $\psi(x) \geq 0$,

$$\mathrm{Re}\left\langle \psi,\, \frac{\overline{u_n}}{u_{n,\varepsilon}}\big([(H_{A,m})^\alpha - m^\alpha]u_n\big)\right\rangle \geq \left\langle \psi,\, [(H_{0,m})^\alpha - m^\alpha](u_{n,\varepsilon} - \varepsilon)\right\rangle$$

for all $\varepsilon > 0$. Here the bilinear inner product $\langle \cdot, \cdot\rangle$ is an integral with respect to the Lebesgue measure dx, and also considered as the bilinear inner product between the dual pair of the test functions and the distributions: $\big\langle C_0^\infty(\mathbb{R}^d),\, \mathcal{D}'(\mathbb{R}^d)\big\rangle$. Therefore,

$$\mathrm{Re}\left\langle [(H_{A,m})^\alpha - m^\alpha]\Big(\frac{\overline{u_n}}{u_{n,\varepsilon}}\psi\Big),\, u_n\right\rangle \geq \left\langle [(H_{0,m})^\alpha - m^\alpha]\psi,\, u_{n,\varepsilon} - \varepsilon\right\rangle.$$

Since we have that $u_n \to u$ and $u_{n,\varepsilon} \to u_\varepsilon$ in $(C^\infty \cap L^2)(\mathbb{R}^d)$ as $n \to \infty$, we have that $\big(\frac{\overline{u_n}}{u_{n,\varepsilon}}\big)\psi \to \big(\frac{\overline{u}}{u_\varepsilon}\big)\psi$. It follows by Lemma 2.2 that

$$[(H_{A,m})^\alpha - m^\alpha]\Big(\frac{\overline{u_n}}{u_{n,\varepsilon}}\Big)\psi \;\;\to\;\; ([H_{A,m})^\alpha - m^\alpha]\Big(\frac{\overline{u}}{u_\varepsilon}\Big)\psi$$

in L^2 as $n \to \infty$, so that

$$\mathrm{Re}\left\langle [(H_{A,m})^\alpha - m^\alpha]\Big(\frac{\overline{u}}{u_\varepsilon}\psi\Big),\, u\right\rangle \geq \left\langle [(H_{0,m})^\alpha - m^\alpha]\psi,\, (u_\varepsilon - \varepsilon)\right\rangle.$$

Thus we obtain

$$\mathrm{Re}\left[\frac{\overline{u(x)}}{u_\varepsilon(x)}\big([(H_{A,m})^\alpha - m^\alpha]u\big)(x)\right] \geq \big([(H_{0,m})^\alpha - m^\alpha](u_\varepsilon - \varepsilon)\big)(x), \tag{3.23}$$

pointwise a.e., and so in distributional sense, and hence (3.17) follows for $u \in (C^\infty \cap L^2)(\mathbb{R}^d)$.

(i-II) *For general* $u \in L^2(\mathbb{R}^d)$ *with* $(H_{A,m})^\alpha u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ $(0 < \alpha < 1)$.

Put $u^\delta = \rho_\delta * u$. Then $u^\delta \in C^\infty \cap L^2$, so by (3.23) in step (i-I) above,

$$\mathrm{Re}\left[\frac{\overline{u^\delta}}{(u^\delta)_\varepsilon}\big([(H_{A,m})^\alpha - m^\alpha]u^\delta\big)\right] \geq [(H_{0,m})^\alpha - m^\alpha]\big((u^\delta)_\varepsilon - \varepsilon\big), \tag{3.24}$$

pointwise a.e., and also in distributional sense, for all $\varepsilon > 0$ and all $\delta > 0$.

For fixed $\varepsilon > 0$, we first let $\delta \downarrow 0$, and next $\varepsilon \downarrow 0$. In fact, if $\delta \downarrow 0$, then $u^\delta \to u$ in L^2 as well as a.e. by passing to a subsequence of $\{u^\delta\}$. Hence $\overline{u^\delta}/(u^\delta)_\varepsilon \to \overline{u}/u_\varepsilon$ a.e. and by Lemma 3.1, $(H_{A,m})^\alpha u^\delta \to (H_{A,m})^\alpha u$ locally in L^1, and therefore also a.e. by passing to a subsequence. Since $\big|\frac{\overline{u^\delta}}{(u^\delta)_\varepsilon}\big| \leq 1$, it follows by the Lebesgue dominated convergence theorem that on the left-hand side of (3.24),

$$\frac{\overline{u^\delta}}{(u^\delta)_\varepsilon}[(H_{A,m})^\alpha - m^\alpha]u^\delta \;\;\to\;\; \frac{\overline{u}}{u_\varepsilon}[(H_{A,m})^\alpha - m^\alpha]u$$

