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On Calderón's Problem for a System of Elliptic Equations

by

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Abstract

We consider Calderón's problem in the case of a partial Dirichlet-to-Neumann map for systems of elliptic equations in a bounded two-dimensional domain. The main result of the paper is as follows: If two systems of elliptic equations generate the same partial Dirichlet-to-Neumann map on some subboundary, then the coefficients can be uniquely determined up to gauge equivalence.

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§1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary, let N be an arbitrarily chosen natural number, let $\widetilde{\Gamma}$ be a relatively open set on $\partial\Omega$ and $\Gamma_0 = \mathrm{Int}(\partial\Omega \setminus \widetilde{\Gamma})$. Consider the following boundary value problem:

$$\begin{aligned} &L(x,D)u = \Delta u + 2A\partial_z u + 2B\partial_{\bar z} u + Qu = 0 \quad \text{in } \Omega, \\ &u|_{\Gamma_0} = 0, \qquad u|_{\widetilde{\Gamma}} = f. \end{aligned} \tag{1.1}$$

Here $u = (u_1, \dots, u_N)$ is an unknown vector-valued function and A, B, Q are $N \times N$ matrices. Consider the following partial Dirichlet-to-Neumann map limited on $\widetilde{\Gamma}$:

$$\begin{aligned} &\Lambda_{A,B,Q} f = \partial_{\vec\nu} u|_{\widetilde{\Gamma}}, \quad \text{where } L(x,D)u = 0 \text{ in } \Omega, \\ &u|_{\Gamma_0} = 0, \qquad u|_{\widetilde{\Gamma}} = f, \end{aligned}$$

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where $\vec{\nu}$ is the outward unit normal to $\partial\Omega$. The inverse problem of determining A, B, Q is a generalization of the so-called Calderón problem (see [2]), which itself is the mathematical realization of *Electrical Impedance Tomography* (EIT).

The uniqueness of the Dirichlet-to-Neumann map on an arbitrarily chosen subboundary for single conductivity equations and the Schrödinger equations was first proved by [6]; see [9] as a related survey paper.

The goal of this paper is to extend the result obtained in [3], which considers elliptic equations in a convex domain in $\mathbb{R}^d$ with $d \geq 3$ and proves that the coefficients of two systems of elliptic equations producing the same Dirichlet-to-Neumann map can be determined up to gauge equivalence. However [3] discusses only the case of $\widetilde{\Gamma} = \partial\Omega$, which means the Dirichlet-to-Neumann map on the whole boundary.

In this paper, for a Dirichlet-to-Neumann map limited to an arbitrarily small subboundary in two dimensions, we prove a necessary and sufficient condition for operators producing the same Dirichlet-to-Neumann map. Our main result is stated as follows.

Theorem 1.1. *Let A_j, $B_j \in C^{5+\alpha}(\bar{\Omega})$, $Q_j \in C^{4+\alpha}(\bar{\Omega})$ with $j = 1, 2$ and some $\alpha \in (0,1)$, and the operators $L_j(x,D)$ are of the form (1.1) with coefficients A_j, B_j, Q_j and the adjoint operators $L_j^*(x,D)$, $j = 1, 2$ to these operators do not have a zero eigenvalue.*

Then $\Lambda_{A_1,B_1,Q_1} = \Lambda_{A_2,B_2,Q_2}$ if and only if

$$A_1 = A_2 \quad \text{and} \quad B_1 = B_2 \quad \text{on } \widetilde{\Gamma}, \tag{1.2}$$

and there exists an invertible matrix $\mathbf{Q} \in C^{6+\alpha}(\bar{\Omega})$ such that

$$\mathbf{Q}|_{\widetilde{\Gamma}} = I, \qquad \partial_{\vec{\nu}}\mathbf{Q}|_{\widetilde{\Gamma}} = 0, \tag{1.3}$$

$$A_2 = 2\mathbf{Q}^{-1}\partial_{\bar{z}}\mathbf{Q} + \mathbf{Q}^{-1}A_1\mathbf{Q} \quad \text{in } \Omega, \tag{1.4}$$

$$B_2 = 2\mathbf{Q}^{-1}\partial_{z}\mathbf{Q} + \mathbf{Q}^{-1}B_1\mathbf{Q} \quad \text{in } \Omega \tag{1.5}$$

and

$$Q_2 = \mathbf{Q}^{-1}Q_1\mathbf{Q} + \mathbf{Q}^{-1}\Delta\mathbf{Q} + 2\mathbf{Q}^{-1}A_1\partial_z\mathbf{Q} + 2\mathbf{Q}^{-1}B_1\partial_{\bar{z}}\mathbf{Q} \quad \text{in } \Omega. \tag{1.6}$$

For a related result, see [4].

The paper is organized as follows. In Section 3 we construct the complex geometric optics solutions for the boundary value problem (1.1). In Section 4 we prove some asymptotics for integrals involving the complex geometric optics solutions for the operators $L_1(x,D)$ and $L_2(x,D)^*$. In Section 5, from the asymptotics relations obtained in Section 4, it is proved that there exists a gauge transformation $\mathbf{Q}$ that preserves the Dirichlet-to-Neumann map and transforms the coefficient $A_1 \to A_2$. After that, for the operators $\mathbf{Q}^{-1}L_1(x,D)\mathbf{Q}$ and $L_2(x,D)$, we obtain some system of integral-differential equations, and we study this system of integral-differential equations and show that the operators $\mathbf{Q}^{-1}L_1(x,D)\mathbf{Q}$ and $L_2(x,D)$ are the same.

Notation. Let $i = \sqrt{-1}$ and $\overline{z}$ be the complex conjugate of $z \in \mathbb{C}$. We set $\partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$, $\partial_{\overline{z}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$ and

$$\partial_{\overline{z}}^{-1} g = -\frac{1}{\pi}\int_\Omega \frac{g(\xi_1,\xi_2)}{\zeta - z}\, d\xi_1\, d\xi_2, \qquad \partial_{z}^{-1} g = -\frac{1}{\pi}\int_\Omega \frac{g(\xi_1,\xi_2)}{\overline{\zeta} - \overline{z}}\, d\xi_1\, d\xi_2$$

(see, e.g., [11]).

Let $\vec{e}_j$, $j = 1,\dots,N$ be the standard basis in $\mathbb{R}^N$. For a holomorphic function Φ, we set $\Phi' = \partial_z\Phi$ and $\bar{\Phi}' = \partial_{\overline{z}}\bar{\Phi}$, $\Phi'' = \partial_z^2\Phi$, $\bar{\Phi}'' = \partial_{\overline{z}}^2\bar{\Phi}$. Let $\vec{\tau} = (\nu_2, -\nu_1)$ be the tangential vector to $\partial\Omega$, and let us set $\partial_{\vec{\nu}} = \partial_{x_1}\nu_1 + \partial_{x_2}\nu_2$ and $\partial_{\vec{\tau}} = \partial_{x_1}\nu_2 - \partial_{x_2}\nu_1$. Let $W_2^{1,\tau}(\Omega)$ be the Sobolev space $W_2^1(\Omega)$ with the norm $\|u\|_{W_2^{1,\tau}(\Omega)} = \|\nabla u\|_{L^2(\Omega)} + |\tau|\|u\|_{L^2(\Omega)}$. Moreover, for a normed space X with norm $\|\cdot\|_X$, by $\lim_{\eta\to\infty}\frac{\|f(\eta)\|_X}{\eta} = 0$ and $\|f(\eta)\|_X \le C\eta$ as $\eta \to \infty$ with some $C > 0$, we define $f(\eta) = o_X(\eta)$ and $f(\eta) = O_X(\eta)$ as $\eta \to \infty$. Let $\beta = (\beta_1, \beta_2)$, $\beta_j \in \mathbb{N}_+ := \mathbb{N}\setminus\{0\}$, $|\beta| = \beta_1 + \beta_2$ and I be the identity matrix. By A^* we denote the adjoint matrix to a matrix A in the space $\mathbb{R}^N$. By $(\cdot,\cdot)_{L^2(\Omega)}$ we denote the $L^2(\Omega)$-scalar product over $\mathbb{R}$, while $(\cdot,\cdot)$ is the scalar product in $\mathbb{R}^2$ if there is no fear of confusion.

§2. Construction of the operators P_B and T_B

Let A, B be $N\times N$ matrices with the elements from $C^{5+\alpha}(\overline{\Omega})$ with $\alpha \in (0,1)$. Consider the boundary value problem:

$$\begin{aligned} \mathcal{K}(x,D)(U_0,T_0) = (2\partial_{\overline{z}}U_0 + AU_0, 2\partial_z T_0 + BT_0) = 0 \quad &\text{in } \Omega,\\ U_0 + T_0 = 0 \quad &\text{on } \Gamma_0. \end{aligned} \tag{2.1}$$

Without loss of generality we assume that $\widetilde{\Gamma}$ is an arc with the endpoints $x_\pm$.

We have

Proposition 2.1. *Let A, $B \in C^{5+\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$, $\Psi \in C^\infty(\partial\Omega)$, $\vec{r_{0,k}},\dots,\vec{r_{5,k}} \in \mathbb{C}^N$ be arbitrarily given and $x_1,\dots,x_{\hat{k}}$ be mutually distinct arbitrary points from the domain Ω. For any positive ϵ there exists a solution $(U_0,T_0) \in C^{6+\alpha}(\overline{\Omega})$ to problem (2.1) such that*

$$\partial_z^j U_0(x_\ell) = \vec{r}_{j,\ell}, \quad \forall\, j \in \{0,\dots,5\} \quad \text{and} \quad \forall\, \ell \in \{1,\dots,\hat{k}\}, \tag{2.2}$$

and

$$\|U_0 - \Psi\|_{C^{5+\alpha}(\bar{\Gamma}_0)} \le \epsilon. \tag{2.3}$$

Before the proof of Proposition 2.1, we prove slightly stronger versions of [7, Proposition 6.1 and Corollary 6.1].

Proposition 2.2. *Let $x_1, \dots, x_{\hat{k}}$ be mutually distinct arbitrary points from the domain Ω and $r_{0,k}, \dots, r_{5,k} \in \mathbb{C}$ be arbitrarily given and $\Psi \in C^\infty(\partial\Omega)$ be a real-valued function. For any positive ϵ, there exists a holomorphic function $a(z)$ depending on ϵ such that*

$$\partial_z^j a(x_\ell) = r_{j,\ell}, \quad \forall\, j \in \{0, \dots, 5\} \quad \textit{and} \quad \forall\, \ell \in \{1, \dots, \hat{k}\}, \tag{2.4}$$

and

$$\|a - \Psi\|_{C^{5+\alpha}(\bar{\Gamma}_0)} \le \epsilon, \quad \operatorname{Im} a|_{\Gamma_0} = 0, \tag{2.5}$$

where $\alpha \in (0, 1)$.

Proof. Since by [7, Corollary 6.1], for each positive ϵ_1 there exists a holomorphic function b such that

$$\|b - \Psi\|_{C^{5+\alpha}(\bar{\Gamma}_0)} \le \epsilon_1, \qquad \operatorname{Im} b|_{\Gamma_0} = 0,$$

it suffices to prove the proposition for the case $\Psi = 0$.

We introduce the operator $\mathcal{R}(x_j) : C^5(\bar{\Omega}) \to \mathbb{C}^6$ by formula $\mathcal{R}(x_j)v = (v(x_j), \dots, \partial_z^5 v(x_j))$. Let us fix some $j \in \{1, \dots, \hat{k}\}$ and let $j_1 \ne j$, $j_1 \in \{1, \dots, \hat{k}\}$. By [7, Corollary 6.1], for each positive ϵ_1 and each $p \in \{1, \dots, 4\}$, there exists a holomorphic function $v_p(z)$ with the following properties:

$$\begin{aligned}
\operatorname{Im} v_p|_{\Gamma_0} = 0, \qquad & \|v_p\|_{C^{5+\alpha}(\bar{\Gamma}_0)} \le \epsilon_1, \\
|v_1(x_{j_1}) - 1| \le \epsilon_1, \qquad & |v_2(x_{j_1}) - \sqrt{-1}| \le \epsilon_1, \\
|v_3(x_{j_1}) - 1/2| \le \epsilon_1, \qquad & |v_4(x_{j_1}) - 2\sqrt{-1}| \le \epsilon_1, \\
|\mathcal{R}(x_j)v_1 - \vec{e}_1| + \sum_{p=2}^{4} & |\mathcal{R}(x_j)v_p| \le \epsilon_1,
\end{aligned}$$

where $\vec{e}_1 = (1, 0, \dots, 0) \in \mathbb{R}^6$. We set $\widetilde{v}_\ell(z) = v_{1+2\ell}(z) - \frac{\operatorname{Im} v_{1+2\ell}(x_{j_1})}{\operatorname{Im} v_{2+2\ell}(x_{j_1})} v_{2+2\ell}(z)$, $\ell \in \{0, 1\}$. Then $\widetilde{v}_\ell(x_j)$ is a real number and there exists a constant C independent of ϵ_1 such that

$$\begin{gathered}
|\widetilde{v}_0(x_{j_1}) - 1| + |\widetilde{v}_1(x_{j_1}) - \sqrt{-1}| \le C\epsilon_1, \\
\operatorname{Im} \widetilde{v}_\ell|_{\Gamma_0} = 0, \quad \|\widetilde{v}_\ell\|_{C^{5+\alpha}(\bar{\Gamma}_0)} \le C\epsilon_1, \quad \ell = 1, 2, \\
|\mathcal{R}(x_j)\widetilde{v}_1 - \vec{e}_1| + |\mathcal{R}(x_j)\widetilde{v}_1| \le C\epsilon_1,
\end{gathered}$$

for all sufficiently small ϵ_1. Taking $v_{j_1,2}(z) = \widetilde{v}_0(z) - \frac{\widetilde{v}_0(x_{j_1})}{\widetilde{v}_1(x_{j_1})} \widetilde{v}_1(z)$, we obtain that $v_{j_1,2}(x_{j_1}) = 0$, $\operatorname{Im} v_{j_1,2}|_{\Gamma_0} = 0$, $\|v_{j_1,2}\|_{C^{5+\alpha}(\bar{\Gamma}_0)} \le C\epsilon$, $|\mathcal{R}(x_j)v_{j_1,2} - \vec{e}_1| \le C\epsilon$. Let

v be a holomorphic function satisfying $\|v\|_{C^{5+\alpha}(\bar\Gamma_0)} \le \epsilon$ and $|\mathcal{R}(x_j)v - \vec r| \le \epsilon$, where $\vec r$ is an arbitrary fixed vector from $\mathbb{R}^6$. The holomorphic function $\hat v_j = v\Pi_{j_1=1, j_1\ne j}^{\hat k} v_{j_1,2}^6$ satisfies

$$\mathcal{R}(x_k)\hat v_j = 0, \quad \forall\, k \in \{1,\dots,j-1,j+1,\dots,\hat k\}; \quad |\mathcal{R}(x_j)\hat v_j - \vec r| \le \epsilon, \tag{2.6}$$

$$\operatorname{Im} \hat v_j|_{\Gamma_0} = 0, \qquad \|\hat v_j\|_{C^{5+\alpha}(\bar\Gamma_0)} \le \epsilon. \tag{2.7}$$

Using the functions $\hat v_j$ and the argument used in the construction of these functions, for any $j \in \{1,\dots,\hat k\}$ we construct a holomorphic function c_j such that $\operatorname{Im} c_j|_{\Gamma_0} = 0$ and

$$\mathcal{R}(x_k)c_j = 0, \quad \forall\, k \in \{1,\dots,j-1,j+1,\dots,\hat k\}; \qquad c_j(x_j) = 0,$$
$$\partial_z c_j(x_j) = 1, \qquad \|c_j\|_{C^{5+\alpha}(\bar\Gamma_0)} \le \epsilon.$$

Indeed let $\widetilde v_j$ satisfy (2.6), (2.7) where $\vec r$ and ϵ are replaced by $(\sqrt{-1},0,\dots,0)$ and $\epsilon/2$, and $\hat v_j$ satisfy (2.6), (2.7) where ϵ is replaced by $\epsilon/2$ and the first coordinate of $\vec r$ is real. Consider the function $\hat w_j = \hat v_j - \frac{\operatorname{Im} \hat v_j(x_j)}{\operatorname{Im} \widetilde v_j(x_j)}\widetilde v_j$. The function $\hat w_j$ satisfies (2.6), (2.7) and $\hat w_j(x_j)$ is a real number.

Next, observe that if the first coordinate of $\vec r$ is zero, then one can take $\hat w_j$ satisfying (2.6), (2.7) and $\hat w_j(x_j) = 0$. We take a function $\hat w_j = \hat v_j - \frac{\hat v_j(x_j)}{\widetilde v_j(x_j)}\widetilde v_j$, where $\widetilde v_j$ satisfies (2.6), (2.7) where $\vec r$ and ϵ are replaced by $(\sqrt{-1},0,\dots,0)$ and $\epsilon/2$, and $\hat v_j$ satisfies (2.6), (2.7) with $\epsilon/2$ instead of ϵ. Moreover, $\widetilde v_j(x_j)$, $\hat v_j(x_j)$ are real numbers.

Let $\widetilde v_j$ satisfy (2.6), (2.7) where $\vec r$ and ϵ are replaced by $(0,\sqrt{-1},0,\dots,0)$ and $\epsilon/2$, and $\hat v_j$ satisfy (2.6), (2.7) where $\vec r$ and ϵ are replaced by $(0,1,0,\dots,0)$ and $\epsilon/2$. Moreover, $\widetilde v_j(x_j) = \hat v_j(x_j) = 0$. Let $\hat w_j = \hat v_j - \frac{\operatorname{Im} \partial_z \hat v_j(x_j)}{\operatorname{Im} \partial_z \widetilde v_j(x_j)}\widetilde v_j$. Then, since $\partial_z \hat w_j(x_j) \ne 0$ for all sufficiently small ϵ, we set $c_j(z) = \hat w_j(z)/\partial_z \hat w_j(x_j)$. Since $\partial_z \hat w_j(x_j)$ is a real number, we have $\operatorname{Im} c_j|_{\Gamma_0} = 0$. The construction of functions c_j is complete.

We set $\widetilde a_{j,\ell}(z) = \frac{1}{\ell!}c_j^\ell(z)$ and $a_{j,5}(z) = \widetilde a_{j,5}(z) = \frac{1}{5!}c_j^5(z)$,

$$a_{j,\ell}(z) = \widetilde a_{j,\ell}(z) - \sum_{k=\ell+1}^{5} \partial_z^\ell \widetilde a_{j,\ell}(x_j)a_{j,\ell}(z), \quad \ell \in \{0,\dots,4\}.$$

Then $\operatorname{Im} a_{j,\ell}|_{\Gamma_0} = 0$ and

$$\partial_z^\ell a_{j,\ell}(x_j) = 1; \qquad \partial_z^m a_{j,\ell}(x_k) = 0, \quad \forall\,(m,k) \in \{0,\dots,5\} \times \{1,\dots,\hat k\} \setminus \{(\ell,j)\},$$
$$\|a_{j,\ell}\|_{C^{5+\alpha}(\bar\Gamma_0)} \le \epsilon_2.$$

For an arbitrary $\epsilon_2 > 0$, we similarly construct holomorphic functions $b_{j,\ell}(z)$ such that $\operatorname{Im} b_{j,\ell}|_{\Gamma_0} = 0$ and

$$\partial_z^\ell b_{j,\ell}(x_j) = \sqrt{-1}; \quad \partial_z^m b_{j,\ell}(x_k) = 0, \quad \forall\,(m,k) \in \{0,\dots,5\} \times \{1,\dots,\hat{k}\} \setminus \{(\ell,j)\};$$
$$\|b_{j,\ell}\|_{C^{5+\alpha}(\bar{\Gamma}_0)} \le \epsilon_2.$$

The holomorphic function

$$a(z) = \sum_{\ell=1}^{\hat{k}} \sum_{j=0}^{5} (\operatorname{Re} r_{j,\ell} a_{j,\ell}(z) + \operatorname{Im} r_{j,\ell} b_{j,\ell}(z))$$

satisfies (2.4) and (2.5) with $C\epsilon$. □

Corollary 2.3. *Let $x_1, \dots, x_{\hat{k}}$ be mutually distinct arbitrary points from the domain Ω, $\alpha \in (0,1)$ and $r_{0,k}, \dots, r_{5,k} \in \mathbb{C}$ be arbitrary and $\Psi \in C^\infty(\partial\Omega)$. For any positive ϵ, there exists a holomorphic function $a(z)$ in general depending on ϵ such that*

$$\partial_z^j a(x_\ell) = r_{j,\ell}, \quad \forall\, j \in \{0,\dots,5\} \;\; \text{and} \;\; \forall\, \ell \in \{1,\dots,\hat{k}\}, \tag{2.8}$$

and

$$\|a - \Psi\|_{C^{5+\alpha}(\bar{\Gamma}_0)} \le \epsilon. \tag{2.9}$$

In order to prove Proposition 2.1, we prove the following proposition.

Proposition 2.4. *Let ϵ be a positive number, $A \in C^{5+\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$, $\Psi \in C^\infty(\bar{\Gamma}_0)$, $\vec{r}_{0,k}, \dots, \vec{r}_{5,k} \in \mathbb{C}^N$ be arbitrary vectors and $x_1, \dots, x_{\hat{k}}$ be mutually distinct arbitrary points from the domain Ω. There exists a solution $U_0 \in C^{6+\alpha}(\overline{\Omega})$ to the problem*

$$2\partial_{\bar{z}} U_0 + AU_0 = 0 \quad \text{in } \Omega \tag{2.10}$$

such that

$$\partial_z^m U_0(x_\ell) = \vec{r}_{m,\ell}, \quad \forall\, m \in \{0,\dots,5\} \;\; \text{and} \;\; \forall\, \ell \in \{1,\dots,\hat{k}\}, \tag{2.11}$$

and

$$\|U_0 - \Psi\|_{C^{5+\alpha}(\bar{\Gamma}_0)} \le \epsilon. \tag{2.12}$$

Proof of Proposition 2.1. We fix some positive ϵ_1. By Proposition 2.4, there exists a solution $\widetilde{U}_0 \in C^{6+\alpha}(\overline{\Omega})$ to problem (2.10) that satisfies (2.11) and

$$\|\widetilde{U}_0 - \Psi\|_{C^{5+\alpha}(\bar{\Gamma}_0)} \le \epsilon_1/4. \tag{2.13}$$

Let $\widetilde{T}_0$ be a solution to the boundary value problem

$$2\partial_z \widetilde{T}_0 + B\widetilde{T}_0 = 0 \quad \text{in } \Omega \tag{2.14}$$

such that

$$\|\widetilde{T}_0 + \Psi\|_{C^{5+\alpha}(\bar\Gamma_0)} \le \epsilon_1/4. \tag{2.15}$$

Then (2.13) and (2.15) yield

$$\|\widetilde{U}_0 + \widetilde{T}_0\|_{C^{5+\alpha}(\bar\Gamma_0)} \le \epsilon_1/2. \tag{2.16}$$

Consider the boundary value problem

$$\mathcal{K}(x,D)(U,T) = 0 \quad \text{in } \Omega, \qquad U + T = g \quad \text{on } \Gamma_0. \tag{2.17}$$

For any $g \in C^{6+\alpha}(\bar\Gamma_0)$, problem (2.17) admits a solution $(U,T) \in C^{6+\alpha}(\bar\Omega) \times C^{6+\alpha}(\bar\Omega)$. It is shown in [12] that problem (2.17) has a solution that satisfies an estimate

$$\|(U,T)\|_{C^{5+\alpha}(\bar\Omega)} \le C\|g\|_{C^{5+\alpha}(\bar\Gamma_0)}.$$

In particular for $g = -\widetilde{U}_0 - \widetilde{T}_0$, we have

$$\|(U,T)\|_{C^{5+\alpha}(\bar\Omega)} \le C\|\widetilde{U}_0 + \widetilde{T}_0\|_{C^{5+\alpha}(\bar\Gamma_0)} \le C\epsilon_1/2.$$

For any $\ell \in \{1,\dots,N\}$ we construct solutions $(U_0(\ell), T_0(\ell))$ to problem (2.17) with $g = 0$ depending on $\epsilon_0 > 0$ such that

$$|U_0(\ell)(x_k) - \vec{e}_\ell| < \epsilon_0, \quad \forall\, \ell \in \{1,\dots,\hat{k}\} \tag{2.18}$$

and

$$\|U_0(\ell) - \vec{e}_\ell\|_{C^{5+\alpha}(\bar\Gamma_0)} \le \epsilon_0, \quad \forall\, \ell \in \{1,\dots,\hat{k}\}. \tag{2.19}$$

Let $\mathbb{U}(\epsilon,x) = [U(1),\dots,U(N)]$ and $\mathbb{T}(\epsilon,x) = [T(1),\dots,T(N)]$ be $N\times N$ matrices. Let $\mathbb{U}$ be a matrix such that $\mathbb{U} = [U_0(1),\dots,U_0(k)]$. The matrix $\mathbb{U}$ is invertible on $\bar\Gamma_0 \cup \{x_0,\dots,x_{\hat{k}}\}$ and $\mathbb{U}^{-1} \in C^{6+\alpha}(\bar\Gamma_0)$. Then solution (U_0,T_0) to problem (2.1) is given by formula

$$(U_0, T_0) = (\mathbb{U}\boldsymbol{a}, \mathbb{T}\bar{\boldsymbol{a}}), \tag{2.20}$$

where $\boldsymbol{a}(z) = (a_1(z),\dots,a_N(z)) \in C^6(\bar\Omega)$ is a holomorphic vector-valued function such that $\operatorname{Im} \boldsymbol{a}|_{\Gamma_0} = 0$. Take the holomorphic function $\boldsymbol{a}(z)$ such that $\boldsymbol{a}(x_k) = \mathbb{U}^{-1}(x_k)U(x_k)$ for all $k \in \{1,\dots,\hat{k}\}$ and any $m \in \{1,\dots,5\}$,

$$\partial_z^m \boldsymbol{a}(x_k) = \mathbb{U}^{-1}(x_k)\left(\partial_z^m U(x_k) - \sum_{p=0}^{m-1}\binom{m}{p}\partial_z^{m-p}\mathbb{U}(x_k)\partial_z^p \boldsymbol{a}(x_k)\right), \quad \forall\, k \in \{1,\dots,\hat{k}\}$$

and

$$\|\boldsymbol{a}\|_{C^{5+\alpha}(\bar\Gamma_0)} \le \epsilon, \qquad \operatorname{Im} \boldsymbol{a}|_{\Gamma_0} = 0,$$

provided that $\epsilon_1 > 0$ is sufficiently small. We note that the existence of the function $\boldsymbol{a}(z)$ has already been proved in Proposition 2.2. Then the pair $(\widetilde{U}_0 + U - \mathbb{U}\boldsymbol{a}, \widetilde{T}_0 + T - \mathbb{T}\overline{\boldsymbol{a}})$ satisfies (2.2) and (2.12). The proof of Proposition 2.1 is complete. □

Henceforth $B^{\ell}_4(\partial\Omega)$ denotes a Besov space (see, e.g., [10]).

