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Γ-Unitaries, Dilation and a Natural Example

by

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Abstract

This note constructs an explicit normal boundary dilation for a commuting pair (S, P) of bounded operators with the symmetrized bidisk

$$\Gamma = \{(z_1 + z_2, z_1 z_2) : |z_1|, |z_2| \leq 1\}$$

as a spectral set. Such explicit dilations have hitherto been constructed only in the unit disk [11], the unit bidisk [3] and in the tetrablock [6]. The dilation is minimal and unique under a suitable condition. This paper also contains a natural example of a Γ-isometry. We compute its associated fundamental operator.

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§1. Introduction

This section contains the background and the statements of two main results.

In 1951, von Neumann proved the inequality

$$\|f(T)\| \leq \sup\{|f(z)| : |z| \leq 1\},$$

where T is a Hilbert space contraction and f is a polynomial. A proof, different from that of von Neumann, emerged when Sz.-Nagy proved his dilation theorem: Every contraction T can be dilated to a unitary U, i.e., if T acts on $\mathcal{H}$, then there is a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a unitary U on $\mathcal{K}$ such that

$$T^n = P_{\mathcal{H}} U^n|_{\mathcal{H}}.$$

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Indeed, the proof of von Neumann's inequality then is

$$\|f(T)\| = \|P_{\mathcal{H}} f(U)|_{\mathcal{H}}\|_{\mathcal{H}} \le \|f(U)\|_{\mathcal{K}} \le \sup\{|f(z)| : |z| \le 1\}$$

because $f(U)$ is a normal operator with $\sigma(f(U)) = \{f(z) : z \in \sigma(U)\} \subset \{f(z) : |z| = 1\}$.

It has long been a theme of research whether the converse direction is possible. This means that one chooses a compact subset K of the plane or of $\mathbb{C}^d$ for $d > 1$, considers a d-tuple $\underline{T} = (T_1, T_2, \ldots, T_d)$ of commuting bounded operators that satisfies

$$\|f(\underline{T})\| \le \sup\{|f(z)| : z \in K\}$$

for all rational functions f with poles off K and tries to see if there is a commuting tuple of bounded normal operators $\underline{N} = (N_1, N_2, \ldots, N_d)$ with $\sigma(\underline{N}) \subset bK$, the distinguished boundary of K, such that

$$f(\underline{T}) = P_{\mathcal{H}} f(\underline{N})|_{\mathcal{H}}.$$

The tuple $\underline{N}$ is then called a normal boundary dilation. An explicit construction of such an $\underline{N}$ has succeeded, apart from in the disk [11], only in the bidisk [3], although the existence of a dilation is abstractly known for an annulus [1].

The (closed) symmetrized bidisk

$$\Gamma = \{(z_1 + z_2, z_1 z_2) : |z_1|, |z_2| \le 1\}$$

is polynomially convex. Then, by the Oka–Weil theorem, a polynomial dilation is the same as a rational dilation. In other words,

$$T_1^{k_1} \cdots T_d^{k_d} = P_{\mathcal{H}} N_1^{k_1} \cdots N_d^{k_d}|_{\mathcal{H}}$$

for $k_1, \ldots, k_d \ge 0$.

Consider the class $A(\Gamma)$ of functions continuous in Γ and holomorphic in the interior of Γ. A boundary of Γ (with respect to A(Γ)) is a subset on which every function in $A(\Gamma)$ attains its maximum modulus. It is known that there is a smallest one among such boundaries. This particular smallest one is called the *distinguished boundary* of the symmetrized bidisk and is denoted by $b\Gamma$. It is well known that $b\Gamma$ is the symmetrization of the torus, i.e., $b\Gamma = \{(z_1 + z_2, z_1 z_2) : |z_1| = 1 = |z_2|\}$.

Definition 1. A Γ-contraction is a commuting pair of bounded operators (S, P) on a Hilbert space $\mathcal{H}$ such that the set Γ is a spectral set for (S, P), i.e.,

$$\|f(S, P)\| \le \sup\{|f(s, p)| : (s, p) \in \Gamma\},$$

for any polynomial f in two variables.

Definition 2. A Γ-unitary (R, U) is a commuting pair of bounded normal operators on a Hilbert space $\mathcal{H}$ such that $\sigma(R, U) \subset b\Gamma$ (this is automatically a Γ-contraction).

Definition 3. A Γ-isometry is the restriction of a Γ-unitary to a joint invariant subspace.

The work of the first author and other co-authors showed in [4] that given a Γ-contraction (S, P), there exists a unique operator $F \in \mathcal{B}(\mathcal{D}_P)$ with numerical radius no greater than 1 that satisfies the fundamental equation

$$S - S^*P = D_P F D_P, \tag{1.1}$$

where $D_P = (I - P^*P)^{1/2}$ is the defect operator of the contraction P and $\mathcal{D}_P = \overline{\operatorname{Ran}}\, D_P$ (the second component of a Γ-contraction is always a contraction). This operator F is called the *fundamental operator* of the Γ-contraction (S, P). Our first major result is the construction of a Γ-unitary dilation of a Γ-contraction explicitly. Let F be the fundamental operator of a Γ-contraction (S, P) on $\mathcal{H}$. The Γ-isometry, discovered in [4], that dilates (S, P) is described below. The space is $\widetilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{D}_P \oplus \mathcal{D}_P \oplus \cdots$, which is the same as the minimal isometric dilation space of the contraction P. In fact, the second component V of the Γ-isometric dilation (T_F, V) is the minimal isometric dilation of P. So

$$V = \left(\begin{array}{c|cccc} P & 0 & 0 & 0 & \cdots \\ \hline D_P & 0 & 0 & 0 & \cdots \\ 0 & I & 0 & 0 & \cdots \\ 0 & 0 & I & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

The first component T_F is

$$\left(\begin{array}{c|cccc} S & 0 & 0 & 0 & \cdots \\ \hline F^*D_P & F & 0 & 0 & \cdots \\ 0 & F^* & F & 0 & \cdots \\ 0 & 0 & F^* & F & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

The Γ-unitary dilation is obtained by extending the Γ-isometry above. Note that by Definition 3, every Γ-isometry is the restriction of a Γ-unitary to a joint invariant subspace. So the existence of a Γ-unitary dilation of (S, P) is guaranteed the moment one produces a Γ-isometric dilation. We construct it below.

The defining criterion of a Γ-contraction implies that the adjoint pair (S^*, P^*) is also a Γ-contraction. Consider its fundamental operator $G \in \mathcal{B}(\mathcal{D}_{P^*})$, where $D_{P^*} = (I - PP^*)^{1/2}$ is the defect operator and $\mathcal{D}_{P^*} = \overline{\text{Ran}}\, D_{P^*}$ is its defect space. This G satisfies

$$S^* - SP^* = D_{P^*} G D_{P^*}. \tag{1.2}$$

Just as the Γ-isometric dilation acts on the space of minimal isometric dilation of P, it turns out that the Γ-unitary dilation acts on the space of minimal unitary dilation of P. For brevity, let us denote $\mathcal{D}_{P^*} \oplus \mathcal{D}_{P^*} \oplus \cdots$ by $l^2(\mathcal{D}_{P^*})$. Note that the isometry V above has a natural unitary extension U on $\widetilde{\mathcal{H}} \oplus l^2(\mathcal{D}_{P^*})$. In operator matrix form it is

$$\begin{pmatrix} V & X' \\ 0 & Y' \end{pmatrix}$$

with respect to the decomposition $\widetilde{\mathcal{H}} \oplus l^2(\mathcal{D}_{P^*})$, where the operators $X' : l^2(\mathcal{D}_{P^*}) \to \widetilde{\mathcal{H}}(= \mathcal{H} \oplus \mathcal{D}_P \oplus \mathcal{D}_P \oplus \cdots)$ and $Y' : l^2(\mathcal{D}_{P^*}) \to l^2(\mathcal{D}_{P^*})$ are given by

$$\begin{pmatrix} D_{P^*} & 0 & 0 & \cdots \\ -P^* & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & I & 0 & \cdots \\ 0 & 0 & I & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \text{respectively.}$$

On the same space, the Γ-unitary dilation acts. Its first component R is the following extension of T_F:

$$\begin{pmatrix} T_F & X \\ 0 & Y \end{pmatrix}$$

with respect to the decomposition $\widetilde{\mathcal{H}} \oplus l^2(\mathcal{D}_{P^*})$, where the operators $X : l^2(\mathcal{D}_{P^*}) \to \widetilde{\mathcal{H}}$ and $Y : l^2(\mathcal{D}_{P^*}) \to l^2(\mathcal{D}_{P^*})$ are given by

$$\begin{pmatrix} D_{P^*}G & 0 & 0 & \cdots \\ -P^*G & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} G^* & G & 0 & 0 & \cdots \\ 0 & G^* & G & 0 & \cdots \\ 0 & 0 & G^* & G & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \text{respectively.}$$

Theorem 4. *The pair (R, U) is a Γ-unitary dilation of (S, P).*

Note the similarity of the construction to Schäffer's construction in [11] of the unitary dilation of a contraction. The crucial inputs are F and G in the construction of R. After we completed this work, we came to know that Pal [9] has independently proved the theorem above.

