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Two-Weight Norm, Poincaré, Sobolev and Stein–Weiss Inequalities on Morrey Spaces

by

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Abstract

We establish two-weight norm inequalities for singular integral operators and fractional integral operators on Morrey spaces. As a consequence of these inequalities, we obtain two-weight Poincaré and Sobolev inequalities on Morrey spaces. Moreover, we also establish the Stein–Weiss inequality, the Hardy inequality and the Rellich inequality on Morrey spaces.

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§1. Introduction

The main results of this paper are the two-weight norm inequalities and some important inequalities in partial differential equations, namely, Poincaré inequalities, Sobolev inequalities, Stein–Weiss inequalities, Hardy inequalities and Rellich inequalities on Morrey spaces.

The two-weight norm inequality is a natural extension of the weighted norm inequality in analysis [14]. The main theme of the two-weight norm inequalities is the characterization of those weight functions such that the corresponding two-weight inequalities for the maximal function [6, 21, 40], singular integral operators [5, 8, 7, 10, 13, 27, 32, 34, 41, 48] and fractional integral operators [24, 26, 33, 42] hold.

On the other hand, the two-weight norm inequality also plays a significant role in the study of partial differential equations. In particular, the two-weight

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Poincaré and Sobolev inequalities are consequences of the two-weight inequalities of fractional integral operators. The two-weight Poincaré and Sobolev inequalities have profound applications for partial differential equations such as eigenvalues estimation [22], the regularity of degenerate second-order elliptic differential operators [4, 12] and unique continuation for differential inequalities [3].

In this paper, we further extend two-weight norm inequalities to Morrey spaces. Morrey spaces were introduced by Morrey for the study of solutions of some quasi-linear elliptic partial differential equations [25]. The introduction of Morrey spaces has been one of the most important extensions of Lebesgue spaces.

Most recently, the study of two-weight norm inequalities has been extended to Morrey spaces. The two-weight norm inequality for Morrey spaces was introduced by Tanaka in [46]. In [46], some conditions on weights so that two-weight norm inequalities for maximal functions are valid on Morrey spaces are identified.

The results in [46] give us the motivation to study two-weight norm inequalities for singular integral operators and fractional integral operators.

In this paper, we find that, roughly speaking, whenever two-weight norm inequalities for singular integral operators and fractional integral operators are valid for Lebesgue spaces, they are also valid for Morrey spaces. Therefore, the main result of this paper shows that two-weight norm inequalities on Morrey spaces follow from the corresponding inequalities on Lebesgue spaces.

By using this principle, we establish two-weight norm inequalities for singular integral operators and fractional integral operators on Morrey spaces. These inequalities yield two-weight Poincaré and Sobolev inequalities on Morrey spaces.

Furthermore, this principle also gives an extension of Stein–Weiss inequalities for fractional integral operators to Morrey spaces. In turn it yields Hardy inequalities and Rellich inequalities on Morrey spaces.

This paper is organized as follows. We give some preliminary results for weighted Lebesgue spaces in Section 2. Two-weight norm inequalities for singular integral operators on Morrey spaces are presented in Section 3. We establish two-weight norm inequalities for fractional integral operators on Morrey spaces in Section 4. The two-weight Poincaré and Sobolev inequalities on Morrey spaces are given in Section 5. In Section 6, we have Stein–Weiss inequalities, Hardy inequalities and Rellich inequalities on Morrey spaces.

§2. Preliminaries

Let $B(z,r) = \{x \in \mathbb{R}^n : |x - z| < r\}$ denote the open ball with center $z \in \mathbb{R}^n$ and radius $r > 0$. Let $\mathbb{B} = \{B(z,r) : z \in \mathbb{R}^n, r > 0\}$. For any Lebesgue-measurable set E, let $|E|$ and χ_E be the Lebesgue measure and the characteristic function of E, respectively.

For any nonnegative locally integrable function u, we call u a weight. The weighted Lebesgue spaces consist of those Lebesgue-measurable functions f satisfying

$$\|f\|_{L_p(u)} = \Big(\int_{\mathbb{R}^n} |f(x)|^p u(x)\, dx \Big)^{1/p} < \infty.$$

Notice that $L_p(u)$ is not necessarily a Banach function space [2, Chapter 1, Definition 1]. More specifically, for any unbounded Lebesgue-measurable set E with finite measure $|E| < \infty$, $\|\chi_E\|_{L_p(u)}$ is not necessarily finite.

Next, we present some duality results for $L_p(u)$ with respect to Lebesgue measure.

By using the Hölder inequality, we find that

$$\begin{aligned}\int_{\mathbb{R}^n} |f(x)g(x)|\, dx &\le \Big(\int_{\mathbb{R}^n} |f(x)|^p u(x)\, dx \Big)^{1/p} \Big(\int_{\mathbb{R}^n} |g(x)|^{p'} u(x)^{-p'/p}\, dx \Big)^{1/p'} \\ &= \|f\|_{L_p(u)} \|g\|_{L_{p'}(u^{-p'/p})},\end{aligned}$$

where p' is the conjugate of p.

Moreover, we also have the norm conjugate formulas for the pair $L_p(u)$ and $L_{p'}(u^{-p'/p})$.

Lemma 2.1. *Let $1 < p < \infty$ and u be a nonnegative locally integrable function. We have*

$$\|f\|_{L_p(u)} = \sup \Big\{ \int_{\mathbb{R}^n} |f(x)g(x)|\, dx : \|g\|_{L_{p'}(u^{-p'/p})} \le 1 \Big\}, \tag{2.1}$$

$$\|f\|_{L_{p'}(u^{-p'/p})} = \sup \Big\{ \int_{\mathbb{R}^n} |f(x)g(x)|\, dx : \|g\|_{L_p(u)} \le 1 \Big\}. \tag{2.2}$$

The above results follow from the norm conjugate formulas for the Lebesgue spaces L_p.

Recall that the Hardy–Littlewood maximal operator M is defined as

$$(Mf)(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|\, dy, \quad f \in L^1_{\mathrm{loc}},$$

where the supremum is taken over all balls $B \in \mathbb{B}$ containing x.

We now define the class of pairs of weights (v, u) such that M is bounded from $L_p(v)$ to $L_p(u)$.

Definition 2.2. Let $1 < p < \infty$. For any nonnegative locally integrable functions u, v, we write $(v, u) \in \mathbb{M}_p$ if the Hardy–Littlewood maximal operator M is bounded from $L_p(v)$ to $L_p(u)$.

For instance, for any $1 < p < \infty$ and any weight u, in view of [14, Chapter II, Theorem 2.12], we have $(Mu, u) \in \mathbb{M}_p$.

For any $1 < p < \infty$, $(u, u) \in \mathbb{M}_p$ is equivalent to $u \in A_p$, where A_p is the Muckenhoupt class of weight functions [44, Chapter V, Section 1]. The reader is referred to [21] for the proof of this result.

We have a precise characterization of the condition $(v, u) \in \mathbb{M}_p$ by Sawyer [40].