Furthermore we show

Proposition 2.5. *Let ϵ be a positive number, $A \in C^{5+\alpha}(\bar\Omega)$ for some $\alpha \in (0,1)$, $\Psi \in C^\infty(\Gamma_0)$, $\vec r_1, \dots, \vec r_{\hat k} \in \mathbb{C}^N$ be arbitrary vectors and $x_1, \dots, x_{\hat k}$ be mutually distinct arbitrary points from the domain Ω. Then there exists a solution $U_0 \in C^{6+\alpha}(\overline{\Omega})$ to the problem*

$$2\partial_{\bar z} U_0 + AU_0 = 0 \quad \textit{in } \Omega \tag{2.21}$$

such that

$$U_0(x_\ell) = \vec r_\ell, \quad \forall\, \ell \in \{1, \dots, \hat k\}, \tag{2.22}$$

and

$$\|U_0 - \Psi\|_{C^0(\bar\Gamma_0)} \le \epsilon. \tag{2.23}$$

Proof. Consider the following extremal problem:

$$J_\epsilon(U) = \|U - \Psi\|^4_{B^{23/4}_4(\Gamma_0)} + \epsilon\|U\|^4_{B^{23/4}_4(\partial\Omega)} + \sum_{j=1}^{\hat k} |U(x_j) - c_j|^2 \to \inf, \tag{2.24}$$

$$2\partial_{\bar z} U + AU = 0 \quad \text{in } \Omega. \tag{2.25}$$

Here ϵ is a positive parameter. We claim that for each $\epsilon > 0$ there exists a unique solution to (2.24) and (2.25), which we denote as $\widehat{U}_\epsilon$. This fact can be proved by standard arguments. Denote by $\mathcal{U}_{\text{ad}}$ the set of admissible elements of the problem (2.24) and (2.25), namely

$$\mathcal{U}_{\text{ad}} = \{U \in W^1_4(\Omega);\, 2\partial_{\bar z} U + AU = 0 \ \text{ in } \Omega\}.$$

Clearly $0 \in \mathcal{U}_{\text{ad}}$ and the set of admissible elements is not empty. Set $\hat J_\epsilon = \inf_{U \in \mathcal{U}_{\text{ad}}} J_\epsilon(U)$. Therefore there exists a minimizing sequence $\{U_k\}_{k=1}^\infty \subset W^1_4(\Omega)$ such that

$$\hat J_\epsilon = \lim_{k \to +\infty} J_\epsilon(U_k).$$

The L^p estimates for the ∂_z-operator and the uniform boundedness of the trace of U_k in $B^{23/4}_4(\partial\Omega)$ imply the boundedness of the sequence $\{U_k\}$ in $W^1_4(\Omega)$. Without loss of generality, we can assume that

$$U_k \to \widehat{U}_\epsilon \quad \text{weakly in } W^1_4(\Omega) \quad \text{and} \quad U_k \to \widehat{U}_\epsilon \quad \text{weakly in } B^{23/4}_4(\partial\Omega).$$

Then, since the norm in the space $B^{23/4}_4(\partial\Omega)$ is lower semicontinuous with respect to the weak convergence, we obtain that

$$J_\epsilon(\widehat{U}_\epsilon) \le \lim_{k\to+\infty} J_\epsilon(U_k) = \hat{J}_\epsilon.$$

Thus the function $\widehat{U}_\epsilon$ is a solution to the extremal problem (2.24) and (2.25). Since the set of admissible elements is convex and the functional J_ϵ is strictly convex, this solution is unique.

By the Lagrange principle (see, e.g., [1]) there exists a multiplier $p \in L^{4/3}(\Omega)$ such that

$$J'_\epsilon(\hat{U}_\epsilon)[\widetilde{\delta}] + \operatorname{Re}\big(p_\epsilon, \overline{2\partial_{\bar z}\widetilde{\delta} + A\widetilde{\delta}}\big)_{L^2(\Omega)} = 0, \quad \forall\, \widetilde{\delta} \in W^1_4(\Omega).$$

This equality can be written in the form

$$\begin{aligned} I'_{\Gamma_0,23/4}(\hat{U}_\epsilon - \Psi)[\widetilde{\delta}] + \epsilon I'_{\partial\Omega,23/4}(\hat{U}_\epsilon)[\widetilde{\delta}] + \operatorname{Re}\big(p_\epsilon, \overline{2\partial_{\bar z}\widetilde{\delta} + A\widetilde{\delta}}\big)_{L^2(\Omega)} \\ + 2\operatorname{Re}\sum_{j=1}^{\hat{k}} (\widehat{U}_\epsilon(x_j) - c_j)\overline{\widetilde{\delta}(x_j)} = 0, \quad \forall\, \widetilde{\delta} \in W^1_4(\Omega), \end{aligned} \tag{2.26}$$

where $I'_{\Gamma^*,\kappa}(\hat{w})$ denotes the derivative of the functional $w \to \|w\|^4_{B^\kappa_4(\Gamma^*)}$ at $\hat{w}$.

Observe that $J_\epsilon(\hat{U}_\epsilon) \le J_\epsilon(0) = \|\Psi\|^4_{B^{23/4}_4(\Gamma_0)} + \sum_{j=1}^{\hat{k}} |c_j|^2$. This implies that the sequence $\{\hat{U}_\epsilon\}$ is bounded in $B^{23/4}_4(\Gamma_0)$, the sequences $\{\hat{U}_\epsilon(x_j) - c_j\}$ are bounded in $\mathbb{C}$, the sequence $\epsilon I'_{\partial\Omega,23/4}(\hat{U}_\epsilon)[\widetilde{\delta}]$ converges to zero for any $\widetilde{\delta}$ from $B^{23/4}_4(\partial\Omega)$. Then (2.26) implies that the sequence $\{p_\epsilon\}_{\epsilon\in(0,1)}$ is bounded in $L^{4/3}(\Omega)$. Passing to the limit in (2.26) we obtain

$$\begin{aligned} I'_{\Gamma_0,23/4}(\hat{U} - \Psi)[\widetilde{\delta}] + \operatorname{Re}\big(p, \overline{2\partial_{\bar z}\widetilde{\delta} + A\widetilde{\delta}}\big)_{L^2(\Omega)} \\ + 2\operatorname{Re}\sum_{j=1}^{\hat{k}} (\hat{U}(x_j) - c_j)\overline{\widetilde{\delta}(x_j)} = 0, \quad \forall\, \widetilde{\delta} \in W^1_4(\Omega). \end{aligned} \tag{2.27}$$

From (2.27), we obtain

$$2\partial_{\bar z} p + A^* p = \sum_{j=1}^{\hat{k}} (\widehat{U}(x_j) - c_j)\delta(x - x_j) \quad \text{in } \Omega, \qquad p|_{\partial\Omega\setminus\Gamma_0} = 0. \tag{2.28}$$

By $p \in L^{4/3}(\Omega)$ and (2.28), the function p belongs to $L^2(\Omega \setminus \mathcal{O})$, where $\mathcal{O}$ is any open set containing $\Gamma_0 \cup \{x_1, \dots, x_{\hat{k}}\}$. Then the uniqueness of the Cauchy problem for the ∂_z-operator yields $p \equiv 0$ on $\Omega \setminus \cup_{j=1}^{\hat{k}} \{x_j\}$. This fact and (2.28) imply that $U(x_j) - c_j = 0$.

Let us fix an arbitrary $\hat{j} \in \{1, \dots, \hat{k}\}$. We proved that for any positive ϵ there exists a solution $U(\ell) \in C^{6+\alpha}(\bar{\Omega})$, depending on ϵ, to the problem

$$2\partial_{\bar{z}} U(\ell) + AU(\ell) = 0 \quad \text{in } \Omega$$

such that

$$\|U(\ell) - \vec{e}_\ell\|_{C^0(\bar{\Gamma}_0)} \le \epsilon, \qquad \sum_{k=1}^{\hat{k}} |U(\ell)(x_k) - \vec{e}_\ell| \le \epsilon. \tag{2.29}$$

Let $\mathbb{U}(\epsilon, x) = [U(1), \dots, U(N)]$ be an $N \times N$ matrix. By (2.29) the matrix $\mathbb{U}(\epsilon, x)$ is invertible on $\bar{\Gamma}_0$ and $\|\mathbb{U}^{-1}\|_{C^{5+\alpha}(\bar{\Gamma}_0)} \le C$ for all sufficiently small ϵ. By Corollary 2.3, for each positive ϵ_1, there exists a holomorphic vector-valued function $\boldsymbol{a}(z)$ such that

$$\|\boldsymbol{a} - \mathbb{U}^{-1}\Psi\|_{C^0(\bar{\Gamma}_0)} \le \epsilon_1, \qquad \boldsymbol{a}(x_k) = \mathbb{U}^{-1}(x_k)\vec{r}_k, \quad \forall\, k \in \{1, \dots, \hat{k}\}.$$

Then function $\mathbb{U}\boldsymbol{a}$ satisfies (2.22) and (2.23). □

Proof of Proposition 2.4. Let us fix some $\epsilon_0 > 0$. First, for any $\ell \in \{1, \dots, N\}$ we construct solutions $U_0(\ell)$ to problem (2.21) depending on $\epsilon_0 > 0$ such that

$$U_0(\ell)(x_k) = \vec{e}_\ell, \quad \forall\, \ell \in \{1, \dots, \hat{k}\} \tag{2.30}$$

and

$$\|U_0(\ell) - \vec{e}_\ell\|_{C^0(\bar{\Gamma}_0)} \le \epsilon_0, \quad \forall\, \ell \in \{1, \dots, \hat{k}\}. \tag{2.31}$$

We set $\mathbb{U} = [U_0(1), \dots, U_0(k)]$. The matrix $\mathbb{U}$ is invertible on $\bar{\Gamma}_0$ and $\mathbb{U}^{-1} \in C^{6+\alpha}(\bar{\Gamma}_0)$. Then the solution U_0 to problem (2.21) is given by the formula

$$U_0 = \mathbb{U}\boldsymbol{a}, \tag{2.32}$$

where $\boldsymbol{a}(z) = (a_1(z), \dots, a_N(z)) \in C^6(\bar{\Omega})$ is a holomorphic vector-valued function. Take the holomorphic function $\boldsymbol{a}(z)$ such that

$$\boldsymbol{a}(x_k) = \vec{r}_{0,k},$$
$$\partial_z^m \boldsymbol{a}(x_k) = \mathbb{U}^{-1}(x_k)\left(\vec{r}_{m,k} - \sum_{p=0}^{m-1} \binom{m}{p} \partial_z^{m-p}\mathbb{U}(x_k)\partial_z^p \boldsymbol{a}(x_k)\right)$$

for each $k \in \{1, \dots, \hat{k}\}$ and $m \in \{1, \dots, 5\}$, and

$$\|\boldsymbol{a} - \mathbb{U}^{-1}\Psi\|_{C^{5+\alpha}(\bar{\Gamma}_0)} \le \epsilon.$$

The existence of such a function $\boldsymbol{a}(z)$ is already proved in Corollary 2.3. Hence (2.11) and (2.12) hold true. The proof of the proposition is complete. ☐

We construct matrices $\mathcal{C}$ and $\mathcal{P}$ satisfying

$$\mathcal{C} = (T_0(1), \dots, T_0(N)), \quad \mathcal{P} = (U_0(1), \dots, U_0(N)) \in C^{6+\alpha}(\bar{\Omega}) \tag{2.33}$$

and for any $j \in \{1, \dots, N\}$,

$$\mathcal{K}(x, D)(U_0(j), T_0(j)) = 0 \quad \text{in } \Omega, \qquad U_0(j) + T_0(j) = 0 \quad \text{on } \Gamma_0. \tag{2.34}$$

Let $\hat{x}$ be some point from Ω. By Proposition 2.1 for equation (2.34), we can construct solutions $(U_0(j), T_0(j))$ such that

$$U_0(j)(\hat{x}) = \vec{e}_j, \quad \forall j \in \{1, \dots, N\}.$$

By $\mathcal{Z}$ we denote the set of zeros of the function q on $\overline{\Omega} : \mathcal{Z} = \{z \in \overline{\Omega};\ q(z) = 0\}$. Obviously $\operatorname{card} \mathcal{Z} < \infty$. By κ we denote the highest order of the zeros of the function q on $\overline{\Omega}$.

By Proposition 2.1 we construct solutions $(U_0^{(j)}, T_0^{(j)})$ to problem (2.34) such that

$$U_0^{(j)}(x) = \vec{e}_j, \quad \forall j \in \{1, \dots, N\} \ \text{ and } \ \forall x \in \mathcal{Z}. \tag{2.35}$$

Set $\widetilde{\mathcal{P}}(x) = (U_0^{(1)}(x), \dots, U_0^{(N)}(x))$ and $\widetilde{\mathcal{C}}(x) = (T_0^{(1)}(x), \dots, T_0^{(N)}(x))$. Then there exists a holomorphic function $\widetilde{q}$ such that $\det \widetilde{\mathcal{P}} = \widetilde{q}(z) e^{-(1/2)\partial_{\bar{z}}^{-1} \operatorname{tr} A}$ in Ω. Let $\widetilde{\mathcal{Z}} = \{z \in \overline{\Omega};\ \widetilde{q}(z) = 0\}$ and $\widetilde{\kappa}$ the highest order of the zeros of the function $\widetilde{q}$.

By (2.35) we see that $\widetilde{\mathcal{Z}} \cap \mathcal{Z} = \emptyset$. Therefore there exists a holomorphic function $r(z)$ such that $r|_{\mathcal{Z}} = 0$ and $(1-r)|_{\widetilde{\mathcal{Z}}} = 0$ and the orders of zeros of the function r on $\mathcal{Z}$ and the function $1 - r$ on $\widetilde{\mathcal{Z}}$ are greater than or equal to $\max\{\kappa, \widetilde{\kappa}\}$.

We set

$$P_A f = \tfrac{1}{2}\mathcal{P}\partial_{\bar{z}}^{-1}(\mathcal{P}^{-1} r f) + \tfrac{1}{2}\widetilde{\mathcal{P}}\partial_{\bar{z}}^{-1}(\widetilde{\mathcal{P}}^{-1}(1-r) f). \tag{2.36}$$

Then

$$P_A^* f = -\tfrac{1}{2} r (\mathcal{P}^{-1})^* \partial_{\bar{z}}^{-1}(\mathcal{P}^* f) - \tfrac{1}{2}(1-r)(\widetilde{\mathcal{P}}^{-1})^* \partial_{\bar{z}}^{-1}(\widetilde{\mathcal{P}}^* f).$$

For any matrix $A \in C^{5+\alpha}(\overline{\Omega})$ with $\alpha \in (0,1)$, the linear operators P_A, $P_A^* \in \mathcal{L}(L^2(\Omega), W_2^1(\Omega))$ solve the differential equations

$$(-2\partial_{\bar{z}} + A^*) P_A^* g = g \quad \text{in } \Omega \quad \text{and} \quad (2\partial_{\bar{z}} + A) P_A g = g \quad \text{in } \Omega.$$

In a similar way, using matrices $\mathcal{C}$, $\widetilde{\mathcal{C}}$ and some antiholomorphic function r_1, we construct the operators

$$S_B f = \tfrac{1}{2}\mathcal{C}\partial_z^{-1}(\mathcal{C}^{-1}r_1 f) + \tfrac{1}{2}\widetilde{\mathcal{C}}\partial_z^{-1}(\widetilde{\mathcal{C}}^{-1}(1-r_1)f)$$

and

$$S_B^* f = -\tfrac{1}{2}r_1(\mathcal{C}^{-1})^*\partial_z^{-1}(\mathcal{C}^* f) - \tfrac{1}{2}(1-r_1)(\widetilde{\mathcal{C}}^{-1})^*\partial_z^{-1}(\widetilde{\mathcal{C}}^* f). \tag{2.37}$$

For any matrix $B \in C^{5+\alpha}(\overline{\Omega})$ with $\alpha \in (0,1)$, the linear operators S_B and S_B^* solve the differential equations

$$(2\partial_z + B)S_B g = g \quad \text{in } \Omega \quad \text{and} \quad (-2\partial_z + B^*)S_B^* g = g \quad \text{in } \Omega.$$

Finally we introduce two operators:

$$\widetilde{\mathcal{R}}_{\tau,B} g = e^{\tau(\overline{\Phi}-\Phi)}S_B(e^{\tau(\Phi-\overline{\Phi})}g) \ \text{ and } \ \mathcal{R}_{\tau,B} g = e^{\tau(\Phi-\overline{\Phi})}P_B(e^{\tau(\overline{\Phi}-\Phi)}g).$$

Here, Φ is given later.

§3. Step 1: Construction of complex geometric optics solutions

For $j = 1, 2$, let $L_j(x, D)$ be operators of the form (1.1) with the coefficients A_j, B_j, Q_j. In this step, we will construct two complex geometric optics solutions u_1 and v respectively for the operators $L_1(x, D)$ and $L_2(x, D)^*$. Here and henceforth $L_2(x, D)^*$ denotes the formal adjoint operator to $L_2(x, D)$.

As the phase function for such a solution, we consider a holomorphic function $\Phi(z)$ such that $\Phi(z) = \varphi(x_1, x_2) + i\psi(x_1, x_2)$ with real-valued functions φ and ψ. For some $\alpha \in (0,1)$ the function Φ belongs to $C^{6+\alpha}(\overline{\Omega})$. Moreover,

$$\partial_{\overline{z}}\Phi = 0 \quad \text{in } \Omega, \qquad \operatorname{Im}\Phi|_{\Gamma_0} = 0. \tag{3.1}$$

Denote by $\mathcal{H}$ the set of all the critical points of the function Φ: $\mathcal{H} = \{z \in \overline{\Omega};\ \Phi'(z) = 0\}$. Assume that Φ has no critical points on $\overline{\widetilde{\Gamma}}$ and that all the critical points are nondegenerate:

$$\Phi''(z) \neq 0, \quad \forall\, z \in \mathcal{H}, \qquad \operatorname{card}\mathcal{H} < \infty. \tag{3.2}$$

Let $\partial\Omega = \cup_{j=1}^{\mathcal{N}}\gamma_j$. The following proposition was proved in [7].

Proposition 3.1. *Let $\widetilde{x}$ be an arbitrary point in domain Ω. There exists a sequence of functions $\{\Phi_\epsilon\}_{\epsilon\in(0,1)} \in C^6(\bar{\Omega})$ satisfying (3.1), (3.2) and there exists a sequence $\{\widetilde{x}_\epsilon\}, \epsilon \in (0,1)$ such that*

$$\widetilde{x}_\epsilon \in \mathcal{H}_\epsilon = \{z \in \overline{\Omega};\ \Phi_\epsilon'(z) = 0\}, \qquad \widetilde{x}_\epsilon \to \widetilde{x} \ \text{ as } \epsilon \to +0. \tag{3.3}$$

Moreover, for any j from $\{1,\dots,\mathcal{N}\}$ we have

$$\mathcal{H}_\epsilon \cap \gamma_j = \emptyset \quad \text{if } \gamma_j \cap \widetilde{\Gamma} \neq \emptyset,$$
$$\mathcal{H}_\epsilon \cap \gamma_j \subset \Gamma_0 \quad \text{if } \gamma_j \cap \widetilde{\Gamma} = \emptyset,$$
$$\operatorname{Im}\Phi_\epsilon(\widetilde{x}_\epsilon) \notin \big\{ \operatorname{Im}\Phi_\epsilon(x);\, x \in \mathcal{H}_\epsilon \setminus \{\widetilde{x}_\epsilon\}\big\} \quad \text{and} \quad \operatorname{Im}\Phi_\epsilon(\widetilde{x}_\epsilon) \neq 0. \tag{3.4}$$

Let the function Φ satisfy (3.1), (3.2) and $\widetilde{x}$ be some point from $\mathcal{H}$. Denote

$$Q_1(1) = -2\partial_z A_1 - B_1 A_1 + Q_1, \qquad Q_2(1) = -2\partial_{\overline{z}} B_1 - A_1 B_1 + Q_1.$$

Let $(U_0, T_0) \in C^{6+\alpha}(\overline{\Omega})$ be a solution to the boundary value problem

$$\begin{aligned} &\mathcal{K}(x,D)(U_0,T_0) = (2\partial_{\overline{z}} U_0 + A_1 U_0, 2\partial_z T_0 + B_1 T_0) = 0 \quad \text{in } \Omega,\\ &U_0 + T_0 = 0 \quad \text{on } \Gamma_0. \end{aligned} \tag{3.5}$$

The complex geometric optics solutions are constructed in [6] and [7]. We recall the main steps of the construction. Let the pair (U_0, T_0) be defined in the following way. Let

$$U_0 = \mathcal{P}_1 \boldsymbol{a}, \qquad T_0 = \mathcal{C}_1 \overline{\boldsymbol{a}}, \tag{3.6}$$

where $\boldsymbol{a}(z) = (a_1(z),\dots,a_N(z)) \in C^{5+\alpha}(\bar{\Omega})$ is a holomorphic vector-valued function such that $\operatorname{Im} \boldsymbol{a}|_{\Gamma_0} = 0$, or

$$U_0 = \mathcal{P}_1 \boldsymbol{a}, \qquad T_0 = -\mathcal{C}_1 \overline{\boldsymbol{a}}, \tag{3.7}$$

where $\boldsymbol{a}(z) = (a_1(z),\dots,a_N(z)) \in C^{5+\alpha}(\bar{\Omega})$ is a holomorphic vector-valued function such that $\operatorname{Re} \boldsymbol{a}|_{\Gamma_0} = 0$, and matrices $\mathcal{C}_1$ and $\mathcal{P}_1$ are constructed by

$$\mathcal{C}_1 = (T_0(1),\dots,T_0(N)), \qquad \mathcal{P}_1 = (U_0(1),\dots,U_0(N)) \in C^{6+\alpha}(\bar{\Omega}) \tag{3.8}$$

and for any $k \in \{1,\dots,N\}$ the functions $(U_0(k), T_0(k))$ solve the boundary value problems

$$\mathcal{K}(x,D)(U_0(k),T_0(k)) = 0 \quad \text{in } \Omega, \qquad U_0(k) + T_0(k) = 0 \quad \text{on } \Gamma_0. \tag{3.9}$$

In order to make a choice of $\mathcal{C}_1$ and $\mathcal{P}_1$, let us fix a small positive number ϵ. By Proposition 2.1 there exist solutions $(U_0(k), T_0(k))$ to problem (3.9) such that

$$\|U_0(k) - \vec{e}_k\|_{C^{5+\alpha}(\bar{\Gamma}_0)} \le \epsilon, \quad \forall\, k \in \{1,\dots,N\}. \tag{3.10}$$

This inequality and the boundary conditions in (3.9) on Γ_0 imply

$$\|T_0(k) + \vec{e}_k\|_{C^{5+\alpha}(\bar{\Gamma}_0)} \le \epsilon, \quad \forall\, k \in \{1,\dots,N\}. \tag{3.11}$$

Let e_1, e_2 be smooth functions such that

$$e_1 + e_2 = 1 \quad \text{on } \Omega, \tag{3.12}$$

and let e_1 vanish in a neighborhood of $\partial\Omega$ and e_2 vanish in a neighborhood of the set $\mathcal{H} \cap \Omega$.