In the case of any dilation, uniqueness is a natural question, i.e., given $\underline{T} = (T_1, T_2, \ldots, T_d)$ acting on $\mathcal{H}$ and a dilation $\underline{N} = (N_1, N_2, \ldots, N_d)$ acting on $\mathcal{K} \supset \mathcal{H}$, is it true that any other dilation, say $\underline{N'} = (N'_1, N'_2, \ldots, N'_d)$ on $\mathcal{K}' \supset \mathcal{H}$ is unitarily equivalent to $\underline{N}$? The answer is yes when the compact set $K = \overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ and the number d of co-ordinates of $\underline{T}$ is 1 under a certain natural condition called minimality. If T is a contraction, N is a unitary dilation and the space $\mathcal{K}$ is minimal, i.e.,

$$\mathcal{K} = \{N^n h : h \in \mathcal{H} \text{ and } n \in \mathbb{Z}\},$$

then any other minimal unitary dilation of T is unitarily equivalent to N. The Γ-unitary dilation constructed above is minimal. Moreover, it is unique in the sense described in the theorem below.

Theorem 5 (Uniqueness). *Let (S, P) be a Γ-contraction on a Hilbert space $\mathcal{H}$ and (R, U), as defined above, be the Γ-unitary dilation of (S, P).*

(i) If $(\tilde{R}, U)$ is another Γ-unitary dilation of (S, P), then $\tilde{R} = R$.

(ii) If $(\tilde{R}, \tilde{U})$, on some Hilbert space $\tilde{\mathcal{K}}$ containing $\mathcal{H}$, is another Γ-unitary dilation of (S, P), where $\tilde{U}$ is a minimal unitary dilation of P, then $(\tilde{R}, \tilde{U})$ is unitarily equivalent to (R, U).

This theorem is special because when $K = \overline{\mathbb{D}} \times \overline{\mathbb{D}}$, then the corresponding minimality condition does not yield unitary equivalence; see [8].

The last section of this paper, i.e., Section 5, has concrete examples of fundamental operators. Fundamental operators are of utmost importance in the study of Γ-contractions, as is clear from the discussion above and also from abundant use of fundamental operators in the literature. A few notable mentions of the uses of the fundamental operator are [5, Prop. 4.3 and Thm. 4.4] and [10, Thm. 3.5]. Computing the fundamental operator of a given Γ-contraction is usually difficult. In Section 5, we explicitly compute the fundamental operators of three natural examples. These examples originate from function theory on the bidisk, which has been a rich source of examples of Γ-contractions; see [4].

§2. Elementary results on Γ-contractions

This section contains certain preliminary results on Γ-contractions. Just as

$$PD_P = D_{P^*}P \tag{2.1}$$

and its adjoint equation

$$D_P P^* = P^* D_{P^*} \tag{2.2}$$

have been known since the time of Sz.-Nagy and Foias, we have a crucial operator equality in the case of a Γ-contraction (S,P) that relates S, P and the fundamental operator F. It is

$$D_P S = F D_P + F^* D_P P. \tag{2.3}$$

The adjoint form of this equality involves the Γ-contraction (S^*,P^*) and its fundamental operator G. It is

$$D_{P^*} S^* = G D_{P^*} + G^* D_{P^*} P^*. \tag{2.4}$$

The next lemma gives a relation between the fundamental operators of the two Γ-contractions (S,P) and (S^*,P^*). This can be found in [5, Prop. 2.3]. Hence we omit the proof.

Lemma 6. *Let (S,P) be a Γ-contraction and F, G are fundamental operators of (S,P) and (S^*,P^*) respectively. Then*

$$P^*G = F^*P^*|_{\mathcal{D}_{P^*}}. \tag{2.5}$$

Remark 7. If one applies Lemma 6 for the Γ-contraction (S^*,P^*) in place of (S,P), then the result is $PF = G^*P|_{\mathcal{D}_P}$.

The next two lemmas give new relations between the fundamental operators of Γ-contractions (S,P) and (S^*,P^*).

Lemma 8. *Let (S,P) be a Γ-contraction on a Hilbert space $\mathcal{H}$. If F and G are fundamental operators of (S,P) and (S^*,P^*) respectively, then*

$$(SD_P - D_{P^*}GP)|_{\mathcal{D}_P} = D_P F. \tag{2.6}$$

Proof. Note that the LHS and the RHS of (2.6) are operators from $\mathcal{D}_P$ to $\mathcal{H}$:

$$\begin{aligned}(SD_P - D_{P^*}GP)D_P h &= S(I-P^*P)h - D_{P^*}GPD_P h\\ &= Sh - SP^*Ph - (D_{P^*}GD_{P^*})Ph\\ &= Sh - SP^*Ph - S^*Ph + SP^*Ph\\ &= Sh - S^*Ph = D_P F D_P h \quad \text{for all } h \in \mathcal{H}.\end{aligned}$$

Since $\mathcal{D}_P = \overline{\operatorname{Ran}} D_P$ and the operators are bounded, we are done. □

Remark 9. If one applies Lemma 8 for the Γ-contraction (S^*,P^*) in place of (S,P), then the result is $S^*D_{P^*} - D_P F P^* = D_{P^*}G$.

Lemma 10. *Let F and G be the fundamental operators of (S,P) and (S^*,P^*) respectively. Then*

$$(F^*D_PD_{P^*} - FP^*)|_{\mathcal{D}_{P^*}} = D_PD_{P^*}G - P^*G^*. \tag{2.7}$$

Proof. Note that the LHS and the RHS of (2.7) are operators from $\mathcal{D}_{P^*}$ to $\mathcal{D}_P$:

$$\begin{aligned}(F^*D_PD_{P^*} - FP^*)D_{P^*}h &= F^*D_P(I - PP^*)h - FP^*D_{P^*}h \\ &= F^*D_Ph - F^*D_PPP^*h - FD_PP^*h \qquad \text{[using (2.2)]} \\ &= F^*D_Ph - (F^*D_PP + FD_P)P^*h \\ &= (F^*D_P - D_PSP^*)h \qquad \text{[using (2.3)]} \\ &= (D_PS^* - P^*G^*D_{P^*})h - D_PSP^*h \qquad \text{[using (2.6)]} \\ &= D_P(S^* - SP^*)h - P^*G^*D_{P^*}h \\ &= D_PD_{P^*}GD_{P^*}h - P^*G^*D_{P^*}h \\ &= (D_PD_{P^*}G - P^*G^*)D_{P^*}h\end{aligned}$$

for all $h \in \mathcal{H}$. Since $\mathcal{D}_{P^*} = \overline{\operatorname{Ran}} D_{P^*}$ and the operators are bounded, we are done. □

§3. Γ-unitary dilation of a Γ-contraction: Proof of Theorem 4

The starting point of the proof of Theorem 4 is the pair (T_F, V) on $\widetilde{\mathcal{H}} = \mathcal{H} \oplus l^2(\mathcal{D}_P)$, where

$$T_F(h \oplus (a_0, a_1, a_2, \dots)) = (Sh \oplus (F^*D_Ph + Fa_0, F^*a_0 + Fa_1, F^*a_1 + Fa_2, \dots))$$

and

$$V(h \oplus (a_0, a_1, a_2, \dots)) = (Ph \oplus (D_Ph, a_0, a_1, a_2, \dots)).$$

We know from [4] that this pair is a Γ-isometric dilation for (S,P). So the job reduces to finding an explicit Γ-unitary extension of (T_F, V). For that, it is natural to consider the minimal unitary extension U of V on $\mathcal{K} = \widetilde{\mathcal{H}} \oplus l^2(\mathcal{D}_{P^*})$. The explicit form of U due to Schäffer [11] is given in Section 1. Schäffer proved that U is the minimal unitary dilation of P.

We shall first prove that (R,U) on $\mathcal{K}$, defined in Section 1, is a Γ-unitary. To be able to do that, we need a tractable characterization of a Γ-unitary. This can be found in [4]: the fourth part of Theorem 2.5 there tells us that a pair of commuting operators (R,U) defined on a Hilbert space $\mathcal{H}$ is a Γ-unitary if and only if U is unitary and (R,U) is a Γ-contraction. So, for our particular (R,U), we shall show that

(i) $RU = UR$ and

(ii) $\|f(R,U)\| \leq \|f\|_{\infty,\Gamma}$, for every polynomial f in two variables.

To show that $R = \begin{pmatrix} T_F & X \\ 0 & Y \end{pmatrix}$ and $U = \begin{pmatrix} V & X' \\ 0 & Y' \end{pmatrix}$ commute, we shall have to show $YY' = Y'Y$ and $XY' + T_F X' = X'Y + VX$.