Theorem 2.3. *Let $1 < p < \infty$. For any nonnegative locally integrable functions u, v, $(v, u) \in \mathbb{M}_p$ if and only if there exists a constant $C > 0$ such that for any cube Q,*

$$\int_Q (M(v^{1-p'}\chi_Q)(x))^p u(x)\, dx \le C \int_Q v(x)^{1-p'}\, dx < \infty. \tag{2.3}$$

Notice that whenever (v, u) satisfies (2.3), $v^{1-p'} = v^{-p'/p}$ is locally integrable.

We now obtain some estimates for the norm of the characteristic function of $B \in \mathbb{B}$ on weighted Lebesgue spaces whenever $(v, u) \in \mathbb{M}_p$. These estimates are crucial for the establishment of two-weight norm inequalities for singular integral operators.

Lemma 2.4. *Let $1 < p < \infty$ and v, u be nonnegative locally integrable functions. If $(v, u) \in \mathbb{M}_p$, then there exists a constant $C > 0$ such that for any $B \in \mathbb{B}$, we have*

$$\|\chi_B\|_{L_{p'}(v^{-p'/p})}\|\chi_B\|_{L_p(u)} \le C|B|. \tag{2.4}$$

Proof. For any $B \in \mathbb{B}$, we consider the projection

$$(P_B g)(y) = \left(\frac{1}{|B|}\int_B |g(x)|\, dx\right)\chi_B(y).$$

There exists a constant $C > 0$ such that for any $B \in \mathbb{B}$, $P_B(f) \le CM(f)$. Hence, $\sup_B \|P_B\|_{L_p(v)\to L_p(u)} \le C\|M\|_{L_p(v)\to L_p(u)}$.

Furthermore, as $(v, u) \in \mathbb{M}_p$, that is, (v, u) satisfies (2.3), we find that $v^{1-p'} = v^{-1/(p-1)} = v^{-p'/p}$ is locally integrable. Therefore, for any $B \in \mathbb{B}$, $\|\chi_B\|_{L_{p'}(v^{-p'/p})}$ is well defined.

Consequently, (2.2) ensures that

$$\begin{aligned}\|\chi_B\|_{L_{p'}(v^{-p'/p})}\|\chi_B\|_{L_p(u)} &= \sup\left\{\int_B |g(x)|\, dx \|\chi_B\|_{L_p(u)} : \|g\|_{L_p(v)} \le 1\right\}\\ &= \sup\left\{\frac{1}{|B|}\int_B |g(x)|\, dx \|\chi_B\|_{L_p(u)} : \|g\|_{L_p(v)} \le 1\right\}|B|\end{aligned}$$

$$\le C \sup \left\{ \|P_B(g)\|_{L_p(u)} : \|g\|_{L_p(v)} \le 1 \right\} |B|$$
$$\le C \sup \left\{ \|M(g)\|_{L_p(u)} : \|g\|_{L_p(v)} \le 1 \right\} |B| \le C|B|.$$

□

We now give the definition of weighted Morrey spaces that is used in this paper.

Definition 2.5. Let $1 < p < \infty$, $\omega : \mathbb{B} \to (0, \infty)$ and u be a nonnegative locally integrable function. The *generalized Morrey space* $\mathcal{M}^p_\omega(u)$ consists of those Lebesgue-measurable functions f satisfying

$$\|f\|_{\mathcal{M}^p_\omega(u)} = \sup_{B \in \mathbb{B}} \frac{1}{\omega(B)} \|f\chi_B\|_{L_p(u)} < \infty.$$

We call ω the Morrey weight for $\mathcal{M}^p_\omega(u)$.

When $u \equiv 1$, $\mathcal{M}^p_\omega(u)$ is the generalized Morrey space introduced by Nakai [28]. Furthermore, whenever $u \equiv 1$ and $\omega(B) = |B|^{(1/p)-(1/q)}$ for some $q \ge p$, $\mathcal{M}^p_\omega(u)$ becomes the classical Morrey space in [25].

§3. Singular integral operator

We obtain two-weight norm inequalities for singular integral operators on Morrey spaces in this section. We first present the conditions imposed on the Morrey weight ω.

Definition 3.1. Let $1 < p < \infty$ and u be a nonnegative locally integrable function. For any $\omega : \mathbb{B} \to (0, \infty)$, we write $\omega \in \mathcal{W}_{p,u}$ if there exists a constant $C > 0$ such that for any $x \in \mathbb{R}^n$ and $r > 0$,

$$\omega(B(x, 2r)) \le C\omega(B(x, r)), \tag{3.1}$$

$$\sum_{j=0}^{\infty} \frac{\|\chi_{B(x,r)}\|_{L_p(u)}}{\|\chi_{B(x,2^{j+1})}\|_{L_p(u)}} \omega(B(x, 2^{j+1}r)) \le C\omega(B(x, r)). \tag{3.2}$$

For any $a \in \mathbb{R}$, whenever $u(x) = |x|^a$, we write $\mathcal{W}_{p,u}$ as $\mathcal{W}_{p,a}$.

Notice that (3.2) is connected to [30, (1.3)]. Moreover, some similar conditions have been used in [16, 17, 18, 19, 20] for studies of vector-valued maximal inequalities, fractional integral operators and singular integral operators on Morrey spaces with variable exponents.

Let $\lambda < 1/p$. When $u \equiv 1$ and $\omega(B)$ satisfies the condition

$$\omega(B(x,2r)) \le 2^{\lambda}\omega(B(x,r))$$

for all $x \in \mathbb{R}^n$ and $r > 0$, then $\omega \in \mathcal{W}_{p,u}$.

We now present one of the main results for this paper: two-weight norm inequalities for singular integral operators on Morrey spaces.

Theorem 3.2. *Let $1 < p < \infty$ and $\omega \in \mathcal{W}_{p,u}$ and $(v,u) \in \mathbb{M}_p$. Let*

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)\,dy,$$

where

$$|K(x,y)| \le C\frac{1}{|x-y|^n}, \quad x \ne y \tag{3.3}$$

for some $C > 0$. If $T : L_p(v) \to L_p(u)$ is bounded, then T can be extended to be a bounded linear operator from $\mathcal{M}^p_\omega(v)$ to $\mathcal{M}^p_\omega(u)$

Proof. Let $f \in \mathcal{M}^p_\omega(v)$. For any $z \in \mathbb{R}^n$ and $r > 0$, write $f(x) = f_0(x) + \sum_{j=1}^{\infty} f_j(x)$, where $f_0 = \chi_{B(z,2r)}f$ and $f_j = \chi_{B(z,2^{j+1}r)\setminus B(z,2^j r)}f$, $j \in \mathbb{N}\setminus\{0\}$.