For any positive ϵ, set $G_\epsilon = \{x \in \Omega; \operatorname{dist}(\operatorname{supp} e_1, x) > \epsilon\}$. The following proposition was proved in [8]:

Proposition 3.2. *Let $B, q \in C^{5+\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$, the function Φ satisfy (3.1), (3.2) and $\widetilde{q} \in W^1_p(\Omega)$ for some $p > 2$. Suppose that $q|_{\mathcal{H}} = \widetilde{q}|_{\mathcal{H}} = 0$. Then the following asymptotic formulae hold true:*

$$\widetilde{\mathcal{R}}_{\tau,B}\left(e_1\left(q + \widetilde{q}/\tau\right)\right)|_{\overline{G}_\epsilon} = e^{\tau(\overline{\Phi}-\Phi)} \sum_{\widetilde{y} \in \mathcal{H}} \left(\frac{m_{+,\widetilde{y}} e^{2i\tau\psi(\widetilde{y})}}{\tau^2} + o_{C^2(\overline{G}_\epsilon)}\left(\frac{1}{\tau^2}\right)\right) \quad \text{as } |\tau| \to +\infty, \tag{3.13}$$

$$\mathcal{R}_{\tau,B}\left(e_1\left(q + \widetilde{q}/\tau\right)\right)|_{\overline{G}_\epsilon} = e^{\tau(\Phi-\overline{\Phi})} \sum_{\widetilde{y} \in \mathcal{H}} \left(\frac{m_{-,\widetilde{y}} e^{-2i\tau\psi(\widetilde{y})}}{\tau^2} + o_{C^2(\overline{G}_\epsilon)}\left(\frac{1}{\tau^2}\right)\right) \quad \text{as } |\tau| \to +\infty. \tag{3.14}$$

Let $\widetilde{x} \in \mathcal{H} \setminus \partial\Omega$. Denote

$$q_1 = P_{A_1}(Q_1(1)U_0) - M_1, \qquad q_2 = S_{B_1}(Q_2(1)T_0) - M_2 \in C^{5+\alpha}(\bar{\Omega}),$$

where the functions $M_1 \in \operatorname{Ker}(2\partial_{\overline{z}} + A_1)$ and $M_2 \in \operatorname{Ker}(2\partial_z + B_1)$ are taken such that

$$q_1(\widetilde{x}) = q_2(\widetilde{x}) = \partial_x^\beta q_1(x) = \partial_x^\beta q_2(x) = 0, \quad \forall\, x \in \mathcal{H} \setminus \{\widetilde{x}\} \ \text{ and } \ \forall\, |\beta| \le 5. \tag{3.15}$$

Moreover, we can assume that

$$\lim_{x \to x_\pm} \frac{|T_0(x)| + |U_0(x)|}{|x - x_\pm|^{98}} = 0. \tag{3.16}$$

Indeed, in order to obtain (3.15) and (3.16) for the function q_1, let us take the pair (U_*, V_*) as a nontrivial solution to problem (3.5) such that for some vectors $\vec{u}$ and $\vec{v}$ either $U_*(\widetilde{x}) = \vec{u}$ or $V_*(\widetilde{x}) = \vec{v}$, and let $a(z)$ be a holomorphic function in Ω such that $a|_{\mathcal{H}\setminus\{\widetilde{x}\}\cup\{x_\pm\}} = 0$, $\operatorname{Im} a|_{\Gamma_0} = 0$ and $a(\widetilde{x}) = 1$. Set $(U_0, V_0) = (a^{100}U_*, \bar{a}^{100}V_*)$ and take the functions $M_1(\bar{z})$ and $M_2(z)$ as polynomials such that $(P_{A_1}(Q_1(1)U_0) - M_1)|_{\mathcal{H}} = (S_{B_1}(Q_2(1)T_0) - M_2)|_{\mathcal{H}} = 0$ and $\partial^j_{\bar{z}}(P_{A_1}(Q_1(1)U_0) - M_1)|_{\mathcal{H}\setminus\{\widetilde{x}\}} = \partial^j_z(S_{B_1}(Q_2(1)T_0) - M_2)|_{\mathcal{H}\setminus\{\widetilde{x}\}} = 0$ for all j from $\{1, \dots, 5\}$. Then we obviously have $q_k|_{\mathcal{H}} = 0$ and $\partial_z^j q_k|_{\mathcal{H}\setminus\{\widetilde{x}\}} = \partial_{\bar z}^j q_k|_{\mathcal{H}\setminus\{\widetilde{x}\}} = 0$ for $k = 1, 2$ and $j \in \{1, \dots, 5\}$. Finally, in order to prove the last two equalities in (3.15), we need to consider the case

$$\partial_z^{\beta_1}\partial_{\bar z}^{\beta 2} q_1(x) = \partial_z^{\beta_1}\partial_{\bar z}^{\beta 2} q_2(x) = 0, \quad \forall\, |\beta| \le 5 \ \text{ and } \ \beta_1 \ne 0, \beta_2 \ne 0, \ \forall\, x \in \mathcal{H}\setminus\{\widetilde{x}\}. \tag{3.17}$$

Let us prove the equality for the function q_1. The proof for the function q_2 is the same. We argue by induction. First we observe that

$$\begin{aligned}\partial_z^{\beta_1}\partial_{\bar z}^{\beta 2} q_1(x) &= \frac{1}{2}\partial_z^{\beta_1-1}\partial_{\bar z}^{\beta 2}(A_1 q_1 + (Q_1(1)a^{100}U_*))(x) \\ &= \frac{1}{2}\partial_z^{\beta_1-1}\partial_{\bar z}^{\beta 2}(A_1 q_1)(x), \quad \forall\, x \in \mathcal{H}\setminus\{\widetilde{x}\}.\end{aligned} \tag{3.18}$$

From this equality, by the assumption in the induction, we have

$$\partial_z\partial_{\bar z} q_1(x) = \frac{1}{2}\partial_{\bar z}(A_1 q_1)(x) = 0, \quad \forall\, x \in \mathcal{H}\setminus\{\widetilde{x}\}.$$

If (3.17) is proved for all $|\beta| \le k-1$, then from equality (3.18) the conclusion holds for all $|\beta| \le k$.

Next we introduce the functions $(U_{-1}, T_{-1}) \in C^{5+\alpha}(\bar\Omega)\times C^{5+\alpha}(\bar\Omega)$ as solutions to the following boundary value problem:

$$\mathcal{K}(x, D)(U_{-1}, T_{-1}) = 0 \quad \text{in } \Omega, \qquad (U_{-1} + T_{-1})|_{\Gamma_0} = \frac{q_1}{2\Phi'} + \frac{q_2}{2\overline{\Phi}'}. \tag{3.19}$$

In order to fix the choice of the operators P_{B_1} and T_{A_1} in formulae (2.36) and (2), we take $\mathcal{C} = \mathcal{C}_1$, $\mathcal{P} = \mathcal{P}_1$ and $\widetilde{\mathcal{C}} = \widetilde{\mathcal{C}}_1$, $\widetilde{\mathcal{P}} = \widetilde{\mathcal{P}_1}$ for appropriately constructed $\widetilde{\mathcal{C}}_2$ and $\widetilde{\mathcal{P}_2}$. We set $p_1 = -Q_2(1)(\frac{e_1 q_1}{2\Phi'} - U_{-1}) + L_1(x, D)(\frac{e_2 q_1}{2\Phi'})$, $p_2 = -Q_1(1)(\frac{e_1 q_2}{2\Phi'} - T_{-1}) + L_1(x, D)(\frac{e_2 q_2}{2\Phi'})$, $\widetilde{q}_2 = S_{B_1} p_2 - \widetilde{M_2}$ and $\widetilde{q}_1 = P_{A_1} p_1 - \widetilde{M}_1 \in C^{5+\alpha}(\bar\Omega)$, where $\widetilde{M}_1 \in \mathrm{Ker}(2\partial_{\bar z} + A_1)$ and $\widetilde{M}_2 \in \mathrm{Ker}(2\partial_z + B_1)$ are taken such that

$$\partial_x^\beta \widetilde{q}_1(x) = \partial_x^\beta \widetilde{q}_2(x) = 0, \quad \forall\, x \in \mathcal{H} \ \text{ and } \ \forall\, |\beta| \le 5. \tag{3.20}$$

By Proposition 3.2, there exist functions $m_{\pm,\widetilde{x}} \in C^{2+\alpha}(\overline{G}_\epsilon)$ such that

$$\widetilde{\mathcal{R}}_{\tau,B_1}\left(e_1\left(q_1 + \widetilde{q}_1/\tau\right)\right)|_{\overline{G}_\epsilon} = e^{\tau(\overline{\Phi}-\Phi)}\left(\frac{m_{+,\widetilde{x}}e^{2i\tau\psi(\widetilde{x})}}{\tau^2} + o_{C^2(\overline{G}_\epsilon)}\Big(\frac{1}{\tau^2}\Big)\right) \quad \text{as } |\tau| \to +\infty \tag{3.21}$$

and

$$\mathcal{R}_{\tau,A_1}\left(e_1\left(q_2 + \widetilde{q}_2/\tau\right)\right)|_{\overline{G}_\epsilon} = e^{\tau(\Phi-\overline{\Phi})}\left(\frac{m_{-,\widetilde{x}}e^{-2i\tau\psi(\widetilde{x})}}{\tau^2} + o_{C^2(\overline{G}_\epsilon)}\Big(\frac{1}{\tau^2}\Big)\right) \quad \text{as } |\tau| \to +\infty. \tag{3.22}$$

The functions $m_{\pm,y}$ with $y \neq \widetilde{x}$ are identically equal to zero, thanks to (3.20). For any $\widetilde{x} \in \mathcal{H}$, we introduce the functions $a_{\pm,\widetilde{x}}, b_{\pm,\widetilde{x}} \in C^{2+\alpha}(\overline{\Omega})$ as solutions to the boundary value problem

$$\mathcal{K}(x,D)(a_{\pm,\widetilde{x}}, b_{\pm,\widetilde{x}}) = 0 \quad \text{in } \Omega, \qquad (a_{\pm,\widetilde{x}} + b_{\pm,\widetilde{x}})|_{\Gamma_0} = m_{\pm,\widetilde{x}}. \tag{3.23}$$

We introduce the functions $a_{\pm,\widetilde{x}}, b_{\pm,\widetilde{x}}$ in the form

$$(a_{\pm,\widetilde{x}}, b_{\pm,\widetilde{x}}) = (\mathcal{P}_1(x)\boldsymbol{a}_{\pm,\widetilde{x}}(z), \mathcal{C}_1(x)\boldsymbol{b}_{\pm,\widetilde{x}}(\bar{z})), \tag{3.24}$$

where $\boldsymbol{a}_{\pm,\widetilde{x}}(z)$ is some holomorphic function and $\boldsymbol{b}_{\pm,\widetilde{x}}(\bar{z})$ is some antiholomorphic function. Let $(U_{-2}, T_{-2}) \in C^{5+\alpha}(\overline{\Omega}) \times C^{5+\alpha}(\overline{\Omega})$ be a solution to the boundary value problem

$$\mathcal{K}(x,D)(U_{-2}, T_{-2}) = 0 \quad \text{in } \Omega, \qquad (U_{-2} + T_{-2})|_{\Gamma_0} = \frac{\widetilde{q}_1}{2\Phi'} + \frac{\widetilde{q}_2}{2\overline{\Phi}'}.$$

We introduce the functions $U_{0,\tau}$, $T_{0,\tau} \in C^{2+\alpha}(\overline{\Omega})$ by

$$U_{0,\tau} = U_0 + \frac{U_{-1} - e_2 q_1/2\Phi'}{\tau} + \frac{1}{\tau^2}\left(e^{2i\tau\psi(\widetilde{x})}a_{+,\widetilde{x}} + e^{-2i\tau\psi(\widetilde{x})}a_{-,\widetilde{x}} + U_{-2} - \frac{\widetilde{q}_1 e_2}{2\Phi'}\right) \tag{3.25}$$

and

$$T_{0,\tau} = T_0 + \frac{T_{-1} - e_2 q_2/2\bar{\Phi}'}{\tau} + \frac{1}{\tau^2}\left(e^{2i\tau\psi(\widetilde{x})}b_{+,\widetilde{x}} + e^{-2i\tau\psi(\widetilde{x})}b_{-,\widetilde{x}} + T_{-2} - \frac{\widetilde{q}_2 e_2}{2\overline{\Phi}'}\right). \tag{3.26}$$

We set $\mathcal{O}_\epsilon = \{x \in \Omega; \mathrm{dist}(x, \partial\Omega) \le \epsilon\}$.

In [7] and [8], it is shown that there exists a function u_{-1} in the complex geometric optics solution satisfying the estimate

$$\sqrt{|\tau|}\|u_{-1}\|_{L^2(\Omega)} + \frac{1}{\sqrt{|\tau|}}\|\nabla u_{-1}\|_{L^2(\Omega)} + \|u_{-1}\|_{W_2^{1,\tau}(\mathcal{O}_\epsilon)} = o\Big(\frac{1}{\tau}\Big) \quad \text{as } |\tau| \to +\infty \tag{3.27}$$

and the function

$$u_1(x) = U_{0,\tau}e^{\tau\Phi} + T_{0,\tau}e^{\tau\overline{\Phi}} - e^{\tau\Phi}\widetilde{\mathcal{R}}_{\tau,B_1}(e_1(q_1 + \widetilde{q}_1/\tau)) - e^{\tau\overline{\Phi}}\mathcal{R}_{\tau,A_1}(e_1(q_2 + \widetilde{q}_2/\tau)) + e^{\tau\varphi}u_{-1} \tag{3.28}$$

solves the boundary value problem

$$L_1(x,D)u_1 = 0 \quad \text{in } \Omega, \qquad u_1|_{\Gamma_0} = 0. \tag{3.29}$$

Similarly, we construct the complex geometric optics solutions for the operator $L_2(x,D)^*$. Let $(V_0, W_0) \in C^{6+\alpha}(\overline{\Omega}) \times C^{6+\alpha}(\overline{\Omega})$ be a solution to the following boundary value problem:

$$\begin{aligned} &\mathcal{M}(x,D)(V_0,W_0) = ((2\partial_z - B_2^*)V_0, (2\partial_{\overline{z}} - A_2^*)W_0) = 0 \quad \text{in } \Omega,\\ &(V_0 + W_0)|_{\Gamma_0} = 0, \end{aligned} \tag{3.30}$$

which satisfies $V_0(\widetilde{x}) = r$ for some $r \in \mathbb{R}^N$ and

$$\lim_{x\to x_\pm} \frac{|V_0(x)|}{|x-x_\pm|^{98}} = \lim_{x\to x_\pm} \frac{|W_0(x)|}{|x-x_\pm|^{98}} = 0. \tag{3.31}$$

Such a pair (V_0, W_0) exists by Propositions 2.1 and 2.2. More specifically let

$$V_0 = \mathcal{C}_2\overline{\boldsymbol{b}}, \qquad W_0 = \mathcal{P}_2\boldsymbol{b}, \tag{3.32}$$

where $\boldsymbol{b}(z) = (b_1(z),\dots,b_N(z)) \in C^{5+\alpha}(\bar{\Omega})$ is a holomorphic vector-valued function such that $\operatorname{Im}\boldsymbol{b}|_{\Gamma_0} = 0$, or

$$V_0 = \mathcal{C}_2\overline{\boldsymbol{b}}, \qquad W_0 = -\mathcal{P}_2\boldsymbol{b}, \tag{3.33}$$

where $\boldsymbol{b}(z) = (b_1(z),\dots,b_N(z)) \in C^{5+\alpha}(\bar{\Omega})$ is a holomorphic vector-valued function such that $\operatorname{Re}\boldsymbol{b}|_{\Gamma_0} = 0$, and the matrices $\mathcal{C}_2$ and $\mathcal{P}_2$ are constructed by

$$\mathcal{C}_2 = (V_0(1),\dots,V_0(N)), \qquad \mathcal{P}_2 = (W_0(1),\dots,W_0(N)), \tag{3.34}$$

and for any $k \in \{1,\dots,N\}$,

$$\mathcal{M}(x,D)(V_0(k),W_0(k)) = 0 \quad \text{in } \Omega, \qquad (V_0(k)+W_0(k))|_{\Gamma_0} = 0. \tag{3.35}$$

Moreover, by Proposition 2.1, there exist solutions $(V_0(k), W_0(k))$ to problem (3.30) such that

$$\|W_0(k) - \vec{e}_k\|_{C^{5+\alpha}(\bar{\Gamma}_0)} \le \epsilon, \quad \forall\, k \in \{1,\dots,N\}. \tag{3.36}$$

This inequality and the boundary conditions in (3.30) on Γ_0 imply

$$\|V_0(k) + \vec{e}_k\|_{C^{5+\alpha}(\bar{\Gamma}_0)} \le \epsilon, \quad \forall\, k \in \{1,\dots,N\}. \tag{3.37}$$

In order to fix the choice of the operators $P_{-B_2^*}$ and $T_{-A_2^*}$, we take $\mathcal{C} = \mathcal{C}_2$, $\mathcal{P} = \mathcal{P}_2$ and $\widetilde{\mathcal{C}} = \widetilde{\mathcal{C}_2}$, $\widetilde{\mathcal{P}} = \widetilde{\mathcal{P}_2}$ for appropriately constructed $\widetilde{\mathcal{C}_2}$, $\widetilde{\mathcal{P}_2}$. We set $q_3 = P_{-A_2^*}(Q_1(2)W_0) - M_3$ and $q_4 = S_{-B_2^*}(Q_2(2)V_0) - M_4 \in C^{5+\alpha}(\bar{\Omega})$, where $Q_1(2) = Q_2^* - 2\partial_{\overline{z}}B_2^* - B_2^*A_2^*$, $\quad Q_2(2) = Q_2^* - 2\partial_z A_2^* - A_2^*B_2^*$, $M_3 \in \operatorname{Ker}(2\partial_{\overline{z}} - A_2^*)$ and $M_4 \in \operatorname{Ker}(2\partial_z - B_2^*)$ are chosen such that for all $x \in \mathcal{H} \setminus \{\widetilde{x}\}$ and for all $|\beta| \le 5$,

$$q_3(\widetilde{x}) = q_4(\widetilde{x}) = \partial_x^\beta q_3(x) = \partial_x^\beta q_4(x) = 0. \tag{3.38}$$

We note that in order to have (3.38), the pair (V_0, W_0) should have zeros of sufficiently large orders on $\mathcal{H}\setminus\{\widetilde{x}\}$. The latter can be achieved by choosing the function $\boldsymbol{b}$ such that it has zeros of sufficiently large orders on $\mathcal{H}\setminus\{\widetilde{x}\}$.

By (3.2) the functions $\frac{q_3}{2\Phi'}, \frac{q_4}{2\overline{\Phi}'}$ belong to the space $C^{5+\alpha}(\overline{\Gamma}_0)$. Therefore we can introduce the functions V_{-1}, $W_{-1} \in C^{5+\alpha}(\overline{\Omega})$ as a solution to the following boundary value problem:

$$\mathcal{M}(x,D)(V_{-1},W_{-1}) = 0 \quad \text{in } \Omega, \quad (V_{-1}+W_{-1})|_{\Gamma_0} = -\Big(\frac{q_3}{2\Phi'} + \frac{q_4}{2\overline{\Phi}'}\Big). \tag{3.39}$$

Let $p_3 = Q_1(2)(\frac{e_1q_3}{2\Phi'} + W_{-1}) + L_2(x,D)^*(\frac{q_3e_2}{2\Phi'})$, $p_4 = Q_2(2)(\frac{e_1q_4}{2\overline{\Phi}'} + V_{-1}) + L_2(x,D)^*(\frac{q_4e_2}{2\overline{\Phi}'})$ and $\widetilde{q}_4 = (S_{-B_2^*}p_4 - \widetilde{M}_3)$, $\widetilde{q}_3 = (P_{-A_2^*}p_3 - \widetilde{M}_4) \in C^{5+\alpha}(\overline{\Omega})$, where $\widetilde{M}_3 \in \mathrm{Ker}(2\partial_{\bar z} - B_2^*)$ and $\widetilde{M}_4 \in \mathrm{Ker}(2\partial_z - A_2^*)$, and $(\widetilde{q}_3, \widetilde{q}_4)$ are chosen such that

$$\partial_x^\beta \widetilde{q}_3(x) = \partial_x^\beta \widetilde{q}_4(x) = 0, \quad \forall\, x \in \mathcal{H} \ \text{ and } \ \forall\, |\beta| \le 5. \tag{3.40}$$

By Proposition 3.2, there exist smooth functions $\widetilde{m}_{\pm,\widetilde{x}} \in C^{2+\alpha}(\overline{G}_\epsilon)$, $\widetilde{x} \in \mathcal{H}$, independent of τ such that

$$\widetilde{\mathcal{R}}_{-\tau,-B_2^*}(e_1(q_3 + \widetilde{q}_3/\tau))|_{\bar G_\epsilon} = \frac{\widetilde{m}_{+,\widetilde{x}} e^{2i\tau(\psi - \psi(\widetilde{x}))}}{\tau^2} + e^{2i\tau\psi} o_{C^2(\overline{G_\epsilon})}\Big(\frac{1}{\tau^2}\Big) \quad \text{as } |\tau| \to +\infty \tag{3.41}$$

and

$$\mathcal{R}_{-\tau,-A_2^*}(e_1(q_4 + \widetilde{q}_4/\tau))|_{\bar G_\epsilon} = \frac{\widetilde{m}_{-,\widetilde{x}} e^{-2i\tau(\psi - \psi(\widetilde{x}))}}{\tau^2} + e^{-2i\tau\psi} o_{C^2(\overline{G_\epsilon})}\Big(\frac{1}{\tau^2}\Big) \quad \text{as } |\tau| \to +\infty. \tag{3.42}$$

Using the functions $\widetilde{m}_{\pm,\widetilde{x}}$, we introduce functions $\widetilde{a}_{\pm,\widetilde{x}}, \widetilde{b}_{\pm,\widetilde{x}} \in C^{2+\alpha}(\overline{\Omega})$ that solve the boundary value problem

$$\mathcal{M}(x,D)(\widetilde{a}_{\pm,\widetilde{x}}, \widetilde{b}_{\pm,\widetilde{x}}) = 0 \quad \text{in } \Omega, \qquad (\widetilde{a}_{\pm,\widetilde{x}} + \widetilde{b}_{\pm,\widetilde{x}})|_{\Gamma_0} = \widetilde{m}_{\pm,\widetilde{x}}. \tag{3.43}$$

We choose $\widetilde{a}_{\pm,\widetilde{x}}, \widetilde{b}_{\pm,\widetilde{x}}$ in the form

$$(\widetilde{a}_{\pm,\widetilde{x}}, \widetilde{b}_{\pm,\widetilde{x}}) = (\mathcal{C}_2(x)\widetilde{\boldsymbol{a}}_{\pm,\widetilde{x}}(\bar z), \mathcal{P}_2(x)\widetilde{\boldsymbol{b}}_{\pm,\widetilde{x}}(z)), \tag{3.44}$$

where $\boldsymbol{a}_{\pm,\widetilde{x}}(\bar z)$ is some antiholomorphic function and $\boldsymbol{b}_{\pm,\widetilde{x}}(z)$ is some holomorphic function. By (3.2) the functions $\frac{\widetilde{q}_3}{2\Phi'}, \frac{\widetilde{q}_4}{2\overline{\Phi}'}$ belong to the space $C^{5+\alpha}(\overline{\Gamma}_0)$. Therefore there exists a pair $(V_{-2}, W_{-2}) \in C^{5+\alpha}(\bar\Omega) \times C^{5+\alpha}(\bar\Omega)$ that solves the boundary value problem

$$\mathcal{M}(x,D)(V_{-2},W_{-2}) = 0 \quad \text{in } \Omega, \qquad (V_{-2}+W_{-2})|_{\Gamma_0} = -\Big(\frac{\widetilde{q}_3}{2\Phi'} + \frac{\widetilde{q}_4}{2\overline{\Phi}'}\Big). \tag{3.45}$$

