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Pull-back of Quasi-Log Structures

by

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Abstract

We prove that the pull-back of a quasi-log scheme by a smooth quasi-projective morphism has a natural quasi-log structure. We treat an application to log Fano pairs. This paper also contains a proof by Kento Fujita of the simple connectedness of log Fano pairs with only log canonical singularities.

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§1. Introduction

The following theorem is the main result of this paper; it is natural but missing from the literature. For a precise statement, see Theorem 3.5 below.

Theorem 1.1 (Pull-back of quasi-log structures). *Let $[X, \omega]$ be a quasi-log scheme and let $h : X' \to X$ be a smooth quasi-projective morphism. Then $[X', \omega']$, where $\omega' = h^*\omega \otimes \omega_{X'/X}$ with $\omega_{X'/X} = \det \Omega^1_{X'/X}$, has a natural quasi-log structure induced by h.*

In particular, if h is a finite étale morphism, then $[X', \omega']$, where $\omega' = h^*\omega$, has a natural quasi-log structure induced by h.

We make an important remark: we do not know whether Theorem 1.1 holds true or not without assuming that h is *quasi-projective*. The following corollary is an easy application of Theorem 1.1.

Corollary 1.2. *Let $[X, \omega]$ be a projective quasi-log canonical pair such that $-\omega$ is ample. Then the algebraic fundamental group of X is trivial, or equivalently, X has no nontrivial finite étale covers.*

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The following conjecture arises naturally from Corollary 1.2.

Conjecture 1.3. *Let $[X, \omega]$ be a projective quasi-log canonical pair such that $-\omega$ is ample. Then X is simply connected.*

In general, there exists an irreducible projective variety whose algebraic fundamental group is trivial and whose topological fundamental group is nontrivial (Example 5.4). The following conjecture is a special case of Conjecture 1.3.

Conjecture 1.4. *Let (X, Δ) be a projective semi-log canonical pair such that $-(K_X + \Delta)$ is ample. Then X is simply connected.*

It is well known that Conjecture 1.4 holds when (X, Δ) is Kawamata log terminal (see [T]). Kento Fujita pointed out that Conjecture 1.4 holds true when (X, Δ) is log canonical.

Theorem 1.5 (Fujita, Theorem 6.1). *Let (X, Δ) be a projective log canonical pair such that $-(K_X + \Delta)$ is ample. Then X is simply connected.*

We work over $\mathbb{C}$, the complex number field, throughout this paper. We recommend [F3] for a gentle introduction to the theory of quasi-log structures. Since [F2] will not be published, we reproduce some of the arguments from it in the current paper. For basic definitions and properties of semi-log canonical pairs, see [F6].

§2. Preliminaries

Notation 2.1. A pair $[X, \omega]$ consists of a scheme X and an $\mathbb{R}$-Cartier divisor (or $\mathbb{R}$-line bundle) ω on X. In this paper, a scheme means a separated scheme of finite type over $\operatorname{Spec} \mathbb{C}$. A variety is a reduced scheme.

Notation 2.2 (Divisors). Let B_1 and B_2 be two $\mathbb{R}$-Cartier divisors on a scheme X. Then B_1 is linearly (resp. $\mathbb{Q}$-linearly, or $\mathbb{R}$-linearly) equivalent to B_2, denoted by $B_1 \sim B_2$ (resp. $B_1 \sim_{\mathbb{Q}} B_2$, or $B_1 \sim_{\mathbb{R}} B_2$) if $B_1 = B_2 + \sum_{i=1}^{k} r_i(f_i)$ such that $f_i \in \Gamma(X, \mathcal{K}_X^*)$ and $r_i \in \mathbb{Z}$ (resp. $r_i \in \mathbb{Q}$, or $r_i \in \mathbb{R}$) for every i. Here, $\mathcal{K}_X$ is the sheaf of total quotient rings of $\mathcal{O}_X$, and $\mathcal{K}_X^*$ is the sheaf of invertible elements in the sheaf of rings $\mathcal{K}_X$. We note that (f_i) is a *principal Cartier divisor* associated to f_i, that is, the image of f_i by $\Gamma(X, \mathcal{K}_X^*) \to \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$, where $\mathcal{O}_X^*$ is the sheaf of invertible elements in $\mathcal{O}_X$.

Let D be a $\mathbb{Q}$-divisor (resp. an $\mathbb{R}$-divisor) on an equidimensional variety X, that is, D is a finite formal $\mathbb{Q}$-linear (resp. $\mathbb{R}$-linear) combination $D = \sum_i d_i D_i$ of irreducible reduced subschemes D_i of codimension one. We define the *round-up*

$\lceil D \rceil = \sum_i \lceil d_i \rceil D_i$ (resp. *round-down* $\lfloor D \rfloor = \sum_i \lfloor d_i \rfloor D_i$), where for every real number x, $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) is the integer defined by $x \le \lceil x \rceil < x+1$ (resp. $x-1 < \lfloor x \rfloor \le x$). The *fractional part* $\{D\}$ of D denotes $D - \lfloor D \rfloor$. We put $D^{<1} = \sum_{d_i<1} d_i D_i$, $D^{\le 1} = \sum_{d_i \le 1} d_i D_i$ and $D^{=1} = \sum_{d_i=1} D_i$. We can define $D^{\ge 1}$, $D^{>1}$ and so on analogously. We call D a *boundary* (resp. *subboundary*) $\mathbb{R}$-divisor if $0 \le d_i \le 1$ (resp. $d_i \le 1$) for every i.

Notation 2.3 (Singularities of pairs). Let X be a normal variety and let Δ be an $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $f : Y \to X$ be a resolution such that $\mathrm{Exc}(f) \cup f_*^{-1}\Delta$, where $\mathrm{Exc}(f)$ is the exceptional locus of f and $f_*^{-1}\Delta$ is the strict transform of Δ on Y, has a simple normal crossing support. We can write $K_Y = f^*(K_X + \Delta) + \sum_i a_i E_i$. We say that (X, Δ) is *sub log canonical* if $a_i \ge -1$ for every i. We usually write $a_i = a(E_i, X, \Delta)$ and call it the *discrepancy coefficient* of E_i with respect to (X, Δ). It is well known that there exists the largest Zariski open set U of X such that $(U, \Delta|_U)$ is sub log canonical. If there exist a resolution $f : Y \to X$ and a divisor E on Y such that $a(E, X, \Delta) = -1$ and $f(E) \cap U \ne \emptyset$, then $f(E)$ is called a *log canonical center* (an *lc center*, for short) with respect to (X, Δ). If (X, Δ) is sub log canonical and Δ is effective, then (X, Δ) is called *log canonical*.

We note that we can define $a(E_i, X, \Delta)$ in more general settings ([K2, Definition 2.4]).