Commutativity of Y and Y' can be verified by direct computation, but perhaps a more elegant way to see it is to note that the space on which these operators act is unitarily equivalent to the space of $\mathcal{D}_{P^*}$-valued Hardy space on the disk. Under conjugation by the same unitary, Y' becomes the backward shift and Y becomes the adjoint of multiplication by the operator-valued function $G + G^*z$ (a so-called co-analytic Toeplitz operator). Thus they commute.

For all $(a_0, a_1, a_2, \dots) \in l^2(\mathcal{D}_{P^*})$ we have

$$\begin{aligned}
&(XY' + T_F X')(a_0, a_1, a_2, \dots) \\
&\quad = X(a_1, a_2, a_3, \dots) + T_F(D_{P^*}a_0 \oplus (-P^*a_0, 0, 0, \dots)) \\
&\quad = (D_{P^*}Ga_1 \oplus (-P^*Ga_1, 0, 0, \dots)) \\
&\qquad + (SD_{P^*}a_0 \oplus ((F^*D_P D_{P^*} - FP^*)a_0, -F^*P^*a_0, 0, 0, \dots)) \\
&\quad = (SD_{P^*}a_0 + D_{P^*}Ga_1) \\
&\qquad \oplus ((F^*D_P D_{P^*} - FP^*)a_0 - P^*Ga_1, -F^*P^*a_0, 0, 0, \dots)
\end{aligned}$$

and

$$\begin{aligned}
&(X'Y + VX)(a_0, a_1, a_2, \dots) \\
&\quad = X'(G^*a_0 + Ga_1, G^*a_1 + Ga_2, G^*a_2 + Ga_3, \dots) \\
&\qquad + V(D_{P^*}Ga_0 \oplus (-P^*Ga_0, 0, 0, \dots)) \\
&\quad = ((D_{P^*}G^*a_0 + D_{P^*}Ga_1) \oplus (-P^*G^*a_0 - P^*Ga_1, 0, 0, \dots)) \\
&\qquad + \ (PD_{P^*}Ga_0 \oplus (D_P D_{P^*}Ga_0, -P^*Ga_0, 0, 0, \dots)) \\
&\quad = ((D_{P^*}G^* + PD_{P^*}G)a_0 + D_{P^*}Ga_1) \\
&\qquad \oplus ((D_P D_{P^*}G - P^*G^*)a_0 - P^*Ga_1, -P^*Ga_0, 0, 0, \dots).
\end{aligned}$$

The lemmas of the previous section will now be useful. By Lemmas 10 and 6 and equation (2.4), it follows that $XY' + T_F X' = X'Y + VX$. Thus the proof of commutativity is complete.

We now prove that R is a normal operator. What we first prove is that $R = R^*U$, because this will imply that R is a normal operator. Establishing the equality $R = R^*U$ is equivalent to showing the following equalities:

(a) $Y = Y^*Y' + X^*X'$;

(b) $X^*V = 0$;

(c) $X = T_F^* X'$; and
(d) $T_F = T_F^* V$.

From the definitions of X and Y, it is easy to check that

$$X^*(h \oplus (a_0, a_1, a_2, \dots)) = (G^* D_{P^*} h - G^* P a_0, \dots)$$

and

$$Y^*(a_0, a_1, a_2, \dots) = (G a_0, G^* a_0 + G a_1, G^* a_1 + G a_2, \dots).$$

Thus

$$\begin{aligned}
&(Y^* Y' + X^* X')(a_0, a_1, a_2, \dots) \\
&\quad = Y^*(a_1, a_2, a_3, \dots) + X^*(D_{P^*} a_0 \oplus (-P^* a_0, 0, 0, \dots)) \\
&\quad = (G a_1, G^* a_1 + G a_2, G^* a_2 + G a_3, \dots) \\
&\qquad + (G^*(I - PP^*) a_0 + G^* P P^* a_0, 0, 0, \dots) \\
&\quad = (G^* a_0 + G a_1, G^* a_1 + G a_2, G^* a_2 + G a_3, \dots) = Y(a_0, a_1, a_2, \dots),
\end{aligned}$$

which establishes (a). To prove (b), we use equation (2.1) and see that

$$\begin{aligned}
X^* V(h \oplus (a_0, a_1, a_2, \dots)) &= X^*(Ph \oplus (D_P h, a_0, a_1, a_2, \dots)) \\
&= ((G^* D_{P^*} P - G^* P D_P) h, 0, 0, 0, \dots) = 0.
\end{aligned}$$

To prove (c), we use Remark 9 and Lemma 6 to get

$$\begin{aligned}
T_F^* X'(a_0, a_1, a_2, \dots) &= T_F^*(D_{P^*} a_0 \oplus (-P^* a_0, 0, 0, \dots)) \\
&= (S^* D_{P^*} a_0 - D_P F P^* a_0) \oplus (-F^* P^*, 0, 0, \dots) \\
&= X(a_0, a_1, a_2, \dots).
\end{aligned}$$

Since (T_F, V) is a Γ-isometry, (d) holds, by [4, Thm. 2.14].

Now we proceed to show that (R, U) satisfies the von Neumann inequality. For any polynomial f in two variables we have

$$f(R, U) = \begin{pmatrix} f(T_F, V) & Z_f \\ 0 & f(Y, Y') \end{pmatrix},$$

where (T_F, V) and $(Y, Y') = (M_{G+G^* z}, M_z)^*$ are Γ-contractions and Z_f is an operator depending on f. We have by [7, Lem. 1] that

$$\sigma(f(R, U)) \subset \sigma(f(T_F, V)) \cup \sigma(f(Y, Y')),$$

which gives

$$\begin{aligned}
r(f(R, U)) \le \max\{r(f(T_F, V)), r(f(Y, Y'))\} &\le \max\{\|f(T_F, V)\|, \|f(Y, Y')\|\} \\
&\le \|f\|_{\infty, \Gamma}.
\end{aligned}$$

Since R is a normal operator, so is $f(R,U)$ and hence $r(f(R,U)) = \|f(R,U)\|$. This completes the proof of part (ii). Hence (R,U) is a Γ-unitary.

To complete the proof of Theorem 4, we need to show that (R,U) dilates (S,P). This is trivial because (R,U) is the extension of (T_F,V), which is a co-extension of (S,P). □

§4. Minimality and uniqueness

In this section we prove Theorem 5. First we remark that the dilation is minimal.

Remark 11 (Minimality). Minimality of a commuting normal boundary dilation $\underline{N} = (N_1, N_2, \dots, N_d)$ on a space $\mathcal{K}$ of a commuting tuple $(T_1, T_2, \dots, T_d)$ of bounded operators on a space $\mathcal{H}$ means that the space $\mathcal{K}$ is no bigger than the closure of the span of the following set:

$$\{N_1^{k_1} N_2^{k_2} \cdots N_d^{k_d} N_1^{*l_1} N_2^{*l_2} \cdots N_d^{*l_d} h : h \in \mathcal{H}, \text{ where } k_i, l_i \in \mathbb{N} \text{ for } i = 1, 2, \dots, d\}.$$

Note that the space $\mathcal{K}$ has to be at least this big. In our construction, the space is just the minimal unitary dilation space of P (which is unique up to unitary equivalence). It is a bit of a surprise that one can find the Γ-unitary dilation of (S,P) on the same space, while one would have normally expected the dilation space to be bigger. Since no dilation of (S,P) can take place on a space smaller than the minimal unitary dilation space of P (because the dilation has to dilate P as well), our construction of Γ-unitary dilation is minimal. Indeed, post facto we know from our dilation that

$$\begin{aligned}\overline{\text{span}}&\{R^{m_1} R^{*m_2} U^n h : h \in \mathcal{H}, m_1, m_2 \in \mathbb{N} \text{ and } n \in \mathbb{Z}\}\\ &= \overline{\text{span}}\{U^n h : h \in \mathcal{H} \text{ and } n \in \mathbb{Z}\}.\end{aligned}$$

Note the absence of R on the right-hand side.

We now prove a weaker version of the uniqueness theorem and then we use it to prove the main result.