Since $T : L_p(v) \to L_p(u)$ is a bounded linear operator, we find that $\|Tf_0\|_{L_p(u)} \le C\|f_0\|_{L_p(v)}$. Consequently, (3.1) ensures that

$$\begin{aligned}\frac{1}{\omega(B(z,r))}\|\chi_{B(z,r)}(Tf_0)\|_{L_p(u)} &\le C\frac{1}{\omega(B(z,2r))}\|\chi_{B(z,2r)}f\|_{L_p(v)}\\ &\le C\sup_{\substack{y\in\mathbb{R}^n\\ R>0}}\frac{1}{\omega(B(y,R))}\|\chi_{B(y,R)}f\|_{L_p(v)}.\end{aligned}\tag{3.4}$$

According to (3.3), there exists a constant $C > 0$ such that for any $r > 0$, $x \in \mathbb{R}^n$ and Lebesgue-measurable function f with $\operatorname{supp} f \subseteq \mathbb{R}^n\setminus B(x,r)$,

$$|Tf(x)| \le C\frac{1}{r^n}\int_{\mathbb{R}^n}|f(y)|\,dy. \tag{3.5}$$

Then, (3.5) guarantees that there is a constant $C > 0$ such that, for any $j \ge 1$,

$$\chi_{B(z,r)}(x)|(Tf_j)(x)| \le C2^{-jn}r^{-n}\chi_{B(z,r)}(x)\int_{B(z,2^{j+1}r)}|f(y)|\,dy. \tag{3.6}$$

The Hölder inequality ensures that

$$\int_{B(z,2^{j+1}r)}|f(y)|\,dy \le C\big\|\chi_{B(z,2^{j+1}r)}f\big\|_{L_p(v)}\|\chi_{B(z,2^{j+1}r)}\|_{L_{p'}(v^{-p'/p})}$$

for some $C > 0$.

Subsequently, applying the norm $\|\cdot\|_{L_p(u)}$ on both sides of (3.6), we have

$$\begin{aligned}\|\chi_{B(z,r)}(Tf_j)\|_{L_p(u)} &\le C2^{-jn}r^{-n}\|\chi_{B(z,r)}\|_{L_p(u)}\big\|\chi_{B(z,2^{j+1}r)}f\big\|_{L_p(v)}\\ &\quad\times\|\chi_{B(z,2^{j+1}r)}\|_{L_{p'}(v^{-p'/p})}.\end{aligned}\tag{3.7}$$

Applying (2.4) with $B=B(z,2^{j+1}r)$, we have

$$\|\chi_{B(z,2^{j+1}r)}\|_{L_{p'}(v^{-p'/p})}\le C\frac{2^{(j+1)n}r^n}{\|\chi_{B(z,2^{j+1}r)}\|_{L_p(u)}}.$$

Using the above inequality on (3.7), we obtain

$$\begin{aligned}\|\chi_{B(z,r)}(Tf_j)\|_{L_p(u)} &\le C2^{-jn}r^{-n}\frac{\|\chi_{B(z,r)}\|_{L_p(u)}\big\|\chi_{B(z,2^{j+1}r)}f\big\|_{L_p(v)}2^{(j+1)n}r^n}{\|\chi_{B(z,2^{j+1}r)}\|_{L_p(u)}}\\ &\le C\frac{\|\chi_{B(z,r)}\|_{L_p(u)}}{\|\chi_{B(z,2^{j+1}r)}\|_{L_p(u)}}\big\|\chi_{B(z,2^{j+1}r)}f\big\|_{L_p(v)}.\end{aligned}$$

Thus,

$$\begin{aligned}&\|\chi_{B(z,r)}(Tf_j)\|_{L_p(u)}\\ &\quad\le C\frac{\|\chi_{B(x,r)}\|_{L_p(u)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L_p(u)}}\frac{\omega(B(z,2^{j+1}r))}{\omega(B(z,2^{j+1}r))}\|\chi_{B(z,2^{j+1}r)}f\|_{L_p(v)}\\ &\quad\le C\frac{\|\chi_{B(x,r)}\|_{L_p(u)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L_p(u)}}\omega(B(z,2^{j+1}r))\sup_{\substack{y\in\mathbb{R}^n\\ R>0}}\frac{1}{\omega(B(y,R))}\|\chi_{B(y,R)}f\|_{L_p(v)}.\end{aligned}\tag{3.8}$$

As $\omega\in\mathcal{W}_{p,u}$, (3.4) and (3.8) yield

$$\begin{aligned}\frac{1}{\omega(B(z,r))}\|\chi_{B(z,r)}(Tf)\|_{L_p(u)} &\le\frac{1}{\omega(B(z,r))}\sum_{j=0}^{\infty}\|\chi_{B(z,r)}(Tf_j)\|_{L_p(u)}\\ &\le C\sup_{\substack{y\in\mathbb{R}^n\\ R>0}}\frac{1}{\omega(B(y,R))}\|\chi_{B(y,R)}f\|_{L_p(v)},\end{aligned}$$

where the constant $C>0$ is independent of r and z. Taking the supremum over $z\in\mathbb{R}^n$ and $r>0$ gives the boundedness of T from $\mathcal{M}^p_\omega(u)$ to $\mathcal{M}^p_\omega(v)$. □

Some similar results are obtained in [19] for vector-valued operators with singular kernels on Morrey spaces built on Banach function spaces. Notice that $L_p(u)$ is not necessarily a Banach function space, therefore, the results in [19] do not apply to $\mathcal{M}^p_\omega(u)$.

Even though we present the result in Theorem 3.2 for singular integral operators, the proof of Theorem 3.2 also applies to the Hardy–Littlewood maximal operator. Therefore, we have the following two-weight norm inequalities for the Hardy–Littlewood maximal operator on Morrey spaces. For brevity, we skip the proof of the following corollary.

Corollary 3.3. *Let $1 < p < \infty$, $(v, u) \in \mathbb{M}_p$ and $\omega \in \mathcal{W}_{p,u}$. Then, there exists a constant $C > 0$ such that for any $f \in \mathcal{M}^p_\omega(v)$, we have*

$$\|Mf\|_{\mathcal{M}^p_\omega(u)} \le C\|f\|_{\mathcal{M}^p_\omega(v)}. \tag{3.9}$$

When $v = u$, the above corollary is connected with the work in [30]. Notice that for the Hardy–Littlewood maximal operator, condition (3.2) can be further relaxed; see [35, 46]. In fact, the above result is a special case of [46, Theorem 3.1]. A detailed description of the two-weight norm inequality for the Hardy–Littlewood maximal operator on Morrey spaces is given in [46]. In addition, our concentration is on two-weight norm inequalities for singular integral operators and fractional integral operators. Therefore, we refer the reader to [46] for the details of two-weight norm inequalities for the Hardy–Littlewood maximal operator on Morrey spaces. We just remark that the characterizations given in [46] use the pre-dual of Morrey spaces [39, 46].

We now apply Theorem 3.2 to Calderón–Zygmund operators. For completeness, we first recall the definition of a Calderón–Zygmund operator.