Let us recall the definition of simple normal crossing pairs.

Definition 2.4 (Simple normal crossing pairs). We say that the pair (X, D) is *simple normal crossing* at a point $a \in X$ if X has a Zariski open neighborhood U of a that can be embedded in a smooth variety Y, where Y has regular system of parameters $(x_1, \ldots, x_p, y_1, \ldots, y_r)$ at $a = 0$ in which U is defined by a monomial equation $x_1 \cdots x_p = 0$ and $D = \sum_{i=1}^r \alpha_i (y_i = 0)|_U$ with $\alpha_i \in \mathbb{R}$. We say that (X, D) is a *simple normal crossing pair* if it is simple normal crossing at every point of X. We say that a simple normal crossing pair (X, D) is *embedded* if there exists a closed embedding $\iota : X \to M$, where M is a smooth variety of $\dim X + 1$. We call M the *ambient space* of (X, D). If $(X, 0)$ is a simple normal crossing pair, then X is called a *simple normal crossing variety*. If X is a simple normal crossing variety, then X has only Gorenstein singularities. Thus, it has an invertible dualizing sheaf ω_X. Therefore, we can define the *canonical divisor* K_X such that $\omega_X \simeq \mathcal{O}_X(K_X)$. It is a Cartier divisor on X and is well defined up to linear equivalence.

Let X be a simple normal crossing variety and let $X = \bigcup_{i \in I} X_i$ be the irreducible decomposition of X. A *stratum* of X is an irreducible component of $X_{i_1} \cap \cdots \cap X_{i_k}$ for some $\{i_1, \ldots, i_k\} \subset I$.

Let X be a simple normal crossing variety and let D be a Cartier divisor on X. If (X, D) is a simple normal crossing pair and D is reduced, then D is called a *simple normal crossing divisor* on X.

Let (X, D) be a simple normal crossing pair. Let $\nu : X^\nu \to X$ be the normalization. We define Θ by the formula $K_{X^\nu} + \Theta = \nu^*(K_X + D)$, that is, Θ is the sum of the inverse images of D and the singular locus of X. Then a *stratum* of (X, D) is an irreducible component of X or the ν-image of a log canonical center of (X^ν, Θ) (Notation 2.3). When $D = 0$, this definition is compatible with the above definition of the strata of X. When D is a boundary $\mathbb{R}$-divisor, W is a stratum of (X, D) if and only if W is a semi-log canonical stratum (an slc stratum, for short) of (X, D) ([F6, Definition 2.5]). Note that (X, D) is semi-log canonical if D is a boundary $\mathbb{R}$-divisor.

Notation 2.5. $\pi_1(X)$ denotes the topological fundamental group of X.

§3. Pull-back of quasi-log structures

In this section, we give a precise statement of Theorem 1.1 (Theorem 3.5). First, let us recall the definition of *globally embedded simple normal crossing pairs* in order to define quasi-log schemes.

Definition 3.1 (Globally embedded simple normal crossing pairs). Let Y be a simple normal crossing divisor on a smooth variety M and let D be an $\mathbb{R}$-divisor on M such that $\mathrm{Supp}(D+Y)$ is a simple normal crossing divisor on M and that D and Y have no common irreducible components. We put $B_Y = D|_Y$ and consider the pair (Y, B_Y). We call (Y, B_Y) a *globally embedded simple normal crossing pair* and M the *ambient space* of (Y, B_Y).

It is obvious that a globally embedded simple normal crossing pair is an embedded simple normal crossing pair in Definition 2.4.

Let us define *quasi-log schemes*. For Ambro's original definition in [A], see Definition A.2 below.

Definition 3.2 (Quasi-log schemes). A *quasi-log scheme* is a scheme X endowed with an $\mathbb{R}$-Cartier divisor (or $\mathbb{R}$-line bundle) ω on X, a proper closed subscheme $X_{-\infty} \subset X$ and a finite collection $\{C\}$ of reduced and irreducible subschemes of X such that there is a proper morphism $f : (Y, B_Y) \to X$ from a globally embedded simple normal crossing pair satisfying the following properties:

(1) $f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$.

(2) The natural map $\mathcal{O}_X \to f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil)$ induces an isomorphism

$$\mathcal{I}_{X_{-\infty}} \xrightarrow{\simeq} f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil - \lfloor B_Y^{>1}\rfloor),$$

where $\mathcal{I}_{X_{-\infty}}$ is the defining ideal sheaf of $X_{-\infty}$.

(3) The collection of subvarieties $\{C\}$ coincides with the images of (Y, B_Y)-strata that are not included in $X_{-\infty}$.

We simply write $[X,\omega]$ to denote the above data $\bigl(X, \omega, f : (Y, B_Y) \to X\bigr)$ if there is no risk of confusion. Note that a quasi-log scheme X is the union of $\{C\}$ and $X_{-\infty}$. We also note that ω is called the *quasi-log canonical class* of $[X,\omega]$, which is defined up to $\mathbb{R}$-linear equivalence. We sometimes simply say that $[X,\omega]$ is a *quasi-log pair*. The subvarieties C are called the *qlc strata* of $[X,\omega]$, $X_{-\infty}$ is called the *non-qlc locus* of $[X,\omega]$ and $f : (Y, B_Y) \to X$ is called a *quasi-log resolution* of $[X,\omega]$.

Remark 3.3. Let $\mathrm{Div}(Y)$ be the group of Cartier divisors on Y and let $\mathrm{Pic}(Y)$ be the Picard group of Y. Let $\delta_Y : \mathrm{Div}(Y)\otimes\mathbb{R} \to \mathrm{Pic}(Y)\otimes\mathbb{R}$ be the homomorphism induced by $A \mapsto \mathcal{O}_Y(A)$ where A is a Cartier divisor on Y. When ω is an $\mathbb{R}$-line bundle in Definition 3.2, $f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$ means $f^*\omega = \delta_Y(K_Y + B_Y)$ in $\mathrm{Pic}(Y)\otimes\mathbb{R}$. Even when ω is an $\mathbb{R}$-line bundle, we use $-\omega$ to denote the inverse of ω in $\mathrm{Pic}(X)\otimes\mathbb{R}$ if there is no risk of confusion. If ω is an $\mathbb{R}$-Cartier divisor on X in Theorem 1.1, $h^*\omega \otimes \det \Omega^1_{X'/X}$ means $\delta_{X'}(h^*\omega)\otimes \det\Omega^1_{X'/X}$ in $\mathrm{Pic}(X')\otimes\mathbb{R}$, where $\delta_{X'} : \mathrm{Div}(X')\otimes\mathbb{R} \to \mathrm{Pic}(X')\otimes\mathbb{R}$.

For various applications, the notion of *qlc pairs* is very useful.