We introduce functions $V_{0,\tau}$, $W_{0,\tau} \in C^{2+\alpha}(\bar\Omega)$ by

$$W_{0,\tau} = W_0 + \frac{W_{-1} + e_2 q_3/2\Phi'}{\tau} + \frac{1}{\tau^2}\left(e^{2i\tau\psi(\widetilde{x})}\widetilde{b}_{+,\widetilde{x}} + e^{-2i\tau\psi(\widetilde{x})}\widetilde{b}_{-,\widetilde{x}} + W_{-2} + \frac{e_2\widetilde{q}_3}{2\Phi'}\right) \tag{3.46}$$

and

$$V_{0,\tau} = V_0 + \frac{V_{-1} + e_2 q_4/2\overline{\Phi}'}{\tau} + \frac{1}{\tau^2}\left(e^{2i\tau\psi(\widetilde{x})}\widetilde{a}_{+,\widetilde{x}} + e^{-2i\tau\psi(\widetilde{x})}\widetilde{a}_{-,\widetilde{x}} + V_{-2} + \frac{e_2\widetilde{q}_4}{2\bar\Phi'}\right). \tag{3.47}$$

The last term v_{-1} in the complex geometric optics solution satisfies the estimate

$$\sqrt{|\tau|}\|v_{-1}\|_{L^2(\Omega)} + \frac{1}{\sqrt{|\tau|}}\|\nabla v_{-1}\|_{L^2(\Omega)} + \|v_{-1}\|_{W_2^{1,\tau}(\mathcal{O}_\epsilon)} = o\Big(\frac{1}{\tau}\Big) \quad \text{as } |\tau| \to +\infty \tag{3.48}$$

and the function

$$v = V_{0,\tau}e^{-\tau\bar\Phi} + W_{0,\tau}e^{-\tau\Phi} - e^{-\tau\Phi}\widetilde{\mathcal{R}}_{-\tau,-B_2^*}\left(e_1\left(q_3 + \widetilde{q}_3/\tau\right)\right) - e^{-\tau\overline{\Phi}}\mathcal{R}_{-\tau,-A_2^*}\left(e_1\left(q_4 + \widetilde{q}_4/\tau\right)\right) + v_{-1}e^{-\tau\varphi} \tag{3.49}$$

solves the boundary value problem

$$L_2(x,D)^*v = 0 \quad \text{in } \Omega, \qquad v|_{\Gamma_0} = 0. \tag{3.50}$$

We close this section with one technical proposition that can be proved similarly to [7, Propositions 5.3 and 5.4]:

Proposition 3.3. *Suppose that the matrices $\mathcal{C}_j$, $\mathcal{P}_j \in C^{6+\alpha}(\bar\Omega)$, $j = 1,2$ with some $\alpha \in (0,1)$ are given by (3.8)–(3.10), (3.34)–(3.36) and satisfy*

$$\int_{\partial\Omega}\left\{(\nu_1 + i\nu_2)\Phi'(\mathcal{P}_1\boldsymbol{a}, \mathcal{P}_2\boldsymbol{b}) + (\nu_1 - i\nu_2)\bar\Phi'(\mathcal{C}_1\bar{\boldsymbol{a}}, \mathcal{C}_2\bar{\boldsymbol{b}})\right\} d\sigma = 0 \tag{3.51}$$

for all the holomorphic vector-valued functions $\boldsymbol{a}, \boldsymbol{b}$ such that $\operatorname{Im}\boldsymbol{a}|_{\Gamma_0} = \operatorname{Im}\boldsymbol{b}|_{\Gamma_0} = 0$. Then there exist a holomorphic function $\Theta \in W_2^{1/2}(\Omega)$ and an antiholomorphic function $\widetilde\Theta \in W_2^{1/2}(\Omega)$ such that

$$\widetilde\Theta|_{\widetilde\Gamma} = \mathcal{C}_2^*\mathcal{C}_1, \qquad \Theta|_{\widetilde\Gamma} = \mathcal{P}_2^*\mathcal{P}_1 \tag{3.52}$$

and

$$\Theta = \widetilde\Theta \quad \text{on } \Gamma_0. \tag{3.53}$$

Proof. First we show that for arbitrary holomorphic vector-valued functions $\boldsymbol{a}, \boldsymbol{b}$ satisfying $\operatorname{Im} \boldsymbol{a}|_{\Gamma_0} = \operatorname{Im} \boldsymbol{b}|_{\Gamma_0} = 0$, there exist a holomorphic function $\widetilde{\Psi}$ and an antiholomorphic function Ψ such that

$$(3.54)\qquad \begin{aligned} (\bar{\Phi}'(\mathcal{C}_1\bar{\boldsymbol{a}}, \mathcal{C}_2\bar{\boldsymbol{b}}) - \Psi)|_{\Gamma_0} &= (\Phi'(\mathcal{P}_1\boldsymbol{a}, \mathcal{P}_2\boldsymbol{b}) - \widetilde{\Psi})|_{\Gamma_0} = 0 \\ \text{and} \quad ((\nu_1 - i\nu_2)\Psi &+ (\nu_1 + i\nu_2)\widetilde{\Psi})|_{\Gamma_0} = 0. \end{aligned}$$

Observe that equality (3.51) implies

$$(3.55)\qquad \mathcal{I} = \int_{\partial\Omega} \{(\nu_1 + i\nu_2)\Phi'(\mathcal{P}_1\boldsymbol{a}, \mathcal{P}_2\boldsymbol{b}) + (\nu_1 - i\nu_2)\bar{\Phi}'(\mathcal{C}_1(-\bar{\boldsymbol{a}}), \mathcal{C}_2\bar{\boldsymbol{b}})\}\, d\sigma = 0,$$

for arbitrary holomorphic vector-valued functions $\boldsymbol{a}$, $\boldsymbol{b}$ satisfying $\operatorname{Re} \boldsymbol{a}|_{\Gamma_0} = \operatorname{Im} \boldsymbol{b}|_{\Gamma_0} = 0$. Indeed,

$$\begin{aligned} \mathcal{I} &= \frac{1}{i} \int_{\partial\Omega} \{(\nu_1 + i\nu_2)\Phi'(\mathcal{P}_1 i\boldsymbol{a}, \mathcal{P}_2\boldsymbol{b}) + (\nu_1 - i\nu_2)\bar{\Phi}'(\mathcal{C}_1(-i\bar{\boldsymbol{a}}), \mathcal{C}_2\bar{\boldsymbol{b}})\}\, d\sigma \\ &= \frac{1}{i} \int_{\partial\Omega} \{(\nu_1 + i\nu_2)\Phi'(\mathcal{P}_1 i\boldsymbol{a}, \mathcal{P}_2\boldsymbol{b}) + (\nu_1 - i\nu_2)\bar{\Phi}'(\mathcal{C}_1(\overline{i\boldsymbol{a}}), \mathcal{C}_2\bar{\boldsymbol{b}})\}\, d\sigma = 0. \end{aligned}$$

Here, in order to obtain the last equality, we used (3.51). In order to prove equalities (3.54), consider the extremal problem

$$(3.56)\qquad J(\Psi, \widetilde{\Psi}) = \|\bar{\Phi}'(\mathcal{C}_1\bar{\boldsymbol{a}}, \mathcal{C}_2\bar{\boldsymbol{b}}) - \Psi\|^2_{L^2(\widetilde{\Gamma})} + \|\Phi'(\mathcal{P}_1\boldsymbol{a}, \mathcal{P}_2\boldsymbol{b}) - \widetilde{\Psi}\|^2_{L^2(\widetilde{\Gamma})} \to \inf,$$

where

$$(3.57)\qquad \frac{\partial \Psi}{\partial z} = 0 \quad \text{in } \Omega, \qquad \frac{\partial \widetilde{\Psi}}{\partial \bar{z}} = 0 \quad \text{in } \Omega, \qquad ((\nu_1 - i\nu_2)\Psi + (\nu_1 + i\nu_2)\widetilde{\Psi})|_{\Gamma_0} = 0.$$

Denote a unique solution to this extremal problem (3.56) and (3.57) by $(\widehat{\Psi}, \widehat{\widetilde{\Psi}}) \in W_2^{1/2}(\Omega) \times W_2^{1/2}(\Omega)$. Applying the Fermat theorem, we obtain

$$(3.58)\qquad \operatorname{Re}(\Phi'(\mathcal{P}_1\boldsymbol{a}, \mathcal{P}_2\boldsymbol{b}) - \widehat{\widetilde{\Psi}}, \delta)_{L^2(\widetilde{\Gamma})} + \operatorname{Re}(\bar{\Phi}'(\mathcal{C}_1\bar{\boldsymbol{a}}, \mathcal{C}_2\bar{\boldsymbol{b}}) - \widehat{\Psi}, \widetilde{\delta})_{L^2(\widetilde{\Gamma})} = 0$$

for any δ, $\widetilde{\delta}$ from $W_2^{1/2}(\Omega)$ such that

$$(3.59)\qquad \frac{\partial \delta}{\partial \bar{z}} = 0 \quad \text{in } \Omega, \qquad \frac{\partial \widetilde{\delta}}{\partial z} = 0 \quad \text{in } \Omega, \qquad ((\nu_1 + i\nu_2)\delta + (\nu_1 - i\nu_2)\widetilde{\delta})|_{\Gamma_0} = 0,$$

and there exist two functions P, $\widetilde{P} \in W_2^{1/2}(\Omega)$ such that

$$(3.60)\qquad \frac{\partial P}{\partial \bar{z}} = 0 \quad \text{in } \Omega, \qquad \frac{\partial \widetilde{P}}{\partial z} = 0 \quad \text{in } \Omega,$$

$$(3.61)\qquad \begin{aligned} (\nu_1 + i\nu_2)P &= \overline{\Phi'(\mathcal{P}_1\boldsymbol{a}, \mathcal{P}_2\boldsymbol{b}) - \widehat{\widetilde{\Psi}}} \quad \text{on } \widetilde{\Gamma}, \\ (\nu_1 - i\nu_2)\widetilde{P} &= \overline{\bar{\Phi}'(\mathcal{C}_1\bar{\boldsymbol{a}}, \mathcal{C}_2\bar{\boldsymbol{b}}) - \widehat{\Psi}} \quad \text{on } \widetilde{\Gamma} \end{aligned}$$

and

$$(P - \widetilde{P})|_{\Gamma_0} = 0. \tag{3.62}$$

Denote $\Psi_0(z) = \frac{1}{2i}(P(z) - \overline{\widetilde{P}(\overline{z})})$ and $\Phi_0(z) = \frac{1}{2}(P(z) + \overline{\widetilde{P}(\overline{z})})$. Equality (3.62) yields

$$\operatorname{Im} \Psi_0|_{\Gamma_0} = \operatorname{Im} \Phi_0|_{\Gamma_0} = 0. \tag{3.63}$$

Hence

$$P = (\Phi_0 + i\Psi_0), \qquad \overline{\widetilde{P}} = (\Phi_0 - i\Psi_0). \tag{3.64}$$

From (3.58), taking $\widetilde{\delta} = \widehat{\Psi}$ and $\delta = \widehat{\widetilde{\Psi}}$, we have

$$\operatorname{Re} \int_{\widetilde{\Gamma}} ((\bar{\Phi}'(\mathcal{C}_1\bar{\boldsymbol{a}}, \mathcal{C}_2\bar{\boldsymbol{b}}) - \widehat{\Psi})\overline{\widetilde{\Psi}} + (\Phi'(\mathcal{P}_1\boldsymbol{a}, \mathcal{P}_2\boldsymbol{b}) - \widehat{\widetilde{\Psi}})\overline{\widetilde{\Psi}})\, d\sigma = 0. \tag{3.65}$$

By (3.60), (3.61) and (3.64), we have

$$\begin{aligned}
\mathbf{H} &= \operatorname{Re} \int_{\widetilde{\Gamma}} ((\overline{\Phi'(\mathcal{P}_1\boldsymbol{a}, \mathcal{P}_2\boldsymbol{b}) - \widehat{\widetilde{\Psi}}})\Phi'(\mathcal{P}_1\boldsymbol{a}, \mathcal{P}_2\boldsymbol{b}) + (\overline{\bar{\Phi}'(\mathcal{C}_1\bar{\boldsymbol{a}}, \mathcal{C}_2\bar{\boldsymbol{b}}) - \widehat{\Psi}})\bar{\Phi}'(\mathcal{C}_1\bar{\boldsymbol{a}}, \mathcal{C}_2\bar{\boldsymbol{b}}))\, d\sigma \\
&= \operatorname{Re} \int_{\widetilde{\Gamma}} ((\nu_1 + i\nu_2)P\Phi'(\mathcal{P}_1\boldsymbol{a}, \mathcal{P}_2\boldsymbol{b}) + (\nu_1 - i\nu_2)\widetilde{P}\bar{\Phi}'(\mathcal{C}_1\bar{\boldsymbol{a}}, \mathcal{C}_2\bar{\boldsymbol{b}}))\, d\sigma \\
&= \operatorname{Re} \int_{\widetilde{\Gamma}} ((\nu_1 + i\nu_2)(\Phi_0 + i\Psi_0)\Phi'(\mathcal{P}_1\boldsymbol{a}, \mathcal{P}_2\boldsymbol{b}) \\
&\quad + (\nu_1 - i\nu_2)(\bar{\Phi}_0 - \overline{i\Psi_0})\bar{\Phi}'(\mathcal{C}_1\bar{\boldsymbol{a}}, \mathcal{C}_2\bar{\boldsymbol{b}}))\, d\sigma.
\end{aligned}$$

By (3.51) and (3.63), we have

$$\operatorname{Re} \int_{\widetilde{\Gamma}} \{((\nu_1 + i\nu_2)\Phi'(\mathcal{P}_1(\Phi_0\boldsymbol{a}), \mathcal{P}_2\boldsymbol{b})) + ((\nu_1 - i\nu_2)\bar{\Phi}'(\mathcal{C}_1\overline{(\Phi_0\boldsymbol{a})}, \mathcal{C}_2\bar{\boldsymbol{b}}))\}\, d\sigma = 0. \tag{3.66}$$

By (3.55) and (3.63), we obtain

$$\begin{aligned}
\operatorname{Re} \int_{\widetilde{\Gamma}} &\{((\nu_1 + i\nu_2)\Phi'(\mathcal{P}_1(i\Psi_0\boldsymbol{a}), \mathcal{P}_2\boldsymbol{b})) \\
&+ \operatorname{Re}((\nu_1 - i\nu_2)\bar{\Phi}'(\mathcal{C}_1\overline{(-i\Psi_0\boldsymbol{a})}, \mathcal{C}_2\bar{\boldsymbol{b}}))\}\, d\sigma = 0.
\end{aligned} \tag{3.67}$$

Then by (3.66), (3.67) and (3.65), we see that $\mathbf{H} = 0$. Taking (3.65) into account, we obtain that $J(\widehat{\Psi}, \widehat{\widetilde{\Psi}}) = 0$. Hence

$$\begin{aligned}
(\mathcal{P}_1\boldsymbol{a}, \mathcal{P}_2\boldsymbol{b})(x) &= (\hat{\tilde{\Psi}}/\Phi')(z) =: \widetilde{\Xi}(z), \\
(\mathcal{C}_1\bar{\boldsymbol{a}}, \mathcal{C}_2\bar{\boldsymbol{b}})(x) &= (\hat{\Psi}/\bar{\Phi}')(\bar{z}) =: \Xi(\bar{z}) \quad \text{on } \widetilde{\Gamma} \setminus \mathcal{H}.
\end{aligned} \tag{3.68}$$

In general, the function Φ may have a finite number of zeros on $\bar\Omega$. At these zeros the functions Ξ, $\widetilde{\Xi}$ may have singularities. On the other hand, observe that Ξ, $\widetilde{\Xi}$ are independent of a particular choice of the function Φ. Making small perturbations of these functions, we can shift the position of the zeros of the function Φ'. Hence there are no poles for the functions Ξ and $\widetilde{\Xi}$. By (3.57), we have $((\nu_1-i\nu_2)\hat\Psi+(\nu_1+i\nu_2)\hat{\widetilde{\Psi}})|_{\Gamma_0}=0$. Next, using the assumption $\operatorname{Im}\Phi|_{\Gamma_0}=0$, by direct computations, we have $((\nu_1+i\nu_2)\Phi'+(\nu_1-i\nu_2)\bar\Phi')|_{\Gamma_0}=0$. Therefore

$$\widetilde{\Xi}(z)=\Xi(\bar z)\quad\text{on }\Gamma_0. \tag{3.69}$$

Consider N holomorphic vector-valued functions $\boldsymbol{b}_j=(b_{j,1},\dots,b_{j,N})$ such that $\operatorname{Im}\boldsymbol{b}_j|_{\Gamma_0}=0$ and the determinant of the square matrix $[\boldsymbol{b}_1,\dots,\boldsymbol{b}_N]$ is not equal to zero at least at one point of domain Ω. The equality (3.68) can be written as

$$(\mathcal{P}_2^*\mathcal{P}_1\boldsymbol{a},\boldsymbol{b}_j)=\widetilde{\Xi}_j(z)\quad\text{and}\quad(\mathcal{C}_2^*\mathcal{C}_1\bar{\boldsymbol{a}},\bar{\boldsymbol{b}}_j)=\Xi_j(\bar z)\quad\text{on }\widetilde{\Gamma}.$$

Then

$$\mathcal{P}_2^*\mathcal{P}_1\boldsymbol{a}=\mathbf{B}^{-1}\vec{\widetilde{\Xi}}\quad\text{and}\quad\mathcal{C}_2^*\mathcal{C}_1\bar{\boldsymbol{a}}=\bar{\mathbf{B}}^{-1}\vec{\Xi}\quad\text{on }\widetilde{\Gamma}.$$

Here $\mathbf{B}$ is the matrix such that the jth row equals $\boldsymbol{b}_j^t$ and $\vec{\widetilde{\Xi}}(z)=(\widetilde{\Xi}_1(z),\dots,\widetilde{\Xi}_N(z))$, $\vec{\Xi}=(\Xi_1(\bar z),\dots,\Xi_N(\bar z))$. Consider N holomorphic vector-valued functions $\boldsymbol{a}_j$ such that $\operatorname{Im}\boldsymbol{a}_j|_{\Gamma_0}=0$. Then

$$\mathcal{P}_2^*\mathcal{P}_1\boldsymbol{a}_j=\mathbf{B}^{-1}\vec{\widetilde{\Xi}}_j\quad\text{and}\quad\mathcal{C}_2^*\mathcal{C}_1\bar{\boldsymbol{a}}_j=\bar{\mathbf{B}}^{-1}\vec{\Xi}_j\quad\text{on }\widetilde{\Gamma}.$$

From this equality, we have

$$\mathcal{P}_2^*\mathcal{P}_1=\mathbf{B}^{-1}\widetilde{\Pi}\mathbf{A}^{-1}\quad\text{and}\quad\mathcal{C}_2^*\mathcal{C}_1=\bar{\mathbf{B}}^{-1}\Pi\bar{\mathbf{A}}^{-1}\quad\text{on }\widetilde{\Gamma}.$$

Here $\mathbf{A}$, $\widetilde{\Pi}$, Π are the matrices such that the jth rows equal $\boldsymbol{a}_j$, $\vec{\Xi}_j$ and $\vec{\widetilde{\Xi}}_j$ respectively. We set

$$\Theta=\mathbf{B}^{-1}\Pi\mathbf{A}^{-1}\quad\text{and}\quad\widetilde{\Theta}=\bar{\mathbf{B}}^{-1}\widetilde{\Pi}\bar{\mathbf{A}}^{-1}.$$

These formulae define the functions Θ, $\widetilde{\Theta}$ correctly except at the points where determinants of the matrices $\mathbf{A}$ and $\mathbf{B}$ are equal to zero. On the other hand, it is obvious that the functions Θ, $\widetilde{\Theta}$ are independent of choices of the matrices $\mathbf{A}$, $\mathbf{B}$. Hence, if we assume that there exists a point of singularity of, say, the function Θ by Proposition 2.1, then we can make a choice of the matrices $\mathbf{A}$, $\mathbf{B}$ such that the determinants of these matrices are not equal to zero at this point and reach a contradiction. The equality (3.53) follows from (3.69) and the fact that $\operatorname{Im}\mathbf{B}|_{\Gamma_0}=\operatorname{Im}\mathbf{A}|_{\Gamma_0}=0$. Indeed,

$$\mathcal{P}_2^*\mathcal{P}_1=\mathbf{B}^{-1}\Pi\mathbf{A}^{-1}=\bar{\mathbf{B}}^{-1}\Pi\bar{\mathbf{A}}^{-1}=\bar{\mathbf{B}}^{-1}\widetilde{\Pi}\bar{\mathbf{A}}^{-1}=\mathcal{C}_2^*\mathcal{C}_1\quad\text{on }\Gamma_0.$$

The proof of the proposition is complete. □

Let u_1 be the complex geometric optics solution given by (3.28) constructed for the operator $L_1(x,D)$. Since the Dirichlet-to-Neumann maps for the operators $L_1(x,D)$ and $L_2(x,D)$ are equal, there exists a solution u_2 to the following boundary value problem:

$$L_2(x,D)u_2 = 0 \quad \text{in } \Omega, \qquad (u_1-u_2)|_{\partial\Omega} = 0, \qquad \partial_{\vec{\nu}}(u_1-u_2) = 0 \quad \text{on } \widetilde{\Gamma}.$$

Setting $u = u_1 - u_2$, $\mathcal{A} = A_1 - A_2$, $\mathcal{B} = B_1 - B_2$ and $\mathcal{Q} = Q_1 - Q_2$, we have

$$L_2(x,D)u + 2\mathcal{A}\partial_z u_1 + 2\mathcal{B}\partial_{\overline{z}} u_1 + \mathcal{Q}u_1 = 0 \quad \text{in } \Omega \tag{3.70}$$

and

$$u|_{\partial\Omega} = 0, \qquad \partial_{\vec{\nu}} u|_{\widetilde{\Gamma}} = 0. \tag{3.71}$$

Let v be the function given by (3.49). Taking the scalar product of (3.70) with v in $L^2(\Omega)$ and using (3.50) and (3.71), we obtain

$$0 = \int_\Omega (2\mathcal{A}\partial_z u_1 + 2\mathcal{B}\partial_{\overline{z}} u_1 + \mathcal{Q}u_1, v)\,dx. \tag{3.72}$$

Denote

$$\begin{aligned} V = V_{0,\tau}e^{-\tau\bar{\Phi}} + W_{0,\tau}e^{-\tau\Phi} - e^{-\tau\Phi}\widetilde{\mathcal{R}}_{-\tau,-B_2^*}(e_1(q_3+\widetilde{q}_3/\tau)) \\ - e^{-\tau\overline{\Phi}}\mathcal{R}_{-\tau,-A_2^*}(e_1(q_4+\widetilde{q}_4/\tau)) \end{aligned} \tag{3.73}$$

and

$$\begin{aligned} U = U_{0,\tau}e^{\tau\Phi} + T_{0,\tau}e^{\tau\overline{\Phi}} - e^{\tau\Phi}\widetilde{\mathcal{R}}_{\tau,B_1}(e_1(q_1+\widetilde{q}_1/\tau)) \\ - e^{\tau\overline{\Phi}}\mathcal{R}_{\tau,A_1}(e_1(q_2+\widetilde{q}_2/\tau)). \end{aligned} \tag{3.74}$$

We have

Proposition 3.4. *Let u_1 be given by (3.28) and v be given by (3.49). Then the following asymptotics holds true:*

$$\int_\Omega (2\mathcal{A}\partial_z u_1 + 2\mathcal{B}\partial_{\overline{z}} u_1 + \mathcal{Q}u_1, v)\,dx = \int_\Omega (2\mathcal{A}\partial_z U + 2\mathcal{B}\partial_{\overline{z}} U + \mathcal{Q}U, V)\,dx + o\Big(\frac{1}{\tau}\Big)$$
$$\text{as } \tau \to +\infty,$$

where the functions U, V are determined by (3.74) and (3.73).

The proof of Proposition 3.4 is exactly the same as the proof of [6, Proposition 5.1].