Lemma 12. *Suppose (S,P) is a Γ-contraction on a Hilbert space $\mathcal{H}$ and (R,U) is the above Γ-unitary dilation of (S,P). If $(\tilde{R},U)$ is another Γ-unitary dilation of (S,P) such that $\tilde{R}$ is an extension of T_F, then $\tilde{R} = R$.*

Proof. Suppose $(\tilde{R},U)$ is another Γ-unitary dilation of (S,P), such that $\tilde{R}$ is an extension of T_F. Since $\tilde{R}$ is an extension of T_F, $\tilde{R}$ is of the form $\begin{pmatrix} T_F & X \\ 0 & Y \end{pmatrix}$ with respect to the decomposition $\mathcal{K} = \widetilde{\mathcal{H}} \oplus l^2(\mathcal{D}_{P^*})$. Since $U = \begin{pmatrix} V & X' \\ 0 & Y' \end{pmatrix}$ is unitary and $\tilde{R}U = U\tilde{R}$, we have, from easy matrix calculations,

$$Y'^* Y' + X'^* X' = I, \qquad X'^* V = 0 \tag{4.1}$$

and

$$\tilde{Y}Y' = Y'\tilde{Y}, \qquad \tilde{X}Y' + T_F X' = X'\tilde{Y} + V\tilde{X}. \tag{4.2}$$

Also since $(\tilde{R}, U)$ is a Γ-unitary, we have $\tilde{R} = \tilde{R}^* U$ and that gives $\tilde{X} = T_F^* X'$. So

$$\begin{aligned}
\tilde{X}(a_0, a_1, a_2, \dots) &= T_F^* X'(a_0, a_1, a_2, \dots) \\
&= T_F^*(D_{P^*}a_0 \oplus (-P^*a_0, 0, 0, \dots)) \\
&= (S^* D_{P^*}a_0 - D_P F P^* a_0) \oplus (-F^* P^* a_0, 0, 0, \dots) \\
&= (D_{P^*}Ga_0 \oplus (-F^* P^* a_0, 0, 0, \dots)) \qquad \text{[by Remark 9]} \\
&= X(a_0, a_1, a_2, \dots).
\end{aligned}$$

Now to find $\tilde{Y}$, we proceed as follows: From the second equation of (4.2) we have

$$\begin{aligned}
& X'\tilde{Y} + V\tilde{X} = \tilde{X}Y' + T_F X' \\
& \quad \Rightarrow X'^* X'\tilde{Y} + X'^* V\tilde{X} = X'^*\tilde{X}Y' + X'^* T_F X' \quad \text{[multiplying } X'^* \text{ from left]} \\
& \quad \Rightarrow (I - Y'^* Y')\tilde{Y} = X'^*\tilde{X}Y' + X'^* T_F X' \qquad \text{[using (4.1)]} \\
& \quad \Rightarrow \tilde{Y}^*(I - Y'^* Y') = Y'^*\tilde{X}^* X' + X'^* T_F^* X'. \qquad (*)
\end{aligned}$$

Note that $(I - Y'^* Y')$ is the orthogonal projection of $l^2(\mathcal{D}_{P^*})$ onto the first component. Let $x = (a_0, a_1, a_2, \dots)$ be in $l^2(\mathcal{D}_{P^*})$. From $(*)$ we get

$$\tilde{Y}^*(a_0, 0, 0, \dots) = Y'^*\tilde{X}^* X'(a_0, a_1, a_2, \dots) + X'^* T_F^* X'(a_0, a_1, a_2, \dots).$$

Thus

$$\begin{aligned}
& Y'^*\tilde{X}^* X'(a_0, a_1, a_2, \dots) + X'^* T_F^* X'(a_0, a_1, a_2, \dots) \\
& \quad = Y'^*\tilde{X}^*(D_{P^*}a_0 \oplus (-P^*a_0, 0, 0, \dots)) + X'^* T_F^*(D_{P^*}a_0 \oplus (-P^*a_0, 0, 0, \dots)) \\
& \quad = Y'^*((D_{P^*}S - PF^* D_P)D_{P^*}a_0 + PFP^*a_0, 0, 0, \dots) \\
& \qquad + \; X'^*((S^* D_{P^*} - D_P F P^*)a_0 \oplus (-F^* P^* a_0, 0, 0, \dots)) \\
& \quad = (0, (D_{P^*}S - PF^* D_P)D_{P^*}a_0 + PFP^*a_0, 0, 0, \dots) \\
& \qquad + \; (D_{P^*}(S^* D_{P^*} - D_P F P^*)a_0 + PF^* P^* a_0, 0, 0, \dots) \\
& \quad = \big(D_{P^*}(S^* D_{P^*} - D_P F P^*)a_0 + PF^* P^* a_0, \\
& \qquad (D_{P^*}S - PF^* D_P)D_{P^*}a_0 + PFP^*a_0, 0, 0, \dots\big).
\end{aligned}$$

Let us denote the operator $(D_{P^*}(S^* D_{P^*} - D_P F P^*) + PF^* P^*)|_{\mathcal{D}_{P^*}}$ by C. Then we have $\tilde{Y}^*(a_0, 0, 0, \dots) = (Ca_0, C^* a_0, 0, 0, \dots)$. Note that C is an operator from $\mathcal{D}_{P^*}$ to $\mathcal{D}_{P^*}$. We shall show that $C = G$, where G is the fundamental operator of the Γ-contraction (S^*, P^*). The following computation establishes that.

For h, h' in $\mathcal{H}$, we have

$$
\begin{aligned}
&\langle CD_{P^*}h, D_{P^*}h'\rangle \\
&\quad= \langle (D_{P^*}(S^*D_{P^*} - D_P F P^*) + PF^*P^*)D_{P^*}h, D_{P^*}h'\rangle \\
&\quad= \langle D_{P^*}S^*(I - PP^*)h - D_{P^*}(D_P F D_P)P^*h + PF^*P^*D_{P^*}h, D_{P^*}h'\rangle \\
&\quad= \langle D_{P^*}S^*h - D_{P^*}S^*PP^*h - D_{P^*}SP^*h + D_{P^*}S^*PP^*h + PF^*P^*D_{P^*}h, \\
&\qquad D_{P^*}h'\rangle \\
&\quad= \langle D_{P^*}S^*h - D_{P^*}SP^*h + PF^*P^*D_{P^*}h, D_{P^*}h'\rangle \\
&\quad= \langle D_{P^*}(S^* - SP^*)h + PF^*P^*D_{P^*}h, D_{P^*}h'\rangle \\
&\quad= \langle D_{P^*}^2 G D_{P^*}h + PF^*P^*D_{P^*}h, D_{P^*}h'\rangle \\
&\quad= \langle (I - PP^*)GD_{P^*}h, D_{P^*}h'\rangle + \langle F^*P^*D_{P^*}h, P^*D_{P^*}h'\rangle \\
&\quad= \langle GD_{P^*}h, D_{P^*}h'\rangle - \langle P^*GD_{P^*}h, P^*D_{P^*}h'\rangle + \langle F^*P^*D_{P^*}h, P^*D_{P^*}h'\rangle \\
&\quad= \langle GD_{P^*}h, D_{P^*}h'\rangle - \langle F^*P^*D_{P^*}h, D_{P^*}h'\rangle + \langle F^*P^*D_{P^*}h, P^*D_{P^*}h'\rangle \\
&\quad= \langle GD_{P^*}h, D_{P^*}h'\rangle.
\end{aligned}
$$

Hence $C = G$ and hence for every a in $\mathcal{D}_{P^*}$,

$$
\tilde{Y}^*(a, 0, 0, 0, \dots) = (Ga, G^*a, 0, 0, \dots).
$$

We want to compute the action of $\tilde{Y}^*$ on an arbitrary vector. Now using the first equation of (4.2), we have for every $n \geq 0$,

$$
\begin{aligned}
\tilde{Y}^*(\overbrace{0, \dots, 0}^{n \text{ times}}, a, 0, \dots) &= \tilde{Y}^* {Y'^*}^n (a, 0, 0, 0, \dots) \\
&= {Y'^*}^n \tilde{Y}^*(a, 0, 0, 0, \dots) \\
&= {Y'^*}^n (Ga, G^*a, 0, 0, \dots) = (\overbrace{0, \dots, 0}^{n \text{ times}}, Ga, G^*a, 0, 0, \dots).
\end{aligned}
$$

Therefore for an arbitrary element $(a_0, a_1, a_2, \dots) \in l^2(\mathcal{D}_{P^*})$, we have

$$
\begin{aligned}
&\tilde{Y}^*(a_0, a_1, a_2, \dots) \\
&\quad= \tilde{Y}^*((a_0, 0, 0, \dots) + (0, a_1, 0, \dots) + (0, 0, a_2, \dots) + \cdots) \\
&\quad= (Ga_0, G^*a_0, 0, 0, \dots) + (0, Ga_1, G^*a_1, 0, 0, \dots) \\
&\qquad + (0, 0, Ga_2, G^*a_2, 0, 0, \dots) + \cdots \\
&\quad= (Ga_0, G^*a_0 + Ga_1, G^*a_1 + Ga_2, \dots).
\end{aligned}
$$