Definition 3.4. Let $[X,\omega]$ be a quasi-log pair. We say that $[X,\omega]$ has only *quasi-log canonical singularities* (*qlc singularities*, for short) or $[X,\omega]$ is a *qlc pair* if $X_{-\infty} = \emptyset$.

Let us state the main theorem of this paper precisely.

Theorem 3.5 (Main theorem). *Let $[X,\omega]$ be a quasi-log pair as in Definition 3.2. Let X' be a scheme and let $h : X' \to X$ be a smooth quasi-projective morphism. Then $[X',\omega']$, where $\omega' = h^*\omega \otimes \omega_{X'/X}$ with $\omega_{X'/X} = \det\Omega^1_{X'/X}$, has a natural quasi-log structure induced by h. More precisely, we have the following:*

(i) (Non-qlc locus). *There is a proper closed subscheme $X'_{-\infty} \subset X'$.*

(ii) (Quasi-log resolution). *There exists a proper morphism $f' : (Y', B_{Y'}) \to X'$ from a globally embedded simple normal crossing pair $(Y', B_{Y'})$ with $f'^*\omega' \sim_{\mathbb{R}} K_{Y'} + B_{Y'}$ that defines a quasi-log structure on $[X',\omega']$ such that $\mathcal{I}_{X'_{-\infty}} = h^*\mathcal{I}_{X_{-\infty}}$.*

(iii) (Qlc strata). *There is a finite collection* $\{C'\}$ *of reduced and irreducible subschemes of* X' *such that* $\{C'\} = \{f^{-1}(C)\}$ *and that the collection of subvarieties* $\{C'\}$ *coincides with the images of* $(Y', B_{Y'})$*-strata that are not included in* $X'_{-\infty}$.

For the definition and basic properties of *quasi-projective morphisms*, see [G, Chapitre II, §5.3 "Morphismes quasi-projectifs"].

§4. On quasi-log structures

Proposition 4.1 ([F2, Proposition 3.50]). *Let* $f : V \to W$ *be a proper birational morphism between smooth varieties and let* B_W *be an* $\mathbb{R}$*-divisor on* W *such that* $\operatorname{Supp} B_W$ *is a simple normal crossing divisor on* W. *Assume that* $K_V + B_V = f^*(K_W + B_W)$ *and that* $\operatorname{Supp} B_V$ *is a simple normal crossing divisor on* V. *Then we have*

$$f_*\mathcal{O}_V(\lceil -(B_V^{<1})\rceil - \lfloor B_V^{>1}\rfloor) \simeq \mathcal{O}_W(\lceil -(B_W^{<1})\rceil - \lfloor B_W^{>1}\rfloor).$$

Furthermore, let S *be a simple normal crossing divisor on* W *such that* $S \subset \operatorname{Supp} B_W^{=1}$. *Let* T *be the union of the irreducible components of* $B_V^{=1}$ *that are mapped into* S *by* f. *Assume that* $\operatorname{Supp} f_*^{-1}B_W \cup \operatorname{Exc}(f)$ *is a simple normal crossing divisor on* V. *Then we have*

$$f_*\mathcal{O}_T(\lceil -(B_T^{<1})\rceil - \lfloor B_T^{>1}\rfloor) \simeq \mathcal{O}_S(\lceil -(B_S^{<1})\rceil - \lfloor B_S^{>1}\rfloor),$$

where $(K_V + B_V)|_T = K_T + B_T$ *and* $(K_W + B_W)|_S = K_S + B_S$.

Proof. By $K_V + B_V = f^*(K_W + B_W)$, we obtain

$$K_V = f^*(K_W + B_W^{=1} + \{B_W\}) + f^*(\lfloor B_W^{<1}\rfloor + \lfloor B_W^{>1}\rfloor) - (\lfloor B_V^{<1}\rfloor + \lfloor B_V^{>1}\rfloor) - B_V^{=1} - \{B_V\}.$$

If $a(\nu, W, B_W^{=1} + \{B_W\}) = -1$ for a prime divisor ν over W, then we can check that $a(\nu, W, B_W) = -1$ by using [KM, Lemma 2.45]. Since

$$f^*(\lfloor B_W^{<1}\rfloor + \lfloor B_W^{>1}\rfloor) - (\lfloor B_V^{<1}\rfloor + \lfloor B_V^{>1}\rfloor)$$

is Cartier, we can easily see that

$$f^*(\lfloor B_W^{<1}\rfloor + \lfloor B_W^{>1}\rfloor) = \lfloor B_V^{<1}\rfloor + \lfloor B_V^{>1}\rfloor + E,$$

where E is an effective f-exceptional divisor. Thus, we obtain

$$f_*\mathcal{O}_V(\lceil -(B_V^{<1})\rceil - \lfloor B_V^{>1}\rfloor) \simeq \mathcal{O}_W(\lceil -(B_W^{<1})\rceil - \lfloor B_W^{>1}\rfloor).$$

Next, we consider the short exact sequence

$$\begin{aligned}0 &\to \mathcal{O}_V(\lceil -(B_V^{<1})\rceil - \lfloor B_V^{>1}\rfloor - T)\\ &\to \mathcal{O}_V(\lceil -(B_V^{<1})\rceil - \lfloor B_V^{>1}\rfloor) \to \mathcal{O}_T(\lceil -(B_T^{<1})\rceil - \lfloor B_T^{>1}\rfloor) \to 0.\end{aligned}$$

Since $T = f^*S - F$, where F is an effective f-exceptional divisor, we can easily see that

$$f_*\mathcal{O}_V(\lceil -(B_V^{<1})\rceil - \lfloor B_V^{>1}\rfloor - T) \simeq \mathcal{O}_W(\lceil -(B_W^{<1})\rceil - \lfloor B_W^{>1}\rfloor - S).$$

We note that

$$(\lceil -(B_V^{<1})\rceil - \lfloor B_V^{>1}\rfloor - T) - (K_V + \{B_V\} + B_V^{=1} - T) = -f^*(K_W + B_W).$$

Therefore, every associated prime of $R^1 f_*\mathcal{O}_V(\lceil -(B_V^{<1})\rceil - \lfloor B_V^{>1}\rfloor - T)$ is the generic point of the f-image of some stratum of $(V, \{B_V\} + B_V^{=1} - T)$ by [F4, Theorem 6.3(i)].