locally in L^1 as $\delta \downarrow 0$. On the other hand, for the right-hand side, since

$$\big|\big((u^\delta)_\varepsilon - \varepsilon\big) - (u_\varepsilon - \varepsilon)\big| \le \big|(u^\delta)_\varepsilon - u_\varepsilon\big| \le \big||u^\delta| - |u|\big| \le |u^\delta - u|,$$

we have $(H_{0,m})^\alpha\big((u^\delta)_\varepsilon - \varepsilon\big) \to (H_{0,m})^\alpha(u_\varepsilon - \varepsilon)$ in $\mathcal{D}'$ (in the distribution sense). This shows that (3.23) holds for $u \in L^2(\mathbb{R}^d)$ with $(H_{A,m})^\alpha u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Next let $\varepsilon \downarrow 0$. Then $\overline{u}/u_\varepsilon \to \operatorname{sgn} u$ a.e. with $|\overline{u}/u_\varepsilon| \le 1$, so that the left-hand side of (3.23) converges to $\mathrm{Re}((\operatorname{sgn} u)[H_{A,m} - m]u)$ a.e., while the right-hand side of (3.23) converges to $[(H_{0,m})^\alpha - m^\alpha]|u|$ in $\mathcal{D}'$. Thus we get (3.1), showing the desired inequality for $0 < \alpha < 1$.

(ii) *The case where $m > 0$ and $\alpha = 1$.*

Once the inequality (3.1) is established for $0 < \alpha < 1$, we let $\alpha \uparrow 1$, with $u \in L^2(\mathbb{R}^d)$ with $H_{A,m}u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Then, as $\alpha \uparrow 1$, by Lemma 3.2 we have $(H_{A,m})^\alpha u \to H_{A,m}u$ in L^1_{loc} and also trivially $m^\alpha \to m$. The left-hand side of (3.1) converges to $\mathrm{Re}((\operatorname{sgn} u)[H_{A,m} - m]u)$ in L^1_{loc}, while the right-hand side converges to $[H_{0,m} - m]|u|$ in distributional sense, so that we have shown the desired inequality (1.3).

(iii) *The case where $m = 0$ and $\alpha = 1$.*

This follows from the case (ii) for $m > 0$, i.e., by letting $m \downarrow 0$ in the equality (1.3) with $m > 0$. To see this, let $u \in L^2(\mathbb{R}^d)$ with $H_{A,0}u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Then, noting that $H_{A,0} = |-i\nabla - A|$, we see by the argument made around (2.1) that the domains of the operators $H_{A,m}$ and $H_{A,0}$ coincide. We also see that $H_{A,0}u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ with $u \in L^2(\mathbb{R}^d)$ implies $H_{A,m}u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. In fact, we can show the following fact.

Lemma 3.3. *Let $u \in L^2(\mathbb{R}^d)$. Then $H_{A,m}u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ if and only if $H_{A,0}u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. In fact, for $\psi \in C_0^\infty(\mathbb{R}^d)$ it holds that*

$$\big|\|\psi H_{A,m}u\|_{L^1} - \|\psi H_{A,0}u\|_{L^1}\big| \le C(d)m^2\|\psi\|_{L^{2d/(d+2)}}\|u\|_{L^2} \tag{3.25}$$

with a constant $C(d)$ depending only on d.

Proof. We have for $\phi \in C_0^\infty(\mathbb{R}^d)$,

$$\begin{aligned} H_{A,m}\phi - H_{A,0}\phi &= ((-\nabla - A)^2 + m^2)^{1/2}\phi - |-i\nabla - A|\phi \\ &= \big[((-\nabla - A)^2 + \theta m^2)^{1/2}\phi\big]_{\theta=0}^{\theta=1} \\ &= \int_0^1 \frac{d}{d\theta}\big[((-\nabla - A)^2 + \theta m^2)^{1/2}\phi\big]\, d\theta \\ &= \frac{m^2}{2}\int_0^1 ((-\nabla - A)^2 + \theta m^2)^{-1/2}\phi\, d\theta. \end{aligned} \tag{3.26}$$