For $(a_0, a_1, a_2, \dots)$ and $(b_0, b_1, b_2, \dots)$ in $l^2(\mathcal{D}_{P^*})$, we have

$$\begin{aligned}
&\langle (a_0, a_1, a_2, \dots), \tilde{Y}^*(b_0, b_1, b_2, \dots)\rangle \\
&\quad = \langle (a_0, a_1, a_2, \dots), (Gb_0, G^*b_0 + Gb_1, G^*b_1 + Gb_2, \dots)\rangle \\
&\quad = \langle a_0, Gb_0\rangle + \langle a_1, G^*b_0 + Gb_1\rangle + \langle a_2, G^*b_1 + Gb_2\rangle + \cdots \\
&\quad = \langle G^*a_0 + Ga_1, b_0\rangle + \langle G^*a_1 + Ga_2, b_1\rangle + \langle G^*a_2 + Ga_3, b_2\rangle + \cdots \\
&\quad = \langle (G^*a_0 + Ga_1, G^*a_1 + Ga_2, G^*a_2 + Ga_3, \dots), (b_0, b_1, b_2, \dots)\rangle.
\end{aligned}$$

Hence, by definition of the adjoint of an operator, we have

$$\tilde{Y}(a_0, a_1, a_2, \dots) = (G^*a_0 + Ga_1, G^*a_1 + Ga_2, G^*a_2 + Ga_3, \dots) = Y(a_0, a_1, a_2, \dots),$$

for every $(a_0, a_1, a_2, \dots) \in l^2(\mathcal{D}_{P^*})$. Therefore $\tilde{R} = R$. Hence the proof is complete. □

Note that when we write the operator U with respect to the decomposition $l^2(\mathcal{D}_P) \oplus \mathcal{H} \oplus l^2(\mathcal{D}_{P^*})$ then this is of the form

$$\begin{pmatrix} U_1 & U_2 & U_3 \\ 0 & P & U_4 \\ 0 & 0 & U_5 \end{pmatrix},$$

where U_1, U_2, U_3, U_4 and U_5 are defined as

$$\begin{gathered}
U_1(a_0, a_1, a_2, \dots) = (0, a_0, a_1, \dots), \qquad U_2(h) = (D_p h, 0, 0, \dots), \\
U_3(b_0, b_0, b_2, \dots) = (-P^*b_0, 0, 0, \dots), \quad U_4(b_0, b_1, b_2, \dots) = D_{P^*}b_0, \\
U_5(b_0, b_1, b_2, \dots) = (b_1, b_2, b_3, \dots)
\end{gathered}$$

for all $h \in \mathcal{H}, (a_0, a_1, a_2, \dots) \in l^2(\mathcal{D}_P)$ and $(b_0, b_0, b_2, \dots) \in l^2(\mathcal{D}_{P^*})$. Note that this is the Schäffer minimal unitary dilation of the contraction P as in [11] (it can also be found in [12, Sect. 5, Chap. 1].

Lemma 13. *Let (R, U) on $\mathcal{K}$ be a dilation of (S, P) on $\mathcal{H}$, where P is a contraction on $\mathcal{H}$, and U on $\mathcal{K}$ is the Schäffer minimal unitary dilation of P. Then R admits a matrix representation of the form*

$$\begin{pmatrix} * & * & * \\ 0 & S & * \\ 0 & 0 & * \end{pmatrix},$$

with respect to the decomposition $\mathcal{K} = l^2(\mathcal{D}_P) \oplus \mathcal{H} \oplus l^2(\mathcal{D}_{P^*})$.

Proof. Let $R = (R_{kl})_{k,l=1}^3$ with respect to $\mathcal{K} = l^2(\mathcal{D}_P) \oplus \mathcal{H} \oplus l^2(\mathcal{D}_{P^*})$. Call $\widetilde{\mathcal{H}} = l^2(\mathcal{D}_P) \oplus \mathcal{H}$. Since U is minimal we have $\mathcal{K} = \bigvee_{m=-\infty}^{\infty} U^m\mathcal{H}$ and $\widetilde{\mathcal{H}} = \bigvee_{m=0}^{\infty} U^m\mathcal{H} = \bigvee_{m=0}^{\infty} V^m\mathcal{H}$, where V is the minimal isometry dilation of P. Note that

$$P_{\mathcal{H}}R(U^m h) = SP^m h = SP_{\mathcal{H}}U^m h \quad \text{for all } h \in \mathcal{H} \text{ and } m \geq 0.$$

Therefore we have $P_{\mathcal{H}}R|_{\widetilde{\mathcal{H}}} = SP_{\mathcal{H}}|_{\widetilde{\mathcal{H}}}$ or equivalently $S^* = P_{\widetilde{\mathcal{H}}}R^*|_{\mathcal{H}}$. This shows that $R_{21} = 0$.

Call $\widetilde{\mathcal{N}} = \mathcal{H} \oplus l^2(\mathcal{D}_{P^*})$, then note that $\widetilde{\mathcal{N}} = \bigvee_{n=0}^{\infty} U^{*n}\mathcal{H}$. We have

$$P_{\mathcal{H}}R^*(U^{*m}h) = S^*P^{*m}h = S^*P_{\mathcal{H}}U^{*m}h \quad \text{for all } h \in \mathcal{H} \text{ and } m \geq 0.$$

This and a similar argument to above give us $S = P_{\widetilde{\mathcal{N}}}R|_{\mathcal{H}}$. Therefore $R_{32} = 0$.

So far, we have shown that R admits a matrix representation of the form

$$\begin{pmatrix} R_{11} & R_{12} & R_{13} \\ 0 & S & R_{23} \\ R_{31} & 0 & R_{33} \end{pmatrix},$$

with respect to the decomposition $\mathcal{K} = l^2(\mathcal{D}_P) \oplus \mathcal{H} \oplus l^2(\mathcal{D}_{P^*})$. To show that $R_{13} = 0$ we proceed as follows:

From the commutativity of R with U we get, by an easy matrix calculation,

$$R_{31}U_1 = U_5R_{31} \quad \text{and} \quad R_{31}U_2 = 0, \tag{4.3}$$

(equating the 31st and 32nd entries of RU and UR respectively). By the definition of U_2, we have $\operatorname{Ran} U_2 = \operatorname{Ran}(I - U_1U_1^*)$. Therefore $R_{31}(I - U_1U_1^*) = 0$, which with the first equation of (4.3) gives $R_{31} = U_5R_{31}U_1^*$, which gives after the nth iteration $R_{31} = U_5^nR_{31}U_1^{*n}$. Now since U_1^{*n} goes to 0 strongly as $n \to \infty$, we have that $R_{31} = 0$. This completes the proof of the lemma. □

Now we are ready to prove Theorem 5, the main result of this section.

Proof of part (i). Since $(\tilde{R}, U)$ is a dilation of (S, P), by Lemma 13 we have $\tilde{R}$ of the form

$$\begin{pmatrix} T & \tilde{R}_{12} \\ 0 & \tilde{R}_{22} \end{pmatrix}$$

with respect to the decomposition $\widetilde{\mathcal{H}} \oplus l^2(\mathcal{D}_{P^*})$, where $T : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$ is of the form

$$\begin{pmatrix} T_{11} & T_{12} \\ 0 & S \end{pmatrix}$$

with respect to the decomposition $l^2(\mathcal{D}_P)\oplus\mathcal{H}$. Since (T,V) on $\tilde{H}$ is the restriction of the Γ-contraction $(\tilde{R},U)$ to $\widetilde{\mathcal{H}}$ and V is an isometry, we have (T,V) is a Γ-isometry. Also note that $T^*|_{\mathcal{H}} = S^*$ and $V^*|_{\mathcal{H}} = P^*$. So (T,V) is a Γ-isometric dilation of (S,P). Also note that V is the Schäffer minimal isometric dilation of P. Now it follows from [4, Thm. 4.3(2)] that $T = T_F$, where T_F is as in Theorem 4. Therefore $\tilde{R}$ is an extension of T_F. Now the proof follows from Lemma 12. □

Proof of part (ii). Since $\tilde{U}$ is a minimal unitary dilation of P, there exists a unitary operator $W : \tilde{\mathcal{K}} \to \mathcal{K}$ such that $W\tilde{U}W^* = U$ and $Wh = h$ for all $h \in \mathcal{H}$. This shows that $(W\tilde{R}W^*, W\tilde{U}W^*)$ is another Γ-unitary dilation of (S,P). But $W\tilde{U}W^* = U$. Hence by part (i) we have $(W\tilde{R}W^*, W\tilde{U}W^*) = (R,U)$. Hence the proof is complete. □

Remark 14. As in the case of Ando's dilation of a commuting pair of contractions, a minimal Γ-unitary dilation of a Γ-contraction need not be unique (up to unitary equivalence). In this section, we constructed a particular Γ-unitary dilation which is the most obvious one because it acts on the minimal unitary dilation space of the contraction P. Moreover, if the Γ-unitary dilation space is no bigger than the minimal unitary dilation space of the contraction P, then the Γ-unitary dilation is unique up to unitary equivalence.