Claim. *No strata of* $(V, \{B_V\} + B_V^{=1} - T)$ *are mapped into* S *by* f*.*

Proof of claim. Assume that there is a stratum C of $(V, \{B_V\} + B_V^{=1} - T)$ such that $f(C) \subset S$. Note that $\operatorname{Supp} f^*S \subset \operatorname{Supp} f_*^{-1}B_W \cup \operatorname{Exc}(f)$ and $\operatorname{Supp} B_V^{=1} \subset \operatorname{Supp} f_*^{-1}B_W \cup \operatorname{Exc}(f)$. Since C is also a stratum of $(V, B_V^{=1})$ and $C \subset \operatorname{Supp} f^*S$, there exists an irreducible component G of $B_V^{=1}$ such that $C \subset G \subset \operatorname{Supp} f^*S$. Therefore, by the definition of T, G is an irreducible component of T because $f(G) \subset S$ and G is an irreducible component of $B_V^{=1}$. So, C is not a stratum of $(V, \{B_V\} + B_V^{=1} - T)$. This is a contradiction. □

On the other hand, $f(T) \subset S$. Therefore, the connecting homomorphism

$$f_*\mathcal{O}_T(\lceil -(B_T^{<1})\rceil - \lfloor B_T^{>1}\rfloor) \to R^1 f_*\mathcal{O}_V(\lceil -(B_Z^{<1})\rceil - \lfloor B_Z^{>1}\rfloor - T)$$

is a zero map by the claim. Thus, we obtain

$$f_*\mathcal{O}_T(\lceil -(B_T^{<1})\rceil - \lfloor B_T^{>1}\rfloor) \simeq \mathcal{O}_S(\lceil -(B_S^{<1})\rceil - \lfloor B_S^{>1}\rfloor)$$

by an easy diagram chasing. We finish the proof. □

It is easy to check the following result.

Proposition 4.2. *In Proposition 4.1, let* C' *be a log canonical center of* (V, B_V) *contained in* T*. Then* $f(C')$ *is a log canonical center of* (W, B_W) *contained in* S *or* $f(C')$ *is contained in* $\operatorname{Supp} B_W^{>1}$*. Let* C *be a log canonical center of* (W, B_W) *contained in* S*. Then there exists a log canonical center* C' *of* (V, B_V) *contained in* T *such that* $f(C') = C$*.*

Theorem 4.3. *In Definition 3.2, we may assume that the ambient space* M *of the globally embedded simple normal crossing pair* (Y, B_Y) *is quasi-projective. In particular,* Y *is quasi-projective.*

Proof. In Definition 3.2, we may assume that $D+Y$ is an $\mathbb{R}$-divisor on a smooth variety M such that $\mathrm{Supp}(D+Y)$ is a simple normal crossing divisor on M, D and Y have no common irreducible components and $B_Y = D|_Y$ as in Definition 3.1. Let $g: M' \to M$ be a projective birational morphism from a smooth quasi-projective variety M' with the following properties:

(i) $K_{M'} + B_{M'} = g^*(K_M + D + Y)$.
(ii) $\mathrm{Supp}\, B_{M'}$ is a simple normal crossing divisor on M'.
(iii) $\mathrm{Supp}\, g_*^{-1}(D+Y) \cup \mathrm{Exc}(g)$ is also a simple normal crossing divisor on M'.

Let Y' be the union of the irreducible components of $B_{M'}^{=1}$ that are mapped into Y by g. We put $(K_{M'} + B_{M'})|_{Y'} = K_{Y'} + B_{Y'}$. Then

$$g_*\mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1})\rceil - \lfloor B_{Y'}^{>1}\rfloor) \simeq \mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil - \lfloor B_Y^{>1}\rfloor)$$

by Proposition 4.1. This implies that $\mathcal{I}_{X_{-\infty}} \xrightarrow{\simeq} f_*g_*\mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1})\rceil - \lfloor B_{Y'}^{>1}\rfloor)$. By construction,

$$K_{Y'} + B_{Y'} = g^*(K_Y + B_Y) \sim_{\mathbb{R}} g^*f^*\omega.$$

By Proposition 4.2, the collection of subvarieties $\{C\}$ in Definition 3.2 coincides with the images of $(Y', B_{Y'})$-strata that are not contained in $X_{-\infty}$. Therefore, by replacing M and (Y, B_Y) with M' and $(Y', B_{Y'})$, we may assume that the ambient space M is quasi-projective. □

Lemma 4.4. *Let (Y, B_Y) be a simple normal crossing pair. Let V be a smooth variety such that $Y \subset V$. Then we can construct a sequence of blow-ups*

$$V_k \to V_{k-1} \to \cdots \to V_0 = V$$

with the following properties:

(1) $\sigma_{i+1} : V_{i+1} \to V_i$ is the blow-up along a smooth irreducible component of $\mathrm{Supp}\, B_{Y_i}$ for every $i \geq 0$.
(2) We put $Y_0 = Y$ and $B_{Y_0} = B_Y$. Let Y_{i+1} be the strict transform of Y_i for every $i \geq 0$.
(3) We define $K_{Y_{i+1}} + B_{Y_{i+1}} = \sigma_{i+1}^*(K_{Y_i} + B_{Y_i})$ for every $i \geq 0$.
(4) There exists an $\mathbb{R}$-divisor D on V_k such that $D|_{Y_k} = B_{Y_k}$.
(5) $\sigma_*\mathcal{O}_{Y_k}(\lceil -(B_{Y_k}^{<1})\rceil - \lfloor B_{Y_k}^{>1}\rfloor) \simeq \mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil - \lfloor B_Y^{>1}\rfloor)$, where $\sigma : V_k \to V_{k-1} \to \cdots \to V_0 = V$.

Proof. It is sufficient to check (5). All the other properties are obvious by the construction of the sequence of blow-ups. By an easy calculation of discrepancy coefficients similar to the proof of Proposition 4.1, we can check that

$$\sigma_{i+1*}\mathcal{O}_{V_{i+1}}(\lceil -(B^{<1}_{Y_{i+1}})\rceil - \lfloor B^{>1}_{Y_{i+1}}\rfloor) \simeq \mathcal{O}_{V_i}(\lceil -(B^{<1}_{Y_i})\rceil - \lfloor B^{>1}_{Y_i}\rfloor)$$

for every i. This implies the desired isomorphism. □

The following lemma is easily checked.

Lemma 4.5. *In Lemma 4.4, let C' be a stratum of (Y_k, B_{Y_k}). Then $\sigma(C')$ is a stratum of (Y, B_Y). Let C be a stratum of (Y, B_Y). Then there is a stratum C' of (Y_k, B_{Y_k}) such that $\sigma(C') = C$.*

The following lemma is easy but very useful (cf. [K2, Proposition 10.59]).

Lemma 4.6. *Let Y be a simple normal crossing variety. Let V be a smooth quasi-projective variety such that $Y \subset V$. Let $\{P_i\}$ be any finite set of closed points of Y. Then we can find a quasi-projective variety W such that $Y \subset W \subset V$, $\dim W = \dim Y + 1$ and W is smooth at P_i for every i.*

For the proof, see, for example, the proof of [F6, Theorem 1.2, step 2]. We note that we cannot always make W smooth in Lemma 4.6.