Multiply by $\psi \in C_0^\infty(\mathbb{R}^d)$ with $\psi(x) \geq 0$, and integrate the absolute value in x; then we have

$$\begin{aligned}\big\|\psi H_{A,m}\phi - \psi H_{A,0}\phi\big\|_{L^1} &\leq \frac{m^2}{2}\int_0^1 \big\|\psi((-\nabla - A)^2 + \theta m^2)^{-1/2}\phi\big\|_{L^1}\, d\theta \\ &\leq \frac{m^2}{2}\int_0^1 \big\|\psi(-\Delta + \theta m^2)^{-1/2}|\phi|\big\|_{L^1}\, d\theta \\ &= \frac{m^2}{2}\int_0^1 \int_{\mathbb{R}^d} \big[\psi(-\Delta + \theta m^2)^{-1/2}|\phi|\big](x)\, dx\, d\theta, \end{aligned} \tag{3.27}$$

where the second inequality is due to Lemma 2.1(i) with $\beta = \frac{1}{2}$ and $p = 1$. Note also that the operator $(-\Delta + m^2)^{-1/2}$ in (3.27) has the following positive integral kernel:

$$(-\Delta + m^2)^{-1/2}(x) = \frac{2m^{d-1}}{(2\pi)^{(d+1)/2}}\frac{K_{(d-1)/2}(m|x|)}{(m|x|)^{(d-1)/2}}, \quad m > 0, \tag{3.28}$$

with $K_\nu(\tau)$ the modified Bessel function of the third kind of order ν, which was also referred to around (2.8)/(2.9). In fact, using the expression (2.9) for the integral kernel of $e^{-tH_{0,m}} = e^{-[-\Delta+m^2]^{-1/2}}$ and integrating it in t on $(0,\infty)$, then we have

$$\begin{aligned}(-\Delta + m^2)^{-1/2}(x) &= \int_0^\infty k_0^{m,1}(t,x)\cdot e^{-mt}\, dt \\ &= \int_0^\infty 2\Big(\frac{m}{2\pi}\Big)^{(d+1)/2}\frac{tK_{(d+1)/2}(m(x^2+t^2)^{1/2})}{(x^2+t^2)^{(d+1)/4}}\, dt.\end{aligned}$$

Change the variables $\tau = m(x^2+t^2)^{1/2}$, so that $2t\, dt = \frac{2\tau}{m^2}d\tau$, and use $\frac{d}{\tau d\tau}\frac{K_\nu(\tau)}{\tau^\nu} = -\frac{K_{\nu+1}(\tau)}{\tau^{\nu+1}}$; then we see that the last line above is equal to

$$\begin{aligned}&\int_{m|x|}^\infty \frac{m^{(d+1)/2}}{(2\pi)^{(d+1)/2}}\frac{K_{(d+1)/2}(\tau)}{(\tau/m)^{(d+1)/2}}\frac{2\tau}{m^2}\, d\tau \\ &\quad = -\frac{1}{(2\pi)^{(d+1)/2}}\int_{m|x|}^\infty m^{d+1}\frac{d}{\tau\, d\tau}\Big[\frac{K_{(d-1)/2}(\tau)}{\tau^{(d-1)/2}}\Big]\frac{2\tau}{m^2}\, d\tau \\ &\quad = \frac{2m^{d-1}}{(2\pi)^{(d+1)/2}}\frac{K_{(d-1)/2}(m|x|)}{(m|x|)^{(d-1)/2}},\end{aligned}$$

which yields (3.28).

Since it holds that $0 < K_\nu(\tau) \leq C[\tau^{-\nu} \vee \tau^{-1/2}]e^{-\tau}$, $\tau > 0$ with a constant $C > 0$ when $\nu > 0$, we obtain

$$\frac{K_{(d-1)/2}(\theta^{1/2}m|x|)}{(\theta^{1/2}m|x|)^{(d-1)/2}} \leq C\frac{1}{(\theta^{1/2}m|x|)^{(d-1)}}.$$

Then we see from (3.27) by the Hardy–Littlewood–Sobolev inequality (see, e.g., [LLos01, Chap. 4, Sec. 4.3]), noting that $p = \frac{2d}{d+2}$ satisfies the relation $\frac{1}{p} + \frac{d-1}{d} + \frac{1}{2} = 2$,

$$\begin{aligned}
&\big\|\psi H_{A,m}\phi - \psi H_{A,0}\phi\big\|_{L^1} \\
&\quad \le \frac{m^{d+1}}{(2\pi)^{(d+1)/2}} \int_0^1 d\theta\, \theta^{(d+1)/2} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \psi(x) \frac{K_{(d-1)/2}(\theta^{1/2}m|x-y|)}{(\theta^{1/2}m|x-y|)^{(d-1)/2}} |\phi(y)|\, dx\, dy \\
&\quad \le \frac{C(d)}{2} \frac{m^{d+1}}{(2\pi)^{(d+1)/2}} \int_0^1 d\theta\, \theta\, m^{-(d-1)} \|\psi\|_{L^{2d/(d+2)}} \|\phi\|_{L^2}
\end{aligned}$$

(3.29)

$$\quad = C(d) \frac{m^2}{(2\pi)^{(d+1)/2}} \|\psi\|_{L^{2d/(d+2)}} \|\phi\|_{L^2},$$

with a constant $C(d) > 0$ depending on d.