We say that T is a Calderón–Zygmund operator if T is a bounded linear operator on L_2 and there exists a $\delta > 0$ such that for any $x, y \in \mathbb{R}^n$ with $x \neq y$ and any z with $|x - z| \le \frac{1}{2}|x - y|$, the Schwartz kernel of T, $K(x, y)$, satisfies

$$|K(x,y)| \le C|x-y|^{-n},$$
$$|K(x,y) - K(z,y)| \le C|x-z|^{\delta}|x-y|^{-n-\delta},$$
$$|K(x,z) - K(x,y)| \le C|y-z|^{\delta}|x-y|^{-n-\delta}.$$

We are now ready to apply Theorem 3.2 to Calderón–Zygmund operators.

Theorem 3.4. *Let $1 < p < \infty$ and u be a nonnegative integrable function. Suppose that $\omega \in \mathcal{W}_{p,M^{[p]+1}u}$, where $M^{[p]+1}$ is the $([p]+1)$-iterate of the Hardy–Littlewood maximal operator M and $[p]$ is the integer part of p.*

If T is a Calderón–Zygmund operator, then, for any nonnegative locally integrable function u, we have

$$\|Tf\|_{\mathcal{M}^p_\omega(u)} \le C\|f\|_{\mathcal{M}^p_\omega(M^{[p]+1}u)}.$$

Proof. In view of [32, Theorem 3.1], we find that $T : L^p(M^{[p]+1}u) \to L_p(u)$ is bounded. Moreover, according to [14, Chapter II, Theorem 2.12], we find that

$$\int_{\mathbb{R}^n} (Mf(x))^p u(x)\,dx \le C \int_{\mathbb{R}^n} |f(x)|^p Mu(x)\,dx \le \int_{\mathbb{R}^n} |f(x)|^p M^{[p]+1}u(x)\,dx$$

for some $C > 0$. Therefore, $(M^{[p]+1}u, u) \in \mathbb{M}_p$. Finally, Theorem 3.2 ensures the boundedness of T from $\mathcal{M}^p_\omega(M^{[p]+1}u)$ to $\mathcal{M}^p_\omega(u)$. □

In particular, Theorem 3.4 also extends the result of the two-weight norm inequality for singular integral operators in [5] to Morrey spaces. Specifically, for any nonnegative locally integrable function u, and any $1 < r < \infty$,

$$u \le (M(u^r)^{1/r}).$$

According to [44, Chapter V, Proposition 8], $(M(u^r))^{1/r} \in A_1$, where A_1 is the class of Muckenhoupt weight functions. The characterization of A_1 [44, Chapter V, Section 5.2] guarantees that

$$M^{[p]+1}u \le M^{[p]+1}(M(u^r)^{1/r}) \le CM(u^r)^{1/r}.$$

As a special case of Theorem 3.4, we have an extension of the Córdoba–Fefferman inequalities on Morrey spaces.

Corollary 3.5. *Let $1 < r < \infty$, $1 < p < \infty$, u be a nonnegative integrable function and $\omega \in \mathcal{W}_{p,M^{[p]+1}u}$. If T is a Calderón–Zygmund operator, then, for any nonnegative locally integrable function u, we have*

$$\|Tf\|_{\mathcal{M}^p_\omega(u)} \le C\|f\|_{\mathcal{M}^p_\omega((M(u^r))^{1/r})}.$$

For some more conditions for which the two-weight norm inequalities for singular integral operators are valid on Lebesgue space and some other related results such as sharp bounds for two-weight norm inequalities, the reader is referred to [5, 8, 7, 10, 13, 27, 32, 34, 48].

Furthermore, one-weight norm inequalities for Calderón–Zygmund operators and their commutators on Morrey spaces with weight ω belonging to A_p, $1 < p < \infty$ are obtained in [23, Theorems 3.3 and 3.4]. Note that Corollary 3.5 gives two-weight norm inequalities for which the weight $(M(u^r))^{1/r}$ belongs to A_1.

§4. Fractional integral operators

In this section, we establish two-weight norm inequalities for fractional integral operators on Morrey spaces.

We first recall the definition of fractional integral operators. For any $0<\alpha<n$, the fractional integral operator I_α is defined by

$$(I_\alpha f)(x)=\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}\,dy.$$

The fractional integral operator I_α is also called the Riesz potential. The corresponding fractional maximal operator is defined by

$$(M_\alpha f)(x)=\sup_{B\ni x}\frac{1}{|B|^{1-\alpha/n}}\int_B|f(y)|\,dy,$$

where the supremum is taken over all balls $B\in\mathbb{B}$ containing x.

For the mapping properties of fractional integral operators on function spaces, the reader is referred to [1, 29]. Studies of fractional integral operators have been extended to generalized fractional integral operators. The reader may consult [36, 37, 38].

Similar to Definition 2.2, the following class consists of those pairs of weights such that the fractional integral operator is bounded from $L_p(v)$ to $L_p(u)$.

Definition 4.1. Let $1<p,q<\infty$ and $0<\alpha<n$. For any nonnegative locally integrable functions v,u, we write $(v,u)\in\mathbb{F}_{p,q,\alpha}$ if $v^{-p'/p}$ is locally integrable and

$$\|I_\alpha f\|_{L_q(u)}\le C\|f\|_{L_p(v)} \tag{4.1}$$

for some $C>0$.

We see that in the above definition, we require that $v^{-p'/p}$ is locally integrable; this condition is crucial for our study since it guarantees that for any $B\in\mathbb{B}$, $\|\chi_B\|_{L_{p'}(v^{-p'/p})}$ is well defined.

According to [42], for $1<p<\infty$ and $v,u^{1-p'}\in A_\infty$, where A_∞ is the Muckenhoupt class of weight functions, (v,u) satisfies (4.1) if and only if

$$\sup_{B\in\mathbb{B}}\frac{1}{|B|^{1-\alpha/n}}\int_B u(y)\,dy\left(\frac{1}{|B|}\int_B v^{1-p'}(y)\,dy\right)^{p-1}<\infty. \tag{4.2}$$

Notice that (4.2) also guarantees that $v^{1-p'}=v^{-p'/p}$ is locally integrable. Thus, for any (v,u) satisfying (4.2), $(v,u)\in\mathbb{F}_{p,q,\alpha}$.

Additionally, in view of [26, 47], for any $1<p<\frac{n}{\alpha}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$ and any weight v, we have $(v^p,v^q)\in\mathbb{F}_{p,q,\alpha}$ if and only if

$$\sup_{B\in\mathbb{B}}\left(\frac{1}{|B|}\int_B v^q(x)\,dx\right)^{1/q}\left(\frac{1}{|B|}\int_B v^{-p'}(x)\,dx\right)^{1/p'}<\infty. \tag{4.3}$$

Also notice that whenever v satisfies the above condition, $(v^p)^{-p'/p} = v^{-p'}$ is locally integrable.

For some more conditions that guarantee $(v,u) \in \mathbb{F}_{p,q,\alpha}$ and some related results such as sharp weighted bounds for fractional integral operators, the reader is referred to [24, 33].