Conditions (3.15), (3.16) and (3.38) may impose some restrictions on the pairs (U_0, V_0) and (T_0, W_0) and this will be inconvenient for us, since in the next section we shall try to establish the identity (3.51). However we can argue as follows. We set

$$u_1 = U_0 e^{\tau\Phi} + V_0 e^{\tau\bar{\Phi}} + u_{\mathrm{cor}} e^{\tau\varphi}, \qquad v = T_0 e^{-\tau\Phi} + W_0 e^{-\tau\bar{\Phi}} + v_{\mathrm{cor}} e^{-\tau\varphi}, \tag{3.75}$$

where

$$\|u_{\mathrm{cor}}\|_{W_2^{1,\tau}(\Omega)} + \|v_{\mathrm{cor}}\|_{W_2^{1,\tau}(\Omega)} \le C. \tag{3.76}$$

From (3.75) and (3.76), we have

$$\begin{aligned}&\int_\Omega (2\mathcal{A}\partial_z u_1 + 2\mathcal{B}\partial_{\bar z} u_1 + \mathcal{Q}u_1, v)\, dx \\ &\quad = \int_\Omega ((2\mathcal{A}\partial_z + 2\mathcal{B}\partial_{\bar z} + \mathcal{Q})(U_0 e^{\tau\Phi} + V_0 e^{\tau\bar\Phi}), T_0 e^{-\tau\Phi} + W_0 e^{-\tau\bar\Phi})\, dx + o(\tau) \\ &\quad \text{as } \tau \to +\infty.\end{aligned}$$

This equality and short computations immediately imply (3.51).

§4. Step 2: Asymptotics

We introduce the functionals

$$\begin{aligned}\mathfrak{F}_{\tau,\widetilde{x}} u = &\frac{\pi}{2|\det \psi''(\widetilde{x})|^{1/2}} \\ &\times \left(\frac{u(\widetilde{x})}{\tau} - \frac{\partial^2_{zz} u(\widetilde{x})}{2\Phi''(\widetilde{x})\tau^2} + \frac{\partial^2_{\bar z\bar z} u(\widetilde{x})}{2\bar\Phi''(\widetilde{x})\tau^2} + \frac{\partial_z u(\widetilde{x})\Phi'''(\widetilde{x})}{2(\Phi''(\widetilde{x}))^2\tau^2} - \frac{\partial_{\bar z} u(\widetilde{x})\bar\Phi'''(\widetilde{x})}{2(\bar\Phi''(\widetilde{x}))^2\tau^2} \right)\end{aligned}$$

and

$$\mathfrak{I}_\tau u = \int_{\partial\Omega} u \frac{\nu_1 - i\nu_2}{2\tau\Phi'} e^{\tau(\Phi - \bar\Phi)}\, d\sigma - \int_{\partial\Omega} \frac{\nu_1 - i\nu_2}{\Phi'} \partial_z \left(\frac{u}{2\tau^2\Phi'} \right) e^{\tau(\Phi-\bar\Phi)}\, d\sigma.$$

Using this notation and the fact that Φ is a harmonic function, we rewrite the classical result of [5, Theorem 7.7.5] as

Proposition 4.1. *Let $\Phi(z)$ satisfy (3.1), (3.2) and $u \in C^{5+\alpha}(\bar\Omega)$, $\alpha \in (0,1)$ be some function that has zeros of order 5 on the set $\mathcal{H} \cap \partial\Omega$. Then the following asymptotic formula is true:*

$$\int_\Omega u e^{\tau(\Phi - \bar\Phi)}\, dx = \sum_{\widetilde{y} \in \mathcal{H}} e^{2i\tau\psi(\widetilde{y})} \mathfrak{F}_{\tau,\widetilde{y}} u + \mathfrak{I}_\tau u + o\Big(\frac{1}{\tau^2}\Big) \quad \text{as } \tau \to +\infty. \tag{4.1}$$

Denote

$$\mathbf{H}(x,\partial_z,\partial_{\overline{z}}) = 2\mathcal{A}\partial_z + 2\mathcal{B}\partial_{\bar z} + \mathcal{Q} \quad\text{and}\quad \mathcal{J}_\tau = \int_\Omega (\mathbf{H}(x,\partial_z,\partial_{\overline{z}})U,V)\,dx,$$

where U and V are given by (3.74) and (3.73) respectively. We have

Proposition 4.2. *The following asymptotic formula holds true:*

$$\begin{aligned}(4.2)\qquad 0 = {} & \sum_{k=-1}^{1}\tau^k J_k + \frac{1}{\tau}\Big((J_+ + I_{+,\Phi} + K_+)(\widetilde{x})e^{2\tau i\psi(\widetilde{x})} \\ & + (J_- + I_{-,\Phi} + K_-)(\widetilde{x})e^{-2\tau i\psi(\widetilde{x})}\Big) \\ & + \int_{\widetilde{\Gamma}}((\nu_1 - i\nu_2)(\mathcal{A}U_0e^{\tau\Phi}, V_0e^{-\tau\bar\Phi}) + (\nu_1 + i\nu_2)(\mathcal{B}T_0e^{\tau\bar\Phi}, W_0e^{-\tau\Phi}))\,d\sigma \\ & + o\Big(\frac{1}{\tau}\Big) \quad \text{as } \tau\to+\infty,\end{aligned}$$

where J_{-1} and J_0 are independent of τ, and

$$(4.3)\qquad J_1 = \int_{\partial\Omega}((\nu_1 - i\nu_2)\bar\Phi'(T_0,V_0) + (\nu_1+i\nu_2)\Phi'(U_0,W_0))\,d\sigma,$$

$$\begin{aligned}(4.4)\qquad J_+(\widetilde{x}) = {} & \frac{\pi}{2|\det\psi''(\widetilde{x})|^{1/2}}\big(-(2\partial_z\mathcal{A}U_0,V_0) - (\mathcal{A}U_0, A_2^*V_0) \\ & - (\mathcal{B}A_1U_0,V_0) + (\mathcal{Q}U_0,V_0)\big)(\widetilde{x}),\end{aligned}$$

$$\begin{aligned}(4.5)\qquad J_-(\widetilde{x}) = {} & \frac{\pi}{2|\det\psi''(\widetilde{x})|^{1/2}}\big(-(\mathcal{A}B_1T_0,W_0) - (2\partial_{\bar z}\mathcal{B}T_0,W_0) \\ & - (\mathcal{B}T_0, B_2^*W_0) + (\mathcal{Q}T_0,W_0)\big)(\widetilde{x}),\end{aligned}$$

$$\begin{aligned}(4.6)\qquad I_{\pm,\Phi}(\widetilde{x}) = {} & -\int_{\partial\Omega}\Big\{(\nu_1 - i\nu_2)((2b_{\pm,\widetilde{x}}\bar\Phi',V_0) + (2\bar\Phi'T_0,\widetilde{a}_{\pm,\widetilde{x}})) \\ & + (\nu_1+i\nu_2)((2a_{\pm,\widetilde{x}}\Phi',W_0) + (2\Phi'U_0,\widetilde{b}_{\pm,\widetilde{x}}))\Big\}\,d\sigma,\end{aligned}$$

$$\begin{aligned}(4.7)\qquad K_+ = {} & \tau\mathfrak{F}_{\tau,\widetilde{x}}(q_1, T_{B_1}^*(B_1^*\mathcal{A}^*V_0) - \mathcal{A}^*V_0 + 2T_{B_1}^*(\partial_z\mathcal{B}^*V_0) \\ & + T_{B_1}^*(\mathcal{B}^*(A_2^*V_0 - 2\tau\bar\Phi'V_0))) \\ & - 2\tau\mathfrak{F}_{\tau,\widetilde{x}}(P_{-A_2^*}^*(\mathcal{A}(\partial_zU_0 + \tau\Phi'U_0) + \mathcal{B}\partial_{\bar z}U_0), q_4),\end{aligned}$$

$$\begin{aligned}(4.8)\qquad K_- = {} & \tau\mathfrak{F}_{-\tau,\widetilde{x}}(q_2, P_{A_1}^*(2\partial_z(\mathcal{A}^*W_0) - 2\tau\Phi'\mathcal{A}^*W_0) - \mathcal{B}^*W_0 \\ & + P_{A_1}^*(A_1^*\mathcal{B}^*W_0)) \\ & - 2\tau\mathfrak{F}_{-\tau,\widetilde{x}}(q_3, T_{-B_2^*}^*(\mathcal{A}\partial_zT_0 + \mathcal{B}(\partial_{\bar z}T_0 + \tau\bar\Phi'T_0))).\end{aligned}$$

Proof. By Proposition 3.4, the following asymptotic formula holds true:

$$\mathcal{J}_\tau = \int_\Omega(\mathbf{H}(x,\partial_z,\partial_{\overline{z}})U,V)\,dx = o\Big(\frac{1}{\tau}\Big) \quad \text{as } \tau\to+\infty.$$

Integrating by parts and using Proposition 4.1, we obtain

$$
\begin{aligned}
\mathcal{M}_1 &= \int_\Omega (2\mathcal{A}\partial_z(U_{0,\tau}e^{\tau\Phi}) + 2\mathcal{B}\partial_{\bar z}(U_{0,\tau}e^{\tau\Phi}), V_{0,\tau}e^{-\tau\bar\Phi})\,dx \\
&= \int_\Omega \Big((-2\partial_z\mathcal{A}U_{0,\tau}e^{\tau\Phi}, V_{0,\tau}e^{-\tau\bar\Phi}) - (2\mathcal{A}U_{0,\tau}e^{\tau\Phi}, \partial_z V_{0,\tau}e^{-\tau\bar\Phi}) \\
&\quad + (2\mathcal{B}\partial_{\bar z}U_{0,\tau}e^{\tau\Phi}, V_{0,\tau}e^{-\tau\bar\Phi})\Big)\,dx \\
&\quad + \int_{\partial\Omega}(\nu_1 - i\nu_2)(\mathcal{A}U_{0,\tau}e^{\tau\Phi}, V_{0,\tau}e^{-\tau\bar\Phi})\,d\sigma \\
&= e^{2i\tau\psi(\widetilde{x})}\mathfrak{F}_{\tau,\widetilde{x}}(-(2\partial_z\mathcal{A}U_0, V_0) - (2\mathcal{A}U_0, \partial_z V_0) + (2\mathcal{B}\partial_{\bar z}U_0, V_0)) \\
&\quad + \mathfrak{I}_\tau(-(2\partial_z\mathcal{A}U_{0,\tau}, V_{0,\tau}) - (2\mathcal{A}U_{0,\tau}, \partial_z V_{0,\tau}) + (2\mathcal{B}\partial_{\bar z}U_{0,\tau}, V_{0,\tau})) \\
&\quad + \int_{\widetilde{\Gamma}}(\nu_1 - i\nu_2)(\mathcal{A}U_0, V_0)e^{\tau(\Phi-\bar\Phi)}\,d\sigma + \kappa_{0,0} + \frac{\kappa_{0,-1}}{\tau} + o\Big(\frac{1}{\tau}\Big),
\end{aligned}
\tag{4.9}
$$

where $\kappa_{0,j}$ are some constants independent of τ.

Integrating by parts, we obtain that there exist constants $\kappa_{1,j}$, independent of τ, such that

$$
\begin{aligned}
&\int_\Omega (2\mathcal{A}\partial_z(T_{0,\tau}e^{\tau\bar\Phi}) + 2\mathcal{B}\partial_{\bar z}(T_{0,\tau}e^{\tau\bar\Phi}), V_{0,\tau}e^{-\tau\bar\Phi})\,dx \\
&= (2\mathcal{A}\partial_z T_{0,\tau}, V_{0,\tau})_{L^2(\Omega)} + (2\mathcal{B}(\partial_z T_{0,\tau} + \tau\bar\Phi' T_{0,\tau}), V_{0,\tau})_{L^2(\Omega)} \\
&= \tau\kappa_{1,1} + \kappa_{1,0} + \frac{\kappa_{1,-1}}{\tau} \\
&\quad + \frac{1}{\tau}(e^{2i\tau\psi(\widetilde{x})}(2\mathcal{B}b_{+,\widetilde{x}}\bar\Phi', V_0)_{L^2(\Omega)} + e^{-2i\tau\psi(\widetilde{x})}(2\mathcal{B}b_{-,\widetilde{x}}\bar\Phi', V_0)_{L^2(\Omega)}) \\
&\quad + \frac{1}{\tau}(e^{2i\tau\psi(\widetilde{x})}(2\mathcal{B}\bar\Phi' T_0, \widetilde{a}_{+,\widetilde{x}})_{L^2(\Omega)} + e^{-2i\tau\psi(\widetilde{x})}(2\mathcal{B}\bar\Phi' T_0, \widetilde{a}_{-,\widetilde{x}})_{L^2(\Omega)}) \\
&\quad + o\Big(\frac{1}{\tau}\Big).
\end{aligned}
\tag{4.10}
$$

Since by (3.5), (3.23), (3.30) and (3.43), we have

$$
\begin{aligned}
(2\mathcal{B}\bar\Phi' T_0, \widetilde{a}_{\pm,\widetilde{x}}) &= -4\partial_z(\bar\Phi' T_0, \widetilde{a}_{\pm,\widetilde{x}}), \\
\text{and}\quad (2\mathcal{B}b_{\pm,\widetilde{x}}\bar\Phi', V_0) &= -4\partial_z(b_{\pm,\widetilde{x}}\bar\Phi', V_0) \quad \text{in } \Omega,
\end{aligned}
$$

from (4.10) we obtain

$$
\begin{aligned}
\mathcal{M}_2 &= \int_\Omega (2\mathcal{A}\partial_z(T_{0,\tau}e^{\tau\bar\Phi}) + 2\mathcal{B}\partial_{\bar z}(T_{0,\tau}e^{\tau\bar\Phi}), V_{0,\tau}e^{-\tau\bar\Phi})\,dx \\
&= \tau\kappa_{1,1} + \kappa_{1,0} + \frac{\kappa_{1,-1}}{\tau} \\
&\quad - \int_{\partial\Omega}\frac{\nu_1 - i\nu_2}{\tau}(e^{2i\tau\psi(\widetilde{x})}(2\mathcal{B}b_{+,\widetilde{x}}\bar\Phi', V_0) + e^{-2i\tau\psi(\widetilde{x})}(2\mathcal{B}b_{-,\widetilde{x}}\bar\Phi', V_0))\,d\sigma
\end{aligned}
\tag{4.11}
$$

$$-\int_{\partial\Omega}\frac{\nu_1 - i\nu_2}{\tau}(e^{2i\tau\psi(\widetilde{x})}(2\bar{\Phi}'T_0, \widetilde{a}_{+,\widetilde{x}}) + e^{-2i\tau\psi(\widetilde{x})}(2\bar{\Phi}'T_0, \widetilde{a}_{-,\widetilde{x}}))\,d\sigma$$
$$+ o\Big(\frac{1}{\tau}\Big).$$

Integrating by parts, we obtain that there exist constants $\kappa_{2,j}$, independent of τ, such that

$$\begin{aligned}(4.12)\quad \mathcal{M}_3 &= \int_\Omega (2\mathcal{A}\partial_z(U_{0,\tau}e^{\tau\Phi}) + 2\mathcal{B}\partial_{\bar{z}}(U_{0,\tau}e^{\tau\Phi}), W_{0,\tau}e^{-\tau\Phi})\,dx \\ &= (2\mathcal{A}(\partial_z U_{0,\tau} + \tau\Phi' U_{0,\tau}) + 2\mathcal{B}\partial_{\bar{z}}U_{0,\tau}, W_{0,\tau})_{L^2(\Omega)} \\ &= \tau\kappa_{2,1} + \kappa_{1,0} + \frac{\kappa_{2,-1}}{\tau} \\ &\quad + \frac{2}{\tau}(e^{2i\tau\psi(\widetilde{x})}(\mathcal{A}a_{+,\widetilde{x}}\Phi', W_0)_{L^2(\Omega)} + e^{-2i\tau\psi(\widetilde{x})}(\mathcal{A}a_{-,\widetilde{x}}\Phi', W_0)_{L^2(\Omega)}) \\ &\quad + \frac{2}{\tau}(e^{2i\tau\psi(\widetilde{x})}(\mathcal{A}\Phi' T_0, \widetilde{b}_{+,\widetilde{x}})_{L^2(\Omega)} + e^{-2i\tau\psi(\widetilde{x})}(\mathcal{A}\Phi' W_0, \widetilde{b}_{-,\widetilde{x}})_{L^2(\Omega)}) \\ &\quad + o\Big(\frac{1}{\tau}\Big).\end{aligned}$$

Since by (3.5), (3.23), (3.30) and (3.43) we have

$$(\mathcal{A}a_{\pm,\widetilde{x}}\Phi', W_0) = -2\partial_{\bar{z}}(a_{\pm,\widetilde{x}}\Phi', W_0)$$
$$\text{and}\quad (\mathcal{A}\Phi' T_0, \widetilde{b}_{\pm,\widetilde{x}}) = -2\partial_{\bar{z}}(\Phi' T_0, \widetilde{b}_{\pm,\widetilde{x}}) \quad \text{in } \Omega,$$

we obtain from (4.12),

$$\begin{aligned}(4.13)\quad \mathcal{M}_3 &= \int_\Omega (2\mathcal{A}\partial_z(U_{0,\tau}e^{\tau\Phi}) + 2\mathcal{B}\partial_{\bar{z}}(U_{0,\tau}e^{\tau\Phi}), W_{0,\tau}e^{-\tau\Phi})\,dx \\ &= \tau\kappa_{2,1} + \kappa_{1,0} + \frac{\kappa_{2,-1}}{\tau} \\ &\quad - \int_{\partial\Omega}(\nu_1 + i\nu_2)\frac{2}{\tau}(e^{2i\tau\psi(\widetilde{x})}(a_{+,\widetilde{x}}\Phi', W_0) + e^{-2i\tau\psi(\widetilde{x})}(a_{-,\widetilde{x}}\Phi', W_0))\,d\sigma \\ &\quad - \int_{\partial\Omega}(\nu_1 + i\nu_2)\frac{2}{\tau}(e^{2i\tau\psi(\widetilde{x})}(\Phi' T_0, \widetilde{b}_{+,\widetilde{x}}) + e^{-2i\tau\psi(\widetilde{x})}(\Phi' T_0, \widetilde{b}_{-,\widetilde{x}}))\,d\sigma \\ &\quad + o\Big(\frac{1}{\tau}\Big).\end{aligned}$$

Integrating by parts, using (3.5) and Proposition 4.1, we obtain that there exist some constants $\kappa_{3,j}$, independent of τ, such that

$$\begin{aligned}(4.14)\quad \mathcal{M}_4 &= \int_\Omega (2\mathcal{A}\partial_z(T_{0,\tau}e^{\tau\bar{\Phi}}) + 2\mathcal{B}\partial_{\bar{z}}(T_{0,\tau}e^{\tau\bar{\Phi}}), W_{0,\tau}e^{-\tau\Phi})\,dx \\ &= \int_\Omega ((2\mathcal{A}\partial_z T_{0,\tau}e^{\tau\bar{\Phi}}, W_{0,\tau}e^{-\tau\Phi}) - (2\partial_{\bar{z}}\mathcal{B}T_{0,\tau}e^{\tau\bar{\Phi}}, W_{0,\tau}e^{-\tau\Phi})\end{aligned}$$

$$
\begin{aligned}
&- (2\mathcal{B}T_{0,\tau}e^{\tau\bar\Phi}, \partial_{\bar z}W_{0,\tau}e^{-\tau\Phi}))\,dx\\
&+ \int_{\partial\Omega}(\nu_1+i\nu_2)(\mathcal{B}T_{0,\tau}e^{\tau\bar\Phi}, W_{0,\tau}e^{-\tau\Phi})\,d\sigma\\
=\ & e^{-2i\tau\psi(\widetilde x)}\mathfrak{F}_{-\tau,\widetilde x}((2\mathcal{A}\partial_z T_0, W_0) - (2\partial_{\bar z}\mathcal{B}T_0, W_0) - (2\mathcal{B}T_0, \partial_{\bar z}W_0))\\
&+ \mathfrak{I}_{-\tau}((2\mathcal{A}\partial_z T_{0,\tau}, W_{0,\tau}) - (2\partial_{\bar z}\mathcal{B}T_{0,\tau}, W_{0,\tau}) - (2\mathcal{B}T_{0,\tau}, \partial_{\bar z}W_{0,\tau}))\\
&+ \int_{\widetilde\Gamma}(\nu_1+i\nu_2)(\mathcal{B}T_0e^{\tau\bar\Phi}, W_0e^{-\tau\Phi})\,d\sigma + \kappa_{3,1} + \frac{\kappa_{3,-1}}{\tau} + o\Big(\frac{1}{\tau}\Big).
\end{aligned}
$$

Integrating by parts and using Proposition 4.1, we obtain

$$
\begin{aligned}
(4.15)\quad \mathcal{M}_5 =\ & -\int_\Omega (2\mathcal{A}\partial_z(\widetilde{\mathcal{R}}_{\tau,B_1}(e_1(q_1+\widetilde q_1/\tau))e^{\tau\Phi})\\
&+ 2\mathcal{B}\partial_{\bar z}(\widetilde{\mathcal{R}}_{\tau,B_1}(e_1(q_1+\widetilde q_1/\tau))e^{\tau\Phi}), V_{0,\tau}e^{-\tau\bar\Phi})\,dx\\
=\ & \int_\Omega (\mathcal{A}(B_1\widetilde{\mathcal{R}}_{\tau,B_1}(e_1(q_1+\widetilde q_1/\tau)) - e_1q_1)e^{\tau\Phi}\\
&+ 2\partial_{\bar z}\mathcal{B}(\widetilde{\mathcal{R}}_{\tau,B_1}(e_1(q_1+\widetilde q_1/\tau))e^{\tau\Phi}), V_{0,\tau}e^{-\tau\bar\Phi})\,dx\\
&- \int_{\partial\Omega}(\nu_1+i\nu_2)(\mathcal{B}\widetilde{\mathcal{R}}_{\tau,B_1}(e_1(q_1+\widetilde q_1/\tau)), V_{0,\tau})e^{\tau(\Phi-\bar\Phi)}\,d\sigma\\
&+ (2\mathcal{B}\widetilde{\mathcal{R}}_{\tau,B_1}(e_1(q_1+\widetilde q_1/\tau)), \partial_{\bar z}(V_{0,\tau}e^{\tau(\Phi-\bar\Phi)}))_{L^2(\Omega)} + o\Big(\frac{1}{\tau}\Big)\\
=\ & \int_\Omega \Big((\mathcal{A}(B_1S_{B_1}(e^{\tau(\Phi-\bar\Phi)}e_1q_1) - e_1q_1)e^{\tau(\Phi-\bar\Phi)}, V_{0,\tau})\\
&+ (2\partial_z\mathcal{B}(S_{B_1}(e^{\tau(\Phi-\bar\Phi)}e_1q_1)), V_{0,\tau})\Big)\,dx\\
&+ (\mathcal{B}S_{B_1}(e^{\tau(\Phi-\bar\Phi)}e_1q_1), \partial_{\bar z}V_{0,\tau} - 2\tau\bar\Phi' V_{0,\tau})_{L^2(\Omega)}\\
&- \int_{\partial\Omega}(\nu_1+i\nu_2)(\mathcal{B}\widetilde{\mathcal{R}}_{\tau,B_1}(e_1(q_1+\widetilde q_1/\tau)), V_{0,\tau})e^{\tau(\Phi-\bar\Phi)}\,d\sigma + o\Big(\frac{1}{\tau}\Big)\\
=\ & e^{2i\tau\psi(\widetilde x)}\mathfrak{F}_{\tau,\widetilde x}(q_1,\, S^*_{B_1}(B_1^*\mathcal{A}^*V_0) - \mathcal{A}^*V_0\\
&+ 2S^*_{B_1}(\partial_z\mathcal{B}^*V_0) + S^*_{B_1}(\mathcal{B}^*(A_2^*V_0 - 2\tau\bar\Phi' V_0)))\\
&- \int_{\partial\Omega}(\nu_1+i\nu_2)(\mathcal{B}\widetilde{\mathcal{R}}_{\tau,B_1}(e_1(q_1+\widetilde q_1/\tau)), V_{0,\tau})e^{\tau(\Phi-\bar\Phi)}\,d\sigma + o\Big(\frac{1}{\tau}\Big)\\
&\text{as } \tau\to+\infty.
\end{aligned}
$$