Now, to show the desired inequality (3.25), let $u \in L^2(\mathbb{R}^d)$ and assume that either $H_{A,m}u$ or $H_{A,0}u$ in $L^1_{\text{loc}}(\mathbb{R}^d)$, consider, for instance, the latter: $H_{A,0}u \in L^1_{\text{loc}}(\mathbb{R}^d)$. There exists a sequence $\{\phi_n\}_{n=1}^\infty$ in $C_0^\infty(\mathbb{R}^d)$ convergent to u in L^2 as $n \to \infty$. We see by (3.29) that $\{(\psi H_{A,m} - \psi H_{A,0})\phi_n\}_{n=1}^\infty$ is a Cauchy sequence in L^1, so that there exists $v \in L^1(\mathbb{R}^d)$ to which it converges in L^1:

$$(\psi H_{A,m} - \psi H_{A,0})\phi_n \to v, \quad n \to \infty.$$

Since $\psi D[H_{A,0}] \subseteq D[H_{A,0}]$, we see that $\{\psi H_{A,0}\phi_n\}$ converges to $\psi H_{A,0}u \in L^1(\mathbb{R}^d)$ in the weak topology defined by the dual pairing $\langle L^1(\mathbb{R}^d), D[H_{A,0}]\rangle$. So $\{\psi H_{A,m}\phi_n\}$ becomes a Cauchy sequence also in this weak topology $\sigma(L^1(\mathbb{R}^d), D[H_{A,0}])$, converging to $v - \psi H_{A,0}u$, which also belongs to $L^1(\mathbb{R}^d)$. Therefore the existing limit of $\{\psi H_{A,m}\phi_n\}$ should be written as $\psi H_{A,m}u$ to satisfy

$$v = \psi H_{A,m}u + \psi H_{A,0}u.$$

Thus we have seen that (3.29) implies

$$\|\psi H_{A,m}u - \psi H_{A,0}u\|_{L^1} \le C(d) m^2 \|\psi\|_{L^{2d/(d+2)}} \|u\|_{L^2}. \tag{3.30}$$

Hence using the triangle inequality $\big||a| - |b|\big| \le \big|a - b\big|$, we obtain (3.25). This shows (3.25) for the general u, ending the proof of Lemma 3.3. □

Finally, we come back to the proof of Theorem 1.1, continuing case (iii), *where* $m = 0$ *and* $\alpha = 1$. We show that, as $m \downarrow 0$, the left-hand side and the right-hand side of (1.3) with $m > 0$ converge to those with $m = 0$.

As for the left-hand side, the sequence $\{ \|[H_{A,m} - m]u\|_{L^2}^2 \}$ of quadratic forms is increasing as m decreases and converges to $\|H_{A,0}u\|_{L^2}^2$ as $m \downarrow 0$, because

$$[H_{A,m} - m] = \frac{(-i\nabla - A)^2}{H_{A,m} + m} \leq \frac{(-i\nabla - A)^2}{H_A^{m'} + m'} = [H_A^{m'} - m'] \leq H_{A,0} = |-i\nabla - A|$$

for $m \geq m' > 0$. This shows the convergence of the left-hand side of (1.3). As for the right-hand side, it is easy to see that, as $m \downarrow 0$, $H_{0,m}|u| \equiv (-\Delta + m^2)^{1/2}|u|$ converges to $H_0^0|u| \equiv (-\Delta)^{1/2}|u|$ in the distribution sense, because one can show that, for any $\psi \in C_0^\infty(\mathbb{R}^n)$, $\{H_{0,m}\psi\}$ converges to $H_0^0\psi$ as $m \downarrow 0$, by using their integral operator representation formula (2.6) with $\alpha = 1$; in fact, it is due to the convergence of the Lévy measure $n^{m,1}(dy)$ to the Lévy measure $n^{0,1}(dy)$ on $\mathbb{R}^d \setminus \{0\}$, which amounts to the same thing as (observing (2.9)) the convergence of density $n^{m,1}(y)$ to density $n^{0,1}(y)$. This shows the case $m = 0$, completing the proof of Theorem 1.1. □

Remark. From the proof of Theorem 1.1 above, in particular, the step (i-II), which relies on Lemma 3.1, we see that Theorem 1.1 (Kato's inequality) also holds for $(H_{A,m})^\alpha$, $(H_{0,m})^\alpha$ in place of $H_{A,m}$, $H_{0,m}$ with $0 < \alpha < 1$; i.e., (3.1) holds for $0 < \alpha < 1$ if $u \in L^2(\mathbb{R}^d)$ with $(H_{A,m})^\alpha u \in [L^1_{\text{loc}}(\mathbb{R}^d)]^d$. As a result, Theorem 1.2 (diamagnetic inequality) also holds for $(H_{A,m})^\alpha$, $(H_{0,m})^\alpha$.