Example 4.7 ([F2, Example 3.62]). Let $V \subset \mathbb{P}^5$ be the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$. In this case, there are no smooth hypersurfaces of $\mathbb{P}^5$ containing V. We can check it as follows.

If there exists a smooth hypersurface S such that $V \subset S \subset \mathbb{P}^5$, then $\rho(V) = \rho(S) = \rho(\mathbb{P}^5) = 1$ by the Lefschetz hyperplane theorem. This is a contradiction because $\rho(V) = 2$.

By the above results, we can prove the final lemma in this section.

Lemma 4.8. *Let (Y, B_Y) be a simple normal crossing pair such that Y is quasi-projective. Then there exist a globally embedded simple normal crossing pair (Z, B_Z) and a morphism $\sigma : Z \to Y$ such that*

$$K_Z + B_Z = \sigma^*(K_Y + B_Y)$$

and

$$\sigma_*\mathcal{O}_Z(\lceil -(B^{<1}_Z)\rceil - \lfloor B^{>1}_Z\rfloor) \simeq \mathcal{O}_Y(\lceil -(B^{<1}_Y)\rceil - \lfloor B^{>1}_Y\rfloor).$$

Moreover, let C' be a stratum of (Z, B_Z). Then $\sigma(C')$ is a stratum of (Y, B_Y) or $\sigma(C')$ is contained in $\operatorname{Supp} B^{>1}_Y$. *Let C be a stratum of (Y, B_Y). Then there exists a stratum C' of (Z, B_Z) such that $\sigma(C') = C$.*

Proof. Let V be a smooth quasi-projective variety such that $Y \subset V$. By Lemmas 4.4 and 4.5, we may assume that there exists an $\mathbb{R}$-divisor D on V such that $D|_Y = B_Y$. Then we apply Lemma 4.6. We can find a quasi-projective variety W such that $Y \subset W \subset V$, $\dim W = \dim Y + 1$, and W is smooth at the generic point of any stratum of $(Y, \operatorname{Supp} B_Y)$. Of course, we can make $W \not\subset \operatorname{Supp} D$ by the proof of Lemma 4.6. We apply Hironaka's resolution to W and use Szabó's resolution lemma (see, for example, [F1, 3.5 Resolution lemma]). More precisely, we take blow-ups outside U, where U is the largest Zariski open set of W such that $(Y, B_Y)|_U$ is a globally embedded simple normal crossing pair. Then we obtain a desired globally embedded simple normal crossing pair (Z, B_Z). Precisely speaking, we can check that (Z, B_Z) has the desired properties by an easy calculation of discrepancy coefficients similar to the proof of Proposition 4.1. □

Theorem 4.9. *In Definition 3.2, it is sufficient to assume that* (Y, B_Y) *is a quasi-projective* (*not necessarily embedded*) *simple normal crossing pair.*

Proof. We assume only that (Y, B_Y) is a simple normal crossing pair in Definition 3.2. We assume that Y is quasi-projective. Then we apply Lemma 4.8 to (Y, B_Y). Let $\sigma : (Z, B_Z) \to (Y, B_Y)$ be as in Lemma 4.8. Then $\big(X, \omega, f \circ \sigma : (Z, B_Z) \to X\big)$ is a quasi-log scheme in the sense of Definition 3.2. □

Proposition 4.10 shows that it is not so easy to apply Chow's lemma directly to make (Y, B_Y) quasi-projective in Definition 3.2.

Proposition 4.10 ([F2, Proposition 3.65]). *There exists a complete simple normal crossing variety* Y *with the following property: If* $f : Z \to Y$ *is a proper surjective morphism from a simple normal crossing variety* Z *such that* f *is an isomorphism over the generic point of any stratum of* Y*, then* Z *is nonprojective.*

Proof. We take a smooth complete nonprojective toric variety X. We put $V = X \times \mathbb{P}^1$. Then V is a toric variety. We consider $Y = V \setminus T$, where T is the big torus of V. We will see that Y has the desired property. By the above construction, there is an irreducible component Y' of Y that is isomorphic to X. Let Z' be the irreducible component of Z mapped onto Y' by f. So, it is sufficient to see that Z' is not projective. On $Y' \simeq X$, there is a torus-invariant effective one cycle C such that C is numerically trivial. By construction and the assumption, $g = f|_{Z'} : Z' \to Y' \simeq X$ is birational and an isomorphism over the generic point of any torus-invariant curve on $Y' \simeq X$. We note that any torus-invariant curve on $Y' \simeq X$ is a stratum of Y. We assume that Z' is projective; then there is a very ample effective divisor A on Z' such that A does not contain any irreducible components of the inverse image of C. Then $B = f_*A$ is an effective Cartier divisor on $Y' \simeq X$ such that $\operatorname{Supp} B$ contains no irreducible components of C. This is a contradiction because $\operatorname{Supp} B \cap C \neq \emptyset$ and C is numerically trivial. □

Proposition 4.10 is the main reason why we proved Theorem 4.3 for the proof of our main theorem (Theorems 1.1 and 3.5). Now the proof of Theorem 1.1 is almost obvious.

Proof of Theorem 3.5. Let $f : (Y, B_Y) \to X$ be a quasi-log resolution as in Definition 3.2. By Theorem 4.3, we may assume that Y is quasi-projective. We consider the fiber product $Y' = Y \times_X X'$.

$$\begin{array}{ccc} Y' & \xrightarrow{h'} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow[h]{} & X \end{array}$$

We put $B_{Y'} = h'^* B_Y$. Then $(Y', B_{Y'})$ is a quasi-projective simple normal crossing pair because h is a smooth quasi-projective morphism and (Y, B_Y) is a quasi-projective simple normal crossing pair. Since $K_Y + B_Y \sim_{\mathbb{R}} f^*\omega$, we have

$$f'^*\omega' = f'h^*\omega \otimes f'^*\omega_{X'/X} = h' f^*\omega \otimes \omega_{Y'/Y} \sim_{\mathbb{R}} h'^*(K_Y + B_Y) \otimes \omega_{Y'/Y} = K_{Y'} + B_{Y'}.$$

Note that $\omega_{X'/X}$ is trivial when h is étale. By the flat base change theorem, we have

$$\begin{aligned} h^*\mathcal{I}_{X_{-\infty}} &= h^* f_* \mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil - \lfloor B_Y^{>1} \rfloor) \\ &\simeq f'_* h'^* \mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil - \lfloor B_Y^{>1} \rfloor) \\ &\simeq f'_* \mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1}) \rceil - \lfloor B_{Y'}^{>1} \rfloor). \end{aligned}$$

Finally, by Theorem 4.9, we may assume that $(Y', B_{Y'})$ is a globally embedded simple normal crossing pair. Therefore, $\bigl(X', \omega', f' : (Y', B_{Y'}) \to X'\bigr)$ gives us the desired quasi-log structure. □

§5. An application to quasi-log canonical Fano varieties

Let us recall the vanishing theorem for projective qlc pairs.