The following proposition gives a crucial estimate on the norm of the characteristic functions of $B \in \mathbb{B}$ whenever $(v,u) \in \mathbb{F}_{p,q,\alpha}$.

Proposition 4.2. *Let $1 < p,q < \infty$ and $0 < \alpha < n$. If $(v,u) \in \mathbb{F}_{p,q,\alpha}$, then there exists a constant $C > 0$ such that for any $B \in \mathbb{B}$,*

$$\|\chi_B\|_{L_{p'}(v^{-p'/p})}\|\chi_B\|_{L_q(u)} \le C|B|^{1-\alpha/n}. \tag{4.4}$$

Proof. We consider the operator $P_{B,\alpha}(g)$, $B = B(x_0,r)$, $x_0 \in \mathbb{R}^n$ and $r > 0$, defined by

$$(P_{B,\alpha}g)(y) = \left(\frac{1}{|B|^{1-\alpha/n}}\int_B |g(x)|\,dx\right)\chi_B(y).$$

The operator $P_{B,\alpha}$ is uniformly dominated by the fractional maximal operator M_α.

In addition, there exists a constant $C > 0$ such that for any nonnegative locally integrable function f, we have

$$M_\alpha f \le C I_\alpha f.$$

Consequently, there exists a constant $C > 0$ such that for any $B = B(x_0,r)$, $P_{B,\alpha}(g) \le I_\alpha(g)$. Hence, $\sup_B \|P_{B,\alpha}\|_{L_p(v)\to L_q(u)} < C\|I_\alpha\|_{L_p(v)\to L_q(u)}$.

As $(v,u) \in \mathbb{F}_{p,q,\alpha}$, $v^{-p'/p}$ is locally integrable. Therefore, $\|\chi_B\|_{L_{p'}(v^{-p'/p})}$ is well defined and we are allowed to use (2.2). The uniform boundedness of $P_{B,\alpha}$ and of (2.2) guarantee that

$$\begin{aligned}
&\|\chi_B\|_{L_{p'}(v^{-p'/p})}\|\chi_B\|_{L_q(u)} \\
&\quad = \sup\left\{\left|\int_B g(x)\,dx\right|\|\chi_B\|_{L_q(u)} : g \in L_p(v),\ \|g\|_{L_p(v)} \le 1\right\} \\
&\quad = \sup\left\{\frac{1}{|B|^{1-\alpha/n}}\left|\int_B g(x)\,dx\right|\|\chi_B\|_{L_q(u)} : g \in L_p(v),\ \|g\|_{L_p(v)} \le 1\right\}|B|^{1-\alpha/n} \\
&\quad \le C\sup\left\{\|P_{B,\alpha}(g)\|_{L_q(u)} : g \in L_p(v),\ \|g\|_{L_p(v)} \le 1\right\}|B|^{1-\alpha/n} \\
&\quad \le C\sup\left\{\|I_\alpha(g)\|_{L_q(u)} : g \in L_p(v),\ \|g\|_{L_p(v)} \le 1\right\}|B|^{1-\alpha/n} \\
&\quad \le C|B|^{1-\alpha/n},
\end{aligned}$$

for some $C > 0$. □

We are now ready to establish the main result of this section.

Theorem 4.3. *Let $1 < p, q < \infty$, $0 < \alpha < n$ and $(v, u) \in \mathbb{F}_{p,q,\alpha}$. If $\omega \in \mathcal{W}_{q,u}$, then there exists a constant $C > 0$ such that for any $f \in \mathcal{M}^p_\omega(v)$*

$$\|I_\alpha f\|_{\mathcal{M}^q_\omega(u)} \le C\|f\|_{\mathcal{M}^p_\omega(v)}.$$

Proof. For any $f \in \mathcal{M}^p_\omega(v)$, $z \in \mathbb{R}^n$ and $r > 0$, write $f(x) = f_0(x) + \sum_{j=1}^\infty f_j(x)$, where $f_0 = \chi_{B(z,2r)} f$ and $f_j = \chi_{B(z,2^{j+1}r)\setminus B(z,2^j r)} f$, $j \in \mathbb{N}\setminus\{0\}$.

As $I_\alpha : L_p(v) \to L_q(u)$ is bounded, we obtain $\|I_\alpha f_0\|_{L_q(u)} \le C\|f_0\|_{L_p(v)}$. In view of (3.1), we find that

$$\begin{aligned}\frac{1}{\omega(B(z,r))}\|\chi_{B(z,r)}(I_\alpha f_0)\|_{L_q(u)} &\le C\frac{1}{\omega(B(z,2r))}\|\chi_{B(z,2r)} f\|_{L_p(v)} \\ &\le C \sup_{\substack{y\in\mathbb{R}^n \\ r>0}} \frac{1}{\omega(B(y,r))}\|\chi_{B(y,r)} f\|_{L_p(v)}.\end{aligned} \tag{4.5}$$

The definition of the fractional integral operator yields a constant $C > 0$ such that, for any $j \ge 1$,

$$\chi_{B(z,r)}(x)|(I_\alpha f_j)(x)| \le C2^{-j(n-\alpha)} r^{-n+\alpha} \chi_{B(z,r)}(x) \int_{B(z,2^{j+1}r)} |f(y)|\,dy. \tag{4.6}$$

By using the Hölder inequality, we have

$$\int_{B(z,2^{j+1}r)} |f(y)|\,dy \le C\big\|\chi_{B(z,2^{j+1}r)} f\big\|_{L_p(v)} \|\chi_{B(z,2^{j+1}r)}\|_{L_{p'}(v^{-p'/p})}$$

for some $C > 0$.

By applying the norm $\|\cdot\|_{L_q(u)}$ on both sides of (4.6), we find that

$$\begin{aligned}\|\chi_{B(z,r)}(I_\alpha f_j)\|_{L_q(u)} \le\ & C2^{-j(n-\alpha)} r^{-n+\alpha} \|\chi_{B(z,r)}\|_{L_q(u)} \big\|\chi_{B(z,2^{j+1}r)} f\big\|_{L_p(v)} \\ & \times \|\chi_{B(z,2^{j+1}r)}\|_{L_{p'}(v^{-p'/p})}.\end{aligned} \tag{4.7}$$

According to (4.4), for any $z \in \mathbb{R}^n$, $j \in \mathbb{N}$ and $r > 0$, we have

$$\|\chi_{B(z,2^{j+1}r)}\|_{L_{p'}(v^{-p'/p})} \le C\frac{2^{(j+1)(n-\alpha)} r^{n-\alpha}}{\|\chi_{B(z,2^{j+1}r)}\|_{L_q(u)}}.$$