Proof of Theorem 1.2. This has already been implicitly shown in the proof of Theorem 1.1. In fact, by the same argument used to get (3.20) from (3.18), (3.19), *even for all* $0 < \alpha \leq 1$, we have for f, $g \in C_0^\infty(\mathbb{R}^d)$,

$$\begin{aligned}
|(f, e^{-t[(H_{A,m})^\alpha - m^\alpha]}g)| &\leq e^{m^\alpha t} \int_0^\infty f_{t,\alpha/2}(\lambda)|(f, e^{-\lambda S}g)|\, d\lambda \\
&\leq e^{m^\alpha t} \int_0^\infty f_{t,\alpha/2}(\lambda)(|f|, |e^{-\lambda S}g|)\, d\lambda \\
&\leq e^{m^\alpha t} \int_0^\infty f_{t,\alpha/2}(\lambda)(|f|, e^{-\lambda(-\Delta+m^2)}|g|)\, d\lambda \\
&= (|f|, e^{-t[(H_{0,m})^\alpha - m^\alpha]}|g|).
\end{aligned}$$

Then this is, of course, also valid for f, $g \in L^2(\mathbb{R}^d)$. □

§4. Concluding remarks

In the literature there are three kinds of relativistic Schrödinger operators for a spinless particle of mass $m \geq 0$ corresponding to the classical relativistic Hamiltonian symbol $\sqrt{(\xi - A(x))^2 + m^2}$ with magnetic vector potential $A(x)$, dependent on *how to quantize this symbol*. Of course, one of them is $H_{A,m}$ in (1.1), treated in this paper, and the other two are defined as pseudo-differential operators, differing from $H_{A,m}$ which is defined as an operator-theoretical square root. In [I12, I13], their common and different properties were discussed mainly in connection with the corresponding path integral representations for their semigroups.

The other two relativistic Schrödinger operators are defined by oscillatory integrals, for $f \in C_0^\infty(\mathbb{R}^d)$, as

$$\begin{aligned}(H^{(1)}_{A,m} f)(x) &:= \frac{1}{(2\pi)^d} \iint_{\mathbb{R}^d \times \mathbb{R}^d} e^{i(x-y)\cdot\xi} \sqrt{\Big(\xi - A\Big(\frac{x+y}{2}\Big)\Big)^2 + m^2}\, f(y)\, dy\, d\xi \\ &= \frac{1}{(2\pi)^d} \iint_{\mathbb{R}^d \times \mathbb{R}^d} e^{i(x-y)\cdot(\xi + A((x+y)/2))} \sqrt{\xi^2 + m^2}\, f(y)\, dy\, d\xi\,; \end{aligned} \tag{4.1}$$

$$\begin{aligned}(H^{(2)}_{A,m} f)(x) &:= \frac{1}{(2\pi)^d} \iint_{\mathbb{R}^d \times \mathbb{R}^d} e^{i(x-y)\cdot\xi} \sqrt{\Big(\xi - \int_0^1 A((1-\theta)x + \theta y)\, d\theta\Big)^2 + m^2} \\ &\qquad \times f(y)\, dy\, d\xi \\ &= \frac{1}{(2\pi)^d} \iint_{\mathbb{R}^d \times \mathbb{R}^d} e^{i(x-y)\cdot(\xi + \int_0^1 A((1-\theta)x+\theta y)\, d\theta)} \sqrt{\xi^2 + m^2}\, f(y)\, dy\, d\xi. \end{aligned} \tag{4.2}$$

Equality (4.1) is a *Weyl pseudo-differential operator* with midpoint prescription given in [ITa86] (also [I89, NaU90]), and (4.2) is a modification of (4.1) given in [IfMP07]. Note that $H^{(1)}_{A,m}$ and $H^{(2)}_{A,m}$ are denoted in [I12, I13] by slightly different notation, $H^{(1)}_A$ and $H^{(2)}_A$, respectively, while our $H_{A,m}$ in (1.1) is denoted by $H^{(3)}_A$.

