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Non-Cocycle-Conjugate E_0-Semigroups on Factors

by

Oliver T. Margetts and R. Srinivasan

Abstract

We investigate E_0-semigroups on general factors that are not necessarily of type I or II_1. We show that several families on the hyperfinite II_∞ factor, which arise as tensor products, consist of mutually non-cocycle-conjugate E_0-semigroups. Using CCR representations associated with quasi-free states, we exhibit, for the first time, uncountable families consisting of mutually non-cocycle-conjugate E_0-semigroups on all type III_λ factors, for $\lambda \in (0, 1]$. They are not cocycle conjugate to the E_0-semigroups constructed using CAR representations.

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§1. Introduction

An E_0-semigroup is a semigroup of normal unital $*$-endomorphisms on a von Neumann algebra that is also σ-weakly continuous. They arise naturally in the study of open quantum systems, the theory of interactions, algebraic quantum field theory and quantum stochastic calculus. The study of E_0-semigroups lead to the study of interesting objects like product systems, super-product systems and C^*-semiflows as its associated invariants.

For E_0-semigroups on type I factors, the subject has grown rapidly since its inception in [PW]. We refer to the monograph [Arv] for an extensive treatment regarding the theory of E_0-semigroups on type I factors. Arveson showed that E_0-semigroups on type I factors are completely classified, up to an identification called

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O. T. Margetts: Department of Mathematics and Statistics, Fylde College, Lancaster University, Lancaster LA1 4YF, UK;
e-mail: `o.margetts@lancaster.ac.uk`
R. Srinivasan: Chennai Mathematical Institute, H1, SIPCOT IT Park, Kelambakkam, Siruseri 603103, India;
e-mail: `vasanth@cmi.ac.in`

cocycle conjugacy, by continuous tensor products of Hilbert spaces called *product systems*. This gives a rough division of E_0-semigroups into three types, namely I, II and III. The type I E_0-semigroups on type I factors are cocycle conjugate to the CCR flows ([Arv]), but there are uncountably many E_0-semigroups of types II and III ([PWIII, BhS, Ts1, IS_1, IS_2, Li]) on type I factors.

There has been relatively little progress regarding the study of E_0-semigroups on type II_1 factors since it was initiated in the 1988 paper [PW]. In [Ale], Alevras introduced an index using Powers' boundary representation ([PW]), and computed the index for several important cases. Still, this did not classify even the simplest examples of E_0-semigroups on the hyperfinite II_1 factor called Clifford flows, since it has yet been not proved that the Powers–Alevras index is a cocycle-conjugacy invariant. The problem of proving non-cocycle-conjugacy for Clifford flows was solved in [MaS], in an indirect way.

In [MaS], four new cocycle-conjugacy invariants for E_0-semigroups on II_1 factors, namely a coupling index, a dimension for the gauge group, a *super-product system* and a C^*-semiflow, were introduced and computed for standard examples. Using the C^*-semiflows and the boundary representation of Powers and Alevras, it was shown that the families of Clifford flows and even Clifford flows contain mutually non-cocycle-conjugate E_0-semigroups.

On the other hand, there has been almost no work done regarding E_0-semigroups on type II_∞ factors and type III factors. In this paper, for the first time we produce uncountable families containing mutually non-cocycle-conjugate E_0-semigroups on the hyperfinite type II_∞ factor and on all type III_λ factors for $\lambda \in (0,1]$.

This paper is structured as follows. In Section 2 we fix our notation and give the basic definitions of E_0-semigroups and the notions of cocycle conjugacy, units and the gauge group. We associate a dual E_0-semigroup to any E_0-semigroup "acting standardly" and show that it is well defined up to cocycle conjugacy. Using this we define the notion of multiunits. Finally we recall the definitions of important families of E_0-semigroups, namely CCR flows, generalized CCR flows, Toeplitz CAR flows on type I factors, and Clifford flows, even Clifford flows on the hyperfinite II_1 factor and some important results regarding these families.

In Section 3, we associate a super-product system to E_0-semigroups on general factors, which was initially defined for E_0-semigroups on type II_1 factors, and show that this association is invariant under cocycle conjugacy. Then we define the coupling index and clarify its relationship to the Powers–Arveson index for E_0-semigroups on type I factors. We also prove that the super-product system of tensor products of E_0-semigroups is the tensor product of the super-product systems of the corresponding E_0-semigroups.

In Section 4, we produce E_0-semigroups on type II_∞ factors by tensoring E_0-semigroups on type I factors with E_0-semigroups on type II_1 factors, and study the problem of non-cocycle-conjugacy. We prove that the tensor product of a CCR flow of index m with a Clifford flow (or with an even Clifford flow) of index n is cocycle conjugate to another such tensor product of a CCR flow of index p with a Clifford flow (or with an even Clifford flow) of index q if and only if $(m,n) = (p,q)$. Then we produce uncountable families of non-cocycle-conjugate E_0-semigroups on type II_∞ factors by fixing either a Clifford flow or an even Clifford flow on hyperfinite II_1 factors and tensoring with many families containing mutually non-cocycle-conjugate type III E_0-semigroups on type I_∞ factors.

In Section 5, we analyze E_0-semigroups on type III factors, constructed using CCR representations associated with a quasi-free state corresponding to a complex linear positive operator $A \geq 1$, such that $A - 1$ is injective. Since they are given by a Toeplitz operator, we call them Toeplitz CCR flows on type III factors. We show that these Toeplitz CCR flows are "equimodular" (as defined in [BISS]) with respect to the invariant vacuum state if and only if the quasi-free state is given by an operator of the form $A = 1 \otimes R$ on $L^2(0,\infty) \otimes \mathsf{k}$. In this simplest case, we call these Toeplitz CCR flows just CCR flows on type III factors given by R. We prove that these CCR flows are canonically extendable (which was defined as extendable in [BISS]), and they canonically extend to CCR flows (on type I factors) of index equal to twice the rank of R. From this it follows that CCR flows associated with operators of the form $A = 1 \otimes R$, with R having different ranks, are not cocycle conjugate.

In Section 6, we further analyze the CCR flows given by positive operators of fixed rank. We prove that two such CCR flows are cocycle conjugate if and only if they are unitarily equivalent. This, in consequence, produces uncountably many mutually non-cocycle-conjugate E_0-semigroups on all type III_λ factors for $\lambda \in (0,1]$.

E_0-semigroups can also be constructed on type III factors using CAR representations. But they are not canonically extendable. Since canonical extendability is a property invariant under cocycle conjugacy, it follows that none of the CAR flows are cocycle conjugate to the canonically extendable CCR flows on type III factors.

§2. Preliminaries

Notation. The set of all natural numbers is denoted by $\mathbb{N}$, and we set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. For any real Hilbert space G, we denote the complexification of G by $G^{\mathbb{C}}$. Throughout this paper, we use the symbol k to denote a separable real

Hilbert space with $\dim(\mathsf{k}) \in \overline{\mathbb{N}}$, except in Sections 5 and 6. In Sections 5 and 6, k is a complex Hilbert space, as mentioned there. For any measurable subset $S \subseteq \mathbb{R}$, $L^2(S, \mathsf{k})$ is the Hilbert space of square integrable functions on S taking values in k.

All our Hilbert spaces are separable. For a complex Hilbert space H, by $\overline{H}$ we denote the dual space antiisomorphic to H. The inner product is always conjugate linear in the first variable and linear in the second variable. For $S \subset H$ a subset of vectors, we shall write $[S]$ for the norm-closed subspace of H spanned by S.

All our von Neumann algebras act on separable Hilbert spaces. For von Neumann algebras M and N, $\mathrm{M} \vee \mathrm{N}$ denotes the von Neumann algebra generated by M and N. For von Neumann algebras $\mathrm{M}_1, \mathrm{M}_2, \mathrm{N}_1, \mathrm{N}_2$ the following relations hold:

$$(\mathrm{M}_1 \otimes \mathrm{N}_1) \vee (\mathrm{M}_2 \otimes \mathrm{N}_2) = (\mathrm{M}_1 \vee \mathrm{M}_2) \otimes (\mathrm{N}_1 \vee \mathrm{N}_2)\,, \tag{1}$$

$$(\mathrm{M}_1 \otimes \mathrm{N}_1) \cap (\mathrm{M}_2 \otimes \mathrm{N}_2) = (\mathrm{M}_1 \cap \mathrm{M}_2) \otimes (\mathrm{N}_1 \cap \mathrm{N}_2)\,. \tag{2}$$

§2.1. E_0-semigroups and dual E_0-semigroups

Definition 2.1. An E_0-semigroup on a von Neumann algebra M is a semigroup $\{\alpha_t : t \geq 0\}$ of normal, unital *-endomorphisms of M satisfying

(i) $\alpha_0 = \mathrm{id}$;
(ii) $\alpha_t(\mathrm{M}) \neq \mathrm{M}$ for all $t > 0$;
(iii) $t \mapsto \rho(\alpha_t(x))$ is continuous for all $x \in \mathrm{M}$, $\rho \in \mathrm{M}_*$.

Definition 2.2. A cocycle for an E_0-semigroup α on M is a strongly continuous family of unitaries $\{U_t : t \geq 0\} \subseteq \mathrm{M}$ satisfying $U_s\alpha_s(U_t) = U_{s+t}$ for all s, $t \geq 0$.

For a cocycle $\{U_t : t \geq 0\}$, we automatically have $U_0 = 1$. Furthermore, the family of endomorphisms $\alpha^U_t(x) := U_t\alpha_t(x)U^*_t$ defines an E_0-semigroup. This leads to the following equivalence relations on E_0-semigroups.

Definition 2.3. Let α and β be E_0-semigroups on von Neumann algebras M and N. Then

(i) α and β are *conjugate* if there exists a *-isomorphism $\theta : \mathrm{M} \to \mathrm{N}$ such that $\beta_t = \theta \circ \alpha_t \circ \theta^{-1}$ for all $t \geq 0$;
(ii) α and β are *cocycle conjugate* if there exists a cocycle $\{U_t : t \geq 0\}$ for α such that β is conjugate to α^U.

Two E_0-semigroups α and β, acting on $\mathrm{M} \subseteq B(H_1)$ and $\mathrm{N} \subseteq B(H_2)$ respectively, are said to be spatially conjugate if there exists a unitary $U : H_1 \mapsto H_2$ satisfying

(i) $U\mathrm{M}U^* = \mathrm{N}$;

(ii) $\beta_t(x) = U\alpha_t(U^*xU)U^*$ for all $t \geq 0$, $x \in \mathrm{N}$.

We say that a von Neumann algebra M is in standard form if $\mathrm{M} \subseteq B(H)$ has a cyclic and separating vector $\Omega \in H$. By picking a faithful normal state, it is easy to verify that every E_0-semigroup α is conjugate to an E_0-semigroup β on a von Neumann algebra N in standard form, which is uniquely determined up to spatial conjugacy. So we assume, without loss of generality, that all our E_0-semigroups are acting on a von Neumann algebra in standard form.

Theorem 2.4 ([Ar_3]). *Let* M *be a von Neumann algebra with cyclic and separating vectors* Ω_1 *and* Ω_2. *If* J_1 *and* J_2 *are the corresponding modular conjugations, then the *-automorphism* $\mathrm{Ad}_{J_1J_2}|\mathrm{M} \to \mathrm{M}$ *is inner.*

Let α be an E_0-semigroup on a factor M with cyclic and separating vector Ω and let J_Ω be the modular conjugation associated to the vector Ω by Tomita–Takesaki theory. Define a complementary E_0-semigroup on M′ by setting

$$\alpha'_t(x') = J_\Omega\alpha_t(J_\Omega x' J_\Omega)J_\Omega \quad (x' \in \mathrm{M}').$$

The following proposition asserts that the cocycle-conjugacy class of α' does not depend on Ω. We call α', which is determined uniquely up to cocycle conjugacy, the *dual E_0-semigroup.*

Proposition 2.5. *Let* M *and* N *be von Neumann algebras acting standardly with respective cyclic and separating vectors* $\Omega_1 \in H_1$ *and* $\Omega_2 \in H_2$. *If the E_0-semigroups* α *on* M *and* β *on* N *are cocycle conjugate, then* α' *and* β'*, defined with respect to* Ω_1 *and* Ω_2*, are cocycle conjugate. Moreover, if* α *and* β *are conjugate, then* α' *and* β' *are spatially conjugate and the implementing unitary can be chosen so that it also intertwines* α *and* β.

Proof. If α is conjugate to β, let $U : H_1 \to H_2$ be the unitary implementing the conjugacy and $\Omega_\theta \in H_1$ be a cyclic separating vector satisfying $Ux\Omega_\theta = \theta(x)\Omega_2$ for all $x \in \mathrm{M}$. It is clear that $UJ_{\Omega_\theta} = J_{\Omega_2}U$ and hence

$$\beta'_t(x) = J_{\Omega_2}U\alpha_t(U^*J_{\Omega_2}xJ_{\Omega_2}U)U^*J_{\Omega_2} = UJ_{\Omega_\theta}\alpha_t(J_{\Omega_\theta}U^*xUJ_{\Omega_\theta})J_{\Omega_\theta}U^*$$

for all $x \in \mathrm{N}'$. It follows from Theorem 2.4 that the *-isomorphism $\mathrm{M}' \to \mathrm{M}'$, $x \mapsto J_{\Omega_\theta}J_{\Omega_1}xJ_{\Omega_1}J_{\Omega_\theta}$ is inner, so let $V \in \mathrm{M}'$ be the implementing unitary. Then the right-hand side becomes

$$UVJ_{\Omega_1}\alpha_t(J_{\Omega_1}V^*U^*xUVJ_{\Omega_1})J_{\Omega_1}V^*U^* = UV\alpha'_t((UV)^*xUV)(UV)^*.$$

So α' and β' are spatially conjugate and, since $V \in \mathrm{M}'$, for all $x \in \mathrm{N}$ we also have

$$UV\alpha_t((UV)^*xUV)(UV)^* = U\alpha_t(U^*xU)U^* = \beta_t(x).$$

For cocycle conjugacy we may assume $\mathrm{M} = \mathrm{N}$, $\Omega = \Omega_1 = \Omega_2$ and $\{U_t : t \geq 0\}$ is an α-cocycle such that $\beta_t(\cdot) = U_t\alpha_t(\cdot)U_t^*$. For $t \geq 0$, let $V_t = J_\Omega U_t J_\Omega$; then $\{V_t : t \geq 0\}$ forms an α'-cocycle and $\beta_t'(m') = V_t\alpha_t'(m')V_t^*$ for all $m' \in \mathrm{M}'$. □

Definition 2.6. Let α be an E_0-semigroup acting standardly on $\mathrm{M} \subseteq B(H)$. A unit for α is a strongly continuous semigroup $T = \{T_t : t \geq 0\}$ of operators in $B(H)$ such that $T_0 = 1$ and $T_t x = \alpha_t(x)T_t$ for all $t \geq 0$, $x \in \mathrm{M}$. Denote the collection of units by $\mathcal{U}_\alpha$.

Definition 2.7. Let α be an E_0-semigroup on the von Neumann algebra M acting standardly on H. A μnit or multiunit for the E_0-semigroup α is a strongly continuous semigroup of bounded operators $(T_t)_{t\geq 0}$ in $B(H)$ satisfying

$$T_t x = \begin{cases} \alpha_t(x)T_t & \text{if } x \in \mathrm{M}, \\ \alpha_t'(x)T_t & \text{if } x \in \mathrm{M}', \end{cases}$$

together with $T_0 = 1$. Denote the collection of multiunits for α by $\mathcal{U}_{\alpha,\alpha'}$. We say that α is multispatial if it admits a multiunit.

When $\langle \alpha_t(m)\Omega, \Omega\rangle = \langle m\Omega, \Omega\rangle$ for all $t \geq 0$, $m \in \mathrm{M}$, there exists a unit S_t, which is the semigroup of isometries determined by $S_t x\Omega := \alpha_t(x)\Omega$. We call $\{S_t : t \geq 0\}$ the *canonical* Ω*-unit* associated to α.

An E_0-semigroup α on a II_1 factor M is automatically multispatial. Indeed, the canonical unit with respect to the trace is a multiunit for α. On the other hand, a type III E_0-semigroup on a type I factor is not multispatial, which follows from Example 2.9. In Section 4, we provide examples of E_0-semigroups on type II_∞ factors that are not multispatial. The following proposition gives a large number of multispatial examples, of which E_0-semigroups on II_1 factors are a special case.

Proposition 2.8. *Let α be an E_0-semigroup acting standardly on a factor* M *with cyclic and separating vector Ω, and φ be the faithful normal state associated with Ω. Then the following are equivalent:*

(i) *φ is an invariant state for (M, α), and the corresponding canonical unit $(S_t)_{t\geq 0}$ is a multiunit.*
(ii) *φ is an invariant state for (M, α), and for all $t \geq 0$, the canonical unit $(S_t)_{t\geq 0}$ and modular conjugation J, with respect to Ω, satisfy $S_t = JS_tJ$.*
(iii) *For all $t \geq 0$, $s \in \mathbb{R}$ the modular group $(\sigma_s)_{s\in\mathbb{R}}$ satisfies $\alpha_t = \sigma_{-s}^\Omega \circ \alpha_t \circ \sigma_s^\Omega$.*

Proof.