Therefore, the above inequality and (4.7) give

$$\begin{aligned}&\|\chi_{B(z,r)}(I_\alpha f_j)\|_{L_q(u)} \\ &\quad\le C2^{-j(n-\alpha)} r^{-n+\alpha} \frac{\|\chi_{B(z,r)}\|_{L_q(u)} \big\|\chi_{B(z,2^{j+1}r)} f\big\|_{L_p(v)} 2^{(j+1)(n-\alpha)} r^{n-\alpha}}{\|\chi_{B(z,2^{j+1}r)}\|_{L_q(u)}} \\ &\quad\le C\frac{\|\chi_{B(z,r)}\|_{L_q(u)}}{\|\chi_{B(z,2^{j+1}r)}\|_{L_q(u)}} \big\|\chi_{B(z,2^{j+1}r)} f\big\|_{L_p(v)}.\end{aligned}$$

Thus,

$$\begin{aligned}(4.8)\quad &\|\chi_{B(z,r)}(I_\alpha f_j)\|_{L_q(u)}\\ &\le C\frac{\|\chi_{B(x,r)}\|_{L_q(u)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L_q(u)}}\frac{\omega(B(z,2^{j+1}r))}{\omega(B(z,2^{j+1}r))}\|\chi_{B(z,2^{j+1}r)}f\|_{L_p(v)}\\ &\le C\frac{\|\chi_{B(x,r)}\|_{L_q(u)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L_q(u)}}\omega(B(z,2^{j+1}r))\sup_{\substack{y\in\mathbb{R}^n\\ R>0}}\frac{1}{\omega(B(y,R))}\|\chi_{B(y,R)}f\|_{L_p(v)}.\end{aligned}$$

In view of $\omega\in\mathcal{W}_{q,u}$, (4.5) and (4.8) we assert that

$$\begin{aligned}\frac{1}{\omega(B(z,r))}\|\chi_{B(z,r)}(I_\alpha f)\|_{L_q(u)} &\le \frac{1}{\omega(B(z,r))}\sum_{j=0}^{\infty}\|\chi_{B(z,r)}(I_\alpha f_j)\|_{L_q(u)}\\ &\le C\sup_{\substack{y\in\mathbb{R}^n\\ R>0}}\frac{1}{\omega(B(y,R))}\|\chi_{B(y,R)}f\|_{L_p(v)},\end{aligned}$$

where the constant $C>0$ is independent of r and z. Taking the supremum over $z\in\mathbb{R}^n$ and $r>0$ ensures the boundedness of T from $\mathcal{M}^q_\omega(u)$ to $\mathcal{M}^p_\omega(v)$. ☐

It is well known that $I_\alpha : L^p\to L^q$ is bounded whenever $\frac{1}{p}=\frac{1}{q}+\frac{\alpha}{n}$. Therefore, by applying Theorem 4.3 with $u\equiv 1$, $v\equiv 1$ and $\omega(B)=|B|^{(1/p)-(1/r)}$, $r\ge p$, we recapture Spanne's result on the mapping properties of fractional integral operators on Morrey spaces [31].

Furthermore, for any $0<\alpha<n$, $1<p<\frac{n}{\alpha}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$, $\omega\in\mathcal{W}_{q,v^q}$ and v satisfying (4.3), we have

$$\sup_{B\in\mathbb{B}}\frac{1}{\omega(B)}\left(\int_B|I_\alpha f(x)v(x)|^q\,dx\right)^{1/q}\le C\sup_{B\in\mathbb{B}}\frac{1}{\omega(B)}\left(\int_B|f(x)v(x)|^p\,dx\right)^{1/p}$$

for some $C>0$. This is a generalization of the result in [26] to Morrey spaces.

The reader is referred to [19] for the mapping properties of fractional integral operators on Morrey spaces built on Banach function spaces. Similarly to the discussion at the end of Section 3, as the weighted Lebesgue space $L_p(u)$ is not necessarily a Banach function space, the results in [19] do not apply to $\mathcal{M}^p_\omega(u)$.

In the next section, we use the results from Theorem 4.3 to establish the two-weight Poincaré and Sobolev inequalities on Morrey spaces.

§5. Poincaré and Sobolev inequalities

In this section, we give some applications of Theorem 4.3 on Poincaré and Sobolev inequalities on Morrey spaces. At the end of this section, as an example of the main results of this section, we present the two-weight Sobolev inequality on Morrey spaces with a Newtonian potential.

The Poincaré and Sobolev inequalities are two fundamental inequalities for partial differential equations.

The Poincaré and Sobolev inequalities are consequences of the mapping properties of fractional integral operators. Therefore, whenever we have the mapping properties of I_α from $\mathcal{M}^p_\omega(v)$ to $\mathcal{M}^p_\omega(u)$, we can obtain the two-weight Poincaré and Sobolev inequalities on Morrey spaces.

We first present the Poincaré inequality when $n > 1$. Since for any $D \in \mathbb{B}$, whenever $\int_D f(x)\,dx = 0$ or $\operatorname{supp} f \subset D$, we have

$$|f(x)| \le CI_1(\chi_D|\nabla f|) = C\int_D \frac{|\nabla f(y)|}{|x-y|^{n-1}}\,dy, \quad \forall x \in D; \tag{5.1}$$

see [9, (4.34) and (4.35)].

Thus, Theorem 4.3 yields the weighted Poincaré inequality on Morrey spaces.

Theorem 5.1 (Poincaré inequality). *Let $n > 1$, $1 < p, q < \infty$, $(v,u) \in \mathbb{F}_{p,q,1}$ and $\omega \in \mathcal{W}_{q,u}$. For any $D \in \mathbb{B}$ and for any once continuously differentiable function f, if either $\int_D f(x)\,dx = 0$ or $\operatorname{supp} f \subset D$, we have*

$$\sup_{B\in\mathbb{B}} \frac{1}{\omega(B)}\Big(\int_{D\cap B} |f(x)|^q u(x)\,dx\Big)^{1/q} \le C \sup_{B\in\mathbb{B}} \frac{1}{\omega(B)}\Big(\int_B |\nabla f(x)|^p v(x)\,dx\Big)^{1/p}$$

for some $C > 0$.

The previous result gives the two-weight Poincaré inequality on Morrey spaces with the assumption that $(v,u) \in \mathbb{F}_{p,q,1}$. We also have another version of the two-weight Poincaré inequality on Morrey spaces where $(v,u) \in \mathbb{M}_p$.

Theorem 5.2 (Poincaré inequality). *Let $n > 1$, $1 < p < \infty$, $(v,u) \in \mathbb{M}_p$ and $\omega \in \mathcal{W}_{p,u}$. For any once continuously differentiable function f, if either $\int_{B(x_0,r)} f(x)\,dx = 0$ or $\operatorname{supp} f \subset B(x_0,r)$, we have*

$$\sup_{B\in\mathbb{B}} \frac{1}{\omega(B)}\Big(\int_{B(x_0,r)\cap B} |f(x)|^p u(x)\,dx\Big)^{1/p} \le Cr \sup_{B\in\mathbb{B}} \frac{1}{\omega(B)}\Big(\int_B |\nabla f(x)|^p v(x)\,dx\Big)^{1/p}, \tag{5.2}$$

for some $C > 0$.