§5. Examples of fundamental operators

§5.1. Hardy space of the bidisk

Consider the Hilbert space

$$H^2(\mathbb{D}^2) = \Big\{ f : \mathbb{D}^2 \to \mathbb{C} : f(z_1,z_2) = \textstyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij} z_1^i z_2^j \\ \text{with } \textstyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} |a_{ij}|^2 < \infty \Big\}$$

with the inner product $\left\langle \sum_{i,j=0}^{\infty} a_{ij} z_1^i z_2^j, \sum_{i,j=0}^{\infty} b_{ij} z_1^i z_2^j \right\rangle = \sum_{i,j=0}^{\infty} a_{ij}\bar{b_{ij}}$. Note that the operator pair $(M_{z_1+z_2}, M_{z_1z_2})$ on $H^2(\mathbb{D}^2)$ is a Γ-isometry, since it is the restriction of the Γ-unitary $(M_{z_1+z_2}, M_{z_1z_2})$ on $L^2(\mathbb{T}^2)$, where $\mathbb{T}$ denotes the unit circle. For brevity, we call the pair $(M_{z_1+z_2}, M_{z_1z_2})$ on $H^2(\mathbb{D}^2)$ by (S,P). In this section, we shall first find the fundamental operator of (S^*,P^*).

Note that every element $f \in H^2(\mathbb{D}^2)$ can be expressed in the matrix form

$$((a_{ij}))_{i,j=0}^{\infty} = \begin{pmatrix} a_{00} & a_{01} & a_{02} & \cdots \\ a_{10} & a_{11} & a_{12} & \cdots \\ a_{20} & a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where the (ij)th entry in the matrix denotes the coefficient of $z_1^i z_2^j$ in $f(z_1,z_2) = \sum_{i=0}^\infty \sum_{j=0}^\infty a_{ij} z_1^i z_2^j$. We shall write the matrix form instead of writing the series. In this notation,

$$S(\ ((a_{ij}))_{i,j=0}^\infty\ ) = \big(a_{(i-1)j} + a_{i(j-1)}\big) \text{ and } P(\ ((a_{ij}))_{i,j=0}^\infty\ ) = \big(a_{(i-1)(j-1)}\big) \tag{5.1}$$

with the convention that a_{ij} is zero if either i or j is negative.

Lemma 15. *The adjoints of the operators S and P are as follows:*

$$S^*\begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & a_{11} & a_{12} & \dots \\ a_{20} & a_{21} & a_{22} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} a_{10}+a_{01} & a_{11}+a_{02} & a_{12}+a_{03} & \dots \\ a_{20}+a_{11} & a_{21}+a_{12} & a_{22}+a_{13} & \dots \\ a_{30}+a_{21} & a_{31}+a_{22} & a_{32}+a_{23} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$P^*\begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & a_{11} & a_{12} & \dots \\ a_{20} & a_{21} & a_{22} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & a_{22} & a_{23} & \dots \\ a_{31} & a_{32} & a_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Proof. This is a matter of straightforward inner product computation. □

Lemma 16. *The defect space of P^* in matrix form is*

$$\mathcal{D}_{P^*} = \left\{ \begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & 0 & 0 & \dots \\ a_{20} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} : |a_{00}|^2 + \sum_{j=1}^\infty |a_{0j}|^2 + \sum_{j=1}^\infty |a_{j0}|^2 < \infty \right\}.$$

The defect space in the function form is $\overline{\text{span}}\{1, z_1^i, z_2^j : i,j \geq 1\}$. The defect operator for P^* is

$$D_{P^*}\begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & a_{11} & a_{12} & \dots \\ a_{20} & a_{21} & a_{22} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & 0 & 0 & \dots \\ a_{20} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Proof. Since P is an isometry, D_{P^*} is a projection onto $\text{Range}(P)^\perp = H^2(\mathbb{D}^2) \ominus \text{Range}(P)$. The rest follows from the formula for P in (5.1). □

Definition 17. Define $B : \mathcal{D}_{P^*} \to \mathcal{D}_{P^*}$ by

$$B \begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & 0 & 0 & \dots \\ a_{20} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} a_{10}+a_{01} & a_{02} & a_{03} & \dots \\ a_{20} & 0 & 0 & \dots \\ a_{30} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{5.2}$$

for all a_{j0}, $a_{0j} \in \mathbb{C}$, $j = 0, 1, 2, \dots$ with $|a_{00}|^2 + \sum_{j=1}^{\infty} |a_{0j}|^2 + \sum_{j=1}^{\infty} |a_{j0}|^2 < \infty$.

Lemma 18. *The operator B as defined in Definition 17 is the fundamental operator of (S^*, P^*).*

Proof. To show that B is the fundamental operator of (S^*, P^*), we shall show that B satisfies the fundamental equation $S^* - SP^* = D_{P^*} B D_{P^*}$. Using Lemma 15, we get

$$\begin{aligned}
&(S^* - SP^*) \begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & a_{11} & a_{12} & \dots \\ a_{20} & a_{21} & a_{22} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\
&= S^* \begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & a_{11} & a_{12} & \dots \\ a_{20} & a_{21} & a_{22} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} - S \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & a_{22} & a_{23} & \dots \\ a_{31} & a_{32} & a_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\
&= \begin{pmatrix} a_{10}+a_{01} & a_{11}+a_{02} & a_{12}+a_{03} & \dots \\ a_{20}+a_{11} & a_{21}+a_{12} & a_{22}+a_{13} & \dots \\ a_{30}+a_{21} & a_{31}+a_{22} & a_{32}+a_{23} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\
&\quad - \begin{pmatrix} 0 & a_{11} & a_{12} & a_{13} & \dots \\ a_{11} & a_{21}+a_{12} & a_{22}+a_{13} & a_{23}+a_{14} & \dots \\ a_{21} & a_{31}+a_{22} & a_{32}+a_{23} & a_{33}+a_{24} & \dots \\ a_{31} & a_{41}+a_{32} & a_{42}+a_{33} & a_{43}+a_{34} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\
&= \begin{pmatrix} a_{10}+a_{01} & a_{02} & a_{03} & \dots \\ a_{20} & 0 & 0 & \dots \\ a_{30} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.
\end{aligned}$$

Using Lemma 16 and Definition 17, we get

$$D_{P^*}BD_{P^*}\begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & a_{11} & a_{12} & \dots \\ a_{20} & a_{21} & a_{22} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = D_{P^*}B\begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & 0 & 0 & \dots \\ a_{20} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$
$$= D_{P^*}\begin{pmatrix} a_{10}+a_{01} & a_{02} & a_{03} & \dots \\ a_{20} & 0 & 0 & \dots \\ a_{30} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} a_{10}+a_{01} & a_{02} & a_{03} & \dots \\ a_{20} & 0 & 0 & \dots \\ a_{30} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Hence the proof is complete. □

Now we shall consider two subspaces of the Hilbert space $H^2(\mathbb{D}^2)$. The first one consists of all symmetric functions in $H^2(\mathbb{D}^2)$, i.e.,

$$H_+ = \{f \in H^2(\mathbb{D}^2) : f(z_1, z_2) = f(z_2, z_1)\},$$

and the second one consists of all antisymmetric functions in $H^2(\mathbb{D}^2)$, i.e.,

$$H_- = \{f \in H^2(\mathbb{D}^2) : f(z_1, z_2) = -f(z_2, z_1)\}.$$

It can be checked that $H^2(\mathbb{D}^2) = H_+ \oplus H_-$. Since both H_+ and H_- are invariant under the pair $(M_{z_1+z_2}, M_{z_1z_2})$, the spaces H_+ and H_- are reducing for $(M_{z_1+z_2}, M_{z_1z_2})$. It can be easily checked from the definition of a Γ-contraction that a restriction of a Γ-contraction to an invariant subspace is again a Γ-contraction. So $(M_{z_1+z_2}, M_{z_1z_2})|_{H_+}$ and $(M_{z_1+z_2}, M_{z_1z_2})|_{H_-}$ are Γ-contractions. Since restriction of an isometry to an invariant subspace is again an isometry, $M_{z_1z_2}|_{H_+}$ and $M_{z_1z_2}|_{H_-}$ are isometries. Hence by [4, Thm. 2.14(2)], the pairs $(M_{z_1+z_2}, M_{z_1z_2})|_{H_+}$ and $(M_{z_1+z_2}, M_{z_1z_2})|_{H_-}$ are Γ-isometries. For brevity, we shall use the notation (S_+, P_+) and (S_-, P_-) for the pairs $(M_{z_1+z_2}, M_{z_1z_2})|_{H_+}$ and $(M_{z_1+z_2}, M_{z_1z_2})|_{H_-}$ respectively. We shall find their fundamental operators.

§5.2. Symmetric case

Every element $f \in H_+$ has the form $f(z_1, z_2) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij} z_1^i z_2^j$, where $a_{ij} \in \mathbb{C}$ and $a_{ij} = a_{ji}$ for all $i, j \geq 0$. So we can write f in the matrix form

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{01} & a_{11} & a_{12} & \dots \\ a_{02} & a_{12} & a_{22} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In what follows, we shall exhibit the fundamental operator of the Γ-isometry (S_+, P_+). The results are collected and stated in two lemmas without proof because the proofs are similar to what we did above.