(i)⇒(ii). For all $m' \in \mathrm{M}'$, $t \geq 0$, we have

$$S_t m'\Omega = \alpha'_t(m')\Omega = J\alpha_t(Jm'J)\Omega = JS_tJm'\Omega,$$

so $S_t = JS_tJ$ for all $t \geq 0$.

(ii)⇒(i). For all $m' \in \mathrm{M}'$, $t \geq 0$, we have

$$S_t m' = JS_tJm'J^2 = J\alpha_t(Jm'J)S_tJ = \alpha'_t(m')S_t.$$

(ii)⇒(iii). For all $t \geq 0$, $m \in \mathrm{M}$,

$$\Delta^{1/2}S_t m\Omega = \Delta^{1/2}\alpha_t(m)\Omega = J\alpha_t(m^*)\Omega = JS_tm^*\Omega = S_tJm^*\Omega = S_t\Delta^{1/2}m\Omega,$$

so $\Delta^{1/2}S_t \supseteq S_t\Delta^{1/2}$. Thus $\sigma_s^\Omega \circ \alpha_t(m)\Omega = \Delta^{is}S_tm\Omega = S_t\Delta^{is}m\Omega = \alpha_t \circ \sigma_s^\Omega(m)\Omega$. (See, e.g., [Cnw, Section X].)

(iii)⇒(ii). From the commutation relation we see that, for all $t \geq 0$, the state $\varphi \circ \alpha_t$ satisfies the KMS condition for σ^Ω. Thus, by uniqueness, $\varphi \circ \alpha_t = \varphi$ for all $t \geq 0$. It also follows from the commutation relation that $\Delta^{is}S_t = S_t\Delta^{is}$ for all $s \in \mathbb{R}$; thus we can infer that $\Delta^{1/2}S_t \supseteq \Delta^{1/2}S_t$ and, by the *-preserving property of α, $JS_t = S_tJ$ for all $t \geq 0$.

□

Example 2.9. Let $\mathrm{M} = B(H)$ and $\overline{H}$ be the dual space of H, with an antiisomorphism $\xi \mapsto \overline{\xi}$ from $H \mapsto \overline{H}$. Consider the standard representation $\pi : \mathrm{M} \to B(H \otimes \overline{H})$, defined by linear extension of $\pi(X)(\xi \otimes \overline{\eta}) = X\xi \otimes \overline{\eta}$, with cyclic and separating vector $\Omega = \sum_{n=1}^{\infty} \frac{1}{n} e_n \otimes \overline{e_n}$, where $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis for H. Then the corresponding modular conjugation is given by

$$J\xi \otimes \overline{\eta} = \eta \otimes \overline{\xi}.$$

If X is an operator on H then let $\overline{X}$ be the operator on the dual space defined by $\overline{X}\overline{\eta} = \overline{X\eta}$, so $J(X \otimes 1)J = 1 \otimes \overline{X}$. Let α be an E$_0$-semigroup on M and denote by β the conjugate semigroup $\pi \circ \alpha \circ \pi^{-1}$ on $\pi(\mathrm{M})$. Then we have

$$\beta'_t(1 \otimes \overline{X}) = J(\alpha_t(X) \otimes 1)J = 1 \otimes \overline{\alpha_t(X)}.$$

Thus the dual E$_0$-semigroup β' is conjugate to an E$_0$-semigroup $\overline{\alpha}$ on $B(\overline{H})$ given by $\overline{\alpha}_t(\overline{X}) = \overline{\alpha_t(X)}$.

Definition 2.10. A gauge cocycle for α is a cocycle $\{U_t : t \geq 0\}$ that satisfies the locality condition $U_t \in \alpha_t(\mathrm{M})' \cap \mathrm{M}$ for all $t \geq 0$. Under the multiplication $(UV)_t := U_tV_t$, the collection of all gauge cocycles forms a group, denoted by

$G(\alpha)$, called the gauge group of α. The group $G(\alpha)$ is an invariant of α under cocycle conjugacy.

Lemma 2.11. *Let α be an E_0-semigroup on a factor $\mathrm{M} \subseteq B(H)$ in standard form. Then there exists a family of isometries $\{U_i(t) : i \in \mathcal{I}\} \subseteq B(H)$ satisfying $\alpha_t(x) = \sum_{i\in\mathcal{I}} U_i(t)xU_i(t)^*$ for all $x \in \mathrm{M}$, in σ-weak topology. When M is a factor of type II_∞ or type III, the indexing set $\mathcal{I}$ is a singleton, and otherwise $\mathcal{I} = \mathbb{N}$.*

Proof. Refer to [Arv, Proposition 2.1.1] when M is a type I factor and to [Ale, Proposition 3.2] when M is type II_1 factor. For any factor M, H can be considered a left module over M with respect to the identity action and also with $x \cdot \xi = \alpha_t(x)\xi$ for $\xi \in H$, and the $\alpha_t(\mathrm{M})$-dimension of $L^2(\mathrm{M})$ is ∞. But when M is of type II_∞ or type III, the M-dimension of the standard module is also infinite, and the existence of a unitary $U_t \in B(H)$ satisfying $\alpha_t(x) = U_t x U_t^*$ is guaranteed. □

Proposition 2.12. *Let α and β be two E_0-semigroups on factors M_1 and M_2 respectively. Then there exists a unique E_0-semigroup $\alpha \otimes \beta$ on $\mathrm{M}_1 \otimes \mathrm{M}_2$ satisfying*

$$(\alpha_t \otimes \beta_t)(m_1 \otimes m_2) = \alpha_t(m_1) \otimes \beta_t(m_2) \quad \forall\, m_1 \in \mathrm{M}_1,\ \ m_2 \in \mathrm{M}_2,\ \ t \geq 0.$$

Proof. For each $t \geq 0$, let $\{U_i(t) : i \in \mathcal{I}\}$ and $\{V_j(t) : j \in \mathcal{J}\}$ be the isometries in Lemma 2.11, implementing α and β respectively. Now the endomorphism $\alpha_t \otimes \beta_t$ is implemented by the family of isometries $\{U_i(t) \otimes V_j(t) : i \in \mathcal{I}, j \in \mathcal{J}\}$. □

§2.2. Generalized CCR flows

For a complex separable Hilbert space K, let $\Gamma_s(K) := \bigoplus_{n=0}^{\infty} K^{\vee n}$ be the symmetric Fock space over K, i.e., the sum of symmetric tensor powers of K, with $K^{\vee 0} = \mathbb{C}$. Define the exponential vectors $\varepsilon(u) := \oplus_{n=0}^{\infty}(u^{\otimes n}/\sqrt{n!})$ for each $u \in K$, and the vacuum vector is $\varepsilon(0)$. The exponential vectors are linearly independent and total in $\Gamma_s(K)$. The well-known isomorphism between $\Gamma_s(K_1) \otimes \Gamma_s(K_2) \to \Gamma_s(K_1 \oplus K_2)$ is given by the extension of $\varepsilon(u) \otimes \varepsilon(v) \mapsto \varepsilon(u + v)$.

Define the Weyl operator by $W_0(u)\varepsilon(v) := e^{-\|u\|^2/2 - \langle u,v\rangle}\varepsilon(u+v)$ for $u, v \in K$, which extends to a unitary operator on $\Gamma_s(K)$. Then $\{W_0(u) : u \in K\}$ satisfies the well-known Weyl commutation relations

$$W_0(u)W_0(v) = e^{-i\,\mathrm{Im}\langle u,v\rangle} W_0(u + v) \quad \forall\, u, v \in K.$$

For a unitary operator U between K_1 and K_2, define the second quantization $\Gamma(U)$ by $\Gamma(U)(\varepsilon(u)) = \varepsilon(Uu)$ for $u \in K$, which extends to a unitary operator between $\Gamma_s(K_1)$ and $\Gamma_s(K_2)$. We can also define the second quantization for antiunitaries in the same way, but extending antilinearly.

Let k be a real Hilbert space. Let $K = L^2((0,\infty),\mathsf{k}^{\mathbb{C}})$ denote square integrable functions taking values in $\mathsf{k}^{\mathbb{C}}$. Throughout this paper we denote by $(T_t)_{t\geq 0}$ the right shift semigroup on $L^2((0,\infty),\mathsf{k}^{\mathbb{C}})$ (or its restriction to $L^2((0,\infty),\mathsf{k})$) defined by

$$(T_t f)(s) = \begin{cases} 0, & s < t, \\ f(s-t), & s \geq t, \end{cases}$$

for $f \in K$. The *CCR flow* of index $\dim \mathsf{k}$ is the E_0-semigroup $\theta = \{\theta_t : t \geq 0\}$ acting on $B(\Gamma_s(L^2((0,\infty),\mathsf{k}^{\mathbb{C}})))$ defined by the extension of

$$\theta_t(W_0(f)) := W_0(S_t f), \quad f \in L^2((0,\infty),\mathsf{k}^{\mathbb{C}}).$$

The CCR flow of index n is cocycle conjugate to the CCR flow of index m if and only if $m = n$ (see [Arv, Corollary 2.6.10]).

Generalized CCR flows are defined in [IS_1] as follows. Let $\{T_t^1\}$ and $\{T_t^2\}$ be two C_0-semigroups acting on a real Hilbert space G. We say that $\{T_t^1\}$ is a perturbation of $\{T_t^2\}$ if the following conditions are satisfied:

(i) ${T_t^1}^* T_t^2 = 1$.

(ii) $T_t^1 - T_t^2$ is a Hilbert Schmidt operator.

Given a perturbation $\{T_t^1\}$ of $\{T_t^2\}$, there exists a unique E_0-semigroup $\theta = \{\theta_t : t \geq 0\}$ on $B(\Gamma_s(G^{\mathbb{C}}))$ defined and extended by

$$\alpha_t(W_0(x+iy)) = W_0(T_t^1 x + i T_t^2 y), \quad x, y \in G.$$

Also, θ is called the *generalized CCR flow* associated with the pair $(\{T_t^1\},\{T_t^2\})$.

§2.3. Toeplitz CAR flows

Let K be a complex Hilbert space. We denote by $\mathcal{A}(K)$ the CAR algebra over K, which is the universal C^*-algebra generated by $\{a(x) : x \in K\}$, where $x \mapsto a(x)$ is an antilinear map satisfying the CAR relations

$$a(x)a(y) + a(y)a(x) = 0,$$
$$a(x)a(y)^* + a(y)^*a(x) = \langle x, y\rangle 1,$$

for all x, $y \in K$. Since $\mathcal{A}(K)$ is known to be simple, any set of operators satisfying the CAR relations generates a C^*-algebra canonically isomorphic to $\mathcal{A}(K)$. The *quasi-free state* ω_A on $\mathcal{A}(K)$, associated with a positive contraction $A \in B(K)$, is the state determined by its $2n$-point function as

$$\omega_A\big(a(x_n)\cdots a(x_1)a(y_1)^*\cdots a(y_m)^*\big) = \delta_{n,m}\det(\langle x_i, Ay_j\rangle),$$

where $\det(\cdot)$ denotes the determinant of a matrix (see [Arv, Chapter 13]). Given a positive contraction, it is a fact that such a state always exists and is uniquely determined by the above relation. We denote by (H_A, π_A, Ω_A) the GNS triple associated with a quasi-free state ω_A on $\mathcal{A}(K)$, and set $\mathrm{M}_A := \pi_A(\mathcal{A}(K))''$.

Now let $K = L^2((0,\infty), \mathsf{k}^{\mathbb{C}})$. An operator $X \in B(L^2((0,\infty), \mathsf{k}^{\mathbb{C}}))$ is said to be Toeplitz if $T_t^* X T_t = X$ for all $t \geq 0$. Let $A \in B(K)$ be a positive Toeplitz contraction satisfying $\mathrm{Tr}(A - A^2) < \infty$. Then M_A is a type I factor and there exists a unique E_0-semigroup $\theta = \{\theta_t : t \geq 0\}$ on M_A, determined by

$$\theta_t(\pi_A(a(f))) = \pi_A(a(T_t f)) \quad \forall\, f \in K.$$

We call θ the *Toeplitz CAR flow* associated with A (see [Arv, Chapter 13] and also [IS_2]).

§2.4. Clifford flows, even Clifford flows

Next we recall some examples of E_0-semigroups on hyperfinite type II_1 factors (see [PW, Ale, MaS] for discussions on these examples). For a real Hilbert space K, let $\Gamma_a(K^{\mathbb{C}}) := \bigoplus_{n=0}^{\infty} (K^{\mathbb{C}})^{\wedge n}$ be the antisymmetric Fock space over $K^{\mathbb{C}}$, i.e., the sum of antisymmetric tensor powers of K. For any $f \in K^{\mathbb{C}}$, the fermionic creation operator $a^*(f)$ is the bounded operator defined by the linear extension of

$$a^*(f)\xi = \begin{cases} f & \text{if } \xi = \Omega, \\ f \wedge \xi & \text{if } \xi \perp \Omega, \end{cases}$$

where Ω is the vacuum vector (1 in the 0-particle space $\mathbb{C}$), and $f \wedge \xi$ is the antisymmetric tensor product. The annihilation operator is defined by $a(f) = a^*(f)^*$. The unital C^*-algebra $\mathrm{Cl}(K)$ generated by the self-adjoint elements

$$\{u(f) = (a(f) + a^*(f))/\sqrt{2} : f \in K\}$$

is the Clifford algebra over K. The vacuum Ω is cyclic and defines a tracial state for $\mathrm{Cl}(K)$, so the weak completion yields a II_1 factor; in fact it is the hyperfinite II_1 factor $\mathcal{R}$.

Now if $K = L^2((0,\infty), \mathsf{k})$, where k is a separable real Hilbert space with dimension $n \in \overline{\mathbb{N}}$ as mentioned before, then there exists a unique E_0-semigroup on $\mathcal{R}$, defined by the extension of

$$\alpha_t^n(u(f_1)\cdots u(f_k)) = u(T_t f_1)\cdots u(T_t f_k), \quad f_1 \cdots f_k \in K,$$

called the *Clifford flow* of rank n. The von Neumann algebra generated by the even products,

$$\mathcal{R}_{\mathrm{e}} = \{u(f_1)u(f_2)\cdots u(f_{2n}) : f_i \in L^2((0,\infty), \mathsf{k}),\ \ n \in \mathbb{N}\},$$

is also isomorphic to the hyperfinite II_1 factor. The restriction of the Clifford flow α^n of rank n to this subfactor is called the *even Clifford flow* of rank n.

Let α be an E_0-semigroup on a II_1 factor. For each $t \geq 0$ let $\mathcal{A}_\alpha(t) := \alpha_t(\mathrm{M})' \cap \mathrm{M}$. These algebras form an increasing filtration. Define the inductive limit C^*-algebra $\mathcal{A}_\alpha := \overline{\bigcup_{t\geq 0} \mathcal{A}_\alpha(t)}^{\|\cdot\|}$, together with a semigroup of *-endomorphisms $\alpha|_{\mathcal{A}_\alpha}$. This is called the C^*-semiflow corresponding to α. Since this is a subalgebra of M, there is a canonical trace on $\mathcal{A}_\alpha$ that we denote by τ_α. Two cocycle-conjugate E_0-semigroups have isomorphic (in the obvious sense of the word) τ-semiflows (see [MaS, Section 9]). We will be using the following fact in Section 4. See [MaS, Theorem 9.6] for details of the proof.

Proposition 2.13. *Any two Clifford flows (or even Clifford flows) are cocycle conjugate if and only if they are conjugate if and only if they have isomorphic τ-semiflows if and only if they have the same rank.*

§3. Super-product systems and coupling index

An E_0-semigroup α on $B(H)$ is completely determined by the invariant (Arveson) product system, defined by $E_t^\alpha = \{X \in B(H) : XT = \alpha_t(T)X \ \ \forall T \in B(H)\}$. A super-product system, of an E_0-semigroup on a general factor, is a generalization of Arveson's product system. However, for the class of non–type I factors, a super-product system is not a complete invariant, as shown in Section 5 for the case of type III factors.

Definition 3.1. A super-product system of Hilbert spaces is a one-parameter family of separable Hilbert spaces $\{H_t : t > 0\}$, together with isometries

$$U_{s,t} : H_s \otimes H_t \mapsto H_{s+t} \quad \text{for } s, t \in (0, \infty),$$

satisfying the following two axioms of associativity and measurability:

(i) Associativity. For any $s_1, s_2, s_3 \in (0, \infty)$,

$$U_{s_1,s_2+s_3}(1_{H_{s_1}} \otimes U_{s_2,s_3}) = U_{s_1+s_2,s_3}(U_{s_1,s_2} \otimes 1_{H_{s_3}}).$$

(ii) Measurability. The space $\mathcal{H} = \{(t, \xi_t) : t \in (0, \infty), \ \ \xi_t \in H_t\}$ is equipped with the structure of a standard Borel space that is compatible with the projection $p : \mathcal{H} \mapsto (0, \infty)$ given by $p((t, \xi_t) = t$, tensor products and the inner products (see [Arv, Remark 3.1.2]).