Proof. According to the Hedberg inequality [15] and [9, Lemma 4.3.19], for any $x \in B(x_0,r)$,

$$\int_{B(x_0,r)} \frac{|f(y)|}{|x-y|^{n-1}}\,dy \le CrM(f)(x).$$

Therefore, (5.1) gives

$$|f(x)| \leq C \int_{B(x_0,r)} \frac{|\nabla f(y)|}{|x-y|^{n-1}}\, dy \leq CrM(\nabla f)(x), \quad \forall x \in B(x_0,r).$$

As $(v,u) \in \mathbb{M}_p$ and $\omega \in \mathcal{W}_{p,u}$, (3.9) yields (5.2). □

Let $1 < p < \infty$, u be a weight and $\omega \in \mathcal{W}_{p,u}$. Theorem 4.3 ensures that for any once continuously differentiable function f, if either $\int_{B(x_0,r)} f(x)\, dx = 0$ or $\operatorname{supp} f \subset B(x_0,r)$, we have

$$\begin{aligned}\sup_{B\in\mathbb{B}} \frac{1}{\omega(B)} \Big(\int_{B(x_0,r)\cap B} |f(x)|^p u(x)\, dx \Big)^{1/p} \\ \leq Cr \sup_{B\in\mathbb{B}} \frac{1}{\omega(B)} \Big(\int_B |\nabla f(x)|^p Mu(x)\, dx \Big)^{1/p}\end{aligned}$$

because $(Mu,u) \in \mathbb{M}_p$.

The above theorem can be further generalized by using the results from [46]. For brevity, we leave the details to the reader.

For the case where $\operatorname{supp} f \subset D$, the Poincaré inequality is also called the Friedrich inequality. Therefore, the above results are generalizations of the Friedrich inequality on Morrey spaces.

The two-weight Poincaré inequality is related to Harnack inequalities and the regularity of the degenerate second-order elliptic differential operator $\nabla A \nabla$ when $A(x)$ is a nonnegative matrix with least and greatest eigenvalues $v(x)$ and $u(x)$, respectively; see [4, 12].

For studies of the Poincaré inequalities on Banach function spaces, the reader is referred to [9, Chapter 4].

Next, we present the Sobolev inequality when $n > 2$. Let Δ denote the Laplacian. Since $f = I_2(\Delta f)$, Theorem 4.3 gives the weighted Sobolev inequality on Morrey spaces.

Theorem 5.3 (Sobolev inequality). *Let $n > 2$, $1 < p, q < \infty$, $(v,u) \in \mathbb{F}_{p,q,2}$ and $\omega \in \mathcal{W}_{q,u}$. For any $D \in \mathbb{B}$ and for any twice continuously differentiable function f with* $\operatorname{supp} f \subseteq D$, *we have*

$$\sup_{B\in\mathbb{B}} \frac{1}{\omega(B)} \Big(\int_{B\cap D} |f(x)|^q u(x)\, dx \Big)^{1/q} \leq C \sup_{B\in\mathbb{B}} \frac{1}{\omega(B)} \Big(\int_B |\Delta f(x)|^p v(x)\, dx \Big)^{1/p}$$

for some $C > 0$.

The two-weight Sobolev inequalities are related to the absence of positive eigenvalues for Schrödinger operator $-\Delta + v$ and the unique continuation for the differential inequality $|\Delta f| \leq u|f|$; see [22, 3], respectively.

We now present an example for the two-weight Sobolev inequality on Morrey spaces.

Corollary 5.4. *Let $n > 2$, $v \in A_\infty$ and $\omega \in \mathcal{W}_{2,v}$. Suppose that there exists a constant $C > 0$ such that for any $B \in \mathbb{B}$,*

$$\frac{1}{|B|^{1-1/n}} \int_B v(y)\,dy < C. \tag{5.3}$$

Then, for any twice continuously differentiable function f with $\operatorname{supp} f \subseteq D$, we have

$$\sup_{B\in\mathbb{B}} \frac{1}{\omega(B)} \Big(\int_{B\cap D} |f(x)|^2 v(x)\,dx \Big)^{1/2} \le C \sup_{B\in\mathbb{B}} \frac{1}{\omega(B)} \Big(\int_B |\Delta f(x)|^2 \frac{1}{v(x)}\,dx \Big)^{1/2} \tag{5.4}$$

for some $C > 0$.

Proof. Since

$$\frac{1}{|B|^{1-2/n}} \int_B v(y)\,dy \left(\frac{1}{|B|} \int_B \left(\frac{1}{v(y)} \right)^{-1} (y)\,dy \right) = \left(\frac{1}{|B|^{2-2/n}} \int_B v(y)\,dy \right)^2,$$

(5.3) ensures that the pair $(\frac{1}{v}, v)$ fulfills (4.2). That is, $(\frac{1}{v}, v) \in \mathbb{F}_{2,2,2}$. Therefore, Theorem 5.3 yields (5.4). □

Condition (5.3) is equivalent to the condition $v \in \mathcal{M}^1_{\omega_0}$, where $\omega_0(B) = |B|^{1-1/n}$. In addition, compared to [36, 37], our result requires only that f is locally in $L^2(v)$ and that v is locally integrable.

In particular, it is well known that the Newtonian potential in $\mathbb{R}^3$, $v_0(y) = |y|^{-1}$ satisfies (5.3). In fact, the monotonicity of Morrey spaces in the local parameter and [11, Lemma 2.4] guarantee that $v_0 \in \mathcal{M}^1_{\omega_0}$.

Since $v_0 \in A_\infty$, we obtain the following weighted Sobolev inequality with Newtonian potential on Morrey spaces.

Let $\omega \in \mathcal{W}_{2,-1}$. For any twice continuously differentiable function f with $\operatorname{supp} f \subseteq D$, we have

$$\sup_{B\in\mathbb{B}} \frac{1}{\omega(B)} \Big(\int_{B\cap D} |f(x)|^2 \frac{1}{|x|}\,dx \Big)^{1/2} \le C \sup_{B\in\mathbb{B}} \frac{1}{\omega(B)} \Big(\int_B |\Delta f(x)|^2 |x|\,dx \Big)^{1/2}$$

for some $C > 0$.

In the next section, we extend the above inequality to some different power-weight functions by using the Stein–Weiss inequality on Morrey spaces.

§6. Stein–Weiss inequalities

In this section, we extend Stein–Weiss inequalities for fractional integral operators to Morrey spaces. As special cases of Stein–Weiss inequalities, we obtain Hardy inequalities and Rellich inequalities on Morrey spaces.

The celebrated Stein–Weiss inequalities [45, Theorem B*] give the mapping properties of fractional integral operators on power-weighted Lebesgue spaces.

Theorem 6.1. *Let $a, b \in \mathbb{R}$ and $1 < p \le q < \infty$. If*

$$a < \frac{np}{p'}, \quad b < n, \quad 0 \le \frac{a}{p} + \frac{b}{q}, \tag{6.1}$$

$$\alpha = \frac{a}{p} + \frac{b}{q} + n\left(\frac{1}{p} - \frac{1}{q}\right) > 0, \tag{6.2}$$

then

$$\left(\int_{\mathbb{R}^n} |I_\alpha f(x)|^q |x|^{-b}\, dx\right)^{1/q} \le C\left(\int_{\mathbb{R}^n} |f(x)|^p |x|^a\, dx\right)^{1/p}, \tag{6.3}$$

for some $C > 0$.