In this section we should like to call attention to the fact that the distributional form of Kato's inequality is *missing* for $H^{(3)}_{A,m}$ or our $H_{A,m}$ in (1.1), although it already exists for the other two, $H^{(1)}_{A,m}$ in (4.1), $H^{(2)}_{A,m}$ in (4.2). Indeed, it has been shown in [I89, ITs92] for $H^{(1)}_{A,m}$ under some suitable conditions on $A(x)$ (which differ from $A \in L^2_{\rm loc}$), and can be shown in the same way for $H^{(2)}_{A,m}$ (cf. [I13]). Therefore, at least the case of Theorem 1.1 with $A = 0$ turns out to be already known.

Let us briefly mention here some known facts for $H^{(1)}_{A,m}$, $H^{(2)}_{A,m}$ and $H^{(3)}_{A,m}$.

1°. With suitable reasonable conditions on $A(x)$, they all define *self-adjoint* operators in $L^2(\mathbb{R}^d)$, which are bounded below. For instance, they become self-adjoint operators defined as quadratic forms, for $H^{(1)}_{A,m}$ and $H^{(2)}_{A,m}$ when $A \in L^{1+\delta}_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ for some $\delta > 0$ (cf. [I89, I13, IfMP07]), and for $H^{(3)}_{A,m}$ when $A \in L^2_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ (e.g., [CFKiSi87, pp. 8–10] or [I13]).
Furthermore, they are bounded below by the *same lower bound*, in particular,

$$H^{(j)}_{A,m} \geq m, \quad j = 1, 2, 3.$$

2°. $H^{(2)}_{A,m}$ and $H^{(3)}_{A,m}$ are covariant under gauge transformation, i.e., for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$ it holds that $H^{(j)}_{A+\nabla\varphi} = e^{i\varphi}H^{(j)}_{A,m}e^{-i\varphi}$, $j = 2, 3$. However, $H^{(1)}_{A,m}$ is not.

3°. All these three operators are different in general, but coincide if $A(x)$ is linear in x; i.e., if $A(x) = \dot{A}\cdot x$ with $\dot{A} : d \times d$ real symmetric *constant matrix*, then $H^{(1)}_{A,m} = H^{(2)}_{A,m} = H^{(3)}_{A,m}$. So, this holds for uniform magnetic fields with $d = 3$.

Appendix A.

Our aim is to derive the following expressions for integral kernel $k_0^{m,\alpha}(t,x)$ of semigroup $e^{-t[(H_{0,m})^\alpha - m^\alpha]}$ and density function $n^{m,\alpha}$ of Lévy measure $n^{m,\alpha}(dy)$ for $0 < \alpha \le 1$, which are mentioned around formulas (2.7), (2.8)/(2.9):

$$\begin{aligned} k_0^{m,\alpha}(t,x) = {}& \frac{e^{m^\alpha t}}{\pi(2\pi)^{d/2}|x|^{d/2-1}} \int_0^\infty e^{-tr^{\alpha/2}\cos(\alpha/2)\pi} \sin(tr^{\alpha/2}\sin(\tfrac{\alpha}{2}\pi)) \\ & \times (m^2+r)^{(d/2-1)/2} K_{d/2-1}((m^2+r)^{1/2}|x|)\, dr, \end{aligned} \tag{A.1}$$

$$n^{m,\alpha} = \frac{2^{1+(\alpha/2)}\sin\left(\frac{\alpha}{2}\pi\right)(2\pi)^{\alpha/2}\Gamma(\frac{\alpha}{2}+1)}{\pi}\left(\frac{m}{2\pi}\right)^{(d+\alpha)/2} \frac{K_{(d+\alpha)/2}(m|x|)}{|x|^{(d+\alpha)/2}}. \tag{A.2}$$

Equality (A.2) is essentially the same as ν^m in [ByMaRy09, (2.7), p. 4877], which is established for the heat semigroup $e^{-t[(-\Delta+m^{\alpha/2})^{\alpha/2}-m]}$ instead of our $e^{-t[(H_{0,m})^\alpha - m^\alpha]}$. Indeed, putting $m = m'^{(1/\alpha)}$ in (A.2) to rewrite it with Euler's reflection formula $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ yields eq. (2.7) in this reference with m replaced by m'.