Lemma 19. *The adjoints of S_+ and P_+ are*

$$S_+^* \begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{01} & a_{11} & a_{12} & \dots \\ a_{02} & a_{12} & a_{22} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 2a_{01} & a_{11}+a_{02} & a_{12}+a_{03} & \dots \\ a_{11}+a_{02} & 2a_{12} & a_{22}+a_{13} & \dots \\ a_{12}+a_{03} & a_{22}+a_{13} & 2a_{23} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$P_+^* \begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{01} & a_{11} & a_{12} & \dots \\ a_{02} & a_{12} & a_{22} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots \\ a_{12} & a_{22} & a_{23} & \dots \\ a_{13} & a_{23} & a_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The defect space of P_+^* in matrix form is

$$\mathcal{D}_{P_+^*} = \left\{ \begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{01} & 0 & 0 & \dots \\ a_{02} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} : a_{0j} \in \mathbb{C}, j \geq 0 \text{ with } |a_{00}|^2 + 2\sum_{j=1}^{\infty} |a_{0j}|^2 < \infty \right\}.$$

The defect space in function form is $\overline{\text{span}}\{z_1^i + z_2^i : i \geq 0\}$. The defect operator is

$$D_{P_+^*} \begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{01} & a_{11} & a_{12} & \dots \\ a_{02} & a_{12} & a_{22} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{01} & 0 & 0 & \dots \\ a_{02} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Definition 20. Define $B_+ : \mathcal{D}_{P_+^*} \to \mathcal{D}_{P_+^*}$ by

$$B_+ \begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{01} & 0 & 0 & \dots \\ a_{02} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 2a_{01} & a_{02} & a_{03} & \dots \\ a_{02} & 0 & 0 & \dots \\ a_{03} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{5.3}$$

for all $a_{0j} \in \mathbb{C}$, $j \geq 0$ with $|a_{00}|^2 + 2\sum_{j=1}^{\infty} |a_{0j}|^2 < \infty$.

Lemma 21. *The operator B_+ defined on $\mathcal{D}_{P_+^*}$ is the fundamental operator of (S_+^*, P_+^*).*

§5.3. Antisymmetric case

Every element $f \in H_-$ has the form $f(z_1, z_2) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} z_1^i z_2^j$, where $a_{ij} \in \mathbb{C}$ and $a_{ij} = -a_{ji}$ for all $i, j \geq 0$. So we can write f in the matrix form

$$\begin{pmatrix} 0 & a_{01} & a_{02} & \dots \\ -a_{01} & 0 & a_{12} & \dots \\ -a_{02} & -a_{12} & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Lemma 22. *The adjoints of S_- and P_- are*

$$S_-^* \begin{pmatrix} 0 & a_{01} & a_{02} & \dots \\ -a_{01} & 0 & a_{12} & \dots \\ -a_{02} & -a_{12} & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 0 & a_{02} & a_{12} + a_{03} & \dots \\ -a_{02} & 0 & a_{13} & \dots \\ -a_{12} - a_{03} & -a_{13} & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$P_-^* \begin{pmatrix} 0 & a_{01} & a_{02} & \dots \\ -a_{01} & 0 & a_{12} & \dots \\ -a_{02} & -a_{12} & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 0 & a_{12} & a_{13} & \dots \\ -a_{12} & 0 & a_{23} & \dots \\ -a_{13} & -a_{23} & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The defect space of P_-^* in matrix form is

$$\mathcal{D}_{P_-^*} = \left\{ \begin{pmatrix} 0 & a_{01} & a_{02} & \dots \\ -a_{01} & 0 & 0 & \dots \\ -a_{02} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} : a_{0j} \in \mathbb{C}, j \geq 1 \text{ with } 2\sum_{j=1}^{\infty} |a_{0j}|^2 < \infty \right\}.$$

The defect space in function form is $\overline{\text{span}}\{z_1^i - z_2^i : i \geq 1\}$ and the defect operator is

$$D_{P_-^*} \begin{pmatrix} 0 & a_{01} & a_{02} & \dots \\ -a_{01} & 0 & a_{12} & \dots \\ -a_{02} & -a_{12} & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 0 & a_{01} & a_{02} & \dots \\ -a_{01} & 0 & 0 & \dots \\ -a_{02} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Definition 23. Define $B_- : \mathcal{D}_{P_-^*} \to \mathcal{D}_{P_-^*}$ by

$$B_- \begin{pmatrix} 0 & a_{01} & a_{02} & \dots \\ -a_{01} & 0 & 0 & \dots \\ -a_{02} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 0 & a_{02} & a_{03} & \dots \\ -a_{02} & 0 & 0 & \dots \\ -a_{03} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{5.4}$$

for all $a_{0j} \in \mathbb{C}$, $j \geq 1$ with $2\sum_{j=1}^{\infty} |a_{0j}|^2 < \infty$.

Lemma 24. *B_- is the fundamental operator of (S_-^*, P_-^*).*

§5.4. Explicit unitary equivalence

The three spaces $H^2(\mathbb{D}^2)$, H_+ and H_- described above provide us with examples of Γ-isometries. The respective operator pairs (S, P), (S_+, P_+) and (S_-, P_-) are pure Γ-isometries. Agler and Young in [2, Thm. 3.2] proved that any pure Γ-isometry is unitarily equivalent to (M_φ, M_z) on $H^2_{\mathcal{E}}(\mathbb{D})$ for some Hilbert space $\mathcal{E}$. Moreover, φ is linear. It was shown later in [5, Thm. 3.1] that $\mathcal{E}$ can be taken to be $\mathcal{D}_{P^*}$ and $\varphi(z) = B^* + Bz$, where $B \in \mathcal{B}(\mathcal{D}_{P^*})$ is the fundamental operator of the Γ-coisometry (S^*, P^*). In the final theorem of this paper, we explicitly find the unitary operators that implement unitary equivalence for the pure Γ-isometries (S, P), (S_+, P_+) and (S_-, P_-).

Theorem 25. *The three unitary operators are described separately below.*

(a) The unitary operator $U : H^2(\mathbb{D}^2) \to H^2_{\mathcal{D}_{P^*}}(\mathbb{D})$ that satisfies $U^* M_{B^*+zB} U = S$ and $U^* M_z U = P$ is $Uf(z) = D_{P^*}(I - zP^*)^{-1} f$.

(b) The unitary operator $U_+ : H_+ \to H^2_{\mathcal{D}_{P_+^*}}(\mathbb{D})$ that satisfies

$$U_+^* M_{B_+^*+zB_+} U_+ = S_+ \quad \text{and} \quad U_+^* M_z U_+ = P_+$$

is simply the restriction of the U above to H_+.

(c) The unitary operator $U_- : H_- \to H^2_{\mathcal{D}_{P_-^*}}(\mathbb{D})$ that satisfies

$$U_-^* M_{B_-^*+zB_-} U_- = S_- \quad \text{and} \quad U_-^* M_z U_- = P_-$$

is the restriction of U to H_-.

Proof. (a) First note that the function $z \mapsto D_{P^*}(I - zP^*)^{-1} f$ is a holomorphic function on $\mathbb{D}$, for every $f \in H^2(\mathbb{D}^2)$. Its Taylor series expansion is

$$\begin{aligned} &D_{P^*}(I - zP^*)^{-1} f \\ &\quad = D_{P^*}(I + zP^* + z^2 P^{*2} + \cdots) f \end{aligned}$$

$$= D_{P^*}f + zD_{P^*}P^*f + z^2 D_{P^*}{P^*}^2 f + \cdots$$

(5.5)
$$= \begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & 0 & 0 & \dots \\ a_{20} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} + z \begin{pmatrix} a_{11} & a_{12} & a_{13} \dots \\ a_{21} & 0 & 0 & \dots \\ a_{31} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} + z^2 \begin{pmatrix} a_{22} & a_{23} & a_{24} \dots \\ a_{32} & 0 & 0 & \dots \\ a_{42} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} + \cdots .$$

To see that U is an isometry, we do a norm computation:

$$\begin{aligned} \|Uf\|^2_{H^2_{\mathcal{D}_{P^*}}(\mathbb{D})} &= \|D_{P^*}f\|^2_{\mathcal{D}_{P^*}} + \|D_{P^*}P^*f\|^2_{\mathcal{D}_{P^*}} + \|D_{P^*}{P^*}^2 f\|^2_{\mathcal{D}_{P^*}} + \cdots \\ &= \|f\|^2 - \lim_{n\to\infty} \|{P^*}^n f\|^2 = \|f\|^2_{H^2(\mathbb{D}^2)} \quad \text{[since } P \text{ is pure]}. \end{aligned}$$

From equation (5.5) it is easy to see that U is onto $H^2_{\mathcal{D}_{P^*}}(\mathbb{D})$. Therefore U is unitary.