A super-product system is an (Arveson) product system if the isometries $U_{s,t}$ are unitaries and further, the axiom of local triviality is satisfied, i.e., there exists a single separable Hilbert space H satisfying $\mathcal{H} \cong (0,\infty) \times H$ as measure spaces (see [Arv, Remark 3.1.2]).

Proposition 3.2. *Let* $\mathrm{M} \subset B(H)$ *be a factor acting standardly with cyclic and separating vector* Ω *and* α *an* E_0*-semigroup on* M*. For each* $t > 0$*, let*

$$H_t^\alpha = \{X \in B(H) : \forall_{m \in \mathrm{M}}\, Xm = \alpha_t(m)X,\ \ \forall_{m' \in \mathrm{M}'}\, Xm' = \alpha_t'(m')X\};$$

then $H^\alpha = \{H_t^\alpha : t > 0\}$ *is a concrete super-product system with respect to the family of isometries* $U_{s,t}\,(X \otimes Y) = XY$*.*

Proof. It is routine to verify that $X^*Y \in (\mathrm{M} \cup \mathrm{M}')' = \mathbb{C}1$ for any X, $Y \in H_t^\alpha$. Clearly each H_t^α is closed under the operator norm, and this coincides with the norm induced by the inner product $\langle X, Y\rangle 1 := X^*Y$, hence each H_t^α is a Hilbert space with respect to this inner product. It is straightforward to check for $X \in H_s^\alpha$, $Y \in H_t^\alpha$ that $XY \in H_{s+t}^\alpha$ and that the map $U_{s,t}(X \otimes Y) = XY$ is an isometry. The measurability axiom can be proved in an exactly similar manner as in the case of product systems, as given in [Arv, Theorem 2.4.7, page 37]. □

Definition 3.3. By an isomorphism between super-product systems $(H_t^1, U_{s,t}^1)$ and $(H_t^2, U_{s,t}^2)$ we mean an isomorphism of Borel spaces $V : \mathcal{H}^1 \mapsto \mathcal{H}^2$ whose restriction to each fiber provides a unitary operator $V_t : H_t^1 \mapsto H_t^2$ satisfying

$$V_{s+t}U_{s,t}^1 = U_{s,t}^2(V_s \otimes V_t).$$

A priori, the super-product system appears to depend upon the chosen state Ω. The following theorem shows that the isomorphism class does not depend on Ω.

Theorem 3.4. *Let* α *and* β *be* E_0*-semigroups acting standardly on respective factors* M *and* N *with cyclic and separating vectors* Ω_1 *and* Ω_2*. If* α *and* β *are cocycle conjugate then the associated respective product systems* H^{α,Ω_1} *and* H^{β,Ω_2} *are isomorphic.*

Proof. First we show that for any two cyclic and separating vectors Ω_1, Ω_2, that H^{α,Ω_1} and H^{α,Ω_2} are isomorphic. By Theorem 2.4 there exists a unitary $V \in \mathrm{M}'$ such that $J_{\Omega_1}J_{\Omega_2}m'J_{\Omega_2}J_{\Omega_1} = Vm'V^*$ for any $m' \in \mathrm{M}'$. We claim the maps $H_t^\alpha \ni X \mapsto VXV^*$ give the required isomorphism. Indeed, VXV^* is clearly an intertwiner for α, and

$$\begin{aligned} VXV^*m' &= VX(V^*m'V)V^* = VJ_{\Omega_1}\alpha_t(J_{\Omega_1}V^*m'VJ_{\Omega_1})J_{\Omega_1}XV^* \\ &= J_{\Omega_2}\alpha_t(J_{\Omega_2}m'J_{\Omega_2})J_{\Omega_2}VXV^*. \end{aligned}$$

It is a direct verification to check that this also provides an isomorphism of super-product systems.

Next, if α and β are conjugate then, letting U be the unitary implementing the conjugacy, we get an isomorphism $\mathrm{Ad}_{J_\Omega U J_\Omega U} : H_t^{\alpha,\Omega} \to H_t^{\beta,U\Omega}$. Last, if β is a cocycle perturbation of α by the cocycle $(U_t)_{t\geq 0}$, then left multiplication by $J_\Omega U_t J_\Omega U_t$ gives the required family of unitaries $H_t^{\alpha,\Omega} \to H_t^{\beta,\Omega}$. □

We will thus talk freely of the (abstract) super-product system $\{H_t^\alpha, U_{s,t}\}$ for α. The multiunits associated with an E_0-semigroup α are precisely the units in the associated super-product system, i.e., measurable sections $\{u_t : u_t \in H_t^\alpha,\ t \geq 0\}$ satisfying $u_{s+t} = U_{s,t}(u_s \otimes u_t)$. Thus, for a multispatial E_0-semigroup α, we can define a covariance function $c : \mathcal{U}_{\alpha,\alpha'} \times \mathcal{U}_{\alpha,\alpha'} \to \mathbb{C}$ by $X_t^* Y_t = e^{c(X,Y)t}1$ for all $t \in \mathbb{R}_+$. Since the covariance function is conditionally positive definite (see [Arv, Proposition 2.5.2]) the assignment $\langle f, g\rangle \mapsto \sum_{X,Y \in \mathcal{U}_{\alpha,\alpha'}} c(X,Y)\overline{f(X)}g(Y)$ defines a positive semidefinite form on the space of finitely supported functions $f : \mathcal{U}_{\alpha,\alpha'} \to \mathbb{C}$ satisfying $\sum_{X\in\mathcal{U}_{\alpha,\alpha'}} f(X) = 0$. Hence, if this space is nonempty, we may quotient and complete to obtain a Hilbert space $H(\mathcal{U}_{\alpha,\alpha'})$.

Let α and β be cocycle-conjugate E_0-semigroups on respective factors M and N acting standardly. Then there is a bijection $\mathcal{U}_{\alpha,\alpha'} \to \mathcal{U}_{\beta,\beta'}$ that preserves the covariance function. In particular, if one E_0-semigroup is multispatial, then so is the other, and we have $H(\mathcal{U}_{\alpha,\alpha'}) \cong H(\mathcal{U}_{\beta,\beta'})$.

Definition 3.5. For a multispatial E_0-semigroup α, define the coupling index $\mathrm{Ind}_c(\alpha)$ as the cardinal $\dim H(\mathcal{U}_{\alpha,\alpha'})$.

Every pair $(X^\alpha, X^\beta) \in \mathcal{U}_{\alpha,\alpha'} \times \mathcal{U}_{\beta,\beta'}$ gives a multiunit $X^\alpha \otimes X^\beta$ for $\alpha \otimes \beta$. As

$$(X_t^\alpha \otimes X_t^\beta)^*(Y_t^\alpha \otimes Y_t^\beta) = e^{(c(X^\alpha,Y^\alpha)+c(X^\beta,Y^\beta))t}1,$$

there exists an isometry $H(\mathcal{U}^\varphi_{\alpha,\alpha'}) \oplus H(\mathcal{U}^\psi_{\beta,\beta'}) \hookrightarrow H(\mathcal{U}^{\varphi\otimes\psi}_{\alpha\otimes\beta,(\alpha\otimes\beta)'})$ (see [Arv, Lemma 3.7.5]). So we have $\mathrm{Ind}_c(\alpha\otimes\beta) \geq \mathrm{Ind}_c(\alpha) + \mathrm{Ind}_c(\beta)$. We do not know whether the equality holds.

Remark 3.6. Let α be an E_0-semigroup on a factor $\mathrm{M} \subseteq B(H)$ in standard form. If there exists an E_0-semigroup σ on $B(H)$ satisfying

$$\sigma_t(x) = \begin{cases} \alpha_t(x) & \text{if } x \in \mathrm{M}, \\ \alpha'_t(x) & \text{if } x \in \mathrm{M}', \end{cases} \quad \forall\, t \geq 0,$$

then the super-product system of α is the Arveson product system of σ, and $\mathrm{Ind}_c(\alpha)$ is equal to the Powers–Arveson index of σ. In this case we say α is canonically extendable. Let β be another E_0-semigroup on a factor $\mathrm{N} \subseteq B(K)$ in standard form, which is cocycle conjugate to α. Let $U : H \mapsto K$ be unitary and $(U_t)_{t\geq 0}$ be a unitary cocycle for α satisfying $\beta_t = \mathrm{Ad}_U \mathrm{Ad}_{U_t} \alpha_t \mathrm{Ad}_{U^*}$ for all $t \geq 0$. By Proposition 2.5 there exists a unitary $V : H \mapsto K$ implementing the conjugacy of both $(\mathrm{Ad}_{U_t} \alpha_t)_{t \geq 0}$ and β as well as the conjugacy of the respective dual E_0-semigroups. Now

$$\theta_t = \mathrm{Ad}_V \mathrm{Ad}_{U_t} \mathrm{Ad}_{J_\Omega U_t J_\Omega} \sigma_t \mathrm{Ad}_{V^*}$$

provides the canonical extension for β. So canonical extendability is a property that is invariant under cocycle conjugacy, and σ is called the canonical extension.

All E_0-semigroups on type I factors are canonically extendable; indeed $\alpha \otimes \bar{\alpha}$ is the canonical extension for an E_0-semigroup α on $B(H)$, as shown in Example 2.9. The super-product system of α is $E_t^\alpha \otimes \bar{E}_t^\alpha$. So by [Arv] α is multispatial if and only if it is spatial in the sense of Powers–Arveson, in which case $\mathrm{Ind}_c(\alpha) = \mathrm{Ind}(\alpha \otimes \overline{\alpha}) = 2\,\mathrm{Ind}(\alpha)$: its coupling index is twice its Powers–Arveson index.

All of our known examples of E_0-semigroups on II_1 factors (and type II_∞ factors) are not canonically extendable (see [MaS]). It is an open problem to construct a canonically extendable E_0-semigroup on a II_1 factor. On type III factors we know examples of both extendable (arising from CCR representations) and nonextendable (arising from CAR representations) type.

The previous example suggests that a better definition for the coupling index would be half the dimension of $H_{\alpha,\alpha'}$. However, by historical accident, our first paper on the coupling index considered only E_0-semigroups on type II_1 factors. We will not attempt to redact the original definition, since we do not have any reason to believe that an arbitrary E_0-semigroup must have an even coupling index, though constructing an example with an odd coupling index is an open problem.

Proposition 3.7. *Let α and β be E_0-semigroups on factors M_1 and M_2; then the super-product system for $\alpha \otimes \beta$ is the tensor product of the super-product systems for α and β.*

Proof. Assume $\mathrm{M}_1 \subseteq B(H_1)$ and $\mathrm{M}_2 \subseteq B(H_2)$ are in standard form with respective cyclic and separating vectors Ω_1 and Ω_2. It is clear that $H_t^\alpha \otimes H_t^\beta \subseteq H_t^{\alpha\otimes\beta}$; we prove the other inclusion as follows. Let $\Omega = \Omega_1 \otimes \Omega_2$. Notice that any operator $X \in H_t^{\alpha\otimes\beta}$ is determined by its value on Ω through the relation

$$X(m_1 \otimes m_2)\Omega = (\alpha_t(m_1) \otimes \beta_t(m_2))\, X\Omega.$$

Suppose $X \in H_t^{\alpha\otimes\beta}$ such that $X \perp H_t^\alpha \otimes H_t^\beta$; then X^* is 0 on $H_t^\alpha H_1 \otimes H_t^\beta H_2$. This implies that the projection of $X\Omega$ onto $H_t^\alpha H_1 \otimes H_t^\beta H_2$ is 0. So it remains to show that the projection of $X\Omega$ onto $(H_t^\alpha H_1 \otimes H_t^\beta H_2)^\perp$ is also 0.

Before proving this remaining assertion, we claim that there does not exist a vector $\xi \in H_2$ such that

$$0 \neq (1 \otimes P_\xi)X\Omega \in (H_t^\alpha H_1)^\perp \otimes \mathbb{C}\xi,$$

where P_ξ is the projection onto the one-dimensional subspace spanned by ξ. Suppose there exists such a vector ξ; let $E_\xi : H_1 \to H_1 \otimes H_2$ denote the isometry $\eta \mapsto \eta \otimes \xi$, and we write $E^\xi : H_1 \otimes H_2 \to H_1$ for its adjoint. Note that $E_\xi E^\xi = 1 \otimes P_\xi$. Define $T \in B(H_1)$ by $T = E^\xi X E_{\Omega_2}$, so that $0 \neq T\Omega_1 = E^\xi X\Omega \in (H_t^\alpha H_1)^\perp$ (where E_Ω is also defined similarly). Then, for all $m_1, m_2 \in \mathrm{M}$,

$$\begin{aligned} Tm_1m_2\Omega_1 &= E^\xi X(m_1m_2 \otimes 1)\Omega = E^\xi(\alpha_t(m_1) \otimes 1)X(m_2 \otimes 1)\Omega \\ &= \alpha_t(m_1)E^\xi X(m_2 \otimes 1)\Omega = \alpha_t(m_1)Tm_2\Omega_1, \end{aligned}$$

so that $Tm_1 = \alpha_t(m_1)T$ and, similarly, $Tm_1' = \alpha_t'(m_1')T$ for all $m' \in \mathrm{M}'$. Thus $T \in H_t^\alpha$, contradicting $T\Omega_1 \in (H_t^\alpha H_1)^\perp$. Hence the claim is proved. Flipping the same argument, by switching the role of the first and second tensor components, we also conclude that there does not exist an $\eta \in H_1$ such that

$$0 \neq (P_\eta \otimes 1)X\Omega \in \mathbb{C}\eta \otimes (H_t^\beta H_2)^\perp.$$

Now assume towards a contradiction that $0 \neq X\Omega \in (H_t^\alpha H_1 \otimes H_t^\beta H_2)^\perp$, i.e.,

$$\begin{aligned} 0 \neq X\Omega \in \big((H_t^\alpha H_1)^\perp \otimes H_t^\beta H_2\big) &\oplus \big((H_t^\alpha H_1)^\perp \otimes (H_t^\beta H_2)^\perp\big) \\ &\oplus \big(H_t^\alpha H_1 \otimes (H_t^\beta H_2)^\perp\big). \end{aligned}$$

There exists a vector $\xi \in H_2$ such that $0 \neq (1 \otimes P_\xi)X\Omega$ and there exists an $\eta \in H_1$ such that $0 \neq (P_\eta \otimes 1)X\Omega$. If $\xi \in H_t^\beta H_2$, then we have $(1 \otimes P_\xi)X\Omega \in (H_t^\alpha H_1)^\perp \otimes \mathbb{C}\xi$, and similarly if $\eta \in H_t^\alpha H_1$, then we have $(P_\eta \otimes 1)X\Omega \in \mathbb{C}\eta \otimes (H_t^\beta H_2)^\perp$. Both are not possible by the claims in the preceding paragraph. By applying a projection if needed, we also conclude that η and ξ cannot have nonzero components in $H_t^\alpha H_1$ and $H_t^\beta H_2$ respectively. Hence $X\Omega \in ((H_t^\alpha H_1)^\perp \otimes (H_t^\beta H_2)^\perp)$, which is also not possible by the same claims. □

§4. E_0-semigroups on II_∞ factors

In this section, by considering tensor products of E_0-semigroups on type I_∞ factors with E_0-semigroups on type II_1 factors, we produce several (both countable and uncountable) families of mutually non-cocycle-conjugate E_0-semigroups on II_∞ factors. Let $\mathcal{R}$ be the hyperfinite II_1 factor, and we always assume $\mathcal{R} \subseteq L^2(\mathcal{R})$ with respect to the tracial state. Let $\mathcal{R}_\infty = B(H) \otimes \mathcal{R}$; then $\mathcal{R}_\infty$ is the hyperfinite II_∞ factor.

Throughout this section, α^n denotes either the Clifford flow or the even Clifford flow of rank n, with $n \in \overline{\mathbb{N}}$, and when n is fixed we simply write α. The super-product systems of α^n are computed in [MaS, Corollary 8.13]. Set

$$H_t^{e,n} = [\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_{2m};\ \ \xi_1, \xi_2, \dots, \xi_{2m} \in L^2((0,t), \mathsf{k}^{\mathbb{C}}),\ \ m \in \mathbb{N}_0]$$

for all $t \geq 0$, and $\dim(\mathsf{k}) = n \in \overline{\mathbb{N}}$. We may write just H_t^e in many instances when n is fixed. The super-product system of the Clifford flow (which is isomorphic to the super-product system of the even Clifford flow) of rank n is described by $H_t^{\alpha^n}\Omega = H_t^{e,n}$ for all $t \geq 0$, where $\Omega \in L^2(\mathcal{R})$ is the vacuum vector. The isometries $U_{s,t} : H_s^{\alpha^n}\Omega \otimes H_t^{\alpha^n}\Omega \mapsto H_{s+t}^{\alpha^n}\Omega$ are given by

$$\begin{aligned} &U_{s,t}((\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_{2m}) \otimes (\eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_{2m'})) \\ &\qquad = \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_{2m} \wedge T_s\eta_1 \wedge T_s\eta_2 \wedge \cdots \wedge T_s\eta_{2m'} \end{aligned}$$

where $\xi_1, \xi_2, \dots, \xi_{2m} \in L^2((0,s), \mathsf{k}^{\mathbb{C}})$, $\eta_1, \eta_2, \dots, \eta_{2m} \in L^2((0,t), \mathsf{k}^{\mathbb{C}})$.