Theorem 5.1 (Vanishing theorem for qlc pairs). *Let $[X, \omega]$ be a projective qlc pair and let L be a Cartier divisor on X such that $L - \omega$ is ample. Then $H^i(X, \mathcal{O}_X(L))$ $= 0$ for every $i > 0$.*

We give a proof of Theorem 5.1 for the readers' convenience.

Proof. Let $f : (Y, B_Y) \to X$ be a quasi-log resolution as in Definition 3.2. Since $[X, \omega]$ is qlc, $B_Y = B_Y^{\leq 1}$ holds. Then

$$f^*(L - \omega) \sim_{\mathbb{R}} f^*L - (K_Y + B_Y) = f^*L + \lceil -(B_Y^{<1}) \rceil - (K_Y + \{B_Y\} + B_Y^{=1})$$

because $B_Y = B_Y^{\leq 1}$. Therefore, we have $H^i(X, f_*\mathcal{O}_Y(f^*L + \lceil -(B_Y^{<1}) \rceil)) = 0$ for every $i > 0$ by [F5, Theorem 1.1(ii)]. Note that

$$f_*\mathcal{O}_Y(f^*L + \lceil -(B_Y^{<1}) \rceil) \simeq \mathcal{O}_X(L) \otimes f_*\mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil) \simeq \mathcal{O}_X(L)$$

because $X_{-\infty} = \emptyset$. This implies that $H^i(X, \mathcal{O}_X(L)) = 0$ for every $i > 0$. □

By combining Theorem 5.1 with Theorem 3.5, we can easily check Corollary 1.2.

Proof of Corollary 1.2. Without loss of generality, we may assume that X is connected. Since $-\omega$ is ample, $H^i(X, \mathcal{O}_X) = 0$ for every $i > 0$ by Theorem 5.1. Therefore, we have $\chi(X, \mathcal{O}_X) = 1$. Assume there exists a nontrivial finite étale morphism $f : \widetilde{X} \to X$ from a connected scheme $\widetilde{X}$. By Theorem 3.5, the pair $[\widetilde{X}, \widetilde{\omega}]$, where $\widetilde{\omega} = f^*\omega$, is a qlc pair such that $-\widetilde{\omega}$ is ample. Thus, $H^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 0$ for every $i > 0$ by Theorem 5.1 again. This implies $\chi(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 1$. By the Riemann–Roch formula (see [Ft, Example 18.3.9]), we have $\chi(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = \deg f \cdot \chi(X, \mathcal{O}_X)$. Therefore, we obtain $\deg f = 1$, a contradiction. This means that X has no nontrivial finite étale covers, or equivalently, the algebraic fundamental group of X is trivial. □

As a direct consequence of Corollary 1.2 and the main theorem of [F6], we have the following result.

Corollary 5.2. *Let (X, Δ) be a projective semi-log canonical pair such that $-(K_X + \Delta)$ is ample. Then the algebraic fundamental group of X is trivial.*

Proof. By [F6], $[X, K_X + \Delta]$ has a natural quasi-log structure with only qlc singularities. Therefore, Corollary 5.2 is a special case of Corollary 1.2. □

Note that a union of some slc strata of a log Fano pair with only semi-log canonical singularities is a quasi-log canonical Fano variety by Example 5.3.

Example 5.3. Let (X, Δ) be a connected projective semi-log canonical pair such that $-(K_X + \Delta)$ is ample. Let W be the union of some slc strata of (X, Δ) with the reduced scheme structure. Then $[W, \omega]$, where $\omega = (K_X + \Delta)|_W$, is a projective qlc pair such that $-\omega$ is ample by adjunction (see [F6, Theorem 1.13]). By [F6, Theorem 1.11], we obtain $H^1(X, \mathcal{I}_W) = 0$ where $\mathcal{I}_W$ is the defining ideal sheaf of W on X. Therefore, we obtain $H^0(W, \mathcal{O}_W) = \mathbb{C}$ by the surjection $\mathbb{C} = H^0(X, \mathcal{O}_X) \to H^0(W, \mathcal{O}_W)$. This implies that W is connected.

The author learned the following example from Tetsushi Ito.

Example 5.4 (Topological versus algebraic). We consider the Higman group G. It is generated by 4 elements a, b, c, d with the relations

$$a^{-1}ba = b^2, \quad b^{-1}cb = c^2, \quad c^{-1}dc = d^2, \quad d^{-1}ad = a^2.$$

It is well known that G has no nontrivial finite quotients. By [S, Theorem 12.1], there is an irreducible projective variety X such that $\pi_1(X) \simeq G$. In this case, the algebraic fundamental group of X, which is the profinite completion of $\pi_1(X)$, is trivial.

Example 5.4 shows that Conjecture 1.3 does not directly follow from Corollary 1.2. We give a nontrivial example of reducible log Fano pairs with only semi-log canonical singularities.

Example 5.5. We consider the lattice $N = \mathbb{Z}^3$. Let n be an integer with $n \geq 3$. We consider a convex polyhedron P in $N_{\mathbb{R}} = N \otimes \mathbb{R} \simeq \mathbb{R}^3$ whose vertices are $v_0, v_1, \ldots, v_n \in N$ such that $v_0 = (0,0,-1)$ and that the third coordinates of $v_1, \ldots, v_n$ are 1. Assume that P contains $(0,0,0)$ in its interior. Then the cones spanned by $(0,0,0)$ and faces of P subdivide $\mathbb{R}^3$ into $n+1$ three-dimensional cones. This subdivision of $\mathbb{R}^3$ corresponds to a complete toric threefold X. Then we have the following properties:

(1) $-K_X$ is ample since P is convex.

(2) $D_0 \sim D_1 + \cdots + D_n$, and D_0 is $\mathbb{Q}$-Cartier, where D_i is the torus-invariant prime divisor on X associated to v_i for every i.

(3) Let $x \in X$ be the torus-invariant closed point associated to the cone spanned by $v_1, v_2, \ldots, v_n$. Then $X \setminus x$ is $\mathbb{Q}$-factorial, but X is not $\mathbb{Q}$-factorial when $n \geq 4$.

(4) We put $\Delta = D_1 + \cdots + D_n$. Then (X, Δ) is a log canonical Fano threefold. Note that $-(K_X + \Delta) \sim D_0$.

(5) We put $W = \lfloor \Delta \rfloor = \Delta$ and $K_W + \Delta_W = (K_X + \Delta)|_W$. Then (W, Δ_W) is a two-dimensional log Fano pair with only semi-log canonical singularities. Note that W is Cohen–Macaulay since W is $\mathbb{Q}$-Cartier.