We now show that $U^*M_zU = P$:

$$\begin{aligned} U^*M_zU \begin{pmatrix} a_{00} & a_{01} & a_{02} \dots \\ a_{10} & a_{11} & a_{12} \dots \\ a_{20} & a_{21} & a_{22} \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} &= U^* \left(z \begin{pmatrix} a_{00} & a_{01} & a_{02} \dots \\ a_{10} & 0 & 0 & \dots \\ a_{20} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} + z^2 \begin{pmatrix} a_{11} & a_{12} & a_{13} \dots \\ a_{21} & 0 & 0 & \dots \\ a_{31} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \right. \\ &\qquad \left. + z^3 \begin{pmatrix} a_{22} & a_{23} & a_{24} \dots \\ a_{32} & 0 & 0 & \dots \\ a_{42} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} + \cdots \right) \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & a_{00} & a_{01} & a_{02} \dots \\ 0 & a_{10} & a_{11} & a_{12} \dots \\ 0 & a_{20} & a_{21} & a_{22} \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = P \begin{pmatrix} a_{00} & a_{01} & a_{02} \dots \\ a_{10} & a_{11} & a_{12} \dots \\ a_{20} & a_{21} & a_{22} \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} . \end{aligned}$$

From the definition of B (Definition 17), one can easily find that for all a_{j0}, $a_{0j} \in \mathbb{C}, j = 0, 1, 2, \dots$ with $|a_{00}|^2 + \sum_{j=1}^{\infty} |a_{0j}|^2 + \sum_{j=1}^{\infty} |a_{j0}|^2 < \infty$,

(5.6)
$$B^* \begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & 0 & 0 & \dots \\ a_{20} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 0 & a_{00} & a_{01} & \dots \\ a_{00} & 0 & 0 & \dots \\ a_{10} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} .$$

To show that $U^* M_{B^*+zB} U = S$, we first calculate $M_{B^*+zB} U$. Now

$$
\begin{aligned}
M_{B^*+zB}U &\begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & a_{11} & a_{12} & \dots \\ a_{20} & a_{21} & a_{22} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\
&= M_{B^*+Bz} \left(\begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & 0 & 0 & \dots \\ a_{20} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} + z \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & 0 & 0 & \dots \\ a_{31} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \right. \\
&\quad \left. +z^2 \begin{pmatrix} a_{22} & a_{23} & a_{24} & \dots \\ a_{32} & 0 & 0 & \dots \\ a_{42} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} + \cdots \right) \\
&= M_{B^*} \left(\begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & 0 & 0 & \dots \\ a_{20} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} + z \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & 0 & 0 & \dots \\ a_{31} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \right. \\
&\quad \left. +z^2 \begin{pmatrix} a_{22} & a_{23} & a_{24} & \dots \\ a_{32} & 0 & 0 & \dots \\ a_{42} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} + \cdots \right) \\
&\quad + M_B \left(z \begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & 0 & 0 & \dots \\ a_{20} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} + z^2 \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & 0 & 0 & \dots \\ a_{31} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \right. \\
&\quad \left. +z^3 \begin{pmatrix} a_{22} & a_{23} & a_{24} & \dots \\ a_{32} & 0 & 0 & \dots \\ a_{42} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} + \cdots \right) \\
&= \left(\begin{pmatrix} 0 & a_{00} & a_{01} & \dots \\ a_{00} & 0 & 0 & \dots \\ a_{10} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} + z \begin{pmatrix} 0 & a_{11} & a_{12} & \dots \\ a_{11} & 0 & 0 & \dots \\ a_{21} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \right.
\end{aligned}
$$

$$+z^2\begin{pmatrix} 0 & a_{22} & a_{23} & \dots \\ a_{22} & 0 & 0 & \dots \\ a_{32} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} + \cdots \Bigg)$$

$$+ \left(z \begin{pmatrix} a_{10}+a_{01} & a_{02} & a_{03} & \dots \\ a_{20} & 0 & 0 & \dots \\ a_{30} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} + z^2 \begin{pmatrix} a_{21}+a_{12} & a_{13} & a_{14} & \dots \\ a_{31} & 0 & 0 & \dots \\ a_{41} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \right.$$

$$\left. +z^3 \begin{pmatrix} a_{32}+a_{23} & a_{24} & a_{25} & \dots \\ a_{42} & 0 & 0 & \dots \\ a_{52} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} + \cdots \right)$$

$$= \begin{pmatrix} 0 & a_{00} & a_{01} & \dots \\ a_{00} & 0 & 0 & \dots \\ a_{10} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} + z \begin{pmatrix} a_{10}+a_{01} & a_{11}+a_{02} & a_{12}+a_{03} & \dots \\ a_{20}+a_{11} & 0 & 0 & \dots \\ a_{30}+a_{21} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$+ z^2 \begin{pmatrix} a_{21}+a_{12} & a_{22}+a_{13} & a_{23}+a_{14} & \dots \\ a_{31}+a_{22} & 0 & 0 & \dots \\ a_{41}+a_{32} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} + \cdots .$$

Therefore

$$U^* M_{B^*+Bz} U \begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & a_{11} & a_{12} & \dots \\ a_{20} & a_{21} & a_{22} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$= U^* \left(\begin{pmatrix} 0 & a_{00} & a_{01} & \dots \\ a_{00} & 0 & 0 & \dots \\ a_{10} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} + z \begin{pmatrix} a_{10}+a_{01} & a_{11}+a_{02} & a_{12}+a_{03} & \dots \\ a_{20}+a_{11} & 0 & 0 & \dots \\ a_{30}+a_{21} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \right.$$

$$\left. + z^2 \begin{pmatrix} a_{21}+a_{12} & a_{22}+a_{13} & a_{23}+a_{14} & \dots \\ a_{31}+a_{22} & 0 & 0 & \dots \\ a_{41}+a_{32} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} + \cdots . \right)$$

(5.7)

$$= \begin{pmatrix} 0 & a_{00} & a_{01} & a_{02} & \cdots \\ a_{00} & a_{10}+a_{01} & a_{11}+a_{02} & a_{12}+a_{03} & \cdots \\ a_{10} & a_{20}+a_{11} & a_{21}+a_{12} & a_{22}+a_{13} & \cdots \\ a_{20} & a_{30}+a_{21} & a_{31}+a_{22} & a_{32}+a_{23} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = S \begin{pmatrix} a_{00} & a_{01} & a_{02} & \cdots \\ a_{10} & a_{11} & a_{12} & \cdots \\ a_{20} & a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Therefore $S = U^* M_{B^*+Bz} U$. Surjectivity of $U|_{H_+}$ and $U|_{H_-}$ can be easily checked. The rest of the argument is as above. □

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References

[1] J. Agler, Rational dilation on an annulus, Ann. of Math. **121** (1985), 537–563. Zbl 0609.47013 MR 0794373

[2] J. Agler and N. J. Young, A model theory for Γ-contractions, J. Operator Theory **49** (2003), 45–60. Zbl 1019.47013 MR 1978320

[3] T. Ando, On a pair of commutative contractions, Acta Sci. Math. (Szeged) **24** (1963), 88–90. Zbl 0116.32403 MR 0155193

[4] T. Bhattacharyya, S. Pal and S. Shyam Roy, Dilations of Γ-contractions by solving operator equations, Adv. Math. **230** (2012), 577–606. Zbl 1251.47010 MR 2914959

[5] T. Bhattacharyya and S. Pal, A functional model for pure Γ-contractions, J. Operator Theory **71** (2014), 327–339. Zbl 1324.47015 MR 3214641

[6] T. Bhattacharyya and H. Sau, Explicit and unique construction of tetrablock unitary dilation in a certain case, Complex Anal. Oper. Theory **10** (2016), 749–768. Zbl 1344.47006 MR 3480602

[7] H. K. Du and P. Jin, Perturbation of spectrum of 2 by 2 operator matrices, Proc. Amer. Math. Soc. **121** (1994), 761–766. Zbl 0814.47016 MR 1185266

[8] W. S. Li and D. Timotin, The central Ando dilation and related orthogonality properties, J. Funct. Anal. **154** (1998), 1–16. Zbl 0913.47008 MR 1616520

[9] S. Pal, From Stinespring dilation to Sz.-Nagy dilation on the symmetrized bidisc and operator models, New York J. Math. **20** (2014), 645–664. Zbl 1314.47016 MR 3262025

[10] S. Pal and O. Shalit, Spectral sets and distinguished varieties in the symmetrized bidisc, J. Funct. Anal. **266** (2014), 5779–5800. Zbl 1311.47008 MR 3182959

[11] J. J. Schäffer, On unitary dilations of contractions, Proc. Amer. Math. Soc. **6** (1955), 322. Zbl 0064.11602 MR 0068740

[12] B. Sz.-Nagy, C. Foias, H. Bercovici and L. Kérchy, *Harmonic analysis of operators on Hilbert space*, 2nd ed., Universitext, Springer, New York, 2010. Zbl 1234.47001 MR 2760647