§4.1. Tensoring with CCR flows

Throughout this subsection, let $\theta^m = \{\theta_t^m : t \geq 0\}$ denote the CCR flow of index $m \in \overline{\mathbb{N}}$ on $B(H^m)$, where $H^m = \Gamma_s(L^2(\mathbb{R}_+, \mathsf{k}^{\mathbb{C}}))$, $\dim(\mathsf{k}) = m$. The (Arveson) product system of Hilbert spaces associated with θ^m is the well-known exponential product system $\{H_t^m : t \geq 0\}$ of index m, which is described as follows: $H_t^m = \Gamma_s(L^2((0,t), \mathsf{k}^{\mathbb{C}}))$ with $\dim(\mathsf{k}) = m$ and the unitaries $U_{s,t} : H_s^m \otimes H_t^m \mapsto H_{s+t}^m$ are the extensions of $\varepsilon(x) \otimes \varepsilon(y) \mapsto \varepsilon(x + T_s y)$.

Theorem 4.1. *The E_0-semigroup $\theta^m \otimes \alpha^n$ is cocycle conjugate to $\theta^p \otimes \alpha^q$ if and only if $(m,n) = (p,q)$.*

Proof. Step 1. Assume $\theta^m \otimes \alpha^n$ is cocycle conjugate to $\theta^p \otimes \alpha^q$.

Thanks to Proposition 3.7, the super-product system of $\theta^m \otimes \alpha^n$ is given by $(H_t^m \otimes \overline{H}_t^m) \otimes H_t^{e,n}$. Notice that the super-product system $H^{e,n}$ can be embedded as a super-product subsystem into the product system corresponding to the CAR flow (on type I factors) of index n. Since any unit in the super-product subsystem is also a unit for the bigger product system, it follows from [Arv] for product systems that units in $(H_t^m \otimes \overline{H}_t^m) \otimes H_t^{e,n}$ are of the form $u_t \otimes v_t$, with u_t a unit for $(H_t^m \otimes \overline{H}_t^m)_{t\geq 0}$ and v_t a unit for $(H_t^{e,n})_{t\geq 0}$. But the super-product system $(H_t^{e,n})_{t\geq 0}$ has the canonical unit as the unique unit up to a scalar (see [MaS, Section 8]). By comparing the coupling index we get $m = p$.

Step 2. Take $\theta^m = \theta^p = \theta$, $H_t^m = H_t$ and assume $\theta \otimes \alpha^n$ is cocycle conjugate to $\theta \otimes \alpha^q$.

Set $\mathrm{M} \cong B(H)$. We assume $\mathcal{R}_\infty = \mathrm{M} \otimes \mathcal{R} \subseteq B(H \otimes \overline{H}) \otimes B(L^2(\mathcal{R}))$ is in standard form, by identifying M with $B(H) \otimes 1$, and (without loss of generality) that both the semigroups act on the same algebra. Suppose that there exists a $\theta \otimes \alpha^n$-cocycle U in $\mathcal{R}_\infty$ and a unitary $V \in B(H \otimes \overline{H} \otimes L^2(\mathcal{R}))$ such that

$$\theta_t \otimes \alpha_t^p = \mathrm{Ad}_{VU_t} \circ (\theta_t \otimes \alpha_t^n) \circ \mathrm{Ad}_{V^*} \quad \forall\, t \geq 0.$$

Let $(S_t^n)_{t\geq 0}, (S_t^p)_{t \geq 0}$ be the canonical units in $B(L^2(\mathcal{R}))$ for α^n and α^p respectively. Notice that θ and its complementary E_0-semigroup θ' extend to $\theta \otimes \theta'$ on $B(H \otimes \overline{H})$, and the super-product system $H_t \otimes \overline{H}_t$ is the product system of Hilbert spaces associated with $\theta \otimes \theta'$. The multiunits of θ are just the units in the product system $H_t \otimes \overline{H}_t$.

Let $u_t \otimes S_t^n$ be a unit for $H_t \otimes \overline{H}_t \otimes H_t^{e,n}$, with u_t a unit for $H_t \otimes \overline{H}_t$. Let $J = J_1 \otimes J_2$, with J_1, J_2 modular conjugations for M and $\mathcal{R}$ with respect to cyclic and separating vectors Ω_1 and Ω_2 respectively. Let $U_t' = JU_tJ$. Then $(VU_t'U_t(u_t \otimes S_t^n)V^*)_{t \geq 0}$ is a unit for $(H_t \otimes \overline{H}_t \otimes H_t^{e,p})_{t\geq 0}$, which is of the form $(v_t \otimes S_t^p)_{t \geq 0}$, for some unit $(v_t)_{t\geq 0}$ for $H_t \otimes \overline{H}_t$. Since the (left) action of $(U_t'U_t)_{t \geq 0}$ and Ad_V on the units preserves the covariance function, the map $u \mapsto v$ also preserves the covariance function. So there is an induced automorphism of $(\mathcal{U}, c)$ (see [Arv, Definition 3.74 and Section 3.8]), where $\mathcal{U}$ is the collection of units for $H_t \otimes \overline{H}_t$ and c is the corresponding covariance function. As proved in [Arv, Section 3.8], this automorphism is given by a gauge cocycle of $\theta \otimes \theta'$; so there exists a gauge cocycle $(W_t)_{t\geq 0}$ of $\theta \otimes \theta'$ satisfying

$$VU_t'U_t(u_t \otimes S_t^n)V^* = W_t u_t \otimes S_t^p \quad \forall\, u_t \in \mathcal{U}. \tag{3}$$

It is also clear that

$$(U_t'U_t)^* V^* (v_t \otimes S_t^p) V = W_t^* v_t \otimes S_t^n \quad \forall\, v_t \in \mathcal{U}.$$

For every choice of units $u_{t_1}, \ldots, u_{t_n}$ in $H_t \otimes \overline{H}_t$, with $t_1, \ldots, t_n \in \mathbb{R}_+$ satisfying $t_1 + \cdots + t_n = t$, we have

$$\begin{aligned} VU_t'U_t((u_{t_1} \cdots u_n) \otimes S_t^n)V^* &= (VU_{t_1}'U_{t_1}(u_{t_1} \otimes S_{t_1}^n)V^*) \cdots (VU_{t_n}'U_{t_n}(u_{t_n} \otimes S_{t_n}^n)V^*) \\ &= (W_{t_1}u_{t_1} \otimes S_{t_1}^p) \cdots (W_{t_n}u_{t_n} \otimes S_{t_n}^p) \\ &= W_t u_{t_1} \cdots u_n \otimes S_t^p, \end{aligned}$$

where we have used the properties of $(U_t)_{t\geq 0}$ and $(W_t)_{t \geq 0}$ being cocycles, $(u_t)_{t\geq 0}$, $(S_t^n)_{t\geq 0}$ and $(S_t^p)_{t\geq 0}$ being units and equation (3). Since the product system of a CCR flow is generated by units (and by a similar argument), we get

$$VU_t'U_t(T \otimes S_t^n)V^* = W_t T \otimes S_t^p, \qquad (U_t'U_t)^* V^*(R \otimes S_t^p)V = W_t^* R \otimes S_t^n \tag{4}$$

for all T, $R \in H_t \otimes \overline{H}_t$.

Now, for any $X \in \theta_t(\mathrm{M})' \cap \mathrm{M}$, $T \in H_t \otimes \overline{H}_t$, we have

$$\begin{aligned} VU_t(X \otimes 1)U_t^* V^*(T \otimes S_t^p) &= VU_t' U_t(X \otimes 1)(U_t' U_t)^* V^*(T \otimes S_t^p) VV^* \\ &= VU_t' U_t(X \otimes 1)(W_t^* T \otimes S_t^n) V^* \\ &= VU_t' U_t(X W_t^* T \otimes S_t^n) V^* \\ &= W_t X W_t^* T \otimes S_t^p, \end{aligned}$$

where we have used equation (4) and the fact that $XW_t^*T \in H_t \otimes \overline{H}_t$. It follows that, for any $\xi \in H \otimes \overline{H}$ and $m' \in \mathcal{R}' \cap B(L^2(\mathcal{R}))$,

$$\begin{aligned} VU_t(X \otimes 1)U_t^* V^*(T\xi \otimes m'\Omega_2) &= (1 \otimes m')VU_t(X \otimes 1)U_t^* V^*(T\xi \otimes S_t^p \Omega_2) \\ &= (1 \otimes m')(W_t X W_t^* T\xi \otimes S_t^p \Omega_2) \\ &= (W_t X W_t^* \otimes 1)(T\xi \otimes m'\Omega_2). \end{aligned}$$

Since $(H_t \otimes \overline{H}_t)(H \otimes \bar{H}) = H \otimes \bar{H}$ and Ω_2 is cyclic for M', we have

$$\mathrm{Ad}_{VU_t}(X \otimes 1) = \mathrm{Ad}_{W_t}(X) \otimes 1 \quad \forall\, X \in \theta_t(\mathrm{M})' \cap \mathrm{M}. \tag{5}$$

Since $U_t \in \mathcal{R}_\infty$ and Ad_V is an automorphism of $\mathcal{R}_\infty$ it follows a fortiori that $\mathrm{Ad}_{W_t}(X) \in \mathrm{M}$ for all $X \in \theta_t(\mathrm{M})' \cap \mathrm{M}$. Now, from the explicit description of gauge cocycles given in [Arv, Section 9.8], it follows that W_t is a product of gauge cocycles of θ and θ', and we assume, without loss of generality, that $(W_t)_{t \geq 0} \subseteq \mathrm{M}$ is a gauge cocycle of θ.

Now we consider the C^*-semiflows associated with these E_0-semigroups. For $i = n, p$, let

$$\mathcal{C}_t^i = ((\theta_t \otimes \alpha_t^i)(\mathcal{R}_\infty))' \cap \mathcal{R}_\infty, \qquad \mathcal{A}_t^i = \alpha_t^i(\mathcal{R})' \cap \mathcal{R}, \quad t \geq 0,$$
$$\mathcal{C}^i = \overline{\bigcup_{t \geq 0} (((\theta_t \otimes \alpha_t^i)(\mathcal{R}_\infty))' \cap \mathcal{R}_\infty)}^{\|\cdot\|}, \qquad \mathcal{A}^i = \overline{\bigcup_{t \geq 0} (\alpha_t^i(\mathcal{R})' \cap \mathcal{R})}^{\|\cdot\|}.$$

The inductive limit ϕ of the maps $\phi_t := \mathrm{Ad}_{VU_t}|\mathcal{C}_t^n \to \mathcal{C}_t^p$ provides an isomorphism between $\mathcal{C}^n$ and $\mathcal{C}^p$ intertwining the C^*-semiflows.

By equation (5), we have

$$\begin{aligned} \mathrm{Ad}_{(W_t^* \otimes 1)VU_t}(1 \otimes Y)(X \otimes 1) &= (W_t^* \otimes 1)VU_t(1 \otimes Y)(X \otimes 1)U_t^* V^*(W_t \otimes 1) \\ &= (X \otimes 1)\mathrm{Ad}_{(W_t^* \otimes 1)VU_t}(1 \otimes Y) \end{aligned}$$

for all $X \in \theta_t(\mathrm{M})' \cap \mathrm{M}$ and $Y \in \mathcal{R}$. Hence, for all $Y \in \mathcal{A}_t^n$,

$$\mathrm{Ad}_{(W_t^* \otimes 1)VU_t}(1 \otimes Y) \in ((\theta_t(\mathrm{M})' \cap \mathrm{M}) \otimes 1)' \cap \mathcal{C}_t^p = 1 \otimes \mathcal{A}_t^p,$$

where the latter equality follows from relation (1) in Section 2. It follows that for each $t \geq 0$, ϕ_t restricts to a map from $1 \otimes \mathcal{A}_t^n$ to $1 \otimes \mathcal{A}_t^p$, and hence ϕ restricts to an isomorphism intertwining the C^*-semiflows for α^n and α^p.

We claim that ϕ intertwines the tracial states on $\mathcal{A}_t^i$ induced by the canonical trace on $\mathcal{R}$. Indeed, by [Ale, Proposition 2.9] each $\mathcal{A}_t^i$ is a II_1 factor and hence the maps ϕ_t intertwine the induced traces on each of the corresponding subalgebras; the statement follows by taking inductive limits. In the terminology of [MaS, Section 9], α^n and α^p have isomorphic τ-semiflows, and hence by Proposition 2.13, $n = p$. □

§4.2. Tensoring with generalized CCR flows

Throughout this subsection, we denote by $\theta = \{\theta_t : t \geq 0\}$ a generalized CCR flow associated with a pair $\big(\{T_t^1\}_{t\geq 0}, \{T_t^2\}_{t\geq 0}\big)$, where $\{T_t^1 : t \geq 0\}$ and $\{T_t^2 : t \geq 0\}$ are two C_0-semigroups that are perturbations of one another (see Section 2.2). In our examples we assume the semigroup $\{T_t^1 : t \geq 0\}$ is the right shift on $L^2(0,\infty)$ with index 1. Basic facts about spectral densities describing off-white noises can be found in [Ts2].

In [IS_1], local algebras associated with product systems were used to distinguish generalized CCR flows given by off-white noises with spectral density converging to 1 at infinity. Here we define and use local algebras associated with super-product systems to study E_0-semigroups on the hyperfinite II_∞ factor, given by tensor products of such generalized CCR flows with α.

A subset $\mathcal{O} \subseteq [0,a]$ is an elementary set if $\mathcal{O} = \cup_{n=1}^N (s_n, t_n)$, a finite disjoint union of open intervals. We assume $s_{n+1} > t_n$. By $\mathcal{O}^c$ we mean the interior of the complement in $[0,a]$. For a Borel set $E \subseteq \mathbb{R}$, $|E|$ denotes the Lebesgue measure of E.

Let $H = (H_t, U_{s,t})$ be any super-product system. Fix an arbitrary $a > 0$. The local algebra $\mathcal{A}^H(I)$ associated with the super-product system H for any interval $I = (s,t) \subseteq [0,a]$ is defined by

$$\mathcal{A}^H(I) = U_I^a \left(\mathbb{C}1_{H_s} \otimes B(H_{t-s}) \otimes \mathbb{C}1_{H_{a-t}}\right)(U_I^a)^*,$$

where U_I^a is the canonical isometry $U_I^a : H_s \otimes H_{t-s} \otimes H_{a-t} \mapsto H_a$ determined uniquely by the associativity axiom. Here we consider $\mathcal{A}^H(I)$ as a von Neumann subalgebra of $B(P_I^a H_a)$, where $P_I^a = U_I^a (U_I^a)^*$.

For an elementary open set $\mathcal{O}_N = \cup_{n=1}^N (s_n, t_n)$, denote by $P_{\mathcal{O}_N}^a$ the projection $U_{\mathcal{O}_N}^a (U_{\mathcal{O}_N}^a)^*$, where

$$U_{\mathcal{O}_N}^a : H_{s_1 - t_0} \otimes H_{t_1 - s_1} \otimes \cdots \otimes H_{s_N - t_{N-1}} \otimes H_{t_N - s_N} \otimes H_{s_{N+1} - t_N} \mapsto H_a \tag{6}$$

is the canonical isometry uniquely determined by the associativity axiom of the super-product system. (Here we have set $t_0 = 0$ and $s_{N+1} = a$.) We just write $U_{\mathcal{O}_N}$ for $U^a_{\mathcal{O}_N}$ and $P_{\mathcal{O}_n}$ for $P^a_{\mathcal{O}_N}$ when a is unambiguously fixed. For $1 \leq k \leq N$, if we define $I_k = (s_k, t_k)$, $\mathcal{O}_{k]} = \cup_{n=1}^{k-1}(s_n, t_n)$, $\mathcal{O}_{[k} = \cup_{n=k+1}^{N}(s_n - t_k, t_n - t_k)$, then using the associativity axiom, it is not difficult to verify that

$$U^a_{\mathcal{O}_N} = U^a_{I_k}\left(U^{s_k}_{\mathcal{O}_{k]}} \otimes 1_{H_{t_k - s_k}} \otimes U^{a-t_k}_{\mathcal{O}_{[k}}\right).$$

Using this we see for $x \in B(H_{t_k - s_k})$,

$$\begin{aligned} P^a_{\mathcal{O}_N} U^a_{I_k}\left(1_{H_{s_k}} \otimes x \otimes 1_{H_{a-t_k}}\right)(U^a_I)^* &= U^a_{\mathcal{O}_N}\left(1_{H_{s_k}} \otimes x \otimes 1_{H_{a-t_k}}\right)(U^a_{\mathcal{O}_N})^* \\ &= U^a_{I_k}\left(1_{H_{s_k}} \otimes x \otimes 1_{H_{a-t_k}}\right)(U^a_I)^* P^a_{\mathcal{O}_N}, \end{aligned}$$

and hence $P^a_{\mathcal{O}_N} \in \mathcal{A}'_{s_k,t_k}$ for all $1 \leq k \leq N$.