To show (A.1) and (A.2), we use another formula (3.18)/(3.19) to express the semigroup $e^{-t(H_{0,m})^\alpha} \equiv e^{-t(-\Delta+m^2)^{(\alpha/2)}}$ $(0 < \alpha \le 1)$ for the fractional power:

$$(e^{-t(H_{0,m})^\alpha}u)(x) = \int_{\mathbb{R}^d}\left(\int_0^\infty f_{t,\alpha/2}(s)e^{-s(-\Delta+m^2)}ds\, u\right)(y)\, dy,$$

where $e^{-t(-\Delta+m^2)}$ is the heat semigroup multiplied by $e^{-m^2 t}$:

$$(e^{-t(-\Delta+m^2)}u)(x) = \frac{1}{(4\pi t)^{d/2}}\int_{\mathbb{R}^d} e^{-m^2 t - \left((x-y)^2/4t\right)}u(y)\, dy.$$

Then $f_{t,\alpha/2}(s)$ in (3.19) is rewritten as

$$f_{t,\alpha/2}(s) = \frac{1}{\pi}\int_0^\infty e^{sr\cos\theta - tr^\alpha\cos(\alpha/2)\theta}\sin(sr\sin\theta - tr^{\alpha/2}\sin\tfrac{\alpha}{2}\theta + \theta)\, dr \qquad (t > 0,\ \ s \ge 0),$$

where the integration path is deformed to the union of two paths $re^{-i\theta}(-\infty < -r < 0)$ and $re^{i\theta}(0 < r < \infty)$, where $\frac{\pi}{2} \le \theta \le \pi$ (see [Y78, IX, 11, pp. 259–263]).

Then we take $\theta = \pi$ to have

$$
\begin{aligned}
&(e^{-t(H_{0,m})^\alpha})(x) \\
&\quad = \int_0^\infty ds\, \frac{1}{\pi} \frac{1}{(4\pi s)^{d/2}} e^{-m^2 s - \left(x^2/4s\right)} \int_0^\infty e^{-sr - tr^{\alpha/2}\cos{(\alpha/2)\pi}} \sin(tr^{\alpha/2} \sin \tfrac{\alpha}{2}\pi)\, dr \\
&\quad = \frac{1}{\pi(4\pi)^{d/2}} \int_0^\infty dr\, e^{-tr^{\alpha/2}\cos{(\alpha/2)\pi}} \sin(tr^{\alpha/2}\sin\tfrac{\alpha}{2}\pi)(m^2+r)^{d/2-1} \\
&\qquad\qquad \times \int_0^\infty \frac{e^{-s-\left((m^2+r)x^2/4s\right)}}{s^{d/2}}\, ds \\
&\quad = \frac{1}{\pi(2\pi)^{d/2}|x|^{d/2-1}} \int_0^\infty e^{-tr^{\alpha/2}\cos{(\alpha/2)\pi}} \sin(tr^{\alpha/2}\sin\tfrac{\alpha}{2}\pi)(m^2+r)^{(d/2-1)/2} \\
&\qquad\qquad \times K_{d/2-1}((m^2+r)^{1/2}|x|)\, dr,
\end{aligned}
$$

where we have used the representation formula of the modified Bessel function of the third kind, $K_\nu(z)$ [GrR94, Sec. 8.432. 6, p. 969]:

$$
K_\nu(z) = \tfrac{1}{2}\left(\tfrac{z}{2}\right)^\nu \int_0^\infty e^{-t-\left(z^2/4t\right)}\, t^{-\nu-1}\, dt, \quad \nu > -\tfrac{1}{2}, \; z > 0.
$$

It follows that the integral kernel $k_0^{m,\alpha}(t,x)$ of the semigroup $e^{-t[(H_{0,m})^\alpha - m^\alpha]}$ turns out to be

$$
\begin{aligned}
k_0^{m,\alpha}(t,x) &:= e^{-t[(H_{0,m})^\alpha - m^\alpha]}(x) \\
&= \frac{e^{m^\alpha t}}{\pi(2\pi)^{d/2}|x|^{d/2-1}} \int_0^\infty e^{-tr^{\alpha/2}\cos{(\alpha/2)\pi}} \sin(tr^{\alpha/2}\sin\tfrac{\alpha}{2}\pi) \\
&\qquad \times (m^2+r)^{(d/2-1)/2} K_{d/2-1}((m^2+r)^{1/2}|x|)\, dr.
\end{aligned}
$$

This shows (A.1).

Next, we have

$$
\begin{aligned}
\frac{d}{dt}k_0^{m,\alpha}(t,x) &= \frac{1}{\pi(2\pi)^{d/2}|x|^{(d-1)/2}} \int_0^\infty dr\, \frac{d}{dt}\Big[e^{t(m^\alpha - r^{\alpha/2}\cos{(\alpha/2)\pi})} \sin(tr^{\alpha/2}\sin\tfrac{\alpha}{2}\pi) \\
&\quad \times (m^2+r)^{(d/2-1)/2} K_{d/2-1}((m^2+r)^{1/2}|x|)\Big] \\
&= \frac{1}{\pi(2\pi)^{d/2}|x|^{(d-1)/2}} \int_0^\infty dr\, \Big[(m^\alpha - r^{\alpha/2}\cos\tfrac{\alpha}{2}\pi)\sin(tr^{\alpha/2}\sin\tfrac{\alpha}{2}\pi) \\
&\quad + r^{\alpha/2}\sin\tfrac{\alpha}{2}\pi\cos(tr^{\alpha/2}\sin\tfrac{\alpha}{2}\pi)\Big] e^{t(m^\alpha - r^{\alpha/2}\cos{(\alpha/2)\pi})} \\
&\quad \times (m^2+r)^{(d/2-1)/2} K_{d/2-1}((m^2+r)^{1/2}|x|).
\end{aligned}
$$