After integration by parts, we have

$$
\begin{aligned}
\mathcal{M}_6 =\ & -\int_\Omega (2\mathcal{A}\partial_z(\widetilde{\mathcal{R}}_{\tau,B_1}(e_1(q_1+\widetilde q_1/\tau))e^{\tau\Phi})\\
&+ 2\mathcal{B}\partial_{\bar z}(\widetilde{\mathcal{R}}_{\tau,B_1}(e_1(q_1+\widetilde q_1/\tau))e^{\tau\Phi}), W_{0,\tau}e^{-\tau\Phi})\,dx\\
=\ & \int_\Omega (\mathcal{A}(B_1\widetilde{\mathcal{R}}_{\tau,B_1}(e_1(q_1+\widetilde q_1/\tau)) - e_1q_1)
\end{aligned}
$$

$$+ 2\partial_{\bar z}\mathcal{B}\widetilde{\mathcal{R}}_{\tau,B_1}(e_1(q_1+\widetilde{q}_1/\tau)), W_{0,\tau})\,dx + o\Big(\frac{1}{\tau}\Big)$$
$$- (2\mathcal{B}\widetilde{\mathcal{R}}_{\tau,B_1}(e_1(q_1+\widetilde{q}_1/\tau)), \partial_{\bar z}W_{0,\tau})_{L^2(\Omega)}$$
$$- \int_{\partial\Omega}(\nu_1+i\nu_2)(\mathcal{B}\widetilde{\mathcal{R}}_{\tau,B_1}(e_1(q_1+\widetilde{q}_1/\tau)), W_{0,\tau})\,d\sigma.$$

Using (3.21), (3.22) and [7, Proposition 8], we obtain that

$$\mathcal{M}_6 = -\int_\Omega (\mathcal{A}q_1, W_{0,\tau})\,dx + o\Big(\frac{1}{\tau}\Big) \quad \text{as } \tau\to+\infty. \tag{4.16}$$

Integrating by parts and using Proposition 4.1, we have

(4.17)

$$\begin{aligned}
&\mathcal{M}_7\\
&= -\int_\Omega (2\mathcal{A}\partial_z(U_{0,\tau}e^{\tau\Phi}) + 2\mathcal{B}\partial_{\bar z}(U_{0,\tau}e^{\tau\Phi}), \mathcal{R}_{-\tau,-A_2^*}(e_1(q_4+\widetilde{q}_4/\tau))e^{-\tau\bar\Phi})\,dx\\
&= -2\int_\Omega (\mathcal{A}(\partial_z U_{0,\tau} + \tau\Phi' U_{0,\tau})e^{\tau\Phi} + \mathcal{B}\partial_{\bar z}U_{0,\tau}e^{\tau\Phi},\\
&\quad \mathcal{R}_{-\tau,-A_2^*}(e_1(q_4+\widetilde{q}_4/\tau))e^{-\tau\bar\Phi})\,dx\\
&= -2\int_\Omega (P^*_{-A_2^*}(\mathcal{A}(\partial_z U_0 + \tau\Phi' U_0) + \mathcal{B}\partial_{\bar z}U_{0,\tau}), e_1 q_4 e^{\tau(\Phi-\bar\Phi)})\,dx + o\Big(\frac{1}{\tau}\Big)\\
&= -2e^{2i\tau\psi(\widetilde{x})}\mathfrak{F}_{\tau,\widetilde{x}}(P^*_{-A_2^*}(\mathcal{A}(\partial_z U_0 + \tau\Phi' U_0) + \mathcal{B}\partial_{\bar z}U_0), q_4) + o\Big(\frac{1}{\tau}\Big)\\
&\quad \text{as } \tau\to+\infty.
\end{aligned}$$

Integrating by parts and using [6, Proposition 8], we have

(4.18)

$$\begin{aligned}
&\mathcal{M}_8\\
&= -\int_\Omega (2\mathcal{A}\partial_z(U_{0,\tau}e^{\tau\Phi}) + 2\mathcal{B}\partial_{\bar z}(U_{0,\tau}e^{\tau\Phi}), \widetilde{\mathcal{R}}_{-\tau,-B_2^*}(e_1(q_3+\widetilde{q}_3/\tau))e^{-\tau\Phi})\,dx\\
&= \int_\Omega (-(-2\partial_z\mathcal{A}U_0 + \mathcal{B}\partial_{\bar z}U_0,\ \widetilde{\mathcal{R}}_{-\tau,-B_2^*}(e_1(q_3+\widetilde{q}_3/\tau)))\\
&\quad - (\mathcal{A}U_{0,\tau},\ B_2^*\widetilde{\mathcal{R}}_{-\tau,-B_2^*}(e_1(q_3+\widetilde{q}_3/\tau)) - e_1q_3))\,dx + o\Big(\frac{1}{\tau}\Big)\\
&\quad - \int_{\partial\Omega}(\nu_1 - i\nu_2)(\mathcal{A}U_0, \widetilde{\mathcal{R}}_{-\tau,-B_2^*}(e_1(q_3+\widetilde{q}_3/\tau)))\,d\sigma\\
&= -\int_\Omega (\mathcal{A}U_{0,\tau}, q_3)\,dx + o\Big(\frac{1}{\tau}\Big) \quad \text{as } \tau\to+\infty
\end{aligned}$$

and

$$
\begin{aligned}
\mathcal{M}_9 &= -\int_\Omega (2\mathcal{A}\partial_z(\mathcal{R}_{\tau,A_1}(e_1(q_2+\widetilde{q}_2/\tau))e^{\tau\bar{\Phi}}) \\
&\quad + 2\mathcal{B}\partial_{\bar{z}}(\mathcal{R}_{\tau,A_1}(e_1(q_2+\widetilde{q}_2/\tau))e^{\tau\bar{\Phi}}), V_{0,\tau}e^{-\tau\bar{\Phi}})\,dx \\
&= \int_\Omega \big((\mathcal{R}_{\tau,A_1}(e_1(q_2+\widetilde{q}_2/\tau)), \partial_z(2\mathcal{A}^*V_{0,\tau})) \\
&\quad + (\mathcal{B}(A_1\mathcal{R}_{\tau,A_1}(e_1(q_2+\widetilde{q}_2/\tau)) - e_1q_2), V_{0,\tau})\big)\,dx + o\Big(\frac{1}{\tau}\Big) \\
&\quad - \int_{\partial\Omega} (\nu_1 - i\nu_2)(\mathcal{A}\mathcal{R}_{\tau,A_1}(e_1(q_2+\widetilde{q}_2/\tau)), V_0)\,d\sigma \\
&= -\int_\Omega (\mathcal{B}q_2, V_{0,\tau})\,dx + o\Big(\frac{1}{\tau}\Big) \quad \text{as } \tau \to +\infty.
\end{aligned} \tag{4.19}
$$

Integrating by parts and using Proposition 4.1, we obtain

$$
\begin{aligned}
\mathcal{M}_{10} &= -\int_\Omega (2\mathcal{A}\partial_z(\mathcal{R}_{\tau,A_1}(e_1(q_2+\widetilde{q}_2/\tau))e^{\tau\bar{\Phi}}) \\
&\quad + 2\mathcal{B}\partial_{\bar{z}}(\mathcal{R}_{\tau,A_1}(e_1(q_2+\widetilde{q}_2/\tau))e^{\tau\bar{\Phi}}),\ W_{0,\tau}e^{-\tau\Phi})\,dx \\
&= \int_\Omega ((-\mathcal{R}_{\tau,A_1}(e_1(q_2+\widetilde{q}_2/\tau)),\ -\partial_z(2\mathcal{A}^*W_{0,\tau}) + 2\tau\Phi'\mathcal{A}^*W_{0,\tau}) \\
&\quad + (\mathcal{B}(A_1\mathcal{R}_{\tau,A_1}(e_1(q_2+\widetilde{q}_2/\tau)) - e_1q_2),\ W_{0,\tau})e^{\tau(\bar{\Phi}-\Phi)})\,dx \\
&\quad - \int_{\partial\Omega} (\nu_1 - i\nu_2)(\mathcal{A}\mathcal{R}_{\tau,A_1}(e_1(q_2+\widetilde{q}_2/\tau)),\ W_{0,\tau})e^{\tau(\bar{\Phi}-\Phi)}\,d\sigma + o\Big(\frac{1}{\tau}\Big) \\
&= \int_\Omega (e_1q_2, P^*_{A_1}(2\partial_z(\mathcal{A}^*W_{0,\tau}) - 2\tau\Phi'\mathcal{A}^*W_0) - \mathcal{B}^*W_0 \\
&\quad + P^*_{A_1}(A_1^*\mathcal{B}^*W_0))e^{\tau(\bar{\Phi}-\Phi)}\,dx \\
&\quad - \int_{\partial\Omega} (\nu_1 - i\nu_2)(\mathcal{A}\mathcal{R}_{\tau,A_1}(e_1(q_2+\widetilde{q}_2/\tau)), W_0)e^{\tau(\bar{\Phi}-\Phi)}\,d\sigma + o\Big(\frac{1}{\tau}\Big) \\
&= e^{-2i\tau\psi(\widetilde{x})}\mathfrak{F}_{-\tau,\widetilde{x}}(q_2,\ P^*_{A_1}(2\partial_z(\mathcal{A}^*W_{0,\tau}) - 2\tau\Phi'\mathcal{A}^*W_0) \\
&\quad - \mathcal{B}^*W_0 + P^*_{A_1}(A_1^*\mathcal{B}^*W_0)) \\
&\quad - \int_{\partial\Omega} (\nu_1 - i\nu_2)(\mathcal{A}\mathcal{R}_{\tau,A_1}(e_1(q_2+\widetilde{q}_2/\tau)), W_0)e^{\tau(\bar{\Phi}-\Phi)}\,d\sigma + o\Big(\frac{1}{\tau}\Big) \\
&\quad \text{as } \tau \to +\infty.
\end{aligned} \tag{4.20}
$$

By (3.15) and Proposition 4.1, we obtain

(4.21)

$$
\begin{aligned}
&\mathcal{M}_{11} \\
&= -\int_\Omega (2\mathcal{A}\partial_z(T_{0,\tau}e^{\tau\bar{\Phi}}) + 2\mathcal{B}\partial_{\bar{z}}(T_{0,\tau}e^{\tau\bar{\Phi}}), \widetilde{\mathcal{R}}_{-\tau,-B_2^*}(e_1(q_3+\widetilde{q}_3/\tau))e^{-\tau\Phi})\,dx
\end{aligned}
$$

$$= -\int_\Omega (2\mathcal{A}\partial_z T_{0,\tau} + 2\mathcal{B}(\partial_{\bar z} T_{0,\tau} + \tau\bar\Phi' T_{0,\tau}), \widetilde{\mathcal{R}}_{-\tau,-B_2^*}(e_1(q_3+\widetilde{q}_3/\tau)))e^{\tau(\bar\Phi-\Phi)}\,dx$$
$$= -\int_\Omega (e_1 q_3, T^*_{-B_2^*}(2\mathcal{A}\partial_z T_{0,\tau} + 2\mathcal{B}(\partial_{\bar z} T_{0,\tau} + \tau\bar\Phi' T_{0,\tau})))e^{\tau(\bar\Phi-\Phi)}\,dx + o\Big(\frac{1}{\tau}\Big)$$
$$= -e^{-2i\tau\psi(\widetilde{x})}\mathfrak{F}_{-\tau,\widetilde{x}}(q_3,\, S^*_{-B_2^*}(2\mathcal{A}\partial_z T_0 + 2\mathcal{B}(\partial_{\bar z}T_0 + \tau\bar\Phi' T_0))) + o\Big(\frac{1}{\tau}\Big)$$
as $\tau \to +\infty$.

By Proposition 4.1, there exist constants $\kappa_{4,j}$, independent of τ, such that

$$(4.22)\qquad \mathcal{M}_{12} = \int_\Omega (\mathcal{Q}(U_{0,\tau}e^{\tau\bar\Phi} + T_{0,\tau}e^{\tau\Phi}),\, V_{0,\tau}e^{-\tau\bar\Phi} + W_{0,\tau}e^{-\tau\Phi})\,dx$$
$$= \kappa_{4,0} + \kappa_{4,-1}/\tau + \frac{\pi}{2\tau|\det\psi''(\widetilde{x})|^{1/2}}$$
$$\times ((\mathcal{Q}U_0, V_0)(\widetilde{x})e^{2i\tau\psi(\widetilde{x})} + (\mathcal{Q}T_0, W_0)(\widetilde{x})e^{-2i\tau\psi(\widetilde{x})})$$
$$+ o\Big(\frac{1}{\tau}\Big) \quad \text{as } \tau \to +\infty.$$

Since $\mathcal{J}_\tau = \sum_{k=1}^{12}\mathcal{M}_k$, the proof of Proposition 4.2 is complete. □

We have

Proposition 4.3. *The matrices A_j and B_j on $\widetilde\Gamma$ satisfy*

$$(4.23)\qquad A_1 - A_2 = B_1 - B_2 = 0 \quad \text{on } \widetilde\Gamma.$$

For any matrices $\mathcal{C}_j$, $\mathcal{P}_j \in C^{5+\alpha}(\bar\Omega)$ satisfying (3.8)–(3.10) and (3.34)–(3.36) with sufficiently small positive ϵ and some $\alpha \in (0,1)$, there exists a holomorphic matrix $\Theta \in C^{6+\alpha}(\bar\Omega)$ such that the matrix $\mathbf{Q} = \mathcal{P}_1\Theta^{-1}\mathcal{P}_2^*$ verifies

$$(4.24)\qquad 2\partial_{\bar z}\mathbf{Q} + A_1\mathbf{Q} - \mathbf{Q}A_2 = 0 \quad \text{in } \Omega\setminus\mathcal{X}, \qquad \mathbf{Q}|_{\widetilde\Gamma} = I, \quad \partial_{\vec\nu}\mathbf{Q}|_{\widetilde\Gamma} = 0,$$

where $\mathcal{X} = \{x \in \bar\Omega;\ \det\Theta(x) = 0\}$ and

$$(4.25)\qquad \mathbf{Q} \in C^{6+\alpha}(\Omega\setminus\mathcal{X}), \qquad \det\mathbf{Q} \neq 0 \quad \text{in } \bar\Omega\setminus\mathcal{X}.$$

Proof. From (4.2), we have $J_0 = J_1 = 0$. All remaining terms on the right-hand side of (4.2) except for $\int_{\widetilde\Gamma}((\nu_1 - i\nu_2)(\mathcal{A}U_0e^{\tau\Phi}, V_0e^{-\tau\bar\Phi}) + (\nu_1+i\nu_2)(\mathcal{B}T_0e^{\tau\bar\Phi}, W_0e^{-\tau\Phi}))\,d\sigma$ are of order $o(\frac{1}{\sqrt{\tau}})$. Let the phase function $\Phi = \varphi + i\psi$ be given by [7, Proposition 2.2]. Let $\widetilde{x}$ be an arbitrary point from $\widetilde\Gamma$ and $\mu \in C_0^5(\widetilde\Gamma)$ be equal to 1 in some neighborhood of $\widetilde{x}$. Thanks to (3.16) and (3.31), the functions U_0, V_0, T_0, W_0 can be chosen such that

$$\lim_{x\to\hat{x}_\pm}\frac{|U_0(x)| + |T_0(x)|}{|x-\hat{x}_\pm|^{98}} = \lim_{x\to\hat{x}_\pm}\frac{|V_0(x)| + |W_0(x)|}{|x-\hat{x}_\pm|^{98}} = 0$$

and

$$\left(\frac{\partial}{\partial\vec{\tau}+0}\right)^6 \operatorname{Im}\Phi(\hat{x}_-)\neq 0, \qquad \left(\frac{\partial}{\partial\vec{\tau}-0}\right)^6 \operatorname{Im}\Phi(\hat{x}_+)\neq 0.$$

Here $\frac{\partial}{\partial\vec{\tau}+0}$ and $\frac{\partial}{\partial\vec{\tau}-0}$ mean the limit from the right and the limit from the left, respectively. Hence we have

$$\begin{aligned}\mathcal{Z} &= \int_{\widetilde{\Gamma}}((\nu_1-i\nu_2)(\mathcal{A}U_0e^{\tau\Phi},V_0e^{-\tau\bar{\Phi}})+(\nu_1+i\nu_2)(\mathcal{B}T_0e^{\tau\bar{\Phi}},W_0e^{-\tau\Phi}))\,d\sigma\\ &= \int_{\widetilde{\Gamma}}\mu((\nu_1-i\nu_2)(\mathcal{A}U_0e^{\tau\Phi},V_0e^{-\tau\bar{\Phi}})+(\nu_1+i\nu_2)(\mathcal{B}T_0e^{\tau\bar{\Phi}},W_0e^{-\tau\Phi}))\,d\sigma\\ &\quad + o\Big(\frac{1}{\sqrt{\tau}}\Big).\end{aligned}$$

For the restriction of the function ψ on $\operatorname{supp}\mu$, the set of the critical points $\mathcal{G}$ is finite and all the points are nondegenerate. Applying the stationary phase argument to the last integral, we obtain

$$\begin{aligned}\mathcal{Z} = \sum_{x\in\mathcal{G}}\frac{\kappa(x)}{\sqrt{\tau}}\big(&(\nu_1-i\nu_2)(x)(\mathcal{A}U_0,V_0)(x)e^{i\tau\psi(x)}\\ &+(\nu_1+i\nu_2)(x)(\mathcal{B}T_0,W_0)(x)e^{-i\tau\psi(x)}\big)+o\Big(\frac{1}{\sqrt{\tau}}\Big).\end{aligned}\tag{4.26}$$

Here κ is some function not vanishing for any $x\in\mathcal{G}$. Since $\psi(\widetilde{x})\neq-\psi(\widetilde{x})+2\pi k$ and $\psi(\widetilde{x})-\psi(x)\neq 0 \operatorname{modulo} 2\pi k$ for all x from $\mathcal{G}\setminus\{\widetilde{x}\}$, by (4.26) and (4.2), we have (4.23).

From Proposition 4.2, for any function Φ satisfying (3.1) and (3.2), we have

$$\int_{\partial\Omega}((\nu_1+i\nu_2)\Phi'(T_0,V_0)+(\nu_1-i\nu_2)\bar{\Phi}'(U_0,W_0))\,d\sigma=0.\tag{4.27}$$

If $\boldsymbol{a}(z)=(a_1(z),\dots,a_N(z))$ and $\boldsymbol{b}(z)=(b_1(z),\dots,b_N(z))$ are holomorphic functions such that $\operatorname{Im}\boldsymbol{a}|_{\Gamma_0}=\operatorname{Im}\boldsymbol{b}|_{\Gamma_0}=0$, then the pairs $(\mathcal{P}_1\boldsymbol{a},\mathcal{C}_1\overline{\boldsymbol{a}})$ and $(\mathcal{P}_2\boldsymbol{b},\mathcal{C}_2\overline{\boldsymbol{b}})$ solve boundary value problems (3.5) and (3.30) respectively. Therefore we can rewrite (4.27) as

$$\int_{\partial\Omega}\{(\nu_1+i\nu_2)\Phi'(\mathcal{P}_1\boldsymbol{a},\mathcal{P}_2\boldsymbol{b})+(\nu_1-i\nu_2)\bar{\Phi}'(\mathcal{C}_1\bar{\boldsymbol{a}},\mathcal{C}_2\bar{\boldsymbol{b}})\}\,d\sigma=0.\tag{4.28}$$

Thanks to (4.28), all the assumptions of Proposition 3.3 hold true. By Proposition 3.3 there exist a holomorphic matrix $\Theta(z)$ and an antiholomorphic matrix $\widetilde{\Theta}(\bar{z})$ on $\overline{\Omega}$ such that

$$\Theta=\mathcal{P}_2^*\mathcal{P}_1 \quad\text{on } \widetilde{\Gamma} \quad\text{and}\quad \widetilde{\Theta}=\mathcal{C}_2^*\mathcal{C}_1 \quad\text{on } \widetilde{\Gamma} \quad\text{and}\quad \Theta,\widetilde{\Theta}\in L^2(\Omega)\tag{4.29}$$

and

$$\Theta - \widetilde{\Theta} = 0 \quad \text{on } \Gamma_0. \tag{4.30}$$

From (4.29) and (4.30), we have

$$\Theta - \widetilde{\Theta} = \begin{cases} \mathcal{P}_2^*\mathcal{P}_1 - \mathcal{C}_2^*\mathcal{C}_1 & \text{if } x \in \widetilde{\Gamma}, \\ 0 & \text{if } x \in \Gamma_0. \end{cases}$$

By (3.8), (3.9), (3.34) and (3.35), we have

$$\Theta - \widetilde{\Theta} = \mathcal{P}_2^*\mathcal{P}_1 - \mathcal{C}_2^*\mathcal{C}_1 \quad \text{on } \partial\Omega. \tag{4.31}$$

From (4.31) and the classical regularity theory of systems of elliptic equations (see, e.g., [12]), we see that Θ, $\widetilde{\Theta} \in C^{6+\alpha}(\bar{\Omega})$. Without loss of generality, we can assume that

$$\det \mathcal{P}_2^* \neq 0 \quad \text{and} \quad \det \mathcal{P}_1 \neq 0 \quad \text{on } \widetilde{\Gamma}. \tag{4.32}$$

Moreover (3.10) and (3.36) yield

$$\det \mathcal{P}_2^* \neq 0 \quad \text{and} \quad \det \mathcal{P}_1 \neq 0 \quad \text{on } \overline{\Gamma}_0.$$

Observe that

$$I = \mathcal{P}_1 \Theta^{-1} \mathcal{P}_2^* \quad \text{on } \widetilde{\Gamma} \tag{4.33}$$

by (4.29).

Since

$$2\partial_{\bar z}\mathcal{P}_1 + A_1\mathcal{P}_1 = 0 \ \text{ in } \Omega \quad \text{and} \quad 2\partial_{\bar z}\mathcal{P}_2^* - \mathcal{P}_2^* A_2 = 0 \ \text{ in } \Omega$$

by the construction of the matrices $\mathcal{P}_j$, and the matrix Θ is holomorphic, we have

$$2\partial_{\bar z}(\mathcal{P}_1\Theta^{-1}) + A_1(\mathcal{P}_1\Theta^{-1}) = 0 \quad \text{in } \Omega \setminus \mathcal{X}$$

and

$$2\partial_{\bar z}(\mathcal{P}_1\Theta^{-1}\mathcal{P}_2^*) + A_1(\mathcal{P}_1\Theta^{-1}\mathcal{P}_2^*) - (\mathcal{P}_1\Theta^{-1}\mathcal{P}_2^*)A_2 = 0 \quad \text{in } \Omega \setminus \mathcal{X}. \tag{4.34}$$

Thus the first equation in (4.24) holds true. By (4.33) the second equation in (4.24) is proved.

By (4.23) and (4.33), we have

$$-2\partial_{\bar z}\mathbf{Q} = A_1\mathcal{P}_1\Theta^{-1}\mathcal{P}_2^* - \mathcal{P}_1\Theta^{-1}\mathcal{P}_2^* A_2 = A_1 I - I A_2 = A_1 - A_2 = 0. \tag{4.35}$$

In order to prove the third equation in (4.24), we observe that there exists a matrix $\Upsilon(x)$ with real-valued entries such that $\det \Upsilon(x) \neq 0$ and $\nabla = \Upsilon(x)(\partial_{\vec\nu}, \partial_{\vec\tau})$.

Therefore $\partial_{\bar z} = \frac{1}{2}((\Upsilon_{11} + i\Upsilon_{21})\partial_{\vec\nu} + (\Upsilon_{12} + i\Upsilon_{22})\partial_{\vec\tau})$. By (4.35) on $\widetilde\Gamma$ the following equation holds:

$$\begin{aligned}\partial_{\bar z}\mathbf{Q} &= \frac{1}{2}((\Upsilon_{11} + i\Upsilon_{21})\partial_{\vec\nu}\mathbf{Q} + (\Upsilon_{12} + i\Upsilon_{22})\partial_{\vec\tau}\mathbf{Q})\\ &= \frac{1}{2}((\Upsilon_{11} + i\Upsilon_{21})\partial_{\vec\nu}\mathbf{Q} + (\Upsilon_{12} + i\Upsilon_{22})\partial_{\vec\tau}I)\\ &= \frac{1}{2}(\Upsilon_{11} + i\Upsilon_{21})\partial_{\vec\nu}\mathbf{Q} = 0.\end{aligned}$$

Since the determinant of the matrix Υ is not equal to zero, we have $(\Upsilon_{11}+i\Upsilon_{21}) \neq 0$. Hence from the above equation, we have $\partial_{\vec\nu}\mathbf{Q} = 0$.