For a general open set $\mathcal{O} \subseteq (0, a)$ with $\mathcal{O} = \cup_{n=1}^{\infty} I_n$ a disjoint union of intervals, define

$$P_{\mathcal{O}} = \bigwedge_{n=1}^{\infty} P_{\mathcal{O}_n},$$

where $\mathcal{O}_n = \cup_{k=1}^{n} I_k$ is an increasing sequence of elementary open sets. The projection $P_{\mathcal{O}}$ does not depend on the choice of the intervals or the elementary open sets $\{\mathcal{O}_n\}_{n=1}^{\infty}$, since $P_{\mathcal{O}_n} \leq P_{\mathcal{O}_m}$ if the elementary sets satisfies $\mathcal{O}_m \subseteq \mathcal{O}_n$. (Caution: The relation $P_{\mathcal{O}_2} \leq P_{\mathcal{O}_1}$ does not hold in general for arbitrary elementary sets satisfying $\mathcal{O}_1 \subseteq \mathcal{O}_2$, but it does hold for sets in this collection, since the interval components of the elementary open subset are a subcollection of the interval components of the bigger elementary open set.) Every $P_{\mathcal{O}_m}$ commutes with $\mathcal{A}(I_n)$ if $I_n \subseteq \mathcal{O}_m$. So $P_{\mathcal{O}}$ also commutes with $\mathcal{A}(I_n)$. Define

$$\mathcal{A}^H(\mathcal{O}) = \bigvee_{n=1}^{\infty} P_{\mathcal{O}} \mathcal{A}^H(I_n),$$

the von Neumann algebra generated by $\{P_{\mathcal{O}} \mathcal{A}^H(I_n)\}_{n=1}^{\infty}$ in $B(P_{\mathcal{O}} H_a)$.

If the family $(V_t)_{t \geq 0}$ provides an isomorphism between two super-product systems $(H_t, U_{s,t})$ and $(H'_t, U'_{s,t})$, then $\mathrm{Ad}(V_a)$ provides an isomorphism between $\mathcal{A}^H(\mathcal{O})$ and $\mathcal{A}^{H'}(\mathcal{O})$. Hence the family of von Neumann algebras $\{\mathcal{A}^H(\mathcal{O}) : \mathcal{O} \subseteq [0, a]\}$ is an invariant for the super-product system $(H_t, U_{s,t})$, and hence for the associated E_0-semigroup.

Lemma 4.2. *Let H be a super-product system and $\mathcal{O} = \bigcup_{n=1}^{\infty} I_n \subseteq [0, a]$ an open set for mutually disjoint open intervals $I_n = (s_n, t_n)$. Then*

(1) if H is spatial, then $\mathcal{A}^H(\mathcal{O})$ has a direct summand that is a type I_∞ factor, and further, if $t_n < s_{n+1}$, then $\mathcal{A}^H(\mathcal{O})$ is a type I factor;

(2) if $H = H^e$ is the super-product system associated with a Clifford flow of any fixed index, then $\mathcal{A}^H(\mathcal{O})$ is a type I_∞ factor for any open set $\mathcal{O} \subseteq [0, a]$.

Proof. (1) Let $(S_t)_{t>0}$ be a unit for H. Without loss of generality we assume that $\|S_t\| = 1$ for all $t > 0$. Notice $P_\mathcal{O} S_a = S_a$. Let $L = \left[\mathcal{A}^H(\mathcal{O})S_a\right] \subseteq P_\mathcal{O} H_a$ and P_L be the projection from $P_\mathcal{O} H_a$ onto L, which belongs to $\mathcal{A}^H(\mathcal{O})'$. We introduce a state ω of $\mathcal{A}^H(\mathcal{O})$ by $\omega(x) = \langle x S_a, S_a\rangle$. We have $\omega(x) = \langle x S_{t_i - s_i}, S_{t_i - s_i}\rangle$ for any $x \in P_\mathcal{O} \mathcal{A}^H(I_i)$. Now for $x_i \in \mathcal{A}^H(I_{n_i})$ $i = 1, 2, \ldots, N$, we have

$$\begin{aligned}
\omega(P_\mathcal{O} x_1 x_2 \cdots x_N) &= \langle P_\mathcal{O} P_{\mathcal{O}_N} x_1 x_2 \cdots x_N S_a, S_a\rangle \\
&= \langle U_{\mathcal{O}_N} \left(x_1 \otimes 1 \otimes \cdots \otimes x_N \otimes 1\right) U^*_{\mathcal{O}_N} S_a, S_a\rangle \\
&= \langle x_1 S_{t_{n_1} - s_{n_1}}, S_{t_{n_1} - s_{n_1}}\rangle \cdots \langle x_N S_{t_{n_N} - s_{n_N}}, S_{t_{n_N} - s_{n_N}}\rangle \\
&= \omega(x_1)\omega(x_2)\cdots\omega(x_N),
\end{aligned}$$

where $H_{\mathcal{O}^c_N} = \otimes_{k=0}^N H_{s_{n_{k+1}} - t_{n_k}}$ with $t_{n_0} = 0$ and $s_{n_{N+1}} = a$. This shows that ω is a product pure state of $\bigotimes_{i=1}^N P_\mathcal{O} \mathcal{A}(I_{n_i}) \subset \mathcal{A}^H(\mathcal{O})$ for all N. Therefore $\mathcal{A}^H(\mathcal{O})P_L$ is a type I_∞ factor.

The other statement, when $t_n < s_{n+1}$, follows from Theorem 4.15.

(2) For an interval I, denote by $H_k(I) = \left[f_1 \wedge f_2 \wedge \cdots \wedge f_k : f_i \in L^2(I, \mathsf{k}^\mathbb{C})\right]$, the k-particle space of the antisymmetric Fock space of $L^2(I)$, and k is the multiplicity space of the Clifford flow. Define

$$H_\mathcal{O} = \left[\Omega, \xi_{n_1} \wedge \xi_{n_2} \wedge \cdots \wedge \xi_{n_N} : \xi_{n_i} \in H_{2k_i}(I_{n_i}), \;\; k_i, n_i, N \in \mathbb{N}\right].$$

It is not difficult to verify that $\mathcal{A}^{H^e}(\mathcal{O})$ is nothing but $B(H_\mathcal{O})$. □

We denote by $\mathcal{A}^\gamma(\mathcal{O})$ the local algebra associated with the super-product system H^γ of an E$_0$-semigroup γ.

Proposition 4.3. *Let γ and β be two E_0-semigroups and $\mathcal{O} \subseteq (0, a)$. Then*

$$\mathcal{A}^{\gamma\otimes\beta}(\mathcal{O}) = \mathcal{A}^\gamma(\mathcal{O}) \otimes \mathcal{A}^\beta(\mathcal{O}).$$

Proof. Thanks to Proposition 3.7, the above proposition holds true for intervals. For elementary sets, it follows from relation (1) in Section 2, and hence for any open set. □

From the above proposition it follows immediately, thanks to Lemma 4.2(2), that $\mathcal{A}^{\theta\otimes\alpha}(\mathcal{O})$ is a type I factor if and only if $\mathcal{A}^\theta(\mathcal{O})$ is a type I factor, for any open $\mathcal{O} \subseteq \; [0, a]$.

It is shown in [IS$_1$] that there exists a "one-parameter continuous family of off-white noises, whose spectral density functions converge to 1 at infinity" such that the associated family of generalized CCR flows $\{\theta^\lambda : \lambda \in (0, \frac{1}{2}]\}$ contains mutually non-cocycle-conjugate E_0-semigroups. This is accomplished by producing an open set $\mathcal{O}$, for any given $\lambda_1, \lambda_2 \in (0, \frac{1}{2}]$ such that $\mathcal{A}^{\theta^{\lambda_1}}(\mathcal{O})$ is a type III factor, but $\mathcal{A}^{\theta^{\lambda_2}}(\mathcal{O})$ is a type I factor (see [IS_1, Theorem 8.8]). In the following theorem let α be any one of the α^n with fixed n, then from the preceding discussions we have the following theorem.

Theorem 4.4. *The family of E_0-semigroups $\{\theta^\lambda \otimes \alpha : \lambda \in (0, \frac{1}{2}]\}$ is mutually non-cocycle-conjugate.*

When the "spectral density" converges to ∞ at ∞, the local algebras $\mathcal{A}(\mathcal{O})$ are not useful in distinguishing the associated generalized CCR flows. Tsirelson used the lim inf and lim sup of subspaces of the sum system, associated with elementary sets, to distinguish those generalized CCR flows (see [Ts1, Section 13]). Tsirelson's invariants can be equivalently described by the lim sup of local von Neumann algebras associated with elementary sets, as shown in [BhS, Section 3] (which was called the lim inf there, as remarked in the following definition).

Definition 4.5. For a sequence of von Neumann algebras $\mathcal{A}_n \subseteq B(H)$, define

$$\limsup \mathcal{A}_n = \Big\{T \in B(H) : \exists\, n_k \uparrow \infty,\ \ T_{n_k} \in \mathcal{A}_{n_k} \text{ such that } \operatorname*{w-lim}_{k \mapsto \infty} T_{n_k} = T\Big\}'',$$

where $\text{w-lim}_{k\mapsto\infty} T_{n_k}$ is the limit in the weak operator topology. (We realized this should be termed lim sup rather than lim inf as initially defined in [BhS, Section 3].) Also define

$$\liminf \mathcal{A}_n = \Big\{T \in B(H) : \exists\, T_n \in \mathcal{A}_n \text{ such that } \operatorname*{s-lim}_{n\to\infty} T_n = T,\ \ \operatorname*{s-lim}_{n\to\infty} T_n^* = T^*\Big\}'',$$

where the limits of the sequences $\{T_n\}$ and $\{T_n^*\}$ are in the strong operator topology.

Since the local algebras $\mathcal{A}^H(\mathcal{O})$ for super-product systems are not proper von Neumann subalgebras of $B(H_a)$, we need to modify the definition slightly. For an elementary open set $\mathcal{O} \subseteq [0,1]$, define

$$\tilde{\mathcal{A}}^H(\mathcal{O}) = \mathcal{A}^H(\mathcal{O})'' \cap B(H_1) = \mathcal{A}^H(\mathcal{O}) \oplus \mathbb{C}\,(1 - P_{\mathcal{O}})\,.$$

For product systems, $\tilde{\mathcal{A}}^H(\mathcal{O}) = \mathcal{A}^H(\mathcal{O})$. Given any sequence of elementary open sets $\mathcal{O}_n \subseteq [0,1]$, $\limsup \tilde{\mathcal{A}}^H(\mathcal{O}_n) \subseteq B(H_1)$ is an invariant for the super-product system $H = (H_t, U_{s,t})$.

Lemma 4.6. *For a sequence of von Neumann algebras $\mathcal{A}_n \subseteq B(H)$,*

$$\limsup \mathcal{A}_n \subseteq (\liminf \mathcal{A}_n')'\,.$$

Proof. Suppose $T \in \limsup \mathcal{A}_n$ and $S \in \liminf A_n'$, so that there exists a subsequence $T_{n_k} \in \mathcal{A}_{n_k}$ such that $T_{n_k} \to T$ weakly, and there exists $S_n \in A_n'$ such that $(S_n, S_n^*) \mapsto (S, S^*)$ strongly. Then for any $\xi, \eta \in H$, we have

$$|\langle TS\xi, \eta\rangle - \langle T_{n_k} S_{n_k}\xi, \eta\rangle| \leq |\langle S\xi, T^*\eta\rangle - \langle S\xi, T_{n_k}^*\eta\rangle| + \|S\xi - S_{n_k}\xi\| \|T_{n_k}^*\eta\|,$$
$$|\langle ST\xi, \eta\rangle - \langle S_{n_k} T_{n_k}\xi, \eta\rangle| \leq |\langle T\xi, S^*\eta\rangle - \langle T_{n_k}\xi, S^*\eta\rangle| + \|S^*\eta - S_{n_k}^*\eta\| \|T_{n_k}\xi\|.$$

Since $\{\|T_{n_k}^*\eta\|\}$ and $\{\|T_{n_k}\xi\|\}$ are bounded we have

$$\langle TS\xi, \eta\rangle = \lim_k \langle T_{n_k} S_{n_k}\xi, \eta\rangle = \lim_k \langle S_{n_k} T_{n_k}\xi, \eta\rangle = \langle ST\xi, \eta\rangle \quad \forall\, \xi, \eta \in H.$$

□

For an open set $\mathcal{O} \subseteq [0,1]$ we denote by $\mathcal{O}^c$ the interior of the complement in $[0,1]$. Since we are dealing with L^2-spaces with respect to Lebesgue measure, end points of the intervals do not matter. As before, H^e denotes the super-product system associated with Clifford flow of any fixed rank.

Proposition 4.7. *Let $\{\mathcal{O}_n : n \in \mathbb{N}\}$ be a sequence of elementary sets contained in $[0,1]$ such that $|\mathcal{O}_n| \to 0$. Then*

$$\liminf \tilde{\mathcal{A}}^{H^e}(\mathcal{O}_n)' = B(H_1) \quad \textit{and} \quad \limsup \tilde{\mathcal{A}}^{H^e}(\mathcal{O}_n) = \mathbb{C}.$$

Proof. Set $\Gamma_a^e(L^2(\mathcal{O}, \mathsf{k}^{\mathbb{C}})) = [\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_{2m};\ \xi_1, \xi_2, \ldots, \xi_{2m} \in L^2(\mathcal{O}, \mathsf{k}^{\mathbb{C}}),\ m \in \mathbb{N}_0]$, and when $m = 0$ the wedge product is just the vacuum vector Ω. The map

$$\begin{aligned} V_{\mathcal{O}}((\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_{2m}) \otimes (\eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_{2m'})) \\ = \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_{2m} \wedge \eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_{2m'}, \end{aligned}$$

where $\xi_1, \xi_2, \ldots, \xi_{2m} \in L^2(\mathcal{O}, \mathsf{k}^{\mathbb{C}})$, $\eta_1, \eta_2, \ldots, \eta_{2m'} \in L^2(\mathcal{O}^c, \mathsf{k}^{\mathbb{C}})$, extends to an isometry between $\Gamma_a^e(L^2(\mathcal{O}, \mathsf{k}^{\mathbb{C}})) \otimes \Gamma_a^e(L^2(\mathcal{O}^c, \mathsf{k}^{\mathbb{C}})) \mapsto H_1^e$. Define

$$\mathcal{B}(\mathcal{O}) = V_{\mathcal{O}} \left(B(\Gamma_a^e(L^2(\mathcal{O}, \mathsf{k}^{\mathbb{C}}))) \otimes 1_{\Gamma_a^e(L^2(\mathcal{O}^c, \mathsf{k}^{\mathbb{C}}))} \right) V_{\mathcal{O}}^* \quad \text{and} \quad \tilde{\mathcal{B}}(\mathcal{O}) = \mathcal{B}(\mathcal{O})''.$$

Since $|\mathcal{O}_n| \to 0$, for any $f \in L^2((0,1), \mathsf{k}^{\mathbb{C}})$, we have $1_{\mathcal{O}_n^c} f \to f$. Using this it is easy to verify that $\liminf \tilde{\mathcal{B}}(\mathcal{O}_n^c) = B(H_1)$.