This W shows that the number of irreducible components of log Fano pairs with only semi-log canonical singularities is not bounded.

We recommend that readers who can read Japanese look at [F7] for some related topics and open problems on singular Fano varieties.

§6. Simple connectedness of log canonical log Fano pairs

In this section, we prove that a log Fano pair with only log canonical singularities is always simply connected. Theorem 6.1 is Fujita's answer to the author's question.

Theorem 6.1 (Fujita). *Let (X, Δ) be a projective log canonical pair such that $-(K_X + \Delta)$ is ample. Then X is simply connected.*

Proof. First of all, we may assume that X is connected. Without loss of generality, we may assume that Δ is a $\mathbb{Q}$-divisor by perturbing Δ slightly. Then, by [HM, Corollary 1.3(2)], X is rationally chain connected. Since X is normal and rationally chain connected, $\pi_1(X)$ is finite (see [K1, Theorem 4.13]). Let $f : \widetilde{X} \to X$ be the universal cover of X. Since $\pi_1(X)$ is finite, f is finite and étale. It is obvious that $(\widetilde{X}, \widetilde{\Delta})$ is log canonical and $-(K_{\widetilde{X}} + \widetilde{\Delta})$ is ample, where $K_{\widetilde{X}} + \widetilde{\Delta} = f^*(K_X + \Delta)$. By [F4, Theorem 8.1], we have $H^i(X, \mathcal{O}_X) = H^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 0$ for every $i > 0$. This implies $\chi(X, \mathcal{O}_X) = \chi(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 1$. On the other hand, by the Riemann–Roch formula, $\chi(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = \deg f \cdot \chi(X, \mathcal{O}_X)$ holds (see [Ft, Example 18.3.9]). Thus we obtain $\deg f = 1$. Therefore, X is simply connected. □

Remark 6.2. By [HM, Corollary 1.3(2)], we can easily see that a log Fano pair with only semi-log canonical singularities is rationally chain connected. However, [K1, Theorem 4.13] does not always hold for *nonnormal* rationally chain connected varieties. Note that a nodal rational curve C is rationally chain connected such that $\pi_1(C)$ is infinite. Therefore, the proof of Theorem 6.1 does not work for log Fano pairs with only semi-log canonical singularities.

The following well-known example shows some subtleties on log Fano pairs with only log canonical singularities. Example 6.3 says that Theorem 6.1 does not always hold when $-(K_X + \Delta)$ is only nef and big.

Example 6.3. Let $C \subset \mathbb{P}^2$ be a smooth cubic curve and let $X \subset \mathbb{P}^3$ be the cone over $C \subset \mathbb{P}^2$. Then X is a Gorenstein log canonical surface such that $-K_X$ is ample. It is easy to see that X is rationally chain connected and that $\pi_1(X) = \{1\}$ by Theorem 6.1. Let $f : Y \to X$ be the blow-up at P, where P is the vertex of X. Then $K_Y + E = f^*K_X$. The pair (Y, E) is purely log terminal and $-(K_Y + E)$ is big and semiample. Note that the exceptional curve E is isomorphic to C and that Y is a $\mathbb{P}^1$-bundle over C. Therefore, it is easy to see that Y is not rationally chain connected and $\pi_1(Y) \neq \{1\}$.

Example 6.4 is a nontrivial example of *irreducible nonnormal* semi-log canonical Fano varieties.

Example 6.4. We put $X = (x^2w - zy^2 = 0) \subset \mathbb{P}^3$. Then X is a Gorenstein Fano variety with only semi-log canonical singularities. Note that X is irreducible and nonnormal. By using the van Kampen theorem, we see that $\pi_1(X) = \{1\}$.

Appendix A. Ambro's original definition

In this section, we prove that our definition of quasi-log schemes (Definition 3.2) is equivalent to Ambro's original definition in [A].

First, let us recall the definition of *normal crossing pairs*. We need it for Ambro's original definition of quasi-log schemes in [A].

Definition A.1 (Normal crossing pairs). A variety X has *normal crossing* singularities if, for every closed point $x \in X$,

$$\widehat{\mathcal{O}}_{X,x} \simeq \frac{\mathbb{C}[[x_0, \ldots, x_N]]}{(x_0 \cdots x_k)},$$

for some $0 \leq k \leq N$, where $N = \dim X$. Let X be a normal crossing variety. We say that a reduced divisor D on X is *normal crossing* if, in the above notation, we have

$$\widehat{\mathcal{O}}_{D,x} \simeq \frac{\mathbb{C}[[x_0, \ldots, x_N]]}{(x_0 \cdots x_k, x_{i_1} \cdots x_{i_l})},$$

for some $\{i_1, \ldots, i_l\} \subset \{k+1, \ldots, N\}$. A *stratum* of X is an irreducible component of X or the ν-image of a log canonical center of (X^ν, Ξ), where $\nu : X^\nu \to X$ is the normalization and $K_{X^\nu} + \Xi = \nu^* K_X$, that is, Ξ is the inverse image of the singular locus of X. A *permissible* Cartier divisor on X is a Cartier divisor on X whose support contains no strata of X. A *permissible* $\mathbb{R}$-Cartier divisor is a finite $\mathbb{R}$-linear combination of permissible Cartier divisors on X. We say that the pair (X, B) is a *normal crossing pair* if the following conditions are satisfied:

(1) X is a normal crossing variety;

(2) B is a permissible $\mathbb{R}$-Cartier divisor whose support is normal crossing on X.

We say that a normal crossing pair (X, B) is *embedded* if there exists a closed embedding $\iota : X \to M$, where M is a smooth variety of dimension $\dim X + 1$. We call M the *ambient space* of (X, B). We put

$$K_{X^\nu} + \Theta = \nu^*(K_X + B),$$

where $\nu : X^\nu \to X$ is the normalization of X, that is, Θ is the sum of the inverse images of B and the singular locus of X. A *stratum* of (X, B) is an irreducible component of X or the ν-image of some log canonical center of (X^ν, Θ) on X.

It is obvious that a *simple normal crossing pair* in Definition 2.4 is a *normal crossing pair* in Definition A.1. Note that the differences between normal crossing varieties and simple normal crossing varieties sometimes cause some subtle trouble. For the details, see, for example, [F1, 3.6 Whitney umbrella].

Let us recall Ambro's original definition of quasi-log schemes in [A].