Then by the fact (2.7), we have, as $t \downarrow 0$,

$$
\begin{aligned}
n^{m,\alpha}(t,x) &= \frac{1}{t} k_0^{m,\alpha}(t,x) \\
&\to \frac{d}{dt} k_0^{m,\alpha}(t,x)\Big|_{t=0} =: n^{m,\alpha}(x) \\
&= \frac{\sin \frac{\alpha}{2}\pi}{\pi(2\pi)^{d/2}|x|^{d/2-1}} \int_0^\infty dr\,(m^2+r)^{(d/2-1)/2} r^{\alpha/2} K_{d/2-1}((m^2+r)^{1/2}|x|).
\end{aligned}
$$

Here, the integral in the last line above is equal to

$$
\begin{aligned}
&\int_0^\infty (m^2+\tau^2)^{(d/2-1)/2}\tau^\alpha K_{d/2-1}((m^2+\tau^2)^{1/2}|x|)\, 2\tau\, d\tau \quad (r := \tau^2) \\
&\quad = 2\int_m^\infty a^{d/2-1}(a^2-m^2)^{(1+\alpha)/2} K_{d/2-1}(a|x|)\, \frac{a}{(a^2-m^2)^{1/2}}\, da \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (a := (m^2+\tau^2)^{1/2}) \\
&\quad = \frac{2}{|x|^{1/2}} \int_m^\infty a^{(d-1)/2}(a^2-m^2)^{\alpha/2} K_{d/2-1}(a|x|)\, (a|x|)^{1/2}\, da.
\end{aligned}
$$

Then we use the following formula [EMOT54, Chap. X (*K-Transforms*), 10.2.(13), p. 129]:

$$
\int_a^\infty x^{1/2-\nu}(x^2-a^2)^\mu K_\nu(xy)(xy)^{1/2}\, dx = 2^\mu a^{\mu-\nu+1} y^{-\mu-1/2}\Gamma(\mu+1) K_{\mu-\nu+1}(ay), \qquad y>0,\ \mu>-1,
$$

with $\mu = \frac{\alpha}{2}$, $-\nu = \frac{d}{2}-1$ and with "ν" in place of "$-\nu$" because $K_{-\nu}(\tau) = K_\nu(\tau)$, to finally obtain

$$
\begin{aligned}
n^{m,\alpha}(x) &= \frac{\sin \frac{\alpha}{2}\pi}{\pi(2\pi)^{d/2}|x|^{d/2-1}} 2^{\alpha/2+1} m^{(d+\alpha)/2}|x|^{-(\alpha/2+1)}\Gamma(\tfrac{\alpha}{2}+1) K_{(d+\alpha)/2}(m|x|) \\
&= \frac{2^{1+\alpha/2}\sin\left(\frac{\alpha}{2}\pi\right)(2\pi)^{\alpha/2}\Gamma(\frac{\alpha}{2}+1)}{\pi}\left(\frac{m}{2\pi}\right)^{(d+\alpha)/2} \frac{K_{(d+\alpha)/2}(m|x|)}{|x|^{(d+\alpha)/2}}.
\end{aligned}
$$

If $\alpha = 1$, this expression reduces to

$$
n^{m,1}(x) = 2\left(\frac{m}{2\pi}\right)^{(d+1)/2} \frac{K_{(d+1)/2}(m|x|)}{|x|^{(d+1)/2}},
$$

which is nothing but the first formula of (2.9), and we see that $n^{m,\alpha}(x)$ tends to $n^{m,1}(x)$, as $\alpha \uparrow 1$ since $\Gamma(\frac{1}{2}+1) = \frac{\pi^{1/2}}{2}$.

Acknowledgements

The work by FH is supported in part by JSPS Grant-in-Aid for Challenging Exploratory Research 15K13445, and the work by TI is supported in part by JSPS Grant-in-Aid for Scientific Research 23540191 and 25400166.

The authors would like to thank an anonymous referee for very careful reading of the manuscript and valuable comments improving the paper.

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