Notice that for any elementary set $\mathcal{O} = \cup_{i=1}^N (s_i, t_i)$,

$$\begin{aligned} &U_{\mathcal{O}} \left(\Phi_{s_1 - t_0} \otimes H_{t_1 - s_1} \otimes \cdots \otimes \Phi_{s_N - t_{N-1}} \otimes H_{t_N - s_N} \otimes \Phi_{s_{N+1} - t_N} \right) \\ &\quad \subseteq V_{\mathcal{O}} \left(\Gamma_a^e(L^2(\mathcal{O}, \mathsf{k}^{\mathbb{C}})) \otimes \Phi \right), \end{aligned}$$

where $\Phi_{s_i - t_i}$ denote the vacuum vectors in H_{s-i-t_i} and $U_{\mathcal{O}}$ is the canonical isometry as in (6). This consequently implies that $\tilde{\mathcal{A}}^{H^e}(\mathcal{O}) \subseteq \tilde{\mathcal{B}}(\mathcal{O})$ for any elementary open set $\mathcal{O} \subseteq [0,1]$. Hence we have

$$\tilde{\mathcal{B}}(\mathcal{O}_n^c) \subseteq \tilde{\mathcal{B}}(\mathcal{O}_n)' \subseteq \tilde{\mathcal{A}}^{H^e}(\mathcal{O}_n)' \quad \forall\, n \in \mathbb{N}.$$

So we have $\liminf \tilde{\mathcal{A}}^{H^e}(\mathcal{O}_n)' = B(H_1)$. Now it follows from Lemma 4.6 that $\limsup \tilde{\mathcal{A}}^{H^e}(\mathcal{O}_n) = \mathbb{C}$. □

The Arveson product system of Hilbert spaces associated with generalized CCR flows is described by sum systems. Since we do not want to recall the definitions and restate the facts, which are used only in the following proposition, we ask the reader to refer to [BhS, Section 1] and [IS_1, Section 3] for the definition of sum systems and for the construction of product systems from sum systems. Also, for definitions/facts/notation regarding the lim inf and lim sup of Hilbert subspaces, which we use in the proof of the following proposition, we ask the reader to refer to [BhS, Section 3] (they were originally defined and used in [Ts1]). The lim inf and lim sup of Hilbert subspaces are defined in [BhS, Definition 3.2]. We provide an exact reference to the facts we use in the following proposition.

For product systems arising from sum systems also, while dealing with local algebras, end points of intervals do not matter (see [BhS, Corollary 25]).

Proposition 4.8. *Let $H = (H_t, U_{s,t})$ be the product system constructed from a sum system $(G_{s,t}, S_t)_{s,t\in(0,\infty)}$. For a sequence of elementary sets $\mathcal{O}_n \subseteq [0,1]$,*

$$\liminf \mathcal{A}^H(\mathcal{O}_n) = \{W_0(x+iy) : x \in \liminf G_{\mathcal{O}_n},\ y \in \liminf G^{\perp}_{\mathcal{O}_n^c}\}'',$$

$$\limsup \mathcal{A}^H(\mathcal{O}_n) = \{W_0(x+iy) : x \in \limsup G_{\mathcal{O}_n},\ y \in \limsup G^{\perp}_{\mathcal{O}_n^c}\}''.$$

Further,

$$\limsup \mathcal{A}^H(\mathcal{O}_n) = \big(\liminf \big(\mathcal{A}^H(\mathcal{O}_n)'\big)\big)' = \big(\liminf \mathcal{A}^H(\mathcal{O}_n^c)\big)'.$$

Proof. For an elementary set $\mathcal{O} \subseteq [0,1]$,

$$\mathcal{A}^H(\mathcal{O}) = \{W_0(x+iy) : x \in G_{\mathcal{O}_n},\ y \in G^{\perp}_{\mathcal{O}_n^c}\}'' \quad \text{and} \quad \mathcal{A}^{\mathcal{H}}(\mathcal{O})' = \mathcal{A}^H(\mathcal{O}^c)$$

(see the discussion just before [BhS, Lemma 3.2]). The strong continuity of the Weyl representation $x \mapsto W_0(x)$ (see [Par, Proposition 20.1]) implies

$$\{W_0(x+iy) : x \in \liminf G_{\mathcal{O}_n},\ y \in \liminf G^{\perp}_{\mathcal{O}_n^c}\}'' \subseteq \liminf \mathcal{A}^H(\mathcal{O}_n).$$

On the other hand,

$$\begin{aligned}
&\{W_0(x+iy) : x \in \liminf G_{\mathcal{O}_n},\ y \in \liminf G^{\perp}_{\mathcal{O}_n^c}\}' \\
&\quad = \big\{W_0(x+iy) : x \in \big(\liminf G^{\perp}_{\mathcal{O}_n^c}\big)^{\perp},\ y \in \big(\liminf G_{\mathcal{O}_n}\big)^{\perp}\big\}'' \\
&\quad = \{W_0(x+iy) : x \in \limsup G_{\mathcal{O}_n^c},\ y \in \limsup G^{\perp}_{\mathcal{O}_n}\}'' && \text{(by [BhS, Lemma 3.1])} \\
&\quad \subseteq \limsup \{W_0(x+iy) : x \in G_{\mathcal{O}_n^c},\ y \in G^{\perp}_{\mathcal{O}_n}\}'' && \text{(by [BhS, Lemma 3.2(i)])} \\
&\quad \subseteq \big(\liminf \{W_0(x+iy) : x \in G_{\mathcal{O}_n^c},\ y \in G^{\perp}_{\mathcal{O}_n}\}'\big)' && \text{(by Lemma 4.6)} \\
&\quad = \big(\liminf \mathcal{A}^H(\mathcal{O}_n)\big)'.
\end{aligned}$$

Hence $\liminf \mathcal{A}^H(\mathcal{O}_n) = \{W_0(x+iy) : x \in \liminf G_{\mathcal{O}_n}, y \in \liminf G^{\perp}_{\mathcal{O}^c_n}\}''$. The proof of the corresponding statement for $\limsup \mathcal{A}^H(\mathcal{O}_n)$ is as follows: one inclusion follows from [BhS, Lemma 3.2(i)] and the other inclusion can be proved by flipping $\liminf \mathcal{A}^H(\mathcal{O}_n)$ with $\limsup \mathcal{A}^H(\mathcal{O}_n)$ in the above arguments. The remaining statements follow from above and [BhS, Lemma 3.1]. □

Let $(\{T^1_t\}, \{T^2_t\})$ be a perturbation pair and θ be the associated generalized CCR flow on $B(\Gamma_s(G^{\mathbb{C}}))$. Let $j : G^{\mathbb{C}} \mapsto G^{\mathbb{C}}$ be the antiunitary $x+iy \mapsto y+ix$ for $x, y \in G$, and $\Gamma(j) : \Gamma_s(G^{\mathbb{C}}) \mapsto \Gamma_s(G^{\mathbb{C}})$ be the second quantization of j defined by $\Gamma(j)(\varepsilon(\xi)) = \varepsilon(j\xi)$ and extended antilinearly to $\Gamma_s(G^{\mathbb{C}})$. Then

$$\Gamma(j)W(x+iy)\Gamma(j) = W(y+ix) \quad \forall\, x, y \in G.$$

By the discussion in Example 2.9, the dual E$_0$-semigroup of θ on $B(\Gamma_s(G^{\mathbb{C}}))$ is conjugate to the E$_0$-semigroup $\overline{\theta}$, given by

$$\begin{aligned}\overline{\theta}_t(W(x+iy)) &= \Gamma(j)\theta_t(\Gamma(j)W(x+iy)\Gamma(j))\Gamma(j)\\ &= W(T^2_t x + iT^1_t y) \quad \forall\, x, y \in G.\end{aligned}$$

So $\overline{\theta}$ is the generalized CCR flow given by the perturbation pair $(\{T^2_t\}, \{T^1_t\})$, and in particular the associated Arveson product system $(\overline{H}_t, \overline{U}_{s,t})$ is also given by a sum system, say $(\overline{G}_{s,t}, \overline{S_t})$.

Corollary 4.9. *Let θ be a generalized CCR flow. Then*

$$\limsup \tilde{\mathcal{A}}^\theta(\mathcal{O}_n) = \big(\liminf\big(\tilde{\mathcal{A}}^\theta(\mathcal{O}_n)'\big)\big)'.$$

Proof. Let $(H_t, U_{s,t})$ be the Arveson product system of θ. By Remark 3.6, the super-product system of θ is given by $(H_t \otimes \overline{H_t}, U_{s,t} \otimes \overline{U_{s,t}})$, which arises from the sum system $\big(G_{s,t} \oplus \overline{G}_{s,t}, S_t \oplus \overline{S_t}\big)$. Also, for any two sequences of Hilbert subspaces $\{G_n\}$ and $\{F_n\}$, it is easy to see that $\liminf(G_n \oplus F_n) = \liminf G_n \oplus \liminf F_n$ and $\limsup(G_n \oplus F_n) = \limsup G_n \oplus \limsup F_n$. Now the corollary follows from the above Proposition 4.8. □

Proposition 4.10. *Let a sequence of elementary sets $\{\mathcal{O}_n \subseteq [0,1] : n \in \mathbb{N}\}$ be such that $|\mathcal{O}_n| \to 0$ and let θ be any generalized CCR flow. Then $\limsup\big(\mathcal{A}^\theta(\mathcal{O}_n) \otimes \mathcal{A}^\alpha(\mathcal{O}_n)\big)$ is $\mathbb{C}1$ if and only if $\limsup \mathcal{A}^\theta(\mathcal{O}_n) = \mathbb{C}1$.*

Proof. Let $\mathcal{A}_n$, $\mathcal{B}_n$ be any two families of von Neumann algebras. It immediately follows if $\limsup(\mathcal{A}_n \otimes \mathcal{B}_n) = \mathbb{C}$, then both $\limsup \mathcal{A}_n = \mathbb{C}1 = \limsup \mathcal{B}_n$, since $\limsup \mathcal{A}_n \otimes \limsup \mathcal{B}_n \subseteq \limsup(\mathcal{A}_n \otimes \mathcal{B}_n)$.

Also, $\liminf \mathcal{A}'_n \otimes \liminf \mathcal{B}'_n \subseteq \liminf(\mathcal{A}'_n \otimes \mathcal{B}'_n)$. So by Lemma 4.6, we have

$$\limsup(\mathcal{A}_n \otimes \mathcal{B}_n) \subseteq (\liminf(\mathcal{A}'_n \otimes \mathcal{B}'_n))' \subseteq (\liminf \mathcal{A}'_n \otimes \liminf \mathcal{B}'_n)'.$$

If $\limsup \mathcal{A}^\theta(\mathcal{O}_n) = \mathbb{C}1$ and $|\mathcal{O}_n| \to 0$ then, thanks to Corollary 4.9 and Proposition 4.7, both $\liminf \big(\mathcal{A}^\theta(\mathcal{O}_n)'\big) = B(H_1^\theta)$ and $\liminf \big(\mathcal{A}^\alpha(\mathcal{O}_n)'\big) = B(H_1^e)$. Hence $\limsup \big(\mathcal{A}^\theta(\mathcal{O}_n) \otimes \mathcal{A}^\alpha(\mathcal{O}_n)\big) = \mathbb{C}$. □

For $r > 0$, let σ_r be a smooth positive even function with $\sigma_r(\lambda) = \log^r |\lambda|$ for large $|\lambda|$. Then σ_r is the spectral density function of an off-white noise, and gives rise to a family of generalized CCR flows $\{\theta^r : r > 0\}$. In [Ts1], a sequence of elementary sets (with Lebesgue measure converging to 0) is produced for any given $r_1 \neq r_2$, so that $\limsup \mathcal{A}^{\theta^{r_1}}(\mathcal{O}_n) = \mathbb{C}$ but $\limsup \mathcal{A}^{\theta^{r_2}}(\mathcal{O}_n)$ is nontrivial. (Tsirelson produced invariants through sum systems, but this is equivalent to the above statement, as explained in [BhS, Section 3].) In the following theorem let α be any one of the α^n with fixed n; then thanks to Proposition 4.10 we have the following theorem.

Theorem 4.11. *The family of E_0-semigroups $\{\theta^r \otimes \alpha : r > 0\}$ is mutually non-cocycle-conjugate.*

§4.3. Tensoring with Toeplitz CAR flows

To distinguish Toeplitz CAR flows discussed in [IS_2], type I factorizations were used as invariants, as defined by Araki and Woods [AW_1]. Here we define these invariants with respect to super-product systems and use them to distinguish E_0-semigroups on hyperfinite II_∞ factors, given by tensor products of Toeplitz CAR flows with α. Throughout this subsection, every index set (indexing a type I factorization) is assumed to be countable.

Definition 4.12. Let H be a Hilbert space. We say that a family of type I subfactors $\{\mathrm{M}_\lambda\}_{\lambda\in\Lambda}$ of $B(H)$ is a *type I factorization* of $B(H)$ if

(i) $\mathrm{M}_\lambda \subset \mathrm{M}'_\mu$ for any λ, $\mu \in \Lambda$ with $\lambda \neq \mu$;

(ii) $B(H) = \bigvee_{\lambda\in\Lambda} \mathrm{M}_\lambda$.

We say that a type I factorization $\{\mathrm{M}_\lambda\}_{\lambda\in\Lambda}$ is a *complete atomic Boolean algebra of type I factors* (abbreviated as CABATIF) if for any subset $\Gamma \subset \Lambda$, the von Neumann algebra $\bigvee_{\lambda\in\Gamma} \mathrm{M}_\lambda$ is a type I factor.

Two type I factorizations $\{\mathrm{M}_\lambda\}_{\lambda\in\Lambda}$ of $B(H)$ and $\{\mathrm{N}_\mu\}_{\mu\in\Lambda'}$ of $B(H')$ are said to be unitarily equivalent if there exist a unitary U from H onto H' and a bijection $\sigma : \Lambda \to \Lambda'$ such that $U\mathrm{M}_\lambda U^* = \mathrm{N}_{\sigma(\lambda)}$.

Let $A = \{a_n\}_{n=0}^\infty$ be a strictly increasing sequence of nonnegative numbers starting from 0 and converging to $a < \infty$. Define $P_N^A = U_N U_N^*$ where

$$U_N : \bigotimes_{n=0}^{N-1} H_{a_{n+1}-a_n} \otimes H_{a-a_N} \mapsto H_a$$

is the canonical isometry uniquely determined by the associativity axiom of the super-product system. Clearly $\{P_N^A : N \in \mathbb{N}\}$ is a decreasing family of projections in N. Define $P^A = \bigwedge_{n=1}^{\infty} P_N^A$. (We write $P^{A,\theta}$ to remember the E_0-semigroup.)

Lemma 4.13. *Let $H = (H_t, U_{s,t})$ be a super-product system that can be embedded into a product system, and let $A = \{a_n\}_{n=0}^{\infty}$ be a strictly increasing sequence of nonnegative numbers starting from 0 and converging to $a < \infty$. Then $\{P^A \mathcal{A}^H((a_n, a_{n+1}))\}_{n=0}^{\infty}$ is a type I factorization of $B(P^A H_a)$.*

Proof. Let $E = (E_t, V_{s,t})$ be a product system where the super-product system H can be embedded. Then $\{\mathcal{A}^E((a_n, a_{n+1}))\}_{n=0}^{\infty}$ is a type I factorization of $B(E(a))$ because

$$B(E_a) = \bigvee_{0<t<a} \mathcal{A}^E((0,t))$$

holds (see [Arv, Proposition 4.2.1]). Let Q^A be the orthogonal projection from E_a onto $P_A H_a$. Then $Q^A \mathcal{A}^E((a_n, a_{n+1}))Q^A = P^A \mathcal{A}^H((a_n, a_{n+1}))$. ☐

The following proposition is immediate, since

$$\bigvee_{\lambda\in\Gamma} (\mathrm{M}_\lambda^1 \otimes \mathrm{M}_\lambda^2) = \Big(\bigvee_{\lambda\in\Gamma} \mathrm{M}_\lambda^1\Big) \otimes \Big(\bigvee_{\lambda\in\Gamma} \mathrm{M}_\lambda^2\Big).$$

Proposition 4.14. *For two type I factorizations $\{\mathrm{M}_\lambda^1\}_{\lambda\in\Lambda}$ and $\{\mathrm{M}_\lambda^2\}_{\lambda\in\Lambda}$, $\{\mathrm{M}_\lambda^1 \otimes \mathrm{M}_\lambda^2\}_{\lambda\in\Lambda}$ is a CABATIF if and only if both $\{\mathrm{M}_\lambda^1\}_{\lambda\in\Lambda}$ and $\{\mathrm{M}_\lambda^2\}_{\lambda\in\Lambda}$ are CABATIFs.*

When $\{\mathrm{M}_\lambda\}_{\lambda\in\Lambda}$ is a type I factorization of $B(H)$, we say that a nonzero vector ξ is *factorizable* if, for any λ, there exists a minimal projection p_λ of M_λ such that $p_\lambda \xi = \xi$. Araki and Woods characterized a CABATIF as a type I factorization with a decomposable vector. One can find the following theorem in [AW_1, Lemma 4.3, Theorem 4.1].

Theorem 4.15 (Araki–Woods). *A type I factorization is a CABATIF if and only if it has a factorizable vector.*

As before, let $A=\{a_n\}_{n=0}^{\infty}$ be a strictly increasing sequence of nonnegative numbers starting from 0 and converging to $a < \infty$. When a super-product system H has a unit $\{S_t : t \geq 0\}$, then $P^A S_a = S_a$ and further, it gives a factorizable vector for the type I factorization $\{P^A \mathcal{A}^H((a_n, a_{n+1}))\}_{n=0}^{\infty}$, which is necessarily a CABATIF thanks to Theorem 4.15. So type I factorization associated with the super-product system of Clifford flow of any rank is a CABATIF for any sequence $A = \{a_n\}$. Now if θ is a Toeplitz CAR flow, then the type I factorization $\{P^{A,\theta\otimes\alpha} \mathcal{A}^{\theta\otimes\alpha}((a_n, a_{n+1}))\}_{n=0}^{\infty}$ is a CABATIF if and only if $\{P^{A,\theta} \mathcal{A}^{\theta}((a_n, a_{n+1}))\}_{n=0}^{\infty}$ is a CABATIF.