We see that condition (6.2) follows from the scaling condition of the mapping properties of fractional integral operators. More precisely, for any $t > 0$, write $D_t f(x) = f(tx)$. We find that

$$I_\alpha(D_t f)(x) = t^{-\alpha} I_\alpha(tx).$$

Additionally,

$$\left(\int_{\mathbb{R}^n} |I_\alpha(D_t f)(x)|^q |x|^{-b}\, dx\right)^{1/q} = t^{-\alpha+(b-n)/q}\left(\int_{\mathbb{R}^n} |I_\alpha f(x)|^q |x|^{-b}\, dx\right)^{1/q}$$

and

$$\left(\int_{\mathbb{R}^n} |(D_t f)_(x)|^p |x|^a\, dx\right)^{1/p} = t^{-(a+n)/p}\left(\int_{\mathbb{R}^n} |f(x)|^p |x|^a\, dx\right)^{1/p}.$$

Therefore, the validity of (6.3) requires that

$$t^{-\alpha+(b-n)/q} = t^{-(a+n)/p}, \quad \forall t > 0.$$

Consequently, (6.2) is necessary for the validity of (6.3).

According to Theorem 4.3, we obtain Stein–Weiss inequalities on Morrey spaces.

Theorem 6.2 (Stein–Weiss inequality for Morrey spaces). *Let $a, b \in \mathbb{R}$ and $1 < p \leq q < \infty$. If $\omega \in \mathcal{W}_{q,-b}$ and a, b, α satisfy (6.1) and (6.2), then there exists a constant $C > 0$ such that*

$$\sup_{B\in\mathbb{B}} \frac{1}{\omega(B)}\left(\int_B |I_\alpha f(x)|^q |x|^{-b}\,dx\right)^{1/q} \leq C \sup_{B\in\mathbb{B}} \frac{1}{\omega(B)}\left(\int_B |f(x)|^p |x|^a\,dx\right)^{1/p}. \tag{6.4}$$

Proof. Since $u(x) = |x|^\theta$ is locally integrable if and only if $\theta > -n$, according to (6.1), we find that $|x|^{a(-p'/p)}$ and $|x|^{-b}$ are locally integrable. Moreover, since a, b, α satisfy (6.1) and (6.2), we find that

$$\alpha = \frac{a}{p} + \frac{b}{q} + n\left(\frac{1}{p} - \frac{1}{q}\right) < \frac{n}{p'} + \frac{n}{q} + \frac{n}{p} - \frac{n}{q} = n.$$

Therefore, Theorem 6.1 ensures that $(|x|^a, |x|^{-b}) \in \mathbb{F}_{p,q,\alpha}$. We are allowed to apply Theorem 4.3 and obtain (6.4). □

In view of (5.1), we obtain the power-weighted Poincaré inequality on Morrey spaces.

Corollary 6.3. *Let $n > 1$, $a, b \in \mathbb{R}$ and $1 < p \leq q < \infty$. Suppose that $\omega \in \mathcal{W}_{q,-b}$ and a, b, α satisfy (6.1) and*

$$1 = \frac{a}{p} + \frac{b}{q} + n\left(\frac{1}{p} - \frac{1}{q}\right). \tag{6.5}$$

For any $D \in \mathbb{B}$ and for any once continuously differentiable function f, if either $\int_D f(x)\,dx = 0$ or $\operatorname{supp} f \subset D$, we have

$$\sup_{B\in\mathbb{B}} \frac{1}{\omega(B)}\left(\int_{D\cap B} |f(x)|^q |x|^{-b}\,dx\right)^{1/q} \leq C \sup_{B\in\mathbb{B}} \frac{1}{\omega(B)}\left(\int_B |\nabla f(x)|^p |x|^a\,dx\right)^{1/p},$$

for some $C > 0$.

In particular, when $a = 0$, $b = 2$ and $p = q = 2$, we have the generalization of the Hardy inequality [43, (6.2.2)] to Morrey spaces.

Corollary 6.4. *Let $n > 2$ and $\omega \in \mathcal{W}_{2,-2}$. For any $D \in \mathbb{B}$ and for any once continuously differentiable function f, if either $\int_D f\,dx = 0$ or $\operatorname{supp} f \subset D$, we have*

$$\sup_{B\in\mathbb{B}} \frac{1}{\omega(B)}\left(\int_{D\cap B} |f(x)|^2 |x|^{-2}\,dx\right)^{1/2} \leq C \sup_{B\in\mathbb{B}} \frac{1}{\omega(B)}\left(\int_B |\nabla f(x)|^2\,dx\right)^{1/2},$$

for some $C > 0$.

Similarly, the identity $f = I_2(\Delta f)$ and Theorem 6.2 yield the power-weighted Sobolev inequalities on Morrey spaces.

Corollary 6.5. *Let $n > 2$, $a, b \in \mathbb{R}$ and $1 < p \leq q < \infty$. Suppose that $\omega \in \mathcal{W}_{q,-b}$ and a, b, α satisfy (6.1) and*

$$2 = \frac{a}{p} + \frac{b}{q} + n\left(\frac{1}{p} - \frac{1}{q}\right). \tag{6.6}$$

For any $D \in \mathbb{B}$ and for any twice continuously differentiable function f with $\operatorname{supp} f \subseteq D$, *we have*

$$\sup_{B\in\mathbb{B}} \frac{1}{\omega(B)}\Big(\int_{B\cap D} |f(x)|^q |x|^{-b}\,dx\Big)^{1/q} \leq C \sup_{B\in\mathbb{B}} \frac{1}{\omega(B)}\Big(\int_B |\Delta f(x)|^p |x|^a\,dx\Big)^{1/p},$$

for some $C > 0$.

In particular, when $a = 0$, $b = 4$ and $p = q = 2$, the above corollary gives the Rellich inequality [43, (6.2.8)] on Morrey spaces.

Corollary 6.6. *Let $n \geq 5$ and $\omega \in \mathcal{W}_{2,-4}$. For any $D \in \mathbb{B}$ and for any twice continuously differentiable function f with* $\operatorname{supp} f \subseteq D$, *we have*

$$\sup_{B\in\mathbb{B}} \frac{1}{\omega(B)}\Big(\int_{D\cap B} |f(x)|^2 |x|^{-4}\,dx\Big)^{1/2} \leq C \sup_{B\in\mathbb{B}} \frac{1}{\omega(B)}\Big(\int_B |\Delta f(x)|^2\,dx\Big)^{1/2}$$

for some $C > 0$.

Notice that (6.5) and (6.6) are inherited from (6.3). Hence they follow from the scaling condition of the mapping properties of fractional integral operators.

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