Definition A.2 (Quasi-log schemes). A *quasi-log scheme* is a scheme X endowed with an $\mathbb{R}$-Cartier divisor (or $\mathbb{R}$-line bundle) ω, a proper closed subscheme $X_{-\infty} \subset X$ and a finite collection $\{C\}$ of reduced and irreducible subschemes of X such that there is a proper morphism $f : (Y, B_Y) \to X$ from an *embedded normal crossing pair* satisfying the following properties:

(1) $f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$.
(2) The natural map $\mathcal{O}_X \to f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil)$ induces an isomorphism
$$\mathcal{I}_{X_{-\infty}} \xrightarrow{\simeq} f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil - \lfloor B_Y^{>1}\rfloor),$$
where $\mathcal{I}_{X_{-\infty}}$ is the defining ideal sheaf of $X_{-\infty}$.
(3) The collection of subvarieties $\{C\}$ coincides with the images of (Y, B_Y)-strata that are not included in $X_{-\infty}$.

In Definition 3.2, we assume that (Y, B_Y) is a *globally embedded simple normal crossing pair*. On the other hand, in Definition A.2, we assume only that (Y, B_Y) is an *embedded normal crossing pair*.

Remark A.3 (Schemes versus varieties). A quasi-log *scheme* is called a quasi-log *variety* in [A]. However, X is not always reduced when $X_{-\infty} \neq \emptyset$. Note that X is reduced when $X_{-\infty} = \emptyset$.

Example A.4 ([A, Examples 4.3.4]). Let X be an effective Cartier divisor on a smooth variety M such that $\operatorname{Supp} X$ is a simple normal crossing divisor. Assume that Y, the reduced part of X, is nonempty. We put $\omega = (K_M + X)|_X$. Let $X_{-\infty}$ be the union of the nonreduced components of X. We put $K_Y + B_Y = (K_M + X)|_Y$. Let $f : Y \to X$ be the closed embedding. Then $\big(X, \omega, f : (Y, B_Y) \to X\big)$ is a quasi-log scheme. Note that X has nonreduced irreducible components if $X_{-\infty} \neq \emptyset$. We also note that f is not surjective if $X_{-\infty} \neq \emptyset$.

Lemma A.5 is essentially the same as Ambro's *embedded log transformations* in [A].

Lemma A.5. *Let (Y, B_Y) be an embedded normal crossing pair and let M be the ambient space of (Y, B_Y). Then there are a projective surjective morphism*

$\sigma : M' \to M$ from a smooth variety M' such that σ is a composition of blow-ups and a simple normal crossing pair (Z, B_Z) embedded into M' with the following properties:

(i) $\sigma : Z \to Y$ is surjective and $K_Z + B_Z = \sigma^*(K_Y + B_Y)$.

(ii) $\sigma_*\mathcal{O}_Z(\lceil -(B_Z^{<1})\rceil - \lfloor B_Z^{>1}\rfloor) \simeq \mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil - \lfloor B_Y^{>1}\rfloor)$.

(iii) Let C' be a stratum of (Z, B_Z). Then $\sigma(C')$ is a stratum of (Y, B_Y) or is contained in Supp $B_Y^{\geq 1}$*. Let C be a stratum of (Y, B_Y). Then there is a stratum C' of (Z, B_Z) such that $\sigma(C') = C$.*

Proof. First, we can construct a sequence of blow-ups $M_k \to M_{k-1} \to \cdots \to M_0 = M$ with the following properties:

(a) $\sigma_{i+1} : M_{i+1} \to M_i$ is the blow-up along a smooth stratum of Y_i for every i.

(b) We put $Y_0 = Y$, $B_{Y_0} = B_Y$ and $Y_{i+1} = \sigma_{i+1}^{-1}(Y_i)$ with the reduced scheme structure.

(c) Y_k is a simple normal crossing divisor on M_k.

We can check that $K_{Y_{i+1}} = \sigma_{i+1}^* K_{Y_i}$ for every i by construction. We can directly check that $R^1\sigma_{i+1*}\mathcal{O}_{M_{i+1}}(-Y_{i+1}) = 0$ and $\sigma_{i+1*}\mathcal{O}_{M_{i+1}}(-Y_{i+1}) \simeq \mathcal{O}_{M_i}(-Y_i)$ for every i. Therefore, by the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{M_i}(-Y_i) & \longrightarrow & \mathcal{O}_{M_i} & \longrightarrow & \mathcal{O}_{Y_i} & \longrightarrow & 0 \\ & & \downarrow\simeq & & \downarrow\simeq & & \downarrow & & \\ 0 & \longrightarrow & \sigma_{i+1*}\mathcal{O}_{M_{i+1}}(-Y_{i+1}) & \longrightarrow & \sigma_{i+1*}\mathcal{O}_{M_{i+1}} & \longrightarrow & \sigma_{i+1*}\mathcal{O}_{Y_{i+1}} & \longrightarrow & 0, \end{array}$$

we obtain $\sigma_{i+1*}\mathcal{O}_{Y_{i+1}} \simeq \mathcal{O}_{Y_i}$ for every i. We put $B_{Y_{i+1}} = \sigma_{i+1}^* B_{Y_i}$ for every i. Then, by replacing (Y, B_Y) and M with (Y_k, B_{Y_k}) and M_k, we may assume that Y is a simple normal crossing divisor on M.

Next, we can construct a sequence of blow-ups $M_k \to M_{k-1} \to \cdots \to M_0 = M$ with the following properties:

(1) $\sigma_{i+1} : M_{i+1} \to M_i$ is the blow-up along a smooth stratum of $(Y_i, \operatorname{Supp} B_{Y_i})$ contained in $\operatorname{Supp} B_{Y_i}$ for every i.

(2) We put $Y_0 = Y$ and $B_{Y_0} = B_Y$. Let Y_{i+1} be the strict transform of Y_i on M_{i+1} for every i.

(3) We put $K_{Y_{i+1}} + B_{Y_{i+1}} = \sigma_{i+1}^*(K_{Y_i} + B_{Y_i})$ for every i.

(4) $\operatorname{Supp} B_{Y_k}$ is a simple normal crossing divisor on Y_k.

Finally, by construction, we can check the properties (i), (ii) and (iii) for $\sigma : M_k \to M$ and (Y_k, B_{Y_k}) by an easy calculation of discrepancy coefficients similar to the proof of Proposition 4.1. □

Proposition A.6. *Assume that* (Y, B_Y) *is an embedded simple normal crossing pair in Definition A.2. Let* M *be the ambient space of* (Y, B_Y)*. Then, by taking some sequence of blow-ups of* M*, we may further assume that* (Y, B_Y) *is a globally embedded simple normal crossing pair in Definition A.2.*

Proof. It is sufficient to apply Lemmas 4.4 and 4.5 by putting $V = M$. If $B_Y = B_Y^{\leq 1}$, then this proposition is nothing but [F6, Lemma 3.3]. □

Therefore, by Lemma A.5 and Proposition A.6, Definition 3.2 is equivalent to Ambro's original definition of quasi-log schemes: Definition A.2.

Theorem A.7. *Definition 3.2 is equivalent to Definition A.2.*

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