If $\det \mathbf{Q}(x_0) = 0$, then $\det \mathcal{P}_1(x_0) \det \mathcal{P}_2(x_0) = 0$. Let matrices $\widehat{\mathcal{P}_j}$ be constructed as $\mathcal{P}_j$ but with a different choice of the pairs $(U_0(k), T_0(k))$, $(V_0(k), W_0(k))$ that are solutions to problems (3.5) and (3.30) respectively, and satisfy (3.10) and (3.37). In such a way, we obtain other matrices $\mathcal{P}_j$, Θ, $\mathbf{Q}$ that satisfy (4.24) with a possibly different set $\mathcal{X}$. We denote such matrices $\mathcal{P}_j$, Θ, $\mathbf{Q}$ by $\hat{\mathcal{P}}_j$, $\hat\Theta$, $\widehat{\mathbf{Q}}$. By the uniqueness of the Cauchy problem for the ∂_z-operator, we have

$$\mathbf{Q} = \widehat{\mathbf{Q}} \quad \text{on } \Omega \setminus \mathcal{X} \cup \widehat{\mathcal{X}} \quad \text{where } \widehat{\mathcal{X}} = \{x \in \bar\Omega;\, \det \widehat\Theta(x) = 0\}.$$

Consequently $\widehat{\mathbf{Q}}(x_0) = 0$. On the other hand, one can choose the matrices $\widehat{\mathcal{P}_j}$ such that $\det \widehat{\mathcal{P}_j}(x_0) \neq 0$. Therefore we reach a contradiction. The proof of the proposition is complete. □

Our next goal is to show that the matrix $\mathbf{Q}$ is regular on $\bar\Omega$.

Now we prove that if the operators $L_j(x, D)$ generate the same Dirichlet-to-Neumann map, then the operators $L_j(x, D)^*$ generate the same Dirichlet-to-Neumann map.

Proposition 4.4. *Let A_j, B_j, $Q_j \in C^{5+\alpha}(\bar\Omega)$, $j = 1, 2$ with some $\alpha \in (0, 1)$. If $\Lambda_{A_1,B_1,Q_1} = \Lambda_{A_2,B_2,Q_2}$, then $\Lambda_{-A_1^*,-B_1^*,R_1} = \Lambda_{-A_2^*,-B_2^*,R_2}$, where $R_j = -\partial_z A_j^* - \partial_{\bar z} B_j^* + Q_j^*$ for $j \in \{1, 2\}$.*

Proof. Let v_j solve

$$L_j(x, D)^* v_j = 0 \quad \text{in } \Omega, \quad v_j|_{\Gamma_0} = 0, \quad v_j|_{\widetilde\Gamma} = g$$

and $\widetilde u_j$ solve

$$L_j(x, D)\widetilde u_j = 0 \quad \text{in } \Omega, \quad \widetilde u_j|_{\Gamma_0} = 0, \quad \widetilde u_j|_{\widetilde\Gamma} = f.$$

By our assumption and the Fredholm alternative for both problems, solutions exist and are unique for any $f, g \in C_0^\infty(\widetilde{\Gamma})$. By the Green formula, we have

$$\begin{aligned}(L_j(x,D)^* v_j, \widetilde{u}_j)_{L^2(\Omega)} &- (v_j, L_j(x,D)\widetilde{u}_j)_{L^2(\Omega)} \\ &= (\partial_{\vec{\nu}} v_j, \widetilde{u}_j)_{L^2(\widetilde{\Gamma})} - (v_j, \partial_{\vec{\nu}}\widetilde{u}_j)_{L^2(\widetilde{\Gamma})} \\ &\quad - (A_j(\nu_1 - i\nu_2)g, f)_{L^2(\widetilde{\Gamma})} - (B_j(\nu_1 + i\nu_2)g, f)_{L^2(\widetilde{\Gamma})}, \quad j = 1, 2.\end{aligned}$$

Subtracting the above formulae for different j, using (4.23) and taking into account that $\Lambda_{A_1,B_1,Q_1} = \Lambda_{A_2,B_2,Q_2}$, we have

$$(\partial_{\vec{\nu}} v_1 - \partial_{\vec{\nu}} v_2, f)_{L^2(\widetilde{\Gamma})} = 0.$$

Since the function $f \in C_0^\infty(\widetilde{\Gamma})$ can be arbitrarily chosen, the proof of the proposition is complete. □

By Proposition 2.1, there exist solutions $(\mathrm{U}_0(k), \mathrm{T}_0(k))$ to the problem

$$(4.36)\qquad \begin{aligned}(2\partial_{\bar z}\mathrm{U}_0(k) - A_1^*\mathrm{U}_0(k), 2\partial_z \mathrm{T}_0(k) - B_1^*\mathrm{T}_0(k)) = 0 \quad &\text{in } \Omega,\\ \mathrm{U}_0(k) + \mathrm{T}_0(k) = 0 \quad &\text{on } \Gamma_0\end{aligned}$$

and solutions $(\mathrm{V}_0(k), \mathrm{W}_0(k))$ to

$$(4.37)\qquad \begin{aligned}(2\partial_{\bar z}\mathrm{V}_0(k) + A_2\mathrm{V}_0(k), 2\partial_z \mathrm{W}_0(k) + B_2\mathrm{W}_0(k)) = 0 \quad &\text{in } \Omega,\\ \mathrm{V}_0(k) + \mathrm{W}_0(k) = 0 \quad &\text{on } \Gamma_0\end{aligned}$$

for $k \in \{1, \dots, N\}$ such that

$$(4.38)\qquad \|\mathrm{U}_0(k) - \vec{e}_k\|_{C^{5+\alpha}(\bar\Gamma_0)} + \|\mathrm{W}_0(k) - \vec{e}_k\|_{C^{5+\alpha}(\bar\Gamma_0)} \le \epsilon, \quad \forall\, k \in \{1, \dots, N\}.$$

This inequality and the boundary conditions in (4.36) and (4.37) imply

$$(4.39)\qquad \|\mathrm{T}_0(k) - \vec{e}_k\|_{C^{5+\alpha}(\bar\Gamma_0)} + \|\mathrm{V}_0(k) - \vec{e}_k\|_{C^{5+\alpha}(\bar\Gamma_0)} \le \epsilon, \quad \forall\, k \in \{1, \dots, N\}.$$

We define matrices $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{R}_1$, $\mathcal{R}_2$ by

$$(4.40)\qquad \begin{aligned}\mathcal{M}_1 &= (\mathrm{T}_0(1), \dots, \mathrm{T}_0(N)), \qquad & \mathcal{R}_1 &= (\mathrm{U}_0(1), \dots, \mathrm{U}_0(N)),\\ \mathcal{M}_2 &= (\mathrm{V}_0(1), \dots, \mathrm{V}_0(N)), \qquad & \mathcal{R}_2 &= (\mathrm{W}_0(1), \dots, \mathrm{W}_0(N)).\end{aligned}$$

By Proposition 3.3, there exists a holomorphic matrix $\mathcal{Y}$ such that the matrix function $\mathbf{G} = \mathcal{M}_1\mathcal{Y}^{-1}\mathcal{M}_2^*$ solves

$$(4.41)\qquad \begin{aligned}&2\partial_{\bar z}\mathbf{G} + \mathbf{G}A_2^* - A_1^*\mathbf{G} = 0 \quad \text{in } \Omega \setminus \{x \in \bar\Omega;\, \det \mathcal{Y}(x) = 0\},\\ &\mathbf{G}|_{\widetilde{\Gamma}} = I, \qquad \partial_{\vec{\nu}}\mathbf{G}|_{\widetilde{\Gamma}} = 0.\end{aligned}$$

Observe that the matrix $\mathbf{Q}^{*-1}$ solves

$$2\partial_{\bar z}\mathbf{Q}^{*-1} + \mathbf{Q}^{*-1}A_2^* - A_1^*\mathbf{Q}^{*-1} = 0 \quad \text{in } \Omega \setminus \{x \in \bar\Omega;\ \det \mathcal{P}_1(x) \det \mathcal{P}_2(x) = 0\} \tag{4.42}$$

and

$$\mathbf{Q}^{*-1}|_{\widetilde\Gamma} = I, \qquad \partial_{\vec\nu}\mathbf{Q}^{*-1}|_{\widetilde\Gamma} = 0. \tag{4.43}$$

Here the matrix $\mathbf{Q}$ is constructed in Proposition 4.3 and we recall that $\mathbf{Q}^*$ is the adjoint matrix in $L^2(\Omega)$ over $\mathbb{R}$.

Let matrices $\widehat{\mathcal{P}_j}$ be constructed as $\mathcal{P}_j$ but with a different choice of the pairs $(U_0(k), T_0(k))$, $(V_0(k), W_0(k))$ that are solutions to problems (3.5) and (3.30) respectively, and satisfy (3.10) and (3.37). In such a way, we obtain another matrix $\mathbf{Q}$ that satisfies (4.24) with a possibly different set $\mathcal{X}$. We denote such a matrix $\mathbf{Q}$ by $\widehat{\mathbf{Q}}$. By the uniqueness of the Cauchy problem for the ∂_z-operator, we have

$$\mathbf{Q} = \widehat{\mathbf{Q}} \quad \text{on } \Omega \setminus \{x \in \bar\Omega;\ \det(\mathcal{P}_1\mathcal{P}_2\hat{\mathcal{P}}_1\hat{\mathcal{P}}_2)(x) = 0\}. \tag{4.44}$$

Let $x_* \in \bar\Omega$ be a point such that $\det(\mathcal{P}_1\mathcal{P}_2)(x_*) = 0$. We choose the matrices $\hat{\mathcal{P}}_j$ such that the determinants of these matrices are not equal to zero in some neighborhood of the point x_*. Then by (4.44) the matrix $\mathbf{Q}^{*-1}$ can be extended in a neighborhood of x_* as a $C^{6+\alpha}$-matrix. Hence

$$2\partial_{\bar z}\mathbf{Q}^{*-1} + \mathbf{Q}^{*-1}A_2^* - A_1^*\mathbf{Q}^{*-1} = 0 \quad \text{in } \Omega. \tag{4.45}$$

By (4.41) and the uniqueness of the Cauchy problem for the ∂_z-operator, we obtain

$$\mathbf{G} = \mathbf{Q}^{*-1} \quad \text{in } \Omega \setminus \{x \in \bar\Omega;\ \det \mathcal{Y}(x) = 0\}.$$

Repeating the above argument, we obtain that the matrix $\mathbf{G}^{-1} \in C^{6+\alpha}(\overline{\Omega})$ can be defined. Therefore the matrix $\mathbf{Q}$ belongs to the space $C^{6+\alpha}(\bar\Omega)$ and solves equation (4.24) in Ω.

The operator $\widetilde{L}_1(x, D) = \mathbf{Q}^{-1}L_1(x, D)\mathbf{Q}$ has the form

$$\widetilde{L}_1(x, D) = \Delta + 2A_2\partial_z + 2\widetilde{B}_1\partial_{\bar z} + \widetilde{Q}_1,$$

where

$$\widetilde{B}_1 = \mathbf{Q}^{-1}(B_1\mathbf{Q} + 2\partial_{\bar z}\mathbf{Q}), \quad \widetilde{Q}_1 = \mathbf{Q}^{-1}(Q_1\mathbf{Q} + \Delta\mathbf{Q} + 2A_1\partial_z\mathbf{Q} + 2B_1\partial_{\bar z}\mathbf{Q}).$$

The Dirichlet-to-Neumann maps of the operators $L_1(x, D)$ and $\widetilde{L}_1(x, D)$ are the same. Let $\widetilde{u}_1$ be the complex geometric optics solution for $\widetilde{L}_1(x, D)$ constructed in the same way as the solution for the operator $L_1(x, D)$. In fact, we can set

$\widetilde{u}_1 = \mathbf{Q}u_1$ where u_1 is the complex geometric optics solution given by (3.28) constructed for the operator $L_1(x, D)$. For the elements of the complex geometric optics solution $\widetilde{u}_1$ such as $U_{0,\tau}, T_{0,\tau}$, we use the same notation as in the construction of the function u_1. Since the Dirichlet-to-Neumann maps for the operators $\widetilde{L}_1(x, D)$ and $L_2(x, D)$ are equal, there exists a solution u_2 to

$$L_2(x, D)u_2 = 0 \quad \text{in } \Omega, \quad (\widetilde{u}_1 - u_2)|_{\partial\Omega} = 0, \quad \partial_{\vec{\nu}}(\widetilde{u}_1 - u_2) = 0 \quad \text{on } \widetilde{\Gamma}.$$

Setting $\widetilde{u} = \widetilde{u}_1 - u_2$, $\widetilde{\mathcal{B}} = \widetilde{B}_1 - B_2$ and $\widetilde{\mathcal{Q}} = \widetilde{Q}_1 - Q_2$, we have

$$L_2(x, D)\widetilde{u} + 2\widetilde{\mathcal{B}}\partial_{\bar{z}}\widetilde{u}_1 + \widetilde{\mathcal{Q}}\widetilde{u}_1 = 0 \quad \text{in } \Omega \tag{4.46}$$

and

$$\widetilde{u}|_{\partial\Omega} = 0, \qquad \partial_{\vec{\nu}}\widetilde{u}|_{\widetilde{\Gamma}} = 0. \tag{4.47}$$

Let v be the function given by (3.49). Taking the scalar product of (4.46) with v in $L^2(\Omega)$ over real numbers and using (3.50) and (4.47), we obtain

$$\int_\Omega (2\widetilde{\mathcal{B}}\partial_{\bar{z}}\widetilde{u}_1 + \widetilde{\mathcal{Q}}\widetilde{u}_1, v)\, dx = \int_\Omega (2\widetilde{\mathcal{B}}\partial_{\bar{z}}U + \widetilde{\mathcal{Q}}U, V)\, dx + o\Big(\frac{1}{\tau}\Big) = 0, \tag{4.48}$$

where the function V is given by (3.73) and

$$U = U_{0,\tau}e^{\tau\Phi} + T_{0,\tau}e^{\tau\overline{\Phi}} - e^{\tau\Phi}\widetilde{\mathcal{R}}_{\tau,\widetilde{B}_1}(e_1(q_1 + \widetilde{q}_1/\tau)) - e^{\tau\overline{\Phi}}\mathcal{R}_{\tau,A_2}(e_1(q_2 + \widetilde{q}_2/\tau)). \tag{4.49}$$

We have

Proposition 4.5. *The following equalities are true:*

$$S^*_{\widetilde{B}_1}(\widetilde{\mathcal{B}}^*V_0) = S^*_{\widetilde{B}_1}(\bar{\Phi}'\widetilde{\mathcal{B}}^*V_0) = S^*_{-B_2^*}(\widetilde{\mathcal{B}}T_0) = S^*_{-B_2^*}(\bar{\Phi}'\widetilde{\mathcal{B}}T_0) = 0 \quad \textit{on } \widetilde{\Gamma} \tag{4.50}$$

and

$$I_{\pm,\Phi}(\widetilde{x}) = 0. \tag{4.51}$$

Proof. Since the matrix $\mathcal{P}_1$ satisfies the equality $2\partial_{\bar{z}}\mathcal{P}_1 + A_2\mathcal{P}_1 = 0$, the matrix $\mathcal{P}_2^*\mathcal{P}_1$ is holomorphic in the domain Ω. Indeed,

$$2\partial_{\bar{z}}(\mathcal{P}_2^*\mathcal{P}_1) = 2(\partial_{\bar{z}}\mathcal{P}_2^*\mathcal{P}_1 + \mathcal{P}_2^*\partial_{\bar{z}}\mathcal{P}_1) = \mathcal{P}_2^*A_2\mathcal{P}_1 - \mathcal{P}_2^*A_2\mathcal{P}_1 = 0. \tag{4.52}$$

In order to obtain the last equality, we used $2\partial_{\bar{z}}\mathcal{P}_2^* = A_2^*\mathcal{P}_2^*$. Equality (4.52) implies

$$\int_{\partial\Omega} (\nu_1 + i\nu_2)\Phi'(\mathcal{P}_1\boldsymbol{a}, \mathcal{P}_2\boldsymbol{b})\, d\sigma = 0. \tag{4.53}$$

By (4.48) the conclusion of Proposition 4.2 holds true, if the operator $L_1(x, D)$ is replaced by the operator $\widetilde{L}_1(x, D)$.

From (4.53) and (3.51), we obtain

$$\int_{\partial\Omega} (\nu_1 - i\nu_2)\bar{\Phi}'(\mathcal{C}_1\bar{\boldsymbol{a}}, \mathcal{C}_2\bar{\boldsymbol{b}})\, d\sigma = 0. \tag{4.54}$$

By Proposition 4.2, there exists an antiholomorphic function $\widetilde{\Theta}$ in Ω such that $\mathcal{C}_2^*\mathcal{C}_1 = \widetilde{\Theta}(\bar{z})$ on $\widetilde{\Gamma}$. Hence

$$\begin{aligned}\int_{\widetilde{\Gamma}} (\nu_1 - i\nu_2)\bar{\Phi}'(\mathcal{C}_2^*\mathcal{C}_1\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}})\, d\sigma &= \int_{\widetilde{\Gamma}} (\nu_1 - i\nu_2)\bar{\Phi}'(\widetilde{\Theta}\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}})\, d\sigma \\ &= -\int_{\Gamma_0} (\nu_1 - i\nu_2)\bar{\Phi}'(\widetilde{\Theta}\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}})\, d\sigma.\end{aligned}$$

We write (4.54) as

$$\int_{\Gamma_0} (\nu_1 - i\nu_2)\bar{\Phi}'((\mathcal{C}_2^*\mathcal{C}_1 - \widetilde{\Theta})\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}})\, d\sigma = 0. \tag{4.55}$$

Therefore, by [7, Corollary 7.1], from (4.55) we obtain

$$\mathcal{C}_2^*\mathcal{C}_1 = \widetilde{\Theta} \quad \text{on } \partial\Omega. \tag{4.56}$$

We observe that for the construction of the function U_0, instead of the matrix $\mathcal{C}_1$, we can also use the matrix $\widetilde{\mathcal{C}}_1$. In that case the equality (4.56) has the form

$$\mathcal{C}_2^*\widetilde{\mathcal{C}}_1 = \widetilde{\Theta}_* \quad \text{on } \partial\Omega, \tag{4.57}$$

where $\widetilde{\Theta}_*$ is some antiholomorphic function in Ω. We define $S^*_{\widetilde{B}_1}(\bar{\Phi}'\widetilde{\mathcal{B}}^*V_0)$ on $\mathbb{R}^2 \setminus \bar{\Omega}$ by formula (2.37). Now let $y = (y_1, y_2) \in \widetilde{\Gamma}$ be an arbitrary point and $z = y_1 + iy_2$. Then, thanks to (4.23), for any sequence $\{y_j\}_{j=1}^\infty \subset \mathbb{R}^2 \setminus \bar{\Omega}$ such that $y_j \to y$, we have

$$S^*_{\widetilde{B}_1}(\bar{\Phi}'\widetilde{\mathcal{B}}^*V_0)(y_j) \to S^*_{\widetilde{B}_1}(\bar{\Phi}'\widetilde{\mathcal{B}}^*V_0)(y) \quad \text{as } j \to +\infty. \tag{4.58}$$

Indeed, by (2.37) and (4.23), there exists a constant C such that

$$|S^*_{\widetilde{B}_1}(\bar{\Phi}'\widetilde{\mathcal{B}}^*V_0)(y_j) - S^*_{\widetilde{B}_1}(\bar{\Phi}'\widetilde{\mathcal{B}}^*V_0)(y)| \le C \int_\Omega \|\widetilde{\mathcal{B}}^*(\xi)\| \left| \frac{1}{z_j - \zeta} - \frac{1}{z - \zeta} \right| d\xi, \tag{4.59}$$

where $z_j = y_{j,1} + iy_{j,2}$. Since $\widetilde{\mathcal{B}}^*(\xi) = 0$, $\xi \in \widetilde{\Gamma}$ by (4.23), the sequence

$$\left\{ \|\widetilde{\mathcal{B}}^*(\xi)\| \left| \frac{1}{z_j - \zeta} - \frac{1}{z - \zeta} \right| \right\}_{j=1}^\infty$$

is bounded in $L^\infty(\Omega)$. Moreover for any positive δ the above sequence converges to zero in $L^\infty(\Omega \setminus B(y, \delta))$. Thus, from these facts and (4.59), we obtain (4.58) immediately.

By (4.56) and (4.57), we have

$$
\begin{aligned}
(4.60)\quad S^*_{\widetilde{B}_1}(\bar\Phi'\widetilde{\mathcal{B}}^*V_0)(y_j) &= \frac{1}{2}(\mathcal{C}_1^{-1}r_{0,1})(y_j)\partial_z^{-1}(\mathcal{C}_1^*\bar\Phi'\widetilde{\mathcal{B}}^*V_0)(y_j)\\
&\quad+\frac{1}{2}(\widetilde{\mathcal{C}}_1^{-1}(1-r_{0,1}))(y_j)\partial_z^{-1}(\widetilde{\mathcal{C}_1}^*\bar\Phi'\widetilde{\mathcal{B}}^*V_0)(y_j)\\
&= -\frac{1}{2\pi}r_{0,1}(\bar z_j)(\mathcal{C}_1^{-1})^*(y_j)\int_\Omega \frac{\partial_z(\bar\Phi'\mathcal{C}_1^*\mathcal{C}_2)\bar{\boldsymbol b}}{\bar z_j-\bar\zeta}\,d\xi\\
&\quad-(1-r_{0,1}(\bar z_j))(\widetilde{\mathcal{C}}_1^{-1})^*(y_j)\frac{1}{2\pi}\int_\Omega \frac{\partial_z(\bar\Phi'\widetilde{\mathcal{C}}_1^*\mathcal{C}_2)\bar{\boldsymbol b}}{\bar z_j-\bar\zeta}\,d\xi\\
&= -\frac{1}{4\pi}r_{0,1}(\bar z_j)(\mathcal{C}_1^{-1})^*(y_j)\int_{\partial\Omega} \frac{(\nu_1-i\nu_2)\widetilde\Theta^*\bar\Phi'\bar{\boldsymbol b}}{\bar z_j-\bar\zeta}\,d\sigma\\
&\quad-(1-r_{0,1}(\bar z_j))(\widetilde{\mathcal{C}}_1^{-1})^*(y_j)\frac{1}{4\pi}\int_{\partial\Omega} \frac{(\nu_1-i\nu_2)\widetilde\Theta^*_*\bar\Phi'\bar{\boldsymbol b}}{\bar z_j-\bar\zeta}\,d\sigma\\
&= 0.
\end{aligned}
$$

Here, in order to obtain the last equality, we used the fact that $z_j\notin\Omega$ and therefore the functions $\frac{\widetilde\Theta^*_*\bar\Phi'\bar{\boldsymbol b}}{\bar z_j-\bar\zeta}$, $\frac{\widetilde\Theta^*\bar\Phi'\bar{\boldsymbol b}}{\bar z_j-\bar\zeta}$ are antiholomorphic in Ω. From (4.58) and (4.60), we have $S^*_{\widetilde{B}_1}(\bar\Phi'\widetilde{\mathcal{B}}^*V_0)|_{\widetilde\Gamma}=0$. The proof of the remaining equalities in (4.50) is the same. Next we show that $I_{\pm,\Phi}(\widetilde x)=0$. By (3.24) and (3.44), we have

$$
\begin{aligned}
(4.61)\qquad I_{\pm,\Phi}(\widetilde x) = \int_{\partial\Omega}\Big\{&(\nu_1-i\nu_2)((2\mathcal{C}_2^*\mathcal{C}_1\boldsymbol{b}_{\pm,\widetilde x}\bar\Phi',\bar{\widetilde{\boldsymbol b}})+(2\bar\Phi'\mathcal{C}_2^*\mathcal{C}_1\overline{\boldsymbol a},\widetilde{\boldsymbol a}_{\pm,\widetilde x}))\\
&+(\nu_1+i\nu_2)((2\mathcal{P}_2^*\mathcal{P}_1\boldsymbol{a}_{\pm,\widetilde x}\Phi',\widetilde{\boldsymbol b})+(2\Phi'\mathcal{P}_2^*\mathcal{P}_1\boldsymbol a,\widetilde{\boldsymbol b}_{\pm,\widetilde x}))\Big\}\,d\sigma.
\end{aligned}
$$

Since by (4.56) the restriction of the function $\mathcal{C}_2^*\mathcal{C}_1$ on $\partial\Omega$ coincides with the restriction of some antiholomorphic function in Ω and by (4.52) the function $\mathcal{P}_2^*\mathcal{P}_1$ is holomorphic in Ω, the equality (4.61) implies (4.51). The proof of the proposition is complete. □

We use the above proposition to prove

Proposition 4.6. *The following equalities hold true:*

$$
(4.62)\qquad \bar\Phi' S^*_{\widetilde{B}_1}(\widetilde{\mathcal{B}}^*V_0) = S^*_{\widetilde{B}_1}(\bar\Phi'\widetilde{\mathcal{B}}^*V_0),
$$

$$
(4.63)\qquad \bar\Phi' S^*_{-B_2^*}(\widetilde{\mathcal{B}}T_0) = S^*_{-B_2^*}(\bar\Phi'\widetilde{\mathcal{B}}T_0).
$$