In [IS$_2$], an uncountable family of mutual non-cocycle-conjugate Toeplitz CAR flows $\{\theta^\nu : \nu \in (0, \frac{1}{4}]\}$ is constructed. This family is distinguished by providing a sequence $A = \{a_n\}_{n=0}^\infty$ for any given $\nu_1, \nu_2 \in (0, \frac{1}{4}]$, so that $\{P^A \mathcal{A}^{\theta^{\nu_1}}((a_n, a_{n+1}))\}_{n=0}^\infty$ is a CABATIF but $\{P^A \mathcal{A}^{\theta^{\nu_2}}((a_n, a_{n+1}))\}_{n=0}^\infty$ is not a CABATIF. From the above discussions we have the following theorem.

Theorem 4.16. *The family of E_0-semigroups $\{\theta^\nu \otimes \alpha : \nu \in (0, \frac{1}{4}]\}$ is mutually non-cocycle-conjugate.*

If a generalized CCR flow (or a Toeplitz CAR flow) is fixed, it is still open to show that it leads to non-cocycle-conjugate E_0-semigroups, when tensored with Clifford flows of different indices.

§5. CCR flows on hyperfinite type III factors

In this section we investigate a class of E_0-semigroups on hyperfinite type III factors arising from quasi-free representations of the CCR algebra. The structure of these representations was worked out in the early papers [Ar$_1$, Ar$_2$, DAn, Hol, ArY, AW$_2$]; in order to make this paper reasonably self-contained we include the relevant details.

For a complex Hilbert space K, there exists a universal C^*-algebra generated by unitaries $\{w_v : v \in \mathsf{K}\}$, subject to

$$w_u w_v = e^{-i\,\mathrm{Im}\langle u,v\rangle} w_{u+v} \quad (u, v \in \mathsf{K}),$$

known as the algebra of canonical commutation relations, or the CCR algebra, denoted by $\mathrm{CCR}(\mathsf{K})$ (see, e.g., [Pet]).

From here onwards, in the last two sections of this paper, k will denote a separable complex Hilbert space, with conjugation j. We denote the conjugation on $\mathsf{K} = L^2(\mathbb{R}_+; \mathsf{k})$ also by j, obtained as $(jf)(s) := jf(s)$ for all $s \geq 0$. Let $A \geq 1$ be a complex linear operator on K such that $T = \frac{1}{2}(A - 1)$ is injective. The state on $\mathrm{CCR}(\mathsf{K})$ determined by

$$\varphi_A(w_f) = e^{-\frac{1}{2}\langle f, Af\rangle} = e^{-\frac{1}{2}\|\sqrt{1+2T}f\|^2}$$

is known as the quasi-free state with symbol A. The corresponding GNS representation, on $\Gamma_s(\mathsf{K}) \otimes \Gamma_s(\mathsf{K})$, is given by

$$\pi_A(w(f)) = W_A(f) := W_0(\sqrt{1+T}f) \otimes W_0(j\sqrt{T}f).$$

It follows from [Ar$_1$] that this representation generates a factor $\mathrm{M}_A = \{\pi_A(w(f)) : f \in \mathsf{K}\}''$, for which the vacuum vector $\Omega = \varepsilon(0) \otimes \varepsilon(0)$ in $\Gamma_s(\mathsf{K}) \otimes \Gamma_s(\mathsf{K}) = \Gamma_s(\mathsf{K} \oplus \mathsf{K})$

is cyclic and separating. Under this representation, Ω induces the state φ_A. The factor is type I if and only if A^2-1 is trace class, and otherwise it is type III (see [Hol]).

Lemma 5.1. *Let $X \in B(L^2(\mathbb{R}_+;\mathsf{k}))$ be a positive, injective Toeplitz operator. Then the operators $\sqrt{X}T_t\sqrt{X}^{-1}$ extend to a family of isometries that is a strongly continuous semigroup.*

Proof. Since $\sqrt{X} \geq 0$ is injective, $\sqrt{X}^{-1}$ is closed and densely defined. For any $f \in \mathrm{Dom}(\sqrt{X}^{-1})$, we have

$$\left\|\sqrt{X}T_t\sqrt{X}^{-1}f\right\|^2 = \left\langle \sqrt{X}^{-1}f, T_t^*XT_t\sqrt{X}^{-1}f\right\rangle = \left\langle \sqrt{X}^{-1}f\sqrt{X}f\right\rangle = \|f\|^2,$$

so that $\sqrt{X}\,T_t\,\sqrt{X}^{-1}$ admits a unique isometric extension Y_t. For any $f \in \mathrm{Dom}(\sqrt{X}^{-1})$ it is clear that $Y_sY_tf = Y_{s+t}f$ and $Y_tf \to f$ as $t \to 0$, so the family $(Y_t)_{t\geq 0}$ is a strongly continuous semigroup of isometries. □

For any Toeplitz operator X, we denote the isometric extension of $\sqrt{X}T_t\sqrt{X}^{-1}$ by T_t^X. The following proposition ensures the existence of E_0-semigroups that we call Toeplitz CCR flows given by the Toeplitz operator A.

Proposition 5.2. *Let $A \geq 1$ be a Toeplitz operator on $L^2(\mathbb{R}_+;\mathsf{k})$ such that $A-1$ is injective. Then there exists a unique E_0-semigroup $\alpha^A = \{\alpha_t^A : t \geq 0\}$ on M_A defined by $\alpha_t^A(W_A(f)) = W_A(T_tf)$, where $(T_t)_{t\geq 0}$ is the semigroup of right shifts on $L^2(\mathbb{R}_+;\mathsf{k})$. Further, $\alpha^{A_1\oplus A_2} = \alpha^{A_1}\otimes\alpha^{A_2}$.*

Proof. Thanks to Lemma 5.1, we have the semigroup of isometries

$$Y = (Y_t)_{t\geq 0} = \begin{bmatrix} T_t^{1+T} & 0 \\ 0 & jT_t^Tj \end{bmatrix}$$

on $\mathsf{K}^{\oplus 2}$. It follows from [Arv, Proposition 2.1.3] that there exists a unique E_0-semigroup σ on $B(H)$ satisfying $\sigma_t(W_0(f)) = W_0(Y_tf)$ for all $f \in \mathsf{K}^{\oplus 2}$. Clearly M_A is an invariant subalgebra for σ, and by the density of $\pi_A(\mathrm{CCR}(\mathsf{K}))$, the restriction $\sigma_t|_{\mathrm{M}_A}$ is the unique E_0-semigroup satisfying the conditions of the proposition. □

Since A is Toeplitz, φ_A is a faithful, normal invariant state for each of these E_0-semigroups, and thus provides a canonical unit $S = (S_t)_{t\geq 0}$ associated with Ω. The modular conjugation with respect to Ω is given by $J_\Omega = \Gamma\left[\begin{smallmatrix} 0 & -j \\ -j & 0\end{smallmatrix}\right]$, and we have $J_\Omega W_A(f)J_\Omega = W_A'(f)$, where $W_A'(f) = W_0(\sqrt{T}f)\otimes W_0(j\sqrt{1+T}f)$. The dual E_0-semigroup α' on M_A' is given by $\alpha_t'(W_A'(f)) = W_A'(T_tf)$. The following proposition characterizes when S is a multiunit.

Proposition 5.3. *If $A \geq 1$ is a Toeplitz operator such that $A-1$ is injective, and α is the E_0-semigroup on M_A satisfying $\alpha_t(W_A(f)) = W_A(T_t f)$, then the canonical unit is a multiunit if and only if $A = 1_{L^2(\mathbb{R})} \otimes R$ for some $R \in B(\mathsf{k})$. Moreover, when this is the case, there exists a CCR flow σ on $B(H)$ extending both α and α'.*

Proof. One way is clear. Suppose S is a multiunit; then the modular group $(\sigma_s^\Omega)_{s\in\mathbb{R}}$ satisfies $\alpha_t = \sigma_{-s}^\Omega \circ \alpha_t \circ \sigma_s^\Omega$ for all $t \geq 0$, $s \in \mathbb{R}$ (see Proposition 2.8(iii)). Here, σ_s^Ω is the Bogoliubov automorphism associated with $T^{is}(1+T)^{-is}$. Hence $T^{is}(1+T)^{-is}$ commutes with T_t for all $t \geq 0$ and so it is of the form $1_{L^2(\mathbb{R}_+)} \otimes U_s$. By considering the (analytic) generator, we infer that $T(1+T)^{-1} = 1_{L^2(\mathbb{R})} \otimes X$, for some densely defined self-adjoint operator X on k. For $f \in \mathrm{Dom}((1+T)^{-1})$ we have

$$(1+T)^{-1} f = (1 - T(1+T)^{-1}) f = 1_{L^2(\mathbb{R}_+)} \otimes (1-X) f.$$

This implies that $(1-X)$ has a bounded inverse and $T = 1_{L^2(\mathbb{R}_+)} \otimes \left(1 - (1-X)^{-1}\right)$.

When $A = 1_{L^2(\mathbb{R})} \otimes R$, then $T_t^{1+T} = T_t = T_t^T$, and the CCR flow given by $T_t \oplus T_t$ extends both α and α'. □

For the rest of the paper we restrict to this class of E_0-semigroups, where the canonical unit is a multiunit. Since the Toeplitz part of A is trivial, these E_0-semigroups are simply called CCR flows. We denote the CCR flow given by $A = 1 \otimes R$ by $\alpha^{(R)}$. Since $L^2(\mathbb{R}_+)$ is infinite-dimensional, $\mathrm{Tr}(I \otimes (R^2 - 1)) < \infty$ if and only if $\mathrm{Tr}(R^2 - 1) = \|\sqrt{R^2-1}\|_{\mathrm{HS}} = 0$, i.e., $R^2 - 1 = 0$. Here, $\|\cdot\|_{\mathrm{HS}}$ refers to the Hilbert–Schmidt norm. By our ($A-1$ is injective) assumption, we get $R \neq 1$, so M_A is a type III factor.

The second half of the above proposition shows that the super-product system for α, α' is isomorphic to the completely spatial product system of index $2\dim \mathsf{k}$; hence $\mathrm{Ind}_c(\alpha) = 2\dim \mathsf{k}$. If $\dim \mathsf{k} = n$, we say that the corresponding E_0-semigroup is a CCR flow on the type III factor M_A of rank n. Since R is injective, $\mathrm{rank}(R) = \dim(\mathsf{k})$. The following corollary is immediate.

Corollary 5.4. *CCR flows on hyperfinite type III factors associated with operators of the form $A_i = 1 \otimes R_i$, $i = 1, 2$ are not cocycle conjugate if R_1 and R_2 have different ranks.*

In order to classify these semigroups further, we must determine when the algebras M_A are isomorphic. For this we require the following lemma, whose proof is easy. Here and elsewhere, $\sigma(X)$ denotes the spectrum of X.

Lemma 5.5. *Let X, Y be closed, densely defined operators of the form $X = \sum_{i=1}^\infty \lambda_i P_i$, $Y = \sum_{j=1}^\infty \mu_j Q_j$, where $\{P_i\}_{i=1}^\infty$ and $\{Q_j\}_{j=1}^\infty$ are families of mutually orthogonal projections. Then $\sigma(X \otimes Y) = \overline{\sigma(X)\sigma(Y)}$.*

The following theorem may be gleaned from [AW$_2$]; for the reader's convenience we include the details.

Theorem 5.6. *Let $A = I \otimes R \geq 1$ be such that $T = (A-1)/2$ is injective. Then there are the following three possibilities:*

(i) A has discrete spectrum and there exists $\lambda \in \sigma\left((1+T)^{-1}T\right) \subseteq (0,1)$ such that the eigenvalues of $(1+T)^{-1}T$ all have the form $\lambda_i = \lambda^{d_i}$ for some $d_i \in \mathbb{N}$.

(ii) A has discrete spectrum, but is not of the form (i).

(iii) A contains nonempty purely continuous spectrum (see [Kat, X.1.1]).

In case (i), M_A is the hyperfinite III$_\lambda$ factor, whereas in all other cases, M_A is the hyperfinite III$_1$ factor.

Proof. By definition, A is one of the three types described above, so it remains to show that the factors are as claimed. In [AW$_2$, Section 12], the following is observed:

(a) If A has discrete spectrum then M_A is an infinite tensor product of factors of type I (ITPFI), and so hyperfinite.

(b) If A has discrete spectrum and λ is a limit point of $\sigma((1+T)^{-1}T)$, then $\lambda \in r_\infty(\mathrm{M}_A)$, the asymptotic ratio set of M_A.

(c) If A has nonempty purely continuous spectrum then M_A is isomorphic to an ITPFI and $r_\infty(\mathrm{M}_A) = \mathbb{R}_+$.

By [Con, Theorem 3.6.1], $r_\infty(\mathrm{M}_A) = S(\mathrm{M}_A)$ for ITPFI factors, so the third point is equivalent to M_A being hyperfinite type III$_1$. If A has discrete spectrum then, as $A = I \otimes R$, all eigenvalues have infinite multiplicity, so all points in the spectrum are limit points. Thus, if A has discrete spectrum and satisfies (ii), then by (b) $r_\infty(\mathrm{M}_A) \neq \{0\} \cup \{\lambda^n : n \in \mathbb{Z}\}$ for any $\lambda \in (0,1)$, and clearly $r_\infty(\mathrm{M}_A) \neq \{0,1\}$, so $S(\mathrm{M}_A) = \mathbb{R}_+$ and M_A is type III$_1$. If A satisfies (i) then, again by (b), $r_\infty(\mathrm{M}_A) \supseteq \{0\} \cup \{\lambda^n : n \in \mathbb{Z}\}$ and we are left to show that the modular spectrum of M contains nothing further. We simply show $\sigma(\Delta_\Omega) \supseteq \{0\} \cup \{\lambda^n : n \in \mathbb{Z}\}$. Since Δ_Ω is the sum of tensor powers of $(1+T)^{-1}T \oplus T^{-1}(1+T)$, by Lemma 5.5 its spectrum is the closure of $\bigcup_{n\in\mathbb{Z}}^\infty \sigma((1+T)^{-1}T)^n$, i.e., $\{0\} \cup \{\lambda^n : n \in \mathbb{Z}\}$. ☐

Remark 5.7. When k is one-dimensional, only (i) can occur. When k has finite dimension, (ii) occurs if and only if $(1+T)^{-1}T$ has eigenvalues λ_i, λ_j with $\log \lambda_i / \log \lambda_j \notin \mathbb{Q}$. In infinite dimensions there are further examples of case (ii) coming from sequences of rational powers with strictly increasing denominators, e.g., $(\lambda^{n/(n+1)})_{n\in\mathbb{N}}$. Clearly, case (iii) can occur only if k is infinite-dimensional.

In particular, thanks to Corollary 5.4, $A = \frac{1+\lambda}{1-\lambda} I_{L^2(\mathbb{R}_+;\mathsf{k})}$ gives infinitely many non-cocycle-conjugate E_0-semigroups on each hyperfinite III_λ factor with $0<\lambda<1$ distinguished by their rank. Distinguishing between two CCR flows of equal rank is more complicated, and we take up a detailed analysis in the next section.

§6. Characterizing cocycle conjugacy for CCR flows

In this section we show that there are uncountably many non-cocycle-conjugate E_0-semigroups on each hyperfinite III_λ factor with $\lambda \in (0, 1]$. The proof relies upon the precise form of the gauge group and a detailed analysis of its fate under cocycle perturbations.

Proposition 6.1. *Let $A = I \otimes R \geq 1$ be such that $A - 1$ is injective and consider the corresponding CCR flow α on M_A. Then every element of the gauge group $G(\alpha)$ has the form*

$$U_t = e^{i\lambda t} W_A(1_{(0,t)} \otimes \xi) \quad (t \geq 0)$$

for some $\lambda \in \mathbb{R}$, $\xi \in \mathsf{k}$. As a topological group, $G(\alpha)$ is isomorphic to the central extension of $(\mathsf{k}, +)$ by the $\mathbb{R}$-valued 2-cocycle $\omega(\xi, \eta) = -\operatorname{Im}\langle \xi, \eta \rangle$.

Proof. Let θ be the CCR flow on $B(H)$ mentioned in Proposition 5.3, which extends both α and α'. Since $\alpha_t(\mathrm{M})' \cap \mathrm{M} \subseteq \theta_t(B(H))'$, every gauge cocycle for α is also a gauge cocycle for θ, and $G(\alpha)$ is the subgroup of $G(\theta)$ consisting of cocycles living in M_A. From [Arv, Section 3.8], it follows that $G(\theta)$ consists of cocycles of the form

$$U_t(\lambda, \xi, V) = e^{i\lambda t} W_0(1_{(0,t)} \otimes \xi)(\Gamma(I_{L^2[0,t]} \otimes V) \otimes \Gamma(I_{L^2([t,\infty);\mathsf{k}^{\oplus 2})})) \quad \forall\, t \geq 0,$$

where $\lambda \in \mathbb{R}$, $\xi \in \mathsf{k}^{\oplus 2}$ and $V \in \mathcal{U}(\mathsf{k}^{\oplus 2})$.

If $U_t(\lambda, \xi, V) \in \mathrm{M}_A$ then for any $\eta \in \mathsf{k}$, we have

$$W'_A(1_{(0,t)} \otimes \eta) U_t(\lambda, \xi, V) = U_t(\lambda, \xi, V) W'_A(1_{(0,t)} \otimes \eta).$$

Evaluating on Ω we get

$$W'_A(1_{(0,t)} \otimes \eta) W_0(1_{(0,t)} \otimes \xi)\Omega = W_0(1_{(0,t)} \otimes \xi) W_0((I \otimes V)\Sigma' \iota(1_{(0,t)} \otimes \eta))\Omega,$$

where $\Sigma' := \begin{bmatrix} \sqrt{T} & 0 \\ 0 & j\sqrt{1+T}j \end{bmatrix}$, $\iota(f) := \begin{pmatrix} f \\ jf \end{pmatrix}$. Thanks to the linear independence of exponential vectors, comparing both sides,

$$\begin{pmatrix} \sqrt{R}\eta \\ j\sqrt{1+R}\eta \end{pmatrix} = V \begin{pmatrix} \sqrt{R}\eta \\ j\sqrt{1+R}\eta \end{pmatrix},$$

$$\left\langle \begin{pmatrix} \sqrt{R}\eta \\ j\sqrt{1+R}\eta \end{pmatrix}, \xi \right\rangle = \left\langle \xi, V \begin{pmatrix} \sqrt{R}\eta \\ j\sqrt{1+R}\eta \end{pmatrix} \right\rangle,$$

for all $\eta \in \mathsf{k}$. The first equation implies that the unitary V is the identity on the real linear subspace $L = \left\{ \begin{pmatrix} \sqrt{R}\eta \\ j\sqrt{1+R}\eta \end{pmatrix} : \eta \in \mathsf{k} \right\}$. But the complex Hilbert space spanned by L is the whole of $\mathsf{k} \oplus \mathsf{k}$ (consider $\begin{pmatrix} \sqrt{R}\eta \\ j\sqrt{1+R}\eta \end{pmatrix} \pm i \begin{pmatrix} \sqrt{R}i\eta \\ j\sqrt{1+R}i\eta \end{pmatrix}$ and range(R) is dense). The other equation implies that the imaginary part of the inner product of ξ with any element in L is 0, which means ξ is of the form $\begin{pmatrix} \sqrt{1+R}\eta' \\ j\sqrt{R}\eta' \end{pmatrix}$ for some $\eta' \in \mathsf{k}$. □

Definition 6.2. A continuous bijective real linear operator $Z : H \mapsto H$ is said to be a symplectic automorphism if $\mathrm{Im}\langle Zf, Zg\rangle = \mathrm{Im}\langle f, g\rangle$ for all $f, g \in H$.

Proposition 6.3. *If $R_1 \neq ZR_2Z^*$ for any symplectic automorphism Z, then the CCR flow corresponding to $A_1 = I \otimes R_1$ is not cocycle conjugate to the CCR flow corresponding to $A_2 = I \otimes R_2$.*

Proof. Let α^1 and α^2 be the CCR flows corresponding to R_1 and R_2 acting standardly on M_1 and M_2 respectively. Suppose that there exists a unitary V and an α^1-cocycle $(W_t)_{t\geq 0}$ implementing cocycle conjugacy so that $\mathrm{Ad}_V W_t \alpha^1_t \mathrm{Ad}_{V^*} = \alpha^2_t$ for all $t \geq 0$. Recall the algebras $\mathcal{A}_{\alpha^i}(t) = \alpha^i_t(\mathrm{M}_i)' \cap \mathrm{M}_i$, $i = 1, 2$ defined at the end of Section 2. Note that the isomorphism $\phi_t = \mathrm{Ad}_{VW_t} : \mathcal{A}_{\alpha^1}(t) \to \mathcal{A}_{\alpha^2}(t)$ is strongly continuous. For $i = 1, 2$ consider the topological group

$$G_t(\alpha^i) := \{(u_s)_{s\in[0,t]} : (u_s)_{s\geq 0} \in G(\alpha^i)\},$$

which is canonically isomorphic to the gauge group. Then the map $(u_s)_{s\in[0,t]} \to (\phi_t(u_s))_{s\in[0,t]}$ induces an isomorphism $G_t(\alpha^1) \to G_t(\alpha^2)$. Indeed, the only nonobvious aspect is to check that $W_t u_s W_t^* = W_s u_s W_s^*$ for each $u \in G_t(\alpha^1)$ and $s \in [0, t]$, which follows from the cocycle property and the fact that $u_s \in \alpha^1_s(\mathrm{M}_1)'$. Denote $c \otimes 1_{(0,t)}$ by $c_{t]}$ for any $c \in \mathsf{k}$, $t \in \mathbb{R}_+$. Since ϕ_t is linear and strongly continuous, there exist continuous maps $\varphi : \mathsf{k} \to \mathbb{R}$ and $Z : \mathsf{k} \to \mathsf{k}$ satisfying $\phi_t(W_{A_1}(c_{s]})) = e^{is\varphi(c)} W_{A_2}(Z(c)_{s]})$ for all $c \in \mathsf{k}$. These induce a group homomorphism, so we must have

$$\begin{aligned} &e^{is(\varphi(c)+\varphi(d)-\mathrm{Im}\langle Z(c),Z(d)\rangle)} W_{A_2}(Z(c)_{s]} + Z(d)_{s]}) \\ &\quad = e^{is(\varphi(c+d)-\mathrm{Im}\langle c,d\rangle)} W_{A_2}(Z(c+d)_{s]}); \end{aligned}$$

hence

$$\varphi(c) + \varphi(d) - \mathrm{Im}\langle Z(c), Z(d)\rangle = \varphi(c + d) - \mathrm{Im}\langle c, d\rangle$$

for all c, $d \in \mathsf{k}$. The imaginary parts of the inner products are antisymmetric under an exchange of c and d, whereas the other terms are clearly symmetric; thus it follows that Z is a symplectic automorphism and f is a real linear functional.

By the Riesz representation theorem there exists $x \in \mathsf{k}$ with $\varphi(c) = \operatorname{Re}\langle x, c\rangle$ for all $c \in \mathsf{k}$, and we can form a functional Ψ on $L^2([0,t];\mathsf{k})$ by setting $\Psi(f) := \operatorname{Re}\langle 1_{[0,t]} \otimes x, f\rangle$. Since $W_{A_j}(c_{[r,s]}) = W_{A_j}(-c_{r]})W_{A_j}(c_{s]})$ for each $j = 1, 2$, $c \in \mathsf{k}$, $0 \le r \le s \le t$, then we have, for any step function $f \in L^2([0,t];\mathsf{k})$, that $\phi_t(W_{A_1}(f)) = e^{i\Psi(f)}W_{A_2}((I \otimes Z)f)$, by the homomorphism property of ϕ_t. Thus, if $f \in L^2([0,t];\mathsf{k})$ is the limit of a sequence of step functions $(f_n)_{n=1}^{\infty}$ then

$$\begin{aligned}\phi_t(W_{A_1}(f)) &= \operatorname*{s\text{-}lim}_{n\to\infty} \phi_t(W_{A_1}(f_n)) \\ &= \operatorname*{s\text{-}lim}_{n\to\infty} e^{i\Psi(f_n)}W_{A_2}((I \otimes Z)f_n) \\ &= e^{i\Psi(f)}W_{A_2}((I \otimes Z)f).\end{aligned}$$

Now using canonical commutation relations we get

$$\left(\operatorname{Ad}(W_{A_2}(-i(1_{[0,t]} \otimes x)/2))\phi_t\right)(W_{A_1}(f)) = W_{A_2}((I \otimes Z)f).$$

Since ϕ_t is normal, this implies that the representations of $\operatorname{CCR}(L^2([0,t];\mathsf{k}))$ given by

$$w_f \mapsto W_{A_1}(f) \quad \text{and} \quad w_f \mapsto W_{A_2}((I \otimes Z)f) \quad (f \in L^2([0,t];\mathsf{k}))$$

are quasi-equivalent. In particular, the restriction of φ_{A_1} to $\operatorname{CCR}(L^2([0,t];\mathsf{k}))$ is quasi-equivalent to the state

$$\operatorname{CCR}(L^2([0,t];\mathsf{k})) \ni w_f \mapsto \langle \Omega, W_{A_2}((I \otimes Z)f)\Omega\rangle = e^{-\frac{1}{2}\operatorname{Re}\langle f,(I\otimes Z^*)A_2(I\otimes Z)f\rangle}.$$

Thus, by [ArY] it must be the case that

$$\sqrt{I_{L^2([0,t])} \otimes R_1} - \sqrt{(I_{L^2([0,t])} \otimes Z^*)(I_{L^2([0,t])} \otimes R_2)(I_{L^2([0,t])} \otimes Z)}$$

is Hilbert–Schmidt. But $L^2([0,t])$ is infinite-dimensional, so we must have $\sqrt{R_1} = \sqrt{Z^* R_2 Z}$, i.e.,

$$R_1 = Z^* R_2 Z,$$

as required. □

This condition suggests that there should be a large number of distinct CCR flows on the hyperfinite III_λ factor, for each rank $n \ge 2$. To show this, we need to analyze the relation $R_1 = Z^* R_2 Z$ in more detail.

As a real Hilbert space, k is isomorphic to a direct sum $\mathsf{k}_{\mathbb{R}} \oplus \mathsf{k}_{\mathbb{R}}$ and under this identification, multiplication by i becomes multiplication by $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. Using this we see that a real linear operator $X = \left[\begin{smallmatrix} X_1 & X_2 \\ X_3 & X_4 \end{smallmatrix}\right] \in B(\mathsf{k}_{\mathbb{R}} \oplus \mathsf{k}_{\mathbb{R}})$ is complex linear if and only if $X_1 = X_4$ and $X_2 = -X_3$, and it is a positive complex linear operator if and only if it is of the form $X = \left[\begin{smallmatrix} X_1 & 0 \\ 0 & X_1 \end{smallmatrix}\right]$, for some positive operator X on $\mathsf{k}_{\mathbb{R}}$. In [Par, see Proposition 22.1] it is shown that for a symplectic automorphism Z there exist unitaries U_1, U_2 on k and a positive operator Z_1 on $\mathsf{k}_{\mathbb{R}}$ such that

$$U_1^* Z U_2^* = \begin{bmatrix} Z_1 & 0 \\ 0 & Z_1^{-1} \end{bmatrix}.$$

Setting $U_1^* R_2 U_1 = \left[\begin{smallmatrix} X & 0 \\ 0 & X \end{smallmatrix}\right]$ and $U_2 R_1 U_2^* = \left[\begin{smallmatrix} Y & 0 \\ 0 & Y \end{smallmatrix}\right]$ we obtain

$$\begin{bmatrix} Y & 0 \\ 0 & Y \end{bmatrix} = \begin{bmatrix} Z_1 & 0 \\ 0 & Z_1^{-1} \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} Z_1 & 0 \\ 0 & Z_1^{-1} \end{bmatrix},$$

i.e., $Y = Z_1 X Z_1$ and $Y = Z_1^{-1} X Z_1^{-1}$, which leads to

$$Z_1^2 X Z_1^2 = X \quad \text{and} \quad Z_1^2 Y Z_1^2 = Y. \tag{7}$$

To analyze these conditions we use the following proposition.

Proposition 6.4. *Suppose B, $R \in B(\mathsf{k})$ are positive invertible operators and $R \geq 1$. If $BRB = R$ then $B = 1$.*

Proof. The assumption implies $R^{1/2} B R^{-1/2}$ is a unitary and it has the same spectrum as B, which is positive. Hence $R^{1/2} B R^{-1/2} = 1$ and hence $B = 1$. □

Now we are able to give a complete classification of CCR flows when $R - 1$ is injective.

Theorem 6.5. *Let R_1, $R_2 \geq 1$ be bounded operators with $R_1 - 1$ and $R_2 - 1$ injective. The CCR flows $\alpha^{(R_1)}$ and $\alpha^{(R_2)}$ are cocycle conjugate if and only if there exists a unitary U such that $R_1 = U R_2 U^*$. When this is true, $\alpha^{(R_1)}$ is conjugate to $\alpha^{(R_2)}$.*

Proof. If R_1 and R_2 give cocycle-conjugate E$_0$-semigroups, thanks to Proposition 6.3, $R_1 = Z^* R_2 Z$ for some symplectic automorphism $Z = U_1 \left[\begin{smallmatrix} Z_1 & 0 \\ 0 & Z_1^{-1} \end{smallmatrix}\right] U_2$, where Z_1 is a positive operator on $\mathsf{k}_{\mathbb{R}}$. As before, if we set $U_1^* R_2 U_1 = \left[\begin{smallmatrix} X & 0 \\ 0 & X \end{smallmatrix}\right]$ and $U_2 R_1 U_2^* = \left[\begin{smallmatrix} Y & 0 \\ 0 & Y \end{smallmatrix}\right]$ then $Z_1^2 X Z_1^2 = X$ and $Z_1^2 Y Z_1^2 = Y$, so by Proposition 6.4, $Z_1^2 = 1$. Since Z_1 is positive it follows $Z_1 = 1$.

Conversely, suppose that there exists a unitary U such that $R_1 = U R_2 U^*$ and let $A_j = I \otimes R_j$ for $j = 1$, 2. Then the quasi-free states given by A_1 and $(I \otimes U^*) A_2 (I \otimes U)$ are quasi-equivalent; indeed they are same states. This implies that the representations of $\mathrm{CCR}(L^2([0,t]; \mathsf{k}))$ given by

$$w_f \mapsto W_{A_1}(f) \quad \text{and} \quad w_f \mapsto W_{A_2}((I \otimes U^*) f) \quad (f \in L^2([0,t]; \mathsf{k}))$$

are quasi-equivalent. Let $\theta : \mathrm{M}_{A_1} \mapsto \mathrm{M}_{A_2}$ be the isomorphism satisfying

$$\theta(W_{A_1}(f)) = W_{A_2}((I \otimes U^*)f);$$

then, since $(I \otimes U)$ commutes with T_t, we have

$$\begin{aligned}\left(\theta\alpha_t^{R_1}\theta^{-1}\right)(W_{A_2}(f)) &= W_{A_2}((I \otimes U^*)T_t(I \otimes U)f) \\ &= W_{A_2}(T_t f) = \alpha_t^{R_2}(W_{A_2}(f)).\end{aligned}$$

□

Remarks 6.6. (1) If $0 < \lambda < 1$ then there exists exactly one rank 1 CCR flow on the hyperfinite III_λ factor.

If $n \geq 2$ then there exist a countable infinity of non-cocycle-conjugate CCR flows on the hyperfinite III_λ factor with rank n. These are given, for instance, by choosing natural numbers $1 = d_1 \leq \cdots \leq d_n$ and then

$$T(1+T)^{-1} = I \otimes \mathrm{diag}(\lambda^{d_1}, \ldots, \lambda^{d_n}),$$

so that the quasi-free representation corresponding to

$$R = \mathrm{diag}\left(\frac{1+\lambda^{d_1}}{1-\lambda^{d_1}}, \ldots, \frac{1+\lambda^{d_n}}{1-\lambda^{d_n}}\right)$$

generates a hyperfine III_λ factor. Each distinct choice of d_i gives different eigenvalues for R by injectivity of the map $[0,1) \to \mathbb{R}_+$, $x \mapsto (1+x)/(1-x)$.

Using a similar argument we see that there exist uncountably many CCR flows of infinite rank on the hyperfinite III_λ factor, one for each distinct sequence of integers $1, d_1, d_2, \ldots$ up to permutations. (To see that this collection is uncountable, note that every strictly increasing sequence gives a different example.)

(2) The hyperfinite III_1 factor admits no CCR flows of rank 1. For any rank $n \geq 2$, the hyperfinite type III_1 factor admits uncountably many non-cocycle-conjugate CCR flows. For n finite this is seen by noting that each distinct sequence of numbers $\lambda_1 \leq \cdots \leq \lambda_n$ in $(0,1)$, for which at least one pair (λ_i, λ_j) satisfies

$$\log(\lambda_i)/\log(\lambda_j) \notin \mathbb{Q},$$

defines a CCR flow on the hyperfinite III_1 factor with

$$R = \mathrm{diag}\left(\frac{1+\lambda_1}{1-\lambda_1}, \ldots, \frac{1+\lambda_n}{1-\lambda_n}\right).$$

When $n = \infty$ there exist further examples, as indicated by Remark 5.7.

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