Proof. Denote $r=\bar\Phi' S^*_{\widetilde{B}_1}(\widetilde{\mathcal{B}}^*V_0)-S^*_{\widetilde{B}_1}(\bar\Phi'\widetilde{\mathcal{B}}^*V_0)$. Then this function satisfies

$$
2\partial_{\bar z}r-\widetilde{B}_1^*r=0\quad\text{in }\Omega.
$$

Proposition 4.5 yields

$$r|_{\widetilde{\Gamma}} = 0.$$

By the uniqueness of the Cauchy problem for the $\partial_{\bar z}$-operator, we obtain $r \equiv 0$. The proof of (4.63) is the same. □

We use Proposition 4.6 to prove

Proposition 4.7. *Under the conditions of Proposition 4.2, we have*

$$-(\widetilde{\mathcal{B}}A_2U_0, V_0) - (\widetilde{Q}_1(1)U_0, S^*_{\widetilde{B}_1}(\widetilde{\mathcal{B}}^*V_0)) + (\widetilde{\mathcal{Q}}U_0, V_0) = 0 \quad in\ \Omega \tag{4.64}$$

and

$$\begin{aligned} 2(\partial_{\bar z}\widetilde{\mathcal{B}}T_0, W_0) + (\widetilde{\mathcal{B}}T_0, B_2^*W_0) - (\widetilde{\mathcal{Q}}T_0, W_0)& \\ - (Q_1(2)W_0, S^*_{-B_2^*}(\widetilde{\mathcal{B}}T_0)) = 0 \quad in\ \Omega.& \end{aligned} \tag{4.65}$$

Proof. We recall that Φ satisfies (3.1), (3.2) and

$$\operatorname{Im}\Phi(\widetilde{x}) \notin \{\operatorname{Im}\Phi(x);\, x \in \mathcal{H}\setminus\{\widetilde{x}\}\}. \tag{4.66}$$

By Proposition 4.2, equality (4.2) holds true. Thanks to (4.66), (4.23) and Proposition 4.6, we can write (4.2) as

$$(J_\pm + K_\pm)(\widetilde{x}) + I_{\pm,\Phi}(\widetilde{x}) = 0.$$

This equality and Proposition 4.5 imply

$$(J_\pm + K_\pm)(\widetilde{x}) = 0. \tag{4.67}$$

By Propositions 4.1 and 4.6, we obtain

$$\begin{aligned} &\mathfrak{F}_{\tau,\widetilde{x}}(q_1, S^*_{\widetilde{B}_1}(\widetilde{B}_1^*\widetilde{\mathcal{A}}^*V_0) - \widetilde{\mathcal{A}}^*V_0 + 2S^*_{\widetilde{B}_1}(\partial_z\widetilde{\mathcal{B}}^*V_0) \\ &\qquad + S^*_{\widetilde{B}_1}(\widetilde{\mathcal{B}}^*(A_2^*V_0 - 2\tau\bar\Phi'V_0))) \\ &= -2\tau\mathfrak{F}_{\tau,\widetilde{x}}(q_1, S^*_{\widetilde{B}_1}(\widetilde{\mathcal{B}}^*\bar\Phi'V_0)) + o\Big(\frac{1}{\tau}\Big) \\ &= -2\tau\mathfrak{F}_{\tau,\widetilde{x}}(q_1, \bar\Phi' S^*_{\widetilde{B}_1}(\widetilde{\mathcal{B}}^*V_0)) + o\Big(\frac{1}{\tau}\Big) \\ &= -\frac{\pi}{2|\det\psi''(\widetilde{x})|^{1/2}}(2\partial_{\bar z}q_1, S^*_{\widetilde{B}_1}(\widetilde{\mathcal{B}}^*V_0))(\widetilde{x}) + o\Big(\frac{1}{\tau}\Big) \\ &= -\frac{\pi}{2|\det\psi''(\widetilde{x})|^{1/2}}(\widetilde{Q}_1(1)U_0, S^*_{\widetilde{B}_1}(\widetilde{\mathcal{B}}^*V_0))(\widetilde{x}) + o\Big(\frac{1}{\tau}\Big) \quad \text{as } \tau \to +\infty \end{aligned} \tag{4.68}$$

and

$$
(4.69)\qquad -2\mathfrak{F}_{\tau,\widetilde{x}}(P^*_{-A_2^*}(\widetilde{\mathcal{A}}(\partial_z U_0+\tau\Phi' U_0))+\widetilde{\mathcal{B}}\partial_{\bar z}U_{0,\tau},q_4)
= -2\mathfrak{F}_\tau(P^*_{-A_2^*}(\widetilde{\mathcal{A}}\tau\Phi' U_0),q_4)+o\Big(\frac{1}{\tau}\Big)=o\Big(\frac{1}{\tau}\Big)\quad \text{as } \tau\to+\infty.
$$

By (4.68) and (4.69), we have

$$
(4.70)\qquad K_+(\widetilde{x}) = -\frac{\pi}{2|\det\psi''(\widetilde{x})|^{1/2}}(\widetilde{Q}_1(1)U_0, S^*_{\widetilde{B}_1}(\widetilde{\mathcal{B}}^*V_0))(\widetilde{x})+o\Big(\frac{1}{\tau}\Big)\quad \text{as } \tau\to+\infty.
$$

In a similar way, we compute $K_-(\widetilde{x})$:

$$
(4.71)\qquad \mathfrak{F}_{-\tau,\widetilde{x}}(q_2,P^*_{A_2}(2\partial_z(\widetilde{\mathcal{A}}^*W_0)-2\tau\Phi'\widetilde{\mathcal{A}}^*W_0)-\widetilde{\mathcal{B}}^*W_0+P^*_{A_2}(A_2^*\widetilde{\mathcal{B}}^*W_0))
= -2\tau\mathfrak{F}_{-\tau,\widetilde{x}}(q_2,P^*_{A_2}(\Phi'\widetilde{\mathcal{A}}^*W_0))+o\Big(\frac{1}{\tau}\Big)=o\Big(\frac{1}{\tau}\Big)\quad \text{as } \tau\to+\infty
$$

and

$$
(4.72)\qquad -2\mathfrak{F}_{-\tau,\widetilde{x}}(q_3,S^*_{-B_2^*}(2\widetilde{\mathcal{A}}\partial_z T_0+2\widetilde{\mathcal{B}}(\partial_{\bar z}T_0+\tau\bar\Phi' T_0)))
= -2\mathfrak{F}_{-\tau,\widetilde{x}}(q_3,S^*_{-B_2^*}(\tau\widetilde{\mathcal{B}}\bar\Phi' T_0))+o\Big(\frac{1}{\tau}\Big)
= \frac{\pi}{2|\det\psi''(\widetilde{x})|^{1/2}}(Q_1(2)W_0,S^*_{-B_2^*}(\widetilde{\mathcal{B}}T_0))+o\Big(\frac{1}{\tau}\Big)\quad \text{as } \tau\to+\infty.
$$

By (4.71) and (4.72), we have

$$
(4.73)\qquad K_-(\widetilde{x}) = \frac{\pi}{2|\det\psi''(\widetilde{x})|^{1/2}}(Q_1(2)W_0,S^*_{-B_2^*}(\widetilde{\mathcal{B}}T_0))+o\Big(\frac{1}{\tau}\Big)\quad \text{as } \tau\to+\infty.
$$

Substituting the right-hand side of formulae (4.70) and (4.73) into (4.67), we obtain (4.64) and (4.65).

Since by (3.4) for any $x\in\Omega$, there exists a sequence $\{x_\epsilon\}_{\epsilon\in(0,1)}$ converging to x, we rewrite equations (4.64) and (4.65) as

$$
(4.74)\qquad -(\widetilde{\mathcal{B}}A_1U_0,V_0)-(\widetilde{Q}_1(1)U_0,S^*_{\widetilde{B}_1}(\widetilde{\mathcal{B}}^*V_0))+(\widetilde{\mathcal{Q}}U_0,V_0)=0\quad \text{in } \Omega
$$

and

$$
-2(\partial_{\bar z}\widetilde{\mathcal{B}}\widetilde{U}_0,W_0)-(\widetilde{\mathcal{B}}\widetilde{U}_0,B_2^*W_0)+(\widetilde{\mathcal{Q}}\widetilde{U}_0,W_0)+(Q_1(2)W_0,S^*_{-B_2^*}(\widetilde{\mathcal{B}}T_0))=0\ \text{ in } \Omega.
$$

The proof of the proposition is complete. □

§5. Step 3: End of the proof

End of the proof. Let $\widetilde{\gamma}$ be a curve that does not intersect itself and passes through the point $\hat{x}$ and a couple of points x_1, $x_2 \in \widetilde{\Gamma}$ such that the set $\widetilde{\gamma} \cap \partial\Omega \setminus \{x_1, x_2\}$ is empty. Denote by Ω_1 the domain bounded by $\widetilde{\gamma}$ and the part of $\partial\Omega$ located between the points x_1 and x_2. Then we set $\Omega_{1,\epsilon} = \{x \in \Omega; \mathrm{dist}(\Omega_1, x) < \epsilon\}$. By Proposition 2.1, for each point $\hat{x}$ from $\Omega_{1,\epsilon}$ one can construct pairs of functions $(U_0^{(k)}, T_0^{(k)})$, $(V_0^{(\ell)}, W_0^{(\ell)})$ satisfying (3.5), (3.30) and

$$T_0^{(k)}(\hat{x}) = \vec{e}_k, \quad W_0^{(\ell)}(\hat{x}) = \vec{e}_\ell, \quad \forall\, k, \ell \in \{1, \dots, N\}.$$

Then for each $\hat{x}$ there exists a positive $\delta(\hat{x})$ such that the matrices $\{T_{0,i}^{(j)}\}$ and $\{W_{0,i}^{(j)}\}$ are invertible for any $x \in \overline{B(\hat{x}, \delta(\hat{x}))}$. From the covering of $\bar{\Omega}_{1,\epsilon}$ by such balls, we take a finite subcovering $\bar{\Omega}_{1,\epsilon} \subset \cup_{k=1}^{\widetilde{N}} B(x_k, \delta(x_k))$. Then from (4.65) we have a differential inequality:

$$|\partial_{\bar{z}} \widetilde{\mathcal{B}}_{ij}| \le C_1(\epsilon) \left(\sum_{k=1}^{N} |S^*_{-B_2^*}(\widetilde{\mathcal{B}}^* T_0^{(k)})| + |\widetilde{\mathcal{B}}| + |\widetilde{\mathcal{Q}}| \right) \quad \text{in } \Omega_{1,\epsilon}, \quad \forall\, i, j \in \{1, \dots, N\}. \tag{5.1}$$

Let $\phi_0 \in C^2(\bar{\Omega})$ satisfy

$$\nabla \phi_0(x) \ne 0 \quad \text{in } \Omega_1, \qquad \partial_{\widetilde{\nu}} \phi_0|_{\widetilde{\gamma}} \le \alpha' < 0, \qquad \phi_0|_{\widetilde{\gamma}} = 0, \tag{5.2}$$

where α' is some constant and $\widetilde{\nu}$ is the outward normal vector to $\Omega_{1,\epsilon}$ and χ_ϵ satisfies

$$\chi_\epsilon \in C^2(\overline{\Omega_{1,\epsilon}}), \quad \chi_\epsilon = 1 \quad \text{in } \Omega_1,$$

and $\chi_\epsilon \equiv 0$ in some neighborhood of the curve $\partial\Omega_{1,\epsilon} \setminus \widetilde{\Gamma}$. From (5.1), (4.23) and (4.50), we have

$$|\partial_{\bar{z}}(\chi_\epsilon \widetilde{\mathcal{B}}_{ij})| \le C_2(\epsilon) \left(\sum_{k=1}^{N} |\chi_\epsilon S^*_{-B_2^*}(\widetilde{\mathcal{B}}^* T_0^{(k)})| + |\chi_\epsilon \widetilde{\mathcal{B}}| + |[\chi_\epsilon, \partial_{\bar{z}}] \widetilde{\mathcal{B}}_{ij}| + |\chi_\epsilon \widetilde{\mathcal{Q}}| \right) \quad \text{in } \Omega_{1,\epsilon}, \quad \forall\, i, j \in \{1, \dots, N\}, \tag{5.3}$$

$$\chi_\epsilon \widetilde{\mathcal{B}}|_{\partial\Omega_{1,\epsilon}} = \partial_{\widetilde{\nu}}(\chi_\epsilon \widetilde{\mathcal{B}})|_{\partial\Omega_{1,\epsilon}} = 0. \tag{5.4}$$

Here we recall that $[\cdot, \cdot]$ is the commutator.

Set $\psi_0 = e^{\lambda\phi_0}$ with sufficiently large positive λ. Applying the Carleman estimate to the boundary value problem (5.3) and (5.4), we see that there exist constants C_3 and τ_0, both independent of τ, such that

$$(5.5)\qquad \int_{\Omega_{1,\epsilon}} e^{2\tau\psi_0}\left(\frac{1}{\tau}|\nabla\chi_\epsilon\widetilde{\mathcal{B}}|^2+\tau|\chi_\epsilon\widetilde{\mathcal{B}}|^2\right)dx \\ \le C_3\int_{\Omega_{1,\epsilon}}\Bigg(\sum_{k=1}^N|\chi_\epsilon S^*_{-B_2^*}(\widetilde{\mathcal{B}}^*T_0^{(k)})|^2+\chi_\epsilon^2(|\widetilde{\mathcal{B}}|^2+|\widetilde{\mathcal{Q}}|^2) \\ +|[\chi_\epsilon,\partial_{\bar z}]\widetilde{\mathcal{B}}|^2\Bigg)e^{2\tau\psi_0}\,dx,\quad \forall\,\tau\ge\tau_0.$$

By the Carleman estimate for the operator ∂_z and (4.50), there exist constants C_4 and τ_0, independent of τ, such that

$$(5.6)\qquad \int_{\Omega_{1,\epsilon}}|\chi_\epsilon S^*_{-B_2^*}(\widetilde{\mathcal{B}}^*T_0^{(k)})|^2e^{2\tau\psi_0}\,dx \\ \le C_4\int_{\Omega_{1,\epsilon}}\left(|[\chi_\epsilon,\partial_z]S^*_{-B_2^*}(\widetilde{\mathcal{B}}^*T_0^{(k)})|^2+|\chi_\epsilon\widetilde{\mathcal{B}}^*T_0^{(k)}|^2\right)e^{2\tau\psi_0}\,dx$$

and

$$(5.7)\qquad \int_{\Omega_{1,\epsilon}}|\chi_\epsilon S^*_{\widetilde{B}_1}(\widetilde{\mathcal{B}}^*V_0^{(k)})|^2e^{2\tau\psi_0}\,dx \\ \le C_4\int_{\Omega_{1,\epsilon}}\left(|[\chi_\epsilon,\partial_z]S^*_{\widetilde{B}_1}(\widetilde{\mathcal{B}}^*V_0^{(k)})|^2+|\chi_\epsilon\widetilde{\mathcal{B}}^*V_0^{(k)}|^2\right)e^{2\tau\psi_0}\,dx$$

for all $\tau\ge\tau_0$.

Combining estimates (5.5) and (5.6), we obtain that there exists a constant C_5, independent of τ, such that

$$(5.8)\qquad \int_{\Omega_{1,\epsilon}} e^{2\tau\psi_0}\left(\frac{1}{\tau}|\nabla(\chi_\epsilon\widetilde{\mathcal{B}})|^2+\tau|\chi_\epsilon\widetilde{\mathcal{B}}|^2\right)dx \\ \le C_5\int_{\Omega_{1,\epsilon}}\Bigg(\chi_\epsilon^2(|\widetilde{\mathcal{B}}|^2+|\widetilde{\mathcal{Q}}|^2)+\sum_{k=1}^N|[\chi_\epsilon,\partial_z]S^*_{-B_2^*}(\widetilde{\mathcal{B}}^*T_0^{(k)})|^2 \\ +|[\chi_\epsilon,\partial_{\bar z}]\widetilde{\mathcal{B}}|^2\Bigg)e^{2\tau\psi_0}\,dx,\quad \forall\,\tau\ge\tau_0.$$

For all sufficiently large τ, the term $\int_{\Omega_{1,\epsilon}}|\chi_\epsilon\widetilde{\mathcal{B}}|^2e^{2\tau\psi_0}\,dx$ can be absorbed into the left-hand side of the inequality (5.8). Moreover, thanks to the choice of the function χ_ϵ, the supports of the coefficients of the commutator operator $[\chi_\epsilon,\partial_{\bar z}]$ are located in the domain $\Omega_{1,\epsilon}\setminus\Omega_{1,\epsilon/2}$. Hence one can write the estimate (5.8) as

$$(5.9)\qquad \int_{\Omega_{1,\epsilon}} e^{2\tau\psi_0}\left(\frac{1}{\tau}|\nabla(\chi_\epsilon\widetilde{\mathcal{B}})|^2+\tau|\chi_\epsilon\widetilde{\mathcal{B}}|^2\right)dx$$

$$\leq C_6\bigg(\int_{\Omega_{1,\epsilon}} \chi_\epsilon^2|\widetilde{\mathcal{Q}}|^2 e^{2\tau\psi_0}\,dx + \int_{\Omega_{1,\epsilon}\setminus\Omega_{1,\epsilon/2}} \Big(\sum_{k=1}^N |[\chi_\epsilon,\partial_z]S^*_{-B_2^*}(\widetilde{\mathcal{B}}^*T_0^{(k)})|^2 + |[\chi_\epsilon,\partial_{\bar z}]\widetilde{\mathcal{B}}|^2\Big)e^{2\tau\psi_0}\,dx\bigg), \quad \forall\,\tau\geq\tau_1.$$

By Proposition 2.1, for each point $\hat{x}\in\Omega$, one can construct pairs of functions $(U_0^{(k)},T_0^{(k)})$, $(V_0^{(\ell)},W_0^{(\ell)})$ satisfying (3.5), (3.30) and

$$U_0^{(k)}(\hat{x}) = \vec{e}_k, \quad V_0^{(\ell)}(\hat{x}) = \vec{e}_\ell, \quad \forall\,k,\ell\in\{1,\dots,N\}.$$

Then for each $\hat{x}\in\bar{\Omega}_{1,\epsilon}$ there exists positive $\delta(\hat{x})$ such that the matrices $\{U_{0,i}^{(j)}\}$ and $\{V_{0,i}^{(j)}\}$ are invertible for any $x\in\overline{B(\hat{x},\delta(\hat{x}))}$. From the covering of $\Omega_{1,\epsilon}$ by such balls, we take a finite subcovering $\bar{\Omega}\subset\cup_{k=\widetilde{N}}^{\widetilde{N}+N^*}B(x_k,\delta(x_k))$. Then there exists $C_7(\epsilon)>0$ such that

$$|\widetilde{\mathcal{Q}}| \leq C_7(\epsilon)\bigg(|\widetilde{\mathcal{B}}| + \sum_{k=\widetilde{N}+1}^{\widetilde{N}+N^*}|S^*_{\widetilde{B}_1}(\widetilde{\mathcal{B}}^*V_0^{(k)})|\bigg) \quad \text{in } \Omega_{1,\epsilon}. \tag{5.10}$$

Combining (5.7), (5.9) and (5.10), we obtain that there exists a constant C_8, independent of τ, such that

$$\begin{aligned}\int_{\Omega_{1,\epsilon}} & e^{2\tau\psi_0}\Big(\frac{1}{\tau}|\nabla(\chi_\epsilon\widetilde{\mathcal{B}})|^2 + \tau|\chi_\epsilon\widetilde{\mathcal{B}}|^2\Big)\,dx \\ &\leq C_8\int_{\Omega_{1,\epsilon}\setminus\Omega_{1,\epsilon/2}}\bigg(\sum_{k=1}^N|[\chi_\epsilon,\partial_z]S^*_{-B_2^*}(\mathcal{B}^*T_0^{(k)})|^2 \\ &\quad + \sum_{k=\widetilde{N}+1}^{\widetilde{N}+N^*}|[\chi_\epsilon,\partial_z]S^*_{\widetilde{B}_1}(\widetilde{\mathcal{B}}^*V_0^{(k)})|^2 + |[\chi_\epsilon,\partial_{\bar z}]\widetilde{\mathcal{B}}|^2\bigg)e^{2\tau\psi_0}\,dx, \quad \forall\,\tau\geq\tau_1.\end{aligned} \tag{5.11}$$

By (5.2), for all sufficiently small positive ϵ, there exists a positive constant $\theta<1$ such that

$$\psi_0(x) < \theta \quad \text{on } \Omega_{1,\epsilon}\setminus\Omega_{1,\epsilon/2}. \tag{5.12}$$

Since $\hat{x}\in\operatorname{supp}\widetilde{\mathcal{B}}\cap\widetilde{\gamma}$ and $\partial_{\widetilde{\nu}}\phi_0|_{\widetilde{\gamma}}\leq\theta'<0$ with some constant θ', there exists a constant $\kappa>0$ such that

$$\kappa e^{2\tau} \leq \int_{\Omega_{1,\epsilon}} e^{2\tau\psi_0}|\chi_\epsilon\widetilde{\mathcal{B}}|^2 e^{2\tau\psi_0}\,dx, \quad \forall\,\tau\geq\tau_1. \tag{5.13}$$

By (5.12), we can estimate the right-hand side of inequality (5.9) as

$$(5.14)\quad C_9 \int_{\Omega_{1,\epsilon}\setminus\Omega_{1,\epsilon/2}} \left(\sum_{k=1}^{N} |[\chi_\epsilon, \partial_z] S^*_{-B_2^*}(\widetilde{\mathcal{B}}^* T_0^{(k)})|^2 + \sum_{k=\widetilde{N}+1}^{\widetilde{N}+N^*} |[\chi_\epsilon, \partial_z] S^*_{\widetilde{B}_1}(\widetilde{\mathcal{B}}^* V_0^{(k)})|^2 \right.$$
$$\left. + |[\chi_\epsilon, \partial_{\bar z}]\widetilde{\mathcal{B}}|^2 \right) e^{2\tau\psi_0}\, dx \le C_{10} e^{2\theta\tau}, \quad \forall \tau \ge \tau_1,$$

where constants $C_9, C_{10} > 0$ are independent of τ. Using (5.13) and (5.14) in (5.9), we obtain that there exists a constant C_{11}, independent of τ, such that

$$\kappa e^{2\tau} \le C_{11} e^{2\theta\tau}, \quad \forall \tau \ge \tau_1.$$

Since $\theta < 1$, we reach a contradiction. Hence

$$\widetilde{\mathcal{B}} = \widetilde{\mathcal{Q}} = 0 \quad \text{on } \Omega.$$

The proof of the theorem is complete. □

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References

[1] V. Alekseev, V. Tikhomirov and S. Fomin, *Optimal control*, Consultants Bureau, New York, 1987. Zbl 0689.49001 MR 0924574

[2] A. P. Calderón, On an inverse boundary value problem, in *Seminar on Numerical Analysis and its Applications to Continuum Physics*, Soc. Brasil. Mat., Río de Janeiro, 1980, 65–73. MR 0590275

[3] G. Eskin, Global uniqueness in the inverse scattering problem for the Schrödinger operator with external Yang-Mills potentials, Commun. Math. Phys. **222** (2001), 503–531. Zbl 0999.35066 MR 1888087

[4] G. Eskin and J. Ralston, Inverse boundary value problems for systems of partial differential equations, in *Recent Development in Theories and Numerics*, Y-C Hon, M. Yamamoto, J. Cheng and J-Y. Lee (eds), World Scientific Publishing, NJ, 2003, 105–113. Zbl 1209.35150 MR 2088194

[5] L. Hörmander, *The analysis of linear partial differential operators I*, Springer, Berlin, 1980. Zbl 1028.35001 MR 1996773

[6] O. Imanuvilov, G. Uhlmann and M. Yamamoto, The Calderón problem with partial data in two dimensions, J. Amer. Math. Soc. **23** (2010), 655–691. Zbl 1201.35183 MR 2629983

[7] ———, Partial Cauchy data for general second order elliptic operators in two dimensions, Publ. Research Institute Math. Sci. **48** (2012), 971–1055. Zbl 1260.35253 MR 2999548

[8] O. Imanuvilov and M. Yamamoto, Inverse problem by Cauchy data on an arbitrary sub-boundary for systems of elliptic equations, Inverse Problems **28** (2012), 095015. Zbl 1250.35184 MR 2972464

[9] ———, Uniqueness for inverse boundary value problems by Dirichlet-to-Neumann map on subboundaries, Milan J. Math. **81** (2013), 187–258. Zbl 1291.35443 MR 3129784

[10] T. Runst and W. Sickel, *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, Walter de Gruyter, Berlin, 1996. Zbl 0873.35001 MR 1419319

[11] I. Vekua, *Generalized analytic functions*, Pergamon Press, Oxford, 1962. Zbl 0100.07603 MR 0150320

[12] W. Wendland, *Elliptic systems in the plane*, Pitman, London, 1979. Zbl 0396.35001 MR 0518816